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New sets of spectral invariants for electro-elastic bodies with one and two families of fibres

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ABSTRACT

New sets of spectral invariants are proposed to study the behaviour of electro-elastic bodies composed of a matrix filled with one or two (in general non-orthogonal) families of 'fibres', where both the matrix and the fibres can react to the presence of electric fields.

1. Introduction

The interaction of electromagnetic fields with continua has attracted the attention of different researchers for many years. The interested reader can see, for example, the monograph by Maugin (1988), the book by Eringen and Maugin (1990) (and the references mentioned therein), Pao (1978), Hutter et al. (2006) and also §3a of Bustamante and Rajagopal (2013) and §3 of Bustamante and Rajagopal (2015) for short summaries of some older works on electro- and magneto-elasticity.

Many new works on the mathematical modelling of some classes of electro-active polymers[1] have appeared in the literature in the last 15 years, see, for example Dorfmann and Ogden (2005, 2006), Goulbourne et al. (2005), McMeeking and Landis (2005), Vu et al. (2007) and Ogden and Steigmann (2011). Some of such polymers can deform due to the application of an external electric field, because of the presence of electro-active particles inside them that are added during the curing process of the rubber-like material (Bossis et al., 2001). Some polymers can deform due to the particular chemical composition of the polymer (see, for example Nam et al. (2005)), and some other types of polymers deform due to the electric forces that are produced between two very thin electrodes paint on the surfaces of a (in general electrically neutral) slab rubber-like material (Pelrine et al., 2000). If an electric field is applied during the curing process of an electro-active polymer, which is produced by mixing a rubber-like material with electro-active particles, then the particles are aligned in the direction of the field. It has been found experimentally that such elastomers present a stronger electrostiction effect than in the case when the particles are randomly distributed inside the rubber (Bossis et al., 2001). A randomly-distributed-particle material can be considered isotropic in the absence of an electric field. However, if an external electric field is applied, a preferred direction appears, and from the point of view of the mathematical modelling the material behaves in a similar way as a transversely isotropic body (Dorfmann and Ogden, 2005, 2006). In the case of a material, where the electro-active particles are aligned inside the rubber (after the curing process) the body behaves as a transversely isotropic material in the absence of an electric field. However if an electric field is applied, where in general the electric field may not be aligned with the particles, the electro-active material behaves very similar to as a body with two (in general non-orthogonal) families of 'fibres' (Bustamante, 2009b; Bustamante and Merodio, 2011).

Biological tissue can also react to the presence of an electric field; an important example is the heart tissue (see Göktepe and Kuhl (2010), Wong et al. (2011), Dal et al. (2013) and some of the references cited therein). In general such soft tissue can be considered as a matrix material filled with two 'families of fibres'. Due to the scope of this paper, the reaction of the this matrix material and/or the fibres to the presence of an electric field will not be discussed here.

The purpose of the this work is to present some constitutive equations considering some new sets of invariants for some of the

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[1] See, for example Bossis et al. (2001), Bar-Cohen (2002), Carpi et al. (2008), O'Halloran et al. (2008), Brochu and Pei (2010) and Varga et al. (2005), where experimental data and applications involving about such electro-active polymers are presented.

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materials discussed above. It is assumed that such materials behave as elastic bodies,[2] and that the stresses and one of the electric variables can be obtained from a total energy function[3] (Dorfmann and Ogden, 2005, 2006). For a nonlinear elastic body with one or two families of fibres interacting with an electric field, it is possible to show that such total energy function may depend on a large number of invariants. We note that many of the classical invariants of Spencer and Rivlin (Spencer, 1971; Zheng, 1994) do not have clear physical meanings; hence, they are not, in general, experimentally friendly. Invariants that facilitate an experiment are, generally, attractive. In view of this, in this communication set of experimentally friendly spectral invariants that have clear physical interpretations are proposed,[4] to characterize the constitutive equations of electro-elastic bodies. The work here is based on the papers of Shariff and co-workers (Shariff, 2011, 2013; Shariff and Bustamante, 2015; Bustamante and Shariff, 2015).

The paper is divided in the following sections: In §2 a short summary is presented considering the main elements of the theory of Dorfmann and Ogden (2005, 2006) for non-linear electro-elastic bodies. In §3 sets of invariants are proposed to study problems of nonlinear anisotropic elastic bodies, with one, two and three preferred directions. In §4 we study the case of an electro-elastic body with one family of 'fibres', presenting the full expressions for the stresses and the electric displacement, and solving some simple boundary value problems. In §5 and §6 a similar analysis is presented for the case of an electro-elastic body with two families of fibres, using the results presented in §3, where the families of fibres can be orthogonal (see §5), or non-orthogonal (see §6). In §7 some final comments and remarks are given.

2. Basic equations

In this sections a summary is presented of the main elements necessary for understanding the theory of nonlinear electro-elastic bodies developed by Dorfmann and Ogden (2005, 2006), which will be used as basis for our work.

2.1. Kinematics

Let $\mathscr{B}$ denotes the electro-elastic body; the position of a particle $X \in \mathscr{B}$ in the un-deformed and unstressed reference configuration $\mathscr{B}_r$ is denoted $\mathbf{X}$, where $\mathbf{X} = \boldsymbol{\kappa}_r(X)$. The position of the same particle in the deformed current configuration $\mathscr{B}_t$ at a time t is denoted $\mathbf{x}$, and we assume there exists a one-to-one mapping $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The deformation gradient, and the right and the left Cauchy-Green deformation tensors are defined, respectively, as:

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{F}}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{1}$$

where it is assumed that $J = \det\mathbf{F} > 0$, and $\mathbf{U}$, $\mathbf{V}$ are the right and left stretch tensors, respectively. More details about the kinematics of deforming bodies can be found, for example, in Chadwick (1999) and Truesdell and Toupin (1960). In the present work time effects are not considered.

2.2. Equations of electrostatics

If there is no interaction with magnetic fields and there is no distribution of free charges, then the simplified forms of the Maxwell equations are (see, for example Kovetz (2000)):

$$\mathrm{curl}\mathbf{E} = 0, \quad \mathrm{div}\mathbf{D} = 0, \tag{2}$$

where $\mathbf{E}$ is the electric field and $\mathbf{D}$ is the electric displacement. In vacuum $\mathbf{E}$ and $\mathbf{D}$ are related through Kovetz (2000)

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{3}$$

where ε_0 is the electric permittivity in vacuum. For condensed matter an extra field is needed, which is called the electric polarization $\mathbf{P}$, where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{4}$$

In the absence of surface electric charges $\mathbf{D}$, $\mathbf{E}$ must satisfy the continuity conditions

$$\mathbf{n} \cdot [\![\mathbf{D}]\!] = 0, \quad \mathbf{n} \times [\![\mathbf{E}]\!] = \mathbf{0}, \tag{5}$$

where $\mathbf{n}$ is the unit normal outward to $\partial\mathscr{B}_t$ (the surface of the body in the current configuration) and for a vector field $\mathbf{a}$ we define $[\![\mathbf{a}]\!] = \mathbf{a}^{\mathrm{o}} - \mathbf{a}^{\mathrm{i}}$, where $\mathbf{a}^{\mathrm{o}}$, $\mathbf{a}^{\mathrm{i}}$ correspond to the field $\mathbf{a}$ evaluated outside and inside the body near $\partial\mathscr{B}_t$, respectively.

2.3. Equation of motion

In the literature on electromagnetic interactions with continuum media, it is possible to see that there are different ways in order to write the equation of motion, see, for example Pao (1978), Hutter et al. (2006) and Bustamante et al. (2009). Different possible definitions for the stresses inside a continuum media interacting with electromagnetic fields have been defined as well (Bustamante et al., 2009). As shown, for example, in Bustamante et al. (2009) and Hutter et al. (2006), all these definitions and formulations are equivalent, if the comparison is made in terms of quantities that can be measured and that have physical meanings. In this work we use the formulation by Dorfmann and Ogden (2005, 2006) due to its clear mathematical simplicity. In that formulation the original equilibrium equation (considering no time effects) of a body interacting with electric fields is

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{f}_{\mathrm{e}} = \mathbf{0}, \tag{6}$$

where $\boldsymbol{\sigma}$ is a stress tensor field that in general is non-symmetric, and $\mathbf{f}_{\mathrm{e}} = (\mathrm{grad}\mathbf{E})^{\mathrm{T}}\mathbf{P}$ is the electric body force (for simplicity, we have assumed that there is no mechanical body force in the body). In Dorfmann and Ogden (2005, 2006) this equilibrium equation is rewritten as

$$\mathrm{div}\boldsymbol{\tau} = \mathbf{0}, \tag{7}$$

where $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\tau}_{\mathrm{m}}$ is known as the total stress tensor, and

[2] Regarding the assumption that the electro-elastic rubber behaves as an elastic body, it is necessary to point out that the presence of the electro-active particles inside the rubber can be related with the appearance of some inelastic phenomena (Coquelle and Bossis, 2006), but in this work we do not consider such problems. The case of the modelling of soft tissue as an elastic body is more complicated, since such tissue contains water, therefore it is expected the appearance of some dissipative phenomena (see, for example Rubin and Bodner (2002), Pioletti and Rakotomanana (2000), §10.1.1-§10.2.6 of Humphrey (2002) and some of the references cited therein), but for the sake of simplicity in the present work we assume that such material behaves approximately as an elastic body.

[3] In the present paper we follow the traditional approach of assuming the stresses as functions of the strains and other variables (Maugin, 1988; Toupin, 1956; Dorfmann and Ogden, 2005, 2006), but it is necessary to indicate that more general approaches have recently appeared in the literature, where the stresses and other variables of interest are assumed to satisfy some implicit constitutive relations (Bustamante and Rajagopal, 2013, 2015).

[4] See also Criscione (2003) about some criticism on the classical invariants by Rivlin and Spencer.

$$\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{D}\otimes\mathbf{E} - \frac{\varepsilon_0}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{I} \tag{8}$$

is the Maxwell stress tensor. The tensor $\boldsymbol{\tau}$ is symmetric and must satisfy the following continuity condition on $\partial\mathscr{B}_t$

$$\boldsymbol{\tau}\mathbf{n} = \boldsymbol{\tau}_{\mathrm{m}}\mathbf{n} + \mathbf{t}_{\mathrm{a}}, \tag{9}$$

where $\boldsymbol{\tau}_{\mathrm{m}}$ in this case is calculated in vacuum near $\partial\mathscr{B}_t$, and $\mathbf{t}_{\mathrm{a}}$ corresponds to the mechanical (non-electric) external traction. A detailed study on more realistic boundary conditions for problems in electro-elasticity can be found in Bustamante (2009a).

2.4. Lagrangian counterparts of the electric variables and the total stress tensor

Considering the global or integral forms of (2), it is possible to define the electric field $\mathbf{E}_{\mathrm{l}}$ and the electric displacement $\mathbf{D}_{\mathrm{l}}$ in the reference configuration as (for details, see, for example Dorfmann and Ogden (2005, 2006))

$$\mathbf{E}_{\mathrm{l}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{D}_{\mathrm{l}} = J\mathbf{F}^{-1}\mathbf{D}. \tag{10}$$

In a similar manner it is possible to define the total nominal stress tensor $\mathbf{T}$ as

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{11}$$

2.5. Constitutive equations

In the formulation of Dorfmann and Ogden (2005, 2006), it is assumed that the total nominal stress tensor $\mathbf{T}$ is a function of $\mathbf{F}$ and one of the electric variables[5] (in their Lagrangian forms) $\mathbf{E}_{\mathrm{l}}$ or $\mathbf{D}_{\mathrm{l}}$, i.e., $\mathbf{T} = \mathbf{F}(\mathbf{F}, \mathbf{E}_{\mathrm{l}})$ (or $\mathbf{T} = \mathbf{G}(\mathbf{F}, \mathbf{D}_{\mathrm{l}})$). In that formulation it is also assumed the existence of a total energy function[6] $\Omega = \Omega(\mathbf{F},\mathbf{E}_{\mathrm{l}})$ such that:

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{D}_{\mathrm{l}} = -\frac{\partial\Omega}{\partial\mathbf{E}_{\mathrm{l}}}. \tag{12}$$

Due to the requirement that Ω must be Galilean frame invariant, Ω must be a function of $\mathbf{C}$ instead of $\mathbf{F}$.

3. Some new invariants for a transversely isotropic body, a body with two-preferred-direction and a body with three-preferred direction elasticity. Elastic deformations without electric interactions

In the case of an electro sensitive (ES) body with a random distribution of particles we have that $\Omega = \Omega(\mathbf{C},\mathbf{E}_{\mathrm{l}})$, whereas if there is one preferred direction where the body behaves differently (a transversely isotropic ES elastomer), we have (see Bustamante (2009b)) $\Omega = \Omega(\mathbf{C},\mathbf{E}_{\mathrm{l}},\mathbf{a}_0)$, where $\mathbf{a}_0$ is the preferred direction. Finally, in the case of an ES body with two preferred directions $\mathbf{a}_0$, $\mathbf{b}_0$, we have that $\Omega = \Omega(\mathbf{C},\mathbf{E}_{\mathrm{l}},\mathbf{a}_0,\mathbf{b}_0)$. It is possible to observe that the invariants for these cases would be similar to the case of elastic bodies (purely elastic deformations with no electric interaction), with one, two and three preferred direction elasticity, respectively (taking note that an extra invariant will be needed since in general $|\mathbf{E}_{\mathrm{l}}| \neq 1$). Because of this, in the present section different sets of invariants are studied for such elastic bodies, with one, two and three preferred directions, in order to have a more general formulation that can be used in any other similar problem.

An elastic energy of a body (without electric interactions) can be written as

$$W_{\mathrm{C}}(\mathbf{C}) = W_{\mathrm{S}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \tag{13}$$

with the symmetrical property

$$W_{\mathrm{S}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = W_{\mathrm{S}}(\lambda_2, \lambda_1, \lambda_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) = W_{\mathrm{S}}(\lambda_3, \lambda_2, \lambda_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) = \text{etc}, \tag{14}$$

where λ_i and $\mathbf{e}_i$ are the principal values and their respective principal unitary directions of $\mathbf{U}$. It is very important to know that W_{S} should be independent of $\mathbf{e}_i$ and $\mathbf{e}_j$ when $\lambda_i = \lambda_j$, $i \neq j$ in order W_{S} to have a unique value, due to the non-unique values of $\mathbf{e}_i$ and $\mathbf{e}_j$ when $\lambda_i = \lambda_j$. Similarly, W_{S} should be independent of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ when $\lambda_1 = \lambda_2 = \lambda_3$. We define this independent property, the *P*-property. We note that all the strain energies given below satisfy the *P*-property.

3.1. An isotropic body

Just as an illustration of the methodology developed by Shariff (2008, 2011) to propose the new set of spectral invariants, we start with the well known case of an isotropic body, and thereafter we study the case of a transversely isotropic body.

W_{S} is an isotropic invariant function of $\mathbf{C}$ if and only if

$$W_{\mathrm{S}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = W_{\mathrm{S}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3) \tag{15}$$

for all proper orthogonal tensors $\mathbf{Q}$ (Spencer, 1971). Since the vector invariants $\mathbf{e}_i\cdot\mathbf{e}_j$ and $\mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k)$, $(i \neq j \neq k)$ (Spencer, 1971) have zero or unit values, we have, for isotropic elasticity that

$$W_{\mathrm{S}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = W_{\mathrm{I}}(\lambda_1, \lambda_2, \lambda_3) = W_{\mathrm{I}}(\lambda_2, \lambda_1, \lambda_3) = W_{\mathrm{I}}(\lambda_3, \lambda_2, \lambda_1) = \text{etc}. \tag{16}$$

3.2. A transversely isotropic body

In the case of a transversely isotropic elastic solid with the preferred direction $\beth$, we can write the strain energy function in the form

$$W_{\mathrm{T}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \beth) \tag{17}$$

with the symmetrical relations

$$W_{\mathrm{T}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \beth) = W_{\mathrm{T}}(\lambda_2, \lambda_1, \lambda_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \beth) = W_{\mathrm{T}}(\lambda_3, \lambda_2, \lambda_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1, \beth) = \text{etc}. \tag{18}$$

For a transversely isotropic elastic solid, W_{T} is an isotropic invariant of $\mathbf{C}$ and $\beth$, hence it is required that

$$W_{\mathrm{T}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \beth) = W_{\mathrm{T}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3, \mathbf{Q}\beth) \tag{19}$$

for all proper orthogonal tensors $\mathbf{Q}$.

The vector invariants required for W_S are[7]

[5] That electric variable is called the independent electric variable.

[6] In this paper for simplicity we assume that $\mathbf{E}_{\mathrm{l}}$ is the independent electric variable.

[7] In this communication all subscripts i, j and k take the values 1, 2 and 3, unless stated otherwise.

$$q_i = \boldsymbol{\beth}\cdot\mathbf{e}_i, \quad i = 1,2,3, \tag{20}$$

taking note that invariants of the form $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ ($\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors representing $\boldsymbol{\beth}$ and $\mathbf{e}_i$, $i = 1,2,3$) are single valued functions of $\boldsymbol{\beth}\cdot\mathbf{e}_i$ and hence we can eliminate them in W_T. Therefore, for a transversely isotropic elastic solid we have

$$W_T(\lambda_1,\lambda_2,\lambda_3,\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1,\boldsymbol{\beth}) = W_t(\lambda_1,\lambda_2,\lambda_3,q_1,q_2,q_3). \tag{21}$$

Let us define the invariants ζ_i as

$$\zeta_i = (\boldsymbol{\beth}\cdot\mathbf{e}_i)^2, \quad i = 1,2,3, \tag{22}$$

we have $\zeta_1+\zeta_2+\zeta_3 = 1$ and $\zeta_i = q_i^2$, hence only five of the invariants λ_i, q_i, $i = 1,2,3$ are independent. If the strain energy function is independent of the sign of $\mathbf{e}_i$ and $\boldsymbol{\beth}$, then it can be written in the form

$$W_t(\lambda_1,\lambda_2,\lambda_3,q_1,q_2,q_3) = \widehat{W}_t(\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\zeta_3) \tag{23}$$

with the symmetrical property

$$\begin{aligned}\widehat{W}_t(\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\zeta_3) &= \widehat{W}_t(\lambda_2,\lambda_1,\lambda_3,\zeta_2,\zeta_1,\zeta_3)\\ &= \widehat{W}_t(\lambda_3,\lambda_2,\lambda_1,\zeta_3,\zeta_2,\zeta_1) = \text{etc.}\end{aligned} \tag{24}$$

A strain energy of the form W_t has been discussed in Shariff (2008).

3.3. Two-preferred-direction elasticity

In this section a summary of some results shown in Shariff and Bustamante (2015) is presented, where an analysis of a body with two preferred direction elasticity is considered.

Let the two preferred directions be $\boldsymbol{\beth}$ and $\boldsymbol{\gimel}$. The strain energy function takes the form

$$W_F(\lambda_1,\lambda_2,\lambda_3,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\boldsymbol{\beth},\boldsymbol{\gimel}) \tag{25}$$

with symmetrical relations similar to (18). Using a similar analysis as in §3.2, the invariants required for the strain energy are λ_i, q_i and

$$r_i = \boldsymbol{\gimel}\cdot\mathbf{e}_i, \quad i = 1,2,3, \quad \kappa_1 = \boldsymbol{\beth}\cdot\boldsymbol{\gimel}. \tag{26}$$

Let us define

$$\xi_i = (\boldsymbol{\gimel}\cdot\mathbf{e}_i)^2, \quad \chi_i = (\boldsymbol{\beth}\cdot\boldsymbol{\gimel})(\boldsymbol{\beth}\cdot\mathbf{e}_i)(\boldsymbol{\gimel}\cdot\mathbf{e}_i) = \kappa_1 q_i r_i, \quad \text{no sum in } i. \tag{27}$$

We have that

$$\sum_{i=1}^{3}\zeta_i = 1, \quad \sum_{i=1}^{3}\xi_i = 1, \quad \sum_{i=1}^{3} q_i r_i = \kappa_1, \tag{28}$$

where $\xi_i = r_i^2$. From the above relations only six of the *strain* invariants λ_i, q_i, r_i, $i = 1,2,3$ are independent. One of the important results presented in Shariff and Bustamante (2015) was to discover that only 6 of the classical invariants listed (Spencer, 1971) for this problem are independent, and this is illustrated below. The classical invariants belonging to a minimal integrity basis (Spencer, 1971) for a two-preferred-direction elastic solids are (see (22), (27)):

$$I_1 = \mathrm{tr}(\mathbf{C}) = \sum_{i=1}^{3}\lambda_i^2, \quad I_2 = \frac{I_1^2 - \mathrm{tr}(\mathbf{C}^2)}{2} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{29}$$

$$I_3 = \det(\mathbf{C}) = (\lambda_1\lambda_2\lambda_3)^2, \quad I_4 = \boldsymbol{\beth}\cdot(\mathbf{C}\boldsymbol{\beth}) = \sum_{i=1}^{3}\lambda_i^2\zeta_i, \quad I_5 = \boldsymbol{\beth}\cdot\left(\mathbf{C}^2\boldsymbol{\beth}\right) = \sum_{i=1}^{3}\lambda_i^4\zeta_i, \tag{30}$$

$$I_6 = \boldsymbol{\gimel}\cdot(\mathbf{C}\boldsymbol{\gimel}) = \sum_{i=1}^{3}\lambda_i^2\xi_i, \quad I_7 = \boldsymbol{\gimel}\cdot\left(\mathbf{C}^2\boldsymbol{\gimel}\right) = \sum_{i=1}^{3}\lambda_i^4\xi_i, \quad I_8 = (\boldsymbol{\beth}\cdot\boldsymbol{\gimel})\boldsymbol{\beth}\cdot(\mathbf{C}\boldsymbol{\gimel}) = \sum_{i=1}^{3}\lambda_i^2\chi_i, \tag{31}$$

$$I_9 = (\boldsymbol{\beth}\cdot\boldsymbol{\gimel})^2, \quad I_{10} = (\boldsymbol{\beth}\cdot\boldsymbol{\gimel})\boldsymbol{\beth}\cdot\left(\mathbf{C}^2\boldsymbol{\gimel}\right) = \sum_{i=1}^{3}\lambda_i^4\chi_i. \tag{32}$$

Since six of the *strain* invariants λ_i, q_i, r_i, $i = 1,2,3$ are independent, we conclude that only six of the nine strain invariants I_j, $j = 1,2,\ldots,8$ and I_{10} are independent. In fact, Shariff and Bustamante (Shariff, 2008) have shown for $\alpha = \kappa_1^2 \neq 0$ without resorting to spectral invariants that there are three syzygies (see Section 3 of Shariff (2008)):

$$\begin{aligned}I_8^2 - I_9(1-I_9)I_2 &= 2I_1I_8I_9 - 2I_9I_{10} + I_9I_4I_6 - I_1I_9(I_4+I_6)\\ &\quad + I_9(I_5+I_7),\end{aligned} \tag{33}$$

$$\begin{aligned}&4I_{13}^2I_{23}^2I_{12}^2(I_4+I_{22})^2 = \\ &\left\{(I_{10}-I_9I_5)^2 - (1-I_9)I_9\left[I_{12}^2(I_4+I_{22})^2 + I_{13}^2I_{23}^2\right]\right\}^2,\end{aligned} \tag{34}$$

$$4I_{12}^2I_{13}^2I_{23}^2 = \left[I_3 + I_4\left(I_{23}^2 - I_{22}I_{33}\right) + I_{22}I_{13}^2 + I_{33}I_{12}^2\right]^2, \tag{35}$$

where

$$I_{12}^2 = \frac{I_8^2I_9^{-1} - 2I_8I_4 + I_4^2I_9}{1-I_9}, \quad I_{22} = \frac{I_6 - 2I_8 + I_9I_4}{1-I_9}, \tag{36}$$

$$I_{33} = I_1 - I_4 - \frac{I_6 - 2I_8 + I_9I_4}{1-I_9}, \tag{37}$$

$$I_{23}^2 = \frac{I_7 - 2I_{10} + I_9I_5}{1-I_9} - \left(\frac{I_6 - 2I_8 + I_9I_4}{1-I_9}\right)^2 - \frac{I_8^2I_9^{-1} - 2I_8I_4 + I_4^2I_9}{1-I_9}, \tag{38}$$

$$I_{13}^2 = I_5 - I_4^2 - \frac{I_8^2I_9^{-1} - 2I_8I_4 + I_4^2I_9}{1-I_9}. \tag{39}$$

This further validates that only six of the nine classical strain invariants are independent.

The invariants $(27)_2$ χ_i, $i = 1,2,3$ play an important role in the construction of the corresponding infinitesimal strain energy function (Shariff and Bustamante, 2015), hence we include them in the nonlinear strain energy function. Therefore, a two-preferred direction strain energy, independent of the signs of $\mathbf{e}_i$, $\boldsymbol{\beth}$ and $\boldsymbol{\gimel}$, can be expressed in the form

$$\begin{aligned}&\widehat{W}_f(\lambda_1,\lambda_2,\lambda_3,q_1,q_2,q_3,r_1,r_2,r_3,\kappa_1)\\ &= W_f\left(\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\zeta_3,\xi_1,\xi_2,\xi_3,\chi_1,\chi_2,\chi_3,\kappa_1^2\right),\end{aligned} \tag{40}$$

where W_f has the symmetrical property similar to (24).

When ב and נ are orthogonal, $\kappa_1 = ב \cdot נ = 0$ the symmetrical strain energy for an orthotropic elastic solid takes the form

$$W_{\mathrm{p}}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3). \tag{41}$$

The strain energy function of form (41) has been discussed in Shariff (2011).

3.4. Three-preferred-direction elasticity

In this last subsection, some new results are presented for the case of a body with three preferred directions. Let the three preferred directions be ב, נ and א. The strain energy function takes the form

$$W_{\mathrm{R}}(\lambda_1, \lambda_2, \lambda_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, ב, נ, א) \tag{42}$$

with symmetrical relations similar to (18). Using a similar analysis as presented in §3.2 and §3.3, the invariants required for the strain energy are λ_i, q_i, r_i, $i = 1,2,3$, κ_1 (see (26))

$$s_i = א \cdot \mathbf{e}_i, \quad i = 1,2,3, \quad \kappa_2 = ב \cdot א \tag{43}$$

and

$$\kappa_3 = נ \cdot א. \tag{44}$$

Let us define

$$\varrho_i = (א \cdot \mathbf{e}_i)^2, \quad \vartheta_i = (ב \cdot א)(ב \cdot \mathbf{e}_i)(א \cdot \mathbf{e}_i) = \kappa_2 q_i s_i, \tag{45}$$

$$\iota_i = (נ \cdot א)(נ \cdot \mathbf{e}_i)(א \cdot \mathbf{e}_i) = \kappa_3 r_i s_i, \quad i = 1,2,3. \tag{46}$$

The following connections hold:

$$\sum_{i=1}^{3} \zeta_i = 1, \quad \sum_{i=1}^{3} \xi_i = 1, \quad \sum_{i=1}^{3} \varrho_i = 1, \quad \sum_{i=1}^{3} q_i r_i = \kappa_1, \quad \sum_{i=1}^{3} q_i s_i = \kappa_2, \quad \sum_{i=1}^{3} r_i s_i = \kappa_3. \tag{47}$$

Hence only six of the twelve spectral *strain* invariants λ_i, q_i, r_i s_i, $i = 1,2,3$ are independent.

Twenty six classical invariants generated from the tensors $\mathbf{C}$, $ב \otimes ב$, $נ \otimes נ$ and $א \otimes א$ (Spencer, 1971) are required to form a minimal integrity basis. They are I_j, $j = 1,2,\ldots,10$ given in (29)–(32) and additionally

$$I_{11} = א \cdot (\mathbf{C}א) = \sum_{i=1}^{3} \varrho_i \lambda_i^2, \quad I_{12} = א \cdot \left(\mathbf{C}^2 א\right) = \sum_{i=1}^{3} \varrho_i \lambda_i^4, \tag{48}$$

$$I_{13} = (נ \cdot א)[נ \cdot (\mathbf{C}א)] = \kappa_3 \sum_{i=1}^{3} r_i s_i \lambda_i^2, \quad I_{14} = (ב \cdot א)[ב \cdot (\mathbf{C}א)] = \kappa_2 \sum_{i=1}^{3} q_i s_i \lambda_i^2, \tag{49}$$

$$I_{15} = (נ \cdot א)\left[נ \cdot \left(\mathbf{C}^2 א\right)\right] = \kappa_3 \sum_{i=1}^{3} r_i s_i \lambda_i^4, \quad I_{16} = (ב \cdot א)\left[ב \cdot \left(\mathbf{C}^2 א\right)\right] = \kappa_2 \sum_{i=1}^{3} q_i s_i \lambda_i^4, \tag{50}$$

$$I_{17} = (ב \cdot א)(ב \cdot נ)[נ \cdot (\mathbf{C}א)] = \kappa_1 \kappa_2 \sum_{i=1}^{3} r_i s_i \lambda_i^2, \tag{51}$$

$$I_{18} = (נ \cdot א)(ב \cdot נ)[ב \cdot (\mathbf{C}א)] = \kappa_1 \kappa_3 \sum_{i=1}^{3} q_i s_i \lambda_i^2, \tag{52}$$

$$I_{19} = (ב \cdot א)(נ \cdot א)[ב \cdot (\mathbf{C}נ)] = \kappa_2 \kappa_3 \sum_{i=1}^{3} q_i r_i \lambda_i^2, \tag{53}$$

$$I_{20} = (ב \cdot א)(ב \cdot נ)\left[נ \cdot \left(\mathbf{C}^2 א\right)\right] = \kappa_1 \kappa_2 \sum_{i=1}^{3} r_i s_i \lambda_i^4, \tag{54}$$

$$I_{21} = (נ \cdot א)(ב \cdot נ)\left[ב \cdot \left(\mathbf{C}^2 א\right)\right] = \kappa_1 \kappa_3 \sum_{i=1}^{3} q_i s_i \lambda_i^4, \tag{55}$$

$$I_{22} = (ב \cdot א)(נ \cdot א)\left[ב \cdot \left(\mathbf{C}^2 נ\right)\right] = \kappa_2 \kappa_3 \sum_{i=1}^{3} q_i r_i \lambda_i^4, \tag{56}$$

$$I_{23} = (ב \cdot נ)(ב \cdot \mathbf{C}א)\left[נ \cdot \left(\mathbf{C}^2 א\right)\right] = \kappa_1 \left(\sum_{i=1}^{3} q_i s_i \lambda_i^2\right)\left(\sum_{i=1}^{3} r_i s_i \lambda_i^4\right), \tag{57}$$

$$I_{24} = (ב \cdot א)^2 = \kappa_2^2, \quad I_{25} = (נ \cdot א)^2 = \kappa_3^2, \quad I_{26} = (ב \cdot נ)(ב \cdot א)(נ \cdot א) = \kappa_1 \kappa_2 \kappa_3. \tag{58}$$

We note that I_9, I_{24}, I_{25} and I_{26} are non-strain invariants. The number of independent spectral invariants suggests that only six of the twenty two classical strain invariants I_j, $j = 1,2,\ldots,8$, I_k, $k = 10,11,\ldots,23$ are independent. Similar to the work of Shariff and Bustamante (2015), we could construct sixteen independent relations among the classical invariants, but we will not do it here, because it is not the intention of this paper.

In (59) and (60), we give a list of spectral invariants that are independent of the signs of $\mathbf{e}_i$, ב, נ and א, which are required to characterise a three-preferred-direction strain energy function. This list, may not be a complete list to describe the general infinitesimal strain energy function when specialised to infinitesimal elasticity,[8] but this list is sufficient for this paper. Additional set of non-independent spectral invariants can be easily constructed if required.

Strain invariants

$$\lambda_i, \quad \zeta_i, \quad \xi_i, \quad \varrho_i, \quad \chi_i, \quad \vartheta_i, \quad \iota_i, \quad i = 1,2,3. \tag{59}$$

Non-strain invariants

$$\kappa_1^2, \quad \kappa_2^2, \quad \kappa_3^2. \tag{60}$$

[8] See for example Shariff and Bustamante (2015) on how to construct a complete list for two-preferred direction elastic solids.

Some non-independent invariants

Below there is a list of possible invariants which will not be included in the strain energy function.

$$\kappa_1\kappa_2 r_i s_i, \quad \kappa_1\kappa_3 q_i s_i, \quad \kappa_2\kappa_3 q_i r_i. \tag{61}$$

In the case of a function that depends on 4 second order tensors namely $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$, from Spencer (1971) the following invariant is considered as well

$$\mathrm{tr}(\boldsymbol{\Gamma}\boldsymbol{\Delta}\boldsymbol{\Lambda}\boldsymbol{\Xi}), \tag{62}$$

and some permutations of the above such as $\mathrm{tr}(\boldsymbol{\Delta}\boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Xi})$, $\mathrm{tr}(\boldsymbol{\Gamma}\boldsymbol{\Delta}\boldsymbol{\Xi}\boldsymbol{\Lambda})$, etc. In the special situation in which $\boldsymbol{\Gamma} = \mathbf{e}_i\otimes\mathbf{e}_i$ (no sum in i), $\Delta = \boldsymbol{\beth}\otimes\boldsymbol{\beth}$, $\boldsymbol{\Lambda} = \boldsymbol{\gimel}\otimes\boldsymbol{\gimel}$ and $\boldsymbol{\Xi} = \boldsymbol{\aleph}\otimes\boldsymbol{\aleph}$ we have the same non-independent invariants presented in (61).

In some problems the preferred directions $\boldsymbol{\gimel}$ and $\boldsymbol{\aleph}$ are orthogonal, and in these cases we have a reduced number of invariants:

Strain invariants

$$\lambda_i, \quad \zeta_i, \quad \xi_i, \quad \varrho_i, \quad \chi_i, \quad \vartheta_i, \quad i = 1,2,3. \tag{63}$$

Non-strain invariants

$$\kappa_1^2, \quad \kappa_2^2. \tag{64}$$

4. An electro-elastic body with one family of fibres

In this section we propose a new set of spectral invariants for an electro-elastic body with one family of 'fibres', and thereafter we obtain the explicit expressions for the total stress $\boldsymbol{\tau}$ and the electric displacement $\mathbf{D}$. The fibres may or may not be electro-active. These 'fibres' could also be used to represent, for example, a family of chains of particles in the case of a rubber-like material filled with electro-active particles, which are aligned (in the undeformed configuration) in a preferred direction (Bustamante, 2009b). In this section we consider the most general case, where both the matrix and the fibres can react to the presence of an electric field. Let $\mathbf{a}_0$ denote the direction of the fibres in the reference undeformed configuration, in this problem we have (see (12)):

$$\Omega = \Omega(\mathbf{C}, \mathbf{E}_l, \mathbf{a}_0). \tag{65}$$

4.1. Constitutive equation

Let us define the unit vector field $\mathbf{f}$ as

$$\mathbf{f} = \frac{1}{E}\mathbf{E}_l, \quad \text{where} \quad E = |\mathbf{E}_l| \neq 0. \tag{66}$$

The relation (65) can be expressed as

$$\Omega = \widehat{W}(\mathbf{C}, \mathbf{f}, \mathbf{a}_0, E). \tag{67}$$

In view of (40) in §3.3 letting $\boldsymbol{\beth} = \mathbf{f}$ and $\boldsymbol{\gimel} = \mathbf{a}_0$, it follows that Ω can be expressed as

$$\Omega = \widehat{\Omega}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \chi_1, \chi_2, \chi_3, \alpha, E), \tag{68}$$

where in this case (see (22), (27))

$$\zeta_i = (\mathbf{f}\cdot\mathbf{e}_i)^2, \quad \xi_i = (\mathbf{a}_0\cdot\mathbf{e}_i)^2, \quad \alpha = \kappa_1^2 = (\mathbf{f}\cdot\mathbf{a}_0)^2, \\ \chi_i = (\mathbf{f}\cdot\mathbf{a}_0)(\mathbf{f}\cdot\mathbf{e}_i)(\mathbf{a}_0\cdot\mathbf{e}_i), \tag{69}$$

where there is no sum in i. For a compressible body from (12), $(10)_2$, (11) we have

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} = 2J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}, \tag{70}$$

$$\mathbf{D} = -J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_l}, \tag{71}$$

whereas for an incompressible body from (70), (71) we have

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{I} = 2\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}} - p\mathbf{I}, \tag{72}$$

$$\mathbf{D} = -\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_l}. \tag{73}$$

The Lagrangian spectral component relations of $\mathbf{C}$ taking into account (68), (69) (see, for example Ogden (1997)) are

$$\left(\frac{\partial\Omega}{\partial\mathbf{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial\widehat{\Omega}}{\partial\lambda_i}, \quad (i \quad \text{not summed}), \tag{74}$$

and the shear components are

$$\left(\frac{\partial\Omega}{\partial\mathbf{C}}\right)_{ij} = \frac{1}{\left(\lambda_i^2 - \lambda_j^2\right)}\left\{\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i} - \frac{\partial\widehat{\Omega}}{\partial\zeta_j}\right)\mathbf{e}_i\cdot(\mathbf{H}\mathbf{e}_j) + \left(\frac{\partial\widehat{\Omega}}{\partial\xi_i} - \frac{\partial\widehat{\Omega}}{\partial\xi_j}\right)\mathbf{e}_i\cdot(\mathbf{A}_0\mathbf{e}_j) + \left(\frac{\partial\widehat{\Omega}}{\partial\chi_i} - \frac{\partial\widehat{\Omega}}{\partial\chi_j}\right)[\mathbf{e}_i\cdot(\mathbf{H}\mathbf{A}_0\mathbf{e}_j) + \mathbf{e}_j\cdot(\mathbf{H}\mathbf{A}_0\mathbf{e}_i)]\right\}, \quad i \neq j, \tag{75}$$

where we have defined the second order tensor

$$\mathbf{A}_0 = \mathbf{a}_0\otimes\mathbf{a}_0, \quad \mathbf{H} = \mathbf{f}\otimes\mathbf{f}. \tag{76}$$

In view of (70), the Eulerian spectral components of $\boldsymbol{\tau}$ for a compressible solid are:

$$\tau_{ii} = \frac{\lambda_i}{J}\frac{\partial\widehat{\Omega}}{\partial\lambda_i}, \quad (i \quad \text{not summed}), \tag{77}$$

$$\tau_{ij} = \frac{2\lambda_i\lambda_j}{J\left(\lambda_i^2 - \lambda_j^2\right)}\left[\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i} - \frac{\partial\widehat{\Omega}}{\partial\zeta_j}\right)\mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j + \left(\frac{\partial\widehat{\Omega}}{\partial\xi_i} - \frac{\partial\widehat{\Omega}}{\partial\xi_j}\right)\mathbf{e}_i\cdot\mathbf{A}_0\mathbf{e}_j + \left(\frac{\partial\widehat{\Omega}}{\partial\chi_i} - \frac{\partial\widehat{\Omega}}{\partial\chi_j}\right)(\mathbf{e}_i\cdot\mathbf{H}\mathbf{A}_0\mathbf{e}_j + \mathbf{e}_j\cdot\mathbf{H}\mathbf{A}_0\mathbf{e}_i)\right]\cdot \quad i \neq j. \tag{78}$$

For an incompressible body we have:

$$\tau_{ii} = \lambda_i\frac{\partial\widehat{\Omega}}{\partial\lambda_i} - p \quad (i \quad \text{not summed}), \tag{79}$$

$$\tau_{ij} = \frac{2\lambda_i\lambda_j}{\left(\lambda_i^2 - \lambda_j^2\right)}\left[\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i} - \frac{\partial\widehat{\Omega}}{\partial\zeta_j}\right)\mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j + \left(\frac{\partial\widehat{\Omega}}{\partial\xi_i} - \frac{\partial\widehat{\Omega}}{\partial\xi_j}\right)\mathbf{e}_i\cdot\mathbf{A}_0\mathbf{e}_j + \left(\frac{\partial\widehat{\Omega}}{\partial\chi_i} - \frac{\partial\widehat{\Omega}}{\partial\chi_j}\right)(\mathbf{e}_i\cdot\mathbf{H}\mathbf{A}_0\mathbf{e}_j + \mathbf{e}_j\cdot\mathbf{H}\mathbf{A}_0\mathbf{e}_i)\right], \quad i \neq j. \tag{80}$$

The spectral expression for the Lagrangian electric displacement (12)$_2$ $\mathbf{D}_l$ is:

$$\mathbf{D}_l = -\frac{\partial\Omega}{\partial\mathbf{E}_l} = -\sum_{k=1}^{3} D_k\mathbf{e}_k, \tag{81}$$

where (see (68), (69))

$$D_k = (\mathbf{f}\cdot\mathbf{e}_k)\left[\frac{\partial\widehat{\Omega}}{\partial E} + \frac{2}{E}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_k} - \sum_{i=1}^{3}\frac{\partial\widehat{\Omega}}{\partial\zeta_i}\zeta_i - \sum_{i=1}^{3}\frac{\partial\widehat{\Omega}}{\partial\chi_i}\chi_i - \alpha\frac{\partial\widehat{\Omega}}{\partial\alpha}\right)\right] + \frac{(\mathbf{a}_0\cdot\mathbf{e}_k)}{E}\left[\sum_{i=1}^{3}(\mathbf{f}\cdot\mathbf{e}_i)(\mathbf{a}_0\cdot\mathbf{e}_i)\frac{\partial\widehat{\Omega}}{\partial\chi_i} + (\mathbf{f}\cdot\mathbf{a}_0)\frac{\partial\widehat{\Omega}}{\partial\chi_k} + 2(\mathbf{f}\cdot\mathbf{a}_0)\frac{\partial\widehat{\Omega}}{\partial\alpha}\right]. \tag{82}$$

The electric displacement in the deformed configuration can simply be expressed by

$$\mathbf{D} = -\sum_{k=1}^{3}\lambda_k D_k\mathbf{v}_k, \tag{83}$$

where $\mathbf{v}_k$ is the principal direction of the left stretch tensor $\mathbf{V}$.

4.2. Solutions of some boundary value problems

In this section, two simple problems, the uniform extension of a cylinder and the shear of a slab are considered.

4.2.1. Uniform extension of a cylinder

In this first problem, we study the behaviour of a cylinder described in the reference configuration as

$$0 \le R \le R_o, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{84}$$

which deforms as

$$r = \lambda_r R, \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{85}$$

where λ_r, λ_z are constants. We have

$$\mathbf{F} = \begin{pmatrix}\lambda_r & 0 & 0\\ 0 & \lambda_r & 0\\ 0 & 0 & \lambda_z\end{pmatrix}. \tag{86}$$

This simple problem is important since it is one of the simplest experiments, which can be considered in order to obtain actual expressions for the function Ω. The cylinder deforms under the influence of an external mechanical force applied on the surface $z=L$. On the surface $R=R_o$, the cylinder is in contact with vacuum. It is assumed that $\mathbf{a}_0=(0{,}0{,}1)^{\mathrm{T}}$, i.e., the fibres or particles are aligned in the axial direction (in the reference configuration). It is assumed that $\mathbf{E}_l=(0{,}0{,}E_0)^{\mathrm{T}}$, where E_0 is a constant. Considering the above assumptions, from (69), (10)$_1$ we have

$$\zeta_1=\zeta_2=\xi_1=\xi_2=\chi_1=\chi_2=0, \quad \zeta_3=\xi_3=\chi_3=1, \quad \mathbf{E}=\begin{pmatrix}0\\0\\ \frac{E_0}{\lambda_z}\end{pmatrix}. \tag{87}$$

4.2.1.1. Compressible electro-elastic bodies. In this case $J=\lambda_r^2\lambda_z$, identifying $\lambda_1=\lambda_r$, $\lambda_2=\lambda_r$ and $\lambda_3=\lambda_z$ from (77), (78), we have

$$\tau_{rr} = \frac{\lambda_r}{J}\frac{\partial\widehat{\Omega}}{\partial\lambda_1}, \quad \tau_{\theta\theta} = \frac{\lambda_r}{J}\frac{\partial\widehat{\Omega}}{\partial\lambda_2}, \quad \tau_{zz} = \frac{\lambda_z}{J}\frac{\partial\widehat{\Omega}}{\partial\lambda_3}, \tag{88}$$

where the rest of the components of the total stress tensor are zero. From (8), (87) the non-zero components of the Maxwell stress in vacuo are:

$$\tau_{m_{zz}} = \frac{\varepsilon_0 E_0^2}{2\lambda_z^2} = -\tau_{m_{rr}} = -\tau_{m_{\theta\theta}}. \tag{89}$$

If no mechanical stress is applied on the cylindrical surface $R=R_o$ from (9) and (89) we have[9]

$$\tau_{rr} = -\frac{\varepsilon_0 E_0^2}{2\lambda_z^2}, \tag{90}$$

which in view of (88) implies that

$$\frac{1}{\lambda_r\lambda_z}\frac{\partial\widehat{\Omega}}{\partial\lambda_1} = -\frac{\varepsilon_0 E_0^2}{2\lambda_z^2}, \quad R=R_o. \tag{91}$$

Eq. (91) can be used to obtain, for example, the value of λ_r in terms of λ_z. Due to the symmetrical property we have

$$\frac{\partial\widehat{\Omega}}{\partial\lambda_1}(\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\zeta_3,\xi_1,\xi_2,\xi_3,\chi_1,\chi_2,\chi_3,\alpha) = \frac{\partial\widehat{\Omega}}{\partial\lambda_2}(\lambda_2,\lambda_1,\lambda_3,\zeta_2,\zeta_1,\zeta_3,\xi_2,\xi_1,\xi_3,\chi_2,\chi_1,\chi_3,\alpha), \tag{92}$$

and in view that $\lambda_1=\lambda_2=\lambda_r$, we have the relation

$$\tau_{rr} = \tau_{\theta\theta}. \tag{93}$$

Finally, from (82) the spectral components for the electric field are:

$$D_1 = D_2 = 0, \quad D_3 = \frac{\partial\widehat{\Omega}}{\partial E} + \frac{2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_3} - \frac{\partial\widehat{\Omega}}{\partial\xi_3}\right). \tag{94}$$

Since $\boldsymbol{\tau}$, $\mathbf{D}$ and $\mathbf{E}_l$ are constant, all the balance Eqs. (2) and (7) are satisfied trivially.

If $R_o \ll L$ then from (5), (87) we have that $\mathbf{E}^o$ (the external field in vacuum) must be of the form $\mathbf{E}^o = \begin{pmatrix}0\\0\\ \frac{E_0}{\lambda_z}\end{pmatrix}$.

4.2.1.2. Incompressible electro-elastic bodies. In this case, $\lambda_r=\frac{1}{\sqrt{\lambda_z}}$ and the non-zero spectral components of the total stress tensor (79), (80) become

$$\tau_{rr} = \lambda_r\frac{\partial\widehat{\Omega}}{\partial\lambda_1} - p, \quad \tau_{\theta\theta} = \lambda_r\frac{\partial\widehat{\Omega}}{\partial\lambda_2} - p, \quad \tau_{zz} = \lambda_z\frac{\partial\widehat{\Omega}}{\partial\lambda_3} - p. \tag{95}$$

Using the same continuity condition for the surface $R=R_o$ as in the previous case, we have

[9] Regarding the continuity conditions (5), we assume that $L \gg R_o$ such that such conditions are considered (as an approximation) only for the surface $R=R_o$.

$$p = \lambda_r \frac{\partial \widehat{\Omega}}{\partial \lambda_1} + \frac{\varepsilon_0 E_0^2}{2\lambda_z^2}. \tag{96}$$

The spectral components of the electric field is given by (94).

4.2.2. Simple shear of a slab

In this problem we only study the case of an incompressible body. Let us consider the slab defined in the reference configuration as

$$-\frac{L_i}{2} \le X_i \le \frac{L_i}{2}, \quad i = 1,2,3, \tag{97}$$

which deforms as

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{98}$$

where $0 \le \gamma$ is commonly called *the amount of shear*. Let θ denote the orientation (in the anticlockwise sense relative to the X_1 axis) of the in plane Lagrangean principal axes. The angle θ is restricted as

$$\frac{\pi}{4} \le \theta < \frac{\pi}{2}. \tag{99}$$

The deformation tensor $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{100}$$

The principal directions $\mathbf{e}_i$, $i = 1,2,3$ are:

$$\mathbf{e}_1 = \begin{pmatrix} c \\ s \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -s \\ c \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{101}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. It can be easily shown that the principal stretches are

$$\lambda_1 = \frac{\gamma + \sqrt{\gamma^2 + 4}}{2} \ge 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2 + 4} - \gamma}{2} \le 1, \quad \lambda_3 = 1, \tag{102}$$

where we have the connections

$$c = \frac{1}{\sqrt{1 + \lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1 + \lambda_1^2}}, \quad c^2 - s^2 = -\gamma cs. \tag{103}$$

Without loss of generality, we consider $\tau_{33} = 0$ since incompressibility allows the superposition of an arbitrary spherical stress without effecting the deformation. In this section we assume $\mathbf{a}_0 = \begin{pmatrix} \bar{c} \\ \bar{s} \\ 0 \end{pmatrix}$ and $\mathbf{E}_l = \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}$, where $\bar{c} = \cos\psi$, $\bar{s} = \sin\psi$ and E_0 is constant. Following the work of Shariff and Bustamante (2015), from (80) we have

$$\tau_{11} = 2\Big[l_1 c^2 + l_2 s^2 - 2l_4 cs + 2\gamma\Big((l_1 - l_2)cs + l_4\Big(c^2 - s^2\Big)\Big) + \gamma^2\Big(l_1 s^2 + l_2 c^2 + 2l_4 cs\Big) - l_3\Big], \tag{104}$$

$$\tau_{22} = 2\Big(l_1 s^2 + l_2 c^2 + 2l_4 cs - l_3\Big), \quad \tau_{13} = \tau_{23} = 0, \tag{105}$$

$$\tau_{12} = 2\Big\{(l_1 - l_2)cs + l_4\Big(c^2 - s^2\Big) + \gamma\Big(l_1 s^2 + l_2 c^2 + 2l_4 cs\Big)\Big\}, \tag{106}$$

where

$$l_k = \frac{1}{2\lambda_k}\frac{\partial \widehat{\Omega}}{\partial \lambda_k}, \quad \text{no sum in } k, \tag{107}$$

and

$$l_4 = \left(\frac{\partial \Omega}{\partial \mathbf{C}}\right)_{12} = \frac{1}{\left(\lambda_1^2 - \lambda_2^2\right)}\left\{\left(\frac{\partial \widehat{\Omega}}{\partial \zeta_1} - \frac{\partial \widehat{\Omega}}{\partial \zeta_1}\right)sc + \left(\frac{\partial \widehat{\Omega}}{\partial \xi_1} - \frac{\partial \widehat{\Omega}}{\partial \xi_2}\right)(c\bar{c} + s\bar{s})(c\bar{s} - s\bar{c}) + \left(\frac{\partial \widehat{\Omega}}{\partial \chi_1} - \frac{\partial \widehat{\Omega}}{\partial \chi_2}\right)\Big(2cs\bar{s}^2 - \bar{c}\bar{s}\gamma cs\Big)\right\}. \tag{108}$$

The spectral invariants (69) have the values

$$\zeta_1 = s^2, \quad \zeta_2 = c^2, \quad \zeta_3 = 0, \quad \xi_1 = (c\bar{c} + s\bar{s})^2, \quad \xi_2 = (c\bar{s} - s\bar{c})^2, \quad \xi_3 = 0, \tag{109}$$

$$\chi_1 = \bar{s}s(c\bar{c} + s\bar{s}), \quad \chi_2 = (c\bar{s} - s\bar{c}), \quad \chi_3 = 0, \quad \alpha = \bar{s}^2. \tag{110}$$

The spectral components of the electric displacement (82) are:

$$D_1 = s\left\{\frac{\partial \widehat{\Omega}}{\partial E} + \frac{2}{E_0}\left(\frac{\partial \widehat{\Omega}}{\partial \zeta_1} - \frac{\partial \widehat{\Omega}}{\partial \zeta_1}\zeta_1 - \frac{\partial \widehat{\Omega}}{\partial \zeta_2}\zeta_2 - \frac{\partial \widehat{\Omega}}{\partial \chi_1}\chi_1 - \frac{\partial \widehat{\Omega}}{\partial \chi_2}\chi_2 - \alpha\frac{\partial \widehat{\Omega}}{\partial \alpha}\right)\right\} + \frac{(c\bar{c} + s\bar{s})}{E_0}\left\{s(c\bar{c} + s\bar{s})\frac{\partial \widehat{\Omega}}{\partial \chi_1} + c(c\bar{s} - s\bar{c})\frac{\partial \widehat{\Omega}}{\partial \chi_2} + \bar{s}\frac{\partial \widehat{\Omega}}{\partial \chi_1} + 2\bar{s}\frac{\partial \widehat{\Omega}}{\partial \alpha}\right\}, \tag{111}$$

$$D_2 = c\left\{\frac{\partial \widehat{\Omega}}{\partial E} + \frac{2}{E_0}\left(\frac{\partial \widehat{\Omega}}{\partial \zeta_2} - \frac{\partial \widehat{\Omega}}{\partial \zeta_1}\zeta_1 - \frac{\partial \widehat{\Omega}}{\partial \zeta_2}\zeta_2 - \frac{\partial \widehat{\Omega}}{\partial \chi_1}\chi_1 - \frac{\partial \widehat{\Omega}}{\partial \chi_2}\chi_2 - \alpha\frac{\partial \widehat{\Omega}}{\partial \alpha}\right)\right\} + \frac{(c\bar{s} - s\bar{c})}{E_0}\left\{s(c\bar{c} + s\bar{s})\frac{\partial \widehat{\Omega}}{\partial \chi_1} + c(c\bar{s} - s\bar{c})\frac{\partial \widehat{\Omega}}{\partial \chi_2} + \bar{s}\frac{\partial \widehat{\Omega}}{\partial \chi_2} + 2\bar{s}\frac{\partial \widehat{\Omega}}{\partial \alpha}\right\}, \tag{112}$$

and $D_3 = 0$.

Regarding the continuity conditions, if we assume, for example that $L_2 \ll L_1$, $L_2 \ll L_3$, as an approximation, such conditions can be analyzed only on the surfaces $X_2 = \pm L_2/2$. From (111), (112) and $(5)_1$, we have that if $\mathbf{D}^o$ is the electric displacement in vacuum, then it is of the form $D_2^o = D_2$ where D_2 is given in (112). About the continuity condition $(5)_2$, from $(10)_1$, (100) and considering the expression for $\mathbf{E}_l$ assumed in this section, it is easy to show that $\mathbf{E} = \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}$, and from $(5)_2$ the conclusion is that there would be no particular direct restriction on $\mathbf{E}^o$; however from (3) and the previous results we have that $E_2^o = \frac{D_2}{\varepsilon_0}$. The external traction can be easily obtained from (9) considering the above results for the components of the total stress tensor, and for brevity such results are not presented here.

5. An electroelastic body with two orthogonal families of fibres

In this second problem we study a situation that (as mentioned in the Introduction) could be interesting in biomechanics, where we have an electro-elastic body with two orthogonal families of fibres. The matrix and the fibres can react to electric fields. As pointed out in the Introduction, such a model could be useful in the modelling of the behaviour of heart tissue (Dal et al., 2013; Wong et al., 2011). Considering the same definition for $\mathbf{f}$ and $\mathbf{a}_0$ presented in §4 and adding the unitarian field $\mathbf{b}_0$ for the direction of the additional family of fibres, we have

$$\Omega = \Omega(\mathbf{C}, \mathbf{E}_l, \mathbf{a}_0, \mathbf{b}_0). \tag{113}$$

The relation (113) can be expressed as

$$\Omega = \widehat{W}(\mathbf{C}, \mathbf{f}, \mathbf{a}_0, \mathbf{b}_0, E). \tag{114}$$

In view of (63) and (64) of §3.4 and taking $\beth = \mathbf{f}$, $\gimel = \mathbf{a}_0$, $\aleph = \mathbf{b}_0$ and $\mathbf{a}_0 \cdot \mathbf{b}_0 = 0$, it follows that Ω can be expressed as

$$\Omega = \widehat{\Omega}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \varrho_1, \varrho_2, \varrho_3, \chi_1, \chi_2, \chi_3, \vartheta_1, \vartheta_2, \vartheta_3, \alpha, \beta, E), \tag{115}$$

where ζ_i, ξ_i, χ_i, $i = 1,2,3$ have been defined in (69), where from $(45)_{1,2}$ and (43), we have

$$\varrho_i = (\mathbf{b}_0 \cdot \mathbf{e}_i)^2, \quad \beta = \kappa_2^2 = (\mathbf{f} \cdot \mathbf{b}_0), \quad \vartheta_i = (\mathbf{f} \cdot \mathbf{b}_0)(\mathbf{f} \cdot \mathbf{e}_i)(\mathbf{b}_0 \cdot \mathbf{e}_i), \quad i = 1,2,3. \tag{116}$$

The Lagrangian spectral components $\left(\frac{\partial \Omega}{\partial \mathbf{C}}\right)_{ii}$ (no sum in i) are the same as in (74) (see (115) and the definitions above), whereas the shear components are given by

$$\begin{aligned}\left(\frac{\partial \Omega}{\partial \mathbf{C}}\right)_{ij} = \frac{1}{\left(\lambda_i^2 - \lambda_j^2\right)} \Bigg\{ & \left(\frac{\partial \widehat{\Omega}}{\partial \zeta_i} - \frac{\partial \widehat{\Omega}}{\partial \zeta_j}\right) \mathbf{e}_i \cdot \mathbf{H} \mathbf{e}_j \\ & + \left(\frac{\partial \widehat{\Omega}}{\partial \xi_i} - \frac{\partial \widehat{\Omega}}{\partial \xi_j}\right) \mathbf{e}_i \cdot \mathbf{A}_0 \mathbf{e}_j + \left(\frac{\partial \widehat{\Omega}}{\partial \varrho_i} - \frac{\partial \widehat{\Omega}}{\partial \varrho_j}\right) \mathbf{e}_i \cdot \mathbf{B}_0 \mathbf{e}_j \\ & + \left(\frac{\partial \widehat{\Omega}}{\partial \chi_i} - \frac{\partial \widehat{\Omega}}{\partial \chi_j}\right) (\mathbf{e}_i \cdot \mathbf{H} \mathbf{A}_0 \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{H} \mathbf{A}_0 \mathbf{e}_i) \\ & + \left(\frac{\partial \widehat{\Omega}}{\partial \vartheta_i} - \frac{\partial \widehat{\Omega}}{\partial \vartheta_j}\right) (\mathbf{e}_i \cdot \mathbf{H} \mathbf{B}_0 \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{H} \mathbf{B}_0 \mathbf{e}_i) \Bigg\}, \quad i \neq j,\end{aligned} \tag{117}$$

where we have defined the second order tensor

$$\mathbf{B}_0 = \mathbf{b}_0 \otimes \mathbf{b}_0. \tag{118}$$

The Eulerian components of total Cauchy stress for an incompressible body can be obtained from the above expressions using (72), which for brevity, are not shown here.

The spectral components for the electric displacement are given by (81) where

$$\begin{aligned} D_k = (\mathbf{f} \cdot \mathbf{e}_k) \Bigg[& \frac{\partial \widehat{\Omega}}{\partial E} + \frac{2}{E} \left(\frac{\partial \widehat{\Omega}}{\partial \zeta_k} - \sum_{i=1}^{3} \frac{\partial \widehat{\Omega}}{\partial \zeta_i} \zeta_i - \sum_{i=1}^{3} \frac{\partial \widehat{\Omega}}{\partial \chi_i} \chi_i - \sum_{i=1}^{3} \frac{\partial \widehat{\Omega}}{\partial \vartheta_i} \vartheta_i \right. \\ & \left. - \alpha \frac{\partial \widehat{\Omega}}{\partial \alpha} - \beta \frac{\partial \widehat{\Omega}}{\partial \beta} \right) \Bigg] + \frac{(\mathbf{a}_0 \cdot \mathbf{e}_k)}{E} \left[\sum_{i=1}^{3} (\mathbf{f} \cdot \mathbf{e}_i)(\mathbf{a}_0 \cdot \mathbf{e}_i) \frac{\partial \widehat{\Omega}}{\partial \chi_i} \right. \\ & \left. + (\mathbf{f} \cdot \mathbf{a}_0) \frac{\partial \widehat{\Omega}}{\partial \chi_k} + 2 (\mathbf{f} \cdot \mathbf{a}_0) \frac{\partial \widehat{\Omega}}{\partial \alpha} \right] + \frac{(\mathbf{b}_0 \cdot \mathbf{e}_k)}{E} \left[\sum_{i=1}^{3} (\mathbf{f} \cdot \mathbf{e}_i)(\mathbf{b}_0 \cdot \mathbf{e}_i) \frac{\partial \widehat{\Omega}}{\partial \vartheta_i} \right. \\ & \left. + (\mathbf{f} \cdot \mathbf{b}_0) \frac{\partial \widehat{\Omega}}{\partial \vartheta_k} + 2 (\mathbf{f} \cdot \mathbf{b}_0) \frac{\partial \widehat{\Omega}}{\partial \beta} \right], \end{aligned} \tag{119}$$

The magnetic field in the deformed configuration can simply be expressed by (83).

We do not solve boundary value problems for this case, which is done in the following section, where a more general case with two, in general non-orthogonal families of 'fibres', is considered.

6. An electroelastic body with two in general non-orthogonal families of fibres

In this last problem, we study the case of an electroelastic body with two (in general non-orthogonal) families of fibres. These fibres and the matrix can react to the presence of electric fields. We use the same notation as in §5 assuming now $\mathbf{a}_0 \cdot \mathbf{b}_0 \neq 0$, in general.

6.1. Constitutive equation

In view of (59) and (60) of §3.4 and taking $\beth = \mathbf{f}$, $\gimel = \mathbf{a}_0$ and $\aleph = \mathbf{b}_0$, it follows that Ω can be expressed as

$$\Omega = \widehat{\Omega}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \varrho_1, \varrho_2, \varrho_3, \chi_1, \chi_2, \chi_3, \vartheta_1, \vartheta_2, \vartheta_3, \iota_1, \iota_2, \iota_3, \alpha, \beta, \gamma, E), \tag{120}$$

where we recall the expressions (69), (116) and from (44) and $(45)_3$ we have

$$\kappa_3 = (\mathbf{a}_0 \cdot \mathbf{b}_0)^2, \quad \gamma = \kappa_3^2, \quad \iota_i = (\mathbf{a}_0 \cdot \mathbf{b}_0)(\mathbf{a}_0 \cdot \mathbf{e}_i)(\mathbf{b}_0 \cdot \mathbf{e}_i), \quad i = 1,2,3. \tag{121}$$

The Lagrangian spectral component relations $\left(\frac{\partial \Omega}{\partial \mathbf{C}}\right)_{ii}$, $i = 1,2,3$ are the same as in (74) and the shear components are

$$\begin{aligned}\left(\frac{\partial \Omega}{\partial \mathbf{C}}\right)_{ij} = \frac{1}{\left(\lambda_i^2 - \lambda_j^2\right)} \Bigg\{ & \left(\frac{\partial \widehat{\Omega}}{\partial \zeta_i} - \frac{\partial \widehat{\Omega}}{\partial \zeta_j}\right) \mathbf{e}_i \cdot \mathbf{H} \mathbf{e}_j + \left(\frac{\partial \widehat{\Omega}}{\partial \xi_i} - \frac{\partial \widehat{\Omega}}{\partial \xi_j}\right) \mathbf{e}_i \cdot \mathbf{A}_0 \mathbf{e}_j \\ & + \left(\frac{\partial \widehat{\Omega}}{\partial \varrho_i} - \frac{\partial \widehat{\Omega}}{\partial \varrho_j}\right) \mathbf{e}_i \cdot \mathbf{B}_0 \mathbf{e}_j + \left(\frac{\partial \widehat{\Omega}}{\partial \chi_i} - \frac{\partial \widehat{\Omega}}{\partial \chi_j}\right) (\mathbf{e}_i \cdot \mathbf{H} \mathbf{A}_0 \mathbf{e}_j \\ & + \mathbf{e}_j \cdot \mathbf{H} \mathbf{A}_0 \mathbf{e}_i) + \left(\frac{\partial \widehat{\Omega}}{\partial \vartheta_i} - \frac{\partial \widehat{\Omega}}{\partial \vartheta_j}\right) (\mathbf{e}_i \cdot \mathbf{H} \mathbf{B}_0 \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{H} \mathbf{B}_0 \mathbf{e}_i) \\ & + \left(\frac{\partial \widehat{\Omega}}{\partial \iota_i} - \frac{\partial \widehat{\Omega}}{\partial \iota_j}\right) (\mathbf{e}_i \cdot \mathbf{A}_0 \mathbf{B}_0 \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{A}_0 \mathbf{B}_0 \mathbf{e}_i) \Bigg\}, \quad i \neq j.\end{aligned} \tag{122}$$

The expression for D_i, $i = 1,2,3$ are the same as in (119).

6.2. Boundary value problems

6.2.1. Simple shear of a slab

In this section we study the problem of the simple shear of a slab, where the reference configuration is the same as that is described in (97). Let us assume that it deforms as (98) and that $\mathbf{a}_0 = \begin{pmatrix} \bar{c} \\ \bar{s} \\ 0 \end{pmatrix}$, $\mathbf{b}_0 = \begin{pmatrix} \widehat{c} \\ \widehat{s} \\ 0 \end{pmatrix}$ and $\mathbf{E}_l = \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}$, where $\widehat{c} = \cos(\varphi)$, $\widehat{s} = \sin(\varphi)$ and E_0 is constant.

Using the results presented in §4.2.2, we have[10]:

$$\tau_{13} = \tau_{23} = 0,$$
$$\tau_{12} = 2\left\{(l_1 - l_2)cs + l_4\left(c^2 - s^2\right) + \gamma\left(l_1 s^2 + l_2 c^2 + 2l_4 cs\right)\right\}, \tag{123}$$

where

$$l_1 = \frac{1}{2\lambda_1}\frac{\partial\widehat{\Omega}}{\partial\lambda_1}, \quad l_2 = \frac{1}{2\lambda_2}\frac{\partial\widehat{\Omega}}{\partial\lambda_2} \tag{124}$$

and

$$l_4 = \frac{1}{\left(\lambda_1^2 - \lambda_2^2\right)}\left\{\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_1} - \frac{\partial\widehat{\Omega}}{\partial\zeta_1}\right)sc + \left(\frac{\partial\widehat{\Omega}}{\partial\xi_1} - \frac{\partial\widehat{\Omega}}{\partial\xi_2}\right)(c\bar{c} + s\bar{s})(c\bar{s} - s\bar{c}) + \left(\frac{\partial\widehat{\Omega}}{\partial\varrho_i} - \frac{\partial\widehat{\Omega}}{\partial\varrho_j}\right)(c\widehat{c} + s\widehat{s})(c\widehat{s} - s\widehat{c}) + \left(\frac{\partial\widehat{\Omega}}{\partial\chi_1} - \frac{\partial\widehat{\Omega}}{\partial\chi_2}\right)\left(2cs\bar{s}^2 - \bar{c}\bar{s}\gamma cs\right) + \left(\frac{\partial\widehat{\Omega}}{\partial\vartheta_i} - \frac{\partial\widehat{\Omega}}{\partial\vartheta_j}\right)\left(2sc\widehat{s}^2 - \widehat{s}\widehat{c}\gamma cs\right) + \left(\frac{\partial\widehat{\Omega}}{\partial\iota_i} - \frac{\partial\widehat{\Omega}}{\partial\iota_j}\right)(\bar{c}\bar{c} + \bar{s}\bar{s})[2cs(\bar{s}\widehat{s} - \bar{c}\widehat{c}) - \gamma cs(\bar{c}\widehat{s} + \widehat{c}\bar{s})]\right\}. \tag{125}$$

The spectral invariants (69), (116) and (121) have the values

$$\zeta_1 = s^2, \quad \zeta_2 = c^2, \quad \zeta_3 = 0, \quad \xi_1 = (c\bar{c} + s\bar{s})^2,$$
$$\xi_2 = (c\bar{s} - s\bar{c})^2, \quad \xi_3 = 0, \tag{126}$$

$$\chi_1 = \bar{s}s(c\bar{c} + s\bar{s}), \quad \chi_2 = (c\bar{s} - s\bar{c}), \quad \chi_3 = 0, \quad \alpha = \bar{s}^2, \tag{127}$$

$$\varrho_1 = (c\widehat{c} + s\widehat{s})^2, \quad \varrho_2 = (c\widehat{s} - s\widehat{c})^2, \quad \varrho_3 = 0, \tag{128}$$

$$\iota_1 = (\bar{c}\widehat{c} + \bar{s}\widehat{s})(c\bar{c} + s\bar{s})(c\widehat{c} + s\widehat{s}),$$
$$\iota_2 = (\bar{c}\widehat{c} + \bar{s}\widehat{s})(\bar{s}c - \bar{c}s)(\widehat{s}c - \widehat{c}s), \quad \iota_3 = 0, \tag{129}$$

$$\vartheta_1 = \widehat{s}s(c\widehat{c} + s\widehat{s}), \quad \vartheta_2 = (c\widehat{s} - s\widehat{c}), \quad \vartheta_3 = 0 \quad \beta = \widehat{s}^2. \tag{130}$$

The spectral components for the electric field (119) are:

$$D_1 = s\left\{\frac{\partial\widehat{\Omega}}{\partial E} + \frac{2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_1} - \frac{\partial\widehat{\Omega}}{\partial\zeta_1}\zeta_1 - \frac{\partial\widehat{\Omega}}{\partial\zeta_2}\zeta_2 - \frac{\partial\widehat{\Omega}}{\partial\chi_1}\chi_1 - \frac{\partial\widehat{\Omega}}{\partial\chi_2}\chi_2 - \frac{\partial\widehat{\Omega}}{\partial\vartheta_1}\vartheta_1 - \frac{\partial\widehat{\Omega}}{\partial\vartheta_2}\vartheta_2 - \alpha\frac{\partial\widehat{\Omega}}{\partial\alpha} - \beta\frac{\partial\widehat{\Omega}}{\partial\beta}\right)\right\} + \frac{(c\bar{c} + s\bar{s})}{E_0}\left\{s(c\bar{c} + s\bar{s})\frac{\partial\widehat{\Omega}}{\partial\chi_1}c(c\bar{s} - s\bar{c})\frac{\partial\widehat{\Omega}}{\partial\chi_2} + \bar{s}\frac{\partial\widehat{\Omega}}{\partial\chi_1} + 2\bar{s}\frac{\partial\widehat{\Omega}}{\partial\alpha}\right\} + \frac{(c\widehat{c} + s\widehat{s})}{E_0}\left\{s(c\widehat{c} + s\widehat{s})\frac{\partial\widehat{\Omega}}{\partial\vartheta_1} + c(c\widehat{s} + s\widehat{c})\frac{\partial\widehat{\Omega}}{\partial\vartheta_1} + \widehat{s}\frac{\partial\widehat{\Omega}}{\partial\vartheta_1} + 2\widehat{s}\frac{\partial\widehat{\Omega}}{\partial\beta}\right\}, \tag{131}$$

$$D_2 = c\left\{\frac{\partial\widehat{\Omega}}{\partial E} + \frac{2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_2} - \frac{\partial\widehat{\Omega}}{\partial\zeta_1}\zeta_1 - \frac{\partial\widehat{\Omega}}{\partial\zeta_2}\zeta_2 - \frac{\partial\widehat{\Omega}}{\partial\chi_1}\chi_1 - \frac{\partial\widehat{\Omega}}{\partial\chi_2}\chi_2 - \frac{\partial\widehat{\Omega}}{\partial\vartheta_1}\vartheta_1 - \frac{\partial\widehat{\Omega}}{\partial\vartheta_2}\vartheta_2 - \alpha\frac{\partial\widehat{\Omega}}{\partial\alpha} - \beta\frac{\partial\widehat{\Omega}}{\partial\beta}\right)\right\} + \frac{(c\bar{s} - s\bar{c})}{E_0}\left\{s(c\bar{c} - s\bar{s})\frac{\partial\widehat{\Omega}}{\partial\chi_1} + c(c\bar{s} - s\bar{c})\frac{\partial\widehat{\Omega}}{\partial\chi_2} + \bar{s}\frac{\partial\widehat{\Omega}}{\partial\chi_2} + 2\bar{s}\frac{\partial\widehat{\Omega}}{\partial\alpha}\right\} + \frac{(c\widehat{s} - s\widehat{c})}{E_0}\left\{s(c\widehat{c} + s\widehat{s})\frac{\partial\widehat{\Omega}}{\partial\vartheta_1} + c(c\widehat{s} - s\widehat{c})\frac{\partial\widehat{\Omega}}{\partial\vartheta_2} + \widehat{s}\frac{\partial\widehat{\Omega}}{\partial\vartheta_2} + 2\widehat{s}\frac{\partial\widehat{\Omega}}{\partial\beta}\right\}, \tag{132}$$

and D_3=0.

6.2.2. Extension and inflation of cylindrical tube with two families of fibres

The behaviour of the cylindrical tube is defined in the reference configuration as

$$0 < R_\mathrm{i} \le R \le R_\mathrm{o}, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L, \tag{133}$$

where we assume it deforms as

$$r^2 - r_\mathrm{i}^2 = \lambda_z^{-1}\left(R^2 - R_\mathrm{i}^2\right), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{134}$$

and $\lambda_z > 0$ is a constant. In this case the deformation gradient has the form

$$\mathbf{F} = \begin{pmatrix} \frac{1}{\lambda\lambda_z} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{135}$$

where we have defined $\lambda = r/R$ (see, for example, §5.3.3 of Ogden (1997)). With respect to the reference cylindrical coordinate system, we consider $\mathbf{a}_0 = \begin{pmatrix} 0 \\ \bar{c} \\ \bar{s} \end{pmatrix}$, $\mathbf{b}_0 = \begin{pmatrix} 0 \\ \widehat{c} \\ \widehat{s} \end{pmatrix}$ and $\mathbf{E}_L = \begin{pmatrix} 0 \\ 0 \\ E_0 \end{pmatrix}$. In this type of deformation, we have

$$\mathbf{e}_1 = \mathbf{E}_r, \quad \mathbf{e}_2 = \mathbf{E}_\theta, \quad \mathbf{e}_3 = \mathbf{E}_z. \tag{136}$$

The spectral invariants (69), (116) and (121) have the forms:

[10] The normal components of the stress are the same as in (104) and (105).

$$\zeta_1 = \zeta_2 = \xi_1 = \varrho_1 = \chi_1 = \chi_2 = \vartheta_1 = \vartheta_2 = \iota_1 = 0, \quad \zeta_3 = 1,$$
$$\xi_3 = \chi_3 = \overline{s}^2, \tag{137}$$

$$\varrho_3 = \vartheta_3 = \widehat{s}^2, \quad \xi_2 = \overline{c}^2, \quad \varrho_2 = \widehat{s}^2, \quad \iota_2 = (\overline{c}\widehat{c} + \overline{s}\widehat{s})\overline{c}\widehat{c},$$
$$\iota_3 = (\overline{c}\widehat{c} + \overline{s}\widehat{s})\overline{s}\widehat{s}. \tag{138}$$

The non-shear spectral components of stress are:

$$\tau_{rr} = \frac{1}{\lambda\lambda_z}\frac{\partial\widehat{\Omega}}{\partial\lambda_1} - p, \quad \tau_{\theta\theta} = \lambda\frac{\partial\widehat{\Omega}}{\partial\lambda_2} - p, \quad \tau_{zz} = \lambda_z\frac{\partial\widehat{\Omega}}{\partial\lambda_3} - p. \tag{139}$$

If there is no mechanical stress applied on the cylindrical surface, then

$$p = \frac{1}{\lambda\lambda_z}\frac{\partial\widehat{\Omega}}{\partial\lambda_1} + \frac{\varepsilon_0 E_0^2}{2\lambda_z^2}. \tag{140}$$

The shear spectral components are

$$\tau_{rz} = \tau_{r\theta} = 0, \tag{141}$$

$$\tau_{\theta z} = \frac{2\lambda\lambda_z}{\left(\lambda^2 - \lambda_z^2\right)}\left\{\left(\frac{\partial\widehat{\Omega}}{\partial\xi_2} - \frac{\partial\widehat{\Omega}}{\partial\xi_3}\right)\overline{cs} + \left(\frac{\partial\widehat{\Omega}}{\partial\varrho_2} - \frac{\partial\widehat{\Omega}}{\partial\varrho_3}\right)\widehat{cs} + \left(\frac{\partial\widehat{\Omega}}{\partial\chi_2} - \frac{\partial\widehat{\Omega}}{\partial\chi_3}\right)\overline{cs} + \left(\frac{\partial\widehat{\Omega}}{\partial\vartheta_2} - \frac{\partial\widehat{\Omega}}{\partial\vartheta_3}\right)\widehat{cs} + \left(\frac{\partial\widehat{\Omega}}{\partial\iota_2} - \frac{\partial\widehat{\Omega}}{\partial\iota_3}\right)(\overline{c}\widehat{c} + \overline{s}\widehat{s})(\overline{c}\widehat{s} + \overline{s}\widehat{c})\right\}, \tag{142}$$

The spectral components for the electric field take the forms:

$$D_1 = 0,$$
$$D_2 = \frac{\overline{cs}}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\chi_2} + \frac{\partial\widehat{\Omega}}{\partial\chi_3} + 2\frac{\partial\widehat{\Omega}}{\partial\alpha}\right) + \frac{\widehat{cs}}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\vartheta_2} + \frac{\partial\widehat{\Omega}}{\partial\vartheta_3} + 2\frac{\partial\widehat{\Omega}}{\partial\beta}\right), \tag{143}$$

$$D_3 = \left\{\frac{\partial\widehat{\Omega}}{\partial E} - \frac{2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\chi_3}\overline{s}^2 + \frac{\partial\widehat{\Omega}}{\partial\vartheta_3}\widehat{s}^2 + \alpha\frac{\partial\widehat{\Omega}}{\partial\alpha} + \beta\frac{\partial\widehat{\Omega}}{\partial\beta}\right)\right\} + \frac{2\overline{s}^2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\chi_3} + \frac{\partial\widehat{\Omega}}{\partial\alpha}\right) + \frac{2\widehat{s}^2}{E_0}\left(\frac{\partial\widehat{\Omega}}{\partial\vartheta_3} + \frac{\partial\widehat{\Omega}}{\partial\beta}\right). \tag{144}$$

7. Further remarks

In the present communication, we have defined a new set of spectral invariants for electro-active bodies with one and two families of 'fibres', where the host material and the fibres can react to the presence of electric fields. When using the classical invariants of Rivlin and Spencer (Spencer, 1971) for the same problems, not only it is necessary to consider some invariants that may not be independent (see, for example Shariff and Bustamante (2015)), but also several of the classical invariants lack clear physical meanings. Due to the large number of classical invariants associated with the modelling of anisotropic electro-elastic bodies, different researchers modelled these materials using subsets of the full set of classical invariants (see, for example Bustamante (2010), for the equivalent problem of modelling anisotropic magneto active elastomers). In the present paper, not only we have proposed invariants that have clearer physical meanings, but also from the results presented in (77)–(82), (119), (122), we can see that in general it is not possible to neglect some invariants from the formulation, because all of them have effects on the stresses that can be seen clearly from the different boundary value problems presented in the previous sections.

Regarding some other possible applications of the invariant theory presented in the present work, new models have appeared in the literature where both electric and magnetic fields are considered (Santapuri et al., 2013), and also implicit theories where relations that can depend on both the stresses, strains and electric or magnetic fields (Bustamante and Rajagopal, 2013, 2015) have been proposed. In such works, it would be more convenient to use invariants similar to the sets presented here, to model the different couplings between the electromagnetic fields and the stresses and straints.

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