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Large mass boundary condensation patterns in the stationary Keller–Segel system

Manuel del Pino [a], Angela Pistoia [b,*], Giusi Vaira [b]

[a] *Departamento de Ingenieria Matemática, Facultad de Ciencias Fisicas y Matemáticas, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile*

[b] *Dipartimento SBAI, Sapienza Università di Roma, via Antonio Scarpa 16, 00161 Roma, Italy*

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Abstract

We consider the boundary value problem

$$\begin{cases} -\Delta u + u = \lambda e^u, & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, $\lambda > 0$ and ν is the inner normal derivative at $\partial\Omega$. This problem is equivalent to the stationary Keller–Segel system from chemotaxis.

We establish the existence of a solution u_λ which exhibits a sharp boundary layer along the entire boundary $\partial\Omega$ as $\lambda \to 0$. These solutions have large mass in the sense that $\int_\Omega \lambda e^{u_\lambda} \sim |\log \lambda|$.

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* Corresponding author.
E-mail addresses: delpino@dim.uchile.cl (M. del Pino), angela.pistoia@uniroma1.it (A. Pistoia), vaira.giusi@gmail.com (G. Vaira).

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1. Introduction and statement of the main result

Chemotaxis is one of the simplest mechanisms for aggregation of biological species. The term refers to a situation where organisms, for instance bacteria, move towards high concentrations of a chemical which they secrete. A basic model in chemotaxis was introduced by Keller and Segel [9]. They considered an advection–diffusion system consisting of two coupled parabolic equations for the concentration of the considered species and that of the chemical released, represented, respectively, by positive quantities $v(x,t)$ and $u(x,t)$ defined on a bounded, smooth domain Ω in $\mathbb{R}^N$ under no-flux boundary conditions. The system reads

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v - \nabla \cdot (v \nabla u) & \text{in} \quad \Omega \\ \tau \frac{\partial u}{\partial t} = \Delta u - u + v & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the unit inner normal to $\partial\Omega$. Steady states of (1.1) are the positive solutions of the system

$$\begin{cases} \Delta v - \nabla \cdot (v \nabla u) = 0 & \text{in} \quad \Omega \\ \Delta u - u + v = 0 & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) can be reduced to a scalar equation. Indeed, testing the first equation against $(\ln v - u)$, an integration by parts shows that a solution of (1.2) satisfies the relation

$$\int_\Omega v |\nabla(\ln v - u)|^2 = 0$$

and hence $v = \lambda e^u$ for some positive constant λ, and thus u satisfies the equation

$$\begin{cases} -\Delta u + u = \lambda e^u, & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Reciprocally, a solution to problem (1.3) produces one of (1.2) after setting $v = \lambda e^u$. In this paper we consider problem (1.3) when $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary and $\lambda > 0$ is a small parameter. By integrating both sides of the equation we see that a necessary condition for existence is $\lambda < 1$.

The analysis of problems (1.1), (1.2) and their corresponding versions in entire space $\mathbb{R}^2$, has a long history, starting with the work by Childress and Percus [3]. The analysis of the steady state problem (1.3) for small λ started with Schaaf [15] in the one-dimensional case. Existence of a radial solution when Ω is a ball, generating a spike shape at the origin when $\lambda \to 0$ was established by Biler [1]. The shape of an unbounded family of solutions u_λ with uniformly bounded masses

$$\limsup_{\lambda \to 0^+} \int_\Omega \lambda e^{u_\lambda} < +\infty$$

was established in [16,17]. As in the classical analysis by Brezis and Merle [2], blow-up of the family is found to occur at most on a finite number of points $\xi_1, \dots, \xi_k \in \Omega$, $\xi_{k+1}, \dots, \xi_{k+l} \in \partial\Omega$. More precisely, in the sense of measures,

$$-\Delta u_\lambda + u_\lambda = \lambda e^{u_\lambda} \rightharpoonup \sum_{i=1}^{k} 8\pi \delta_{\xi_i} + \sum_{i=k+1}^{k+l} 4\pi \delta_{\xi_i} \tag{1.4}$$

as $\lambda \to 0$. Here δ_ξ denotes the Dirac mass at the point ξ. Correspondingly, away from those points the leading behavior of u_λ is given by

$$u_\lambda(x) \to \sum_{i=1}^{k} 8\pi G(x, \xi_i) + \sum_{i=k+1}^{k+l} 4\pi G(x, \xi_i) \tag{1.5}$$

where $G(\cdot, \xi)$ is the Green function for the problem

$$\begin{cases} -\Delta G + G = \delta_\xi, & \text{in} \quad \Omega, \\ \frac{\partial G}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

For each given non-negative numbers k and l, a solution u_λ with the properties (1.4) and (1.5) for suitable points ξ_i is proven to exist in [7]. Near each point $\xi = \xi_i$ the leading concentration behavior is given by

$$u_\lambda(x) \sim \omega(|x - \xi|)$$

where ω is a radially symmetric solution of the equation

$$-\Delta \omega = \lambda e^{\omega} \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

namely a function of the form

$$\omega(r) = \ln \frac{8\delta^2}{(\delta^2 + r^2)^2} - \ln \lambda,$$

where δ is a suitable scalar dependent on λ and the point ξ.

Since u_λ is uniformly bounded away from the points ξ_i, this forces for the parameter δ to satisfy $\delta^2 \sim \lambda$. We observe that all solutions ω of (1.7) satisfy

$$\int_{\mathbb{R}^2} \lambda e^{\omega} = 8\pi.$$

Thus, consistently with (1.4), masses are quantized as

$$\int_{\Omega} \lambda e^{u_\lambda} \to 4\pi(2k + l). \tag{1.8}$$

A natural question is that of analyzing of solutions with large mass, namely solutions u_λ of (1.2) with

$$\int_\Omega \lambda e^{u_\lambda} \to +\infty \quad \text{as } \lambda \to 0.$$

It is natural to seek for solutions with property which concentrate not just at points but on a *larger-dimensional set*. The purpose of this paper is to prove the existence of a family of solutions to (1.2) with a *boundary condensation property*, exhibiting a boundary layer behavior along the entire $\partial\Omega$. These solutions satisfy

$$\lim_{\lambda\to 0} \frac{1}{|\ln\lambda|} \int_\Omega \lambda e^{u_\lambda} > 0.$$

Let us formally derive the asymptotic shape of these solutions. Let us parametrize points of space in a sufficiently small neighborhood of $\partial\Omega$ in the form $x = \gamma(\theta) + y\nu(\theta)$, where $\gamma(\theta)$ is a parametrization by θ, arclength of $\partial\Omega$, and $\nu(\theta)$ a corresponding unit inner normal, so that $\dot\nu(\theta) = -\kappa(\theta)\dot\gamma(\theta)$, where κ designates inner normal curvature. We get the following expansion for the Euclidean Laplacian in these coordinates

$$\Delta = \partial_{yy} + \frac{1}{1-\kappa(\theta)y}\frac{\partial}{\partial\theta}\left(\frac{1}{1-\kappa(\theta)y}\frac{\partial}{\partial\theta}\right) - \frac{\kappa(\theta)}{1-\kappa(\theta)y}\frac{\partial}{\partial y}$$

The solution we look for has a boundary layer, thus large derivatives along the normal and a comparatively smooth behavior along the tangent direction. It is then reasonable to take near $\partial\Omega$ as a first approximation of a solution $u_\lambda(\theta, y)$ of the equation (1.3) a solution of the ordinary differential equation

$$w_\mu'' + \lambda e^{w_\mu} = 0, \quad w_\mu'(0) = 0, \tag{1.9}$$

which is

$$w_\mu(y) - \ln\lambda = w(y/\mu) - 2\ln\mu - \ln\lambda, \tag{1.10}$$

where $w(y) = \ln 4\frac{e^{\sqrt{2}y}}{\left(1+e^{\sqrt{2}y}\right)^2}$ and the concentration parameter μ satisfies

$$\mu(\theta) = \varepsilon\hat\mu_\varepsilon(\theta) \sim \varepsilon\hat\mu_0(\theta).$$

Here $\varepsilon = \varepsilon(\lambda)$ is a small positive number which we shall choose below and $\hat\mu_0(\theta)$ is a uniformly positive and bounded smooth function.

Let $\varphi \in C(\bar\Omega)$ compactly supported near the boundary of Ω. A direct computation yields

$$\varepsilon \int_\Omega \lambda e^{w_\mu}\varphi = \sqrt{2}\int_{\partial\Omega} \varphi\hat\mu_0^{-1}\, d\theta + O(\varepsilon)\ ,$$

since $\int_0^{+\infty} e^{w(y)}dy = \sqrt{2}$. Thus,

$$\varepsilon\lambda e^{w_\mu} \rightharpoonup \sqrt{2}\hat{\mu}_0^{-1}\delta_{\partial\Omega}$$

where $\delta_{\partial\Omega}$ is the Dirac measure on the curve $\partial\Omega$.

Then we expect that, globally, $\sqrt{2}\,\mathcal{U} = \varepsilon u_\lambda$ satisfies approximately

$$-\Delta\mathcal{U} + \mathcal{U} = \hat{\mu}_0^{-1}\delta_{\partial\Omega},$$

which means in the limit

$$-\Delta\mathcal{U} + \mathcal{U} = 0 \text{ in } \Omega, \quad \partial_\nu\mathcal{U} = -\hat{\mu}_0^{-1} \text{ on } \partial\Omega.$$

Now, from our ansatz (1.10), we should have that close to the boundary

$$\sqrt{2}\,\mathcal{U}(\theta, y) \approx \varepsilon w(y/\mu) - 2\varepsilon\ln\varepsilon\hat{\mu}_\varepsilon - \varepsilon\ln\lambda$$

and hence, in particular

$$\sqrt{2}\,\mathcal{U}(\theta, 0) \approx -\varepsilon\ln\lambda - 2\varepsilon\ln\varepsilon$$

By maximum principle and $\partial_\nu\mathcal{U} = -\hat{\mu}_0^{-1} < 0$ the latter relation is consistent in the limit if the constant $\varepsilon\ln\lambda$ approaches a negative number. If we choose $\mathcal{U} = 1$ on the boundary of Ω, then we take ε such that

$$-\varepsilon\ln\lambda - 2\varepsilon\ln\varepsilon \approx \sqrt{2}$$

so that

$$\varepsilon \approx -\frac{\sqrt{2}}{\ln\lambda}.$$

Hence the limiting $\mathcal{U}$ equals $\mathcal{U}_0$, the unique solution of the problem

$$-\Delta\mathcal{U}_0 + \mathcal{U}_0 = 0 \text{ in } \Omega, \quad \mathcal{U}_0 = 1 \text{ on } \partial\Omega. \tag{1.11}$$

We observe that by maximum principle and Hopf's Lemma, we have that $\partial_\nu\mathcal{U}_0 < 0$, and hence this fixes our choice of $\hat{\mu}_0(\theta)$ as

$$\hat{\mu}_0(\theta) = -\frac{1}{\partial_\nu\mathcal{U}_0} \quad \text{on } \partial\Omega.$$

Our main result asserts the existence of a solution with exactly the profile above for all λ sufficiently small which remains suitably away from a sequence of critical small values where certain resonance phenomenon occurs.

Theorem 1.1. *Suppose that Ω is a smooth bounded domain of $\mathbb{R}^2$. Then there exists a sequence of positive small numbers $\lambda = \lambda_m$ converging to 0 as $m \to +\infty$ such that the problem (1.3) has a solution u_λ such that*

$$0 < \lim_{\lambda \to 0} \frac{1}{|\ln \lambda|} \int_\Omega \lambda e^{u_\lambda}\, dx < +\infty.$$

Moreover, if $\varepsilon_\lambda = \varepsilon_{\lambda_m}$ is the parameter defined by

$$\ln \frac{4}{\varepsilon_\lambda^2} - \ln \lambda = \frac{\sqrt{2}}{\varepsilon_\lambda} \tag{1.12}$$

then

$$\lim_{\lambda \to 0} \varepsilon_\lambda u_\lambda = \sqrt{2}\, \mathcal{U}_0 \qquad C^0\text{-uniformly on compact sets of } \Omega$$

and, in the sense of measures,

$$\varepsilon_\lambda \lambda e^{u_\lambda} \rightharpoonup -\sqrt{2}\, \partial_\nu \mathcal{U}_0\, \delta_{\partial\Omega}.$$

We actually believe that problem (1.3) has a solution which concentrates along the entire boundary, also in the higher-dimensional case $\Omega \subset \mathbb{R}^N$ with $N \geqslant 3$. This fact has been established in the radial case, when Ω is a ball, in [14].

Remark 4.3 below assures the existence of small numbers $\lambda > 0$ for which the problem (1.3) has a solution with the desired behavior. In fact, a more general condition on ε_λ (and then on λ) defined as in (1.12) is provided there. This type of condition, known as non-resonance condition, were imposed to establish the presence of higher dimensional concentration patterns without rotational symmetries in several works in the literature, starting with the pioneering works by Malchiodi and Montenegro [11,12], who prove existence of a concentrating solution u_ε along the boundary for the classical Neumann problem

$$\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega \tag{1.13}$$

with $p > 1$. See also [4,10,13] for related results.

A major difference between our problem and (1.13) is that the limiting profile is highly localized in the sense that the limiting solution has an exponentially sharp boundary layer $O(e^{-\frac{d}{\varepsilon}})$ where d designates distance to the boundary. Instead, in our setting the interaction with the inner part of the domain is much stronger. The interaction inner–outer problem makes the improvement of approximations considerably more delicate. The construction of an inverse for the approximate linearized operator is in fact quite different because of the presence of slow decay elements in the kernel of the asymptotic linearization.

The proof of our result relies on an infinite-dimensional form of Lyapunov–Schmidt reduction.

We look for a solution to (1.3) of the form $U_\lambda + \Phi_\lambda$ where U_λ, *the main term*, is a suitably constructed first approximation and Φ_λ is *the remainder term*. Then Problem (1.3) can be rewritten

as

$$L(\Phi_\lambda) = S_\lambda + N(\Phi_\lambda) \text{ in } \Omega, \tag{1.14}$$

where

$$L(\Phi) := \Delta\Phi - \Phi + \lambda e^{U_\lambda}\Phi, \tag{1.15}$$

$$S_\lambda(U_\lambda) := -\Delta U_\lambda + U_\lambda - \lambda e^{U_\lambda} \tag{1.16}$$

and

$$N(\Phi) := -\lambda e^{U_\lambda}\left[e^{\Phi} - 1 - \Phi\right]. \tag{1.17}$$

The strategy consists of finding an accurate first approximation U_λ (Section 2) so that the error term $S_\lambda(U_\lambda)$ be small in a suitably chosen norm (Section 3). Then an invertibility theory for associated linearized operator L (Section 5) allows to solve equation (1.14) for term Φ_λ via a fixed point argument (Section 4).

The main term U_λ looks like $w_\mu - \ln\lambda$ close to the boundary, with w_μ defined in (1.10) solves the ODE (1.9) and concentration parameter $\mu := \mu(\lambda)$ approaches 0 as λ goes to 0. The profile of U_λ in the inner part of the domain looks like $\tau\, \mathcal{U}_0$ where $\mathcal{U}_0$ solves the Dirichlet boundary problem (1.11) and the dilation parameter $\tau := \tau(\lambda)$ approaches $+\infty$ as λ goes to 0. The concentration parameter $\mu(\lambda)$ and the dilation parameter $\tau(\lambda)$ have to be chosen so that the two profiles match accurately close to the boundary. This is the most delicate part of the paper and it is carried out in sub-section 2.5.

2. The main term

2.1. The problem close to the boundary

Let us parametrize $\partial\Omega$ by the arc length

$$\gamma(\theta) := (\gamma_1(\theta), \gamma_2(\theta)), \qquad \theta \in [0, \ell]$$

where $\ell := |\partial\Omega|$. The tangent vector and the inner normal vector to the point $\gamma(\theta) \in \partial\Omega$ are given by

$$\tau(\theta) := (\dot\gamma_1(\theta), \dot\gamma_2(\theta)); \qquad \nu(\theta) := (-\dot\gamma_2(\theta), \dot\gamma_1(\theta))$$

respectively.

If $\delta > 0$ is small enough, let

$$\mathcal{D}_\delta := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) \le \delta\}$$

be a neighborhood of the curve $\partial\Omega$.

Then for any $x \in \mathcal{D}_\delta$ there exists a unique $(\theta, y) \in [0, \ell] \times [-\delta, 0]$ such that

$$x = \gamma(\theta) + y\nu(\theta) = (\gamma_1(\theta) - y\dot\gamma_2(\theta), \gamma_2(\theta) + y\dot\gamma_1(\theta)).$$

We remark that in these coordinates the points of the boundary take the form $(\theta, 0)$. If $u(\theta, y)$ is a function defined in $[0, \ell] \times [-\delta, 0]$ we can define the function $u(x) = u(\theta(x), y(x))$ (we use the same symbol for sake of simplicity) for $x \in \mathcal{D}_\delta$ and hence close to the boundary the equation (1.3) takes the form

$$\begin{cases} -\dfrac{1}{(1-y\kappa(\theta))^2}\partial^2_{\theta\theta}u - \partial^2_{yy}u - \dfrac{y\dot{\kappa}(\theta)}{(1-y\kappa(\theta))^3}\partial_\theta u + \dfrac{\kappa(\theta)}{1-y\kappa(\theta)}\partial_y u + u = \lambda e^u \quad \text{in } \mathcal{D}_\delta, \\ \partial_y u(\theta, 0) = 0 \end{cases} \tag{2.1}$$

where $\kappa(\theta)$ is the curvature at the point $\gamma(\theta) \in \partial\Omega$.

It is useful to introduce the spaces $C^0_\ell(\mathbb{R})$ and $C^2_\ell(\mathbb{R})$ of ℓ-periodic C^0-functions and C^2-functions, respectively.

2.2. *The scaled problem close to the boundary*

Now, let us introduce an extra parameter $\varepsilon := \varepsilon_\lambda$ such that

$$\ln\frac{4}{\varepsilon_\lambda^2} - \ln\lambda = \frac{\sqrt{2}}{\varepsilon_\lambda}, \qquad \text{i.e.} \quad \lambda = \frac{4}{\varepsilon_\lambda^2}e^{-\frac{\sqrt{2}}{\varepsilon_\lambda}}. \tag{2.2}$$

It is easy to check that $\varepsilon_\lambda \to 0$ as $\lambda \to 0$. We agree that in the following we will use indifferently the two parameters ε and λ to get the necessary estimates.

Moreover, let us choose the concentration parameter $\mu(\theta) := \mu(\lambda, \theta)$ in (1.10) as

$$\mu(\theta) := \varepsilon\hat{\mu}(\theta), \text{ where } \hat{\mu}(\theta) := \hat{\mu}_\varepsilon(\theta) \in C^2_\ell(\mathbb{R}). \tag{2.3}$$

The function $\hat{\mu}$ will be defined in Lemma 2.8.

Finally, let us set

$$\hat{\mu}_0(\theta) := -\frac{1}{\partial_\nu \mathcal{U}_0}\Big|_{\partial\Omega} = -\frac{1}{\partial_y \mathcal{U}_0(\theta, 0)}. \tag{2.4}$$

We note that by maximum principle and Hopf's lemma, μ_0 is a strictly positive C^2-function.

Now, let us scale problem (2.1). In $\mathcal{D}_\delta$ it is natural to consider the change of variables

$$u(\theta, y) = \tilde{u}\left(\frac{\theta}{\varepsilon}, \frac{y}{\mu}\right), \text{ with } \tilde{u} = \tilde{u}(s, t). \tag{2.5}$$

It is clear that

$$(\theta, y) \in \mathcal{D}_\delta \text{ if and only if } (s, t) \in \left[0, \frac{\ell}{\varepsilon}\right] \times \left[-\frac{\delta}{\mu}, 0\right].$$

Let $\tilde{u} = \tilde{u}(s,t)$, then we can compute

$$\begin{aligned}
\partial_\theta u &= \varepsilon^{-1}\partial_s \tilde{u} - \dot{\mu}\mu^{-1} t \partial_t \tilde{u} \\
\partial^2_{\theta\theta} u &= \varepsilon^{-2}\partial^2_{ss}\tilde{u} - 2\varepsilon^{-1}\dot{\mu}\mu^{-1} t\partial^2_{st}\tilde{u} - \ddot{\mu}\mu^{-1} t\partial_t\tilde{u} + 2\dot{\mu}^2\mu^{-2} t\partial_t\tilde{u} + \dot{\mu}^2\mu^{-2}t^2\partial^2_{tt}\tilde{u} \\
\partial_y u &= \mu^{-1}\partial_t\tilde{u} \\
\partial^2_{yy} u &= \mu^{-2}\partial^2_{tt}\tilde{u}
\end{aligned}$$

where the dot stands for the derivative with respect to θ.

Hence, problem (2.1) can be written as

$$\begin{cases}
\hat{\mu}_0^2(\varepsilon s)\partial^2_{ss}\tilde{u} + \partial^2_{tt}\tilde{u} + \tilde{\mathcal{A}}(\tilde{u}) + \lambda\mu^2 e^{\tilde{u}} = 0 & \text{in } \mathcal{C}_\delta, \\
\partial_t \tilde{u} = 0 & \text{on } \partial\mathcal{C}_\delta \cap \{t=0\} \\
\tilde{u}\left(s + \dfrac{\ell}{\varepsilon}, t\right) = \tilde{u}(s,t), \quad \text{i.e. } \tilde{u} \text{ is } \frac{\ell}{\varepsilon}\text{-periodic in } s
\end{cases} \tag{2.6}$$

where $\mathcal{C}_\delta := \mathbb{R}^+ \times \left[-\frac{\delta}{\mu}, 0\right]$ and the linear operator $\tilde{\mathcal{A}}$ is defined by

$$\begin{aligned}
\tilde{\mathcal{A}}(\tilde{u}) := {} & \underbrace{\left[\frac{\hat{\mu}^2}{(1-\mu t\kappa(\varepsilon s))^2} - \hat{\mu}_0^2\right]}_{b_0(s,t)} \partial^2_{ss}\tilde{u} + \underbrace{\frac{\dot{\mu}^2 t^2}{(1-\mu t\kappa(\varepsilon s))^2}}_{b_1(s,t)} \partial^2_{tt}\tilde{u} \\
& - \underbrace{\frac{2\varepsilon^{-1}\mu\dot{\mu}t}{(1-\mu t\kappa(\varepsilon s))^2}}_{b_2(s,t)} \partial^2_{st}\tilde{u} + \underbrace{\frac{\mu^3\varepsilon^{-1}t\dot{\kappa}(\varepsilon s)}{(1-\mu t\kappa(\varepsilon s))^3}}_{b_4(s,t)} \partial_s\tilde{u} - \mu^2\tilde{u} \\
& + \underbrace{\left[-\frac{\ddot{\mu}\mu t}{(1-\mu t\kappa(\varepsilon s))^2} - \frac{\mu^2\dot{\mu}t^2\dot{\kappa}(\varepsilon s)}{(1-\mu t\kappa(\varepsilon s))^3} + \frac{2\dot{\mu}^2 t}{(1-\mu t\kappa(\varepsilon s))^2} - \frac{\mu\kappa(\varepsilon s)}{1-\mu t\kappa(\varepsilon s)}\right]}_{b_3(s,t)} \partial_t\tilde{u}.
\end{aligned} \tag{2.7}$$

It is important to point out that the linear operator $\tilde{\mathcal{A}}$ is a perturbation term since all b_i's are uniformly small when λ is small (because of (2.3)).

2.3. A linear theory close to the boundary

Let us read the first order term of u_λ close to the boundary in the scaled variables: since u_λ looks like $w_\mu - \ln\lambda$ where the one-dimensional bubble w_μ is defined in (1.10), it turns out that the first order term of $\tilde{u}_\lambda$ is nothing but $w - \ln\lambda$ where $w \equiv w_1$, namely

$$w(t) := \ln 4 \frac{e^{\sqrt{2}t}}{\left(1 + e^{\sqrt{2}t}\right)^2}, \quad t \in \mathbb{R} \tag{2.8}$$

which solves

$$w'' + e^w = 0, \qquad \text{in} \quad \mathbb{R}. \tag{2.9}$$

Therefore, it is important to develop a linear theory for the linear operator $\mathcal{L}$ which comes from the linearization of equation (2.6) around the bubble $w - \ln\lambda$, namely

$$\mathcal{L}(\tilde{\phi}) := \hat{\mu}_0^2 \partial_{ss}^2 \tilde{\phi} + \partial_{tt}^2 \tilde{\phi} + \tilde{\mathcal{A}}(\tilde{\phi}) + e^w \tilde{\phi}. \tag{2.10}$$

In order to study $\mathcal{L}$, an important role is played by the linear operator

$$\hat{\mathcal{L}}(\tilde{\phi}) := \partial_{tt}^2 \tilde{\phi} + e^w \tilde{\phi} \tag{2.11}$$

which is nothing but the linearized operator around w of equation (2.9).

Lemma 2.1. *Let us consider the associated linearized eigenvalue problem*

$$\hat{\mathcal{L}}(\tilde{\phi}) = \Lambda \tilde{\phi} \quad \text{in } \mathbb{R}.$$

(i) $\Lambda = 0$ *is an eigenvalue with associated eigenfunctions* $Z_1(t) = w'(t) = \sqrt{2}\frac{1-e^{\sqrt{2}t}}{1+e^{\sqrt{2}t}}$ *and* $Z_2(t) = 2 + t w'(t)$. *We point out that* Z_1 *behaves like a constant at infinity and that* Z_2 *is not a bounded function.*

(ii) *There exists a positive eigenvalue* Λ_1 *with associated radial, positive and bounded eigenfunction* $Z_0 = Z_0(t)$ *with* L^2*-norm equal to one. Moreover,* Z_0 *decays exponentially at infinity as* $O\left(e^{-\sqrt{\Lambda_1}|t|}\right)$.

Proof. (i) has been proved in [8]. (ii) can be proved arguing as in Section 3 in [6]. □

We consider the following projected problem: *given a bounded function h, which is* $\frac{\ell}{\varepsilon}$*-periodic in s, find a bounded* $\frac{\ell}{\varepsilon}$*-periodic function* $c_0(s)$ *and* $\tilde{\phi}$ *such that*

$$\begin{cases} \mathcal{L}(\tilde{\phi}) = h + c_0(s) Z_0(t) & \text{in } \mathcal{C}_\delta, \\ \partial_t \tilde{\phi} = 0 & \text{on } \partial\mathcal{C}_\delta \cap \{t = 0\}, \\ \tilde{\phi}\left(s + \dfrac{\ell}{\varepsilon}, t\right) = \tilde{\phi}(s,t) & \\ \displaystyle\int_{-\frac{2\delta}{\mu}}^{0} \tilde{\phi}(s,t) Z_0(t)\, dt = 0 & \forall\, s \in \mathbb{R}^+. \end{cases} \tag{2.12}$$

In Section 5, we will establish existence and a priori estimates for problem (2.12) in the following norms:

$$\|\phi\|_* := \sup_{\mathcal{C}_\delta}(1+|t|^\sigma)|\phi| + \sup_{\mathcal{C}_\delta}(1+|t|^{\sigma+1})|\nabla\phi|, \ \|h\|_{**} := \sup_{\mathcal{C}_\delta}(1+|t|^{\sigma+2})|h|, \ \text{for } \sigma \in (0,1). \tag{2.13}$$

More precisely, we prove that

Proposition 2.2. *There exist $\lambda_0 > 0$ and a constant $C > 0$, such that for any $\lambda \in (0, \lambda_0)$ and for any h with $\|h\|_{**} < +\infty$, there exists a unique $\phi = \mathcal{T}(h)$ bounded solution of the problem (2.12) such that*

$$\|\phi\|_* \le C\|h\|_{**}. \tag{2.14}$$

2.4. The main term close to the boundary

The function $w_\mu - \ln\lambda$ is the main term of the approximated solution close to the boundary. We need to add some correction terms, which improve the main term.

More precisely, we let

$$u_\lambda(\theta, y) = \underbrace{w_\mu(\theta, y) - \ln\lambda + \alpha_\mu(\theta, y)}_{\text{1st-order}} + \underbrace{v_\mu(\theta, y) + \beta_\mu(\theta, y)}_{\text{2nd-order}} + \underbrace{z_\mu(\theta, y)}_{\text{3rd-order}} + \underbrace{e_0^\varepsilon(\theta) Z_0^\mu(y)}_{\text{unknown!}} \tag{2.15}$$

where

- $\alpha_\mu(\theta, y)$ is defined in Lemma 2.3,
- $v_\mu(\theta, y)$ is defined in Lemma 2.4 and $\beta_\mu(\theta, y)$ is defined in Lemma 2.5,
- $z_\mu(\theta, y)$ is defined in Lemma 2.6,
- $Z_0^\mu(y) = Z_0(\frac{y}{\mu})$, where Z_0 is defined in Lemma 2.1 and the function $e_0^\varepsilon(\theta)$ is defined as follows

$$e_0^\varepsilon(\theta) = \varepsilon^{\frac{3}{2}} e_0(\theta) \text{ with } e_0 \in C_\ell^2(\mathbb{R}). \tag{2.16}$$

We point out that the function e_0 is unknown: it is playing the role of one parameter and it will be chosen in Section 4.3 as solution of an ordinary differential equation. We assume that e_0 has uniformly bounded $\|\cdot\|_\varepsilon$-norm, i.e.

$$\|e_0\|_\varepsilon := \|\varepsilon^2 \ddot{e}_0\|_\infty + \|\varepsilon \dot{e}_0\|_\infty + \|e_0\|_\infty \leqslant M_0, \tag{2.17}$$

for some large fixed number M_0.

The first term we have to add is a sort of projection of the function w_μ, namely the function α_μ given in the next lemma.

Lemma 2.3.

(i) *The Cauchy problem*

$$\begin{cases} -\partial^2_{yy}\alpha_\mu + \dfrac{\kappa(\theta)}{1-y\kappa(\theta)}\partial_y\alpha_\mu = -\dfrac{\kappa(\theta)}{1-y\kappa(\theta)}\partial_y w_\mu - w_\mu + \ln\lambda + \dfrac{1}{(1-y\kappa(\theta))^2}\partial^2_{\theta\theta}w_\mu \\ \alpha_\mu(\theta,0) = \partial_y\alpha_\mu(\theta,0) = 0 \end{cases} \tag{2.18}$$

has the solution

$$\alpha_\mu(\theta,y) = -\int_0^y \frac{1}{1-\sigma\kappa(\theta)}\int_0^\sigma (1-\rho\kappa(\theta))\left[-\frac{\kappa(\theta)}{1-\rho\kappa(\theta)}\partial_y w_\mu(\rho) - w_\mu(\rho) + \ln\lambda\right] d\sigma\, d\rho$$
$$-\int_0^y \frac{1}{1-\sigma\kappa(\theta)}\int_0^\sigma \frac{1}{1-\rho\kappa(\theta)}\partial^2_{\theta\theta}w_\mu(\rho)\, d\rho\, d\sigma$$

(ii) *For any* $(\theta, y) \in \mathcal{D}_{2\delta} \setminus \mathcal{D}_\delta$ *it holds:*

$$\alpha_\mu(\theta,y) := (\kappa(\theta)\ln 4)y + \frac{y^2}{2}\left[\kappa^2(\theta)\ln 4 + \frac{d^2}{d\theta^2}\ln\hat{\mu}^2 - \frac{\sqrt{2}}{\mu}\kappa(\theta) + \left(\ln\frac{4}{\mu^2} - \ln\lambda\right)\right] +$$
$$+\frac{y^3}{6}\left[-\frac{2\sqrt{2}}{\mu}\kappa^2(\theta) - \frac{\sqrt{2}}{\mu} + \kappa(\theta)\left(\ln\frac{4}{\mu^2} - \ln\lambda\right) + \frac{\sqrt{2}}{\varepsilon}\frac{d^2}{d\theta^2}\frac{1}{\hat{\mu}}\right]$$
$$+O\left(|y|^3\right) + O\left(\frac{|y|^4}{\varepsilon}\right).$$

(iii) *Moreover, via the change of variables* $\theta = \varepsilon s$ *and* $y = \mu t$*, the function* $\tilde{\alpha}_\mu(s,t) := \alpha_\mu(\varepsilon s, \mu t)$ *solves the problem*

$$\begin{cases} -\partial^2_{tt}\tilde{\alpha}_\mu + \dfrac{\mu(\varepsilon s)\kappa(\varepsilon s)}{1-t\mu(\varepsilon s)\kappa(\varepsilon s)}\partial_t\tilde{\alpha}_\mu = -\dfrac{\mu(\varepsilon s)\kappa(\varepsilon s)}{1-t\mu(\varepsilon s)\kappa(\varepsilon s)}\partial_t w - \mu^2\left(\ln\dfrac{1}{\mu^2} - \ln\lambda + w\right) \\ +\dfrac{1}{(1-\mu t\kappa(\theta))^2}\left[\hat{\mu}^2\partial^2_{ss}\ln\dfrac{1}{4\hat{\mu}^2} + t\partial_t w\left(-\dfrac{\ddot{\mu}}{\mu} + \dfrac{2\dot{\mu}^2}{\mu^2}\right)\mu^2 + \dot{\mu}^2 t^2\partial^2_{tt}w\right] \\ \tilde{\alpha}_\mu(\varepsilon s, 0) = \partial_t\tilde{\alpha}_\mu(\varepsilon s, 0) = 0. \end{cases} \tag{2.19}$$

(iv) *The following expansion holds*

$$\tilde{\alpha}_\mu(s,t) := \alpha_\mu(\varepsilon s, \mu t) = \mu\alpha_1(\varepsilon s, t) + \mu^2\alpha_2(\varepsilon s, t) + O(\varepsilon^3 t^4), \qquad \textit{for } |t| \leqslant \frac{2\delta}{\mu}$$

where

$$\alpha_1(\varepsilon s, t) = \kappa(\varepsilon s) \int_0^t w(\sigma)\, d\sigma + \frac{\sqrt{2}}{2} \hat{\mu}(\varepsilon s) t^2 \tag{2.20}$$

and

$$\begin{aligned} \alpha_2(\varepsilon s, t) = \kappa^2(\varepsilon s) \int_0^t \sigma w(\sigma)\, d\sigma + \frac{t^2}{2} \ln \frac{1}{\hat{\mu}^2(\theta)} \\ + \int_0^t \int_0^\sigma (w(\rho) - \ln 4)\, d\rho\, d\sigma + \frac{\sqrt{2}}{6} \hat{\mu} \kappa(\theta) t^3 + \frac{d^2}{d\theta^2} \ln \hat{\mu}^2 \frac{t^2}{2}. \end{aligned} \tag{2.21}$$

Proof. We argue as in Lemma 3.1 of [14]. □

Now, let us construct the second order term of our approximated solution.

Lemma 2.4.

(i) *There exists v solution of the linear problem (α_1 is given in (2.20))*

$$-\partial^2_{yy} v - e^w v = e^w \alpha_1(\theta, y) \tag{2.22}$$

such that

$$v(\theta, y) = \nu_1(\theta) y + \nu_2(\theta) + O(e^{-|y|}) \qquad |y| \to +\infty$$

where

$$\nu_1(\theta) := 2\kappa(\theta)(1 - \ln 2) + \ln 4 \hat{\mu}(\theta) \tag{2.23}$$

and

$$\nu_2(\theta) := -\int_{-\infty}^0 \left(\frac{2}{1 - e^{\sqrt{2}y}} + \frac{y}{\sqrt{2}} \right) \alpha_1(\theta, y) \partial_y w(y) e^{w(y)}\, dy. \tag{2.24}$$

(ii) *In particular, the function $v_\mu(\theta, y) := \mu v\left(\theta, \frac{y}{\mu}\right)$ solves the problem*

$$-\partial^2_{yy} v_\mu - e^{w_\mu} v_\mu = \mu e^{w_\mu} \alpha_1\left(\theta, \frac{y}{\mu}\right).$$

(iii) *Moreover, via the change of variables* $\theta = \varepsilon s$ *and* $y = \mu t$, *the function* $\tilde{v}_\mu(s,t) := v_\mu(\varepsilon s, \mu t) = \mu(\varepsilon s) v(\varepsilon s, t)$ *solves the problem*

$$-\partial^2_{tt}\tilde{v}_\mu - e^w \tilde{v}_\mu = \mu e^w \alpha_1(\varepsilon s, t)$$

and the following expansion holds (see (2.3))

$$\tilde{v}_\mu(s,t) := \varepsilon \nu_1(\varepsilon s)\hat{\mu}(\varepsilon s)t + \varepsilon \nu_2(\varepsilon s)\hat{\mu}(\varepsilon s) + O\left(\varepsilon e^{-|t|}\right) \text{ as } |t| \to +\infty.$$

Proof. We apply Lemma 2.7. □

As we have done for the function w_μ, we have to add the projection of the function v_μ, namely the function β_μ given in the next lemma.

Lemma 2.5.

(i) *The Cauchy problem (v_μ is given in Lemma 2.4)*

$$\begin{cases} -\partial^2_{yy}\beta_\mu + \dfrac{\kappa(\theta)}{1 - y\kappa(\theta)}\partial_y \beta_\mu = -\dfrac{\kappa(\theta)}{1 - y\kappa(\theta)}\partial_y v_\mu(\theta, y) \\ \beta_\mu(\theta, 0) = \partial_y \beta_\mu(\theta, 0) = 0 \end{cases} \tag{2.25}$$

has the solution

$$\beta_\mu(\theta, y) = \int_0^y \frac{\kappa(\theta)}{1 - \sigma\kappa(\theta)} \int_0^\sigma \partial_y v_\mu(\theta, \rho)\, d\rho\, d\sigma.$$

(ii) *For any* $(\theta, y) \in \mathcal{D}_{2\delta} \setminus \mathcal{D}_\delta$ *we have:*

$$\beta_\mu(\theta, y) = \nu_1(\theta)\kappa(\theta)\frac{y^2}{2} + O(|y|^3).$$

(iii) *Moreover, via the change of variables* $\theta = \varepsilon s$ *and* $y = \mu t$, *the function* $\tilde{\beta}_\mu(s,t) := \beta_\mu(\varepsilon s, \mu t)$ *solves the problem*

$$\begin{cases} -\partial^2_{tt}\tilde{\beta}_\mu + \dfrac{\mu(\varepsilon s)\kappa(\varepsilon s)}{1 - t\mu(\varepsilon s)\kappa(\varepsilon s)}\partial_t \tilde{\beta}_\mu = -\dfrac{\mu(\varepsilon s)\kappa(\varepsilon s)}{1 - t\mu(\varepsilon s)\kappa(\varepsilon s)}\partial_t \tilde{v}_\mu \\ \tilde{\beta}_\mu(\varepsilon s, 0) = \partial_t \tilde{\beta}_\mu(\varepsilon s, 0) = 0 \end{cases} \tag{2.26}$$

(iv) *The following expansion holds:*

$$\tilde{\beta}_\mu(s,t) := \beta_\mu(\varepsilon s, \mu t) = \mu^2 \beta_1(\varepsilon s, t) + O(\varepsilon^3 t^3)$$

where

$$\beta_1(\varepsilon s, t) = \kappa(\varepsilon s) \int_0^t \int_0^\sigma \partial_y v(\varepsilon s, \rho)\, d\rho\, d\sigma. \tag{2.27}$$

Proof. We argue as in Lemma 3.4 of [14]. □

Finally, we build the third order term of our approximated solution.

Lemma 2.6.

(i) *There exists z solution of the linear problem (α_1, α_2 and β_1 are given in (2.20), (2.21) and (2.27) respectively, and v is given in Lemma 2.4)*

$$-\partial^2_{yy} z - e^w z = e^w \left[\alpha_2(\theta, y) + \beta_1(\theta, y) + \frac{1}{2} (\alpha_1(\theta, y) + v(\theta, y))^2 \right]$$

such that

$$z(\theta, y) = \zeta_1(\theta) y + \zeta_2(\theta) + O(e^{-|y|}) \qquad |y| \to +\infty$$

where

$$\zeta_1(\theta) := \frac{1}{\sqrt{2}} \int_{-\infty}^0 h(\theta, y) \partial_y w(y) e^{w(y)}\, dy$$

and

$$\zeta_2(\theta) := -\int_{-\infty}^0 \left(\frac{2}{1 - e^{\sqrt{2}y}} + \frac{y}{\sqrt{2}} \right) h(\theta, y) \partial_y w(y) e^{w(y)}\, dy,$$

with

$$h(\theta, y) = \alpha_2(\theta, y) + \beta_1(\theta, y) + \frac{1}{2} (\alpha_1(\theta, y) + v(\theta, y))^2 .$$

(ii) *In particular, the function $z_\mu(\theta, y) := \mu^2 z\left(\theta, \frac{y}{\mu}\right)$ solves the problem*

$$-\partial^2_{yy} z_\mu - e^{w_\mu} z_\mu = \mu^2 e^{w_\mu} \left[\alpha_2\left(\theta, \frac{y}{\mu}\right) + \beta_1\left(\theta, \frac{y}{\mu}\right) + \frac{1}{2} \left(\alpha_1\left(\theta, \frac{y}{\mu}\right) + v\left(\theta, \frac{y}{\mu}\right) \right)^2 \right].$$

(iii) *Moreover, via the change of variables $\theta = \varepsilon s$ and $y = \mu t$, the function $\tilde{z}_\mu(s, t) := z_\mu(\varepsilon s, \mu t) = \mu^2(\varepsilon s) z(\varepsilon s, t)$ solves the problem*

$$-\partial^2_{tt} \tilde{z}_\mu - e^w \tilde{z}_\mu = \mu^2 e^w \left[\alpha_2(\varepsilon s, t) + \beta_1(\varepsilon s, t) + \frac{1}{2} (\alpha_1(\varepsilon s, t) + v(\varepsilon s, t))^2 \right]$$

and the following expansion holds (see (2.3))

$$\tilde{z}_\mu(s,t) := \varepsilon^2\zeta_1(\varepsilon s)\hat{\mu}^2(\varepsilon s)t + \varepsilon^2\zeta_2(\varepsilon s)\hat{\mu}^2(\varepsilon s) + O\left(\varepsilon^2 e^{-|t|}\right) \text{ as } |t| \to +\infty.$$

Proof. We apply Lemma 2.7. □

Lemma 2.7. *(Lemma 4.1 [8]) Let $\mathfrak{h} \in C^0(\mathbb{R})$ such that $\int_{\mathbb{R}} \mathfrak{h}(y)w'(y)e^{w(y)}dy < +\infty$. Then the function*

$$\mathfrak{U}(y) := w'(y)\int_0^y \frac{1}{(w'(\sigma))^2}\int_\sigma^0 \mathfrak{h}(\tau)w'(\tau)e^{w(\tau)}d\tau d\sigma$$

solves the ordinary differential equation

$$-\mathfrak{U}'' - e^w\mathfrak{U} = e^w\mathfrak{h} \text{ in } \mathbb{R}.$$

In particular,

$$\mathfrak{U}(y) = \mathfrak{a}y + \mathfrak{b} + O\left(e^{-c|y|}\right) \text{ in } C^1(\mathbb{R}) \text{ as } y \to -\infty$$

where

$$\mathfrak{a} := \frac{1}{\sqrt{2}}\int_{-\infty}^0 \mathfrak{h}(\tau)w'(\tau)e^{w(\tau)}d\tau \text{ and } \mathfrak{b} := -\int_{-\infty}^0 \left(\frac{2}{1-e^{\sqrt{2}\tau}} + \frac{\tau}{\sqrt{2}}\right)\mathfrak{h}(\tau)w'(\tau)e^{w(\tau)}d\tau$$

and

$$\mathfrak{U}(y) = \mathfrak{c}y + \mathfrak{d} + O\left(e^{-c|y|}\right) \text{ in } C^1(\mathbb{R}) \text{ as } y \to +\infty$$

where

$$\mathfrak{c} := \frac{1}{\sqrt{2}}\int_0^{+\infty} \mathfrak{h}(\tau)w'(\tau)e^{w(\tau)}d\tau \text{ and } \mathfrak{d} := -\int_0^{+\infty} \left(\frac{2}{1-e^{\sqrt{2}\tau}} + \frac{\tau}{\sqrt{2}}\right)\mathfrak{h}(\tau)w'(\tau)e^{w(\tau)}d\tau.$$

2.5. *How to match the main term close to the boundary with the main term in inner part*

The solution u_λ in the inner part of the domain looks like $\tau\mathcal{U}_0$ where $\mathcal{U}_0$ solves (1.11) and the dilation parameter $\tau := \tau(\lambda)$ approaches $+\infty$ as λ goes to 0. The function u_λ (and its derivative) built in (2.15) in a neighborhood of the boundary has to match with the function $\tau(\lambda)\mathcal{U}_0$ (and its derivative). To this aim it is necessary to choose the dilation parameter $\tau = \frac{\sqrt{2}}{\varepsilon}$ and most of all it is essential to modify the profile of the solution in the inner part of the domain by building a new function $\mathcal{U}_\varepsilon$ which approaches $\mathcal{U}_0$ as ε goes to zero and such that its value on the boundary

together with the value of its normal derivative coincide with the value of u_λ and its normal derivative. The main tool here is the Dirichlet-to-Neumann map and the key ingredient is the choice of the concentration parameter $\hat{\mu}$ as showed in the next crucial lemma.

Lemma 2.8. *There exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ there exist a function $\hat{\mu}_\varepsilon \in C^2(\partial\Omega)$ and a solution $\mathcal{U}_\varepsilon$ to the problem*

$$\begin{cases} -\Delta \mathcal{U}_\varepsilon + \mathcal{U}_\varepsilon = 0, & \text{in} \quad \Omega, \\ \mathcal{U}_\varepsilon = 1 - \frac{\varepsilon}{\sqrt{2}} \left(\ln \hat{\mu}_\varepsilon^2 - \varepsilon \hat{\mu}_\varepsilon \nu_2 - \varepsilon^2 \hat{\mu}^2 \zeta_2(\theta)\right) & \text{on} \quad \partial\Omega, \\ \partial_\nu \mathcal{U}_\varepsilon = -\frac{1}{\hat{\mu}_\varepsilon} + \frac{\varepsilon}{\sqrt{2}} \left(2\kappa + \hat{\mu}_\varepsilon \ln 4 + \varepsilon \hat{\mu} \zeta_1(\theta)\right) & \text{on} \quad \partial\Omega. \end{cases} \tag{2.28}$$

Moreover ($\hat{\mu}_0$ is given in (2.4))

$$\hat{\mu}_\varepsilon = \hat{\mu}_0 + O(\varepsilon) \quad \text{in } C^1(\partial\Omega) \text{ as } \varepsilon \to 0 \tag{2.29}$$

and

$$\mathcal{U}_\varepsilon = \mathcal{U}_0 + O(\varepsilon) \quad \text{in } C^2(\overline{\Omega}) \text{ as } \varepsilon \to 0. \tag{2.30}$$

Proof. Let us apply the Dirichlet-to-Neumann map, which maps the value on $\partial\Omega$ of a harmonic function $\mathcal{U}$ to the value of its normal derivative $\partial_\nu \mathcal{U}$ on $\partial\Omega$, i.e. $F\left(\mathcal{U}_{|\partial\Omega}\right) = \partial_\nu \mathcal{U}$. Therefore, we are going to find a function $\hat{\mu} \in C^2(\partial\Omega)$ such that

$$F\left(1 - \frac{\varepsilon}{\sqrt{2}} \left(\ln \hat{\mu}^2 - \varepsilon \hat{\mu} \nu_2 - \varepsilon^2 \hat{\mu}^2 \zeta_2(\theta)\right)\right) = -\frac{1}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}} \left(2\kappa + \hat{\mu} \ln 4 + \varepsilon \hat{\mu} \zeta_1(\theta)\right). \tag{2.31}$$

Let

$$H(\varepsilon, \hat{\mu}) = F\left(1 - \frac{\varepsilon}{\sqrt{2}} \left(\ln \hat{\mu}^2 - \varepsilon \hat{\mu} \nu_2 - \varepsilon^2 \hat{\mu}^2 \zeta_2(\theta)\right)\right) + \frac{1}{\hat{\mu}} - \frac{\varepsilon}{\sqrt{2}} \left(2\kappa(\theta) + \hat{\mu} \ln 4 + \varepsilon \hat{\mu} \zeta_1(\theta)\right).$$

We have that

$$H(0, \hat{\mu}_0) = F(1) + \frac{1}{\hat{\mu}_0} = 0, \quad \text{since } F(1) = \partial_\nu \mathcal{U}_0 \text{ and } \hat{\mu}_0 = -\tfrac{1}{\partial_\nu \mathcal{U}_0} \text{ (see (2.4))}.$$

Moreover

$$\frac{\partial H}{\partial \hat{\mu}}(0, \hat{\mu}_0) = -\frac{1}{\hat{\mu}_0^2} \neq 0.$$

Hence by the Implicit Function Theorem, there exists a unique $\hat{\mu} = \hat{\mu}_\varepsilon(\theta) \in C^2(\partial\Omega)$ such that $H(\varepsilon, \hat{\mu}_\varepsilon) = 0$, namely (2.31) holds. Estimates (2.29) and (2.30) follow by elliptic standard regularity theory. $\square$

Lemma 2.9. *Let $\mathcal{U}_\varepsilon$ be given in Lemma 2.8. Then there exists ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$u_\lambda(\theta, y) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\theta, y) = O\left(\varepsilon|y|^2\right) + O\left(\frac{|y|^4}{\varepsilon}\right) \text{ uniformly in } \mathcal{D}_{2\delta} \setminus \mathcal{D}_\delta \tag{2.32}$$

and

$$\partial_y\left[u_\lambda(\theta, y) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\theta, y)\right] = O(\varepsilon|y|) + O\left(\frac{|y|^3}{\varepsilon}\right) \text{ uniformly in } \mathcal{D}_{2\delta} \setminus \mathcal{D}_\delta. \tag{2.33}$$

Proof. Let us prove the estimate (2.32). The proof of (2.33) is similar.

Let $\mathcal{U}$ be a generic harmonic function, namely

$$-\Delta\mathcal{U} + \mathcal{U} = 0 \text{ in } \Omega. \tag{2.34}$$

Then the expansion of $\frac{\sqrt{2}}{\varepsilon}\mathcal{U}$ on the boundary reads as

$$\frac{\sqrt{2}}{\varepsilon}\mathcal{U}(\theta, y) = \frac{\sqrt{2}}{\varepsilon}\left[\mathcal{U}(\theta, 0) + y\partial_y\mathcal{U}(\theta, 0) + \frac{y^2}{2}\partial^2_{yy}\mathcal{U}(\theta, 0) + \frac{y^3}{6}\partial^3_{yyy}\mathcal{U}(\theta, 0)\right] + O\left(\frac{|y|^4}{\varepsilon}\right). \tag{2.35}$$

Now, let us write the expansion of the function u_λ close to the boundary. In $\mathcal{D}_{2\delta} \setminus \mathcal{D}_\delta$ we get

$$w_\mu(y) - \ln\lambda = \ln\frac{4}{\mu^2} - \ln\lambda - \frac{\sqrt{2}y}{\mu} + O(e^{-\sqrt{2}\frac{|y|}{\mu}}) = \frac{\sqrt{2}}{\varepsilon} - \ln\hat{\mu}^2 - \frac{\sqrt{2}y}{\mu} + O(e^{-\sqrt{2}\frac{|y|}{\mu}}),$$

because $\mu = \varepsilon\hat{\mu}$ and (2.2) holds. Therefore, by Lemmas 2.3, 2.4, 2.5 and 2.6 we deduce

$$\begin{aligned}
u_\lambda(\theta, y) &= w_\mu(y) - \ln\lambda + \alpha_\mu(\theta, y) + v_\mu(\theta, y) + \beta_\mu(\theta, y) + z_\mu(\theta, y) + e_0^\varepsilon(\theta)Z_0^\mu(y) \\
&= \frac{\sqrt{2}}{\varepsilon}\underbrace{\left[1 - \frac{\varepsilon}{\sqrt{2}}\left(\ln\hat{\mu}^2 - \varepsilon\hat{\mu}v_2(\theta) - \varepsilon^2\hat{\mu}^2\zeta_2(\theta)\right)\right]}_{\sim\,\mathcal{U}(\theta,0)} \\
&\quad + \frac{\sqrt{2}}{\varepsilon}y\underbrace{\left[-\frac{1}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}}(2\kappa(\theta) + \hat{\mu}\ln 4 + \varepsilon\hat{\mu}\zeta_1(\theta))\right]}_{\sim\,\partial_y\mathcal{U}(\theta,0)} \\
&\quad + \frac{\sqrt{2}}{\varepsilon}\frac{y^2}{2}\underbrace{\left[1 - \frac{\kappa(\theta)}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}}\left(\frac{d^2}{d\theta^2}\ln\hat{\mu}^2 - \ln\hat{\mu}^2 + 2\kappa^2(\theta) + \kappa(\theta)\hat{\mu}\ln 4\right)\right]}_{\sim\,\partial^2_{yy}\mathcal{U}(\theta,0)} \\
&\quad + \frac{\sqrt{2}}{\varepsilon}\frac{y^3}{6}\underbrace{\left[-\frac{1}{\hat{\mu}} - \frac{2}{\hat{\mu}}\kappa^2(\theta) + \kappa(\theta) + \frac{d^2}{d\theta^2}\frac{1}{\hat{\mu}}\right]}_{\sim\,\partial^3_{yyy}\mathcal{U}(\theta,0)}
\end{aligned}$$

$$+ O\left(|y|^3\right) + O\left(\frac{|y|^4}{\varepsilon}\right) + O(e^{-c\frac{|y|}{\varepsilon}}), \tag{2.36}$$

for some $c > 0$. Let us compare (2.35) with (2.36): the first four terms have to be equal! In particular, it means that we have to find an harmonic function $\mathcal{U}$ such that the value of $\mathcal{U}$ and the value of its normal derivative $\partial_\nu \mathcal{U}$ on the boundary have to be equal to $\left[1 - \frac{\varepsilon}{\sqrt{2}}\left(\ln \hat{\mu}^2 - \varepsilon\hat{\mu}\nu_2 - \varepsilon^2\hat{\mu}^2\zeta_2(\theta)\right)\right]$ and $\left[-\frac{1}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}}(2\kappa + \hat{\mu}\ln 4 + \varepsilon\hat{\mu}\zeta_1(\theta))\right]$, respectively. This is done in Lemma 2.8. Therefore, let us replace in (2.35) and (2.36) the generic harmonic function $\mathcal{U}$ with the function $\mathcal{U}_\varepsilon$ which solves problem (2.28). The first two terms coincide. Now, let us check what happens with the higher order terms, namely terms which involve the second and third derivatives of $\mathcal{U}_\varepsilon$. The function $\mathcal{U}_\varepsilon$ solves equation (2.28) which in a neighborhood of the boundary reads as

$$-\frac{1}{(1-y\kappa)^2}\partial^2_{\theta\theta}\mathcal{U}_\varepsilon - \partial^2_{yy}\mathcal{U}_\varepsilon - \frac{y\dot{\kappa}}{(1-y\kappa)^3}\partial_\theta\mathcal{U}_\varepsilon + \frac{\kappa}{(1-y\kappa)}\partial_y\mathcal{U}_\varepsilon + \mathcal{U}_\varepsilon = 0 \text{ in } \mathcal{D}_{2\delta}. \tag{2.37}$$

We have then on the boundary

$$\partial^2_{yy}\mathcal{U}_\varepsilon(\theta,0) = -\partial^2_{\theta\theta}\mathcal{U}_\varepsilon(\theta,0) + \kappa(\theta)\partial_y\mathcal{U}_\varepsilon(\theta,0) + \mathcal{U}_\varepsilon(\theta,0)$$

and

$$\begin{aligned}\partial^3_{yyy}\mathcal{U}_\varepsilon(\theta,0) = &-2\kappa(\theta)\partial^2_{\theta\theta}\mathcal{U}_\varepsilon(\theta,0) - \partial^3_{\theta\theta y}\mathcal{U}_\varepsilon(\theta,0) - \dot{\kappa}(\theta)\partial_\theta\mathcal{U}_\varepsilon(\theta,0)\\ &+ \kappa^2(\theta)\partial_y\mathcal{U}_\varepsilon(\theta,0) + \kappa(\theta)\partial^2_{yy}\mathcal{U}_\varepsilon(\theta,0) + \partial_y\mathcal{U}_\varepsilon(\theta,0)\end{aligned}$$

Then differentiating twice with respect to θ the value $\mathcal{U}_\varepsilon$ on the boundary and the values of $\partial_\nu\mathcal{U}_\varepsilon$ on the boundary we get

$$\begin{aligned}\partial^2_{\theta\theta}\mathcal{U}_\varepsilon(\theta,0) &= -\frac{\varepsilon}{\sqrt{2}}\left(\frac{d^2}{d\theta^2}\ln\hat{\mu}^2 - \varepsilon\frac{d^2}{d\theta^2}(\hat{\mu}\nu_2(\theta)) - \varepsilon^2\frac{d^2}{d\theta^2}(\hat{\mu}^2\zeta_2(\theta))\right)\\ &= -\frac{\varepsilon}{\sqrt{2}}\frac{d^2}{d\theta^2}\ln\hat{\mu}^2 + O(\varepsilon^2)\\ \partial^3_{\theta\theta y}\mathcal{U}_\varepsilon(\theta,0) &= -\frac{d^2}{d\theta^2}\frac{1}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}}\left(2\ddot{\kappa}(\theta) + \ddot{\hat{\mu}}\ln 4 + \varepsilon\frac{d^2}{d\theta^2}(\hat{\mu}\zeta_1)\right) = -\frac{d^2}{d\theta^2}\frac{1}{\hat{\mu}} + O(\varepsilon)\end{aligned} \tag{2.38}$$

By (2.37) and (2.38) we deduce

$$\begin{aligned}\frac{\sqrt{2}}{\varepsilon}\partial^2_{yy}\mathcal{U}_\varepsilon(\theta,0) &= \frac{\sqrt{2}}{\varepsilon}\left[1 - \frac{\kappa(\theta)}{\hat{\mu}} + \frac{\varepsilon}{\sqrt{2}}\left(\frac{d^2}{d\theta^2}\ln\hat{\mu}^2 - \ln\hat{\mu}^2 + 2\kappa^2(\theta) + \kappa(\theta)\hat{\mu}\ln 4\right)\right] + O(\varepsilon)\\ \frac{\sqrt{2}}{\varepsilon}\partial^3_{yyy}\mathcal{U}_\varepsilon(\theta,0) &= \frac{\sqrt{2}}{\varepsilon}\left[-\frac{1}{\hat{\mu}} - \frac{2}{\hat{\mu}}\kappa^2(\theta) + \kappa(\theta) + \frac{d^2}{d\theta^2}\frac{1}{\hat{\mu}}\right] + O(1).\end{aligned} \tag{2.39}$$

Finally, by (2.35), (2.36) and (2.39) the claim follows. □

2.6. The main term in the whole domain

The main term of the solution is given by

$$U_\lambda(x) = \eta_\delta(y(x))u_\lambda(\theta(x), y(x)) + \Big(1 - \eta_\delta(y(x))\Big)\frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(x), \tag{2.40}$$

where u_λ is defined in (2.15), $\mathcal{U}_\varepsilon$ is defined in (2.28) and $\eta_\delta(x) = \eta_\delta(y(x))$ is a cut-off function such that $\eta_\delta = 1$ in $\mathcal{D}_\delta$, $\eta_\delta = 0$ in $\Omega \setminus \mathcal{D}_{2\delta}$, $0 \leqslant \eta_\delta \leqslant 1$ and $|\eta'_\delta| \leqslant \frac{1}{\delta}$ and $|\eta''_\delta| \leqslant \frac{1}{\delta^2}$. We choose (see Lemma 3.1)

$$\delta := \varepsilon^a \qquad a \in \left(\frac{13}{14}, 1\right). \tag{2.41}$$

3. The error estimate

In this section we study the error term

$$S_\lambda(U_\lambda) := -\Delta U_\lambda + U_\lambda - \lambda e^{U_\lambda}, \qquad \text{in } \Omega. \tag{3.1}$$

3.1. Estimate of the error close to the boundary

It is useful to scale the problem. After the change of variables (2.5), in a neighborhood of the curve, we get that the error term is given by

$$\mathcal{R}(\tilde{U}_\lambda) := \hat{\mu}_0^2(\varepsilon s)\partial_{ss}^2\tilde{U}_\lambda + \partial_{tt}^2\tilde{U}_\lambda + \tilde{\mathcal{A}}(\tilde{U}_\lambda) + \lambda\mu^2 e^{\tilde{U}_\lambda} \qquad \text{in } \mathcal{C}_{2\delta} \tag{3.2}$$

where $\tilde{\mathcal{A}}$ is the operator defined in (2.7) and $\tilde{U}_\lambda$ is defined as follows:

$$\tilde{U}_\lambda(s,t) := \begin{cases} \tilde{u}_\lambda(s,t) & \text{in } \mathcal{C}_\delta \\ \tilde{\eta}_\delta(t)\tilde{u}_\lambda(s,t) + (1 - \tilde{\eta}_\delta(t))\frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t) & \text{in } \mathcal{C}_{2\delta} \setminus \mathcal{C}_\delta \end{cases} \tag{3.3}$$

where $\tilde{u}_\lambda$ is the scaled function u_λ defined in (2.15), i.e.

$$\begin{aligned} \tilde{u}_\lambda(s,t) := \ln\frac{4}{\varepsilon^2} - \ln\lambda + \ln\frac{1}{4\hat{\mu}(\varepsilon s)^2} + w(t) + \tilde{\alpha}_\mu(s,t) + \tilde{v}_\mu(s,t) + \tilde{\beta}_\mu(s,t) \\ + \tilde{z}_\mu(s,t) + e_0^\varepsilon(\varepsilon s)Z_0(t). \end{aligned} \tag{3.4}$$

Here $\tilde{\eta}_\delta(t) = \eta_\delta(\mu t)$ is the cut-off function η_δ scaled, which is 1 inside $\mathcal{C}_\delta$ and 0 outside $\mathcal{C}_{2\delta}$. It is only necessary to compute the rate of the error part $\tilde{\mathcal{R}}(\tilde{U}_\lambda)$ defined as

$$\tilde{\mathcal{R}}(\tilde{U}_\lambda) := \mathcal{R}(\tilde{U}_\lambda) - \tilde{\eta}_\delta[\varepsilon^2\hat{\mu}_0^2\ddot{e}_0^\varepsilon(\varepsilon s) + \Lambda_1 e_0^\varepsilon(\varepsilon s)]Z_0(t) \tag{3.5}$$

Lemma 3.1. *There exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we get*

$$\|\tilde{\mathcal{R}}(\tilde{U}_\lambda)\|_{**} \le C\varepsilon^{\frac{5}{2}}.$$

Proof. For sake of simplicity, let $v_{2,\lambda} := \tilde{\eta}_\delta v_{1,\lambda} + (1-\tilde{\eta}_\delta)v_{3,\lambda}$ where $v_{1,\lambda}(s,t) = \tilde{u}_\lambda(s,t)$ and $v_{3,\lambda}(s,t) := \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t)$. We are going to estimate $\|\tilde{\mathcal{R}}(v_{i,\lambda})\|_{**}$ for $i = 1, 2, 3$.

It is useful to point out that the weight $1 + |t|^{2+\sigma}$ present in the weighted norm $\|\cdot\|_{**}$ in $\mathcal{C}_{2\delta}$ has the following growth

$$\sup_{\mathcal{C}_{2\delta}}(1 + |t|^{\sigma+2}) = O\left(\varepsilon^{-(1-a)(\sigma+2)}\right). \tag{3.6}$$

Claim 1: $\|\tilde{\mathcal{R}}(v_{1,\lambda})\|_{**} \le C\varepsilon^{\frac{5}{2}}$.

For sake of simplicity, set

$$\tilde{h}_\mu(s,t) := h_\mu(\varepsilon s, \mu(\varepsilon s)t) \text{ and } h_\mu(\theta, y) := \alpha_\mu(\theta, y) + v_\mu(\theta, y) + \beta_\mu(\theta, y) + z_\mu(\theta, y).$$

We have to take into account that $\mu = \varepsilon\hat{\mu}$. Therefore, a direct computation proves that

$$\partial_t \tilde{h}_\mu = \varepsilon\hat{\mu}\partial_y h_\mu \text{ and } \partial_s \tilde{h}_\mu(s,t) = \varepsilon\partial_\theta h_\mu + \varepsilon^2\dot{\hat{\mu}}t\partial_y h_\mu, \tag{3.7}$$

$$\partial^2_{tt}\tilde{h}_\mu = \varepsilon^2\hat{\mu}^2\partial^2_{yy}h_\mu, \tag{3.8}$$

$$\partial^2_{ss}\tilde{h}_\mu = \varepsilon^2\partial^2_{\theta\theta}h_\mu + 2\varepsilon^3\dot{\hat{\mu}}t\partial^2_{\theta y}h_\mu + \varepsilon^3\ddot{\hat{\mu}}t\partial_y h_\mu + \varepsilon^4\dot{\hat{\mu}}^2t^2\partial^2_{yy}h_\mu \tag{3.9}$$

and

$$\partial^2_{st}\tilde{h}_\mu = \varepsilon^2\dot{\hat{\mu}}\partial_y h_\mu + \varepsilon^2\hat{\mu}\partial^2_{\theta y}h_\mu + \varepsilon^3\hat{\mu}\dot{\hat{\mu}}t\partial^2_{yy}h_\mu. \tag{3.10}$$

A straightforward computation together with Lemmas 2.3, 2.4, 2.5 and 2.6 lead to

$$\begin{aligned}
\tilde{\mathcal{R}}(v_{1,\lambda}) &= \mathcal{R}\left(\ln\frac{4}{\varepsilon^2} - \ln\lambda + \ln\frac{1}{4\hat{\mu}^2} + w + \tilde{h}_\mu + e_0^\varepsilon(\varepsilon s)Z_0(t)\right) \\
&\quad - [\varepsilon^2\hat{\mu}_0^2\ddot{e}_0^\varepsilon(\varepsilon s) + \Lambda_1 e_0^\varepsilon(\varepsilon s)]Z_0(t) \\
&= \frac{\hat{\mu}^2}{(1-\mu t\kappa)^2}\partial^2_{ss}\tilde{h}_\mu + \frac{\dot{\mu}^2t^2}{(1-\mu t\kappa)^2}\partial^2_{tt}\tilde{h}_\mu - \frac{2\varepsilon^{-1}\mu\dot{\mu}t}{(1-\mu t\kappa)^2}\partial^2_{st}\tilde{h}_\mu - \frac{\mu^2\dot{\mu}t^2\dot{\kappa}}{(1-\mu t\kappa)^3}\partial_t w \\
&\quad - \frac{\mu\kappa}{1-\mu t\kappa}\partial_t\tilde{z}_\mu + \frac{\mu^3\varepsilon^{-1}t\dot{\kappa}}{(1-\mu t\kappa)^3}\partial_s\left[\ln\frac{1}{4\hat{\mu}^2} + \tilde{h}_\mu\right] - \mu^2\tilde{h}_\mu \\
&\quad + \left(-\frac{\ddot{\mu}\mu t}{(1-\mu t\kappa)^2} - \frac{\mu^2\dot{\mu}t^2\dot{\kappa}}{(1-\mu t\kappa)^3} + \frac{2\dot{\mu}^2t}{(1-\mu t\kappa)^2}\right)\partial_t\tilde{h}_\mu + \tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s)Z_0(t)\right) \\
&\quad + \underbrace{e^w\left(e^{\tilde{h}_\mu + e_0^\varepsilon Z_0} - 1 - \tilde{v}_\mu - \mu\alpha_1 - \tilde{z}_\mu - \mu^2\left(\alpha_2 + \beta_1 + \frac{1}{2}(\alpha_1 + v)^2\right) - e_0^\varepsilon Z_0\right)}_{\mathcal{S}_0}.
\end{aligned} \tag{3.11}$$

Now

$$\begin{aligned}\tilde{\mathcal{R}}(v_{1,\lambda}) - \tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s)Z_0(t)\right) - \mathcal{S}_0 = {} & O\left(\partial_{ss}^2\tilde{h}_\mu\right) + O\left(\varepsilon^2|t|^2\partial_{tt}^2\tilde{h}_\mu\right) + O\left(\varepsilon|t|\partial_{st}^2\tilde{h}_\mu\right)\\ & + O\left(\varepsilon^3|t|^2\partial_t w\right) + O\left((\varepsilon^2|t| + \varepsilon^3|t|^2)\partial_t\tilde{h}_\mu\right)\\ & + O\left(\varepsilon^2|t|\partial_s\tilde{h}_\mu\right) + O\left(\varepsilon\partial_t\tilde{z}_\mu\right)\\ & + O\left(\varepsilon^2\tilde{h}_\mu\right) + O\left(\varepsilon^3|t|\right) \end{aligned} \tag{3.12}$$

By using the estimates of Lemma 3.2 together with the derivatives of the function $\tilde{h}_\mu$ computed in (3.7), (3.8), (3.9) and (3.10), we get

$$\|\tilde{\mathcal{R}}(v_{1,\lambda}) - \tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s)Z_0(t)\right) - \mathcal{S}_0\|_{**} \le c\varepsilon^{4a-1-(1-a)(\sigma+2)}.$$

Now

$$\begin{aligned}\tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s)Z_0(t)\right) := {} & b_0(s,t)\varepsilon^2\ddot{e}_0^\varepsilon Z_0 + b_1(s,t)e_0^\varepsilon\partial_{tt}^2 Z_0 + b_2(s,t)\varepsilon\dot{e}_0^\varepsilon\partial_t Z_0\\ & + b_3(s,t)\left[e_0^\varepsilon\partial_t Z_0\right] + b_4(s,t)\left[\varepsilon\dot{e}_0^\varepsilon Z_0\right] - \mu^2 e_0^\varepsilon Z_0\end{aligned}$$

where the coefficients $b_j(s,t)$ are defined in (2.7). Then we deduce immediately

$$\|\tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s)Z_0(t)\right)\|_{**} \le c\varepsilon^{\frac{5}{2}}$$

Finally,

$$\begin{aligned}\mathcal{S}_0 = {} & \underbrace{e^w\left[e^{\tilde{h}_\mu} - 1 - \tilde{v}_\mu - \mu\alpha_1 - \tilde{z}_\mu - \mu^2\left(\alpha_2 + \beta_1 + \frac{1}{2}(\alpha_1 + v)^2\right)\right]}_{\mathcal{S}_0^1}\\ & \underbrace{+ e^{w+\tilde{h}_\mu}\left[e^{e_0^\varepsilon Z_0} - 1 - e_0^\varepsilon Z_0\right]}_{\mathcal{S}_0^2} + \underbrace{e^w\left(e^{\tilde{h}_\mu} - 1\right)e_0^\varepsilon Z_0}_{\mathcal{S}_0^3}\end{aligned}$$

and hence

$$\|\mathcal{S}_0^1\|_{**} \le C\varepsilon^3$$

and it is independent of e_0 while

$$\|\mathcal{S}_0^2\|_{**} \le C\varepsilon^3$$

and it is quadratic in e_0 and finally

$$\|\mathcal{S}_0^3\|_{**} \le C\varepsilon^{\frac{5}{2}}$$

and it is linear in e_0. Since $a > \frac{13}{14}$ the claim follows.

Claim 2: $\|\tilde{\mathcal{R}}(v_{3,\lambda})\|_{**} \le Ce^{-\frac{c}{\varepsilon}}$ for some positive constant c.

Since

$$\tilde{\mathcal{R}}(v_{3,\lambda}) = \mathcal{R}(v_{3,\lambda}) = \lambda\mu^2 e^{v_{3,\lambda}},$$

we get

$$\|\mathcal{R}(v_{3,\lambda})\|_{**} \le c \sup_{\mathcal{C}_{2\delta}\setminus\mathcal{C}_\delta} \left| \frac{4}{\varepsilon^2} e^{-\frac{\sqrt{2}}{\varepsilon}} e^{v_{3,\lambda}} \right| (1+|t|^{\sigma+2}) \le ce^{-\frac{c}{\varepsilon}}.$$

Claim 3: $\|\tilde{\mathcal{R}}(v_{2,\lambda})\|_{**} \le C\varepsilon^{\frac{5}{2}}$.

By making some tedious computations, one gets that

$$\tilde{\mathcal{R}}(v_{2,\lambda}) = \tilde{\eta}_\delta \tilde{\mathcal{R}}(v_{1,\lambda}) + (1-\tilde{\eta}_\delta)\mathcal{R}(v_{3,\lambda}) + \mathcal{R}_2 \tag{3.13}$$

where

$$\begin{aligned}\mathcal{R}_2 := {} & \underbrace{\hat{\mu}_0^2 \partial_{ss}^2 \tilde{\eta}_\delta (v_{1,\lambda} - v_{3,\lambda}) + 2\hat{\mu}_0^2 \partial_s \tilde{\eta}_\delta \partial_s (v_{1,\lambda} - v_{3,\lambda}) + \partial_{tt}^2 \tilde{\eta}_\delta (v_{1,\lambda} - v_{3,\lambda}) + 2\partial_t \tilde{\eta}_\delta \partial_t (v_{1,\lambda} - v_{3,\lambda})}_{\mathcal{B}_1} \\ & + \underbrace{\tilde{\mathcal{A}}(\tilde{\eta}_\delta v_{1,\lambda} + (1-\tilde{\eta}_\delta) v_{3,\lambda}) - \tilde{\eta}_\delta \tilde{\mathcal{A}}(v_{1,\lambda}) - (1-\tilde{\eta}_\delta)\tilde{\mathcal{A}}(v_{3,\lambda})}_{\mathcal{B}_2} \\ & + \underbrace{\lambda\mu^2 \left(e^{\tilde{\eta}_\delta v_{1,\lambda} + (1-\tilde{\eta}_\delta) v_{3,\lambda}} - \tilde{\eta}_\delta e^{v_{1,\lambda}} - (1-\tilde{\eta}_\delta) e^{v_{3,\lambda}} \right)}_{\mathcal{B}_3}.\end{aligned}$$

By using the expansion (3.13) and the result of claim 1 and claim 2 we get that

$$\|\tilde{\mathcal{R}}(v_{2,\lambda})\|_{**} \le c\varepsilon^{\frac{5}{2}} + ce^{-\frac{c}{\varepsilon}} + \|\mathcal{R}_2\|_{**}.$$

So we are left to estimate $\|\mathcal{R}_2\|_{**}$. Let us prove that

$$\|\mathcal{R}_2\|_{**} \leqslant C\varepsilon^{3a-(1-a)(\sigma+2)}. \tag{3.14}$$

Now we take into account that

$$\partial_s \tilde{\eta}_\delta = O(\varepsilon), \ \partial_t \tilde{\eta}_\delta = O\left(\frac{\varepsilon}{\delta}\right), \ \partial_{ss}^2 \tilde{\eta}_\delta = O\left(\varepsilon^2\right), \ \partial_{st}^2 \tilde{\eta}_\delta = O\left(\frac{\varepsilon^2}{\delta}\right), \ \partial_{tt}^2 \tilde{\eta}_\delta = O\left(\frac{\varepsilon^2}{\delta^2}\right).$$

By (2.32), (2.33) and (3.28), we immediately deduce that for any $(s,t) \in \mathcal{C}_{2\delta} \setminus \mathcal{C}_\delta$ (remember that $\mu = \varepsilon\hat{\mu}$)

$$v_{1,\lambda}(s,t) - v_{3,\lambda}(s,t) = \hat{u}_\lambda(\varepsilon s, \mu t) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t) = O(\varepsilon^3|t|^2) + O(\varepsilon^3|t|^4),$$

$$\partial_t\left(v_{1,\lambda}(s,t) - v_{3,\lambda}(s,t)\right) = \mu\partial_y\left(\hat{u}_\lambda(\varepsilon s, \mu t) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t)\right) = O(\varepsilon^3|t|) + O(\varepsilon^3|t|^3)$$

and by using (3.28)

$$\begin{aligned}\partial_s\left(v_{1,\lambda}(s,t) - v_{3,\lambda}(s,t)\right) &= \varepsilon\partial_\theta\left(u_\lambda(\varepsilon s, \mu t) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t)\right)\\ &\quad + \varepsilon^2\dot{\hat{\mu}}t\partial_y\left(u_\lambda(\varepsilon s, \mu t) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t)\right)\\ &= O(\varepsilon^2|t|) + O(\varepsilon^3|t|^5) + O(\varepsilon^4|t|^2).\end{aligned}$$

Then

$$\begin{aligned}\mathcal{B}_1 &:= O\left(\varepsilon^2|v_{1,\lambda} - v_{3,\lambda}|\right) + O\left(\varepsilon|\partial_s(v_{1,\lambda} - v_{3,\lambda})|\right)\\ &\quad + O\left(\frac{\varepsilon^2}{\delta^2}|v_{1,\lambda} - v_{3,\lambda}|\right) + O\left(\frac{\varepsilon}{\delta}|\partial_t(v_{1,\lambda} - v_{3,\lambda})|\right)\\ &= O(\varepsilon^5|t|^4) + O(\varepsilon^3|t|) + O\left(\frac{\varepsilon^5}{\delta^2}|t|^4\right) + O\left(\frac{\varepsilon^4}{\delta}|t|^3\right)\end{aligned}$$

from which it follows that

$$\|\mathcal{B}_1\|_{**} \le c\varepsilon^{2a+1-(1-a)(\sigma+2)}.$$

Now

$$\begin{aligned}\mathcal{B}_2 &:= b_0\partial^2_{ss}\tilde{\eta}_\delta(v_{1,\lambda} - v_{3,\lambda}) + 2b_0\partial_s\tilde{\eta}_\delta\partial_s(v_{1,\lambda} - v_{3,\lambda}) + b_1\partial^2_{tt}\tilde{\eta}_\delta(v_{1,\lambda} - v_{3,\lambda})\\ &\quad + 2b_1\partial_t\tilde{\eta}_\delta\partial_t(v_{1,\lambda} - v_{3,\lambda}) + b_2\partial^2_{st}\tilde{\eta}_\delta(\hat{v}_{1,\lambda} - v_{3,\lambda}) + b_2\partial_s\tilde{\eta}_\delta\partial_t(v_{1,\lambda} - v_{3,\lambda})\\ &\quad + b_2\partial_t\tilde{\eta}_\delta\partial_s(v_{1,\lambda} - v_{3,\lambda}) + b_2\partial_s\tilde{\eta}_\delta\partial_t(v_{1,\lambda} - v_{3,\lambda})\\ &\quad + b_3\partial_t\tilde{\eta}_\delta(v_{1,\lambda} - v_{3,\lambda}) + b_4\partial_s\tilde{\eta}_\delta\left(v_{1,\lambda} - v_{3,\lambda}\right)\end{aligned}$$

and straightforward computations show that

$$\|\mathcal{B}_2\|_{**} \leqslant C\varepsilon^{3a-(1-a)(\sigma+2)}.$$

Finally,

$$\mathcal{B}_3 := \lambda\mu^2 e^{v_{1,\lambda}}\left(e^{(1-\tilde{\eta}_\delta)(v_{3,\lambda}-v_{1,\lambda})}-1\right)+\lambda\mu^2 e^{v_{1,\lambda}}(1-\tilde{\eta}_\delta)\left(1-e^{v_{3,\lambda}-v_{1,\lambda}}\right)$$

and so

$$\mathcal{B}_3 = O\left(e^{w}|v_{1,\lambda}-v_{3,\lambda}|\right)$$

and hence

$$\|\mathcal{B}_3\|_{**} \leqslant C\varepsilon^3.$$

Putting together all these estimates (3.14) follows by using the fact that $a < 1$. The result of the claim follows since $a > \frac{13}{14}$.

That concludes the proof. □

Lemma 3.2. *Let α_μ, v_μ, β_μ and z_μ as in Lemmas 2.3, 2.4, 2.5 and 2.6, respectively. It holds true that uniformly with respect to $y \in \mathcal{D}_{2\delta}$*

$$\begin{cases} \alpha_\mu(\theta,y) = O\left(\frac{|y|^2}{\varepsilon}\right)+O\left(\frac{|y|^4}{\varepsilon^2}\right) \\ \partial_\theta\alpha_\mu(\theta,y) = O\left(\frac{|y|^2}{\varepsilon}\right)+O\left(\frac{|y|^3}{\varepsilon^2}\right)+O\left(\frac{|y|^5}{\varepsilon^3}\right), \\ \partial^2_{\theta\theta}\alpha_\mu(\theta,y) = O\left(\frac{|y|^2}{\varepsilon}\right)+O\left(\frac{|y|^3}{\varepsilon^2}\right)+O\left(\frac{|y|^4}{\varepsilon^3}\right)+O\left(\frac{|y|^6}{\varepsilon^4}\right), \\ \partial_y\alpha_\mu(\theta,y) = O\left(\frac{|y|}{\varepsilon}\right), \\ \partial^2_{yy}\alpha_\mu(\theta,y) = O\left(\frac{1}{\varepsilon}\right)+O\left(\frac{|y|^2}{\varepsilon^2}\right), \\ \partial^2_{\theta y}\alpha_\mu(\theta,y) = O\left(|y|\right), \end{cases} \tag{3.15}$$

$$\begin{cases} v_\mu(\theta,y) = O\left(|y|\right), \\ \partial_\theta v_\mu(\theta,y) = O\left(|y|\right), \\ \partial^2_{\theta\theta} v_\mu(\theta,y) = O(|y|)+O\left(\frac{|y|^4}{\varepsilon^3}\right), \\ \partial_y v_\mu = O\left(1\right), \\ \partial^2_{yy} v_\mu = O\left(\frac{|y|^2}{\varepsilon^3}\right), \\ \partial^2_{\theta y} v_\mu(\theta,y) = O\left(\frac{|y|^3}{\varepsilon^3}\right), \end{cases} \tag{3.16}$$

$$\begin{cases} \beta_\mu(\theta, y) = O(|y|^2), \\ \partial_\theta \beta_\mu(\theta, y) = O\left(|y|^2\right) + O\left(\frac{|y|^5}{\varepsilon^3}\right), \\ \partial^2_{\theta\theta} \beta_\mu(\theta, y) = O\left(|y|^2\right) + O\left(\frac{|y|^5}{\varepsilon^3}\right) + O\left(\frac{|y|^6}{\varepsilon^4}\right), \\ \partial_y \beta_\mu(\theta, y) = O\left(|y|\right), \\ \partial^2_{yy} \beta_\mu(\theta, y) = O\left(|y|\right) \\ \partial^2_{\theta y} \beta_\mu(\theta, y) = O\left(|y|\right), \end{cases} \tag{3.17}$$

and

$$\begin{cases} z_\mu(\theta, y) = O(\varepsilon|y|), \\ \partial_\theta z_\mu(\theta, y) = O\left(\varepsilon|y|\right), \\ \partial^2_{\theta\theta} z_\mu(\theta, y) = O(\varepsilon|y|) + O\left(\frac{|y|^4}{\varepsilon^2}\right), \\ \partial_y z_\mu(\theta, y) = O\left(\varepsilon\right), \\ \partial^2_{yy} z_\mu(\theta, y) = O\left(\frac{|y|^2}{\varepsilon^2}\right), \\ \partial^2_{\theta y} z_\mu(\theta, y) = O\left(\frac{|y|^3}{\varepsilon^2}\right) + O\left(\varepsilon\right). \end{cases} \tag{3.18}$$

Proof. Let us remind that $\mu = \varepsilon\hat{\mu}(\theta)$. Then

$$w_\mu(y) = \ln\frac{1}{\varepsilon^2} + \ln\frac{1}{\hat{\mu}^2} + w\left(\frac{y}{\varepsilon\hat{\mu}}\right). \tag{3.19}$$

It is easy to check that (taking into account (2.3))

$$w_\mu(y) - \ln\lambda = \frac{\sqrt{2}}{\varepsilon} + \ln\frac{1}{\hat{\mu}^2} - \frac{\sqrt{2}}{\varepsilon\hat{\mu}}y + O\left(1\right). \tag{3.20}$$

Moreover, some straightforward computations show that

$$\partial_\theta w_\mu(y) = -2\frac{\dot{\hat{\mu}}}{\hat{\mu}} - \frac{\dot{\hat{\mu}}}{\hat{\mu}^2}w'\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y}{\varepsilon} = O\left(\frac{|y|}{\varepsilon}\right), \tag{3.21}$$

$$\partial_y w_\mu(y) = w'\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{1}{\varepsilon\hat{\mu}} = O\left(\frac{1}{\varepsilon}\right) \tag{3.22}$$

and

$$\partial^2_{\theta y} w_\mu(y) = -\frac{\dot{\hat{\mu}}}{\hat{\mu}^2} w'\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{1}{\varepsilon} - \frac{\dot{\hat{\mu}}}{\hat{\mu}^3} w''\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y}{\varepsilon^2} = O\left(\frac{1}{\varepsilon}\right) + O\left(\frac{|y|}{\varepsilon^2}\right), \tag{3.23}$$

$$\partial^2_{\theta\theta} w_\mu(y) = -2\frac{d}{d\theta}\left(\frac{\dot{\hat{\mu}}}{\hat{\mu}}\right) - \frac{d}{d\theta}\left(\frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\right) w'\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y}{\varepsilon} + \left(\frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\right)^2 w''\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y^2}{\varepsilon^2} = O\left(\frac{|y|^2}{\varepsilon^2}\right), \tag{3.24}$$

and analogously

$$\partial^3_{\theta\theta\theta} w_\mu(y) = O\left(\frac{|y|^3}{\varepsilon^3}\right) \tag{3.25}$$

and

$$\partial^4_{\theta\theta\theta\theta} w_\mu(y) = O\left(\frac{|y|^4}{\varepsilon^4}\right). \tag{3.26}$$

Moreover

$$\begin{aligned}\partial^3_{\theta\theta y} w_\mu(y) &= -\frac{d}{d\theta}\left(\frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\right) w''\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y}{\varepsilon^2\hat{\mu}} + \left(\frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\right)^2 w'''\left(\frac{y}{\varepsilon\hat{\mu}}\right)\frac{y^2}{\varepsilon^3\hat{\mu}} \\ &= O\left(\frac{1}{\varepsilon}\right) + O\left(\frac{|y|}{\varepsilon^2}\right) + O\left(\frac{|y|^2}{\varepsilon^3}\right).\end{aligned} \tag{3.27}$$

Let us estimate α_μ and its derivatives.

By using (3.20), (3.21), (3.22), (3.23), (3.24), (3.25), (3.26) and (3.27) and using the expression of α_μ given in Lemma 2.3 we immediately deduce the first three estimates in (3.15). The last three estimates in (3.15) follow by the mean value theorem taking into account the initial value data in (2.18) and by using the equation satisfied by α_μ.

Let us estimate v_μ and its derivatives. Since $v_\mu(\theta, y) = \varepsilon\hat{\mu} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right)$ we get immediately

$$v_\mu = O(|y|); \qquad \partial_y v_\mu = O(1); \qquad \partial^2_{yy} v_\mu = O\left(\frac{|y|^2}{\varepsilon^3}\right).$$

Moreover

$$\partial_\theta v_\mu(\theta, y) = \varepsilon\dot{\hat{\mu}} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) + \varepsilon\hat{\mu}\partial_\theta v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) - \frac{\dot{\hat{\mu}}}{\hat{\mu}}\partial_y v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y = O(|y|),$$

$$\begin{aligned}\partial^2_{\theta y} v_\mu(\theta, y) &= \frac{\dot{\hat{\mu}}}{\hat{\mu}}\partial_y v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) + \partial^2_{\theta y} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) - \frac{\dot{\hat{\mu}}}{\varepsilon\hat{\mu}^2}\partial^2_{yy} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y - \frac{\dot{\hat{\mu}}}{\hat{\mu}}\partial_y v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) \\ &= O\left(\frac{|y|^3}{\varepsilon^3}\right),\end{aligned}$$

and

$$\begin{aligned}\partial^2_{\theta\theta} v_\mu(\theta, y) &= \varepsilon\ddot{\hat{\mu}} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) + 2\varepsilon\dot{\hat{\mu}}\partial_\theta v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) - \frac{\dot{\hat{\mu}}^2}{\hat{\mu}^2}\partial_y v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y \\ &\quad + \varepsilon\hat{\mu}\partial^2_{\theta\theta} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) - 2\frac{\dot{\hat{\mu}}}{\hat{\mu}}\partial^2_{\theta y} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y \\ &\quad - \frac{d}{d\theta}\frac{\dot{\hat{\mu}}}{\hat{\mu}}\partial_y v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y + \frac{\dot{\hat{\mu}}^2}{\varepsilon\hat{\mu}^3}\partial^2_{yy} v\left(\theta, \frac{y}{\varepsilon\hat{\mu}}\right) y^2 \\ &= O(|y|) + O\left(\frac{|y|^4}{\varepsilon^3}\right).\end{aligned}$$

We have used the following facts. Since v solves equation (2.22), the functions $\partial_\theta v$ and $\partial^2_{\theta\theta} v$ solve the equations

$$-\partial^2_{yy}\partial_\theta v - e^w\partial_\theta v = e^w\partial_\theta\alpha_1(\theta, y) \text{ in } \mathbb{R}$$

and

$$-\partial^2_{yy}\partial^2_{\theta\theta} v - e^w\partial^2_{\theta\theta} v = e^w\partial^2_{\theta\theta}\alpha_1(\theta, y) \text{ in } \mathbb{R}.$$

Therefore we apply Lemma 2.7 and we deduce that v, $\partial_\theta v$ and $\partial^2_{\theta\theta} v$ have a linear growth, namely they satisfy for any $y \in \mathbb{R}$ and $\theta \in [0, \ell]$, the inequalities

$$|v(\theta, y)|, |\partial_\theta v(\theta, y)|, |\partial^2_{\theta\theta} v(\theta, y)| \leqslant c_1|y| + c_2$$

and

$$|\partial_y v(\theta, y)|, |\partial^2_{\theta y} v(\theta, y)|, |\partial^3_{\theta\theta y} v(\theta, y)| \leqslant c_3$$

for some positive constants c_1, c_2 and c_3. We also remark that by equation (2.22) we deduce that

$$|\partial^2_{yy} v\,(\theta, y)\,| \leqslant a_1|y|^2 + a_2|y| + a_3 \text{ for any } y \in \mathbb{R} \text{ and } \theta \in [0, \ell],$$

for some positive constants a_1, a_2 and a_3.

Arguing in a similar way, we prove estimates involving the functions β_μ and z_μ. □

Lemma 3.3. *Let $\mathcal{U}_\varepsilon$ be given in Lemma 2.8. Then if ε is small enough*

$$\partial_\theta\left[\hat{u}_\lambda(\theta, y) - \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\theta, y)\right] = O\left(|y|\right) + O\left(\frac{|y|^2}{\varepsilon}\right) + O\left(\frac{|y|^3}{\varepsilon^2}\right) + O\left(\frac{|y|^5}{\varepsilon^3}\right) \text{ uniformly in } \mathcal{D}_{2\delta}. \tag{3.28}$$

Proof. First of all, by mean value theorem we get for some $\bar{y} \in [0, y]$

$$\begin{aligned}\partial_\theta \mathcal{U}_\varepsilon(\theta, y) &= \partial_\theta \mathcal{U}_\varepsilon(\theta, 0) + y\partial_y\left(\partial_\theta \mathcal{U}_\varepsilon\right)(\theta, 0) + \frac{\sqrt{2}}{\varepsilon}\partial^2_{yy}\partial_\theta \mathcal{U}_\varepsilon(\theta, \bar{y})y^2 \\ &= -2\frac{\dot{\hat{\mu}}}{\hat{\mu}^2} + \frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\frac{\sqrt{2}}{\varepsilon}y + O\left(|y|\right) + O\left(\frac{|y|^2}{\varepsilon}\right).\end{aligned}$$

Here we use the boundary condition in (2.28) and the fact that $\partial^2_y y\left(\partial_\theta \mathcal{U}_\varepsilon\right)$ is uniformly bounded because of (2.30).

Now let us compute

$$\begin{aligned}\partial_\theta u_\lambda(\theta, y) &= \partial_\theta w_\mu(y) + \partial_\theta \alpha_\mu(\theta, y) + \partial_\theta v_\mu(\theta, y) + \partial_\theta \beta_\mu(\theta, y) + \partial_\theta z_\mu(\theta, y) + e^0_\varepsilon Z_0 \\ &= \partial_\theta w_\mu(y) + O\left(|y|\right) + O\left(\frac{|y|^2}{\varepsilon}\right) + O\left(\frac{|y|^3}{\varepsilon^2}\right) + O\left(\frac{|y|^5}{\varepsilon^3}\right) \\ &= -2\frac{\dot{\hat{\mu}}}{\hat{\mu}^2} + \frac{\dot{\hat{\mu}}}{\hat{\mu}^2}\frac{\sqrt{2}}{\varepsilon}y + O\left(|y|\right) + O\left(\frac{|y|^2}{\varepsilon}\right) + O\left(\frac{|y|^3}{\varepsilon^2}\right) + O\left(\frac{|y|^5}{\varepsilon^3}\right).\end{aligned} \tag{3.29}$$

We take into account estimate (3.21) together with the first estimates in (3.15), (3.16), (3.17) and (3.18). Then the claim follows. □

3.2. *Estimate of the error in the inner part*

Lemma 3.4. *There exist $c > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\|S_\lambda(U_\lambda)\|_{\infty, \Omega\setminus D_{2\delta}} \leq e^{-\frac{c}{\varepsilon}}. \tag{3.30}$$

Proof. Since $\mathcal{U}_\varepsilon$ solves (2.28) we have

$$S_\lambda(U_\lambda) = S_\lambda\left(\frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon\right) = -\lambda e^{\frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon} = -\frac{4}{\varepsilon^2}e^{\frac{\sqrt{2}}{\varepsilon}(\mathcal{U}_\varepsilon - 1)} = -\frac{4}{\varepsilon^2}e^{\frac{\sqrt{2}}{\varepsilon}(\mathcal{U}_\varepsilon - \mathcal{U}_0)}e^{\frac{\sqrt{2}}{\varepsilon}(\mathcal{U}_0 - 1)}.$$

Now, by the fact that $\partial_\nu \mathcal{U}_0 < 0$ we deduce that $\mathcal{U}_0(x) - 1 \leqslant c < 0$ if $x \in \Omega \setminus \mathcal{D}_{2\delta}$ for some constant c. Moreover, by (2.30) we also deduce that $|\mathcal{U}_\varepsilon(x) - \mathcal{U}_0(x)| \leqslant c\varepsilon$ for any $x \in \Omega \setminus \mathcal{D}_{2\delta}$ for some constant c. Therefore, the claim follows. □

3.3. *The projection of the error along Z_0*

We are going to compute the component of the scaled error $\mathcal{R}(\tilde{U}_\lambda)$ given in (3.2) along Z_0.

Lemma 3.5. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the following expansion holds:*

$$\begin{aligned}\int_{-\frac{2\delta}{\mu}}^{0} \mathcal{R}(\tilde{U}_\lambda)Z_0\,dt &= \varepsilon^{\frac{3}{2}}\left[\varepsilon^2\left(a_0(\varepsilon s)\ddot{e}_0(\varepsilon s) + a^\varepsilon_1(\varepsilon s)\dot{e}_0\right) + a_2(\varepsilon s)e_0\right] + \varepsilon^3 M_0(\varepsilon s) \\ &\quad + \varepsilon^3 H_0(e_0, \dot{e}_0, \ddot{e}_0) \text{ for any } s \in [0, \tfrac{\ell}{\varepsilon}],\end{aligned} \tag{3.31}$$

where

$$a_0(\varepsilon s) = \hat{\mu}_0^2 + \varepsilon a_0^\varepsilon(\varepsilon s) \tag{3.32}$$

and

$$a_2(\varepsilon s) = \Lambda_1 + \varepsilon a_2^\varepsilon(\varepsilon s) \tag{3.33}$$

with a_i^ε $i = 0, 1, 2$ *explicit smooth functions, uniformly bounded in* ε. *Moreover in* (3.31)

- M_0 *is a sum of explicit smooth functions of the form, uniformly bounded in* ε*;*
- H_0 *denotes a sum of functions of the form*

$$h_0(\varepsilon s)\,[h_1(e_0) + o(1)h_2(e_0, \dot{e}_0, \ddot{e}_0)]$$

  - h_0 *is a smooth function uniformly bounded in* ε*;*
  - h_1 *and* h_2 *is a smooth function of its arguments, uniformly bounded in* ε *when* e_0 *satisfies* (2.17)*;*
  - $o(1) \to 0$ *as* $\varepsilon \to 0$ *uniformly when* e_0 *satisfies* (2.17).

Proof. For sake of simplicity, let $v_{2,\lambda} := \tilde{\eta}_\delta v_{1,\lambda} + (1 - \tilde{\eta}_\delta) v_{3,\lambda}$ where $v_{1,\lambda}(s,t) = \tilde{u}_\lambda(s,t)$ and $v_{3,\lambda}(s,t) := \frac{\sqrt{2}}{\varepsilon}\mathcal{U}_\varepsilon(\varepsilon s, \mu t)$.

First of all we get that by using (3.5) and (3.13)

$$\int_{-\frac{2\delta}{\mu}}^{0} \mathcal{R}(\tilde{U}_\lambda) Z_0\,dt = \underbrace{\int_{-\frac{2\delta}{\mu}}^{-\frac{\delta}{\mu}} \mathcal{R}_2 Z_0\,dt}_{I_1^0} + \underbrace{\int_{-\frac{2\delta}{\mu}}^{-\frac{\delta}{\mu}} \mathcal{R}(v_{3,\lambda}) Z_0\,dt}_{I_2^0} + \underbrace{\int_{-\frac{2\delta}{\mu}}^{0} \tilde{\eta}_\delta \tilde{\mathcal{R}}(v_{1,\lambda}) Z_0(t)\,dt}_{I_3^0}$$

$$\underbrace{+\left[\varepsilon^2 \hat{\mu}_0^2 \ddot{e}_0^\varepsilon(\varepsilon s) + \Lambda_1 e_0^\varepsilon(\varepsilon s)\right] \int_{-\frac{2\delta}{\mu}}^{0} \tilde{\eta}_\delta Z_0^2(t)\,dt}_{I_4^0}.$$

We remark that

$$\int_{-\frac{\delta}{\mu}}^{0} Z_0^2(t)\,dt = \frac{1}{2} + O\left(e^{-\sqrt{\Lambda_1}\frac{\delta}{\mu}}\right)$$

and hence

$$I_4^0 := \frac{1}{2}\varepsilon^{\frac{3}{2}}\left[\varepsilon^2 \hat{\mu}_0^2 \ddot{e}_0(\varepsilon s) + \Lambda_1 e_0(\varepsilon s)\right] + O\left(e^{-\sqrt{\Lambda_1}\frac{\delta}{\mu}}\right).$$

Moreover by using Claim 2 of Lemma 3.1 we get that

$$I_2^0 = O\left(e^{-c\frac{1}{\varepsilon}}\right)$$

for some positive c and similarly, by using Claim 3 of Lemma 3.1 and also the exponential decay of Z_0 it follows that

$$I_1^0 = O\left(e^{-c\frac{1}{\varepsilon}}\right)$$

for some positive c. It remains to evaluate only I_3^0. By using (3.11)

$$\begin{aligned}
I_3^0 &= \int_{-\frac{\delta}{\mu}}^{0} \tilde{\mathcal{R}}(v_{1,\lambda}) Z_0(t)\,dt + O\left(e^{-\frac{c}{\varepsilon}}\right) \\
&= \int_{-\frac{\delta}{\mu}}^{0} \frac{\hat{\mu}^2}{(1-\mu t\kappa)^2}\partial_{ss}^2 \tilde{h}_\mu Z_0(t)\,dt + \int_{-\frac{\delta}{\mu}}^{0} \frac{\dot{\mu}^2 t^2}{(1-\mu t\kappa)^2}\partial_{tt}^2 \tilde{h}_\mu Z_0(t)\,dt \\
&- \int_{-\frac{\delta}{\mu}}^{0} \frac{2\varepsilon^{-1}\mu\dot{\mu}t}{(1-\mu t\kappa)^2}\partial_{st}^2 \tilde{h}_\mu Z_0(t)\,dt \\
&- \int_{-\frac{\delta}{\mu}}^{0} \frac{\mu^2\dot{\mu}t^2\dot{\kappa}}{(1-\mu t\kappa)^3}\partial_t w Z_0(t)\,dt - \int_{-\frac{\delta}{\mu}}^{0} \frac{\mu\kappa}{1-\mu t\kappa}\partial_t \tilde{z}_\mu Z_0(t)\,dt \\
&+ \int_{-\frac{\delta}{\mu}}^{0} \frac{\mu^3\varepsilon^{-1}t\dot{\kappa}}{(1-\mu t\kappa)^3}\partial_s\left[\ln\frac{1}{4\hat{\mu}^2} + \tilde{h}_\mu\right] Z_0(t)\,dt \\
&- \int_{-\frac{\delta}{\mu}}^{0} \mu^2\tilde{h}_\mu Z_0(t)\,dt + \int_{-\frac{\delta}{\mu}}^{0} \left(-\frac{\ddot{\mu}\mu t}{(1-\mu t\kappa)^2} - \frac{\mu^2\dot{\mu}t^2\dot{\kappa}}{(1-\mu t\kappa)^3} + \frac{2\dot{\mu}^2 t}{(1-\mu t\kappa)^2}\right)\partial_t \tilde{h}_\mu Z_0(t)\,dt \\
&+ \int_{-\frac{\delta}{\mu}}^{0} \mathcal{S}_0^1 Z_0(t)\,dt + \int_{-\frac{\delta}{\mu}}^{0} \tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s) Z_0(t)\right) Z_0(t)\,dt + \int_{-\frac{\delta}{\mu}}^{0} \mathcal{S}_0^2 Z_0(t)\,dt \\
&+ \int_{-\frac{\delta}{\mu}}^{0} \mathcal{S}_0^3 Z_0(t)\,dt + O\left(e^{-\frac{c}{\varepsilon}}\right) \\
&= \varepsilon^3 M_0(\varepsilon s)(1+o(1)) + \int_{-\frac{\delta}{\mu}}^{0} \tilde{\mathcal{A}}\left(e_0^\varepsilon(\varepsilon s) Z_0(t)\right) Z_0(t)\,dt + \int_{-\frac{\delta}{\mu}}^{0} \mathcal{S}_0^2 Z_0(t)\,dt
\end{aligned}$$

$$+\int_{-\frac{\delta}{\mu}}^{0} \mathcal{S}_0^3 Z_0(t)\,dt + O\left(e^{-\frac{c}{\varepsilon}}\right)$$

where $M_0(\varepsilon s)$ is a sum of smooth functions uniformly bounded in ε that does not depend on e_0. Now

$$\begin{aligned}\int_{-\frac{\delta}{\mu}}^{0} \tilde{\mathcal{A}}\left(e_0^{\varepsilon}(\varepsilon s)Z_0(t)\right)Z_0(t)\,dt &= \varepsilon^{\frac{5}{2}}\left[\varepsilon^2\ddot{e}_0\underbrace{\left(\frac{1}{2}\frac{\partial\hat{\mu}_\varepsilon}{\partial\varepsilon}_{|\varepsilon=0} + 2\kappa(\varepsilon s)\hat{\mu}^2(\varepsilon s)\int_{-\infty}^{0} tZ_0^2(t)\,dt\right)}_{a_0^{\varepsilon}(\varepsilon s)}\right]\\ &+\varepsilon^{\frac{5}{2}}\left[\varepsilon\dot{e}_0\underbrace{\left(-2\hat{\mu}\dot{\hat{\mu}}\int_{-\infty}^{0} t\partial_t Z_0 Z_0\,dt\right)}_{a_1^{\varepsilon}(\varepsilon s)}\right]\\ &+\varepsilon^{\frac{5}{2}}\left[e_0\underbrace{\left(-\hat{\mu}\kappa\int_{-\infty}^{0}\partial_t Z_0 Z_0\,dt\right)}_{\hat{a}_2^{\varepsilon}(\varepsilon s)}\right]\\ &+\varepsilon^{\frac{7}{2}}h(e_0,\dot{e}_0,\ddot{e}_0)(1+o(1))\end{aligned}$$

where $h(e_0,\dot{e}_0,\ddot{e}_0)$ is a sum of functions depending linearly on $e_0,\dot{e}_0,\ddot{e}_0$. Now

$$\int_{-\frac{\delta}{\mu}}^{0}\mathcal{S}_0^2 Z_0\,dt = \varepsilon^3 F(e_0)(1+o(1)); \qquad \int_{-\frac{\delta}{\mu}}^{0}\mathcal{S}_0^3 Z_0\,dt = \varepsilon^{\frac{3}{2}}\left[e_0\underbrace{\int_{-\infty}^{0} e^w Z_0^2(\alpha_1+v)\,dt}_{\tilde{a}_2^{\varepsilon}(\varepsilon s)}\right](1+o(1))$$

where F is quadratic in e_0.

Putting together all these estimates we get

$$\begin{aligned}\int_{-\frac{2\delta}{\mu}}^{0}\mathcal{R}(\tilde{U}_\lambda)Z_0\,dt &= \varepsilon^{\frac{3}{2}}\left[\varepsilon^2\underbrace{\left(\frac{1}{2}\hat{\mu}_0^2+\varepsilon a_0^{\varepsilon}\right)}_{a_0(\varepsilon s)}\ddot{e}_0 + a_1^{\varepsilon}(\varepsilon s)\dot{e}_0 + \underbrace{\left(\frac{1}{2}\Lambda_1+\varepsilon a_2^{\varepsilon}\right)}_{a_2(\varepsilon s)}e_0\right]+\varepsilon^3 M_0(\varepsilon s)\\ &+\varepsilon^3 F(e_0)(1+o(1))+\varepsilon^{\frac{7}{2}}h(e_0,\dot{e}_0,\ddot{e}_0)(1+o(1))\end{aligned}$$

and the result follows. □

4. The remainder term

We split the remainder term Φ_λ in (1.14) as

$$\Phi_\lambda = \eta_{2\delta}\phi_\lambda + \psi_\lambda, \tag{4.1}$$

where ϕ_λ solves a linear problem defined in a neighborhood of the boundary and ψ_λ solves a linear problem defined in the whole domain. More precisely, we are led to consider the couple of linear problems

$$\begin{cases} \Delta\psi - \psi + (1-\eta_{2\delta})\lambda e^{U_\lambda}\psi = -(1-\eta_{2\delta})S_\lambda(U_\lambda) - (1-\eta_{2\delta})N(\eta_{2\delta}\phi + \psi) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 2\nabla\eta_{2\delta}\nabla\phi - \phi\Delta\eta_{2\delta} \qquad \text{in } \Omega \\ \partial_\nu\psi = 0 \text{ on } \partial\Omega \end{cases} \tag{4.2}$$

and

$$\begin{cases} L(\phi) = -S_\lambda(U_\lambda) - N(\eta_{2\delta}\phi + \psi) - \lambda e^{U_\lambda}\psi \text{ in } \mathcal{D}_{2\delta} \\ \partial_\nu\phi = 0 \qquad \text{on } \partial\mathcal{D}_{2\delta}. \end{cases} \tag{4.3}$$

4.1. The remainder term in the whole domain

Given a function ϕ defined in a neighborhood of the boundary, let us find a function ψ which solves problem (4.2).

First of all, it is useful to point out that for any $g \in L^\infty(\Omega)$ there exists a unique ψ solution to the linear problem

$$\begin{cases} \Delta\psi - \psi + (1-\eta_{2\delta})\,\lambda e^{U_\lambda}\psi = g \qquad \text{in } \Omega \\ \partial_\nu\psi = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{4.4}$$

with

$$\|\psi\|_\infty \le C\|g\|_\infty. \tag{4.5}$$

It is enough to show that the linear perturbation term $(1-\eta_{2\delta})\,\lambda e^{U_\lambda}\psi$ is small as ε goes to zero. Indeed, arguing as in Lemma 3.4, we have

$$\|(1-\eta_{2\delta})\lambda e^{u_\lambda}\|_\infty \le e^{-\frac{c}{\varepsilon}}$$

for some positive constant c.

Now, let us split the remainder $\psi = \psi^1 + \psi^2$ where ψ^1 solves a linear problem and ψ^2 solves a nonlinear problem. More precisely, ψ^1 solves (4.4) with

$$g := -(1-\eta_{2\delta})S_\lambda(U_\lambda) - (1-\eta_{2\delta})N(\eta_{2\delta}\phi) - 2\nabla\eta_{2\delta}\nabla\phi - \phi\Delta\eta_{2\delta} \tag{4.6}$$

and ψ^2 solves (4.4) with

$$g := -(1-\eta_{2\delta})[N(\eta_{2\delta}\phi + \psi^1 + \psi^2) - N(\eta_{2\delta}\phi)]. \tag{4.7}$$

It is clear that for any function ϕ there exists a unique ψ^1 solution to (4.4) with the R.H.S. as in (4.6). Let us prove that

$$\|\psi^1_\lambda\|_\infty \le c\varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_*. \tag{4.8}$$

By (4.5) we need to estimate the L^∞-norm of R.H.S. given in (4.6). First of all, in Lemma 3.4 we have

$$\|(1-\eta_{2\delta})S_\lambda(U_\lambda)\|_\infty \le e^{-\frac{c}{\varepsilon}}$$

for some positive constant c. Moreover

$$\begin{aligned}\|(1-\eta_{2\delta})N(\eta_{2\delta}\phi)\|_\infty &\le c\|(1-\eta_{2\delta})\lambda e^{U_\lambda}\eta_{2\delta}^2\phi^2\|_\infty \le c\|(1-\eta_{2\delta})\lambda e^{u_\lambda}\|_\infty\|(1-\eta_{2\delta})\eta_{2\delta}\phi\|_\infty^2 \\ &\le ce^{-\frac{c}{\varepsilon}}\varepsilon^{2\sigma(1-a)}\|\tilde{\phi}\|_*\end{aligned}$$

since

$$\|(1-\eta_{2\delta})\eta_{2\delta}\phi\|_\infty \le c\left(\sup_{\mathcal{C}_{2\delta}\setminus\mathcal{C}_\delta}\frac{1}{1+|t|^\sigma}\right)\|\tilde{\phi}\|_* \le c\varepsilon^{\sigma(1-a)}\|\tilde{\phi}\|_*,$$

where we agree that $\tilde{\phi}$ is nothing but the scaled function $\phi(\varepsilon s, \mu t)$. Finally

$$\|\nabla\eta_{2\delta}\nabla\phi\|_\infty \le \frac{\varepsilon}{\delta}\left(\frac{\varepsilon}{\delta}\right)^{\sigma+1}\|\tilde{\phi}\|_* \le c\varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_*$$

and

$$\|\phi\Delta\eta_{2\delta}\|_\infty \le \frac{\varepsilon^2}{\delta^2}\left(\frac{\varepsilon}{\delta}\right)^{\sigma}\|\tilde{\phi}\|_* \le c\varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_*.$$

Moreover it is possible to show that the nonlinear operator ψ_λ is Lipschitz such that

$$\|\psi_\lambda(\phi_{\lambda,1}) - \psi_\lambda(\phi_{\lambda,2})\|_\infty \leqslant c\varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}_1 - \tilde{\phi}_2\|_*.$$

Once we have found the function ψ^1, we solve equation (4.4) with R.H.S. (4.7). A simple contraction mapping argument (the nonlinear term N is quadratic) yields the existence of a function ψ^2 such that

$$\|\psi^2\|_\infty \le \|(1-\eta_{2\delta})\lambda e^{u_\lambda}\|_\infty\|\psi^1\|_\infty \leqslant e^{-\frac{c}{\varepsilon}}\|\psi^1\|_\infty \tag{4.9}$$

for some positive constant c.

4.2. The remainder term close to the boundary: a nonlinear projected problem

In order to solve problem (4.3), it is necessary to solve a nonlinear projected problem naturally associate with it. Since it is defined in a neighborhood of the boundary, it is useful to scale it. Then we are led to study the problem: *given μ which satisfies (2.3) and e_0 which satisfies (2.17), find a function $c_0(s)$ and a function $\tilde{\phi}$ so that*

$$\begin{cases} \mathcal{L}(\tilde{\phi}) = -\mathcal{R}(\tilde{U}_\lambda) - \mathcal{N}_1(\tilde{\phi}) + c_0 Z_0 \text{ in } \mathcal{C}_{2\delta}, \\ \partial_t \tilde{\phi} = 0, \qquad \text{on } \partial\mathcal{C}_{2\delta} \cap \{t = 0\}, \\ \tilde{\phi}\left(s + \dfrac{\ell}{\varepsilon}, t\right) = \tilde{\phi}(s,t) \\ \displaystyle\int_{-2\frac{\delta}{\mu}}^{0} \tilde{\phi}(s,t) Z_0(t) dt = 0, \end{cases} \tag{4.10}$$

where $\mathcal{L}$ is defined in (2.10), $\mathcal{R}(\tilde{U}_\lambda)$ is defined in (3.3) and the superlinear term $\mathcal{N}_1(\tilde{\phi})$ is defined by

$$\mathcal{N}_1(\tilde{\phi}) = \lambda\mu^2 e^{\tilde{U}_\lambda}\left[e^{\tilde{\eta}_{2\delta}\tilde{\phi} + \tilde{\psi}(\phi)} - 1 - (\tilde{\eta}_{2\delta}\tilde{\phi} + \tilde{\psi}(\phi))\right] + \lambda\mu^2 e^{\tilde{U}_\lambda}\tilde{\psi}(\phi) + \left(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w\right)\tilde{\phi}. \tag{4.11}$$

Here $\tilde{\psi}(\phi)$ is the scaled function $\left[\psi(\phi)\right](\varepsilon s, \mu t)$ and $\psi(\phi)$ is the solution to the problem (4.2). In (4.10) the terms which contain e_1^ε and e_2^ε in $\mathcal{R}(\tilde{U}_\lambda)$ are encoded in the last sum (see (3.5)).

By Proposition 2.2 $\mathcal{L}$ is invertible. Hence solving (4.10) together with boundary, the periodic and orthogonality conditions reduce to solve a fixed point problem, namely

$$\tilde{\phi} = \mathcal{T}(-\tilde{\mathcal{R}}(\tilde{U}_\lambda) - \mathcal{N}_1(\tilde{\phi})) = \mathcal{M}(\tilde{\phi}) \tag{4.12}$$

where $\mathcal{T}$ is the operator defined in Proposition 2.2.

We will prove the following result.

Proposition 4.1. *There exist $c > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ and for any e_0 satisfying (2.17), the problem (4.10) has a unique solution $\tilde{\phi} = \tilde{\phi}(e_0)$ and $c_0 = c_0(e_0)$, which satisfies*

$$\|\tilde{\phi}\|_* \le c\varepsilon^{\frac{5}{2}}. \tag{4.13}$$

Proof. Let us consider the set

$$\mathcal{E} := \left\{\tilde{\phi} \ : \ \|\tilde{\phi}\|_* \le c\varepsilon^{\frac{5}{2}}\right\}$$

for a certain positive constant c. We first show that $\mathcal{M}$ maps $\mathcal{E}$ into itself.

Let $\tilde{\phi} \in \mathcal{E}$. Then by using Lemma 3.1

$$\|\mathcal{M}(\tilde{\phi})\|_* \le C\|\tilde{\mathcal{R}}(\tilde{U}_\lambda) + \mathcal{N}_1(\tilde{\phi})\|_{**} \le c\varepsilon^{\frac{5}{2}} + c\|\mathcal{N}_1(\tilde{\phi})\|_{**}.$$

We evaluate $\|\mathcal{N}_1(\tilde{\phi})\|_{**}$.

$$\begin{aligned}\|\mathcal{N}_1(\tilde{\phi})\|_{**} &\le \|\lambda\mu^2 e^{\tilde{U}_\lambda}(\tilde{\eta}_\delta\tilde{\phi} + \tilde{\psi}(\phi))^2\|_{**} + \|\lambda\mu^2 e^{\tilde{U}_\lambda}\tilde{\psi}_\lambda(\phi)\|_{**} + \|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)\tilde{\phi}\|_{**} \\ &\le \|e^w\tilde{\phi}^2\|_{**} + \|e^w\tilde{\psi}_\lambda^2(\phi)\|_{**} + \|e^w\tilde{\phi}\tilde{\psi}(\phi)\|_{**} + \|e^w\tilde{\psi}(\phi)\|_{**} \\ &\quad + \|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)\tilde{\phi}\|_{**}.\end{aligned}$$

Now

$$\|e^w\tilde{\phi}^2\|_{**} \le \|\tilde{\phi}\|_*^2 \sup_{\mathcal{C}_{2\delta}} e^w \frac{1+|t|^{\sigma+2}}{(1+|t|^\sigma)^2} \le \|\tilde{\phi}\|_*^2$$

$$\|e^w\tilde{\psi}^2(\phi)\|_{**} \le \|\tilde{\psi}(\phi)\|_\infty^2 \sup_{\mathcal{C}_{2\delta}} e^w(1+|t|^{\sigma+2}) \le \varepsilon^{2(\sigma+2)(1-a)}\|\tilde{\phi}\|_*^2$$

$$\|e^w\tilde{\phi}\tilde{\psi}(\phi)\|_{**} \le \|\tilde{\psi}(\phi)\|_\infty\|\tilde{\phi}\|_* \sup_{\mathcal{C}_{2\delta}} e^w \frac{1+|t|^{\sigma+2}}{1+|t|^\sigma} \le \varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_*^2$$

analogously

$$\|e^w\tilde{\psi}(\phi)\|_{**} \le \varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_*$$

and finally

$$\|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)\tilde{\phi}\|_{**} \le \|e^w(\tilde{\alpha}_\mu + \tilde{\beta}_\mu + \tilde{v}_\mu + \tilde{z}_\mu + e_0^\varepsilon Z_0)\tilde{\phi}\|_{**} \le \varepsilon\|\tilde{\phi}\|_*.$$

Putting together all these computations we find that

$$\|\mathcal{N}_1(\tilde{\phi})\|_{**} \le \varepsilon^{(\sigma+2)(1-a)}\|\tilde{\phi}\|_* \tag{4.14}$$

and the first claim is proved.

We next prove that $\mathcal{M}$ is a contraction, so that the fixed point problem (4.12) can be uniquely solved in $\mathcal{E}$.

Indeed, for any $\tilde{\phi}_1, \tilde{\phi}_2 \in \mathcal{M}$ we get (setting $\tilde{\psi}_1 := \tilde{\psi}(\phi_1)$ and $\tilde{\psi}_2 := \tilde{\psi}(\phi_2)$)

$$\begin{aligned}\|\mathcal{M}(\tilde{\phi}_1) - \mathcal{M}(\tilde{\phi}_2)\|_* &\le C\|\mathcal{N}_1(\tilde{\phi}_1) - \mathcal{N}_1(\tilde{\phi}_2)\|_{**} \\ &\le \|e^w \cdot e^{\tilde{\eta}_{2\delta}\tilde{\phi}_1 + \tilde{\psi}_1}\Big(1 - e^{\tilde{\eta}_{2\delta}(\tilde{\phi}_2 - \tilde{\phi}_1) + \tilde{\psi}_2 - \tilde{\psi}_1} + (\tilde{\eta}_{2\delta}(\tilde{\phi}_2 - \tilde{\phi}_1) + \tilde{\psi}_2 - \tilde{\psi}_1)\Big)\|_{**} \\ &\quad + \|e^w\Big(\tilde{\eta}_{2\delta}(\tilde{\phi}_2 - \tilde{\phi}_1) + \tilde{\psi}_2 - \tilde{\psi}_1\Big)(e^{\tilde{\eta}_{2\delta}\tilde{\phi}_1 + \tilde{\psi}_1} - 1)\|_{**} \\ &\quad + \|e^w(\tilde{\psi}_1 - \tilde{\psi}_2)\|_{**} + \|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)(\tilde{\phi}_2 - \tilde{\phi}_1)\|_{**} \\ &\le \|e^w\Big(\tilde{\eta}_{2\delta}(\tilde{\phi}_2 - \tilde{\phi}_1) + (\tilde{\psi}_2 - \tilde{\psi}_1)\Big)^2\|_{**}\end{aligned}$$

$$
\begin{aligned}
&+ \|e^w \left(\tilde{\eta}_{2\delta}(\tilde{\phi}_2 - \tilde{\phi}_1) + \tilde{\psi}_2 - \tilde{\psi}_1\right)(\tilde{\eta}_{2\delta}\tilde{\phi}_1 + \tilde{\psi}_1)\|_{**} \\
&+ \|e^w(\tilde{\psi}_1 - \tilde{\psi}_2)\|_{**} + \|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)(\tilde{\phi}_2 - \tilde{\phi}_1)\|_{**} \\
\le\ & \|e^w(\tilde{\phi}_2 - \tilde{\phi}_1)^2\|_{**} + \|e^w(\tilde{\psi}_2 - \tilde{\psi}_1)^2\|_{**} + \|e^w(\tilde{\phi}_2 - \tilde{\phi}_1)(\tilde{\psi}_2 - \tilde{\psi}_1)\|_{**} \\
&+ \|e^w(\tilde{\phi}_2 - \tilde{\phi}_1)(\tilde{\eta}_{2\delta}\tilde{\phi}_1 + \tilde{\psi}_1)\|_{**} + \|e^w(\tilde{\psi}_2 - \tilde{\psi}_1)(\tilde{\eta}_{2\delta}\tilde{\phi}_1 + \tilde{\psi}_1)\|_{**} \\
&+ \|e^w(\tilde{\psi}_1 - \tilde{\psi}_2)\|_{**} + \|(\lambda\mu^2 e^{\tilde{U}_\lambda} - e^w)(\tilde{\phi}_2 - \tilde{\phi}_1)\|_{**} \\
\le\ & \max_{j=1,2} \|\tilde{\phi}_j\|_* \|\tilde{\phi}_2 - \tilde{\phi}_1\|_* + \max_{j=1,2} \|\tilde{\psi}_j\|_\infty \|\tilde{\psi}_2 - \tilde{\psi}_1\|_\infty \\
&+ \max_{j=1,2} \|\tilde{\psi}_j\|_\infty \|\tilde{\phi}_2 - \tilde{\phi}_1\|_* + \|\tilde{\phi}_1\|_* \|\tilde{\phi}_2 - \tilde{\phi}_1\|_\infty + \|\tilde{\psi}_1\|_\infty \|\tilde{\phi}_2 - \tilde{\phi}_1\|_* \\
&+ \|\tilde{\phi}_1\|_* \|\tilde{\psi}_2 - \tilde{\psi}_1\|_\infty + \|\tilde{\psi}_1\|_\infty \|\tilde{\psi}_2 - \tilde{\psi}_1\|_\infty \\
\le\ & \varepsilon^{\frac{5}{2}} \|\tilde{\phi}_2 - \tilde{\phi}_1\|_*.
\end{aligned}
$$

Hence $\mathcal{M}$ is a contraction and the proof is complete. □

4.3. *Proof of Theorem 1.1 completed*

It only remains to find the function e_0 to get the coefficient c_0 in (4.10) identically equal to zero. To do this, we multiply equation (4.10) by Z_0 and we integrate in t. Thus the equation

$$c_0(\theta) = 0 \text{ for any } \theta \in [0, \ell] \quad (\text{here } \varepsilon s = \theta)$$

is equivalent to

$$\int_{-\frac{2\delta}{\mu}}^{0} \left[\mathcal{R}(\tilde{U}_\lambda) + \mathcal{L}(\tilde{\phi}) + \mathcal{N}_1(\tilde{\phi})\right] Z_0 \, dt = 0 \text{ for any } s \in \left[0, \frac{\ell}{\varepsilon}\right]. \tag{4.15}$$

We first remark that, by using (4.14), it follows that

$$\int_{-\frac{2\delta}{\mu}}^{0} \left[\mathcal{L}(\tilde{\phi}) + \mathcal{N}_1(\tilde{\phi})\right] Z_0 \, dt = \varepsilon^{(\sigma+2)(1-a)+\frac{5}{2}} r \tag{4.16}$$

where r is the sum of functions of the form

$$h_0(\varepsilon s)\,[h_1(e_0, \dot{e}_0) + o(1) h_2(e_0, \dot{e}_0, \ddot{e}_0)]$$

where h_0 is a smooth function uniformly bounded in ε, h_1 depends smoothly on e_0 and on $\dot{e}_0$ and it is bounded in the sense that

$$\|h_1\|_\infty \leqslant c\|e_0\|_\varepsilon$$

and it is compact, as a direct application of Ascoli–Arzelá Theorem shows.

The function h_2 depends on $e_0, \dot{e}_0, \ddot{e}_0$ and it depends linearly on $\ddot{e}_0$ and it is Lipschitz with

$$\|h_2(e_0^1) - h_2(e_0^2)\|_\infty \leqslant o(1)\|e_0^1 - e_0^2\|_\varepsilon.$$

By using (3.31) it follows that (4.15) is equivalent to the following ODE

$$\varepsilon^2 (a_0(\varepsilon s)\ddot{e}_0 + a_1(\varepsilon s)\dot{e}_0) + a_2(\varepsilon s)e_0 = \varepsilon^{\frac{3}{2}} M_0(\varepsilon s) + \varepsilon^{\frac{3}{2}} H_0 + \varepsilon^{(\sigma+2)(1-a)+1} r \tag{4.17}$$

where $a_i(\varepsilon s)$, $i = 0, 1, 2$, M_0, F_0 and H_0 are as in Lemma 3.5 and r is as in (4.16). Our goal is to find a smooth periodic function e_0 which solves (4.17).

In order to do this we introduce an auxiliary problem.

Suppose that $p_0(\theta)$ is a positive $C^2(0, \ell)$ function, $p_1(\theta)$ is a $C^2(0, \ell)$ function and $\varepsilon > 0$ is a parameter small enough.

Given an arbitrary function $f \in C^0(0, \ell)$ let us consider the problem

$$\begin{cases} \varepsilon^2 (\ddot{x} + p_1(\theta)\dot{x}) + p_0(\theta)x = f & \text{in } (0, \ell) \\ x(0) = x(\ell) \;\; \dot{x}(0) = \dot{x}(\ell) \end{cases} \tag{4.18}$$

Lemma 4.2. *Let*

$$\Lambda_{p_0(\theta)} = \left(\int_0^\ell \sqrt{p_0(t)}\, dt \right)^2.$$

There is a small number $\varepsilon_0 = \varepsilon_0(p_0, \ell) > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ satisfies the gap condition

$$|4\pi^2 m^2 \varepsilon^2 - \Lambda_{p_0}| \geqslant \tilde{c}_0 \varepsilon \qquad \text{for any } m \in \mathbb{N} \cup \{0\} \tag{4.19}$$

with $\tilde{c}_0$ is small enough, then there exists a constant $C > 0$ such that problem (4.18) has a unique solution which satisfies

$$\varepsilon^2 \|\ddot{x}\|_\infty + \varepsilon \|\dot{x}\|_\infty + \|x\|_\infty \leqslant \frac{C}{\varepsilon} \|f\|_\infty \tag{4.20}$$

for any $f \in C^0(0, \ell)$. Moreover, if in addition $f \in C^2(0, \ell)$, the unique solution to problem (4.18) satisfies

$$\varepsilon^2 \|\ddot{x}\|_\infty + \varepsilon \|\dot{x}\|_\infty + \|x\|_\infty \leqslant C \left(\|\ddot{f}\|_\infty + \|\dot{f}\|_\infty + \|f\|_\infty \right). \tag{4.21}$$

Proof. Although similar results were obtained in [6], we sketch the proof to illustrate why condition (4.19) is required.

We take

$$\Lambda_{p_0(\theta)} = \left(\int_0^\ell \sqrt{p_0(t)}\, dt \right)^2; \qquad s(\theta) = \frac{\pi}{\sqrt{\Lambda_{p_0}}} \int_0^\theta \sqrt{p_0(t)}\, dt, \qquad y(s) = x(\theta).$$

Then (4.18) is transformed into

$$\begin{cases} \varepsilon^2 (\ddot{y} + q(s)\dot{y}) + \nu_0 y = \tilde{f}(s) & \text{in } (0,\pi) \\ y(0) = y(\pi) \;\; \dot{y}(0) = \dot{y}(\pi) \end{cases} \tag{4.22}$$

with

$$q(s) = \frac{\dot{p}_0}{2p_0} + \frac{1}{\pi\sqrt{\Lambda_{p_0}}}\frac{p_1}{\sqrt{p_0}}; \qquad \nu_0 = \frac{\Lambda_{p_0}}{\pi^2}; \qquad \tilde{f}(s) = \frac{\Lambda_{p_0}}{\pi^2}\frac{f}{p_0}.$$

It is a standard fact that the eigenvalue problem

$$\begin{cases} \ddot{y} + q(s)\dot{y} + \nu y = 0 & \text{in } (0,\pi) \\ y(0) = y(\pi) \;\; \dot{y}(0) = \dot{y}(\pi) \end{cases} \tag{4.23}$$

has an infinite sequence of eigenvalues $(\nu_m)_m \subset \mathbb{R}$ such that

$$\sqrt{\nu_m} = 2m + O\left(\frac{1}{m^3}\right) \qquad \text{as } m \to \infty$$

with associated eigenfunctions $y_m(s)$ that forms an orthonormal basis in $L^2(0,\pi)$.

Thus, if $\nu_0 \neq \varepsilon^2\nu_m$ for all $m \geqslant 0$ the problem (4.18) is solvable. In such a case the solution (4.18) can be described as

$$y(s) = \sum_{m=0}^{\infty} \frac{\tilde{f}_m}{\nu_0 - \varepsilon^2\nu_m} y_m(s)$$

where $\tilde{f}_m(s) := \int_0^\pi \tilde{f}(s) y_m(s)\, ds$.

Since $y \in C^2(0,\pi)$ the above expression holds in $C^2(0,\pi)$. From (4.19) we find that

$$|\nu_0 - \nu_m\varepsilon^2| \geqslant \frac{c}{2}\varepsilon$$

if ε is sufficiently small. Next we notice that, by using Cauchy–Schwarz inequality and Parseval's identity we have

$$\|y\|_\infty \leqslant \sum_{m=0}^{\infty} \left| \frac{\tilde{f}_m y_m(s)}{\nu_0 - \varepsilon^2\nu_m} \right| \leqslant \left(\sum_{m=0}^{\infty} \tilde{f}_m^2 y_m^2 \right)^{\frac{1}{2}} \left(\sum_{m=0}^{\infty} \frac{1}{(\nu_0 - \nu_m\varepsilon^2)^2} \right)^{\frac{1}{2}} \leqslant \frac{c}{\varepsilon}\|\tilde{f}\|_\infty \tag{4.24}$$

Coming back to the original variable

$$\|x\|_\infty = \|y\|_\infty \leqslant \frac{c}{\varepsilon}\left\| \frac{\Lambda_{p_0}}{p_0} \right\|_\infty \|f\|_\infty \leqslant \frac{C}{\varepsilon}\|f\|_\infty.$$

In this way, one can also estimate the $L^\infty(0,\pi)$-norms of $\dot{y}$ and $\ddot{y}$. Therefore the result holds. For a more detailed treatment of this and estimate (4.21) one can see [6, Lemma 8.2]. □

In view of system (4.18), it is natural to consider a perturbation of the equation in (4.18), namely

$$\begin{cases} \varepsilon^2(\ddot{x} + p_1(\theta)\dot{x}) + \big(p_0(\theta) + \varepsilon\tilde{p}_{0,\varepsilon}(\theta)\big)x = f & \text{in } (0,\ell) \\ x(0) = x(\ell) \qquad \dot{x}(0) = \dot{x}(\ell) \end{cases} \tag{4.25}$$

where $\{\tilde{p}_{0,\varepsilon}(\theta)\}_{\varepsilon>0}$ is a family of $C^2(0,\ell)$ functions such that

$$\sup_{\varepsilon>0} \|\tilde{p}_{0,\varepsilon}\|_{C^2(0,\ell)} < C \tag{4.26}$$

and

$$\sup_{\varepsilon>0} \varepsilon \left\| \frac{\partial \tilde{p}_{0,\varepsilon}}{\partial \varepsilon} \right\|_\infty < C. \tag{4.27}$$

Then we have a constant $M > 0$ and a family $\{\Lambda_\varepsilon\}_\varepsilon \subset \mathbb{R}$ such that

$$\Lambda_{p_0(\theta)+\varepsilon\tilde{p}_{0,\varepsilon}(\theta)} = \Lambda_{p_0(\theta)} + \varepsilon\Lambda_\varepsilon$$

and

$$|\Lambda_\varepsilon| + \varepsilon \left| \frac{\partial \Lambda_\varepsilon}{\partial \varepsilon} \right| \leqslant M. \tag{4.28}$$

We observe that if there exists a small $\varepsilon > 0$ such that

$$|4\pi^2 m^2 \varepsilon^2 - \big(\Lambda_{p_0} + \varepsilon\Lambda_\varepsilon\big)| \geqslant \tilde{c}_0\varepsilon \qquad m = 0, 1, 2, \ldots \tag{4.29}$$

for some small $\tilde{c}_0 > 0$, then (4.19) holds after Λ_{p_0} is substituted by $\Lambda_{p_0+\varepsilon\tilde{p}_{0,\varepsilon}}$ and hence existence of a unique solution to (4.25) satisfying a priori bounds (4.20) and (4.21) is guaranteed.

Moreover (4.26) allows us to choose the constant $C > 0$ in (4.20) and (4.21) to be independent of ε.

Remark 4.3.

(i) First we deduce a sufficient condition of $\varepsilon > 0$ for which inequality (4.29) holds. Notice that (4.29) means that if

$$4\pi^2 m^2 \varepsilon^2 \geqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon \qquad (\text{or } 4\pi^2 m^2 \varepsilon^2 \leqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon)$$

then it should be

$$4\pi^2 m^2 \varepsilon^2 - \tilde{c}_0\varepsilon \geqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon \qquad (\text{or } 4\pi^2 m^2 \varepsilon^2 + \tilde{c}_0\varepsilon \leqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon)$$

for a sufficiently small $c > 0$ and for every $m \in \mathbb{N} \cup \{0\}$. Given any small number $\varepsilon > 0$, let us write

$$4\pi^2\varepsilon^2 = \frac{\Lambda_{p_0} + \varepsilon\Lambda_\varepsilon}{(m_0 + a_0)^2} \tag{4.30}$$

with some $m_0 \in \mathbb{N}$ large and $a_0 \in [0, 1)$. Assume $a_0 \neq 0$. Then the least $m \in \mathbb{N}$ satisfying $4\pi^2 m^2\varepsilon^2 \geqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon$ is $m = m_0 + 1$. Besides, for $m \geqslant m_0 + 1$ we have

$$\begin{aligned}
4\pi^2 m^2\varepsilon^2 - \tilde{c}_0\varepsilon &\geqslant \left(\Lambda_{p_0} + \varepsilon\Lambda_\varepsilon\right)\left[\left(\frac{m_0+1}{m_0+a_0}\right)^2 - \tilde{c}_0\frac{\left(\Lambda_{p_0} + \varepsilon\Lambda_\varepsilon\right)^{-\frac{1}{2}}}{2\pi(m_0+a_0)}\right] \\
&\geqslant \left(\Lambda_{p_0} + \varepsilon\Lambda_\varepsilon\right)\left[1 + \frac{2(1-a_0)}{m_0} - \frac{\tilde{c}_0}{2\pi\sqrt{\Lambda_{p_0}}}\frac{1}{m_0} + o\left(\frac{1}{m_0^2}\right)\right] \\
&\geqslant \Lambda_{p_0} + \varepsilon\Lambda_\varepsilon
\end{aligned}$$

provided $a_0 \leqslant 1 - \frac{\tilde{c}_0}{2\pi\sqrt{\Lambda_{p_0}}}$ choosing $\tilde{c}_0 < 2\pi\sqrt{\Lambda_{p_0}}$.

(ii) Let us show the existence of a sequence of small positive numbers $\varepsilon > 0$ converging to zero satisfying (4.29) provided (4.27) holds.
Indeed it is easy to see that the equation (4.30) has a unique pair (m_0, a_0) for any $\varepsilon \in (0, \varepsilon_1)$ where $\varepsilon_1 > 0$ is determined by Λ_{p_0} and M in (4.28).

We come back to the original problem.
Let us introduce the linear operator

$$L_0(e_0) := \varepsilon^2\left(a_0(\varepsilon s)\ddot{e}_0 + a_1(\varepsilon s)\dot{e}_0\right) + a_2(\varepsilon s)e_0.$$

The following result holds.

Lemma 4.4. *We have a positive number $\Lambda_{\hat{\mu}_0}$ and a number $\{\Lambda_{\hat{\mu}_0,\varepsilon}\}_\varepsilon$ such that if*

$$|4\pi^2 m^2\varepsilon^2 - (\Lambda_{\hat{\mu}_0} + \varepsilon\Lambda_{\hat{\mu}_0,\varepsilon})| \geqslant \tilde{c}_0\varepsilon \quad m = 1, 2\ldots \tag{4.31}$$

for some positive and sufficiently small constant $\tilde{c}_0$, then for any $f \in C^0_\ell(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique $e_0 \in C^2_\ell(\mathbb{R})$ solution of $L_0(e_0) = f$. Moreover, there exists $C > 0$ such that

$$\|e_0\|_\varepsilon = \varepsilon^2\|\ddot{e}_0\|_\infty + \varepsilon\|\dot{e}_0\|_\infty + \|e_0\| \leq \frac{C}{\varepsilon}\|f\|_\infty.$$

Finally, if $f \in C^2_\ell(\mathbb{R})$, then

$$\|e_0\|_\varepsilon = \varepsilon^2\|\ddot{e}_0\|_\infty + \varepsilon\|\dot{e}_0\|_\infty + \|e_0\| \leq C\left[\|f\|_\infty + \|\dot{f}\|_\infty + \|\ddot{f}\|_\infty\right].$$

Proof. The equation $\varepsilon^2 (a_0(\varepsilon s)\ddot{e}_0 + a_1(\varepsilon s)\dot{e}_0) + a_2(\varepsilon s)e_0 = f$ can be written as

$$\varepsilon^2 (\ddot{e}_0 + p_1(\theta)\dot{e}_0) + (p_0(\theta) + \varepsilon \tilde{p}_{0,\varepsilon})e_0 = g$$

with

$$p_1(\theta) = \frac{a_1(\varepsilon s)}{a_0(\varepsilon s)}; \quad p_0(\theta) = \frac{\Lambda_1}{\hat{\mu}_0^2(\theta)}; \quad \tilde{p}_{0,\varepsilon}(\theta) = \frac{a_2^\varepsilon(\theta)\hat{\mu}_0^2(\theta) - a_0^\varepsilon \Lambda_1}{\hat{\mu}_0^4(\theta)} + \varepsilon q_\varepsilon(\theta); \quad g = \frac{f(\theta)}{a_0(\theta)}.$$

It is clear that $p_0(\theta) > 0$ is a $C^2(0, \ell)$ function and $\tilde{p}_{0,\varepsilon} \in C^2(0, \ell)$ function and (4.26) and (4.27) hold. Then we let

$$\Lambda_{\hat{\mu}_0} = \left(\int_0^\ell \sqrt{\frac{\Lambda_1}{\hat{\mu}_0^2}}\, dt \right)^2$$

and hence there exist numbers $\Lambda_{\hat{\mu}_0,\varepsilon}$ such that

$$\Lambda_{p_0+\varepsilon\tilde{p}_{0,\varepsilon}} = \Lambda_{\hat{\mu}_0} + \varepsilon \Lambda_{\hat{\mu}_0,\varepsilon}$$

and the result comes from the above discussions. □

Proof of Theorem 1.1. By Lemma 4.4 it follows that there exists a sequence of small $\varepsilon = \varepsilon_m > 0$ converging to zero as $m \to +\infty$ such that the operator $L_0(e_0)$ is invertible with bounds for $L_0(e_0) = h$ given by

$$\|e_0\|_\varepsilon \le C\varepsilon^{-1}\|h\|_\infty,$$

for some positive constant C. Finally, by Contraction Mapping Argument using the properties of the right-hand side of (4.17), it follows that, the problem (4.17) has a unique solution with

$$\|e_0\|_\varepsilon < c\varepsilon^{(\sigma+2)(1-a)}$$

and that concludes the proof. □

5. The linear theory

In this section we give the proof of Proposition 2.2. We need a couple of preliminary results.

Lemma 5.1. *Assume $\xi \notin \{0, \pm\sqrt{\Lambda_1}\}$. Then given $h \in L^\infty(\mathbb{R}^2)$, there exists a unique bounded solution of*

$$(\hat{\mathcal{L}} - \xi^2)\psi = h \text{ in } \mathbb{R}^2. \tag{5.1}$$

Moreover

$$\|\psi\|_\infty \le C_\xi \|h\|_\infty \tag{5.2}$$

for some constant $C_\xi > 0$ only depending on ξ.

Proof. We argue as in Lemma 3.1 of [6]. □

Lemma 5.2. *Let ϕ a bounded solution of $\hat{\mathcal{L}}(\phi) + \partial^2_{ss}\phi = 0$ in $\mathbb{R}^2$. Then $\phi(s,t)$ is a linear combination of the functions $Z_1(t)$, $Z_0(t)\cos(\sqrt{\Lambda_1}s)$, $Z_0(t)\sin(\sqrt{\Lambda_1}s)$.*

Proof. We argue as in Lemma 7.1 of [5]. □

Proof of Proposition 2.2. The proof will be carried out in three steps.

Step 1: A priori bound (special case) Let us assume for the moment that in problem (4.10) the function c_0 is identically zero.

We will prove that there exists $C > 0$ so that for any h with $\|h\|_{**} < +\infty$ and any ϕ solution of problem

$$\begin{cases} \mathcal{L}(\phi) = h & \text{in } \mathcal{C}_{2\delta} \\ \partial_\nu \phi = 0 & \text{on } \partial\mathcal{C}_{2\delta} \cap \{t = 0\} \\ \phi\left(s + \frac{\ell}{\varepsilon}, t\right) = \phi(s,t) & \\ \displaystyle\int_{-\frac{2\delta}{\mu}}^{0} \phi(s,t) Z_0(t)\, dt = 0 & \forall\, s \in \mathbb{R}^+. \end{cases} \tag{5.3}$$

with $\|\phi\|_* < +\infty$ we have

$$\|\phi\|_* \le C\|h\|_{**}.$$

By contradiction we assume that there exist sequences $\lambda_n \to 0$, $(h_n)_n$ and $(\phi_n)_n$ solutions of (5.3) where

$$\delta_n = \varepsilon_n^a \text{ for some } a \in (0,1) \text{ and } \mu_n(\varepsilon_n s) = \varepsilon_n \hat{\mu}(\varepsilon_n s)$$

such that

$$\|\phi_n\|_* = 1 \qquad\qquad \|h_n\|_{**} \to 0.$$

To achieve a contradiction we will first show that

$$\|\phi_n\|_\infty \to 0. \tag{5.4}$$

If this was not the case then we may assume that there is a positive number c for which $\|\phi_n\|_{\infty,\mathcal{C}_{2\delta_n}} > c$. Since we also know that

$$|\phi_n(s,t)| \le \frac{c}{(1+|t|)^\sigma},$$

we conclude that for some $A > 0$

$$\|\phi_n\|_{L^\infty(|t|\le A)} \ge c.$$

Let us fix an s_n such that

$$\|\phi_n(s_n,\cdot)\|_{L^\infty(|t|\le A)} \ge \frac{c}{2}.$$

By elliptic estimates, compactness of Sobolev embeddings and the fact that the coefficients of $\tilde{\mathcal{A}}(\phi_n)$ tend to zero as $\lambda_n \to 0$, we see that we may assume that the sequence of functions $\tilde{\phi}_n(s,t) := \phi_n(s+s_n,t)$ converges uniformly over compact subsets of $\mathbb{R}^2$, to a nontrivial, bounded solution of

$$\hat{\mu}_0^\infty \partial^2_{ss}\tilde{\phi} + \partial^2_{tt}\tilde{\phi} + e^w\tilde{\phi} = 0 \qquad \text{in } \mathbb{R}^2$$

where $\hat{\mu}_0^\infty$ is a positive constant, which with no loss of generality via scaling, we may assume equal to one. By virtue of Lemma 5.2 then $\tilde{\phi}$ is a linear combination of Z_0 and Z_1. Moreover by the decay behavior and the orthogonality conditions assumed, which pass to the limit thanks to the Dominated Convergence, we find then that $\tilde{\phi} \equiv 0$. This is a contradiction that shows the validity of (5.4).

Let us conclude now the result of Step 1.

Since $\|\phi_n\|_* = 1$, there exists (s_n,t_n) with $r_n := |t_n| \to +\infty$ such that

$$r_n^\sigma|\phi_n(s_n,t_n)| + r_n^{\sigma+1}|D\phi_n(s_n,t_n)| \ge c > 0.$$

Let us consider now the scaled function

$$\tilde{\phi}_n(z_0,z) = r_n^\sigma\phi_n(s_n + r_n z_0, r_n z)$$

defined on $\bar{\mathcal{D}}$ given by

$$\bar{\mathcal{D}} := \left\{(z_0,z)\ :\ -r_n^{-1}s_n \le z_0 \le r_n^{-1}\left(\frac{\ell}{\varepsilon_n} - s_n\right);\quad -\frac{2\delta_n r_n^{-1}}{\mu_n(\varepsilon_n s)} < z < 0\right\}.$$

Then we have

$$|\tilde{\phi}_n(z_0,z)| + |z||D\tilde{\phi}_n(z_0,z)| \le |z|^{-\sigma} \qquad \text{in } \bar{\mathcal{D}}$$

and for some z_n with $|z_n| = 1$

$$|\tilde{\phi}_n(0,z_n)| + |D\tilde{\phi}_n(0,z_n)| \ge c > 0.$$

Moreover $\tilde{\phi}_n$ satisfies

$$\tilde{\mu}^2_{0,n}\partial^2_{z_0z_0}\tilde{\phi}_n + \partial^2_{zz}\tilde{\phi}_n + o(1)\tilde{C}(\tilde{\phi}_n) = \tilde{h}_n \qquad \text{in } \bar{\mathcal{D}}$$

where $\tilde{h}_n(z_0,z) = r_n^{\sigma+2}h_n(s_n + r_n z_0, r_n z)$, $\tilde{\mu}_{0,n} = \hat{\mu}_0^n(s_n + r_n z_0)$ and $\tilde{C}(\tilde{\phi}_n)$ is bounded.

Since

$$\|\partial_s \hat{\mu}_{0,n}^2\|_{\infty,\bar{D}} = O(\varepsilon_n); \qquad \left\|\partial_s\left(\frac{r_n^{-1}s_n}{\mu_n(\varepsilon_n(s_n+r_n z_0))}\right)\right\|_{\infty,\bar{D}} = O(\varepsilon_n^a);$$
$$\left\|\partial_{ss}^2\left(\frac{r_n^{-1}s_n}{\mu_n(\varepsilon_n(s_n+r_n z_0))}\right)\right\|_{\infty,\bar{D}} = O(\varepsilon_n^{1+a})$$

then, we may assume that

$$\tilde{\mu}_{0,n}^2 \to \hat{\mu}^* > 0$$

and that the function $\tilde{\phi}_n$ converges uniformly, in C^1-sense over compact subsets of $\mathcal{D}_*$, to $\tilde{\phi}$ which satisfies

$$\hat{\mu}^* \partial_{z_0 z_0}\tilde{\phi} + \partial_{zz}\tilde{\phi} = 0 \qquad \text{in } \mathcal{D}_* \tag{5.5}$$

where

$$\mathcal{D}_* := (0,\infty)\times(-\infty,0)$$

and $\tilde{\phi}$ satisfies

$$|\tilde{\phi}(z_0,z)| + |z||D\tilde{\phi}(z_0,z)| \le |z|^{-\sigma} \tag{5.6}$$

with the boundary condition. With no loss of generality, we may assume that $\hat{\mu}^* = 1$.

Hence $\tilde{\phi}$ is weakly harmonic in $\mathcal{D}_*$ and hence $\tilde{\phi} \equiv const$. Moreover since it satisfies (5.6), it follows that $\tilde{\phi} \equiv 0$.

This is a contradiction.

Step 2: A priori bound (general case) We claim that the a priori estimate obtained in Step 1 is valid for the full problem (4.10). We conclude from Step 1 that

$$\|\phi\|_* \leqslant c\,[\|h\|_{**} + \|c_0 Z_0\|_{**}] \leqslant C\,[\|h\|_{**} + \|c_0\|_\infty] \tag{5.7}$$

for any h with $\|h\|_{**} < \infty$ and solution ϕ of problem (4.10). To conclude we have to find a bound for the coefficient $c_0(s)$.

Testing the equation in (4.10) with Z_0 and integrating with respect to dt, we get

$$c_0(s)\int_{-\frac{2\delta}{\mu}}^{0} Z_0^2\,dt = \int_{-\frac{2\delta}{\mu}}^{0} \mathcal{L}(\phi)Z_1\,dt - \int_{-\frac{2\delta}{\mu}}^{0} hZ_1\,dt \tag{5.8}$$

Since Z_0 decays exponentially

$$\int_{-\frac{2\delta}{\mu}}^{0} Z_0^2\,dt = \frac{1}{2} + O(e^{-\sqrt{\Lambda_1}\frac{\delta}{\mu}})$$

Hence from (5.8) it follows that

$$c_0(s)\left(\frac{1}{2} + O\left(e^{-\sqrt{\Lambda_1}\frac{1}{\varepsilon}}\right)\right) = \int_{-\frac{2\delta}{\mu}}^{0} \hat{\mu}_0^2 \partial_{ss}^2 \phi Z_0\,dt + \int_{-\frac{2\delta}{\mu}}^{0} \left(\partial_{tt}^2\phi + e^w\phi\right) Z_0\,dt + \int_{-\frac{2\delta}{\mu}}^{0} \tilde{\mathcal{A}}(\phi) Z_0\,dt$$
$$- \int_{-\frac{2\delta}{\mu}}^{0} h Z_0\,dt. \tag{5.9}$$

It is easy to see that

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} h Z_0\,dt\right| \le \|h\|_{**} \int_{-\frac{2\delta}{\mu}}^{0} \frac{1}{(1+|t|)^{\sigma+2}}\,dt \le C\|h\|_{**}. \tag{5.10}$$

Now by using the boundary condition, the orthogonality condition and the radial symmetry of Z_0 we get

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} (\partial_{tt}^2\phi + e^w\phi) Z_0\,dt\right| \le O(e^{-\frac{c}{\varepsilon}})\|\phi\|_*. \tag{5.11}$$

Now, since

$$\int_{-\frac{2\delta}{\mu}}^{0} \phi Z_0\,dt = 0$$

if we make twice the s-derivative and we make some computations, we immediately get that

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} \partial_{ss}^2 \phi Z_j\,dt\right| \leqslant O\left(e^{-\frac{c}{\varepsilon}}\right)\|\phi\|_*.$$

Moreover we get

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} \tilde{\mathcal{A}}(\phi) Z_0\, dt\right| = \left|\int_{-\frac{2\delta}{\mu}}^{0} b_0(s,t)\partial^2_{ss}\phi Z_0\, dt + \int_{-\frac{2\delta}{\mu}}^{0} b_1(s,t)\partial^2_{tt}\phi Z_0\, dt \right.$$
$$\left. + \int_{-\frac{2\delta}{\mu}}^{0} b_2(s,t)\partial^2_{st}\phi Z_0\, dt + \int_{-\frac{2\delta}{\mu}}^{0} b_3(s,t)\partial_t\phi Z_0, dt + \int_{-\frac{2\delta}{\mu}}^{0} b_4(s,t)\partial_s\phi Z_0\, dt\right|$$

where b_j are defined in (2.7).

Now reasoning as before

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} b_0(s,t)\partial^2_{ss}\phi Z_0\, dt\right| \le \varepsilon \int_{-\frac{2\delta}{\mu}}^{0} |\partial^2_{ss}\phi Z_0|\, dt \le \varepsilon O\left(e^{-\frac{c}{\varepsilon}}\right)\|\phi\|_*$$

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} b_1(s,t)\partial^2_{tt}\phi Z_0\, dt\right| \le O(e^{-\frac{c}{\varepsilon}})\|\phi\|_* + C\varepsilon^2\|\phi\|_* \leqslant C\varepsilon^2\|\phi\|_*$$

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} b_2(s,t)\partial^2_{st}\phi Z_0\, dt\right| \le \varepsilon\|\phi\|_*; \qquad \left|\int_{-\frac{2\delta}{\mu}}^{0} b_3(s,t)\partial_t\phi Z_0\, dt\right| \le \varepsilon\|\phi\|_*;$$

$$\left|\int_{-\frac{2\delta}{\mu}}^{0} b_4(s,t)\partial_s\phi Z_0\, dt\right| \le \varepsilon^2\|\phi\|_*$$

hence by (5.9)

$$\|c_0\|_\infty \le C\|h\|_{**} + c\varepsilon\|\phi\|_*. \tag{5.12}$$

Combining (5.12) with (5.7) the result follows.

Step 3: (Existence part) We establish now the existence of a solution ϕ for problem (4.10).

We consider the case in which $h(s,t)$ is a T-periodic function in s, for an arbitrarily and large but fixed T. We then look for a weak solution ϕ to (4.10) in H_T defined as the subspace of functions ψ which are in $H^1(B)$ for any B bounded subset of $\mathcal{C}_{2\delta}$, which are T-periodic in s, such that $\partial_\nu\phi = 0$ on $\partial\mathcal{C}_{2\delta} \cap \{t = 0\}$ and so that

$$\int_{-\frac{2\delta}{\mu}}^{0} \psi Z_0 \, dt = 0 \qquad \forall \ \psi \in H^1(B).$$

Let $\mathcal{D}_T := \{t \in \left[-\frac{2\delta}{\mu}, 0\right] \ : \ s \in (0, T)\}$ and the bilinear form in H_T:

$$\mathcal{B}(\phi, \psi) := \int_{\mathcal{D}_T} \psi \mathcal{L}(\phi) \, dt \qquad \forall \ \psi \in H_T.$$

Then problem (4.10) gets weakly formulated as that of finding $\phi \in H_T$ such that

$$\mathcal{B}(\phi, \psi) = \int_{\mathcal{D}_T} h\phi \, dt \qquad \forall \ \psi \in H_T.$$

If h is smooth, elliptic regularity yields that a weak solution is a classical one.

The weak formulation can be readily put into the form

$$\phi + \mathcal{K}(\phi) = h \qquad \text{in } H_T$$

where h is a linear operator of h and $\mathcal{K}$ is compact.

The a priori estimate of Step 2 yields that for $h = 0$ only the trivial solution is present. Fredholm alternative thus applies yielding that problem (4.10) is thus solvable in the periodic setting. This is enough for our purpose. However we remark that if we approximate a general h by periodic functions of increasing period and we use uniform estimate we obtain in the limit a solution to the problem. □

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