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Strategies of Number Sense in Pre-service Secondary Mathematics Teachers

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Abstract This paper presents some results of an investigation on the number sense of a group of pre-service secondary teachers from Spain. The objective of this research was to analyze students' use of strategies associated to number sense and compare them with those obtained in a previous study with pre-service primary teachers in Taiwan, (Yang, Reys & Reys, *International Journal of Science and Mathematics Education*, *7*, 383–403, 2009). Pre-service secondary teachers showed lower success than pre-service primary teachers in a number sense test. Nevertheless, these last based their reasoning mainly on rules and algorithms, while pre-service secondary teachers used more strategies of number sense. In an attempt to delve into the low success shown by of Spanish pre-service secondary teachers, some students were interviewed. Those interviews showed that a deeper work on number sense in the training of these students is needed.

Keywords Number sense · Pre-service secondary teachers · Pre-service primary teachers · Strategies · Degree in mathematics

Introduction

Number sense refers to a person's general understanding of numbers and operations, the ability to use numbers in a flexible way, showing different strategies for handling

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numbers and operations, and the ability to assess the validity of results (McIntosh, Reys & Reys, 1992). Sowder (1992) defines number sense as a well organized conceptual network which allows relating numbers, operations, and their properties to solve numerical problems in a creative and flexible way. The term number sense appears in the curriculum of different countries where the mathematical learning is proposed to be an activity which "makes sense," and number sense has been established as one of the aspects to be covered in compulsory education (Australian Education Council, 1990; National Council of Teachers of Mathematics [NCTM], 2000). Despite the efforts of curriculum to incorporate a different learning of numbers, so much time is still spent on the learning with numerical closed tasks with a unique way of resolution. Textbooks represent one of the problems for developing number sense, since they strongly emphasize on written rule-based methods, and practically do not pay attention to number sense. This view contributes to build in learners the idea that algorithmic methods are the accepted way to solve problems in an academic context. To change this practice, teachers first have to believe in and recognize the usefulness of developing different tasks which promote number sense. In order to get this, it is important to allow a space for reflection on number sense both in primary and secondary teachers' training.

In Spain, compulsory education ranges 10 years, six for primary level (6–12 years old) and four for compulsory secondary level (12–16 years old). The mathematical training of primary and secondary teachers is very different. Primary teachers have a short university formation in basic mathematics (usually one course) and several courses about mathematics education. On the contrary, secondary teachers' training is mostly in pure mathematics, and they complete their formation with a postgraduate in mathematics education. In the Spanish curriculum (published in the Official Bulletin of the State from Spain, BOE), numerical learning in primary education addresses whole numbers, decimals, and fractions, while in secondary education, it is extended to integers and real numbers. The curriculums of both levels highlight the importance of *number sense*. Specifically, the curriculum of secondary level (Real Decreto, 1631, p. 750) states: "the development of number sense started in primary education continues in secondary education with the extension to the new numerical sets and the consolidation of those already studied." What is important at this stage is the understanding of operations, in parallel with the development of the ability to estimate and mental calculation. Therefore, teachers from both educational levels must know what it means and which aspects number sense involves, besides knowing strategies to develop it in the classroom.

In Spain, there is a lack of research on this topic, specifically related to teachers. Internationally, there have been an important amount of research on the use of number sense by pre-service primary teachers (Alajmi & Reys, 2007; Ghazali, Othman, Alias & Saleh, 2010; Tsao, 2004; Veloo, 2010; Yang, Reys & Reys, 2009), where it has been observed that their knowledge on this topic is close to a secondary level student, and they tend to use algorithms. We wonder whether the same occurs with pre-service secondary teachers. The type of academic training of the pre-service primary teachers of these studies is similar to the Spanish case, thus we decided to study pre-service teachers from the secondary level to delve into a new field and compare the results between both levels.

The objective of this research is to analyze the use of strategies associated to number sense shown by a group of pre-service secondary teachers and compare them with those shown by a group of pre-service primary teachers in a previous investigation made in Taiwan (Yang et al., 2009), in order to determine whether the greater mathematical training of pre-service secondary teachers is manifested by numerically richer strategies.

Background for the Study

Different authors have pointed that number sense is a difficult term to describe, although it is recognizable in the action itself when solving numerical tasks. Greeno (1991) characterized number sense in terms of developing flexible mental computation, performing good numerical estimates, and judging and making inferences about quantities. Other researchers have characterized number sense by components (McIntosh et al., 1992; Reys & Yang, 1998; Yang, 2005): (1) understanding the meaning of numbers, (2) recognizing the relative and absolute size of the magnitudes of numbers, (3) using benchmarks, (4) being able to compose and decompose numbers, (5) using several representations of numbers and operations, (6) understanding the relative effect of operations, and (7) developing appropriate strategies and evaluating the reasonableness of an answer. During the course of this investigation, we have used these seven components as the framework in which to support both the design and the analysis of data, since it covers the components proposed in previous studies.

Number sense investigations have been conducted mainly with students from primary level (Alsawaie, 2011; Mohamed & Johnny, 2010; Sengul & Gulbagci, 2012; Veloo, 2010, Yang, 2005), while there has been less emphasis on secondary level (Markovits & Sowder, 1994; Reys & Yang, 1998; Veloo, 2010). Researchers have concluded that most students tend to use rules and algorithms to solve numerical problems and they have great difficulties with estimation. Results also show that there is a relationship between number sense and good mental computation, against the absence of relationship with a good written computation.

Investigations analyzing classroom methodologies which promote number sense indicate that an adequate instruction produces more meaningful learning than traditional methodologies, although this improvement is achieved in a long term (Markovits & Sowder, 1994; Yang, Hsu & Huang, 2004; Veloo, 2010). Teachers' knowledge is key in the development of number sense (Alsawaie, 2011). However, limited skills associated to the number sense in both in-service and pre-service primary teachers have been found (Tsao, 2004). Alajmi & Reys (2007) studied how secondary teachers from Kuwait evaluated the reasonableness of a numerical answer. According to these teachers, an answer is reasonable if the computations done are correct and the response is not far from the exact result. Despite their recognition of the utility of estimation in everyday life, they consider that learning must focus in the acquirement of other kind of numerical knowledge. Yang et al. (2009) focused on the evaluation of number sense in pre-service primary teachers in Taiwan. They found out that most of the participants based their reasoning on rules. The authors indicate that these results are a consequence of the mathematics education program in Taiwan. This study is highly relevant to our research because it has been the starting point for the design and the analysis of data which will be discussed below, making a comparison between both studies.

Objectives

Studies on mathematics teachers have shown how their knowledge of content influences the choices they make about classroom practice. This research analyzes the number sense of pre-service secondary mathematics teachers, as part of the knowledge they need to master. It was carried through a sample of students of the Degree in Mathematics from the University of La Laguna (Tenerife, Spain). The objectives of this research are to (1) analyze the number sense strategies of pre-service secondary teachers, whether the high mathematical training of these students is manifested in strategies numerically richer, and to (2) compare the success and strategies of these students with pre-service primary teachers, through data from a previous study (Yang et al., 2009).

Methodology

Participants

This research was conducted with 67 students of the Degree in Mathematics from the University of La Laguna (Tenerife, Spain), whose mathematical background corresponds to high-level mathematics in analysis, numerical analysis, algebra, geometry, topology, and statistics. In Spain, the main profile of secondary mathematics teachers comes from people who have done this degree or at least they have a similar training.

Number Sense Test

A written test was developed with ten items (initial test), nine from Yang et al. (2009) and one from Yang (2005). There were selected items which a priori were able to be solved using the seven components of number sense. The relationship between each item and the components is detailed in Table 1. We are aware that there are several options to solve the items, thus it may result to other types of arguments or the use of several components simultaneously.

The complete written test was used for a first study in which the number sense of pre-service secondary teachers was analyzed (Almeida, Bruno & Perdomo-Díaz, 2014). The second study, presented in this paper, is the comparison of the results with a group of pre-service primary teachers, but Yang et al. (2009) only present the results for five

Table 1 Relation between items and number sense components

Item	1	2	3	4	5	6	7	8	9	10
Components	1 2 3	2 3 4	3 6	6 7	6 7	6	4 5 6	2 3	3 5 6	2 3

of the items (items 1 to 5 from Table 1), thus those will be the items to be compared, since we have no more information about the remaining. In the comparison, we focus on success and strategies used by both groups.

The test was given to each student on separate sheets and with the following instructions: (1) Solve the items without performing exact calculations; (2) Justify your answer and explain what you thought to get the solution; and (3) You will have 3 min for each question. The restriction on time was established to follow the same condition as Yang et al. in order to be able to make comparison.

For the analysis, responses were coded with 1 (correct) or 0 (otherwise), and the type of reasoning used to justify was distinguished by the following categories: number sense based (NS), when using one or several components of the number sense framework; rule based (RB), if they only made use of algorithms or memorized rules; partially number sense based (PNS), if they combined the use of components of number sense with memorized rules and/or algorithms; high-level mathematical reasoning (HM), when using reasoning of a higher level than required (generalization, sequences, limit…); wrong reasoning (WR), when using mathematically incorrect arguments; unclear or no explanation (Unclear), students did not provide sufficient grounds to identify what reasons led them to the answer(s) or if there was no justification; and blank (B), if there was no answer to the question. At first, the categories used by Yang et al. (2009) were considered. During the coding phase, the need to add two new categories arose: *blank* and *high-level mathematical reasoning*. In those responses categorized as *NS* based, we also identified which component of the seven presented in "Background for the Study" section was used. The process of correction was independently performed by three researchers. Subsequently, they shared their results reaching a consensus in the few cases which they presented any difference. This triangulation provides validity to the qualitative study performed. Furthermore, to strengthen the validity and reliability, it was decided to conduct an interview which allowed confirming and delving into the results of the written test.

Interviews

For this second study, focused in the comparison between the responses of pre-service primary and secondary teachers, the need to delve into some unexpected results arose: the high number of incorrect responses from some students of the Degree in Mathematics, the unclear reasoning of others or, at the other extreme, the use of mathematical reasoning with a higher level than required to respond to the items. For this reason, we interviewed three students who met any of these profiles; we will name them as S1, S2, and S3. S1 and S2 were selected because around half of their test answers were classified as *RB*. In the other half of the test, they had some differences: S1's answers were classified as *WR* while S2's were *Unclear* or *B*. The goal was to see if they were able to follow *NS* arguments, when they were not constrained by time resolution. S3 showed a good use of *NS* but also advanced mathematics arguments, higher than required. The objective was to know if he could solve the items with simpler *NS* reasoning. The interviewed students were again presented to the same items (repeated test), focused in the five analyzed, after 1 year with the same instructions, except for the condition of time (they had unlimited time). After the resolution of the repeated test, they were asked to verbally explain the reasoning followed in each item. When they did not use *NS*, they were requested to think about other arguments.

Results

Results will be presented in two sections. The first one shows the comparison between the responses of pre-service primary and secondary teachers (responses of 67 students) in the 5 items exposed in Yang et al. (2009). The second one presents the results of the interviews (3 selected students).

Pre-service Primary Teachers Versus Pre-service Secondary Teachers

The answers of our 67 students of the Degree in Mathematics will be compared with the results showed by Yang et al. (2009), who conducted a study with 280 university students with a previous training in mathematics, "Basic Mathematics" and "Mathematics Teaching."

Item 1. Sort fractions and decimals.

> Tomás walked 0.4828 km, Juan walked $\frac{13}{38}$ km, María walked $\frac{8}{15}$ km, Julia walked $\frac{17}{16}$ km, David walked 0.966 km, and José walked $\frac{7}{29}$ km. Sort the distances traveled from farthest to closest.

The purpose of item 1 is the use of benchmarks to compare the given numbers without expressing the measures as decimal number, or fractions with common denominator. This item has the biggest difference on success, 42 % in pre-service secondary teachers, compared to 91 % from primary teachers. On the other hand, the opposite occurs when analyzing the responses of *NS*, where pre-service primary teachers did not get any response, unlike the pre-service secondary teachers showed 43 % of responses (see Table 2).

Table 2 Percentages item 1

	Pre-service secondary teachers			Pre-service primary teachers		
Reasoning	Right	Wrong	Total	Right	Wrong	Total
NS	30	13	43	0	0	0
PSN	0	0	0	23.9	0	23.9
RB	7.5	7.5	15	63.9	6.8	70.7
WR	–	12	12	–	0	0
Unclear	4.5	19.5	24	3.2	2.2	5.4
B	–	6	6	–	–	–
Total	42	58		91	9	

Note: Data in columns 5–7 are from Yang et al. (2009)

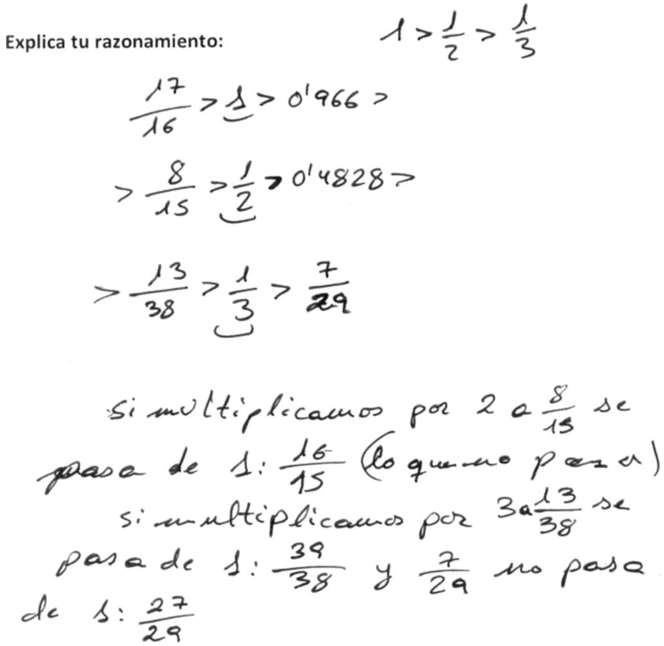

Fig. 1 Pre-service secondary teacher's response to item 1

If we multiply $\frac{8}{15}$ *by 2, it passes from 1,* $\frac{16}{15}$.
If we multiply $\frac{13}{38}$ *by 3 it passes from 1,* $\frac{39}{38}$ *, and* $\frac{7}{29}$ *does not pass 1,* $\frac{27}{29}$.

Students were expected to use the benchmarks $1, \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ to order measures (components 2 and 3), without having to convert fractions into decimals or otherwise, as follows: $\frac{17}{16} > 1 > 0.966 > \frac{8}{15} > \frac{1}{2} > 0.4828 > \frac{13}{38} > \frac{1}{3} > \frac{1}{4} > \frac{7}{29}$. The group of pre-service secondary teachers used this strategy but also other with less benchmarks (see Fig. 1). The students from Yang et al. (2009) used 1 and $\frac{1}{2}$, but neither $\frac{1}{3}$ nor $\frac{1}{4}$, in which case they used algorithms to switch fractions to decimal and determine the order. These answers were classified as *PNS* (24 % of responses). On the other hand, rule-based reasoning was of two types: express all measurements as decimals or as fractions, dominating the first one. Again, there are major differences between both groups about the use of these strategies: 70.7 % of pre-service primary teachers compared to 15 % of secondary. In addition, 12 % of replies from pre-service secondary teachers were classified as *WR*. This shows a low level in the responses for students with a high mathematic level, for example, the answer of a student states: "I sort them by the distance between the numerator and denominator. When the difference is larger, the number is smaller."

Item 2. Colored ribbons.

Victoria and María used colored ribbons for a class assignment. Victoria used $\frac{30}{31}$ m and María $\frac{36}{37}$ m. Who used more ribbon?

Table 3 Percentages item 2

	Pre-service secondary teachers			Pre-service primary teachers		
Reasoning	Right	Wrong	Total	Right	Wrong	Total
NS	34	4.5	38.5	34.3	0	34.3
RB	15	12	27	46.4	4	50.4
HM	3	0	3	–	–	–
WR	–	3	3	14.6	0.7	15.3
Unclear	6	18	24	0	0	0
B	–	4.5	4.5	–	–	–
Total	58	42		95.3	4.7	

Note: Data in columns 5–7 are from Yang et al. (2009)

In this item, students are expected to compare $\frac{30}{31}$ and $\frac{36}{37}$ considering 1 as benchmark (component 3) and comparing the residues of the fractions (component 2), $\frac{1}{31}$ and $\frac{1}{36}$, as a decomposition of the unit (component 4).

This item obtained the best results in Yang et al. (2009). The success was considerably higher than pre-service secondary teachers who obtained 58 % against 95.3 % (see Table 3) but pre-service primary teacher's arguments were again mainly *RB* (50.4 %) while pre-service secondary teachers chose *NS* reasoning (38.5 %).

Responses classified as *NS* based by Yang et al. (2009) were those expected a priori which compare the residuals of fractions (previously explained). This reasoning is also

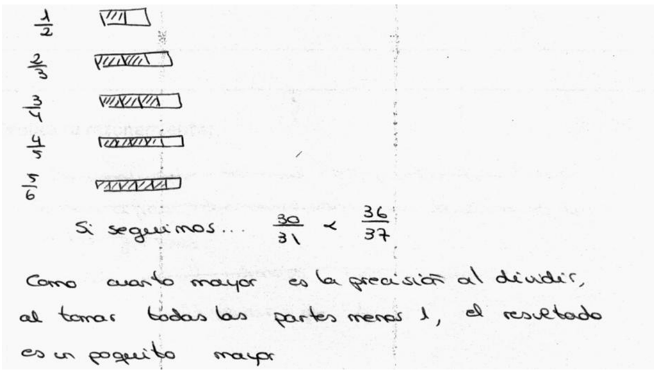

Fig. 2 Pre-service secondary teacher's response to item 2

As the higher the accuracy at divide, to take all parts except 1, the result is slightly higher.

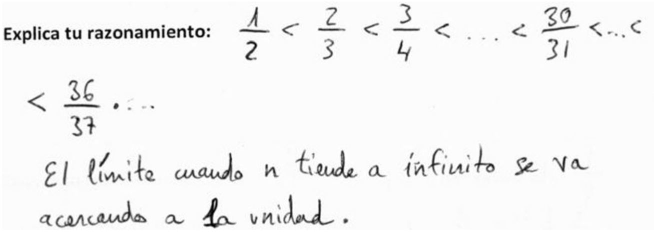

Fig. 3 Pre-service secondary teacher's response to item 2

The limit when n tends to infinity is getting closer to the unit.

found in the sample of pre-service secondary teachers, but the variety of *NS* strategies for the pre-service secondary teachers is wider, for example, the student of Fig. 2 uses different fractions easier to handle as benchmarks (component 3) and the graphic representation (component 5) of them to find a pattern and decide which fraction is bigger.

There were even responses from a higher mathematics level using the concept of sequences, the limit (see Fig. 3) or the monotony (see Fig. 4).

These students related the fractions with a numerical sequence. In the first case, applying the concept of limit, and the second proving the sequence was increasing. This type of reasoning did not arise in pre-service primary teachers who used mainly *RB* strategies, (reduce to a common denominator or perform an algorithm to express them as decimals), which were similar to those of pre-service secondary teachers classified in the same category.

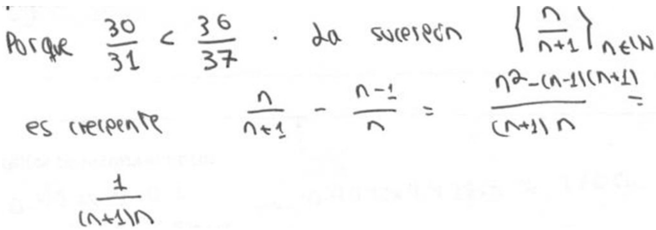

Fig. 4 Pre-service secondary teacher's response to item 2

Because $\frac{30}{31} < \frac{36}{37}$. *The sequence* $\left\{\frac{n}{n+1}\right\}_{n\in\mathbb{N}}$ *is increasing* $\frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2-(n-1)(n+1)}{(n+1)n} = \frac{1}{(n+1)n}$

Table 4 Percentages item 3

	Pre-service secondary teachers			Pre-service primary teachers		
Reasoning	Right	Wrong	Total	Right	Wrong	Total
NS	30	3	33	24.6	0	24.6
PSN	1.5	7.5	9	2.5	12.5	15
RB	0	43	43	1.8	52.2	54
WR	–	1.5	1.5	–	0	0
Unclear	4.5	9	13.5	2.5	3.9	6.4
Total	36	64		31.4	68.6	

Note: Data in columns 5–7 are from Yang et al. (2009)

Item 3. Place the decimal point.

> Carlos used a calculator to perform the operation $0.4975 \times 9428.8 = 4690828$ but he forgot to write down the decimal point. Use estimation to find the place of the decimal.
>
> (a) 46.90828, (b) 469.0828, **(c) 4690.828**, (d) 46908.28, (e) I cannot choose the answer without doing the exact calculation.

The purpose of this item is to analyze the use of benchmarks and the understanding of the effect of operations on numbers to estimate a product of decimal numbers.

This is the only item in which the pre-service secondary teachers slightly exceeded to those of primary (see Table 4) in terms of success, although the percentages were very low for both groups, less than 40 %. Students were expected to choose the correct answer (c), using arguments associated with good *NS* as:

A. 0.4975 *is near to* 0.5 *then*, $0.4975 \times 9428.8 \cong 0.5 \times 9428.8 \cong \frac{9400}{2} = 4700$.
B. *Approaching* 9428.8 *by* $10{,}000$, *a good estimation of the product is* $0.4975 \times 10{,}000 = 4975$.

Both strategies make use of the components 3 and 6 of the framework. The only difference is that the benchmarks are different in each case and in A, the component 6 is related with the relation between multiplication and division, while in B, it is related with the relative effect of operation. These kinds of strategies were used by both groups of pre-service teachers as occurs with those coded as *PNS* or *RB*. There is a high percentage of students who used the rule of "shift the decimal point," i.e., to relate the number of decimal of the result with the number of decimal places in each factor. It is the case of a group of students who chose the answer 46.90828 and stated: "0.4975 has four decimals; 9428.8 has one decimal; in total five decimals (for the result)." This type of error reflects a weak understanding of place value and the decimal point, which is hidden when the work is done with written algorithms. In this case, this rule can be misleading if one considers the outcome of the final product ends in two zeros. In fact,

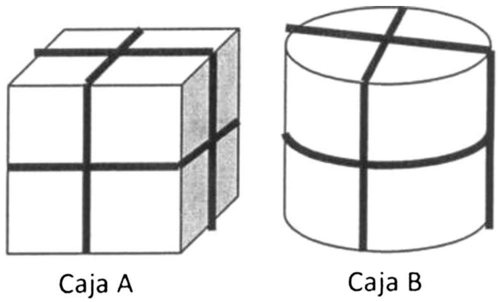

Fig. 5 Boxes item 4

an example of response from a *PNS* strategy in both groups was the following: "The result of the multiplication must be five decimal places; however, 75 × 8 = 600, the last two digits must be eliminated. Therefore, the answer is 4690.828." The results of this item showed that some students did not check the result and did not evaluate whether the result chosen is reasonable.

Item 4. Boxes.

> We have two gift boxes and we want to wrap them with tape as shown in the picture. A box is a cube of side 10 cm. The height and diameter of the case B are also 10 cm. What box needs more tape? (see Fig. 5).

In this item, it is intended that students *develop appropriate strategies for evaluating the reasonableness of an answer* (component 7), considering that it is only needed to compare the central ribbon around the figures, since the other measurements are equal. Success on this item is lower for pre-service secondary teachers than for primary with a difference about 20 % (see Table 5). Moreover, regarding the type of strategy, pre-service secondary teachers' answers are distributed mainly in *NS* (38.5 %), *WR* (19.5 %), and *unclear* (24 %). In the case of pre-service primary teachers, the type of

Table 5 Percentages item 4

	Pre-service secondary teachers			Pre-service primary teachers		
Reasoning	Right	Wrong	Total	Right	Wrong	Total
NS	35.5	3	38.5	21.4	0	21.4
PSN	0	3	3	0	0	0
RB	3	6	9	55.4	12.5	67.9
WR	–	19.5	19.5	–	7.5	7.5
Unclear	21	3	24	2.5	0.7	3.2
B	–	6	6	–	–	–
Total	59.5	40.5		79.3	20.7	

Note: Data in columns 5–7 are from Yang et al. (2009)

reasoning is concentrated in two groups, *NS* (21.4 %) and *RB* (67.9 %). Again, *NS*-based reasoning prevail for pre-service secondary teachers while *RB* reasoning dominate pre-service primary teachers' reasoning.

The reasoning classified as *NS* that predominated in both groups corresponds to answers as showed in Fig. 6. These students used the components 6 and 7 of *NS* to develop an appropriate strategy and understanding the effect of operations in numbers.

Other strategy of *NS* was given, although taking elements of the spatial or geometric sense: "Since an edge of box A is the diameter of box B, then we can introduce box B in box A, and not otherwise. So we can see the dimensions in the cube are bigger than in box B." We have classified this kind of strategy as using component 7 since they showed flexibility finding an appropriate strategy to the situation; even though we are aware, it seems to be more a geometrical strategy than a numerical one. The only strategy classified as *RB* in both groups is the one calculating the total amount of ribbon needed by each box.

Wrong arguments are surprising and away from those which were expected for pre-service secondary teachers who have been trained in advanced mathematics. For example, there were found responses of those students which relate the amount of ribbon needed with the volume of the boxes, or compare the number of faces of both boxes by saying: "a cylinder has two faces and the cube has six, therefore the second one needs more ribbon."

Item 5. Bottles of water.

> A bottle of water of 600 ml costs 18 cents, whereas a bottle of water of 1500 ml costs 35 cents. Estimate which bottles is more profitable.

This item was expected to be solved using proportionality and estimation.

Success in the responses (see Table 6) is similar for both samples of students (85 and 87.9 %) but there is an important difference between the two groups of students: most pre-service secondary teachers' correct answers were *NS* based (68.5 %) while pre-service primary teachers' correct answers were mostly *RB* (60 %).

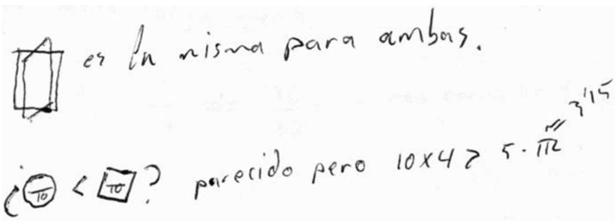

Fig. 6 Pre-service secondary teacher's response to item 4

Is the same for both.
Similar but $10 \cdot 4 > 5 \cdot \pi$

Table 6 Percentages item 5

	Pre-service secondary teachers			Pre-service primary teachers		
Reasoning	Right	Wrong	Total	Right	Wrong	Total
NS	68.5	1.5	70	25.4	0	25.4
PSN	1.5	0	1.5	0	0	0
RB	9	3	12	60	10.3	70.3
WR	–	4.5	4.5	–	0	0
Unclear	6	1.5	7.5	2.5	1.8	4.3
B	–	4.5	4.5	–	–	–
Total	85	15		87.9	12.1	

Note: Data in columns 5–7 are from Yang et al. (2009)

The ones classified as *NS* were those whom based their justification on proportionality, without calculating the exact price of the same amount of water for both bottles (*RB*). It means the use of component 6 from the framework of *NS*. The most common example in both studies was the following: “The small bottle of 600 ml costs 18 cents, then 1200 ml will cost 36 cents. The large bottle of 1500 ml only costs 35 cents, then the big bottle is more profitable.” Other strategy classified as *NS* was the following given by a student who supposed the same price for both bottles (the price of the small one) to see which would be the price of the big bottle (see Fig. 7).

This student made use of two components: component 4 to decompose quantities and component 6 to use proportionality.

A third strategy classified as *NS* was also found. In this case, the student used an appropriate strategy to evaluate the reasonableness of his answer (component 7) comparing the difference in quantity of water with the difference of price, i.e., “how much more water would I obtain with the biggest bottle and at what price?” The student established: “By a difference of 17 cents, you obtain 900 ml more. Hence, the bottle of 1500 ml is more profitable since 600 ml cost 18 cents and 900 ml cost 17 cents.”

In other category, we found the *RB* strategies. These are those which calculate the price of the milliliter for each bottle or the number of milliliters per cent.

Discussion About Pre-service Teachers’ Answers to the Initial Test

Overall, results of the five items analyzed in this work, and extending to the rest of the items, show success in the answers lower in the case of pre-service secondary teachers than for primary teachers (except for item 4). However, from the point of view of the type of reasoning used, there is a different interpretation of the results as the percentage of responses based on *NS* is better in pre-service secondary teachers in all items. The opposite occurs in *RB* reasoning, being higher percentages for students in Taiwan. We also note that *NS* strategies of the secondary group are more varied than those of the

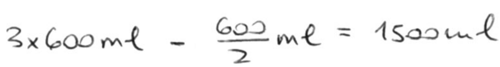

Fig. 7 Pre-service secondary teacher’s response to item 5

group of primary, reaching in some cases to a higher level than required. Table 7 summarizes the differences in the success and the type of reasoning used by both groups.

Pre-service secondary teachers have shown, in general, that they have other strategies, different from *RB*. This has made us think that, since they are aware that there is another way to solve the items, for example with *NS*, they followed the instruction of "do not use algorithms or rules" and sometimes they tried to find others strategies which led them to give responses classified as *Unclear*. On the other hand, pre-service primary teachers showed a worse performance in the use of *NS*. However, their success was better than the group of secondary, although they disobeyed the instructions using *RB* reasoning.

Case Studies

Once the results of the written test were analyzed in deep, some additional questions arose to our mind: (i) Could pre-service secondary teachers perform this activities in various ways?; (ii) Would they answer differently without time limitation?; (iii) About the students who solve an activity using rules, or with wrong arguments, do they know *NS* strategies?; and (iv) About those students who solve tasks with advanced mathematical justifications, could they give simpler reasoning more suited to the task level? To respond to these questions, we interviewed three pre-service secondary teachers, taking into account the nature of their answers. The most relevant features of these interviews in the five items analyzed in this work is briefly shown, especially those in which there were changes. As a reminder, the meaning of codes used such as number sense (NS), rule based (RB), partially number sense (PNS), high-level mathematical reasoning (HM), wrong reasoning (WR), unclear/no explanation (Unclear), and blank (B), respectively, is presented again.

Case 1: Student S1

In the completed initial test, S1 answered correctly only three items, and seven items were incorrect. Regarding to strategies, only one item was classified as NS, one as PNS, four as RB, three as WR, and one as B. This student was interviewed because of his low rate of success and the lack of *NS* strategies. Table 8 shows the results of the initial test, repeated test, and interview for the five items which are contrasted in this paper. In

Table 7 Summary of the results of both groups

Pre-service teachers	Percentages								
	Correct					Incorrect			
	NS	PNS	BR	HM	Other	NS	PNS	RB	Other
Primary	21.1	5.3	45.5	0	5.1	0	2.5	17.1	3.3
Secondary	39.7	0.6	6.9	0.6	8.3	5.1	2.1	14.3	22.4

Note: Data in row 4 are from Yang et al. (2009)

the case of item 5, where there was no change from the initial interview to the final interview, an analysis of it is not shown. We have shadowed the cases where there was a change from the previous step. S1 showed slight changes in the strategies from the initial to repeated test, only in items 1 and 4. He only was able to find a new *NS* strategy in item 2, at the interview.

Item 1

Item 1 was answered in both tests changing fractions to decimals. The first time with miscalculations and the second time correctly. The student argued that there was no another way to order fractions without changing them into their decimal expression because fractions were complex. We suggested drawing some kind of graph but he did not know how to do it.

Item 2

Both at the initial and at the repeated test, S1 justified item 2 expressing fractions into their decimal expression. During the interview, he said he did not know another way to do it. However, when he was asked by the interviewer about whether he could apply what he knew about fractions, he compared both ribbons explaining the idea of part-whole (components 1 and 2); hence, it was observed he could use this strategy of *NS*, although he showed he was more confident with the first strategy.

Item 3

In the two applications of the test, S1 used the rule of "shift the decimal point" in item 3, choosing the wrong answer 46.90828. When the interviewer suggested finding another strategy, he insisted he did not know another strategy since it was hard to do mental computation.

Item 4

S1 did not answer correctly the item 4 in any of the, twice he responded, test. The explanations given in the interview suggest the use of visual intuitions but mathematically weak reasoning. For example, he said: "I think it takes more tape," "Box A needs

Table 8 Student S1 responses to the five items

	Initial test		Repeated test		Interview	
Item 1	0	RB	1	RB	1	RB
Item 2	1	RB	1	RB	1	NS
Item 3	0	RB	0	RB	0	RB
Item 4	0	WR	0	Unclear	0	Unclear
Item 5	1	NS	1	NS	1	NS

more tape because a square has more sides, therefore requires more ribbon," "I cannot do it because I do not remember the formula for the perimeter of the circle."

Case 2: Student S2

The results of S2 in the initial test were low in success and in the use of *NS*, he only obtained four correct items out of ten. Strategies used in each item were 1 with SN, 4 RB, 3 *unclear*, and 2 *B*. Table 9 shows the specific results of the items we are comparing, which shows slight changes in the strategies used during the initial and the repeated test. S2 answered in the same way the two attempts, except in item 4. However, during the interview, he found new strategies in three items (classified as *NS*). Furthermore, the results of items 1 and 4 are shown. In item 5, his answer was NS based for both tests and the interview with strategies as those seen in previous section.

Item 1

In the initial test, the student answered correctly expressing all the numbers in their decimal form (algorithm of division). In the repeated test, he also used a *RB* method but with a different rule which states: "cross-multiply the numerator and denominator of the two fractions to see which result is greater", i.e., $\frac{a}{b} > \frac{c}{d}$ if $a \cdot b > b \cdot c$. When asked if he knew another method not based on rules, he used a *NS* strategy using the concept of part-whole of fractions (components 1 and 2), reaching the correct answer.

Item 2

S2 answered this item in the same way both at the initial and at the repeated test. He ordered the fractions using a memorized rule: "cross-multiply the numerator and denominator of the two fractions" (as in item 1). When he was asked whether he knew another way to respond without rules, he associated the size of the parts into which the unit is divided (components 1 and 2), and compared the two fractions drawing two segments (in detailed, see Fig. 8).

Item 3

The student answered similarly in both occasions, that he replied the test with the rule of "shift the decimal point." When he was asked for another way of reasoning, he used

Table 9 Student S2 response to the five items

	Initial test		Repeated test		Interview	
Item 1	1	RB	1	RB	1	NS
Item 2	1	RB	1	RB	1	NS
Item 3	0	RB	0	RB	1	NS
Item 4	0	B	1	Unclear	1	Unclear
Item 5	1	NS	1	NS	1	NS

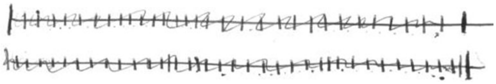

Fig. 8 Student S2 response to item 2

the benchmark 0.5 (component 3) and applied the effect of multiplying by $\frac{1}{2}$ (component 6), choosing the right answer. With two different responses, he decided that the last was the correct one and assessed that the initial response was unreasonable. On the other hand, he ratified: "the rule of shift the decimal point is right, but it may be that the rule has an exception."

Item 4

In this item, S2 shifted from *B* to *Unclear* from the initial test to the repeated test. The second time, he responded correctly, but with poor arguments. In the interview, the student argued that box A needed more ribbon than box B using intuitive ideas as: "It seems to me it takes more tape." He did not make use of any mathematical argument to prove it (neither formula nor graphical development). In fact, when he was asked to try to make the development of the figure, he had difficulties to imagine the position of the ribbons in the picture.

Case 3. Student S3

S3 was characterized in its initial test by a high use of *NS*. Specifically, this student answered all ten items correctly with the following classification: 1 as HM, 6 as NS, and 3 as Unclear. Table 10 shows that the student did not change his responses and arguments from the initial to the repeated test using NS strategies already described in the previous section. When he was asked for a different reasoning, he found different ones in item 2 and 5 which are shown below.

Item 2

In item 2 of the initial test, S3 associated fractions to a sequence which he proved it was increasing (see Fig. 4). He used the same argument in the repeated test. When asked if he knew a way to explain it closer to the insights of a secondary student, he stated the strategy which compares the residuals of fractions previously explained (components 2, 3, and 4, see description item 2 in previous section). He

Table 10 Student S3 responses to the five items

	Initial test		Repeated test		Interview	
Item 1	1	NS	1	NS	1	NS
Item 2	1	HM	1	HM	1	NS
Item 3	1	NS	1	NS	1	NS
Item 4	1	NS	1	NS	1	NS
Item 5	1	NS	1	NS	1	BR

explained that the last answer was not “the first thing which came into his head” and the first time he read the task he related these fractions with the sequence arguing he tried to solve the exercise for the general case, because “that’s the way I work every day.”

Item 5

This student made use of a correct *NS* in all items but in the case of item 5, he changed to a *RB* argument. He first used the proportional reasoning which most of their colleagues did (component 6, see description of item 5 in previous section). When he was asked whether he knew another way to find out the answer, he calculated the price of the milliliter for each bottle. This new reasoning was not the first thought of this student, but he showed the ability to use different strategies and evaluate which one was more appropriate.

Interview Discussion

In the items analyzed, students tended to maintain the type of reasoning they used the first time, although they had unlimited time in the repeated test. It was observed that the students had a preference or tendency toward a kind of reasoning. Interviews show that the first strategy given by students is not always the only strategy they know, since in some cases, they changed their strategy when asked to find a new one.

S1 tended to use rules and was not able to expand his strategies to others related to *NS*. When he was asked to find new strategies, he stayed with those of *RB* and seemed to have forgotten basic mathematics concepts useful to solve the tasks with arguments closer to *NS* as, for example, an adequate mental computation. He is shown as a student with a lack of *NS*, insecure with responses, and with poor flexibility to change the method because of the absence of elementary mathematical concepts; S2 tended to use *RB* methods in their responses and, although he demonstrated knowing strategies of *NS*, he showed he was more confident using rules. The assertion “the rule of shift the decimal point is right, but it could be that the rule has an exception” showed a preference for the use of rules. We are facing a student with a low tendency to the use of *NS* and very insecure using other strategies; S3 answered the test with good mathematics reasoning, all well justified. When he was asked for new strategies, he changed to *RB* method, and in the case of the high-level mathematical reasoning, he found a strategy with basic mathematical concepts. His justifications showed the search for solutions for the generalized problem, a consequence of his academic level. S3 also demonstrated security in their mathematical explanations and flexibility to change strategies.

The case study opens an interesting field of research which determines profiles of *NS* for pre-service teachers, both primary and secondary. The interviews carried out in this study have shown that the knowledge of teachers could be closer to *NS* than the one used in written tests, although a lack of flexibility to accept new methods is also shown. This attitude could be an obstacle to the development of these abilities in their future students.

Conclusion

Generally, pre-service secondary teachers that participated in this research presented lower success than pre-service primary teachers of Yang's study answering questions related with numbers in the sense that they had more incorrect answers than Yang's study primary teachers. Although pre-service secondary teachers' everyday mathematical activity is far from this type of work, a higher success rate and a greater use of *NS* resources would be expected. Otherwise, pre-service secondary teachers made more use of *NS* than pre-service primary teachers, and the replies were more varied. We have found a variety of answers among which we find very low-level responses, *WR*, or answers with a higher level than required, leading to a generalization of mathematics itself. This last case did not appear in Yang et al. (2009). This shows that the higher the mathematical training is, the more variety of strategies they have to face in this kind of situations. This result agrees with Dowker, Flood, Griffiths, Harris & Hook (1996), comparing professional mathematicians, accountants, and college students in psychology and English, making estimations of operations with decimals. Mathematicians outperformed the other groups in the variety of strategies. However, unlike the pre-service secondary teachers of our study, professional mathematicians showed themselves as good estimators and with high levels of accuracy. Pre-service primary teachers answered all items with a high percentage of *RB* responses, which was always the most used reasoning. Also, pre-service secondary teachers used rules and in some cases, they did not reflect on whether results made sense. In other cases, pre-service secondary teachers made miscalculations, maybe caused by the limited time they had to respond the test. Also noteworthy are percentages of responses classified as wrong or unclear. When asked not to use algorithms, we have seen intuitive ideas or basic concepts or procedures with some errors appeared which have not changed during their training in advanced mathematics.

The case study conducted with the repetition of the test and the subsequent interview suggests students know different ways of solving activities but have some inclination toward a way of thinking. It is the case of the rigidity of student who said he did not find another way to solve the activities but using the algorithms. The high background in mathematics which pre-service secondary teachers receive in our country should enable them to use proper *NS* to each activity. We believe it is necessary to receive educational training on this topic, making explicit the importance to promote *NS* from school classrooms. However, if future teachers are inclined to use rules or follow traditional methods, they will hardly encourage the development of *NS* in future high school students.

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