COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **15**, Number **3**, May **2016**

doi:10.3934/cpaa.2016.15.965

pp. **965–989**

INFINITELY MANY SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEM WITH NON-SYMMETRIC POTENTIALS

Weiwei Ao

Department of Mathematics, University of British Columbia
Vancouver, B.C., Canada, V6T1Z2

Liping Wang*

Department of Mathematics and Shanghai Key Laboratory of PMMP
East China Normal University, 500 Dong Chuan Road, Shanghai 200241, China

Wei Yao

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático
(UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

(Communicated by Zhi-qiang Wang)

Abstract. Without any symmetric conditions on potentials, we proved the following nonlinear Schrödinger system

$$\begin{cases} \Delta u - P(x)u + \mu_1 u^3 + \beta u v^2 = 0, & \text{in} \quad \mathbb{R}^2 \\ \Delta v - Q(x)v + \mu_2 v^3 + \beta v u^2 = 0, & \text{in} \quad \mathbb{R}^2 \end{cases}$$

has infinitely many non-radial solutions with suitable decaying rate at infinity of potentials $P(x)$ and $Q(x)$. This is the continued work of [8]. Especially when $P(x)$ and $Q(x)$ are symmetric, this result has been proved in [18].

1. Introduction.

We consider the following nonlinear Schrödinger system

$$\begin{cases} \Delta u - P(x)u + \mu_1 u^3 + \beta u v^2 = 0, & \text{in} \quad \mathbb{R}^2 \\ \Delta v - Q(x)v + \mu_2 v^3 + \beta v u^2 = 0, & \text{in} \quad \mathbb{R}^2 \end{cases} \tag{1}$$

where $P(x), Q(x)$ are positive potentials, $\mu_1, \mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant.

This type of system arises when one considers the standing wave solutions of the time dependent $M-$coupled Schrödinger system of the form with $M = 2$

$$\begin{cases} -i\dfrac{\partial \Phi_j}{\partial t} = \Delta \Phi_j - V_j(x)\Phi_j + \mu_j |\Phi_j|^2 \Phi_j + \Phi_j \displaystyle\sum_{l=1, l\neq j}^{M} \beta_{jl} |\Phi_l|^2, & \text{in} \quad \mathbb{R}^N \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad t > 0, \quad j = 1, \cdots, M, \end{cases} \tag{2}$$

where μ_j and $\beta_{jl} = \beta_{lj}$ are constants. The system (2) arises in applications of many physical problems, especially in the study of incoherent solitons in nonlinear optics and in Bose-Einstein condensates. Physically, the solution Φ_j denotes the

2000 *Mathematics Subject Classification.* Primary: 35B35, 35J25; Secondary: 35B40, 92D25.

Key words and phrases. Schrödinger system, non-symmetric potentials, infinitely many solutions, intermediate reduction method.

*Corresponding author.

j-th component of the beam in Kerr-like photo refractive media. The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β_{jl} is the interaction between the j-th and the l-th components of the beam. Physically, if $\beta_{jl} > 0$ then the interaction is attractive, while the interaction is repulsive if $\beta_{jl} < 0$.

Problem (1) also arises in the Hartee-Fock theory for a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, see for example [19, 20]. Physically, Φ_1, Φ_2 are the wave functions of the corresponding condensates, μ_i and β are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length β determines whether the interspecies of states are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other, which is so-called synchronization. And in the repulsive case, the components of a vector solution tend to segregate with each other leading to phase separations, which is so-called segregation. These phenomena have been documented in experiments and numeric simulations, see [5, 10, 15] and reference therein.

Mathematical work on nonlinear Schrödinger system has been studied extensively in recent years. If the domain is bounded, under the Dirichlet boundary condition, many properties are considered, e.g. local and global bifurcation structure of positive solutions ([2]), a priori bounds for positive solution and multiple existence ([7]), infinitely many positive solutions ([9]. For whole space, there are also many results. For two-component, existence of ground state is obtained in [1] and existence of two continua of bound state solutions is founded in [3]. For multiple existence, one may see [4]. For high components case, existence and non-existence are established in [11] and $k, k \in \mathbb{N}$ pairs of nontrivial spherically symmetric solutions are proved in [13]. Phase separation has been proved in several cases with constant potentials such as the work [12, 24, 25] for two components and [6, 17, 20] for high components as the coupling constant β tends to negative infinity. In constant case $P(x) = Q(x) = 1, \beta$ is positive but small enough, then uniqueness is proved in [27]. Especially, for the case of $\mu_1 = \mu_2$, [25] gives infinitely many non-radial positive solutions for $\beta \leq -1$ which are potentially segregated type. For $N = 3$, Peng and Wang [18] proved the existence of infinitely many solutions of both synchronized and segregated types to (1) for *radially symmetric* positive potentials $P(|x|), Q(|x|)$ with the following algebraic decaying conditions:

(P): There are constants $a \in \mathbb{R}$, $m > 1$ and $\theta > 0$ such that as $r \to \infty$

$$P(r) = 1 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right).$$

(Q): There are constants $b \in \mathbb{R}$, $n > 1$ and $\sigma > 0$ such that as $r \to \infty$

$$Q(r) = 1 + \frac{b}{r^n} + O\left(\frac{1}{r^{n+\sigma}}\right).$$

The constants a, b, m, n and coupling constant β should satisfy some further conditions depending on whether the solutions are synchronized or segregated type, e.g. for segregated type, it needs $m = n, a > 0, b > 0$. In Remark 4.1 of [18], the authors point out that the synchronized result can be extended to the case of $N = 2$ with little change but they don't know what's the segregated case. *Natural questions come: are there infinitely many segregated solutions with arbitrarily large energy for $N = 2$? If so, can we remove the symmetric conditions on $P(x)$ and $Q(x)$? In this paper, we will give affirmative answer.*

Recall the key of their proof in [18] is the symmetry of the potentials $P(|x|), Q(|x|)$ and in the spirit of the work [26] where the authors consider the following nonlinear Schrödinger equation:

$$\Delta u - V(|x|)u + u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{3}$$

where $1 < p < \frac{N+2}{N-2}$ and $V(|x|)$ is positive. Using the number of bubbles as parameter, with Lyapunov-Schmidt reduction at hand, they proved that problem (3) has infinitely many positive non-radial solutions if there are constants $V_\infty > 0$, $a > 0$, $m > 1$ and $\sigma > 0$ such that

$$V(r) = V_\infty + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\sigma}}\right).$$

Such idea is used very widely, see [21, 22, 23] and so on.

Recently, del Pino, J. Wei and the third author, see [8], considered the nonlinear Schrödinger equation (3) for $N = 2$ and proved the existence of infinitely many solutions when the symmetry requirement of the potential V is lifted. More precisely, with so-called intermediate reduction method they proved the existence of infinitely many non-radial solutions when the potential V satisfies

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\sigma}}\right) \quad \text{as} \quad |x| \to \infty,$$

where

$$a > 0, \qquad m \min\left\{1, \frac{p-1}{2}\right\} > 2, \qquad \sigma > 2.$$

Based on their work, we can continue to consider system case. In the following we always assume P, Q satisfy the following decaying rate as $|x| \to +\infty$:

$$P(x) = 1 + \frac{a}{|x|^m} + O\left(\frac{1}{|x|^{m+\theta}}\right), \qquad Q(x) = 1 + \frac{b}{|x|^n} + O\left(\frac{1}{|x|^{n+\sigma}}\right). \tag{4}$$

Our main result in this paper is the following:

Theorem 1.1. *Suppose $P(x)$, $Q(x)$ satisfy (4) and*

$$a, b > 0,\ m \neq n,\ m, n > 2,\ 2\min\{m, n\} > \max\{m, n\} + 2,\ \theta, \sigma > 2. \tag{5}$$

Then there exists $\beta^* > 0$ such that for $\beta < \beta^*$, problem (1) has infinitely many non-radial positive segregated solutions whose energy can be arbitrarily large.

Remark 1. It is obvious that in (4), the constant 1 can be replaced by any positive constant and Theorem 1.1 is still true.

Remark 2. In [18], for symmetric potentials they constructed segregated solutions through the different angles of concentrating points. Without the help of symmetry, to finish the reduction, we need to adjust angles, which means that we can't determine angles in advance. Hence we get the segregated solutions through the different radii of two circles for concentration under the assumption $m \neq n$. We believe this is technical and make the following conjecture:
Suppose $P(x), Q(x)$ satisfy (4) and $a > 0$, $b > 0$, $m = n > 2$, $\theta > 2$, $\sigma > 2$. Then problem (1) has infinitely many non-radial positive solutions whose energy can be arbitrarily large.

Remark 3. For system, new obstruction appears due to interaction uv^2 and vu^2. To get the reduction work, $2\min\{m, n\} > \max\{m, n\} + 2$ is needed, see also Remark 7.

Remark 4. The smallness of β is to make sure the solutions are positive and the linear system (35) is non-degenerated. For the non-degeneracy, we can take $\beta^* = \min\{w_{\mu_1}^{-2}(0), w_{\mu_2}^{-2}(0)\}$ by (36) where w_μ is defined in (6).

Remark 5. Our result can be stated and proved for the case of $\mathbb{R}^3$ with little change to the proof. We leave the proofs for interested readers.

Remark 6. In the following of the paper, without loss of generality, we assume that $m > n$.

Throughout the paper, we make use of the following notations and conventions:

• For quantities A_K and B_K, we write $A_K \sim B_K$ to denote that A_K/B_K goes to 1 as K goes to infinity; $A_K = O(B_K)$ means that $|A_K/B_K|$ are uniformly bounded while $A_K = o(B_K)$ denotes $A_K/B_K \to 0$ as K tends to infinity.

• For simplicity, the letter C denotes various generic constant which is independent of K. It is allowed to vary for different lines.

• We will use the same $|y| = \|y\|_2$ for the Euclidean norm in various Euclidean space $\mathbb{R}^2$ and denote the inner product of vectors a and b by $a \cdot b$.

In the next section we will show the procedure of construction and main idea in each step.

2. **Description of the construction.** In fact we will construct infinitely many non-radial positive solutions for system (1) to prove Theorem 1.1. So in this section let us introduce the approximation and describe the main steps of the proof briefly.

Let w_μ be the unique positive radial solution of the following problem:

$$\begin{cases} \Delta w - w + \mu w^3 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2, \\ w(0) = \max\limits_{x\in\mathbb{R}^2} w(x), \quad w \in H^1(\mathbb{R}^2). \end{cases} \tag{6}$$

It is well known that

$$\lim_{r\to\infty} r^{\frac{1}{2}} e^r w_\mu(r) = \mu^{-\frac{1}{2}}\omega_0, \qquad \lim_{r\to\infty} \frac{w_\mu'(r)}{w_\mu(r)} = -1, \qquad w_\mu'(r) < 0 \tag{7}$$

where ω_0 is a uniform constant independent of μ. The non-degeneracy of w_μ will play the key role in the following proof. We will use (w_{μ_1}, w_{μ_2}) to build up the approximate solutions. Namely, the solutions we construct will be small perturbations of the sum of copies of (w_{μ_1}, w_{μ_2}).

Let x_0^j be defined as

$$x_0^j = (R\cos\theta_j, R\sin\theta_j), \qquad \theta_j = \alpha_1 + (j-1)\frac{2\pi}{K}$$

and y_0^j be defined as

$$y_0^j = (\rho\cos\theta_j', \rho\sin\theta_j'), \qquad \theta_j' = \alpha_2 + (j-1)\frac{2\pi}{K}$$

for $j = 1, \cdots, K$. Here α_1, α_2 are two parameters dealing with the degeneracy due to rotations, and R, ρ are two positive numbers which will satisfy the following so-called balancing condition:

$$a_0 m R^{-m-1} = 2\sin\frac{\pi}{K}\Psi_1(d_1), \qquad b_0 n \rho^{-n-1} = 2\sin\frac{\pi}{K}\Psi_2(d_2), \tag{8}$$

where

$$a_0 = \frac{a}{2}\int_{\mathbb{R}^2} w_{\mu_1}^2 dx, \quad b_0 = \frac{b}{2}\int_{\mathbb{R}^2} w_{\mu_2}^2, \quad d_1 = 2R\sin\frac{\pi}{K}, \quad d_2 = 2\rho\sin\frac{\pi}{K} \tag{9}$$

and Ψ_i are the interaction functions defined as follows:

$$\Psi_i(s) = -\mu_i \int_{\mathbb{R}^2} w_{\mu_i}(x - s\vec{e}) div(w^3_{\mu_i}(x)\vec{e})dx.$$

Here $\vec{e}$ can be any unit vector in $\mathbb{R}^2$, see [14, 16].

We will see (Lemma 3.2) that as $K \to \infty$,

$$R \sim \frac{m}{2\pi} K \ln K, \qquad \rho \sim \frac{n}{2\pi} K \ln K,$$

$$\min_{l \neq j} |x_0^l - x_0^j| = d_1 \sim m \ln K, \quad \min_{l \neq j} |y_0^l - y_0^j| = d_2 \sim n \ln K,$$

and

$$\min_{l,j=1,\cdots,K} |x_0^l - y_0^j| \geq (R - \rho) \sim \frac{m-n}{2\pi} K \ln K.$$

Next we define a small neighborhood of $\mathbf{Q}^0 = (x_0^1, \cdots, x_0^K, y_0^1, \cdots, y_0^K)$ on $\mathbb{R}^{4K}$ in a suitable norm. To be made precise we introduce other parameters. Let $f_{ij}, g_{ij} \in \mathbb{R}$ $i = 1, 2,\ j = 1, \cdots, K$, we define

$$x^j = x_0^j + f_{1j}\vec{n}_{1j} + g_{1j}\vec{t}_{1j}, \qquad y^j = y_0^j + f_{2j}\vec{n}_{2j} + g_{2j}\vec{t}_{2j}, \tag{10}$$

where

$$\vec{n}_{1j} = (\cos\theta_j, \sin\theta_j), \qquad \vec{t}_{1j} = (-\sin\theta_j, \cos\theta_j),$$
$$\vec{n}_{2j} = (\cos\theta_j', \sin\theta_j'), \qquad \vec{t}_{2j} = (-\sin\theta_j', \cos\theta_j').$$

Denote by

$$\mathbf{Q} = (Q_1, \cdots, Q_{2K}) = (x^1, \cdots, x^K, y^1, \cdots, y^K). \tag{11}$$

A trivial but important fact is that these points are 2π periodic in α_1 and α_2.

We can now introduce the other parameters $\mathbf{q}_1, \mathbf{q}_2$ and define the norms. Denote

$\mathbf{q}_1 = (f_{11}, f_{12}, \cdots, f_{1K}, g_{11}, g_{12}, \cdots, g_{1K})$, $\mathbf{q}_2 = (f_{21}, f_{22}, \cdots, f_{2K}, g_{21}, g_{22}, \cdots, g_{2K})$

and

$$\dot{\mathbf{q}}_i = (\dot{f}_{i1}, \cdots, \dot{f}_{iK}, \dot{g}_{i1}, \cdots, \dot{g}_{iK}), \qquad \ddot{\mathbf{q}}_i = (\ddot{f}_{i1}, \cdots, \ddot{f}_{iK}, \ddot{g}_{i1}, \cdots, \ddot{g}_{iK}),$$

where

$$\dot{f}_{ij} = (f_{i,j+1} - f_{ij})\frac{K}{2\pi}, \qquad \ddot{f}_{ij} = (f_{i,j+1} - 2f_{ij} + f_{i,j-1})\frac{K^2}{4\pi^2},$$
$$\dot{g}_{ij} = (g_{i,j+1} - g_{ij})\frac{K}{2\pi}, \qquad \ddot{g}_{ij} = (g_{i,j+1} - 2g_{ij} + g_{i,j-1})\frac{K^2}{4\pi^2},$$
$$f_{K+1} = f_1, \quad f_0 = f_K, \qquad g_{K+1} = g_1, \quad g_0 = g_K.$$

With these notations, we can define the configuration space by

$$\Lambda_K = \left\{ \mathbf{Q} = (Q_1, \cdots, Q_{2K}) \in \mathbb{R}^{4K} \,\middle|\, x^j,\ y^j \text{ defined by (10) and } \|\mathbf{q}_i\|_* \leq 1 \right\}, \tag{12}$$

where $\|\mathbf{q}_i\|_* = \|\mathbf{q}_i\|_\infty + \|\dot{\mathbf{q}}_i\|_\infty + \|\ddot{\mathbf{q}}_i\|_\infty$ is a norm on $\mathbb{R}^{2K}$.

Now we define our approximate solution to be

$$(\bar{U}, \bar{V}) = \left(\sum_{j=1}^K w_{\mu_1}(x - x^j), \sum_{j=1}^K w_{\mu_2}(x - y^j) \right). \tag{13}$$

Note that for $\mathbf{Q} \in \Lambda_K$, by Corollary 3.6. in [8] and (7), $0 < \bar{U}, \bar{V} \leq C$,

$$\min_{j \neq l} |x^j - x^l| = d_1 + O(K^{-1}), \qquad \min_{j \neq l} |y^j - y^l| = d_2 + O(K^{-1}),$$

and

$$\min_{j,l} |x^j - y^l| \geq R - \rho - 4.$$

We will use Lyapunov-Schmidt reduction method to solve this problem. Let

$$S \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Delta u - P(x)u + \mu_1 u^3 + \beta u v^2 \\ \Delta v - Q(x)v + \mu_2 v^3 + \beta v u^2 \end{pmatrix}.$$

Then we want to find solutions of the form $(u, v) = (\bar{U} + \phi, \bar{V} + \psi)$. It is equivalent to solving the following problem:

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0,$$

where

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi - P(x)\phi + 3\mu_1 \bar{U}^2\phi + \beta\bar{V}^2\phi + 2\beta\bar{U}\bar{V}\psi \\ \Delta\psi - Q(x)\psi + 3\mu_2\bar{V}^2\psi + \beta\bar{U}^2\psi + 2\beta\bar{U}\bar{V}\phi \end{pmatrix}, \tag{14}$$

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = S \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} \Delta\bar{U} - P(x)\bar{U} + \mu_1\bar{U}^3 + \beta\bar{U}\bar{V}^2 \\ \Delta\bar{V} - Q(x)\bar{V} + \mu_2\bar{V}^3 + \beta\bar{V}\bar{U}^2 \end{pmatrix}, \tag{15}$$

and

$$N \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 3\mu_1\bar{U}\phi^2 + \mu_1\phi^3 + \beta(2\bar{V}\psi\phi + \psi^2\bar{U} + \psi^2\phi) \\ 3\mu_2\bar{V}\psi^2 + \mu_2\psi^3 + \beta(2\bar{U}\psi\phi + \phi^2\bar{V} + \psi\phi^2) \end{pmatrix}. \tag{16}$$

Remark 7. In the later computations, the term $\beta\psi^2\bar{U}$ can't be avoided in the first component of $N \begin{pmatrix} \phi \\ \psi \end{pmatrix}$. Roughly speaking, some norm of ψ is controlled by $K^{-n}\ln^{-\frac{1}{2}} K$, so in the estimate of $N \begin{pmatrix} \phi \\ \psi \end{pmatrix}$, the term $K^{-2n}\ln^{-1} K$ will appear. To keep $K^{-2n}\ln^{-1} K$ be higher order of K^{-m-2}, the condition $2n > m+2$ in Theorem 1.1 is necessary.

Now we sketch the proof of the main theorem. We will follow the idea in [8] to prove our main result. Certainly system involves more details and computations.

Step 1. Solving the projected problem.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{R}^{4K}$, we look for a solution (ϕ, ψ) and some multipliers $\widehat{\beta}_1, \widehat{\beta}_2 \in \mathbb{R}^{2K}$ such that

$$\begin{cases} L \begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_1 \cdot \dfrac{\partial\bar{U}}{\partial\mathbf{q}_1} \\ \widehat{\beta}_2 \cdot \dfrac{\partial\bar{V}}{\partial\mathbf{q}_2} \end{pmatrix}, \\ \displaystyle\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \\ \displaystyle\int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \ \forall j = 1, \cdots, K, \end{cases}$$

where the vector fields Z_{x^j}, Z_{y^j} are defined by

$$Z_{x^j} = \nabla w_{\mu_1}(x - x^j), \qquad Z_{y^j} = \nabla w_{\mu_2}(x - y^j).$$

This is the first step in Lyapunov-Schmidt reduction. It is done in Section 4 through some a priori estimate and contraction mapping theorem. A required element in this step is the non-degeneracy of w_{μ_i} and the smallness of β. It is worth pointing out that the function (ϕ, ψ) and the multipliers $\widehat{\beta}_1, \widehat{\beta}_2$ found in Step 1 depend on the parameters α and $\mathbf{q}$. Hence we write $\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi(\alpha,\mathbf{q}) \\ \psi(\alpha,\mathbf{q}) \end{pmatrix}$ and $\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\beta}_1(\alpha,\mathbf{q}) \\ \widehat{\beta}_2(\alpha,\mathbf{q}) \end{pmatrix}$.

Step 2. Solving the reduced problem.

First by direct calculation,

$$\frac{\partial \bar{U}}{\partial \alpha_1} = (R\mathbf{q}_{10} + \mathbf{q}_1^{\perp}) \cdot \frac{\partial \bar{U}}{\partial \mathbf{q}_1}, \qquad \frac{\partial \bar{V}}{\partial \alpha_2} = (\rho \mathbf{q}_{20} + \mathbf{q}_2^{\perp}) \cdot \frac{\partial \bar{V}}{\partial \mathbf{q}_2}$$

where $\mathbf{q}_{10} = \mathbf{q}_{20} = (0, \cdots, 0, 1, \cdots, 1)$ and $\mathbf{q}^{\perp} = (-\vec{g}, \vec{f})$ for $\mathbf{q} = (\vec{f}, \vec{g})$.

We define

$$\vec{\beta}_1 = \widehat{\beta}_1 - \gamma_1(R\mathbf{q}_{10} + \mathbf{q}_1^{\perp}), \qquad \vec{\beta}_2 = \widehat{\beta}_2 - \gamma_2(\rho\mathbf{q}_{20} + \mathbf{q}_2^{\perp}),$$

then the new multiplier $\vec{\beta} = (\vec{\beta}_1, \vec{\beta}_2)$ depends on α, $\mathbf{q}$ and $\gamma = (\gamma_1, \gamma_2)$. By Lyapunov-Schmidt reduction, the step of solving $\widehat{\beta}_i = 0$ will be divided into two steps.

Step 2A: Solving $\vec{\beta}_i = 0$ by adjusting $\mathbf{q}$ and γ. In this step, for each $\alpha \in \mathbb{R}^2$, we are going to find $(\mathbf{q}, \gamma)$ such that

$$\vec{\beta}_i = 0, \qquad \mathbf{q}_i \perp \mathbf{q}_{i0},$$

for $i = 1, 2$.

We denote the solution obtained in this step by $\gamma(\alpha), \mathbf{q}(\alpha)$. Then the original problem is reduced to $\gamma(\alpha) = 0$.

Step 2B: Solving $\gamma_i = 0$ by choosing α_1, α_2.

At this last step, we want to find $\alpha \in \mathbb{R}^2$ such that $\gamma(\alpha) = 0$. As a result, the function $(\bar{U} + \phi, \bar{V} + \psi)$ is a genuine solution of (1).

This step is the second step of solving the reduced problem in the secondary Lyapunov-Schmidt reduction. To achieve this, by **Step 2A**, the function (ϕ, ψ) found in **Step 1** solves the following problem:

$$\begin{cases} L\begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \gamma_1 \cdot \dfrac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \cdot \dfrac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}, \\ \displaystyle\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \ \int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \quad \forall\, j = 1, \cdots, K. \end{cases}$$

To solve $\gamma_i(\alpha_1, \alpha_2) = 0$, we first apply the variational reduction to show that $\gamma_i = 0$ has a solution if the reduced energy function $F(\alpha_1, \alpha_2) = \mathcal{E}(\bar{U} + \phi, \bar{V} + \psi)$ has a critical point, where $\mathcal{E}$ is the corresponding energy functional:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + P(x)u^2 + |\nabla v|^2 + Q(x)v^2 - \frac{1}{4}\int_{\mathbb{R}^2} \mu_1|u|^4 + \mu_2|v|^4 - \frac{\beta}{2}\int_{\mathbb{R}^2} u^2v^2.$$

Secondly, it is easy to check that $F(\alpha_1, \alpha_2)$ is 2π periodic and C^1 in α_1, α_2, hence it has critical points.

Finally, the paper is organized as follows. Some preliminary facts and estimates are explained in Section 3. In Section 4, we apply the standard Lyapunov-Schmidt reduction for Step 1. In Section 5, we further reduce the problem to a two-dimensional one. In Section 6 we carry out Step 2B and complete the proof of Theorem 1.1.

3. **Preliminaries.** In this section, we present some preliminary facts, useful estimates whose proof can be found in [8] and the expansion of $\mathcal{E}(\bar{U},\bar{V})$.

First recall the definition of $\Psi_i(s)$

$$\Psi_i(s) = -\mu_i \int_{\mathbb{R}^2} w_{\mu_i}(x - s\vec{e}) div(w^3_{\mu_i}(x)\vec{e})dx$$

and we have

Lemma 3.1 (Lemma 3.2. [8])**.** *For s sufficiently large,*

$$\Psi_i(s) = c_{\mu_i} s^{-\frac{1}{2}} e^{-s}\left(1 + O(s^{-1})\right)$$

where $c_{\mu_i} > 0$ are constants depending only on μ_i.

Next we study the balancing condition (8).

Lemma 3.2 (Lemma 3.3. [8])**.** *For K sufficiently large,*

$$\begin{cases} d_1 = m\ln K + (m+\frac{1}{2})\ln(m\ln K) + O(1),\\ R = \frac{m}{2\pi}K\ln K + \frac{1}{2\pi}(m+\frac{1}{2})K\ln(m\ln K) + O(K),\\ d_2 = n\ln K + (n+\frac{1}{2})\ln(n\ln K) + O(1),\\ \rho = \frac{n}{2\pi}K\ln K + \frac{1}{2\pi}(n+\frac{1}{2})K\ln(n\ln K) + O(K). \end{cases} \tag{17}$$

If we denote

$$\mathcal{E}_V(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V(x)u^2\right)dx - \frac{1}{4}\int_{\mathbb{R}^2} u_+^4,$$

then according to Lemma 3.9 in [8], we obtain

$$\begin{aligned}\mathcal{E}_P(\bar{U}) =& KA_1 + (a_0 + o(1))\sum_{j=1}^{K}|x^j|^{-m} - \frac{1}{2}\sum_{l\neq j}(\lambda_1 + o(1))w_{\mu_1}(|x^l - x^j|)\\ &+ O\left(KR^{-2m} + Ke^{-2d_1}d_1^{\frac{1}{2}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_Q(\bar{V}) =& KA_2 + (b_0 + o(1))\sum_{j=1}^{K}|y^j|^{-n} - \frac{1}{2}\sum_{l\neq j}(\lambda_2 + o(1))w_{\mu_2}(|y^l - y^j|)\\ &+ O\left(K\rho^{-2n} + Ke^{-2d_2}d_2^{\frac{1}{2}}\right).\end{aligned}$$

Here $A_i = \frac{1}{4}\int_{\mathbb{R}^2}\mu_i w^4_{\mu_i}dx,\ \lambda_i = \mu_i\int_{\mathbb{R}^2} w^3_{\mu_i}e^{-x_1}dx$, for $i = 1, 2$ and a_0, b_0 are defined in (9).

Obviously,

$$\mathcal{E}(\bar{U},\bar{V}) = \mathcal{E}_P(\bar{U}) + \mathcal{E}_Q(\bar{V}) - \frac{\beta}{2}\int_{\mathbb{R}^2}\left(\sum_{l=1}^{K} w_{\mu_1}(x - x^l)\right)^2\left(\sum_{j=1}^{K} w_{\mu_2}(x - y^j)\right)^2.$$

Hence in order to get the expression of $\mathcal{E}(\bar{U},\bar{V})$, we just need to estimate the interaction term which actually is higher order. Indeed, for any $l, j = 1, \ldots, K$, if

$|x-x^l| \le \frac{1}{2}|x^l-y^j|$ then $|x-y^j| \ge \frac{1}{2}|x^l-y^j|$. Hence

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\sum_{l=1}^K w_{\mu_1}(x-x^l)\right)^2 \left(\sum_{j=1}^K w_{\mu_2}(x-y^j)\right)^2 \\
\le& K^2 \int_{\mathbb{R}^2} \left(\sum_{l=1}^K w_{\mu_1}^2(x-x^l)\right) \left(\sum_{j=1}^K w_{\mu_2}^2(x-y^j)\right) \\
=& K^2 \sum_{l,j=1}^K \int_{\mathbb{R}^2} w_{\mu_1}^2(x-x^l) w_{\mu_2}^2(x-y^j) \\
=& K^2 \sum_{l,j=1}^K \left(\int_{\{x:|x-x^l|\le\frac{1}{2}|x^l-y^j|\}} + \int_{\{x:|x-x^l|\ge\frac{1}{2}|x^l-y^j|\}}\right) w_{\mu_1}^2(x-x^l) w_{\mu_2}^2(x-y^j) \\
=& K^2 \sum_{l,j=1}^K O\left(e^{-|x^l-y^j|}\frac{1}{|x^l-y^j|}\int_{\mathbb{R}^2} w_{\mu_1}^2 + e^{-|x^l-y^j|}\frac{1}{|x^l-y^j|}\int_{\mathbb{R}^2} w_{\mu_2}^2\right) \\
=& K^4 O\left(e^{-\frac{(m-n)}{2\pi}K\ln K}\frac{1}{K\ln K}\right) = o(K^{-m-n}).
\end{aligned} \tag{18}$$

In conclusion, we have the following energy expansion to finish this section.

Proposition 3.3. *For K sufficiently large, for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\mathbf{q}$ satisfies (12),*

$$\begin{aligned}
\mathcal{E}(\bar{U}, \bar{V}) =& KA_0 + (b_0 + o(1)) \sum_{j=1}^K |y^j|^{-n} - \frac{1}{2}\sum_{l\ne j}(\lambda_2 + o(1)) w_{\mu_2}(y^l - y^j) \\
&+ O\left(KR^{-m} + Ke^{-d_1}d_1^{-\frac{1}{2}} + K\rho^{-2n} + Ke^{-2d_2}d_2^{\frac{1}{2}}\right),
\end{aligned}$$

where b_0 is defined in (9) and

$$A_0 = \frac{1}{4}\int_{\mathbb{R}^2} \mu_1 w_{\mu_1}^4 + \mu_2 w_{\mu_2}^4.$$

Remark 8. Let us recall the we assume $m > n$. Hence in the expansion of energy, interaction between $w_{\mu_2}(x-y^j), j = 1, \dots, K$ plays main role comparing to that of $w_{\mu_1}(x-x^j), j = 1, \dots, K$. It is obvious that if $m < n$, then

$$\begin{aligned}
\mathcal{E}(\bar{U}, \bar{V}) =& KA_0 + (a_0 + o(1)) \sum_{j=1}^K |x^j|^{-m} - \frac{1}{2}\sum_{l\ne j}(\lambda_1 + o(1)) w_{\mu_1}(x^l - x^j) \\
&+ O\left(K\rho^{-n} + Ke^{-d_2}d_2^{-\frac{1}{2}} + KR^{-2m} + Ke^{-2d_1}d_1^{\frac{1}{2}}\right),
\end{aligned}$$

where a_0 is defined in (9).

4. **The Lyapunov-Schmidt reduction.**

The aim of this section is to achieve Step 1 in the procedure of our construction described in Section 2.

We first introduce some notations. Let $0<\eta<1$ be a constant to be determined later. For $h=(h_1(x),h_2(x))$, we define the following weighted norm:

$$\|h\|_{**}=\sup_{x\in\mathbb{R}^2,\, i=1,2}\left|\left(\sum_{j=1}^{K}e^{-\eta|x-x^j|}+\sum_{j=1}^{K}e^{-\eta|x-y^j|}\right)^{-1}h_i(x)\right|,$$

where x^j,y^j are defined in Section 2. In what follows, we always assume that

$$(x^1,\cdots,x^K,y^1,\cdots,y^K)\in\Lambda_K.$$

For $f=\begin{pmatrix}f_1\\ f_2\end{pmatrix}, g=\begin{pmatrix}g_1\\ g_2\end{pmatrix}$, we denote by $\langle f,g\rangle=\int_{\mathbb{R}^2}f_1g_1+f_2g_2$.

Now we state our main result in this section.

Proposition 4.1. *Suppose $P(x)$ and $Q(x)$ satisfy (4) and (5). Then there is a $\beta_*>0$ and a positive integer K_0 such that for $\beta<\beta_*$ and all $K\ge K_0$, every $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$ and $\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2)$ satisfying (12), there exists a unique function $(\phi,\psi)\in(H^2(\mathbb{R}^2))^2\cap\mathcal{B}_K$ and a unique multiplier $(\widehat{\beta}_1,\widehat{\beta}_2)\in\mathbb{R}^{4K}$ such that*

$$\begin{cases} L\begin{pmatrix}\phi\\ \psi\end{pmatrix}+E+N\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\widehat{\beta}_1\cdot\dfrac{\partial\bar U}{\partial\mathbf{q}_1}\\ \widehat{\beta}_2\cdot\dfrac{\partial\bar V}{\partial\mathbf{q}_2}\end{pmatrix},\\ \displaystyle\int_{\mathbb{R}^2}\phi Z_{x^j}=0,\ \int_{\mathbb{R}^2}\psi Z_{y^j}=0,\ \forall\, j=1,\cdots,K,\end{cases}\tag{19}$$

where

$$\mathcal{B}_K=\left\{(\phi,\psi)\in(L^\infty(\mathbb{R}^2))^2:\|(\phi,\psi)\|_{**}\le C_0K^{-n}(\ln K)^{-\frac12}\right\}.$$

Here $C_0>0$ is a constant independent of K. Moreover, $(\alpha,\mathbf{q})\to(\phi(x;\alpha,\mathbf{q}),\psi(x;\alpha,\mathbf{q}))$ is of class C^1 and

$$\sum_{i=1}^{2}(R^{-1}+\rho^{-1})\left\|\frac{\partial(\phi,\psi)}{\partial\alpha_i}\right\|_{**}+\left\|\frac{\partial(\phi,\psi)}{\partial\mathbf{q}}\right\|_{**}\le CK^{-n}(\ln K)^{-\frac12}.\tag{20}$$

4.1. **Linear analysis.** Let M denote the $4K\times4K$ matrix defined as follows:

$$M_{ij}=\left\langle\frac{\partial W}{\partial q_i},\frac{\partial W}{\partial q_j}\right\rangle,\qquad i,j=1,\cdots,4K.\tag{21}$$

where $W=\begin{pmatrix}\bar U\\ \bar V\end{pmatrix}$ and $\mathbf{q}=(\mathbf{q}_1,\mathbf{q}_2):=(q_1,\cdots,q_{4K})$. With definition of $\bar U,\bar V$ and similar computations as Lemma 4.2 in [8], one can obtain

$$M_{jj}=\int_{\mathbb{R}^2}\left(\frac{\partial w_{\mu_1}}{\partial x_1}\right)^2dx=c_0,\qquad M_{jl}=O(R^{-m}),\ j\ne l, j,l=1,\dots,2K,\tag{22}$$

$$M_{jj}=\int_{\mathbb{R}^2}\left(\frac{\partial w_{\mu_2}}{\partial x_1}\right)^2dx=c_1,\qquad M_{jl}=O(\rho^{-n}),\ j\ne l, j,l=2K+1,\dots,4K,\tag{23}$$

and

$$M_{lj}=M_{jl}=0,\qquad\forall\ \ j=1,\dots,2K,\ l=2K+1,\dots,4K.\tag{24}$$

From the computations of (18) and the second part of Lemma 3.8 in [8], it is obtained that for any $p > 1$

$$\begin{aligned}
&\int_{\mathbb{R}^2} w_{\mu_1}(x-x^l)w_{\mu_2}(x-y^j)dx = O\left(e^{-\frac{m-n}{4\pi}K\ln K}|x^l-y^j|^{-\frac{1}{2}}\right), \quad \forall\, j,l,\\
&\int_{\mathbb{R}^2} w_{\mu_1}(x-x^j)w_{\mu_1}(x-x^l)dx = O\left(e^{-|x^j-x^l|}|x^j-x^l|^{\frac{1}{2}}\right), \qquad \forall\, j\neq l, \qquad (25)\\
&\int_{\mathbb{R}^2} w_{\mu_1}(x-x^j)w^p_{\mu_1}(x-x^l)dx = O\left(w_{\mu_1}(x^j-x^l)\right), \qquad \forall\, j\neq l.
\end{aligned}$$

Based on (25) and similar proofs of Lemma 4.2 in [8], we can deduce the following linear result.

Lemma 4.2. *For K large, given any vector $\vec{b}\in\mathbb{R}^{4K}$, there exists a unique vector $\widehat{\beta}\in\mathbb{R}^{4K}$ such that $M\widehat{\beta}=\vec{b}$. Moreover,*

$$\|\widehat{\beta}\|_\infty \le C\|\vec{b}\|_\infty,$$

for some $C>0$ independent of K.

We can now prove the following a priori estimate.

Lemma 4.3. *Under the assumption in Proposition 4.1, there exists a $\beta_*>0$ and a positive integer K_0 such that for all $\beta<\beta_*, K\ge K_0, \alpha\in\mathbb{R}^2, \mathbf{q}$ in (12), and all $h=(h_1,h_2)$ with $\|h\|_{**}<\infty$, there exist a unique vector function $(\phi,\psi)\in(H^2(\mathbb{R}^2))^2$ and a unique multiplier $\widehat{\beta}=(\widehat{\beta}_1,\widehat{\beta}_2)\in\mathbb{R}^{4K}$ such that*

$$\begin{cases}
L\begin{pmatrix}\phi\\ \psi\end{pmatrix} = h + \begin{pmatrix}\widehat{\beta}_1\cdot\dfrac{\partial\bar{U}}{\partial\mathbf{q}_1}\\[2mm] \widehat{\beta}_2\cdot\dfrac{\partial\bar{V}}{\partial\mathbf{q}_2}\end{pmatrix},\\[4mm]
\displaystyle\int_{\mathbb{R}^2}\phi Z_{x^j}=0,\quad \int_{\mathbb{R}^2}\psi Z_{y^j}=0,\quad \forall\, j=1,\cdots,K.
\end{cases}\tag{26}$$

Moreover, we have the following estimate:

$$\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**} + \left\|\begin{pmatrix}\widehat{\beta}_1\\ \widehat{\beta}_2\end{pmatrix}\right\|_\infty \le C\|h\|_{**}\tag{27}$$

for some $C>0$ independent of K.

Proof. Multiply the first equation of (26) by $\frac{\partial W}{\partial\mathbf{q}}$ and integrate over $\mathbb{R}^2$ to obtain

$$M\widehat{\beta} = \left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \frac{\partial W}{\partial\mathbf{q}}\right\rangle - \left\langle h, \frac{\partial W}{\partial\mathbf{q}}\right\rangle,$$

where M is the $4K\times 4K$ matrix defined in (21).

Integration by parts, we have for $j=1,\cdots,K$,

$$\begin{aligned}
\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}Z_{x^j}\\ 0\end{pmatrix}\right\rangle = &\int_{\mathbb{R}^2}\left[-(P(x)-1)+3\mu_1\left(\bar{U}^2-w^2_{\mu_1}(x-x^j)\right)+\beta\bar{V}^2\right]\\
&\times\phi\nabla w_{\mu_1}(x-x^j)+2\beta\bar{U}\bar{V}\psi\nabla w_{\mu_1}(x-x^j).
\end{aligned}$$

By the assumption (4) on $P(x)$, (17), (25) and $\int_{\mathbb{R}^2} \phi Z_{x^j} = 0$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} (P(x)-1)\nabla w_{\mu_1}(x-x^j)\phi\right| \\
=&\left|\left(\int_{\{x:|x-x^j|\le\frac{1}{2}|x^j|\}} + \int_{\{x:|x-x^j|\ge\frac{1}{2}|x^j|\}}\right)(P(x)-1)\nabla w_{\mu_1}(x-x^j)\phi\right| \\
\le& C\left(|x^j|^{-m-1}\ln K + |x^j|^{-m-\theta} + e^{-\frac{1}{2}|x^j|}|x^j|^{\frac{1}{2}}\right)\|\phi\|_\infty \\
\le& C\left(R^{-m-1}\ln K + R^{-m-\theta}\right)\|\phi\|_\infty
\end{aligned}$$

By mean value theorem, for $|x-x^j| < 2m\ln K$,

$$|\bar{U}^2 - w_{\mu_1}^2(x-x^j)| \le C w_{\mu_1}(x-x^j)\sum_{l\ne j} w_{\mu_1}(x-x^l).$$

See [8]. Thus

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\left(\bar{U}^2 - w_{\mu_1}^2(x-x^j)\right)\nabla w_{\mu_1}(x-x^j)\phi\right| \\
\le& C\int_{|x-x^j|<2m\ln K} w_{\mu_1}(x-x^j)\sum_{l\ne j} w_{\mu_1}(x-x^l)\left|\nabla w_{\mu_1}(x-x^j)\phi\right| \\
&+ \int_{|x-x^j|\ge 2m\ln K} 2\sum_{k=1}^K\sum_{l\ne j} w_{\mu_1}(x-x^k) w_{\mu_1}(x-x^l)\left|\nabla w_{\mu_1}(x-x^j)\phi\right| \\
\le& C\|\phi\|_\infty\left[\int_{\mathbb{R}^2} w_{\mu_1}^2(x-x^j)\sum_{l\ne j} w_{\mu_1}(x-x^l)dx + K^2 w_{\mu_1}(2m\ln K)\int_{\mathbb{R}^2} w_{\mu_1}(x)dx\right] \\
\le& C\left(\sum_{l\ne j} w_{\mu_1}(x^j-x^l) + K^2(\ln K)^{-\frac{1}{2}}e^{-2m\ln K}\right)\|\phi\|_\infty \le C d_1^{-\frac{1}{2}}e^{-d_1}\|\phi\|_\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \beta\bar{V}^2\nabla w_{\mu_1}(x-x^j)\phi + 2\beta\bar{U}\bar{V}\nabla w_{\mu_1}(x-x^j)\psi\right| \\
\le& CK\sum_{l=1}^K\int_{\mathbb{R}^2}\left(w_{\mu_2}^2(x-y^l)|\phi| + w_{\mu_2}(x-y^l)|\psi|\right)|\nabla w_{\mu_1}(x-x^j)| \\
\le& CK\sum_{l=1}^K e^{-\frac{1}{2}|x^j-y^l|}\|(\phi,\psi)\|_\infty \le CK^2 e^{-\frac{1}{2}(R-\rho)}\|(\phi,\psi)\|_\infty = o\left(R^{-m-5}\right)\|(\phi,\psi)\|_\infty.
\end{aligned}$$

Thus we have

$$\left|\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix} Z_{x^j}\\ 0\end{pmatrix}\right\rangle\right| \le C d_1^{-\frac{1}{2}}e^{-d_1}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_\infty. \tag{28}$$

Similarly, we have

$$\left|\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix} 0\\ Z_{y^j}\end{pmatrix}\right\rangle\right| \le C d_2^{-\frac{1}{2}}e^{-d_2}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_\infty. \tag{29}$$

By the exponentially decay of w_{μ_i} at infinity, we have

$$\left|\left\langle h, \frac{\partial W}{\partial q_i}\right\rangle\right| \le C\|h\|_{**}. \tag{30}$$

Combining the above estimates (28), (30), Lemma 4.2, and recall that $d_1 > d_2$, we get

$$\|\widehat{\beta}\|_\infty \le C\left(d_2^{-\frac{1}{2}}e^{-d_2}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_\infty + \|h\|_{**}\right). \tag{31}$$

Now we prove (27). We argue by contradiction. Assume there exist $\begin{pmatrix}\phi^{(K)}\\ \psi^{(K)}\end{pmatrix}$, $h^{(K)}$ solution to (26) and

$$\|h^{(K)}\|_{**} \to 0, \qquad \left\|\begin{pmatrix}\phi^{(K)}\\ \psi^{(K)}\end{pmatrix}\right\|_{**} = 1, \tag{32}$$

as $K \to \infty$. For simplicity, we drop K in the superscript.

First by the exponential decay of w_{μ_i}, we can make further computations ([8]). For any $x \in \mathbb{R}^2 \setminus \cup_{j=1}^K B(x^j, \tau)$ large independent of K

$$\bar{U} \le w_{\mu_1}(\tau) + Ce^{-\frac{d_1}{2}} \le 2\mu_1^{-\frac{1}{2}}c_0\tau^{-\frac{1}{2}}e^{-\tau} + Ce^{-\frac{d_1}{2}},$$

where c_0 is defined in (7). So does $\bar{V}$.

With the assumption on P, Q, one can check carefully for τ large, e.g. $\frac{24c_0^2+\eta}{\tau} < \frac{1+\eta-2\eta^2}{4}$, then

$$L_1(W_\pm) := \Delta W_\pm - P(x)W_\pm + 3\mu_1\bar{U}^2W_\pm \le -\frac{1-\eta}{2}W_\pm,$$

and

$$L_2(W_\pm) := \Delta W_\pm - Q(x)W_\pm + 3\mu_2\bar{V}^2W_\pm \le -\frac{1-\eta}{2}W_\pm,$$

in $\mathbb{R}^2 \setminus (\cup_{j=1}^K B(x^j,\tau) \cup \cup_{j=1}^K B(y^j,\tau))$, where $W_\pm = \sum_{j=1}^K e^{\pm\eta|x-x^j|} + \sum_{j=1}^K e^{\pm\eta|x-y^j|}$. Using maximum principle in the domain $\mathbb{R}^2 \setminus (\cup_{j=1}^K B(x^j,\tau) \cup \cup_{j=1}^K B(y^j,\tau))$, we have the following:

$$|\phi(x)| \le C\Big(\|L_1(\phi)\|_{**} + \sup_{j=1}^{2K}\|\phi\|_{L^\infty(B(Q_j,\tau))}\Big)\sum_{l=1}^{2K} e^{-\eta|x-Q_l|}$$

and

$$|\psi(x)| \le C\Big(\|L_2(\psi)\|_{**} + \sup_{j=1}^{2K}\|\psi\|_{L^\infty(B(Q_j,\tau))}\Big)\sum_{l=1}^{2K} e^{-\eta|x-Q_l|},$$

where $Q_j = x^j$ for $j = 1,\dots,K$ and $Q_j = y^{j-K}$ for $j = K+1,\dots,2K$, see also (11).

By the equation satisfied by $\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ and τ large,e.g. $\frac{\beta}{\tau}(\frac{1}{\mu_1}+\frac{1}{\mu_2})$ small, we have

$$\begin{aligned}|\phi(x)| + |\psi(x)| \le & C\left(\left\|L\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**} + e^{-\frac{d_2}{2}}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**}\right.\\ &\left. + \sup_{j=1}^{2K}\|\phi\|_{L^\infty(B(Q_j,\tau))}\right)\sum_{l=1}^{2K} e^{-\eta|x-Q_l|}.\end{aligned} \tag{33}$$

By (31), the assumption (32) and the above estimate (33), there exists a subsequence of Q_j such that

$$\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^\infty(B(Q_j,\tau))} \geq C > 0, \tag{34}$$

for some constant C independent of K. Using elliptic estimates together with Ascoli-Arzela's theorem, without loss of generality, we can find a subsequence x^j such that $\begin{pmatrix} \phi(\cdot + x^j) \\ \psi(\cdot + x^j) \end{pmatrix}$ will converge (on any compact set) to $\begin{pmatrix} \phi_\infty \\ \psi_\infty \end{pmatrix}$ bounded by a constant times $e^{-\eta|x|}$ solving

$$\begin{cases} \Delta\phi_\infty - \phi_\infty + 3\mu_1 w_{\mu_1}^2 \phi_\infty = 0, \\ \Delta\psi_\infty - \psi_\infty + \beta w_{\mu_1}^2 \psi_\infty = 0, \\ \int_{\mathbb{R}^2} \phi_\infty \nabla w_{\mu_1} dy = 0 \end{cases} \tag{35}$$

Since w_{μ_1} is non-degenerate and ϕ_∞ satisfies the orthogonality condition, one has easily $\phi_\infty = 0$. By energy analysis, if

$$\beta < \min\{w_{\mu_1}^{-2}(0), w_{\mu_2}^{-2}(0)\}, \tag{36}$$

the only possibility is $\psi_\infty = 0$, in contradiction with (34). Thus we get the a priori estimate.

Consider the space

$$\mathcal{H} = \left\{ u = (u_1, u_2) \in (H^1(\mathbb{R}^2))^2 \, : \, \int_{\mathbb{R}^2} u_1 Z_{x^j} = 0, \, \int_{\mathbb{R}^2} u_2 Z_{y^j} = 0, \quad j = 1, \dots, K \right\}.$$

Notice that the problem (26) in (ϕ, ψ) gets re-written as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \mathcal{K} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \bar{h} \quad \text{in} \quad \mathcal{H} \tag{37}$$

where $\bar{h}$ is defined by duality and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (26) has a unique solution for each $\bar{h}$ is equivalent to showing that (37) has only trivial solution when $\bar{h} = 0$, which in turn follows from a priori estimate (27). Furthermore, by the standard elliptic regularity result and imbedding theorem, $(\phi, \psi) \in (H^2(\mathbb{R}^2))^2$ is a strong solution. This concludes the proof of Lemma 4.3. □

4.2. **Nonlinear analysis.** Before we give the complete proof of Proposition 4.1, we first show the estimate of the error. Recall the definition of Λ_K in (12).

Lemma 4.4. *Given $(Q_1, \cdots, Q_{2K}) \in \Lambda_K$, then for any $0 < \eta < 1$ and K large enough, there is a constant C independent of K such that*

$$\|E\|_{**} \leq CK^{-n} (\ln K)^{-\frac{1}{2}}.$$

Proof. First recall

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = S \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} \Delta\bar{U} - P(x)\bar{U} + \mu_1 \bar{U}^3 + \beta\bar{U}\bar{V}^2 \\ \Delta\bar{V} - Q(x)\bar{V} + \mu_2\bar{V}^3 + \beta\bar{V}\bar{U}^2 \end{pmatrix}.$$

First let us compute E_1.

$$E_1 = -(P(x) - 1)\bar{U} + \mu_1 \left(\bar{U}^3 - \sum_{j=1}^{K} w_{\mu_1}^3 (x - x^j) \right) + \beta\bar{V}^2\bar{U} := I_1 + I_2 + I_3.$$

According to Lemma 4.4 in [8], we have

$$\|I_1\|_{**} \le CR^{-m} \le CK^{-m}(\ln K)^{-m},$$

and

$$\|I_2\|_{**} \le Cd_1^{-\frac{1}{2}}e^{-d_1} \le CK^{-m}(\ln K)^{-\frac{1}{2}}.$$

Next we deduce the estimate for I_3. First we define for any $\ell \in \mathbb{N}$

$$\Omega_j^\ell = \left\{x \in \mathbb{R}^2, |x - Q_j| = \min_{1\le l\le 2K} |x - Q_l| \le \frac{\ell d_2}{2}\right\}, \qquad j = 1, \dots, 2K$$

and

$$\Omega_{2K+1}^\ell = \mathbb{R}^2 \setminus \left(\cup_{j=1}^{2K} \Omega_j^\ell\right).$$

For $x \in \Omega_{2K+1}^\ell$,

$$\begin{aligned}|I_3| &\le CK \sum_{l,j=1,\cdots,K} w_{\mu_2}^2(x - y^l)w_{\mu_1}(x - x^j)\\ &\le CK \sum_{j=1}^{2K} e^{-|x-Q_j|} \le CKe^{-(1-\eta)\frac{\ell d_2}{2}} \sum_{j=1}^{2K} e^{-\eta|x-Q_j|},\end{aligned}$$

thus one can choose ℓ large enough but independent of K such that

$$Ke^{-(1-\eta)\frac{\ell d_2}{2}} \le CK^{-m-3}.$$

For $x \in \Omega_j^\ell, j = 1, \dots, K$, Corollary 3.6 in [8] tells us

$$\sum_{l=1}^{K} w_{\mu_i}(x - x^l) \le C\ell w_{\mu_i}(x - x^j),$$

which leads to

$$|I_3| \le Cw_{\mu_1}(x - x^j)\bar{V}^2 \le CK^2e^{-(R-\rho)}e^{-\eta|x-x^j|} \le CR^{-m-3}e^{-\eta|x-x^j|},$$

because of $|x - y^l| \ge |x^j - y^l| - |x - x^j| \ge \frac{1}{2}(R - \rho)$.

For $x \in \Omega_j^\ell, j = K + 1, \dots, 2K$, similarly

$$|I_3| \le C \sum_{l=1}^{K} w_{\mu_1}(x - x^l)w_{\mu_2}^2(x - y^j) \le CR^{-m-3}e^{-\eta|x-y^j|}.$$

Hence

$$|E_1| \le CK^{-m}(\ln K)^{-\frac{1}{2}} \sum_{j=1}^{2K} e^{-\eta|x-Q_j|}.$$

Similarly,

$$|E_2| \le CK^{-n}(\ln K)^{-\frac{1}{2}} \sum_{j=1}^{2K} e^{-\eta|x-Q_j|}.$$

In conclusion,

$$\|E\|_{**} \le CK^{-n}(\ln K)^{-\frac{1}{2}}.$$

□

We are now in the position to give the proof of Proposition 4.1. Let C_0 be a positive constant to be determined later, we define

$$\mathcal{B}_K = \left\{ (\phi,\psi) \in (L^\infty(\mathbb{R}^2))^2 : \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{**} \le C_0 K^{-n} (\ln K)^{-\frac{1}{2}} \right\}.$$

Then $\mathcal{B}_K$ is non-empty. Now we define a map $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = L^{-1}\left[-E - N\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right].$$

Now solving equation (19) is equivalent to finding a fixed point for the map $\mathcal{A}$. Since $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is uniformly bounded for $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{B}_K$, by the mean value theorem, there is a positive constant C such that for all $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{B}_K$,

$$\left| N\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right| \le C \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_\infty^2,$$

and

$$\left| N\begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - N\begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right| \le C \sum_{i=1}^{2} \left\| \begin{pmatrix} \phi^{(i)} \\ \psi^{(i)} \end{pmatrix} \right\|_\infty \left\| \begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - \begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_\infty,$$

which leads to

$$\left\| N\begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - N\begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_{**} \le C \sum_{i=1}^{2} \left\| \begin{pmatrix} \phi^{(i)} \\ \psi^{(i)} \end{pmatrix} \right\|_{**} \left\| \begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - \begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_{**}.$$

Hence we obtain

$$\left\| \mathcal{A}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{**} \le C\left(\|E\|_{**} + \left\| N\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{**} \right) \le CK^{-n}(\ln K)^{-\frac{1}{2}} + C\left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{**}^2,$$

and

$$\begin{aligned} \left\| \mathcal{A}\begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - \mathcal{A}\begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_{**} &\le C \left\| N\begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - N\begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_{**} \\ &\le \frac{1}{2} \left\| \begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} - \begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} \right\|_{**}, \end{aligned}$$

which show that $\mathcal{A}$ is a contraction mapping on $\mathcal{B}_K$. Hence there is a unique $(\phi,\psi) \in \mathcal{B}_K$ such that (19) is solved.

For the C^1 regularity of $(\alpha, \mathbf{q}) \to ((\phi,\psi), \widehat{\beta})$, the proof is the same as that of Proposition 4.1 in [8]. Following the same argument and using the estimate on (ϕ,ψ), one can get the estimate (20), we omit the details here.

5. **The reduced problem.** The main purpose of this section, is to achieve Step 2A. As we mentioned in the introduction, we define

$$\vec{\beta}_1 = \widehat{\beta}_1 - \gamma_1(R\mathbf{q}_{10} + \mathbf{q}_1^\perp), \qquad \vec{\beta}_2 = \widehat{\beta}_2 - \gamma_2(\rho\mathbf{q}_{20} + \mathbf{q}_2^\perp), \tag{38}$$

for some $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$.

Then equation (19) becomes

$$L\begin{pmatrix}\phi\\ \psi\end{pmatrix}+E+N\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\vec{\beta}_1\cdot\dfrac{\partial\bar{U}}{\partial\mathbf{q}_1}+\gamma_1\dfrac{\partial\bar{U}}{\partial\alpha_1}\\ \vec{\beta}_2\cdot\dfrac{\partial\bar{V}}{\partial\mathbf{q}_2}+\gamma_2\dfrac{\partial\bar{V}}{\partial\alpha_2}\end{pmatrix}. \tag{39}$$

Note that (ϕ,ψ) does not depend on γ, while $\vec{\beta}=(\vec{\beta}_1,\vec{\beta}_2)$ depends on the parameters $\alpha,\mathbf{q}$ and γ, so we write it as $\vec{\beta}=\vec{\beta}(\alpha,\mathbf{q},\gamma)$.

In this section, we are going to solve $\vec{\beta}=0$ for each α by adjusting γ and $\mathbf{q}$. Multiplying (39) by $\frac{\partial W}{\partial\mathbf{q}}$ and integrating over $\mathbb{R}^2$, we have

$$\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}+E+N\begin{pmatrix}\phi\\ \psi\end{pmatrix},\frac{\partial W}{\partial\mathbf{q}}\right\rangle=M\vec{\beta}+\left\langle\begin{pmatrix}\gamma_1\dfrac{\partial\bar{U}}{\partial\alpha_1}\\ \gamma_2\dfrac{\partial\bar{V}}{\partial\alpha_2}\end{pmatrix},\frac{\partial W}{\partial\mathbf{q}}\right\rangle.$$

By Lemma 4.2, solving $\vec{\beta}=0$ amounts to solve

$$\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}+E+N\begin{pmatrix}\phi\\ \psi\end{pmatrix},\frac{\partial W}{\partial\mathbf{q}}\right\rangle=\left\langle\begin{pmatrix}\gamma_1\dfrac{\partial\bar{U}}{\partial\alpha_1}\\ \gamma_2\dfrac{\partial\bar{V}}{\partial\alpha_2}\end{pmatrix},\frac{\partial W}{\partial\mathbf{q}}\right\rangle.$$

We will first compute the projection of error and the terms involving (ϕ,ψ).

5.1. Projections.

We first compute $\langle E,\frac{\partial W}{\partial\mathbf{q}}\rangle$.

Lemma 5.1. *Under the assumption of Proposition 4.1, for sufficiently large K, the following estimates hold:*

$$\begin{aligned}\int_{\mathbb{R}^2}E_1Z_{x^j}=&-a_0m|x^j|^{-m-1}\frac{x^j}{|x^j|}-\sum_{l\neq j}\Psi_1(x^j-x^l)\frac{x^l-x^j}{|x^l-x^j|}\\&+R^{-m-\theta}\Pi_{1k}(\alpha,\mathbf{q})+R^{-m-3}\Pi_{2k}(\alpha,\mathbf{q})+R^{-(2-\eta)m}\Pi_{3k}(\alpha,\mathbf{q}),\end{aligned}\tag{40}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2}E_2Z_{y^j}=&-b_0n|y^j|^{-n-1}\frac{y^j}{|y^j|}-\sum_{l\neq j}\Psi_2(y^j-y^l)\frac{y^l-y^j}{|y^l-y^j|}\\&+\rho^{-n-\sigma}\Pi_{4k}(\alpha,\mathbf{q})+\rho^{-n-3}\Pi_{5k}(\alpha,\mathbf{q})+\rho^{-(2-\eta)n}\Pi_{6k}(\alpha,\mathbf{q}),\end{aligned}\tag{41}$$

where $a_0=\frac{a}{2}\int_{\mathbb{R}^2}w_{\mu_1}^2(y)dy$, and $b_0=\frac{b}{2}\int_{\mathbb{R}^2}w_{\mu_2}^2(y)dy,\eta$ is a small positive constant chosen later and $\Pi_{lk}(\alpha,\mathbf{q}),l=1,\dots,6$ are smooth vector valued functions which are uniformly bounded as $K\to\infty$.

Proof. By definition, we can easily deduce

$$\begin{aligned}\int_{\mathbb{R}^2} E_1 Z_{x^j} =& -\sum_{l=1}^K \int_{\mathbb{R}^2} (P(x)-1) w_{\mu_1}(x-x^l) \nabla w_{\mu_1}(x-x^j) \\ &+ \mu_1 \int_{\mathbb{R}^2} \left(\left(\sum_{l=1}^K w_{\mu_1}(x-x^l) \right)^3 - \sum_{l=1}^K w_{\mu_1}^3(x-x^l) \right) \nabla w_{\mu_1}(x-x^j) \\ &+ \beta \int_{\mathbb{R}^2} \left(\sum_{l=1}^K w_{\mu_2}(x-y^l) \right)^2 \left(\sum_{l=1}^K w_{\mu_1}(x-x^l) \right) \nabla w_{\mu_1}(x-x^j) \\ =& - J_1 + J_2 + J_3.\end{aligned}$$

From Lemma 5.1 in [8], we have

$$J_1 = a_0 m |x^j|^{-m-1} \frac{x^j}{|x^j|} + R^{-m-\theta} \tilde{\Pi}_{1k}(\alpha, \mathbf{q}) + R^{-m-3} \tilde{\Pi}_{2k}(\alpha, \mathbf{q}) + R^{-2m} \tilde{\Pi}_{3k}(\alpha, \mathbf{q}),$$

and

$$J_2 = -\sum_{l \neq j} \Psi_1(x^l - x^j) \frac{x^l - x^j}{|x^l - x^j|} + R^{-(2-\eta)m} \tilde{\Pi}_{4k}(\alpha, \mathbf{q}).$$

Using for any $j, l = 1, \ldots, K, |x^j - y^l| \geq R - \rho - 4 \sim \frac{m-n}{2\pi} K \ln K$, one can easily check that

$$|J_3| \leq C K^2 \int_{\mathbb{R}^2} \sum_{l=1}^K w_{\mu_2}^2(x-y^l) \left| \nabla w_{\mu_1}(x-x^j) \right| dx = K^{-m-5} \tilde{\Pi}_{5k}(\alpha, \mathbf{q}).$$

Combining the above three estimates, we obtain (40)

Similarly, we can get the estimate (41). □

Now we can analyze $\langle E, \frac{\partial W}{\partial \mathbf{q}} \rangle$. Before we start, we define the following:

$$\widehat{d_1} = -\frac{\Psi_1'(d_1)}{\Psi_1(d_1)} d_1 = d_1 + O(1), \qquad \widehat{d_2} = -\frac{\Psi_2'(d_2)}{\Psi_2(d_2)} d_2 = d_2 + O(1)$$

Then by Lemma 5.2 in [8], Lemma 5.1 and

$$\frac{\partial \bar{U}}{\partial \mathbf{q}} = (\frac{\partial \bar{U}}{\partial \mathbf{q}_1}, \vec{0}) = -(Z_{x^1} \cdot \vec{n}_{11}, \ldots, Z_{x^K} \cdot \vec{n}_{1K}, Z_{x^1} \cdot \vec{t}_{11}, \ldots, Z_{x^K} \cdot \vec{t}_{1K}, \vec{0})^T,$$

$$\frac{\partial \bar{V}}{\partial \mathbf{q}} = (\vec{0}, \frac{\partial \bar{V}}{\partial \mathbf{q}_2}) = -(\vec{0}, Z_{y^1} \cdot \vec{n}_{21}, \ldots, Z_{y^K} \cdot \vec{n}_{2K}, Z_{y^1} \cdot \vec{t}_{21}, \ldots, Z_{y^K} \cdot \vec{t}_{2K})^T$$

we have the following estimates:

Lemma 5.2. *Under the assumption of Proposition 4.1, for K large enough, we can get the following estimates:*

$$\begin{aligned}\int_{\mathbb{R}^2} E_1 \frac{\partial \bar{U}}{\partial \mathbf{q}_1} =& a_0 m R^{-m-2} T_1 \mathbf{q}_1 + R^{-m-\theta} \Pi_{1k} + R^{-m-3} \Pi_{2k} + R^{-(2-\eta)m} \Pi_{3k} \\ &+ R^{-m-3} (\ln K)^2 \Pi_{4k}(\alpha, \mathbf{q}_2, \mathbf{q}_1, \dot{\mathbf{q}}_1, \ddot{\mathbf{q}}_1),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} E_2 \frac{\partial \bar{V}}{\partial \mathbf{q}_2} =& b_0 n \rho^{-n-2} T_2 \mathbf{q}_2 + \rho^{-n-\sigma} \Pi_{5k} + \rho^{-n-3} \Pi_{6k} + \rho^{-(2-\eta)n} \Pi_{7k} \\ &+ \rho^{-n-3} (\ln K)^2 \Pi_{8k}(\alpha, \mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2, \ddot{\mathbf{q}}_2)\end{aligned}$$

where Π_{ik} are uniformly bounded smooth vector functions with

$$\Pi_{4k}(\alpha,\mathbf{q}_2,0,0,0)=0,\qquad \Pi_{8k}(\alpha,\mathbf{q}_1,0,0,0)=0$$

and T_1, T_2 are $2K\times 2K$ matrix defined by the following:

$$T_1=\begin{pmatrix} c_1A_1+c_4I & c_2A_2 \\ -c_2A_2 & c_3A_1 \end{pmatrix}, \tag{42}$$

$$T_2=\begin{pmatrix} \tilde{c}_1A_1+\tilde{c}_4I & \tilde{c}_2A_2 \\ -\tilde{c}_2A_2 & \tilde{c}_3A_1 \end{pmatrix}. \tag{43}$$

Here I is the $K\times K$ identity matrix, and the matrix A_1, A_2 are both $K\times K$ circulant matrices given by the following:

$$A_1=\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix},$$

and $c_1, c_2, c_3, c_4, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4$ are constants given by

$$c_1=\frac{K^2}{4\pi^2},\ c_2=(\widehat{d}_1-1)\frac{K}{4\pi},\ c_3=-\widehat{d}_1\frac{K^2}{4\pi^2},\ c_4=\widehat{d}_1-m-1,$$
$$\tilde{c}_1=\frac{K^2}{4\pi^2},\ \tilde{c}_2=(\widehat{d}_2-1)\frac{K}{4\pi},\ \tilde{c}_3=-\widehat{d}_2\frac{K^2}{4\pi^2},\ \tilde{c}_4=\widehat{d}_2-n-1.$$

Next we consider the terms involving (ϕ,ψ).

Lemma 5.3. *Under the assumption of Proposition 4.1, for K large enough, the following estimates hold:*

$$\left|\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \frac{\partial W}{\partial \mathbf{q}}\right\rangle\right| \le CK^{-2n}(\ln K)^{-1}\Pi_{9,k}(\alpha,\mathbf{q}),$$

and

$$\left|\left\langle N\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \frac{\partial W}{\partial \mathbf{q}}\right\rangle\right| \le C\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**}^2 \le CK^{-2n}(\ln K)^{-1}\Pi_{10,k}(\alpha,\mathbf{q}) \tag{44}$$

where $\Pi_{9,k}, \Pi_{10,k}$ are uniformly bounded smooth vector functions.

Proof. Integrating by parts, with Proposition 4.1 and (28), (29), we can deduce

$$\left|\left\langle L\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \frac{\partial W}{\partial \mathbf{q}}\right\rangle\right| \le Cd_2^{-\frac{1}{2}}e^{-d_2}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{\infty} \le CK^{-2n}(\ln K)^{-1}.$$

For (44),

$$\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**} \le CK^{-n}(\ln K)^{-\frac{1}{2}},\qquad \left|N\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right| \le C\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{\infty}^2,$$

give us

$$\left|\left\langle N\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \frac{\partial W}{\partial \mathbf{q}}\right\rangle\right| \le C\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{**}^2 \le CK^{-2n}(\ln K)^{-1}.$$

□

5.2. **The invertibility of T_i.** In this subsection, we study the linear problem $T_i\mathbf{q}_i = \mathbf{b}_i$. First by Lemma 8.3 and Lemma 8.5 in [8], we have the following:

Lemma 5.4. *There exists $K_0 > 0$ such that for all $K \ge K_0$, and every $\mathbf{b} \in \mathbb{R}^{2K}$, there exist unique vectors $\mathbf{q}_1, \mathbf{q}_2$ and unique constants γ_1, γ_2 such that*

$$T_i\mathbf{q}_i = \mathbf{b} + \gamma_i\mathbf{q}_{i0}, \qquad \mathbf{q}_i \perp \mathbf{q}_{i0}, \quad i = 1, 2.$$

Moreover

$$\|\mathbf{q}_i\|_2 \le C\|\mathbf{b}\|_2, \quad \|\dot{\mathbf{q}}_i\|_2 \le C(\ln K)^{\frac{1}{2}}\|\mathbf{b}\|_2, \quad \|\ddot{\mathbf{q}}_i\|_2 \le C(\ln K)^{\frac{3}{2}}\|\mathbf{b}\|_2,$$

and

$$\|\mathbf{q}_i\|_* \le C(\ln K)^2\|\mathbf{b}\|_\infty.$$

Here $\|\mathbf{q}_i\|_* = \|\mathbf{q}_i\|_\infty + \|\dot{\mathbf{q}}_i\|_\infty + \|\ddot{\mathbf{q}}_i\|_\infty$.

With Lemma 5.1, we can conclude

Lemma 5.5. *There exists $K_0 > 0$ such that for $K \ge K_0$, and every $\mathbf{b}_i \in \mathbb{R}^{2K}$, there exist unique vector $\mathbf{q}_i \in \mathbb{R}^{2K}$ and unique constant $\gamma_i \in \mathbb{R}$ such that*

$$T_i\mathbf{q}_i = \mathbf{b}_i + \gamma_i\mathbf{q}^i, \qquad \mathbf{q}_i \perp \mathbf{q}_{i0},$$

where

$$\begin{pmatrix}\mathbf{q}^1\\ \mathbf{q}^2\end{pmatrix} = \begin{pmatrix}\displaystyle\int_{\mathbb{R}^2}\frac{\partial \bar U}{\partial \alpha_1}\frac{\partial \bar U}{\partial \mathbf{q}_1}\\ \displaystyle\int_{\mathbb{R}^2}\frac{\partial \bar V}{\partial \alpha_2}\frac{\partial \bar V}{\partial \mathbf{q}_2}\end{pmatrix} = M\begin{pmatrix}R\mathbf{q}_{10} + \mathbf{q}_1^\perp\\ \rho\mathbf{q}_{20} + \mathbf{q}_2^\perp\end{pmatrix}. \tag{45}$$

Moreover,

$$\|\mathbf{q}_i\|_* \le C(\ln K)^2\|\mathbf{b}_i\|_\infty. \tag{46}$$

Proof. To prove Lemma 5.5, it suffices to prove a priori estimate (46). Using the definition of $\mathbf{q}^1, \mathbf{q}^2$ in (45) and (22), (23), we find that

$$R^{-1}\mathbf{q}^1 = c_0\mathbf{q}_{10} + O(KR^{-1}), \qquad \rho^{-1}\mathbf{q}^2 = c_1\mathbf{q}_{20} + O(K\rho^{-1}),$$

which imply that

$$\|R^{-1}\mathbf{q}^1\|_\infty \le C, \qquad \|\rho^{-1}\mathbf{q}^2\|_\infty \le C$$

and

$$|R^{-1}\mathbf{q}^1 \cdot \mathbf{q}_{10}| \ge CK, \qquad |\rho^{-1}\mathbf{q}^2 \cdot \mathbf{q}_{20}| \ge CK.$$

Hence take

$$\gamma_i = -\frac{\mathbf{b}_i \cdot \mathbf{q}_{i0}}{\mathbf{q}^i \cdot \mathbf{q}_{i0}},$$

then

$$\|\gamma_1\mathbf{q}^1\|_\infty = \left\|\frac{\mathbf{b}_1 \cdot \mathbf{q}_{10}}{R^{-1}\mathbf{q}^1 \cdot \mathbf{q}_{10}}R^{-1}\mathbf{q}^1\right\|_\infty \le C\|\mathbf{b}_1\|_\infty,$$

So does $\gamma_2\mathbf{q}^2$. Therefore, by Lemma 5.4, we have

$$\|\mathbf{q}_i\|_* \le C(\ln K)^2\|\mathbf{b}_i + \gamma_i\mathbf{q}^i\|_\infty \le C(\ln K)^2\|\mathbf{b}_i\|_\infty.$$

□

Denote the inverse of T_i by T_i^{-1} and $\mathbf{q}_i = T_i^{-1}(\mathbf{b}_i)$.

5.3. **Reduction to two dimensions.**

Proposition 5.6. *Under the assumption of Theorem 1.1, there exists an integer $K_0 > 0$ such that for all $K > K_0$ and for each $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, there exists a unique $(\mathbf{q}, \gamma)$ such that $\vec{\beta} = 0$. As a result $(\phi, \psi), \gamma$ satisfy the equation:*

$$\begin{cases} L\begin{pmatrix} \phi \\ \psi \end{pmatrix} + E + N\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \gamma_1 \dfrac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \dfrac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix} \\ \displaystyle\int_{\mathbb{R}^2} \phi Z_{x^j} = 0, \qquad \int_{\mathbb{R}^2} \psi Z_{y^j} = 0, \qquad j = 1, \dots, K. \end{cases}$$

Moreover, the function (ϕ, ψ) is C^1 in α, and satisfies the following:

$$\|(\phi,\psi)\|_{**} \le C_0 K^{-n} (\ln K)^{-\frac{1}{2}}, \quad \sum_{i=1}^{2} \left[(R^{-1} + \rho^{-1}) \left\| \frac{\partial \mathbf{q}}{\partial \alpha_i} \right\|_* + \|\mathbf{q}_i\|_* \right] \le K^{-\mu} \ln^2 K. \tag{47}$$

for some $\mu > 0$ small enough but independent of K.

To prove Proposition 5.6, it suffices to solve $\vec{\beta}_i = 0$ for each (α_1, α_2). By the results in Lemma 5.1, Lemma 5.2 and Lemma 5.3, we can rewrite this equation as follows:

Lemma 5.7. *For every $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, the equation $\vec{\beta}_i(\alpha, \mathbf{q}, \gamma) = 0$ is equivalent to*

$$a_0 m R^{-m-2} T_1 \mathbf{q}_1 + \Phi_1(\alpha, \mathbf{q}) = \gamma_1 \mathbf{q}^1, \;\; b_0 n \rho^{-n-2} T_2 \mathbf{q}_2 + \Phi_2(\alpha, \mathbf{q}) = \gamma_2 \mathbf{q}^2 \tag{48}$$

where T_i are the $2K \times 2K$ matrix defined in (42) and (43), Φ_i denotes the remaining terms and $\mathbf{q}^1, \mathbf{q}^2$ are given in (45).

Here

$$\begin{aligned} \Phi_1(\alpha, \mathbf{q}) =& R^{-m-\theta} \Pi_{1k} + R^{-m-3} \Pi_{2k} + R^{-(2-\eta)m} \Pi_{3k} + K^{-2n} (\ln K)^{-1} \Pi_{9k}(\alpha, \mathbf{q}) \\ &+ R^{-m-3} (\ln K)^2 \Pi_{4k}(\alpha, \mathbf{q}_2, \mathbf{q}_1, \dot{\mathbf{q}}_1, \ddot{\mathbf{q}}_1), \end{aligned}$$

and

$$\begin{aligned} \Phi_2(\alpha, \mathbf{q}) =& \rho^{-n-\sigma} \Pi_{5k} + \rho^{-n-3} \Pi_{6k} + \rho^{-(2-\eta)n} \Pi_{7k} + K^{-2n} (\ln K)^{-1} \Pi_{10k}(\alpha, \mathbf{q}) \\ &+ \rho^{-n-3} (\ln K)^2 \Pi_{8k}(\alpha, \mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2, \ddot{\mathbf{q}}_2) \end{aligned}$$

where Π_i are uniformly bounded smooth vector functions, and

$$\Pi_{4k}(\alpha, \mathbf{q}_2, 0, 0, 0) = 0, \quad \Pi_{8k}(\alpha, \mathbf{q}_1, 0, 0, 0) = 0.$$

We are now going to prove Proposition 5.6.

Proof of Proposition 5.6. By Lemma 5.5, equation (48) is equivalent to

$$\mathbf{q}_1 = (a_0 m)^{-1} T_1^{-1} (R^{m+2} \Phi_1) = \mathcal{F}_1(\mathbf{q}), \qquad \mathbf{q}_2 = (b_0 n)^{-1} T_2^{-1} (\rho^{n+2} \Phi_2) = \mathcal{F}_2(\mathbf{q}).$$

By Lemma 5.7 and the assumption $m, n > 2, 2n > m + 2, \theta, \sigma > 2$, we can choose η small enough such that $(1 - \eta)n > 2$, then there exists μ small enough, but independent of K, such that

$$R^{m+2} \Phi_1(\alpha, \mathbf{q}) = K^{-\mu} \tilde{\Pi}_1 + (K^{-1} \ln^2 K) \tilde{\mathbb{E}}_1, \;\; \rho^{n+2} \Phi_2(\alpha, \mathbf{q}) = K^{-\mu} \tilde{\Pi}_2 + (K^{-1} \ln^2 K) \tilde{\mathbb{E}}_2$$

where $\tilde{\Pi}_1, \tilde{\Pi}_2, \tilde{\mathbb{E}}_1, \tilde{\mathbb{E}}_2$ are smooth bounded vector functions, $\tilde{\mathbb{E}}_1(\alpha, \mathbf{q}_2, 0, 0, 0) = 0$ and $\tilde{\mathbb{E}}_2(\alpha, \mathbf{q}_1, 0, 0, 0) = 0$.

Hence by Lemma 5.5, for $\|\mathbf{q}_1\|_* + \|\mathbf{q}_2\|_* < \frac{1}{2}$, we have

$$\|\mathcal{F}_i(\mathbf{q})\|_* \le C\left(K^{-\mu}\ln^2 K + K^{-1}\ln^4 K\right) \le CK^{-\mu}\ln^2 K,$$

and

$$\begin{aligned}\|\mathcal{F}_i(\mathbf{q}) - \mathcal{F}_i(\mathbf{q}^\circ)\|_* &\le C\left(K^{-\mu}\ln^2 K + K^{-1}\ln^4 K\right)(\|\mathbf{q}_1 - \mathbf{q}_1^\circ\|_* + \|\mathbf{q}_2 - \mathbf{q}_2^\circ\|_*)\\ &\le \frac{1}{2}(\|\mathbf{q}_1 - \mathbf{q}_1^\circ\|_* + \|\mathbf{q}_2 - \mathbf{q}_2^\circ\|_*).\end{aligned}$$

Therefor $\mathcal{F}_1, \mathcal{F}_2$ are contraction mappings. By Banach fixed point theorem, the result follows and so does the estimate (47).

Moreover, to show the differentiability of $\mathbf{q}(\alpha)$, consider the map $\mathcal{T}(\alpha,\mathbf{q}) = \mathbf{q} - (\mathcal{F}_1,\mathcal{F}_2) : \mathbb{R}^2\times\mathbb{R}^{4K} \to \mathbb{R}^{4K}$ which is of class C^1. Since $\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial \mathbf{q}} = O(K^{-\mu-1}\ln^2 K)$, $\frac{\partial\mathcal{T}}{\partial\mathbf{q}} = I + o(1)$ is invertible, we get the differentiability of $\mathbf{q}(\alpha)$.

Next we study the dependence of $\mathbf{q}$ on α. Assume that we have two solutions corresponding to two sets of parameters. One of them denoted by

$$\mathbf{q} = \left((a_0 m)^{-1}T_{1,\mathbf{q}}^{-1}\left[R^{m+2}\Phi_1(\alpha,\mathbf{q})\right], (b_0 n)^{-1}T_{2,\mathbf{q}}^{-1}\left[\rho^{n+2}\Phi_2(\alpha,\mathbf{q})\right]\right),$$

corresponding to α, the other denote by

$$\mathbf{q}^\circ = \left((a_0 m)^{-1}T_{1,\mathbf{q}^\circ}^{-1}\left[R^{m+2}\Phi_1(\alpha^\circ,\mathbf{q}^\circ)\right], (b_0 n)^{-1}T_{2,\mathbf{q}^\circ}^{-1}\left[\rho^{n+2}\Phi_2(\alpha^\circ,\mathbf{q}^\circ)\right]\right),$$

corresponding to α°. Assume that $R|\alpha_1^\circ - \alpha_1| + \rho|\alpha_2^\circ - \alpha_2| \le \frac{1}{2}$, we have

$$\|\mathbf{q}^\circ - \mathbf{q}\|_* \le CK^{-\mu}(\ln K)^2(R|\alpha_1^\circ - \alpha_1| + \rho|\alpha_2^\circ - \alpha_2|),$$

from which we get the desired result. □

6. **Proof of Theorem 1.1.**

In this section, we prove the main theorem. To solve $\gamma(\alpha) = 0$, we will apply the variational reduction. Let $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2$ and $\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\phi(\alpha,\mathbf{q}(\alpha))\\ \psi(\alpha,\mathbf{q}(\alpha))\end{pmatrix}$ be the function given in Proposition 5.6, we define the reduced energy function by

$$F(\alpha) = \mathcal{E}\begin{pmatrix}\bar U + \phi\\ \bar V + \psi\end{pmatrix} :\ \mathbb{R}^2 \to \mathbb{R}.$$

Here $(\bar U, \bar V)$ and (ϕ,ψ) are 2π periodic in α_1,α_2. Hence by Proposition 5.6, the reduced energy have the following property:

Lemma 6.1. *The functional $F(\alpha)$ is of class C^1 and 2π periodic in α_1,α_2.*

Next lemma shows that if $F(\alpha)$ has a critical point then $\gamma(\alpha) = 0$ has a solution. In other words, after Lyapunov-Schmidt reduction, the following lemma concerns the relationship between the critical points of $F(\alpha)$ and those of the energy functional $\mathcal{E}\begin{pmatrix}u\\ v\end{pmatrix}$.

Lemma 6.2. *Under the assumption of Proposition 5.6, there exists $K_0 > 0$, such that for all $K > K_0$, if α_0 is a critical point of $F(\alpha)$, then $\gamma(\alpha_0) = 0$, and the corresponding function $\begin{pmatrix}\bar U + \phi\\ \bar V + \psi\end{pmatrix}$ is a solution of (1).*

Proof. By Proposition 5.6 for K large and $\alpha \in \mathbb{R}^2$, ϕ satisfies the equation:

$$S\begin{pmatrix} \bar{U}+\phi \\ \bar{V}+\psi \end{pmatrix} = \begin{pmatrix} \gamma_1 \dfrac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \dfrac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}. \tag{49}$$

By the definition of F, obviously,

$$\nabla F(\alpha_1,\alpha_2) = \left\langle S\begin{pmatrix} \bar{U}+\phi \\ \bar{V}+\psi \end{pmatrix}, \nabla_\alpha \begin{pmatrix} \bar{U}+\phi \\ \bar{V}+\psi \end{pmatrix} \right\rangle,$$

where for $i=1,2$

$$\begin{cases} \partial_{\alpha_i}\left(\bar{U}+\phi\right) = \dfrac{\partial\left(\bar{U}+\phi\right)}{\partial \alpha_i} + \dfrac{\partial\left(\bar{U}+\phi\right)}{\partial \mathbf{q}} \cdot \dfrac{\partial \mathbf{q}}{\partial \alpha_i}, \\ \partial_{\alpha_i}\left(\bar{V}+\psi\right) = \dfrac{\partial\left(\bar{V}+\psi\right)}{\partial \alpha_i} + \dfrac{\partial\left(\bar{V}+\psi\right)}{\partial \mathbf{q}} \cdot \dfrac{\partial \mathbf{q}}{\partial \alpha_i}. \end{cases}$$

Thus using (49), we obtain

$$\partial_{\alpha_i} F(\alpha_1,\alpha_2) = \left\langle \begin{pmatrix} \gamma_1 \dfrac{\partial \bar{U}}{\partial \alpha_1} \\ \gamma_2 \dfrac{\partial \bar{V}}{\partial \alpha_2} \end{pmatrix}, \begin{pmatrix} \partial_{\alpha_i}(\bar{U}+\phi) \\ \partial_{\alpha_i}(\bar{V}+\psi) \end{pmatrix} \right\rangle.$$

If α_0 is a critical point of F, that is, $\nabla F(\alpha_0)=0$, then it is easily observed that $(\gamma_1,\gamma_2)=0$ is equivalent to the non-degeneracy of the following matrix

$$\begin{pmatrix} \dfrac{\partial \bar{U}}{\partial \alpha_1}\partial_{\alpha_1}(\bar{U}+\phi) & \dfrac{\partial \bar{V}}{\partial \alpha_2}\partial_{\alpha_1}(\bar{V}+\psi) \\ \dfrac{\partial \bar{U}}{\partial \alpha_1}\partial_{\alpha_2}(\bar{U}+\phi) & \dfrac{\partial \bar{V}}{\partial \alpha_2}\partial_{\alpha_2}(\bar{V}+\psi) \end{pmatrix}. \tag{50}$$

With definitions in (10) and (13), one may check that

$$\frac{\partial \bar{U}}{\partial \alpha_1} = (R\mathbf{q}_{10} + \mathbf{q}_1^{\perp}) \cdot \frac{\partial \bar{U}}{\partial \mathbf{q}_1}, \quad \frac{\partial \bar{U}}{\partial \alpha_2} = 0, \quad \frac{\partial \bar{V}}{\partial \alpha_2} = (\rho\mathbf{q}_{20} + \mathbf{q}_2^{\perp}) \cdot \frac{\partial \bar{V}}{\partial \mathbf{q}_2}, \quad \frac{\partial \bar{V}}{\partial \alpha_1} = 0.$$

By (25) and Proposition 5.6, direct computations give us that

$$\begin{cases} K^{-1}R^{-2}\dfrac{\partial \bar{U}}{\partial \alpha_1}\partial_{\alpha_1}(\bar{U}+\phi) = (1+o(1))\displaystyle\int_{\mathbb{R}^2} \left(\partial_{x_1} w_{\mu_1}\right)^2 dx, \\ K^{-1}\rho^{-2}\dfrac{\partial \bar{V}}{\partial \alpha_2}\partial_{\alpha_1}(\bar{V}+\psi) = o(1), \\ K^{-1}R^{-2}\dfrac{\partial \bar{U}}{\partial \alpha_1}\partial_{\alpha_2}(\bar{U}+\phi) = o(1), \\ K^{-1}\rho^{-2}\dfrac{\partial \bar{V}}{\partial \alpha_2}\partial_{\alpha_2}(\bar{V}+\psi) = (1+o(1))\displaystyle\int_{\mathbb{R}^2} \left(\partial_{x_1} w_{\mu_2}\right)^2 dx, \end{cases}$$

which imply that (50) is non-degenerate and complete the proof. □

Proof of Theorem 1.1. By Lemma 6.1, $F(\alpha)$ is 2π periodic in α_1, α_2 and of class C^1. Hence it has a critical point in $[0,2\pi)\times[0,2\pi)$. Therefore Theorem 1.1 follows. □

Acknowledgments. The authors thank Prof. J. Wei for motivation of the original problem. L. Wang is supported by STCSM 14ZR1412800. The research of W. Yao is supported by Fondecyt Grant 3130543.

REFERENCES

[1] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Acad. Sci. Paris Ser.*, **1342** (2006), 453–458.

[2] T. Bartsch, N. Dancer and Z. Q. Wang, A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Cal. Var. Partial Differential Equations.*, **37** (2010), 345–361.

[3] T. Bartsch, Z. Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, *J. Fixed Point Theory Appl.*, **2** (2007), 353–367.

[4] J. Y. Byeon and M. Tanaka, Semiclassical standing waves with clustering peaks for nonlinear Schrödinger equations, *Memoirs of the American Mathematical Society,* **229** (2014), 89pp.

[5] K. Chow, Periodic solutions for a system of four coupled nonlinear Schrödinger equations, *Phys. Rev. Lett. A*, **285** (2001), 319–326.

[6] M. Conti, S. Terracini and G. Verzini, Nehari's problem and competing species systems, *Ann. Inst. H. Poincar Anal. Non Linaire*, **19** (2002), 871–888.

[7] N. Dancer, J. C. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincar Anal. Non Linaire*, **27** (2010), 953–969.

[8] M. del Pino, J. C. Wei and W. Yao, Intermediate reduction methods and infinitely many positive solutions of nonlinear Schrödinger equations with non-symmetric potentials, *Car. Var. PDE.,* **53** (2015), 473–523.

[9] Y. Guo and J. Wei, Infinitely many positive solutions for nonlinear Schrödinger system with non-symmetric first order, preprint.

[10] F. Hioe and T. Salter, Special set and solution of coupled nonlinear Schrödinger equations, *J. Phys. A: Math. Gen.,* **35** (2002), 8913–8928.

[11] T. C. Lin and J. C. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Comm. Math Phys.*, **255** (2005), 629–653.

[12] T. C. Lin and J. C. Wei, Solitary and self-similar solutions of two-component system of nonlinear Schrödinger equations, *Phy. D*, **220** (2006), 99–115.

[13] Z. Liu and Z. Q. Wang, Multiple bound states of nonlinear Schrödinger system, *Comm. math. Phys.*, **282** (2008), 721–731.

[14] A. Malchiodi, Some new entire solutions of semilinear elliptic equations on $\mathbb{R}^N$, *Adv. Math.*, **221** (2009), 1843–1909.

[15] M. Mitchell and M. Segev, Self-trapping of inconherent white light, *Nature*, **387** (1997), 880–883.

[16] M. Musso, F. Pacard and J. Wei, Finite-energy sign-changing solutions with dihedral symmetry for the stationary nonlinear Schrödinger equation, *J. Eur. Math. Soc.*, **14** (2012), 1923–1953.

[17] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, *Comm. Pure Appl. Math.*, **63** (2010), 267–302.

[18] S. J. Peng and Z. Q. Wang, Segregated and synchronized vector solutions for nonlinear Schrödinger systems, *Arch. Rational. Mech. Anal.*, **208** (2013), 305–339.

[19] E. Timmermans, Phase seperation of Bose Einstein condensates, *Phys. Rev. Lett.*, **81** (1998), 5718–5721.

[20] S. Terracini and G. Verzini, Multipulse phase in $k-$mixtures of Bose-Einstein condenstates, *Arch. Rat. Mech. Anal.*, **194** (2009), 717–741.

[21] L. Wang, J. Wei and S. Yan, A Neumann problem with critical exponent in nonconvex domains and Lin-Ni's conjecture, *Trans. Amer. Math. Soc.*, **362** (2010), 4581–4615.

[22] L. Wang, J. Wei and S. Yan, On Lin-Ni's conjecture in convex domains, *Proc. Lond. Math. Soc.*, **102** (2011), 1099–1126.

[23] L. Wang and C. Zhao, Solutions with clustered bubbles and a boundary layer of an elliptic problem, *Discrete Contin. Dyn. Syst.*, **34** (2014), 2333–2357.

[24] J. C. Wei and T. Weth, Nonradial symmetric bound states for system of two coupled Schrödinger equations, *Rend. Lincei Mat. Appl.*, **18** (2007), 279–293.

[25] J. C. Wei and T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Rat. Mech. Anal.*, **190** (2008), 83–106.

[26] J. C. Wei and S. S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^n$, *Calc. Var. Partial Differential Equations*, **37** (2010), 423–439.

[27] J. Wei and W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, *CPAA*, **11** (2012), 1003–1011.

Received October 2015; revised November 2015.

E-mail address: wwao@math.ubc.ca
E-mail address: lpwang@math.ecnu.edu.cn
E-mail address: wyao.cn@gmail.com