INFINITELY MANY SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEM WITH NON-SYMMETRIC POTENTIALS

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Abstract. Without any symmetric conditions on potentials, we proved the following nonlinear Schrödinger system

\[
\begin{align*}
\Delta u - P(x)u + \mu_1 u^3 + \beta uv^2 &= 0, & \text{in } \mathbb{R}^2 \\
\Delta v - Q(x)v + \mu_2 v^3 + \beta vu^2 &= 0, & \text{in } \mathbb{R}^2
\end{align*}
\]

has infinitely many non-radial solutions with suitable decaying rate at infinity of potentials \( P(x) \) and \( Q(x) \). This is the continued work of [8]. Especially when \( P(x) \) and \( Q(x) \) are symmetric, this result has been proved in [18].

1. Introduction. We consider the following nonlinear Schrödinger system

\[
\begin{align*}
\Delta u - P(x)u + \mu_1 u^3 + \beta uv^2 &= 0, & \text{in } \mathbb{R}^2 \\
\Delta v - Q(x)v + \mu_2 v^3 + \beta vu^2 &= 0, & \text{in } \mathbb{R}^2
\end{align*}
\]

(1)

where \( P(x), Q(x) \) are positive potentials, \( \mu_1, \mu_2 > 0 \) and \( \beta \in \mathbb{R} \) is a coupling constant.

This type of system arises when one considers the standing wave solutions of the time dependent \( M \)-coupled Schrödinger system of the form with \( M = 2 \)

\[
\begin{align*}
-i \frac{\partial \Phi_j}{\partial t} &= \Delta \Phi_j - V_j(x) \Phi_j + \mu_j |\Phi_j|^2 \Phi_j + \Phi_j \sum_{l=1, l \neq j}^{M} \beta_{jl} |\Phi_l|^2, & \text{in } \mathbb{R}^N \\
\Phi_j = \Phi_j(x, t) &\in \mathbb{C}, & t > 0, & j = 1, \cdots, M
\end{align*}
\]

(2)

where \( \mu_j \) and \( \beta_{jl} = \beta_{lj} \) are constants. The system (2) arises in applications of many physical problems, especially in the study of incoherent solitons in nonlinear optics and in Bose-Einstein condensates. Physically, the solution \( \Phi_j \) denotes the


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$j$-th component of the beam in Kerr-like photo refractive media. The positive constant $\mu_j$ is for self-focusing in the $j$-th component of the beam. The coupling constant $\beta_{jl}$ is the interaction between the $j$-th and the $l$-th components of the beam. Physically, if $\beta_{jl} > 0$ then the interaction is attractive, while the interaction is repulsive if $\beta_{jl} < 0$.

Problem (1) also arises in the Hartee-Fock theory for a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, see for example [19, 20]. Physically, $\Phi_1, \Phi_2$ are the wave functions of the corresponding condensates, $\mu_i$ and $\beta$ are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length $\beta$ determines whether the interspecies of states are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other, which is so-called synchronization. And in the repulsive case, the components of a vector solution tend to segregate with each other leading to phase separations, which is so-called segregation. These phenomena have been documented in experiments and numeric simulations, see [5, 10, 15] and reference therein.

Mathematical work on nonlinear Schrödinger system has been studied extensively in recent years. If the domain is bounded, under the Dirichlet boundary condition, many properties are considered, e.g. local and global bifurcation structure of positive solutions ([2]), a priori bounds for positive solution and multiple existence ([7]), infinitely many positive solutions ([9]). For whole space, there are also many results. For two-component, existence of ground state is obtained in [1] and existence of two continua of bound state solutions is founded in [3]. For multiple existence, one may see [4]. For high components case, existence and non-existence are established in [11] and $k, k \in \mathbb{N}$ pairs of nontrivial spherically symmetric solutions are proved in [13]. Phase separation has been proved in several cases with constant potentials such as the work [12, 24, 25] for two components and [6, 17, 20] for high components as the coupling constant $\beta$ tends to negative infinity. In constant case $P(x) = Q(x) = 1, \beta$ is positive but small enough, then uniqueness is proved in [27]. Especially, for the case of $\mu_1 = \mu_2$, [25] gives infinitely many non-radial positive solutions for $\beta \leq -1$ which are potentially segregated type. For $N = 3$, Peng and Wang [18] proved the existence of infinitely many solutions of both synchronized and segregated types to (1) for radially symmetric positive potentials $P(|x|), Q(|x|)$ with the following algebraic decaying conditions:

(P): There are constants $a \in \mathbb{R}$, $m > 1$ and $\theta > 0$ such that as $r \to \infty$

$$P(r) = 1 + \frac{a}{r^m} + O \left( \frac{1}{r^{m+\theta}} \right).$$

(Q): There are constants $b \in \mathbb{R}$, $n > 1$ and $\sigma > 0$ such that as $r \to \infty$

$$Q(r) = 1 + \frac{b}{r^n} + O \left( \frac{1}{r^{n+\sigma}} \right).$$

The constants $a, b, m, n$ and coupling constant $\beta$ should satisfy some further conditions depending on whether the solutions are synchronized or segregated type, e.g. for segregated type, it needs $m = n, a > 0, b > 0$. In Remark 4.1 of [18], the authors point out that the synchronized result can be extended to the case of $N = 2$ with little change but they don’t know what’s the segregated case. Natural questions come: are there infinitely many segregated solutions with arbitrarily large energy for $N = 2$? If so, can we remove the symmetric conditions on $P(x)$ and $Q(x)$? In this paper, we will give affirmative answer.
Recall the key of their proof in [18] is the symmetry of the potentials $P(|x|), Q(|x|)$ and in the spirit of the work [26] where the authors consider the following nonlinear Schrödinger equation:

$$\Delta u - V(|x|)u + u^p = 0 \text{ in } \mathbb{R}^N,$$

where $1 < p < \frac{N+2}{N-2}$ and $V(|x|)$ is positive. Using the number of bubbles as parameter, with Lyapunov-Schmidt reduction at hand, they proved that problem (3) has infinitely many positive non-radial solutions if there are constants $V_\infty > 0, a > 0, m > 1$ and $\sigma > 0$ such that

$$V(r) = V_\infty + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\sigma}}\right).$$

Such idea is used very widely, see [21, 22, 23] and so on.

Recently, del Pino, J. Wei and the third author, see [8], considered the nonlinear Schrödinger equation (3) for $N = 2$ and proved the existence of infinitely many solutions when the symmetry requirement of the potential $V$ is lifted. More precisely, with so-called intermediate reduction method they proved the existence of infinitely many non-radial solutions when the potential $V$ satisfies

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^m+\sigma}\right) \text{ as } |x| \to \infty,$$

where $a > 0, m \min\left\{1, \frac{p-1}{2}\right\} > 2, \sigma > 2.$

Based on their work, we can continue to consider system case. In the following we always assume $P, Q$ satisfy the following decaying rate as $|x| \to +\infty$:

$$P(x) = 1 + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\theta}}\right), \quad Q(x) = 1 + \frac{b}{|x|^n} + O\left(\frac{1}{|x|^{n+\sigma}}\right).$$

Our main result in this paper is the following:

**Theorem 1.1.** Suppose $P(x), Q(x)$ satisfy (4) and $a, b > 0, m \neq n, m, n > 2, 2 \min\{m, n\} > \max\{m, n\} + 2, \theta, \sigma > 2.$

Then there exists $\beta^* > 0$ such that for $\beta < \beta^*$, problem (1) has infinitely many non-radial positive segregated solutions whose energy can be arbitrarily large.

**Remark 1.** It is obvious that in (4), the constant 1 can be replaced by any positive constant and Theorem 1.1 is still true.

**Remark 2.** In [18], for symmetric potentials they constructed segregated solutions through the different angles of concentrating points. Without the help of symmetry, to finish the reduction, we need to adjust angles, which means that we can’t determine angles in advance. Hence we get the segregated solutions through the different radii of two circles for concentration under the assumption $m \neq n$. We believe this is technical and make the following conjecture:

Suppose $P(x), Q(x)$ satisfy (4) and $a > 0, b > 0, m = n > 2, \theta > 2, \sigma > 2.$ Then problem (1) has infinitely many non-radial positive solutions whose energy can be arbitrarily large.

**Remark 3.** For system, new obstruction appears due to interaction $uv^2$ and $v^2u$. To get the reduction work, $2 \min\{m, n\} > \max\{m, n\} + 2$ is needed, see also Remark 7.
Remark 4. The smallness of $\beta$ is to make sure the solutions are positive and the linear system (35) is non-degenerated. For the non-degeneracy, we can take $\beta^* = \min\{w_{\mu_1}^2(0), w_{\mu_2}^2(0)\}$ by (36) where $w_{\mu}$ is defined in (6).

Remark 5. Our result can be stated and proved for the case of $\mathbb{R}^3$ with little change to the proof. We leave the proofs for interested readers.

Remark 6. In the following of the paper, without loss of generality, we assume that $m > n$.

Throughout the paper, we make use of the following notations and conventions:

- For quantities $A_K$ and $B_K$, we write $A_K \sim B_K$ to denote that $A_K/B_K$ goes to $1$ as $K$ goes to infinity; $A_K = O(B_K)$ means that $|A_K/B_K|$ are uniformly bounded while $A_K = o(B_K)$ denotes $A_K/B_K \to 0$ as $K$ tends to infinity.

- For simplicity, the letter $C$ denotes various generic constant which is independent of $K$. It is allowed to vary for different lines.

- We will use the same $|y| = \|y\|_2$ for the Euclidean norm in various Euclidean space $\mathbb{R}^2$ and denote the inner product of vectors $a$ and $b$ by $a \cdot b$.

In the next section we will show the procedure of construction and main idea in each step.

2. Description of the construction. In fact we will construct infinitely many non-radial positive solutions for system (1) to prove Theorem 1.1. So in this section let us introduce the approximation and describe the main steps of the proof briefly.

Let $w_{\mu}$ be the unique positive radial solution of the following problem:

\[
\begin{aligned}
& \Delta w - w + \mu w^3 = 0, \quad w > 0 \text{ in } \mathbb{R}^2, \\
& w(0) = \max_{x \in \mathbb{R}^2} w(x), \quad w \in H^1(\mathbb{R}^2).
\end{aligned}
\]  

(6)

It is well known that

\[\lim_{r \to \infty} r^{\frac{4}{2} - 1} w_\mu(r) = \mu^{-\frac{1}{2}} \omega_0, \quad \lim_{r \to \infty} \frac{w'_\mu(r)}{w_\mu(r)} = -1, \quad w'_\mu(r) < 0 \]  

(7)

where $\omega_0$ is a uniform constant independent of $\mu$. The non-degeneracy of $w_\mu$ will play the key role in the following proof. We will use $(w_{\mu_1}, w_{\mu_2})$ to build up the approximate solutions. Namely, the solutions we construct will be small perturbations of the sum of copies of $(w_{\mu_1}, w_{\mu_2})$.

Let $x^j_0$ be defined as

\[x^j_0 = (R \cos \theta_j, R \sin \theta_j), \quad \theta_j = \alpha_1 + (j - 1) \frac{2\pi}{K}\]

and $y^j_0$ be defined as

\[y^j_0 = (\rho \cos \theta'_j, \rho \sin \theta'_j), \quad \theta'_j = \alpha_2 + (j - 1) \frac{2\pi}{K}\]

for $j = 1, \cdots, K$. Here $\alpha_1, \alpha_2$ are two parameters dealing with the degeneracy due to rotations, and $R, \rho$ are two positive numbers which will satisfy the following so-called balancing condition:

\[a_0 m R^{-m-1} = 2 \sin \frac{\pi}{K} \Psi_1(d_1), \quad b_0 n \rho^{-n-1} = 2 \sin \frac{\pi}{K} \Psi_2(d_2), \]

(8)

where

\[a_0 = \frac{a}{2} \int_{\mathbb{R}^2} w_{\mu_1}^2 \, dx, \quad b_0 = \frac{b}{2} \int_{\mathbb{R}^2} w_{\mu_2}^2, \quad d_1 = 2 R \sin \frac{\pi}{K}, \quad d_2 = 2 \rho \sin \frac{\pi}{K}\]

(9)
and $\Psi_i$ are the interaction functions defined as follows:

$$\Psi_i(s) = -\mu_i \int_{\mathbb{R}^2} w_{\mu_i}(x - s\vec{e}) \text{div}(w_{\mu_i}^2(x)\vec{e}) dx.$$ 

Here $\vec{e}$ can be any unit vector in $\mathbb{R}^2$, see [14, 16].

We will see (Lemma 3.2) that as $K \to \infty$,

$$R \sim \frac{m}{2\pi} K \ln K, \quad \rho \sim \frac{n}{2\pi} K \ln K,$$

$$\min_{i \neq j} |x_0^i - x_0^j| = d_1 \sim m \ln K, \quad \min_{i \neq j} |y_0^i - y_0^j| = d_2 \sim n \ln K,$$

and

$$\min_{i,j=1,\ldots,K} |x_0^i - y_0^j| \geq (R - \rho) \sim \frac{m - n}{2\pi} K \ln K.$$

Next we define a small neighborhood of $Q^0 = (x_0^1, \ldots, x_0^K, y_0^1, \ldots, y_0^K)$ on $\mathbb{R}^{4K}$ in a suitable norm. To be made precise we introduce other parameters. Let $f_{ij}, g_{ij} \in \mathbb{R}$ for $i = 1, 2, j = 1, \ldots, K$, we define

$$x^j = x_0^j + f_{ij} \bar{n}_{1j} + g_{ij} \bar{t}_{1j}, \quad y^j = y_0^j + f_{ij} \bar{n}_{2j} + g_{ij} \bar{t}_{2j},$$

where

$$\bar{n}_{1j} = (\cos \theta_j, \sin \theta_j), \quad \bar{t}_{1j} = (-\sin \theta_j, \cos \theta_j),$$

$$\bar{n}_{2j} = (\cos \theta_j', \sin \theta_j'), \quad \bar{t}_{2j} = (-\sin \theta_j', \cos \theta_j').$$

Denote by

$$Q = (Q_1, \ldots, Q_{2K}) = (x^1, \ldots, x^K, y^1, \ldots, y^K).$$

A trivial but important fact is that these points are $2\pi$ periodic in $\alpha_1$ and $\alpha_2$.

We can now introduce the other parameters $q_1, q_2$ and define the norms. Denote $q_1 = (f_{11}, f_{12}, \ldots, f_{1K}, g_{11}, g_{12}, \ldots, g_{1K})$, $q_2 = (f_{21}, f_{22}, \ldots, f_{2K}, g_{21}, g_{22}, \ldots, g_{2K})$ and

$$\tilde{q}_1 = (\tilde{f}_{11}, \ldots, \tilde{f}_{1K}, \tilde{g}_{11}, \ldots, \tilde{g}_{1K}), \quad \tilde{q}_2 = (\tilde{f}_{11}, \ldots, \tilde{f}_{1K}, \tilde{g}_{11}, \ldots, \tilde{g}_{1K}),$$

where

$$\tilde{f}_{ij} = (f_{i,j+1} - f_{ij}) \frac{K}{2\pi}, \quad \tilde{g}_{ij} = (g_{i,j+1} - g_{ij}) \frac{K}{2\pi},$$

$$f_{K+1} = f_1, \quad f_0 = f_K, \quad g_{K+1} = g_1, \quad g_0 = g_K.$$

With these notations, we can define the configuration space by

$$\Lambda_K = \{Q = (Q_1, \ldots, Q_{2K}) \in \mathbb{R}^{4K} \mid x^j, y^j \text{ defined by (10)} \text{ and } \|q_i\|_* \leq 1\},$$

where $\|q_i\|_* = \|q_i\|_\infty + \|\tilde{q}_i\|_\infty + \|\hat{q}_i\|_\infty$ is a norm on $\mathbb{R}^{2K}$.

Now we define our approximate solution to be

$$(\bar{U}, \bar{V}) = \left( \sum_{j=1}^{K} w_{\mu_1}(x - x^j), \sum_{j=1}^{K} w_{\mu_2}(x - y^j) \right).$$

Note that for $Q \in \Lambda_K$, by Corollary 3.6. in [8] and (7), $0 < \bar{U}, \bar{V} \leq C, \min_{j \neq l} |x^j - x^l| = d_1 + O(K^{-1}), \min_{j \neq l} |y^j - y^l| = d_2 + O(K^{-1}),$
prove our main result. Certainly system involves more details and computations.

1.1 is necessary.

Then we want to find solutions of the form \((u,v)\) and 

\[
K^\hat{\cdot}
\]

some multipliers \(K\).

\[\text{Remark 7.}\] In the later computations, the term \(K^\hat{\cdot}\) and the multipliers \(K\).

where \(K\).

\[\text{Step 1.}\] Solving the projected problem.

Let \(\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2\) and \(q = (q_1, q_2) \in \mathbb{R}^{4K}\), we look for a solution \((\phi, \psi)\) and some multipliers \(\beta_1, \beta_2 \in \mathbb{R}^{2K}\) such that

\[
\begin{align*}
L \left( \frac{\phi}{\psi} \right) + E + N \left( \frac{\phi}{\psi} \right) &= \left( \begin{array}{c}
\beta_1 \\
\beta_2
\end{array} \right) \cdot \left( \begin{array}{c}
\frac{\partial U}{\partial q_1} \\
\frac{\partial U}{\partial q_2}
\end{array} \right), \\
\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \\
\int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \quad \forall j = 1, \ldots, K,
\end{align*}
\]

where the vector fields \(Z_{x^j}, Z_{y^j}\) are defined by

\[
Z_{x^j} = \nabla w_{\mu_1}(x - x^j), \quad Z_{y^j} = \nabla w_{\mu_2}(x - y^j).
\]

This is the first step in Lyapunov-Schmidt reduction. It is done in Section 4 through some a priori estimate and contraction mapping theorem. A required element in this step is the non-degeneracy of \(w_{\mu_1}\) and the smallness of \(\beta\). It is worth pointing out that the function \((\phi, \psi)\) and the multipliers \(\beta_1, \beta_2\) found in Step 1
depend on the parameters $\alpha$ and $q$. Hence we write \((\phi, \psi) = (\phi(\alpha, q), \psi(\alpha, q))\) and 
\[
\begin{pmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2
\end{pmatrix} = \begin{pmatrix}
\hat{\beta}_1(\alpha, q) \\
\hat{\beta}_2(\alpha, q)
\end{pmatrix}.
\]

**Step 2.** Solving the reduced problem.

First by direct calculation,
\[
\frac{\partial U}{\partial \alpha_1} = (Rq_{10} + q^1_1), \quad \frac{\partial U}{\partial \alpha_2} = (\rho q_{20} + q^2_1), \quad \frac{\partial V}{\partial q_{q_1}} = (\rho q_{20} + q^1_2),
\]
where $q_{10} = q_{20} = (0, \cdots, 0, 1, \cdots, 1)$ and $q^i = (-\tilde{q}, \tilde{f})$ for $q = (\tilde{f}, \tilde{g})$.

We define
\[
\hat{\beta}_1 = \beta_1 - \gamma_1(Rq_{10} + q^1), \quad \hat{\beta}_2 = \beta_2 - \gamma_2(\rho q_{20} + q^1),
\]
then the new multiplier $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$ depends on $\alpha$, $q$ and $\gamma = (\gamma_1, \gamma_2)$. By Lyapunov-Schmidt reduction, the step of solving $\hat{\beta}_i = 0$ will be divided into two steps.

**Step 2A:** Solving $\hat{\beta}_i = 0$ by adjusting $q$ and $\gamma$. In this step, for each $\alpha \in \mathbb{R}^2$, we are going to find $(q, \gamma)$ such that
\[
\bar{\beta}_i = 0, \quad q_i \perp q_{10},
\]
for $i = 1, 2$.

We denote the solution obtained in this step by $\gamma(\alpha), q(\alpha)$. Then the original problem is reduced to $\gamma(\alpha) = 0$.

**Step 2B:** Solving $\gamma_i = 0$ by choosing $\alpha_1, \alpha_2$.

At this last step, we want to find $\alpha \in \mathbb{R}^2$ such that $\gamma(\alpha) = 0$. As a result, the function $(U + \phi, \tilde{V} + \psi)$ is a genuine solution of (1).

This step is the second step of solving the reduced problem in the secondary Lyapunov-Schmidt reduction. To achieve this, by **Step 2A**, the function $(\phi, \psi)$ found in **Step 1** solves the following problem:

\[
\left\{
\begin{array}{l}
L \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} + E + N \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} = \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}, \\
\int_{\mathbb{R}^2} \phi Z_{x_j} = 0, \quad \int_{\mathbb{R}^2} \psi Z_{y_j} = 0, \quad \forall j = 1, \cdots, K.
\end{array}
\right.
\]

To solve $\gamma_i(\alpha_1, \alpha_2) = 0$, we first apply the variational reduction to show that $\gamma_i = 0$ has a solution if the reduced energy function $F(\alpha_1, \alpha_2) = \mathcal{E}(U + \phi, \tilde{V} + \psi)$ has a critical point, where $\mathcal{E}$ is the corresponding energy functional:
\[
\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + P(x)u^2 + |\nabla v|^2 + Q(x)v^2 - \frac{1}{4} \int_{\mathbb{R}^2} \mu_1 |u|^4 + \mu_2 |v|^4 - \frac{\beta}{2} \int_{\mathbb{R}^2} u^2 v^2.
\]

Secondly, it is easy to check that $F(\alpha_1, \alpha_2)$ is $2\pi$ periodic and $C^1$ in $\alpha_1, \alpha_2$, hence it has critical points.

Finally, the paper is organized as follows. Some preliminary facts and estimates are explained in Section 3. In Section 4, we apply the standard Lyapunov-Schmidt reduction for Step 1. In Section 5, we further reduce the problem to a two-dimensional one. In Section 6 we carry out Step 2B and complete the proof of Theorem 1.1.
3. Preliminaries. In this section, we present some preliminary facts, useful estimates whose proof can be found in [8] and the expansion of $\mathcal{E}(\bar{U}, \bar{V})$.

First recall the definition of $\Psi_i(s)$

$$\Psi_i(s) = -\mu_i \int_{\mathbb{R}^2} w_{\mu_i}(x - s\vec{e}) \text{div}(w_{\mu_i}^3(x)\vec{e}) dx$$

and we have

**Lemma 3.1** (Lemma 3.2. [8]). For $s$ sufficiently large,

$$\Psi_i(s) = c_{\mu_i} s^{\frac{1}{2}} e^{-s} \left(1 + O(s^{-1})\right)$$

where $c_{\mu_i} > 0$ are constants depending only on $\mu_i$.

Next we study the balancing condition (8).

**Lemma 3.2** (Lemma 3.3. [8]). For $K$ sufficiently large,

$$\begin{aligned}
    d_1 &= m \ln K + (m + \frac{1}{2}) \ln(m \ln K) + O(1), \\
    R &= \frac{m}{2\pi} K \ln K + \frac{1}{2\pi} (m + \frac{1}{2}) K \ln(m \ln K) + O(K), \\
    d_2 &= n \ln K + (n + \frac{1}{2}) \ln(n \ln K) + O(1), \\
    \rho &= \frac{n}{2\pi} K \ln K + \frac{1}{2\pi} (n + \frac{1}{2}) K \ln(n \ln K) + O(K).
\end{aligned}$$

(17)

If we denote

$$\mathcal{E}_V(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x) u^2) dx - \frac{1}{4} \int_{\mathbb{R}^2} u_+^4,$$

then according to Lemma 3.9 in [8], we obtain

$$\mathcal{E}_P(\bar{U}) = KA_1 + \left(a_0 + o(1)\right) \sum_{j=1}^{K} |x|^m - \frac{1}{2} \sum_{l \neq j} (\lambda_1 + o(1)) w_{\mu_1}(|x_l - x_j|)$$

$$+ O\left(KR^{-2m} + Ke^{-2d_1} d_1^2\right)$$

and

$$\mathcal{E}_Q(\bar{V}) = KA_2 + (b_0 + o(1)) \sum_{j=1}^{K} |y|^n - \frac{1}{2} \sum_{l \neq j} (\lambda_2 + o(1)) w_{\mu_2}(|y_l - y_j|)$$

$$+ O\left(K\rho^{-2n} + Ke^{-2d_2} d_2^2\right).$$

Here $A_i = \frac{1}{4} \int_{\mathbb{R}^2} \mu_i w_{\mu_i}^3 dx$, $\lambda_i = \mu_i \int_{\mathbb{R}^2} w_{\mu_i}^3 e^{-x_1} dx$, for $i = 1, 2$ and $a_0, b_0$ are defined in (9).

Obviously,

$$\mathcal{E}(\bar{U}, \bar{V}) = \mathcal{E}_P(\bar{U}) + \mathcal{E}_Q(\bar{V}) - \beta \int_{\mathbb{R}^2} \left( \sum_{i=1}^{K} w_{\mu_i}(x - x_i) \right)^2 \left( \sum_{j=1}^{K} w_{\mu_2}(x - y_j) \right)^2.$$

Hence in order to get the expression of $\mathcal{E}(\bar{U}, \bar{V})$, we just need to estimate the interaction term which actually is higher order. Indeed, for any $l, j = 1, \ldots, K$, if
4. The Lyapunov-Schmidt reduction. The aim of this section is to achieve Step 1 in the procedure of our construction described in Section 2.
We first introduce some notations. Let \( 0 < \eta < 1 \) be a constant to be determined later. For \( h = (h_1(x), h_2(x)) \), we define the following weighted norm:

\[
\| h \|_{**} = \sup_{x \in \mathbb{R}^2, i=1,2} \left| \left( \sum_{j=1}^{K} e^{-\eta|x-x^j|} + \sum_{j=1}^{K} e^{-\eta|x-y^j|} \right)^{-\frac{1}{2}} h_i(x) \right|
\]

where \( x^j, y^j \) are defined in Section 2. In what follows, we always assume that

\[
(x^1, \ldots, x^K, y^1, \ldots, y^K) \in \Lambda_K.
\]

For \( f = \left( f_1 \atop f_2 \right) \), \( g = \left( g_1 \atop g_2 \right) \), we denote by \( \langle f, g \rangle = \int_{\mathbb{R}^2} f_1 g_1 + f_2 g_2 \).

Now we state our main result in this section.

**Proposition 4.1.** Suppose \( P(x) \) and \( Q(x) \) satisfy \((4)\) and \((5)\). Then there is a \( \beta_* > 0 \) and a positive integer \( K_0 \) such that for \( \beta < \beta_* \) and all \( K \geq K_0 \), every \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) and \( q = (q_1, q_2) \) satisfying \((12)\), there exists a unique function \((\phi, \psi) \in (H^2(\mathbb{R}^2))^2 \cap \mathcal{B}_K\) and a unique multiplier \((\hat{\beta}_1, \hat{\beta}_2) \in \mathbb{R}^{4K}\) such that

\[
L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) + E + N \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \begin{pmatrix} \hat{\beta}_1 & \frac{\partial U}{\partial q_1} \\ \hat{\beta}_2 & \frac{\partial V}{\partial q_2} \end{pmatrix},
\]

\[
\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \quad \int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \quad \forall \ j = 1, \ldots, K,
\]

where

\[
\mathcal{B}_K = \left\{ (\phi, \psi) \in (L^\infty(\mathbb{R}^2))^2 : \| (\phi, \psi) \|_{**} \leq C_0 K^{-n} (\ln K)^{-\frac{1}{2}} \right\}.
\]

Here \( C_0 > 0 \) is a constant independent of \( K \). Moreover, \((\alpha, q) \to (\phi(x; \alpha, q), \psi(x; \alpha, q)) \) is of class \( C^1 \) and

\[
\sum_{i=1}^{2} (R^{-1} + \rho^{-1}) \left\| \frac{\partial (\phi, \psi)}{\partial \alpha_i} \right\|_{**} + \left\| \frac{\partial (\phi, \psi)}{\partial q} \right\|_{**} \leq C K^{-n} (\ln K)^{-\frac{1}{2}}.
\]

**4.1. Linear analysis.** Let \( M \) denote the \( 4K \times 4K \) matrix defined as follows:

\[
M_{ij} = \left\langle \frac{\partial W}{\partial q_i}, \frac{\partial W}{\partial q_j} \right\rangle, \quad i, j = 1, \ldots, 4K.
\]

where \( W = \left( \begin{array}{c} U \\ V \end{array} \right) \) and \( q = (q_1, q_2) := (q_1, \ldots, q_{4K}) \). With definition of \( U, V \) and similar computations as Lemma 4.2 in [8], one can obtain

\[
M_{jj} = \int_{\mathbb{R}^2} \left( \frac{\partial w_{1j}}{\partial q_1} \right)^2 dx = c_0, \quad M_{jl} = O(R^{-m}), \quad j \neq l, j, l = 1, \ldots, 2K,
\]

\[
M_{jj} = \int_{\mathbb{R}^2} \left( \frac{\partial w_{2j}}{\partial q_2} \right)^2 dx = c_1, \quad M_{jl} = O(\rho^{-n}), \quad j \neq l, j, l = 2K + 1, \ldots, 4K,
\]

and

\[
M_{ij} = M_{ji} = 0, \quad \forall \ j = 1, \ldots, 2K, \ l = 2K + 1, \ldots, 4K.
\]
Lemma 4.2. For linear result. obtained that for any \( p > \beta \in \mathbb{R} \),
\[
M \text{ is the 4} \times 4 \text{matrix defined in (21).}
\]

Moreover, we have the following estimate:
\[
\int_{\mathbb{R}^2} w_{\mu_1}(x-x^j)w_{\mu_2}(x-y^j)dx = O \left( e^{-\frac{m-n}{4K}lnK|x^j-y^j|^{1/2}} \right), \quad \forall j, l,
\]
\[
\int_{\mathbb{R}^2} w_{\mu_1}(x-x^j)w_{\mu_1}(x-x^l)dx = O \left( e^{-\|x^j-x^l\|_1^{1/2}} \right), \quad \forall j \neq l,
\]
\[
\int_{\mathbb{R}^2} w_{\mu_1}(x-x^j)w_{\mu_1}^p(x-x^l)dx = O \left( w_{\mu_1}(x^j-x^l) \right), \quad \forall j \neq l.
\]

Based on (25) and similar proofs of Lemma 4.2 in [8], we can deduce the following linear result.

Lemma 4.3. Under the assumption in Proposition 4.1, there exists a unique vector \( \hat{\beta} \in \mathbb{R}^{4K} \) such that \( M\hat{\beta} = b \). Moreover,
\[
\|\hat{\beta}\|_{\infty} \leq C\|\hat{b}\|_{\infty},
\]
for some \( C > 0 \) independent of \( K \).

We can now prove the following a priori estimate.

Lemma 4.4. For \( \hat{\beta} \in \mathbb{R}^{4K} \), there exists a unique vector \( \hat{\beta} \in \mathbb{R}^{4K} \) such that \( M\hat{\beta} = b \). Moreover,
\[
\|\phi\|_{\infty} + \|\psi\|_{\infty} \leq C\|h\|_{\infty},
\]
for some \( C > 0 \) independent of \( K \).

Proof. Multiply the first equation of (26) by \( \frac{\partial W}{\partial q} \) and integrate over \( \mathbb{R}^2 \) to obtain
\[
M\hat{\beta} = \left< L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right> - \left( h, \frac{\partial W}{\partial q} \right),
\]
where \( M \) is the \( 4K \times 4K \) matrix defined in (21).

Integration by parts, we have for \( j = 1, \cdots, K \),
\[
\left< L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \left( \begin{array}{c} Z_{x^j} \\ 0 \end{array} \right) \right> = \int_{\mathbb{R}^2} \left[ -(P(x) - 1) + 3\mu_1(U^2 - w_{\mu_1}(x-x^j)) + \beta V^2 \right]
\times \phi \nabla w_{\mu_1}(x-x^j) + 2\beta U V \psi \nabla w_{\mu_1}(x-x^j).
\]
By mean value theorem, for 

\[ |P(x) - 1)\nabla w_{\mu_1}(x - x^j)| \]

we have

\[
\left| \int_{\mathbb{R}^2} (P(x) - 1)\nabla w_{\mu_1}(x - x^j) \phi \right|
\]

\[
= \left| \left( \int_{\{x:|x-x^j| \leq \frac{1}{4}|x^j|\}} + \int_{\{x:|x-x^j| \geq \frac{1}{4}|x^j|\}} \right) (P(x) - 1)\nabla w_{\mu_1}(x - x^j) \phi \right|
\]

\[
\leq C \left( |x^j|^{-m-1} \ln K + |x^j|^{-m-\theta} + e^{-\frac{1}{2}|x^j||x^j|^\frac{1}{2}} \right) \|\phi\|_\infty
\]

\[
\leq C (R^{-m-1} \ln K + R^{-m-\theta}) \|\phi\|_\infty
\]

By mean value theorem, for \(|x - x^j| < 2m \ln K\),

\[
|\bar{U}^2 - w_{\mu_1}^2(x - x^j)| \leq C w_{\mu_1}(x - x^j) \sum_{l \neq j} w_{\mu_1}(x - x^l).
\]

See [8]. Thus

\[
\left| \int_{\mathbb{R}^2} (\bar{U}^2 - w_{\mu_1}^2(x - x^j)) \nabla w_{\mu_1}(x - x^j) \phi \right|
\]

\[
\leq C \int_{|x-x^j| < 2m \ln K} w_{\mu_1}(x - x^j) \sum_{l \neq j} w_{\mu_1}(x - x^l) |\nabla w_{\mu_1}(x - x^j) \phi|
\]

\[
+ \int_{|x-x^j| \geq 2m \ln K} 2 \sum_{k=1}^{K} \sum_{l \neq j} w_{\mu_1}(x - x^k)w_{\mu_1}(x - x^l) |\nabla w_{\mu_1}(x - x^j) \phi|
\]

\[
\leq C ||\phi||_\infty \left[ \int_{\mathbb{R}^2} w_{\mu_1}^2(x - x^j) \sum_{l \neq j} w_{\mu_1}(x - x^l)dx + K^2 w_{\mu_1}(2m \ln K) \int_{\mathbb{R}^2} w_{\mu_1}(x)dx \right]
\]

\[
\leq C \left( \sum_{l \neq j} w_{\mu_1}(x^j - x^l) + K^2(\ln K)^{-\frac{1}{2}} e^{-2m \ln K} \right) ||\phi||_\infty \leq C d^1 \phi e^{-d_1} ||\phi||_\infty,
\]

and

\[
\left| \int_{\mathbb{R}^2} \beta \bar{V}^2 \nabla w_{\mu_1}(x - x^j) \phi + 2\beta \bar{U} \bar{V} \nabla w_{\mu_1}(x - x^j) \phi \right|
\]

\[
\leq C K \sum_{i=1}^{K} \int_{\mathbb{R}^2} (w_{\mu_2}(x - y^i)|\phi| + w_{\mu_2}(y^i)|\psi|) |\nabla w_{\mu_1}(x - x^j)|
\]

\[
\leq C K \sum_{i=1}^{K} e^{-\frac{1}{2}|x^j - y^i|||(\phi, \psi)||_\infty} \leq C K e^{-\frac{1}{2}(R-\rho)} ||(\phi, \psi)||_\infty = o (R^{-m-5}) ||(\phi, \psi)||_\infty.
\]

Thus we have

\[
\left| \left< L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \left( \begin{array}{cc} Z_{x^j} \\ 0 \end{array} \right) \right> \right| \leq C d_1^\phi e^{-d_1} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty.
\]

Similarly, we have

\[
\left| \left< L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \left( \begin{array}{cc} 0 \\ Z_{\psi} \end{array} \right) \right> \right| \leq C d_2^\phi e^{-d_2} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty.
\]
By the exponentially decay of \( w_{\mu_1} \) at infinity, we have
\[
\left| \left< h, \frac{\partial W}{\partial q_j} \right> \right| \leq C \| h \|_{**}.
\] (30)

Combining the above estimates (28), (30), Lemma 4.2, and recall that \( d_1 > d_2 \), we get
\[
\| \hat{\beta} \|_{\infty} \leq C \left( d_2^{-\frac{1}{2}} e^{-d} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{\infty} + \| h \|_{**} \right).
\] (31)

Now we prove (27). We argue by contradiction. Assume there exist \( \left( \frac{\phi(K)}{\psi(K)} \right) \), \( h^{(K)} \) solution to (26) and
\[
\| h^{(K)} \|_{**} \to 0, \quad \left\| \left( \begin{array}{c} \phi(K) \\ \psi(K) \end{array} \right) \right\|_{**} = 1, \tag{32}
\]
as \( K \to \infty \). For simplicity, we drop \( K \) in the superscript.

First by the exponential decay of \( w_{\mu_1} \), we can make further computations ([8]). For any \( x \in \mathbb{R}^2 \setminus (\bigcup_{j=1}^{K} B(x^j, \tau) \cup \bigcup_{j=1}^{K} B(y^j, \tau)) \), where \( W_\pm = \sum_{j=1}^{K} e^{|x-x^j|} + \sum_{j=1}^{K} e^{|x-y^j|} \).

Using maximum principle in the domain \( \mathbb{R}^2 \setminus (\bigcup_{j=1}^{K} B(x^j, \tau) \cup \bigcup_{j=1}^{K} B(y^j, \tau)) \), we have the following:
\[
|\phi(x)| \leq C \left( \| L_1(\phi) \|_{**} + \sup_{j=1}^{2K} \| \phi \|_{L^\infty(B(Q_j, \tau))} \right) \sum_{l=1}^{2K} e^{-\eta|x-Q_l|}
\]
and
\[
|\psi(x)| \leq C \left( \| L_2(\psi) \|_{**} + \sup_{j=1}^{2K} \| \psi \|_{L^\infty(B(Q_j, \tau))} \right) \sum_{l=1}^{2K} e^{-\eta|x-Q_l|},
\]
where \( Q_j = x^j \) for \( j = 1, \ldots, K \) and \( Q_j = y^{j-K} \) for \( j = K+1, \ldots, 2K \), see also (11).

By the equation satisfied by \( \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \) and \( \tau \) large,e.g. \( \bar{\beta}(\frac{1}{\mu_1} + \frac{1}{\mu_2}) \) small, we have
\[
|\phi(x)| + |\psi(x)| \leq C \left( \left\| L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**} + e^{-d_2} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**} + \sup_{j=1}^{2K} \| \phi \|_{L^\infty(B(Q_j, \tau))} \right) \sum_{l=1}^{2K} e^{-\eta|x-Q_l|}.
\] (33)
By (31), the assumption (32) and the above estimate (33), there exists a subsequence of $Q_j$ such that
\[
\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(B(Q_j, \tau))} \geq C > 0,
\]
for some constant $C$ independent of $K$. Using elliptic estimates together with Ascoli-Arzela’s theorem, without loss of generality, we can find a subsequence $x_j$ such that \( \phi \left( \cdot + x^j \right) \) will converge (on any compact set) to \( \phi_\infty \) bounded by a constant times \( e^{-\eta|x|} \) solving
\[
\begin{cases}
\Delta \phi_\infty - \phi_\infty + 3 \mu_1 w^{2}_{\mu_1} \phi_\infty = 0, \\
\Delta \psi_\infty - \psi_\infty + \beta w^{2}_{\mu_2} \psi_\infty = 0, \\
\int_{\mathbb{R}^2} \phi_\infty \nabla w_{\mu_1} dy = 0
\end{cases}
\]
(35)
Since $w_{\mu_1}$ is non-degenerate and $\phi_\infty$ satisfies the orthogonality condition, one has easily $\phi_\infty = 0$. By energy analysis, if
\[
\beta < \min \{ w^{−2}_{\mu_1}(0), w^{−2}_{\mu_2}(0) \},
\]
the only possibility is $\psi_\infty = 0$, in contradiction with (34). Thus we get the a priori estimate.

Consider the space
\[
\mathcal{H} = \left\{ u = (u_1, u_2) \in (H^1(\mathbb{R}^2))^2 : \int_{\mathbb{R}^2} u_1 Z_{x^j} = 0, \int_{\mathbb{R}^2} u_2 Z_{y^j} = 0, \quad j = 1, \ldots, K \right\}.
\]
Notice that the problem (26) in $(\phi, \psi)$ gets re-written as
\[
\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \mathcal{K} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mathcal{h} \quad \text{in} \quad \mathcal{H}
\]
(37)
where $\mathcal{h}$ is defined by duality and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm’s alternative, showing that equation (26) has a unique solution for each $\mathcal{h}$ is equivalent to showing that (37) has only trivial solution when $\mathcal{h} = 0$, which in turn follows from a priori estimate (27). Furthermore, by the standard elliptic regularity result and imbedding theorem, $(\phi, \psi) \in (H^2(\mathbb{R}^2))^2$ is a strong solution. This concludes the proof of Lemma 4.3.

4.2. Nonlinear analysis. Before we give the complete proof of Proposition 4.1, we first show the estimate of the error. Recall the definition of $\Lambda_K$ in (12).

Lemma 4.4. Given $(Q_1, \ldots, Q_{2K}) \in \Lambda_K$, then for any $0 < \eta < 1$ and $K$ large enough, there is a constant $C$ independent of $K$ such that
\[
\|E\|_{**} \leq CK^{-n}(\ln K)^{-\frac{1}{2}}.
\]
Proof. First recall
\[
E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = S \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} \Delta \bar{U} - P(x)\bar{U} + \mu_1 \bar{U}^3 + \beta \bar{U} \bar{V}^2 \\ \Delta \bar{V} - Q(x)\bar{V} + \mu_2 \bar{V}^3 + \beta \bar{V} \bar{U}^2 \end{pmatrix}.
\]
First let us compute $E_1$.
\[
E_1 = -(P(x) - 1)\bar{U} + \mu_1 \left( \bar{U}^3 - \sum_{j=1}^{K} w^{3}_{\mu_1}(x - x^j) \right) + \beta \bar{V}^2 \bar{U} := I_1 + I_2 + I_3.
\]
According to Lemma 4.4 in [8], we have
\[ \|I_1\|_{**} \leq CR^{-m} \leq CK^{-m}(\ln K)^{-m}, \]
and
\[ \|I_2\|_{**} \leq Cd_1^{\frac{1}{2}}e^{-d_1} \leq CK^{-m}(\ln K)^{-\frac{1}{2}}. \]
Next we deduce the estimate for \(I_3\). First we define for any \(\ell \in \mathbb{N}\)
\[ \Omega_{j}^\ell = \left\{ x \in \mathbb{R}^2, |x - Q_j| = \min_{1 \leq l \leq 2K} |x - Q_l| \leq \frac{\ell d_2}{2} \right\}, \quad j = 1, \ldots, 2K \]
and
\[ \Omega_{2K+1}^\ell = \mathbb{R}^2 \setminus \bigcup_{j=1}^{2K} \Omega_{j}^\ell. \]
For \(x \in \Omega_{2K+1}^\ell\),
\[ |I_3| \leq CK \sum_{l,j=1,\ldots,K} w_{\mu_2}(x - y^l)w_{\mu_1}(x - x^l) \]
\[ \leq CK \sum_{j=1}^{2K} e^{-|x - Q_j|} \leq CKe^{-(1-\eta)\frac{\ell d_2}{2}} \sum_{j=1}^{2K} e^{-\eta|x - Q_j|}, \]
thus one can choose \(\ell\) large enough but independent of \(K\) such that
\[ Ke^{-(1-\eta)\frac{\ell d_2}{2}} \leq CK^{-m-3}. \]
For \(x \in \Omega_{j}^\ell, j = 1, \ldots, K\), Corollary 3.6 in [8] tells us
\[ \sum_{l=1}^{K} w_{\mu_1}(x - x^l) \leq C\ell w_{\mu_1}(x - x^j), \]
which leads to
\[ |I_3| \leq Cw_{\mu_1}(x - x^j)V^2 \leq CK^2e^{-(R-\rho)}e^{-\eta|x - x^j|} \leq CR^{-m-3}e^{-\eta|x - x^j|}, \]
because of \(|x - y^l| \geq |x^j - y^l| - |x - x^j| \geq \frac{1}{2}(R - \rho)\).
For \(x \in \Omega_{j}^\ell, j = K + 1, \ldots, 2K\), similarly
\[ |I_3| \leq C \sum_{l=1}^{K} w_{\mu_1}(x - x^l)w_{\mu_2}^2(x - y^l) \leq CR^{-m-3}e^{-\eta|x - y^l|}. \]
Hence
\[ |E_1| \leq CK^{-m}(\ln K)^{-\frac{1}{2}} \sum_{j=1}^{2K} e^{-\eta|x - Q_j|}. \]
Similarly,
\[ |E_2| \leq CK^{-n}(\ln K)^{-\frac{1}{2}} \sum_{j=1}^{2K} e^{-\eta|x - Q_j|}. \]
In conclusion,
\[ \|E\|_{**} \leq CK^{-n}(\ln K)^{-\frac{1}{2}}. \]
We are now in the position to give the proof of Proposition 4.1. Let $C_0$ be a positive constant to be determined later, we define

$$B_K = \left\{ (\phi, \psi) \in (L^\infty(\mathbb{R}^2))^2 : \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**} \leq C_0 K^{-n}(\ln K)^{-\frac{1}{2}} \right\}.$$ 

Then $B_K$ is non-empty. Now we define a map $A$ by

$$A \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = L^{-1} \left[ -E - N \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right].$$

Now solving equation (19) is equivalent to finding a fixed point for the map $A$. Since $\left( \begin{array}{c} \phi \\ \psi \end{array} \right)$ is uniformly bounded for $\left( \begin{array}{c} \phi \\ \psi \end{array} \right) \in B_K$, by the mean value theorem, there is a positive constant $C$ such that for all $\left( \begin{array}{c} \phi \\ \psi \end{array} \right) \in B_K$,

$$\left| N \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right| \leq C \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{\infty}^2,$$

and

$$\left| N \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - N \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right| \leq C \sum_{i=1}^2 \left\| \left( \begin{array}{c} \phi^{(i)} \\ \psi^{(i)} \end{array} \right) \right\|_{\infty} \left\| \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right\|_{\infty},$$

which leads to

$$\left\| N \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - N \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right\|_{**} \leq C \sum_{i=1}^2 \left\| \left( \begin{array}{c} \phi^{(i)} \\ \psi^{(i)} \end{array} \right) \right\|_{**} \left\| \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right\|_{**}.$$

Hence we obtain

$$\left\| A \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**} \leq C \left( \left\| E \right\|_{**} + \left\| N \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**} \right) \leq C K^{-n}(\ln K)^{-\frac{1}{2}} + C \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_{**}^2,$$

and

$$\left\| A \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - A \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right\|_{**} \leq \frac{1}{2} \left\| \left( \begin{array}{c} \phi^{(1)} \\ \psi^{(1)} \end{array} \right) - \left( \begin{array}{c} \phi^{(2)} \\ \psi^{(2)} \end{array} \right) \right\|_{**},$$

which show that $A$ is a contraction mapping on $B_K$. Hence there is a unique $(\phi, \psi) \in B_K$ such that (19) is solved.

For the $C^1$ regularity of $(\alpha, q) \rightarrow ((\phi, \psi), \hat{\beta})$, the proof is the same as that of Proposition 4.1 in [8]. Following the same argument and using the estimate on $(\phi, \psi)$, one can get the estimate (20), we omit the details here.

5. The reduced problem. The main purpose of this section, is to achieve Step 2A. As we mentioned in the introduction, we define

$$\hat{\beta}_1 = \hat{\beta}_1 - \gamma_1 (Rq_{10} + q_{1}^\perp), \quad \hat{\beta}_2 = \hat{\beta}_2 - \gamma_2 (\rho q_{20} + q_{2}^\perp), \quad (38)$$

for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$. 
Then equation (19) becomes

\[
L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) + E + N \left( \begin{array}{c} \phi \\ \psi \end{array} \right) = \left( \begin{array}{c} \vec{\beta}_1 \cdot \nabla U + \gamma_1 \frac{\partial U}{\partial \alpha_1} \\ \vec{\beta}_2 \cdot \nabla V + \gamma_2 \frac{\partial V}{\partial \alpha_2} \end{array} \right).
\] (39)

Note that \((\phi, \psi)\) does not depend on \(\gamma\), while \(\vec{\beta} = (\vec{\beta}_1, \vec{\beta}_2)\) depends on the parameters \(\alpha, q, \gamma\), so we write it as \(\vec{\beta} = \vec{\beta}(\alpha, q, \gamma)\).

In this section, we are going to solve \(\vec{\beta} = 0\) for each \(\alpha\) by adjusting \(\gamma\) and \(q\). Multiplying (39) by \(\frac{\partial W}{\partial q}\) and integrating over \(\mathbb{R}^2\), we have

\[
\left\langle L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) + E + N \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right\rangle = M\vec{\beta} + \left\langle \left( \begin{array}{c} \gamma_1 \frac{\partial U}{\partial \alpha_1} \\ \gamma_2 \frac{\partial V}{\partial \alpha_2} \end{array} \right), \frac{\partial W}{\partial q} \right\rangle.
\]

By Lemma 4.2, solving \(\vec{\beta} = 0\) amounts to solve

\[
\left\langle L \left( \begin{array}{c} \phi \\ \psi \end{array} \right) + E + N \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right\rangle = \left\langle \left( \begin{array}{c} \gamma_1 \frac{\partial U}{\partial \alpha_1} \\ \gamma_2 \frac{\partial V}{\partial \alpha_2} \end{array} \right), \frac{\partial W}{\partial q} \right\rangle.
\]

We will first compute the projection of error and the terms involving \((\phi, \psi)\).

5.1. **Projections.** We first compute \(\langle E, \frac{\partial W}{\partial q} \rangle\).

**Lemma 5.1.** Under the assumption of Proposition 4.1, for sufficiently large \(K\), the following estimates hold:

\[
\int_{\mathbb{R}^2} E_1 Z_{x^j} = -a_0 m |x^j|^{-m-1} \frac{x^j}{|x^j|} - \sum_{l \neq j} \Psi_1(x^j - x^l) \frac{x^l - x^j}{|x^l - x^j|} + R^{-m-\eta} \Pi_{1k}(\alpha, q) + R^{-m-3} \Pi_{2k}(\alpha, q) + R^{-(2-\eta) m} \Pi_{3k}(\alpha, q),
\] (40)

and

\[
\int_{\mathbb{R}^2} E_2 Z_{y^j} = -b_0 n |y^j|^{-n-1} \frac{y^j}{|y^j|} - \sum_{l \neq j} \Psi_2(y^j - y^l) \frac{y^l - y^j}{|y^l - y^j|} + \rho^{-n-\sigma} \Pi_{4k}(\alpha, q) + \rho^{-n-3} \Pi_{5k}(\alpha, q) + \rho^{-(2-\eta) n} \Pi_{6k}(\alpha, q),
\] (41)

where \(a_0 = \frac{\alpha}{2} \int_{\mathbb{R}^2} w_{\mu_1}^2(y) dy\), and \(b_0 = \frac{\beta}{2} \int_{\mathbb{R}^2} w_{\mu_2}^2(y) dy\). \(\eta\) is a small positive constant chosen later and \(\Pi_{lk}(\alpha, q), l = 1, \ldots, 6\) are smooth vector valued functions which are uniformly bounded as \(K \to \infty\).
Proof. By definition, we can easily deduce
\[ \int_{\mathbb{R}^2} E_1 z_{x^j} = -\sum_{l=1}^{K} \int_{\mathbb{R}^2} (P(x) - 1) w_{\mu_1} (x - x^j) \nabla w_{\mu_1} (x - x^j) \]

\[ + \mu_1 \int_{\mathbb{R}^2} \left( \sum_{l=1}^{K} w_{\mu_1} (x - x^j) \right)^2 \nabla w_{\mu_1} (x - x^j) \]

\[ + \beta \int_{\mathbb{R}^2} \left( \sum_{l=1}^{K} w_{\mu_2} (x - y^l) \right) \left( \sum_{l=1}^{K} w_{\mu_1} (x - x^j) \right) \nabla w_{\mu_1} (x - x^j) \]

\[ = - J_1 + J_2 + J_3. \]

From Lemma 5.1 in [8], we have
\[ J_1 = a_0 m |x|^{|m-1| \frac{n}{|x^j|}} + R^{-m-\theta} \tilde{\Pi}_{1k} (\alpha, q) + R^{-m-\theta} \tilde{\Pi}_{2k} (\alpha, q) + R^{-2m} \tilde{\Pi}_{3k} (\alpha, q), \]

and
\[ J_2 = - \sum_{l \neq j} \Psi_1 (x^j - x^j) \frac{x^j - x^j}{|x^j - x^j|} + R^{-(2-\eta)m} \tilde{\Pi}_{4k} (\alpha, q). \]

Using for any \( j, l = 1, \ldots, K, |x^j - y^l| \geq R - \rho - 4 \sim \frac{m-\alpha}{2\pi} K \ln K, \) one can easily check that
\[ |J_3| \leq C K^2 \int_{\mathbb{R}^2} \sum_{l=1}^{K} w_{\mu_1}^2 (x - y^l) |\nabla w_{\mu_1} (x - x^j)| dx = K^{-m-5} \tilde{\Pi}_{5k} (\alpha, q). \]

Combining the above three estimates, we obtain (40).

Similarly, we can get the estimate (41). \( \square \)

Now we can analyze \( (E_1, \frac{\partial W}{\partial q}). \) Before we start, we define the following:
\[ \tilde{d}_1 = -\frac{\Psi_1 (d_1)}{\Psi_1 (d_1)} d_1 = d_1 + O(1), \]

\[ \tilde{d}_2 = -\frac{\Psi_2 (d_2)}{\Psi_2 (d_2)} d_2 = d_2 + O(1) \]

Then by Lemma 5.2 in [8], Lemma 5.1 and
\[ \frac{\partial \hat{U}}{\partial q} = \frac{\partial U}{\partial q_1}, \]

\[ \hat{q}_1 = - (Z_{x^j} \cdot \bar{n}_{11}, \ldots, Z_{x^j} \cdot \bar{n}_{1K}, Z_{x^j} \cdot \bar{t}_{11}, \ldots, Z_{x^j} \cdot \bar{t}_{1K}, \bar{\tilde{q}}_1)^T, \]

\[ \frac{\partial \hat{V}}{\partial q} = (\bar{\tilde{q}}_2, \frac{\partial \hat{V}}{\partial q_2}) = - (\bar{\tilde{q}}_2, \bar{\tilde{q}}_2, \bar{\tilde{q}}_2, \bar{\tilde{q}}_2)^T \]

we have the following estimates:

**Lemma 5.2.** Under the assumption of Proposition 4.1, for \( K \) large enough, we can get the following estimates:
\[ \int_{\mathbb{R}^2} E_1 \frac{\partial \hat{U}}{\partial q_1} = a_0 m R^{-m-2} T_1 q_1 + R^{-m-\theta} \Pi_{1k} + R^{-m-3} \Pi_{2k} + R^{-(2-\eta) \xi} \]

\[ + R^{-m-3} (\ln K)^2 \Pi_{4k} (\alpha, q_1, q_1, \bar{\tilde{q}}_1), \]

and
\[ \int_{\mathbb{R}^2} E_2 \frac{\partial \hat{V}}{\partial q_2} = b_0 n R^{-n-2} T_2 q_2 + R^{-n-\sigma} \Pi_{5k} + R^{-n-3} \Pi_{6k} + R^{-(2-\eta) \xi} \Pi_{7k} + R^{-n-3} (\ln K)^2 \Pi_{8k} (\alpha, q_1, q_2, \bar{\tilde{q}}_2). \]
where \( \Pi_{ik} \) are uniformly bounded smooth vector functions with
\[
\Pi_{4k}(\alpha, q_2, 0, 0, 0) = 0, \quad \Pi_{8k}(\alpha, q_1, 0, 0, 0) = 0
\]
and \( T_1, T_2 \) are \( 2K \times 2K \) matrix defined by the following:
\[
T_1 = \begin{pmatrix}
c_1A_1 + c_4I & c_2A_2 \\
-c_2A_2 & c_3A_1
\end{pmatrix}, \quad (42)
\]
\[
T_2 = \begin{pmatrix}
\hat{c}_1A_1 + \hat{c}_4I & \hat{c}_2A_2 \\
-\hat{c}_2A_2 & \hat{c}_3A_1
\end{pmatrix}, \quad (43)
\]
Here \( I \) is the \( K \times K \) identity matrix, and the matrix \( A_1, A_2 \) are both \( K \times K \) circulant matrices given by the following:
\[
A_1 = \begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 & 1 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
1 & 0 & \cdots & 0 & 1 & -2
\end{pmatrix},
\]
\[
A_2 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & -1 \\
-1 & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & -1 & 0 & 1 \\
1 & 0 & \cdots & 0 & -1 & 0
\end{pmatrix},
\]
and \( c_1, c_2, c_3, c_4, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4 \) are constants given by
\[
c_1 = \frac{K^2}{4\pi^2}, \quad c_2 = (\hat{d}_1 - 1) \frac{K}{4\pi}, \quad c_3 = -\hat{d}_1 \frac{K^2}{4\pi^2}, \quad c_4 = \hat{d}_1 - m - 1,
\]
\[
\hat{c}_1 = \frac{K^2}{4\pi^2}, \quad \hat{c}_2 = (\hat{d}_2 - 1) \frac{K}{4\pi}, \quad \hat{c}_3 = -\hat{d}_2 \frac{K^2}{4\pi^2}, \quad \hat{c}_4 = \hat{d}_2 - n - 1.
\]
Next we consider the terms involving \((\phi, \psi)\).

**Lemma 5.3.** Under the assumption of Proposition 4.1, for \( K \) large enough, the following estimates hold:
\[
\left\| \left\langle L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right\rangle \right\| \leq CK^{-2n}(\ln K)^{-1} \Pi_{9,k}(\alpha, q),
\]
and
\[
\left\| \left\langle N \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right\rangle \right\| \leq C \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty^2 \leq CK^{-2n}(\ln K)^{-1} \Pi_{10,k}(\alpha, q) \quad (44)
\]
where \( \Pi_{9,k}, \Pi_{10,k} \) are uniformly bounded smooth vector functions.

**Proof.** Integrating by parts, with Proposition 4.1 and (28), (29), we can deduce
\[
\left\| \left\langle L \left( \begin{array}{c} \phi \\ \psi \end{array} \right), \frac{\partial W}{\partial q} \right\rangle \right\| \leq Cd_2^{-\frac{1}{2}}e^{-d_2} \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty \leq CK^{-2n}(\ln K)^{-1}.
\]
For (44),
\[
\left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty \leq CK^{-n}(\ln K)^{-\frac{1}{2}}, \quad \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\| \leq C \left\| \left( \begin{array}{c} \phi \\ \psi \end{array} \right) \right\|_\infty^2,
\]
give us
\[
\left\| N \left( \phi, \psi \right) \right\| \leq C \left\| \phi, \psi \right\|_{**} \leq CK^{-2n}(\ln K)^{-1}.
\]

5.2. The invertibility of $T_i$. In this subsection, we study the linear problem $T_i q_i = b_i$. First by Lemma 8.3 and Lemma 8.5 in [8], we have the following:

**Lemma 5.4.** There exists $K_0 > 0$ such that for all $K \geq K_0$, and every $b, q \in \mathbb{R}^{2K}$, there exist unique vectors $q_1, q_2$ and unique constants $\gamma_1, \gamma_2$ such that
\[
T_i q_i = b + \gamma_i q_i, \quad q_i \perp q_{i0}, \quad i = 1, 2.
\]
Moreover
\[
\|q_i\|_2 \leq C\|b\|_2, \quad \|q_i\|_2 \leq C(\ln K)^{1/2}\|b\|_2, \quad \|q_i\|_2 \leq C(\ln K)^2\|b\|_2,
\]
and
\[
\|q_i\|_* \leq C(\ln K)^2\|b\|_*.
\]

Here $\|q_i\|_* = \|q_i\|_\infty + \|q_i\|_* + \|q_i\|_\infty$.

With Lemma 5.1, we can conclude

**Lemma 5.5.** There exists $K_0 > 0$ such that for $K \geq K_0$, and every $b, q, \in \mathbb{R}^{2K}$, there exist unique vector $q_i \in \mathbb{R}^{2K}$ and unique constant $\gamma_i \in \mathbb{R}$ such that
\[
T_i q_i = b_i + \gamma_i q_i, \quad q_i \perp q_{i0},
\]
where
\[
\left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right) = \left( \begin{array}{cc}
\int_{\mathbb{R}^2} \frac{\partial U}{\partial \alpha_1} \frac{\partial q_1}{\partial \alpha_1} & \int_{\mathbb{R}^2} \frac{\partial U}{\partial \alpha_2} \frac{\partial q_1}{\partial \alpha_2} \\
\int_{\mathbb{R}^2} \frac{\partial V}{\partial \alpha_1} \frac{\partial q_2}{\partial \alpha_1} & \int_{\mathbb{R}^2} \frac{\partial V}{\partial \alpha_2} \frac{\partial q_2}{\partial \alpha_2}
\end{array} \right) = M \left( \begin{array}{c}
R q_{10} + q_1 \\
\rho q_{20} + q_2
\end{array} \right).
\]

Moreover,
\[
\|q_i\|_* \leq C(\ln K)^2\|b_i\|_\infty.
\]

**Proof.** To prove Lemma 5.5, it suffices to prove a priori estimate (46). Using the definition of $q_1, q_2$ in (45) and (22), (23), we find that
\[
R^{-1} q_1 = c_0 q_{10} + O(KR^{-1}), \quad \rho^{-1} q_2 = c_1 q_{20} + O(K\rho^{-1}),
\]
which imply that
\[
\|R^{-1} q_1\|_\infty \leq C, \quad \|\rho^{-1} q_2\|_\infty \leq C
\]
and
\[
|R^{-1} q_1 \cdot q_{10}| \geq CK, \quad |\rho^{-1} q_2 \cdot q_{20}| \geq CK.
\]
Hence take
\[
\gamma_i = -\frac{b_i \cdot q_{i0}}{q_i \cdot q_{i0}},
\]
then
\[
\|\gamma_i q_i\|_\infty = \left\| \frac{b_i \cdot q_{i0}}{R^{-1} q_i \cdot q_{i0}} R^{-1} q_i \right\|_\infty \leq C\|b_i\|_\infty,
\]
So does $\gamma_i q_i$. Therefore, by Lemma 5.4, we have
\[
\|q_i\|_* \leq C(\ln K)^2\|b_i + \gamma_i q_i\|_\infty \leq C(\ln K)^2\|b_i\|_\infty.
\]

Denote the inverse of $T_i$ by $T^{-1}_i$ and $q_i = T^{-1}_i(b_i)$.
Lemma 5.7. For every follows: results in Lemma 5.1, Lemma 5.2 and Lemma 5.3, we can rewrite this equation as

\[
\left\{ \begin{array}{l}
L \begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \gamma_1 \partial U \\ \gamma_2 \partial V \end{pmatrix}, \\
\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \quad \int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \quad j = 1, \ldots, K.
\end{array} \right.
\]

Under the assumption of Theorem 1.1, there exists an integer Proposition 5.6.

5.3. Reduction to two dimensions.

Proof of Proposition 5.6. Under the assumption of Theorem 1.1, there exists an integer $K_0 > 0$ such that for all $K > K_0$ and for each $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, there exists a unique $(q, \gamma)$ such that $\beta = 0$. As a result $(\phi, \psi), \gamma$ satisfy the equation:

\[
\left\{ \begin{array}{l}
L \begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \gamma_1 \partial U \\ \gamma_2 \partial V \end{pmatrix}, \\
\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \quad \int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \quad j = 1, \ldots, K.
\end{array} \right.
\]

Moreover, the function $(\phi, \psi)$ is $C^1$ in $\alpha$, and satisfies the following:

\[
\|(\phi, \psi)\|_{\ast \ast} \leq C_0 K^{-n}(\ln K)^{-\frac{1}{2}}, \quad \sum_{i=1}^{2} (R^{-1} + \rho^{-1}) \| \frac{\partial q_i}{\partial \alpha_i} \|_{\ast} + \| q_i \|_{\ast} \leq K^{-\mu} \ln^2 K.
\]

(47)

for some $\mu > 0$ small enough but independent of $K$.

To prove Proposition 5.6, it suffices to solve $\tilde{\beta}_i = 0$ for each $(\alpha_1, \alpha_2)$. By the results in Lemma 5.1, Lemma 5.2 and Lemma 5.3, we can rewrite this equation as follows:

Lemma 5.7. For every $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, the equation $\tilde{\beta}_i(\alpha, q, \gamma) = 0$ is equivalent to

\[
a_0 m R^{-m-2} T_1 q_1 + \Phi_1(\alpha, q) = \gamma_1 q_1^1, \quad b_0 n \rho^{-n-2} T_2 q_2 + \Phi_2(\alpha, q) = \gamma_2 q_2^2
\]

(48)

where $T_i$ are the $2K \times 2K$ matrix defined in (42) and (43), $\Phi_i$ denotes the remaining terms and $q_1^1, q_2^2$ are given in (45).

Here

\[
\Phi_1(\alpha, q) = R^{-m-\eta} P_{ik} + R^{-m-3} P_{2k} + R^{-(2-\eta)n} P_{3k} + K^{-2n}(\ln K)^{-1} P_{5k}(\alpha, q) + R^{-m-3}(\ln K)^2 P_{4k}(\alpha, q_1, q_2, q_1, q_2),
\]

and

\[
\Phi_2(\alpha, q) = \rho^{-n-\sigma} P_{5k} + \rho^{-n-3} P_{6k} + \rho^{-(2-\eta)n} P_{7k} + K^{-2n}(\ln K)^{-1} P_{10k}(\alpha, q) + \rho^{-n-3}(\ln K)^2 P_{8k}(\alpha, q_1, q_2, q_2, q_2)
\]

where $P_i$ are uniformly bounded smooth vector functions, and

\[
P_{4k}(\alpha, q_2, 0, 0, 0) = 0, \quad P_{5k}(\alpha, q_1, 0, 0, 0) = 0.
\]

We are now going to prove Proposition 5.6.

Proof of Proposition 5.6. By Lemma 5.5, equation (48) is equivalent to

\[
q_1 = (a_0 m)^{-1} T_1^{-1}(R^{m+2} \Phi_1) = F_1(q), \quad q_2 = (b_0 n)^{-1} T_2^{-1}(\rho^{n+2} \Phi_2) = F_2(q).
\]

By Lemma 5.7 and the assumption $m, n > 2, 2n > m + 2, \theta, \sigma > 2$, we can choose $\eta$ small enough such that $(1 - \eta)n > 2$, then there exists $\mu$ small enough, but independent of $K$, such that

\[
R^{m+2} \Phi_1(\alpha, q) = K^{-\eta} \Pi_1 + (K^{-1} \ln^2 K)^{\tilde{E}_1}, \quad \rho^{n+2} \Phi_2(\alpha, q) = K^{-\mu} \Pi_2 + (K^{-1} \ln^2 K)^{\tilde{E}_2}
\]

where $\Pi_1, \Pi_2, \tilde{E}_1, \tilde{E}_2$ are smooth bounded vector functions, $\tilde{E}_1(\alpha, q_2, 0, 0, 0) = 0$ and $\tilde{E}_2(\alpha, q_1, 0, 0, 0) = 0$. 
Hence by Lemma 5.5, for \( \|q_1\|_\ast + \|q_2\|_\ast < \frac{1}{2} \), we have
\[
\|\mathcal{F}_i(q)\|_\ast \leq C \left( K^{-\mu} \ln^2 K + K^{-1} \ln^4 K \right) \leq CK^{-\mu} \ln^2 K,
\]
and
\[
\|\mathcal{F}_i(q) - \mathcal{F}_i(q^\circ)\|_\ast \leq C \left( K^{-\mu} \ln^2 K + K^{-1} \ln^4 K \right) (\|q_1 - q_1^\circ\|_\ast + \|q_2 - q_2^\circ\|_\ast)
\]
\[
\leq \frac{1}{2} (\|q_1 - q_1^\circ\|_\ast + \|q_2 - q_2^\circ\|_\ast).
\]
Therefor \( \mathcal{F}_1, \mathcal{F}_2 \) are contraction mappings. By Banach fixed point theorem, the result follows and so does the estimate (47).

Moreover, to show the differentiability of \( q(\alpha) \), consider the map \( T(\alpha, q) = q - (F_1, F_2) : \mathbb{R}^2 \times \mathbb{R}^{4K} \to \mathbb{R}^{4K} \) which is of class \( C^1 \). Since \( \frac{\partial(F_1, F_2)}{\partial q} = O(K^{-\mu-1} \ln^2 K) \), \( \frac{\partial T}{\partial q} = I + o(1) \) is invertible, we get the differentiability of \( q(\alpha) \).

Next we study the dependence of \( q \) on \( \alpha \). Assume that we have two solutions corresponding to two sets of parameters. One of them denoted by
\[
q = \left( (\alpha_0m)^{-1}T_{1,q}^{-1} \left[R^{m+2}\Phi_1(\alpha, q)\right], (b_0n)^{-1}T_{2,q}^{-1} \left[\rho^{n+2}\Phi_2(\alpha, q)\right]\right),
\]
corresponding to \( \alpha \), the other denote by
\[
q^\circ = \left( (\alpha_0m)^{-1}T_{1,q}^{-1} \left[R^{m+2}\Phi_1(\alpha^\circ, q^\circ)\right], (b_0n)^{-1}T_{2,q}^{-1} \left[\rho^{n+2}\Phi_2(\alpha^\circ, q^\circ)\right]\right),
\]
corresponding to \( \alpha^\circ \). Assume that \( R|\alpha_1^\circ - \alpha_1| + \rho|\alpha_2^\circ - \alpha_2| \leq \frac{1}{2} \), we have
\[
\|q^\circ - q\|_\ast \leq CK^{-\mu}(\ln K)^2(R|\alpha_1^\circ - \alpha_1| + \rho|\alpha_2^\circ - \alpha_2|),
\]
from which we get the desired result. \( \square \)

6. Proof of Theorem 1.1. In this section, we prove the main theorem. To solve \( \gamma(\alpha) = 0 \), we will apply the variational reduction. Let \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \) and
\[
\left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = \left( \begin{array}{c}
\phi(\alpha, q(\alpha)) \\
\psi(\alpha, q(\alpha))
\end{array} \right)
\]
be the function given in Proposition 5.6, we define the reduced energy function by
\[
F(\alpha) = \mathcal{E} \left( \begin{array}{c}
\bar{U} + \phi \\
\bar{V} + \psi
\end{array} \right) : \mathbb{R}^2 \to \mathbb{R}.
\]
Here \( (\bar{U}, \bar{V}) \) and \( (\phi, \psi) \) are \( 2\pi \) periodic in \( \alpha_1, \alpha_2 \). Hence by Proposition 5.6, the reduced energy have the following property:

**Lemma 6.1.** The functional \( F(\alpha) \) is of class \( C^1 \) and \( 2\pi \) periodic in \( \alpha_1, \alpha_2 \).

Next lemma shows that if \( F(\alpha) \) has a critical point then \( \gamma(\alpha) = 0 \) has a solution. In other words, after Lyapunov-Schmidt reduction, the following lemma concerns the relationship between the critical points of \( F(\alpha) \) and those of the energy functional \( \mathcal{E} \left( \begin{array}{c}
u \\
v
\end{array} \right) \).

**Lemma 6.2.** Under the assumption of Proposition 5.6, there exists \( K_0 > 0 \), such that for all \( K > K_0 \), if \( \alpha_0 \) is a critical point of \( F(\alpha) \), then \( \gamma(\alpha_0) = 0 \), and the corresponding function \( \left( \begin{array}{c}
\bar{U} + \phi \\
\bar{V} + \psi
\end{array} \right) \) is a solution of (1).
Proof. By Proposition 5.6 for $K$ large and $\alpha \in \mathbb{R}^2$, $\phi$ satisfies the equation:

$$S \left( \bar{U} + \phi \atop \bar{V} + \psi \right) = \begin{pmatrix} \gamma_1 \frac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \frac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}. \tag{49}$$

By the definition of $F$, obviously,

$$\nabla F(\alpha_1, \alpha_2) = \left\langle S \left( \bar{U} + \phi \atop \bar{V} + \psi \right), \nabla \alpha \left( \bar{U} + \phi \atop \bar{V} + \psi \right) \right\rangle,$$

where for $i = 1, 2$

$$\begin{cases} 
\partial_{\alpha_i} (\bar{U} + \phi) = \frac{\partial (\bar{U} + \phi)}{\partial \alpha_i} + \frac{\partial (\bar{U} + \phi)}{\partial q} \cdot \frac{\partial q}{\partial \alpha_i}, \\
\partial_{\alpha_i} (\bar{V} + \psi) = \frac{\partial (\bar{V} + \psi)}{\partial \alpha_i} + \frac{\partial (\bar{V} + \psi)}{\partial q} \cdot \frac{\partial q}{\partial \alpha_i}.
\end{cases}$$

Thus using (49), we obtain

$$\partial_{\alpha_i} F(\alpha_1, \alpha_2) = \left\langle \begin{pmatrix} \gamma_1 \frac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \frac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}, \begin{pmatrix} \partial_{\alpha_i} (\bar{U} + \phi) \\ \partial_{\alpha_i} (\bar{V} + \psi) \end{pmatrix} \right\rangle.$$

If $\alpha_0$ is a critical point of $F$, that is, $\nabla F(\alpha_0) = 0$, then it is easily observed that $(\gamma_1, \gamma_2) = 0$ is equivalent to the non-degeneracy of the following matrix

$$\begin{pmatrix} \frac{\partial \bar{U}}{\partial \alpha_1} & \frac{\partial \bar{V}}{\partial \alpha_1} & \frac{\partial \bar{V}}{\partial \alpha_2} \\
\frac{\partial \bar{U}}{\partial \alpha_2} & \frac{\partial \bar{V}}{\partial \alpha_2} & \frac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}. \tag{50}$$

With definitions in (10) and (13), one may check that

$$\frac{\partial \bar{U}}{\partial \alpha_1} = (Rq_{10} + q_1^1), \quad \frac{\partial \bar{U}}{\partial q_1} = 0, \quad \frac{\partial \bar{V}}{\partial \alpha_2} = (\rho q_{20} + q_2^1), \quad \frac{\partial \bar{V}}{\partial q_2} = 0.$$

By (25) and Proposition 5.6, direct computations give us that

$$\begin{cases} 
K^{-1} R^{-2} \frac{\partial \bar{U}}{\partial \alpha_1} \partial_{\alpha_1} (\bar{U} + \phi) = (1 + o(1)) \int_{\mathbb{R}^2} (\partial_x, w_{\mu_1})^2 dx, \\
K^{-1} \rho^{-2} \frac{\partial \bar{V}}{\partial \alpha_2} \partial_{\alpha_2} (\bar{V} + \psi) = o(1), \\
K^{-1} R^{-2} \frac{\partial \bar{U}}{\partial \alpha_1} \partial_{\alpha_2} (\bar{U} + \phi) = o(1), \\
K^{-1} \rho^{-2} \frac{\partial \bar{V}}{\partial \alpha_2} \partial_{\alpha_2} (\bar{V} + \psi) = (1 + o(1)) \int_{\mathbb{R}^2} (\partial_x, w_{\mu_2})^2 dx,
\end{cases}$$

which imply that (50) is non-degenerate and complete the proof. □

Proof of Theorem 1.1. By Lemma 6.1, $F(\alpha)$ is $2\pi$ periodic in $\alpha_1, \alpha_2$ and of class $C^1$. Hence it has a critical point in $[0, 2\pi) \times [0, 2\pi)$. Therefore Theorem 1.1 follows. □

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