

CRITICAL SHEAR STRESS FOR INCIPIENT MOTION OF NON-COHESIVE PARTICLES IN OPEN-CHANNEL FLOWS OF PSEUDOPLASTIC FLUIDS

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The results of a theoretical and experimental study on the conditions of incipient motion of non-cohesive solid particles in laminar flows of pseudoplastic fluids are presented in the article. It is concluded that, for practical purposes, using a modified particle Reynolds number that includes the consistency coefficient and the flow index, the motion threshold condition is well defined by the Shields diagram, with a slight modification to that presented by Mantz for low Reynolds numbers. The range of particle Reynolds numbers covered in the study was from 0.026–12.56.

Keywords: incipient motion, Shields stress, critical shear stress, pseudoplastic fluid

INTRODUCTION

The motion of non-cohesive particles forming the bed of a non-Newtonian fluid laminar flow can be found as much in nature as in industrial applications.^[1–3] Usually, the non-Newtonian fluid is a rheological idealization of the mixture generated by water and fine solid particles.^[4] Depending on the physicochemical characteristics of the solids, their concentration, and the characteristics of the water, the non-Newtonian nature of the mixture can be modelled as an equivalent homogeneous pseudoplastic fluid. Mudflows are examples of these flows in nature.^[5] In the mining industry, this rheological behaviour is found in ore concentrates and tailings.^[6–8] In such solid-water mixtures, commonly the solid particles cover a wide range of sizes, and the larger particles are conveyed by the mixture formed by the fines and water, which may be interpreted as an equivalent fluid, with a rheology different from that of water.^[6] Ideally, the larger particles should be transported as a suspension, but accidents or programmed detentions of the line may generate a deposit of the larger particles. The need for a subsequent restart of the system imposes the requirement to re-suspend both the fine and the coarse fractions, and it is highly desirable to rely solely on the flow itself for this purpose. Thus, it is of interest to know the flow conditions under which the coarse solid particles are set into motion. As a first step in this regard, an experimental study aimed to determine the threshold conditions for the motion of non-cohesive solid particles under non-Newtonian fluid flows has been implemented, and the results concerning the conditions for incipient motion of pseudoplastic fluid flows are presented in this paper.

LITERATURE REVIEW

Traditionally, two approaches have been developed to determine the threshold conditions at which the particles forming the bed of the channel or conduit can be set into motion: one based on a critical velocity and the other expressed in terms of a critical shear stress. The critical velocity approach was the first to be proposed, dating

back to Du Buat's 1816 work.^[9] In 1926, Fortier and Scobey^[10] published a paper in which permissible velocities for erodible canals are given, depending on the type of soil. Hjulsdröm^[11,12] provided a curve relating the mean velocity of the flow with the size of natural sediments for two conditions, erosion and deposition. Later, Sundborg^[13] complemented Hjulsdröm's results by generating a graphical relationship, which is widely used by sedimentologists. The above relationships have the serious drawbacks that they are not presented in dimensionless form and they should be applied only in the range of variables and conditions for which they were obtained. This has been improved by including dimensionless groups in a number of contributions, e.g. Neill^[14,15] and Yang.^[16]

The shear stress approach was first proposed in 1936 by Shields in his doctoral thesis.^[17] Here, a dimensionless shear stress (known now as a Shields parameter) was defined as follows:

$$\tau_* = \frac{\tau_0}{(\rho_s - \rho)gd} \quad (1)$$

where τ_0 is the bottom shear stress; ρ_s and ρ are the sediment and fluid densities, respectively; g is the magnitude of the acceleration due to gravity; and d is the particle size. Shields presented his results in a diagram as a function of a particle Reynolds number, $Re_* = \frac{u_* d}{\nu}$, where the shear velocity u_* is defined as $\sqrt{\frac{\tau_0}{\rho}}$ and ν is

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the kinematic viscosity of the liquid. Shields did not fit a curve in order to determine a critical dimensionless shear stress, τ_{*c} . Instead, he presented a band, thus making explicit the uncertainty and the arbitrariness associated with the experimental determination of the critical condition. Shields reported his own data and some from other authors, with Re_s in the range between 2–200, within the turbulent flow regime. Later, Rouse^[18] defined a curve which is contained in the band presented by Shields in his original work. The Shields diagram that we are familiarized with nowadays is the Rouse's curve of 1939, with some modifications resulting from additional data and analysis. Myriad research has been published since Shields' work, adding more data and incorporating other effects. In particular, the effect on the critical Shields stress due to non-horizontal slopes and steep channels, shallow flows, accelerated flows, non-uniform bed material, coarse material, bed turbulence, etc. has been studied. Comprehensive reviews of the critical Shields stress have been reported by Buffington and Montgomery^[19] and Dey and Papanicolaou.^[20] Most of the published studies have been carried out with turbulent flows. Exceptions are Mantz,^[21] who presents data in the laminar or laminar-turbulent regime; Yalin and Karahan^[22] which includes data in both laminar and turbulent regimes; Pilotti and Menduni,^[23] who worked with laminar flows and hydraulically smooth turbulent flows; and more recently, Ouriemi et al.^[24] All of the above studies have been performed with a Newtonian fluid, and mostly water. Exceptions are the studies with mudflows performed by Daido^[25] and Wan.^[26] Both authors modelled the clay-water mixture as a Bingham plastic fluid.

THEORETICAL APPROACH

Uniform Laminar Flow of Pseudoplastic Fluids

The Ostwald-de Waele, or Power-law, is one of the most popular models for pseudoplastic fluids and it is given by the following:^[27]

$$\tau_{ij} = K\dot{\gamma}^{n-1}\dot{\gamma}_{ij} \quad (2)$$

where τ_{ij} is the viscous stress tensor, $\dot{\gamma}_{ij}$ is the rate-of-strain tensor, and $\dot{\gamma}$ is its magnitude, consisting of the total contraction of the rate-of-strain tensor. Using Einstein notation, it is written as $\dot{\gamma} = \sqrt{\frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}}$. The parameters K and n are the consistency coefficient and flow index, respectively. Equation (2) is greatly simplified in two-dimensional uniform flows, becoming the following:

$$\tau_{xy} = K\left(\frac{du}{dy}\right)^n \quad (3)$$

where u is the velocity in the flow direction (x) and y is the coordinate axis normal to it.

Consider the free-surface, two-dimensional (or wide rectangular channel), steady, uniform flow along a channel with a bed forming an angle θ with the horizontal. The equilibrium between the component of the weight along the flow direction and the resistance force due to the shear stress acting on the bed, τ_{oU} , is as follows:

$$\tau_{oU} = \rho g H \sin \theta \quad (4)$$

where ρ is the fluid density, and H is the flow depth. The subscript U stands for uniform. For a laminar flow, the equilibrium of forces acting in a control volume limited by the free surface and a plane located at a distance y from the bottom is written as follows:

$$\rho g(H-y)\sin \theta = K\left(\frac{du}{dy}\right)^n \quad (5)$$

where the velocity distribution is as follows:

$$u = \frac{n}{n+1}H^{\frac{n+1}{n}}\left(\frac{\rho g \sin \theta}{K}\right)^{\frac{1}{n}}\left[1 - \left(1 - \frac{y}{H}\right)^{\frac{n+1}{n}}\right] \quad (6)$$

The mean flow velocity is thus given by the following:

$$\bar{u} = \frac{n}{2n+1}H^{\frac{n+1}{n}}\left(\frac{\rho g \sin \theta}{K}\right)^{\frac{1}{n}} \quad (7)$$

From the latter expression, the following dimensionless frictional law is found:

$$Fr^2 = \left(\frac{n}{2n+1}\right)^n Re_K \sin \theta \quad (8)$$

where the squares of the Froude number and of the Reynolds number are defined, respectively, as follows:

$$Fr^2 = \frac{\bar{u}^2}{gH}, \quad Re_K = \frac{\rho \bar{u}^{2-n} H^n}{K} \quad (9)$$

After dividing Equation (6) by Equation (7), the velocity distribution is expressed in terms of the mean velocity:

$$\frac{u}{\bar{u}} = \frac{2n+1}{n+1}\left[1 - \left(1 - \frac{y}{H}\right)^{\frac{n+1}{n}}\right] \quad (10)$$

Gradually Varied Laminar Flow of Pseudoplastic Fluids

In general, the strain rate of a two-dimensional flow is given by $\dot{\gamma} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$, where (u,v) are the local velocities in the (x,y) directions. A gradually varied flow is a slender flow and an order of magnitude analysis allows us to neglect $\frac{\partial v}{\partial x}$ when compared with $\frac{\partial u}{\partial y}$. Using this assumption in Equation (2), the same expression as that for the uniform flow is obtained (Equation (3)). The bottom shear stress in a gradually varied flow, τ_0 , can be related to that acting in a uniform flow with the same flow depth (but different velocity) assuming self-similarity of the velocity distribution, as it is shown below. Self-similarity means that it is possible to write the following:

$$\frac{u(x,y)}{U(x)} = f\left(\frac{y}{H(x)}\right) \quad (11)$$

where U is the mean velocity at the streamwise location x , computed from $U(x) = q/H(x)$, with q the discharge per unit width. In a uniform flow, $U = \bar{u}$, and Equation (11) should be equal to Equation (10). This suggests defining the function f as follows:

$$f = \frac{2n+1}{n+1}\left[1 - \left(1 - \frac{y}{H}\right)^{\frac{n+1}{n}}\right] \quad (12)$$

Then, the bottom shear stress can be computed as follows:

$$\tau_0 = K\left(\frac{du}{dy}\right)^n \Big|_{y=0} = K\left(\frac{U}{H}\right)^n \left(\frac{df}{d\left(\frac{y}{H}\right)}\right)^n \Big|_{\frac{y}{H}=0} \quad (13)$$

Obtaining the following:

$$\tau_0 = K \left(\frac{2n+1}{n} \right)^n \left(\frac{q}{H^2} \right)^n \quad (14)$$

Therefore, from Equations (4) and (14), the ratio between the bottom shear stress in a gradually varied flow and that in a uniform flow is as follows:

$$\frac{\tau_0}{\tau_{0U}} = \left(\frac{2n+1}{n} \right)^n \frac{Fr^2}{Re_K \sin \theta} \quad (15)$$

where both the Reynolds and Froude numbers depend on the flow depth $H(x)$ and the local velocity $U(x)$. Note that for a uniform flow $Fr^2 = \left(\frac{n}{2n+1} \right)^n Re_K \sin \theta$ (Equation (8)), becoming $\tau_0 = \tau_{0U}$.

Equilibrium Condition for a Non-Cohesive Particle Exposed on the Bed

The analysis below follows the Ikeda-Coleman-Iwagaki model presented by Ikeda^[28] and Wiberg and Smith^[29] for a horizontal bed. It considers that a non-cohesive particle of the bed exposed to the flow can be destabilized due to the action of hydrodynamic forces. According to Figure 1, the forces acting on the particle are its weight ($W = \rho_s g V_p$), buoyancy ($B = \rho g V_p$), friction (Coulombic) forces ($F_R = \mu N$), drag (F_D), and lift (F_L). Here, V_p is the particle volume and ρ_s its density, μ is the Coulombic friction coefficient between the particle and the bed, and N is the resulting force normal to the bed.

Considering that the hydrodynamic forces are related to a characteristic flow velocity, u_f , the drag and lift forces are given by $F_D = \frac{1}{2} C_D \rho A_D u_f^2$ and $F_L = \frac{1}{2} C_L \rho A_L u_f^2$, respectively. The drag and lift forces are applied to the projected areas A_D and A_L , which are normal to the flow and planform, respectively. The drag coefficients associated to the drag and lift forces are C_D and C_L , respectively. Assuming a spherical particle, the condition of incipient motion found from the force balance is as follows:

$$\frac{u_f^2}{g \left(\frac{\rho_s}{\rho} - 1 \right) d} = \frac{4 \mu \cos \theta - \sin \theta}{3 C_D + \mu C_L} \quad (16)$$

The hydrodynamic coefficients C_D and C_L depend on a particle Reynolds number, Re_p , modified in such a way that it takes into account the pseudoplastic condition of the fluid:

$$Re_p = \frac{\rho u_f^{2-n} d^n}{K} \quad (17)$$

Although it is possible to find extensive literature on formulations for C_D for non-Newtonian fluids (see for example the recent review by Kelbaliyev^[30]), the opposite happens with C_L .

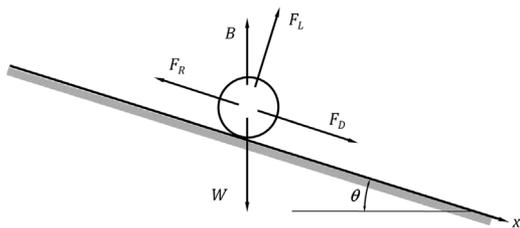


Figure 1. Forces acting in a particle on the bottom.

It is assumed herein that the bottom stress, and thus the shear velocity, are related to a value of the flow velocity evaluated somewhere between $y=0$ and $y=d$. Thus, the velocity u_f is assumed to be the velocity of the undisturbed flow at βd ($0 < \beta < 1$) from the bottom. Noting that the shear velocity, $\sqrt{\frac{\tau_0}{\rho}}$, corresponds in this case to $u_* = \sqrt{gH \sin \theta}$, from Equation (6) results:

$$u_f = \frac{n}{n+1} H \left(\frac{\rho u_*^2}{K} \right)^{\frac{1}{n}} \left[1 - \left(1 - \beta \frac{d}{H} \right)^{\frac{n+1}{n}} \right] \quad (18)$$

Replacing Equation (18) into Equation (16), the critical Shields shear stress is obtained as follows:

$$\tau_{*c} = \frac{\left(\frac{n+1}{n} \right)^2 \left(\frac{d}{H} \right)^2}{Re_{*p}^{\frac{2}{n}} \left[1 - \left(1 - \beta \frac{d}{H} \right)^{\frac{n+1}{n}} \right]^2} \frac{4 \mu \cos \theta - \sin \theta}{3 C_D + \mu C_L} \quad (19)$$

where $Re_{*p} = \frac{\rho u_*^{2-n} d^n}{K}$. Among the many relationships of C_D , that from Ceylan et al.^[31] was chosen, valid for a sphere in a flow without wall effects in the range $10^{-3} < Re_p < 10^3$:

$$C_D = \frac{24}{Re_p} \left[3^{2n-3} \left(\frac{n^2 - n + 3}{n^{3n}} \right) + \frac{4n^4}{24 Re_p^{\frac{n-3}{n}}} \right] \quad (20)$$

O'Neill^[32] studied the drag force acting on a spherical particle in contact with a plane due to the laminar shear flow of a Newtonian fluid, with a linear velocity profile. From a theoretical analysis, O'Neill obtained that the force is given by the Stokes solution, amplified by a coefficient $f_{ON} = 1.7009$. This is equivalent to saying that the drag coefficient of a sphere in an unbounded flow has to be amplified by f_{ON} when applied to a sphere in contact with a rigid surface.

Curiously, there is more information regarding the lift exerted on a small sphere attached to a flat surface under a Newtonian laminar flow than for its turbulent counterpart. In this regard, the works by Saffman,^[33,34] Leighton and Acrivos,^[35] Dandy and Dwyer,^[36] and Mei^[37] can be mentioned. These studies consider the drag acting on the sphere in a simple shear flow (linear velocity distribution). Dandy and Dwyer^[36] computed the $\frac{C_L}{C_D}$ as a function of a particle Reynolds number, Re , ratio that has a minimum for $Re = 50$ and grows for smaller values. The lowest Re in their computations is 1, with $\frac{C_L}{C_D} \approx 0.066$. Leighton and Acrivos^[35] concluded that for small Reynolds numbers, the effect of the lift force on the particle motion is negligible compared to the drag. This conclusion agrees with the observations made by Pilotti and Menduni^[23] who indicate that in their experiments the inception of grain motion never happened with a mechanism different from rolling or sliding. Thus, the sum $C_D + \mu C_L$ from the divisor of Equation (19) is reduced to $f_{ON} C_D (1 + \mu \alpha)$, with α a being constant, which is not relevant in the final analysis of the critical Shield stress.

To use the above results, obtained for simple shear flows of Newtonian fluids, in gravity-driven flows of Ostwald-de Waele fluids does not constitute a deficiency of the analysis if the condition $\frac{d}{H} \ll 1$ is kept. In this case, the relevant portion of the velocity profile (Equations (11, 12)) can be simplified as $\frac{u}{U} \approx \frac{(2n+1)y}{n} \frac{y}{H}$, resulting in a linear relationship between u and y , differing only in the proportionality constant from that for a simple shear flow of a Newtonian fluid.

The critical Shields shear stress τ_{*C} given by Equation (19) is plotted in Figure 2 in terms of particle Reynolds number Re_{*p} for an almost horizontal flume ($\theta \approx 0$), $\frac{d}{H} = 0.01$, $\beta = 0.5$, $f_{ON} = 1.7$, $\mu = 0.8$, and $\alpha = \frac{C_d}{C_D} = 0.07$, for three different flow indexes n : 0.5, 0.75, and 1 (Newtonian fluid). For comparison, in the same figure the data obtained for laminar flows of Newtonian fluids obtained by White,^[21] Yalin and Karahan,^[22] and Pilotti and Menduni^[23] are also included. It is observed that the analytical result obtained in this paper underpredicts the critical condition, which can be explained as a result of the assumptions made in its development, in which an over-exposed, isolated particle has been analyzed, without the stabilizing effect caused by neighbouring particles.

It is observed that for low Re_{*p} , the critical Shields shear stress approaches a constant value given by the following:

$$\tau_{*C} = \frac{1}{2 \times 3^{2n-1}} \frac{n^{3n}}{n^2 - n + 3} \left(\frac{n+1}{n}\right)^n \frac{\mu \cos \theta - \sin \theta}{1 + \mu\alpha} \left(\frac{H}{d}\right)^{-n} \left[1 - \left(1 - \beta \frac{d}{H}\right)^{\frac{n+1}{n}}\right]_{-n} \quad (21)$$

For values $\frac{d}{H} \ll 1$, Equation (21) is simplified to the following:

$$\tau_{*C} = \frac{1}{2 \times 3^{2n-1}} \frac{n^{3n}}{n^2 - n + 3} \frac{1}{\beta^n} \frac{\mu \cos \theta - \sin \theta}{1 + \mu\alpha} \quad (22)$$

For a channel with negligible slope, Equation (22) is further reduced to the following:

$$\tau_{*C} = \frac{1}{2 \times 3^{2n-1}} \frac{n^{3n}}{n^2 - n + 3} \frac{1}{\beta^n} \frac{\mu}{1 + \mu\alpha} \quad (23)$$

Given the results of experiments with highly viscous fluids and low Re_{*p} , White^[21] proposed that the critical shear stress should be a constant. Pilotti and Menduni^[23] arrived at the same conclusion considering a model with some parameters that behave randomly, even in viscous-dominated flows.

The critical Shields stress given by Equation (22) presents a minimum for the value of n that satisfies the following transcendental equation:

$$(n^2 - n + 3)(\ln \beta - 3 \ln n + 2 \ln 3) - 3n^2 + 5n - 10 = 0 \quad (24)$$

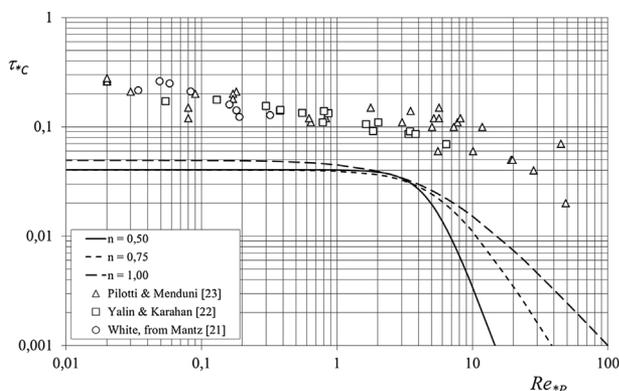


Figure 2. Theoretical curves for the incipient motion of non-cohesive solid particles in laminar flows of Ostwald-de Waele fluids. Data is from Newtonian fluids.

Although for $\frac{d}{H}$ and $Re_{*p} \ll 1$, τ_{*C} depends on the channel slope, particle-bed friction, and lift-to-drag forces ratio, the flow index value that minimizes τ_{*C} is only determined by the relative location β at which the velocity u_f is evaluated, usually $\beta = 0.5$.^[36] For this value, $n = 0.625$ is obtained. From Figure 2, it is also noted that for large Re_{*p} , but within the laminar flow regime, the critical Shields stress decays monotonically for growing Reynolds numbers, approaching a straight line with slope $\frac{5}{3} - \frac{4}{n}$ on a log-log plot given $0.5 \leq n \leq 1$.

It is stressed that the above equations have been deduced for the simplified configuration of Figure 1, and any change in the geometry, isolation of the particle, etc. can increase its stability. The shape of the curve τ_{*C} versus Re_{*p} depends essentially on n and the magnitude of the parameters related to the force equilibrium. Thus, when $f = 1$ and $\alpha = 0$ are used (i.e. drag corresponding to an isolated sphere far from a wall and lift negligible),^[33] the curve corresponding to a Newtonian fluid ($n = 1$) is shifted up, becoming a lower boundary of the Newtonian experimental data.

EXPERIMENTAL

Setup and Materials

Experiments were carried out in a rectangular cross-sectional tilting flume. Fluids were recirculated by means of a stainless steel centrifugal pump connected to a 0.20 m³ tank at the suction side. A heat exchanger was installed in the recirculation loop to control the fluid temperature, and hence the fluid viscosity. A 1.5 × 0.15 × 0.12 m³ (length × width × height) channel was used at the beginning of the study, and it was later modified to 3.0 × 0.21 × 0.15 m³. Upstream of the head reservoir, a small tank to ensure a constant discharge flow was installed. The channel slope could be adjusted between horizontal and 45°. The flow height could be controlled by means of two gates, one located in the upstream end of the flume, following the head reservoir (to control supercritical flows); and the second gate located in the downstream end (to control subcritical flows). A trap was located at the end of the flume to collect the particles transported by the flow. The solid particles that formed the bed were sand and glass spheres, and they were deposited in a 1 cm-thick layer, which was flattened using a smoothing board at the beginning of the experiments. The particle characteristics used in the experiments are summarized in Table 1. Aqueous solutions of sodium carboxymethyl cellulose (CMC) and carbopol were considered for measurements. They both behaved as shear-thinning fluids and were transparent, allowing visualization of the bed particles. All the mixtures were carefully blended to avoid the formation of solid lumps in the mixture. To this purpose, they were first prepared in 800 mL beakers by slowly mixing the powder at the required concentrations with warm water. For carbopol solutions, pH was controlled by adding triethanolamine (TEA), to ensure that carbopol effectively acted as a rheology modifier. The density of

Table 1. Physical properties of the particles used in the experiments

Type	Size range (mm)	d (mm)	ρ_s (kg/m ³)
1	1.0–2.0	1.5	2600
2	2.0–3.5	2.8	2610
3	3.0–5.0	4.0	2630
A	0.05–0.160	0.075	2600
B	0.065–0.180	0.106	2600
C	0.110–0.420	0.202	2600

Table 2. Rheology and density of the solutions used in the experiments

Solution	K (Pa·s ^{<i>n</i>})	n	ρ (kg/m ³)	Bed particles
CMC	0.020	0.940	1005.8	Sand
CMC	1.300	0.690	1007.9	Sand
CMC	0.030	0.850	1003.6	Sand
CMC	0.040	0.820	1005.0	Sand
CMC	0.016	0.930	1005.0	Sand
CMC	0.018	0.900	1005.0	Sand
CMC	0.060	0.830	1007.0	Sand
CMC	0.026	0.930	1007.0	Sand
CMC	0.028	0.920	1007.0	Sand
CMC	0.330	0.690	1012.2	Sand
CMC	0.240	0.730	1012.2	Sand
CMC	0.200	0.760	1010.0	Sand
CMC	0.059	0.900	1010.0	Sand
Carbopol	0.420	0.600	1003.0	Sand
Carbopol	0.420	0.600	1003.0	Sand
Carbopol	0.410	0.610	1003.0	Sand
Carbopol	0.600	0.520	1005.0	Sand
Carbopol	0.410	0.640	1005.0	Sand
Carbopol	0.370	0.65	1005.0	Sand
CMC	0.023	0.86	1003.5	Glass
CMC	0.048	0.82	1005.0	Glass
CMC	0.108	0.75	1010.5	Glass

the solutions was measured by a Gamma R.T.M. Dr. J. Ambrus densimeter, with a scale span from 1.00–1.05 relative to distilled water at 101 kPa and 15 °C. Resulting densities were in the range of 1003–1012.3 kg/m³. Rheology was determined using an Anton Paar Rheolab QC concentric cylinder rheometer instrument before and after each experiment. Both measurements were considered to fit the parameters of the Ostwald-de Waele rheological model, which are summarized in Table 2.

Visualizations and measurements were carried out in the last third of the flume. Entrance length was estimated from the de Collins and Schowalter relation for pseudoplastic fluid flows,^[38] where most of the experimental runs had an entrance length shorter than 2 m. The cases in which the length needed to have a fully developed flow at flume distances larger than 2 m correspond to those with strong generalized motion of the particles, i.e. when the condition of incipient motion was already surpassed and it was not important for the definition of the particle's mobility condition.

Flow Conditions

The experimental conditions are in the following ranges: channel slope: 0.29–3.2°; flow discharge: 0.065–8.3 L/s; flow depth: 0.5–9 cm; mean velocity: 0.8–65 cm/s. The variables that define the flow regimes, Froude and modified Reynolds numbers, Fr and Re_K , are in the ranges 0.06–1.25 and 0.5–770, respectively.

To define the flow regime, the Haldenwang et al. criterion^[39] was used. The criterion defines a modified Reynolds number given

by $Re = \frac{8\rho\bar{u}^2}{K\left(\frac{8\bar{u}}{D_h}\right)^n}$, where $D_h = \frac{4A}{P}$ is the hydraulic diameter, A is the

flow area, and P is the wetted perimeter. The flow regime is laminar when Re is less than a transitional Reynolds number obtained from the following:

$$Re_C = 853.1 \left(\frac{\mu|\dot{\gamma}_{100s^{-1}}}{\mu_w} \right)^{-0.21} Fr + 1.263 \times 10^4 \left(\frac{\mu|\dot{\gamma}_{100s^{-1}}}{\mu_w} \right)^{-0.75} \quad (25)$$

where $\mu|\dot{\gamma}_{100s^{-1}}$ is the equivalent dynamic viscosity at a shear rate of 100 s⁻¹ and μ_w is the dynamic viscosity for water. In the experiments reported in this paper, the transitional Reynolds number, Re_C , took values between 180–2965, whereas the Reynolds number Re was in the range of 2–2107. Strictly speaking, Equation (25) should be applied at flow shear rates equal to 100 s⁻¹. An estimation of the maximum shear rate can be obtained considering Equation (10), from where $\dot{\gamma}|_0 = ((2n+1)/n)(\bar{u}/H)$. In the experiments, the lowest value of $\dot{\gamma}|_0$ was 0.6 s⁻¹, associated to $Re = 66$, $Fr = 0.014$, and $Re_C = 2049$, whereas the highest value of $\dot{\gamma}|_0$ was for 185 s⁻¹, $Re = 1558$, $Fr = 1.8$, and $Re_C = 2964$. The average value of $\dot{\gamma}|_0$ was 60 s⁻¹. Although the computed shear rate $\dot{\gamma}|_0$ is only an estimation, it is not far from the specific shear rate used to obtain the transitional Reynolds number given by Equation (25). The validity of the above criterion to define the flow regime was verified by means of visualization, injecting dye (fluorescein) with a hypodermic needle. The trajectory followed by the dye was the typical undisturbed pattern of laminar motion.

Criterion to Define Incipient Motion

For the experiments with a bed formed by sand particles, three modes of particle motion were defined: no motion, incipient, and generalized. The corresponding mode of particle motion was identified visually after video recording through a 15 × 15 cm² window located 60 cm from the downstream end of the flume. When a single particle started its motion leaving the window, i.e. without coming back to the repose state, the corresponding transport mode was labelled as incipient. When a small amount of particles (< 5 % of the total of the window) left the window without going back to the repose state, the transport mode was labelled as generalized. Although this criterion is arbitrary, it was found to be robust: on changing the window and limiting percentages for motion mode assessment, results did not change significantly. In the experiments with glass spheres forming the channel bed a much simpler method was used, not involving video recording, but just observing if the particles remained on the bed or they moved under different flow conditions (the main goal of these experiments was to study bedforms). Because of the laminar nature of the flow, secondary flows were not generated close to the corners of the flume, and there were no effects due to secondary suspension that could alter the measurement of the incipient motion of the particles.

Results

The data obtained from the experiments were processed and they are condensed in a Shields-type diagram, i.e. the critical dimensionless shear stress τ_{sC} is plotted against the modified Reynolds number of the particle, $Re_{s,p}$, as shown in Figure 3. The shear stress was not corrected by wall effects. For most of the data, the effect of the shear acting on the vertical walls accounts for < 1 % of the shear stress force acting along the wetted perimeter. In order to have a quantitative estimation of the wall effects, the shear stresses were computed for some flow conditions using the CFD software COMSOL Multiphysics[®]. From the numerical simulations, the largest influence of the wall was 6.7 %, corresponding to a width/height aspect ratio of 2.3. For an aspect ratio of 3.5, the effect decreased to 1.9 %. Figure 3 also displays some curves for the condition of incipient motion. The grey lines correspond to those obtained before for a laminar regime with flow index n of 0.5 and 1, already plotted in

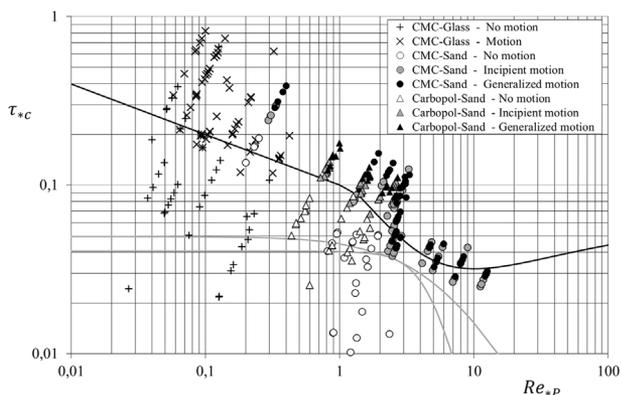


Figure 3. Experimental results in a Shields-type diagram.

Figure 2. The black line corresponds to the incipient motion condition in turbulent flow according the parameterization of Brownlie,^[40] for $Re_{*P} > 1.5$, and the correction by Mantz^[21] for $Re_{*P} < 1$, united by a smooth curve. The scattering of the data is a common feature in the definition of the Shields curve, as discussed elsewhere.^[19,20,41] In spite of the scatter of the data defining incipient motion or the boundary between motion and no motion, the data follow the general trend of the Shields curve, including the (non-monotonic) dip portion corresponding to the smooth-rough transition, for Re_{*P} between 1–100 in Figure 3. This result seems to contradict the previous analysis made for laminar flow, whereby the $\tau_{*C} - Re_{*P}$ curve monotonically decreases. The aforementioned scatter may be explained by the natural dispersion between measurements as reported elsewhere and by the effect of the flow regime, given by Re_K . To the best of the authors' knowledge, a quantitative description of the latter effect remains unexplored. If the flow effect is effectively relevant, then a proper Shields diagram should have a third axis Re_K as an additional variable. The curve proposed by Mantz, $\tau_{*C} = 0.1Re_{*P}^{-0.3}$, in the range $0.03 < Re_{*P} < 1$,^[21] seems to define reasonably well the boundaries of no motion-incipient motion and no motion-motion of the experiments, although strictly speaking, there is no reason why a relationship obtained from data for transitional laminar-turbulent or turbulent flows should apply for laminar flows.

To facilitate comparing the data obtained in the present study, data points corresponding to motion and incipient motion were plotted in Figure 4, together with the experimental data by

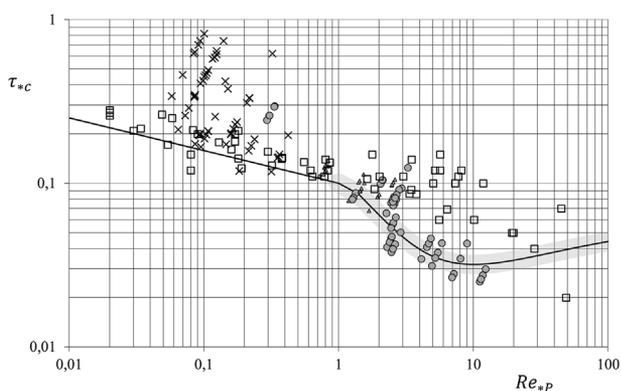


Figure 4. Comparison with data obtained with Newtonian fluids in laminar flows. Grey zone corresponds to Shields' original data band.

White,^[21] Yalin and Karahan,^[22] and Pilotti and Menduni^[23] (all of them represented by squares). It is observed from the figure that the data present lower scatter for $Re_{*P} < 1$ (the lower boundary of motion (\times) has been considered) than those for higher Reynolds numbers. For $Re_{*P} < 1$, the slope of the lower boundary of the critical condition is smaller than Mantz's and the incipient motion condition is given by $\tau_{*C} = 0.1Re_{*P}^{-0.2}$, approximately. Because of the large dispersion of the data for $Re_{*P} > 1$, and the absence of a quantitative analysis to determine the influence of the flow Reynolds number (Re_K) and possibly the Froude number, it seems safe to keep as a lower limit of the incipient condition that determined for turbulent flows.

CONCLUSIONS

The incipient motion of a spherical particle under the laminar flow of an Ostwald-de Waele ("Power-law") fluid was obtained from a force balance on a single spherical particle placed on the topmost layer of the bed. The shear stress that marks the onset of the incipient motion depends on the flow and particle characteristics. The dimensionless version of the latter critical condition, the Shields stress τ_{*C} , depends on a modified particle Reynolds number, Re_{*P} , and other parameters resulting from the geometry, flow characteristics, relative particle size, and forces acting on the particle. The algebraically cumbersome set of equations defining the critical Shields stress is greatly simplified when it is applied to small particle Reynolds numbers, resulting in an expression for τ_{*C} that does not depend on Re_{*P} . This simplified version is further reduced for small relative particle sizes ($\frac{d}{H} \ll 1$). A constant value of the critical Shields stress for small particle Reynolds in laminar flows of Newtonian fluids was proposed by White^[21] in 1940. The critical Shields stress deduced theoretically is below the experimental data encompassing Newtonian and non-Newtonian fluids. The underprediction of the critical condition resulting from the theoretical analysis is a consequence of the simplifying assumptions made, the most important being the unfavourable exposition of an isolated particle when compared with those in the experiments, which are protected by the neighbouring particles. A more complete analysis should take into account the presence of other particles, and, following the approach of Pilotti and Menduni,^[23] the non-deterministic aspect of some of the parameters involved in the equations, like the location where the forces are acting, the repose angle of the erodible bed, etc. The difference between the qualitative behaviour of the analytical relation between τ_{*C} and Re_{*P} for a laminar regime with respect to the turbulent one is attributed to a dependency on the flow Reynolds number, a topic that should be approached in the future. Additionally, the effect of the Froude number needs to be explicitly addressed.

The experimental work carried out in this research has provided an important set of data regarding the condition of incipient motion for non-Newtonian fluids in laminar flows. However, the large scatter of the data does not allow for defining a precise curve. When the data corresponding to Newtonian fluids were incorporated in the analysis, a critical condition was defined by $\tau_{*C} = 0.1Re_{*P}^{-0.2}$, $Re_{*P} < 1$, a relation that gives smaller values of the Shield stress than that obtained by Mantz.^[21] The large scatter of the data for $Re_{*P} > 1$ forces us to define a lower envelope for the critical condition which is coincident with that obtained for turbulent flows.

Thus, as a general conclusion, given the current state of experimental support regarding the condition for incipient motion of non-cohesive particles in laminar flows of non-Newtonian

fluids, for practical purposes the classical Shields curve (modified by Rouse and others) can be used, considering the modified particle Reynolds number that incorporates the non-Newtonian characteristics of the fluid, $Re_{*P} = \frac{\rho u_*^{2-n} d^n}{K}$.

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NOMENCLATURE

A	flow area (m ²)
A _D	particle projected area normal to the flow (m ²)
A _L	particle planform area (m ²)
B	buoyancy of the particle (N)
C _D	drag coefficient
C _L	lift coefficient
d	particle diameter (m)
D _h = $\frac{4A}{P}$	hydraulic diameter (m)
f	function defining the self-similar velocity profile (Equation (12))
f _{ON}	coefficient that amplifies the drag force acting on a particle in contact with a plane
F _D	drag force (N)
F _L	lift force (N)
F _R	friction (Coulombic) force (N)
Fr = $\frac{\bar{u}}{\sqrt{gh}}$	Froude number
g	magnitude of gravity acceleration (m/s ²)
H	flow depth
K	consistency coefficient in the Ostwald-de Waele rheological model (Pa · s ⁿ)
n	flow index in the Ostwald-de Waele rheological model
N	force normal to the bed (N)
P	wetted perimeter (m)
q	discharge per unit width (m ³ /s/m)
Re = $\frac{8\rho\bar{u}^2}{K\left(\frac{8\bar{u}}{D_h}\right)^n}$	flow Reynolds number used to determine the flow regime
Re _C	critical flow Reynolds number used to determine the flow regime
Re _P = $\frac{\rho u_*^{2-n} d^n}{K}$	modified particle Reynolds number
Re _K = $\frac{\rho \bar{u}^{2-n} H^n}{K}$	modified flow Reynolds number
Re _* = $\frac{u_* d}{\nu}$	particle Reynolds number based on shear velocity
Re _{*P} = $\frac{\rho u_*^{2-n} d^n}{K}$	modified particle Reynolds number based on shear velocity
u	velocity in the x direction (m/s)
U = $\frac{q}{H}$	mean flow velocity (m/s)
\bar{u}	mean flow velocity (m/s)
u _f	characteristic flow velocity used to compute the hydrodynamic forces on the particle (m/s)
u _* = $\sqrt{\frac{\tau_0}{\rho}}$	shear velocity (m/s)
x	coordinate in the flow direction (m)
y	coordinate normal to the flow direction (m)
V _P	volume of the particle (m ³)
W	weight of the particle (N)

Greek Letters

$\alpha = \frac{C_L}{C_D}$	ratio of lift and drag coefficients
β	relative location used to evaluate the hydrodynamic forces acting on the particle
$\dot{\gamma}_{ij}$	rate-of-strain tensor (s ⁻¹)
$\dot{\gamma} = \sqrt{\frac{1}{2} \dot{\gamma}_{ij} \dot{\gamma}_{ij}}$	(s ⁻¹)
θ	angle formed by the channel bottom with the horizontal
μ	Coulombic friction coefficient
$\mu \Big _{\dot{\gamma} 100 \text{ s}^{-1}}$	equivalent dynamic viscosity at a shear rate of 100 s ⁻¹ (Pa·s)
μ_w	dynamic viscosity for water (Pa·s)
ν	kinematic viscosity (m ² /s)
ρ	fluid density (kg/m ³)
ρ_s	particles density (kg/m ³)
$\tau_* = \frac{\tau_0}{(\rho_s - \rho)gd}$	dimensionless shear stress (Shields stress)
τ_C	critical Shields stress
τ_0	bottom shear stress (Pa)
τ_{oU}	bottom shear stress for the uniform flow (Pa)
τ_{ij}	viscous stress tensor (Pa)

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