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Topological phase transition of a fractal spin system: The relevance of the network complexity

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A new type of collective excitations, due to the topology of a complex random network that can be characterized by a fractal dimension D_F, is investigated. We show analytically that these excitations generate phase transitions due to the non-periodic topology of the $D_F > 1$ complex network. An Ising system, with long range interactions, is studied in detail to support the claim. The analytic treatment is possible because the evaluation of the partition function can be decomposed into closed factor loops, in spite of the architectural complexity. The removal of the infrared divergences leads to an unconventional phase transition, with spin correlations that are robust against thermal fluctuations. [http://dx.doi.org/10.1063/1.4942826]

For a long time phase transitions have attracted significant attention. In this context, the analytic achievement known as the Onsager[1] solution of the two-dimensional (2D) Ising model, and the Mermin-Wagner theorem (MWT)[2,3] have been important milestones in the development of the field. Here we investigate analytically a complex network with Ising nearest-neighbor (*nn*) plus random long-range interactions, and develop a mechanism that provides a local enhancement of the correlation length. We find that, as a consequence of the inclusion of frustrated long-range interactions, and due to the fractal nature of the non-periodic complex network, inversion symmetry is broken. Consequently, when the effective fractal dimension $D_F > 1$ acquires non-integer values, incoherent fluctuations are strongly suppressed.

The dimensionality restrictions imposed by the MWT,[2–5] which for integer $D < 3$ is responsible for the suppression of correlations by thermal fluctuations, has posed a challenging problem in areas like condensed matter and high energy physics. During the last two decades the strong dependence of the collective behavior of embedded topologies has given impulse to the study of magnetic systems on more intricate architectures. This way complex networks have become a fertile ground in the study of these non-ideal systems[6–8] since, in spite of their compact structure, the spatial fluctuations do give rise to a wide range of critical phenomena.

We notice that in these systems the underlying commonality is the emergence of phase transitions produced by the topological structure of the complex network. Hence we show, mainly using analytic tools, that the local correlation length can increase in *small-world* complex networks due to the randomly frustrated interactions brought about by the topology of the network. Indeed, when the fractal dimension $D_F > 1$ the complexity of the network can yield "long range" correlations of the collective excitations.

We face this challenge by means of two approaches. First, we construct an Ising model with the spins located at the nodes of the network, and allow them to interact through *nn* and long range interactions. The network in this specific case is of a *small-world* type, which has implications, that we will discuss below, for the correlation length and the nature of the phase transition itself.[9] The partition function of the model is calculated analytically, so that all the thermodynamic properties can be obtained. Moreover, we show that this phase transition of the Ising model on such a network is robust against thermal fluctuations, and consequently that an unconventional second order phase transition does occur. Our theory reproduces qualitatively the results of Chen et al.[10] that confirmed, through simulations on a 2D eight-state Potts model, that the presence of randomly distributed ferromagnetic bonds changes the phase transition from first to second order. The same analysis can be applied to results that were obtained by Theodorakis et al.[11] with a 2D Blume-Capel model embedded in a triangular lattice, and by Fytas et al.[12] for an antiferromagnetic Ising model with next nearest neighbor interactions. And second, we use a qualitative description that suggests how the topological complexity of the network can produce such a phase transition. This approach, in essence, generalizes Landau's phenomenological theory for critical phenomena to complex networks.

We start by introducing a ring Ising model, in order to analyze and illustrate the characteristics of this phase transition. As is well known, in the absence of random links, the 1D Ising model obeys the MWT and does not exhibit phase transitions[2] due to the periodicity of the interactions. To the 1D ring Ising model composed of N nodes, with only *nn* interactions, we add the possibility of long range interactions with other particles on the ring, thus generating a complex network. The Ising Hamiltonian, plus random long range interactions, is given by

$$\mathscr{H} = -\sum_i \Big(J\sigma_i\sigma_{i+1} + h\sigma_i + J_0\sigma_i\sigma_{r_i}\Big), \tag{1}$$

where $\sigma_i = \pm 1$; $J > 0$ and J_0 are the exchange constants between *nn* and long range neighbors, respectively, and $|J_0| < |J|$; g is the gyromagnetic ratio; $h = g\mu_B H_0$; μ_B is the Bohr magneton; and H_0 is the uniform applied magnetic field. For simplicity, we assume that a particle at site i of the network has a single long range interaction with particle r_i, selected at random. Consequently, the first term in Eq. (1) describes the *nn* exchange, the second the Zeeman interaction, and the third the exchange between particle i and another one located at r_i.

In order to calculate effects due to the random long range interactions on the phase transition we implement the transfer matrix method, to calculate analytically the partition function. We start defining the matrices

$$\begin{aligned} T_{\sigma_i,\sigma_j} &= \langle \sigma_i | \, e^{\epsilon_{i,j}/k_B T} \, |\sigma_j\rangle, \\ R_{\sigma_i,\sigma_j} &= \langle \sigma_i | \, e^{\Delta_{i,j}/k_B T} \, |\sigma_j\rangle, \\ M_{\sigma_i,\sigma_j} &= \sum_{\sigma_\ell} T_{\sigma_i,\sigma_\ell} R_{\sigma_\ell,\sigma_j}, \end{aligned} \tag{2}$$

where $J_0 \neq 0$, $\epsilon_{i,j} = J\sigma_i\sigma_j + h(\sigma_i + \sigma_j)/2$, and $\Delta_{i,j} = J_0\sigma_i\sigma_j$. J and J_0 describe the *nn* and long range interactions, respectively. $\Delta_{i,j}$ describes the interaction between the magnetic moment at sites $i+1$ and r_i through site i. These definitions allow to write the partition function of the system as

$$\begin{aligned} Z &= \sum_{\{\sigma_1,\sigma_2,\dots\}} \sum_{\{\sigma'_1,\sigma'_2,\dots\}} \prod_i T_{\sigma_{i+1},\sigma'_i} R_{\sigma'_i,\sigma_{r_i}}, \\ &= \sum_{\{\sigma_1,\sigma_2,\dots\}} \prod_i M_{\sigma_{i+1},\sigma_{r_i}} = \prod_k \mathrm{tr}(M^{n_k}). \end{aligned} \tag{3}$$

where n_k is the number of vertices of each loop of nodes that is formed as we follow the random links along the network. Hence, Eq. (3) specifies the cluster decomposition of the transfer matrix in *closed loop factors*. This decomposition is illustrated in Fig. 1 for an $N = 20$ configuration.

The $J_0 = 0$ limit is recovered recalling that $Z = \mathrm{tr}(T^N) = \lambda_+^N + \lambda_-^N$, where $\lambda_+ > \lambda_-$ are the T matrix eigenvalues. In the $N \to \infty$ limit $Z = \lambda_+^N$. However, when long range interactions are incorporated the spatial fluctuations are decomposed into *closed loop factors*, which in the $N \to \infty$ limit regularize the infrared divergences because we now have a multiplication of traces which correspond to the *closed loop factors*, so that $\mathrm{tr}(M^{n_0}) = \lambda_+^{n_0} + \lambda_-^{n_0}$ must be kept in full, for some $n_0 = \min\{n_k\} \ll N$.

For $h = 0$ and at finite temperatures, the average magnetic moment per particle $m_z = (N\beta)^{-1}$ $\partial \ln Z/\partial h$ vanishes for the Ising model, implying that long range order is destroyed by thermal

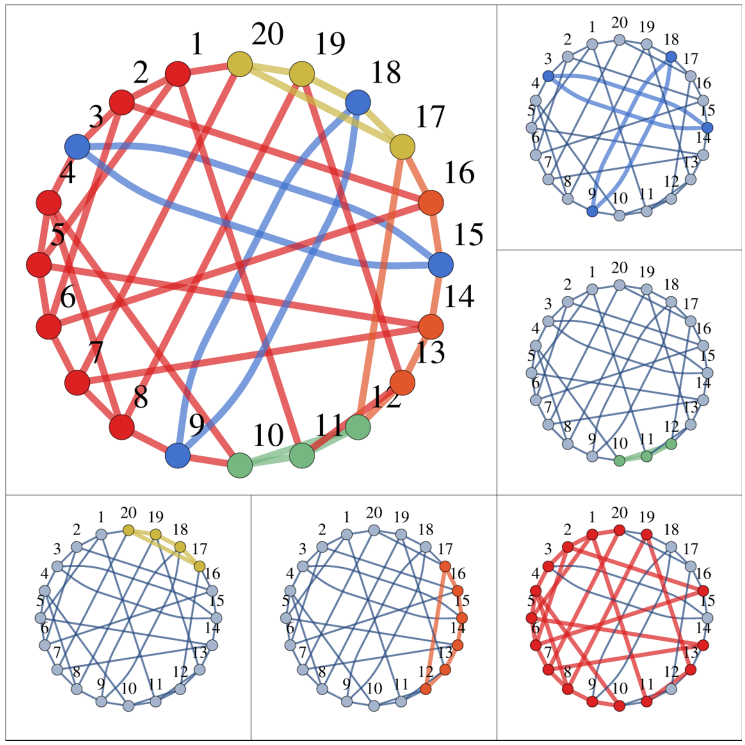

FIG. 1. Geometric representation of the decomposition into closed factor loops. The *closed loop factors* correspond to nodes $\{3,15,18,9\}$ (blue), $\{10,11,12\}$ (green), $\{17,18,19,20\}$ (yellow), $\{12,13,14,15,16,17\}$ (orange), and $\{1,2,3,4,5,6,7,8,9,10,11,13,14,16\}$ (red). Following Eqs. (2) and (3) they correspond to the factorization $Z = \mathrm{tr}(M^2)\mathrm{tr}(M^2)\mathrm{tr}(M^3)\mathrm{tr}(M^4)\mathrm{tr}(M^6)\mathrm{tr}(M^{23})$.

fluctuations. However, as the degree of interconnectivity between nodes grows, the fluctuations are suppressed. This in turn leads to an unconventional continuous phase transition for $J_0/J < 0$, *i.e.*, as the magnetic configuration is determined by a competition of short and long range interactions of different signs. Due to the local frustration this could lead to a spin glass arrangement. However, as a consequence of the *small-world* network topology, below the critical temperature a locally ordered magnetic structure distribution is favored. This is illustrated in Fig. 2, where we display the magnetization *vs.* temperature for an $N = 20$ complex network with $D_F = 1.47$, $J = 1$, $J_0 = -0.5$, and

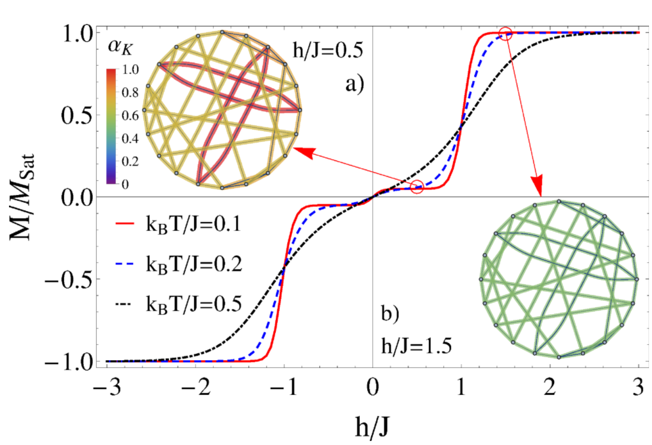

FIG. 2. m_z vs. h/J loops of an $N = 20$ node 1D Ising ring with one randomly distributed long range link per node, for $k_BT/J = 0.1, 0.2, 0.5$. The correlation length α_K of the given loops, as a function of cluster size, is given by the colors of the links of the a) and b) insets, for $h/J = 0.5$ and 1.5, respectively.

$h/J = 0.1$. We remark that the dependence of m_z on T is highly nonlinear on the fractal dimension D_F. This fact is also reflected in the $m_z \neq 0$ values shown in Fig. 2, even in the $h \to 0$ limit.

In order to describe the effects of the topology on the phase transition we study the correlation length α_K of the K-th closed loop $C(n_K)$, with n_K vertices, and

$$\begin{aligned} F(n_K) &= \sum\nolimits_{i,\delta \subseteq C(n_K)} \langle \sigma_i \sigma_{i+\delta} \rangle / n_K \\ &= \sum\nolimits_{\delta \subseteq C(n_K)} \mathrm{tr}(\sigma M^{\delta} \sigma M^{n_K - \delta}) / \mathrm{tr}(M^{n_K}) \,, \end{aligned} \tag{4}$$

where $F(n_K)$ is the two-point correlation function over a closed loop $C(n_K)$. On the basis of the above analytic expression, the correlation length α_K can be obtained from the exponential dependence[13] of $F(n_K) \propto e^{n_K/\alpha_K}$. The collective behavior of the system is dominated by the local correlation of magnetically ordered clusters. In order to describe this local correlation let us consider the 1D Ising ring system with one long range link per node (over $N = 20$ nodes) with a *small-world* network structure, as shown in Figs. 1 and 2. Fig. 2 displays the magnetization *vs.* applied field and the cluster decomposition into local conformations in configuration space. For small clusters the strong correlation is quite apparent, which also shows that the magnetic ordering is robust against thermal fluctuations.

As the magnetic field decreases the correlation length is reduced, and the large cluster magnetic order is destroyed by thermal fluctuations, but it is preserved in small ones. This induces a residual magnetization, where the continuous line represents the $J_0 = -|J|/2\,(J > 0)$ situation, and the dashed line the $J_0 = |J|/2\,(J < 0)$ case. It is also worth mentioning that this continuous phase transition resembles ferrimagnetic or canted anti-ferromagnetic ordering, but its microscopic configuration is created by random long range interactions through different closed loops, instead of locally induced magnetic domains.

It is relevant to notice that the concept of "long range" correlations to induce phase transitions requires some discussion. In general, we expect to observe a phase transition, and its associated critical phenomena, when the correlation length becomes large, e.g., effectively of the size of the system. In the case of the Ising model on a *small-world* network, the correlation length is about the size of the system. However, in this case it is the topology that reduces the effective distance between nodes, allowing the system to become of the size of the correlation length. This is why we have used a *small-world* type network, which allows a large system, in terms of the number of nodes, to have small average correlation length. This observation should be quite general, and be applicable to a number of other systems, as suggested by the qualitative characterization of the general phase transition problem in complex networks that we now present. Particular attention is given to complex networks of *small-world* type. The phase a system adopts, in thermal equilibrium, is characterized by spontaneous symmetry breaking[14] and by the ratio of the order parameter and the characteristic length of the system. Our main assumption is that the addition of the *small-world* structure (long range interactions) does not change significantly the correlation length but that, due to cluster decomposition, it reduces its size-scale and consequently increases the local correlation. Following Landau[14] we assume invariance under time reversal inversion and discrete symmetries to write, neglecting higher order terms of the order parameter ψ, the free energy $\Phi_0(\mathbf{x}, T)$ as $\Phi_0(\mathbf{x}, T) = A(T)|\nabla\psi|^2 + B(T)|\psi|^2$, where $\mathbf{x}$ is the spatial coordinate, $\psi = \psi_0 e^{-x/\xi}$, $\xi = \sqrt{A(T)/B(T)}$ is the correlation length, and A and B are functions of the temperature T. For $T > T_c$, $A(T) \ll B(T)$, and consequently $\xi \ll 1$. On the contrary, below the critical temperature $T < T_c$, $A(T) \gg B(T)$, and $\xi \gg 1$. This way the phase transition is characterized analytically. In order to estimate the critical dimensionality, we use the energy equipartition theorem which, including spatial fluctuations of the classical ground state, takes the form

$$\int d^D x \, \Phi_0(\mathbf{x}, T < T_c) \geq k_B T \,, \tag{5}$$

where D is the dimensionality of the system and k_B is the Boltzmann constant, and the inequality is due to quantum fluctuations.

Using the explicit expression for the order parameter in Eq. (5), we obtain (for $0 < T < T_c$)

$$|\psi_0|^2 \geq \frac{\rho k_B T}{A(T)(\pi\xi)^D\,\Gamma(D)}\left.\frac{k^{D-2}}{D-2}\right|_{k\to 0}^{k_0}, \tag{6}$$

where $1/\rho = \int d^D k/(2\pi)^D$, $\Gamma(y)$ is the Gamma function, and $1/k_0$ is the range of the interaction in units of the *nn* distance. This leads to a $1/k|_{k\to 0}$ infrared divergence for $D = 1$, and a $\ln k|_{k\to 0}$ divergence for $D = 2$, a result that is equivalent to the MWT. It is worth mentioning that our phenomenological model recovers results derived from a microscopic theory, based on the Bogoliubov inequality.[15]

Now we focus our attention on how this phenomenological theory can be applied to a complex network with both *nn* plus long range random links. The complexity of this topology can be characterized, for example, by an effective fractal dimension D_F that can be defined as the scaling $\langle N\rangle \sim d^{D_F}$ of the average number $\langle N\rangle$ of nodes within a radius d of a given node. Therefore, by means of a box-counting procedure, the inversion symmetry breaking may be characterized by an effective fractal dimension, which is expected to couple the free energy to a current density axial vector term, defined as

$$\mathbf{J}(\mathbf{x}) = i(\psi(\mathbf{x})\nabla\psi^*(\mathbf{x}) - \psi^*(\mathbf{x})\nabla\psi(\mathbf{x}))/2\,. \tag{7}$$

This axial vector $\mathbf{J}(\mathbf{x})$ is the simplest term that one can incorporate to break inversion symmetry. The above equation implies that $|\nabla\psi|^2 \sim J^2/|\psi|^2$, where $J(\mathbf{x}) = |\mathbf{J}(\mathbf{x})|$. This is a higher order term that, for $T < T_C$, can be neglected compared to $\Phi_1(\mathbf{x},T) = \gamma(T)J(\mathbf{x})$, where $\gamma(T) \sim k_B(T_C - T)$. When this last term is included in the free energy, we obtain

$$|\psi_0|^2 \geq \frac{\rho k_B T}{\gamma(T)(\pi\xi)^D\,\Gamma(D)}\left.\frac{k^{D-1}}{D-1}\right|_{k\to 0}^{k_0}. \tag{8}$$

This way the infrared divergences are removed for $D > 1$ since $|\psi_0|^2 \sim k^{D-1}/(D-1)$, so that the restrictions on phase transitions for $D < 3$ are removed. With the insight gained from dimensional regularization of the phenomenological approach, we now show how random exchange interactions lead to long range magnetic order. Our starting point, or unperturbed Hamiltonian, is a 1D periodic system (*i.e.* a ring with only *nn* interactions), and the random link interactions are added perturbatively *i.e.* $J_0/J \ll 1$, the expansion in momentum eigenstates is valid. But, this random link distribution brings about a dimensionality change, from integer to fractal ($D \to D_F$), and the breaking of the periodicity of the system (which is essential for the MWT to hold). In Fig. 3 we show the temperature and fractal dimension D_F dependence of the order parameter, for $k_0 \sim 1$ and assuming that the elementary excitation density, of correlation length ξ^{D_F}/ρ, remains constant. It is thus apparent that the introduction of random links removes the infrared divergences, and that phase transitions are in principle now allowed.

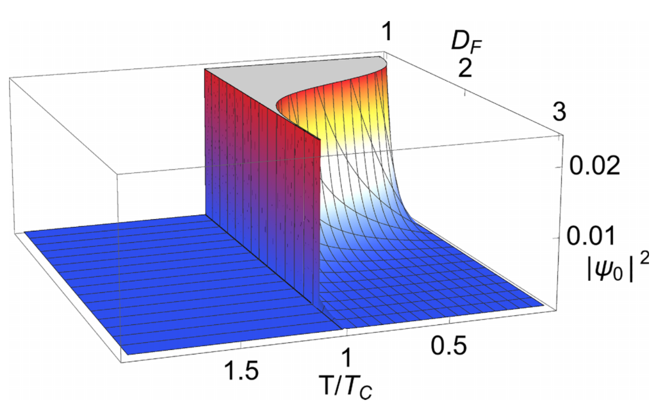

FIG. 3. Regularization of infrared divergences when the fractal dimension $D_F > 1$.

In conclusion, we have shown that random links remove infrared divergences by breaking the inversion symmetry of a complex network. This way, the dimensionality restrictions imposed by the MWT, which assumes translational invariance, are completely removed. Instead, an unconventional second order phase transition emerges due solely to the topology of the system. This topologically induced phase transition displays long range magnetic order in the absence of external fields through interconnected clusters, as suggested by Graß et al.[16] Moreover, the *small-world* network topology guarantees that spin correlations are robust against thermal fluctuations.

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