A New Extragradient Method for Strongly Pseudomonotone Variational Inequalities *

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Abstract. This paper proposes a new extragradient solution method for strongly pseudomonotone variational inequalities. A detailed analysis of the convergence of the iterative sequences and the range of application of the method is given.

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1 Introduction

Variational inequality (VI for brevity) is a useful mathematical model which unifies such important concepts in applied mathematics as systems of nonlinear equations, necessary optimality conditions for optimization problems, complementarity problems, obstacle problems, network equilibrium problems. It is well known that VI has a tight connection with fixed point problems. In order to solve a VI, various solution methods have been proposed. For strongly monotone VIs, the projection method is effectively applicable. For monotone VIs, the Tikhonov regularization method (TRM) and proximal point algorithm (PPA) are known as standard solution methods (see [1, Sections 12.1 and 12.2]). In applying the TRM and the PPA to pseudomonotone VIs, one faces with a huge obstacle: the auxiliary problems may be more difficult than the original one in the sense that the given pseudomonotonicity

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can be destroyed completely (see [2, p. 260 and p. 265]). This circumstance obviously gives
a favor to the extragradient method (EGM) suggested by G. M. Korpelevich [3] which pro-
duces a convergence sequence of easily computed iteration points. In [4, 5] we have analyzed
the convergence and the rate of convergence of a modified EGM where the variant stepsizes
are taken from a fixed closed interval of positive real numbers.

This paper proposes a new EGM where the stepsizes form a non-summable diminishing
sequence of positive real numbers. Note that the subgradient method of N. Z. Shor (see
[6, pp. 138-141]) uses such sequences of stepsizes. The new EGM carries on the double-
projection procedure from Korpelevich’s original paper [3] from step to step. The method
works well for strongly pseudomonotone VIs. For pseudomonotone VIs (or even for monotone
VIs) it may produce a disconvergent iteration sequence. In comparison with the cited papers
[3, 4, 5], the main feature of our method is that the knowledge of the Lipschitz constant of
the given vector field is not required.

The remaining part of the paper has three sections. Section 2 gives some preliminaries.
Section 3 describes the new extragradient method for strongly pseudomonotone variational
inequalities and establishes the convergence and the rate of convergence of the iterative
sequences. The range of application of this solution method is studied in Section 4.

2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle . , . \rangle$ and induced norm $\| . \|$, and $K$ a
nonempty closed convex subset of $H$. For each $u \in H$, there exists a unique point in $K$, denoted by $P_K(u)$, such that

$$\|u - P_K(u)\| \leq \|u - v\| \quad \forall v \in K$$

(see [7]). We now recall some properties of the metric projection $P_K : H \rightarrow K$.

**Proposition 2.1** The following statements are valid.

(a) For any $u \in K$ and $v \in H$,

$$\langle u - P_K(u), v - P_K(u) \rangle \leq 0. \quad (1)$$

(b) For any $u \in H$ and $v \in K$,

$$\|v - P_K(u)\|^2 \leq \|u - v\|^2 - \|u - P_K(u)\|^2. \quad (2)$$
Property (a) can be found in [7]. Property (b) in a finite dimensional setting can be found in [1]. The proof for the infinite dimensional case is identical.

Let $F : K \to H$ be a mapping. The variational inequality problem $VI(K, F)$ defined by $K$ and $F$ is that of finding a point $u^* \in K$ such that

$$\langle F(u^*), u - u^* \rangle \geq 0 \quad \forall u \in K. \quad (3)$$

The solution set of the problem is abbreviated to $\text{Sol}(K, F)$.

If there exists a constant $L > 0$ such that

$$\|F(u) - F(v)\| \leq L\|u - v\| \quad \forall u, v \in K, \quad (4)$$

then $F$ is said to be a Lipschitz continuous mapping.

One often considers (3) with $F$ possessing a certain monotonicity property.

**Definition 2.1** (See [8]) The vector field $F$ is said to be

(a) **strongly monotone on** $K$ if there exists $\gamma > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \gamma \|u - v\|^2 \quad \forall u, v \in K;$$

(b) **monotone on** $K$ if

$$\langle F(u) - F(v), u - v \rangle \geq 0 \quad \forall u, v \in K;$$

(c) **strongly pseudomonotone on** $K$ if if there exists $\gamma > 0$ such that

$$\langle F(u), v - u \rangle \geq 0 \implies \langle F(v), v - u \rangle \geq \gamma \|v - u\|^2$$

for all $u, v \in K$;

(d) **pseudomonotone on** $K$ if

$$\langle F(u), v - u \rangle \geq 0 \implies \langle F(v), v - u \rangle \geq 0$$

for all $u, v \in K$.

The implications (a)$\implies$(b), (a)$\implies$(c), (c)$\implies$(d) and (b)$\implies$(d) are evident. It is well known and easily verified that a strongly pseudomonotone VI can have at most one solution.
3 A New Extragradient Method

Inspired by the extragradient method of Korpelevich [3] and subgradient method of Shor (see [6, pp. 138-141]), we introduce the following solution method for VI($K,F$).

Algorithm 3.1

Data: Select an initial point $u^0 \in K$ and a sequence of stepsizes $\{\alpha_k\}_{k=0}^{\infty} \subset \mathbb{R}_+$ with

$$\sum_{k=0}^{\infty} \alpha_k = +\infty, \quad \lim_{k \to \infty} \alpha_k = 0. \quad (5)$$

Step 0: Set $k = 0$.

Step 1: Compute

$$\bar{u}^k = P_K(u^k - \alpha_k F(u^k)), \quad u^{k+1} = P_K(u^k - \alpha_k F(\bar{u}^k)). \quad (6)$$

Step 2: Check $\bar{u}^k = u^k$. If Yes then Stop. Else set $k \leftarrow k + 1$ and go to Step 1.

If the computation terminates at a step $k$, then one puts $u^k = u^k$ for all $k' \geq k + 1$. Thus, Algorithm 3.1 produces an iterative sequence.

The convergence and the convergence rate of $\{u^k\}$ can be stated as follows.

Theorem 3.2 If $F : K \to H$ is Lipschitz continuous and strongly pseudomonotone on $K$, and if $\text{VI}(K,F)$ has a unique solution $u^*$, then the sequence $\{u^k\}$ generated by Algorithm 3.1 converges in norm to $u^*$. Moreover, there exists an index $k_0 \in \mathbb{N}$ such that $\gamma \alpha_k < 1$ for all $k \geq k_0$ and

$$\|u^{k+1} - u^*\| \leq \sqrt{\prod_{j=k_0}^{k} (1 - \gamma \alpha_j)} \|u^{k_0} - u^*\|, \quad (7)$$

where $\gamma > 0$ is the strong pseudomonotonicity constant of $F$. In addition,

$$\lim_{k \to \infty} \sqrt{\prod_{j=k_0}^{k} (1 - \gamma \alpha_j)} = 0. \quad (8)$$

Proof. Let $L > 0$ be such that (4) is satisfied. By (2) we have

$$\|v - P_K(u)\|^2 \leq \|u - v\|^2 - \|u - P_K(u)\|^2, \quad \forall v \in K, \ \forall u \in H.$$  

Substituting $v = u^* \in K, u = u^k - \alpha_k F(\bar{u}^k)$ into the last inequality and using the relation $u^{k+1} = P_K(u^k - \alpha_k F(\bar{u}^k))$ in (6) yields

$$\|u^* - u^{k+1}\|^2 \leq \|u^k - \alpha_k F(\bar{u}^k) - u^*\|^2 - \|u^k - \alpha_k F(\bar{u}^k) - u^{k+1}\|^2$$

$$= \|(u^k - u^*) - \alpha_k F(\bar{u}^k)\|^2 - \|(u^k - u^{k+1}) - \alpha_k F(\bar{u}^k)\|^2$$

$$= \|u^k - u^*\|^2 - \|u^k - u^{k+1}\|^2 + 2\alpha_k \langle F(\bar{u}^k), u^* - u^{k+1} \rangle. \quad (9)$$
Since $u^* \in \text{Sol}(K, F)$, for all $u \in K$ it holds that $\langle F(u^*), u - u^* \rangle \geq 0$. Thus by the strong pseudomonotonicity property of $F$ we have $\langle F(u), u - u^* \rangle \geq \gamma \|u - u^*\|^2$ for all $u \in K$. For $u = \bar{u}^k \in K$, the latter inequality implies that $\langle F(\bar{u}^k), u^* - \bar{u}^k \rangle \leq -\gamma \|\bar{u}^k - u^*\|^2$. Hence

$$\langle F(\bar{u}^k), u^* - u^{k+1} \rangle = \langle F(\bar{u}^k), u^* - \bar{u}^k \rangle + \langle F(\bar{u}^k), \bar{u}^k - u^{k+1} \rangle \leq -\gamma \|\bar{u}^k - u^*\|^2 + \langle F(\bar{u}^k), \bar{u}^k - u^{k+1} \rangle. \quad (10)$$

By (9) and (10),

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \|u^k - u^{k+1}\|^2 + 2\alpha_k \langle F(\bar{u}^k), \bar{u}^k - u^{k+1} \rangle - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2$$

$$= \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 - \|\bar{u}^k - u^{k+1}\|^2 - 2 \langle u^k - \bar{u}^k, \bar{u}^k - u^{k+1} \rangle + 2\alpha_k \langle F(\bar{u}^k), \bar{u}^k - u^{k+1} \rangle - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2$$

$$= \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 - \|\bar{u}^k - u^{k+1}\|^2 + 2 \langle u^k - \bar{u}^k - \alpha_k F(\bar{u}^k), u^{k+1} - \bar{u}^k \rangle - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2.$$

(11)

To estimate the scalar product in the last expression, we break it into the sum of two other scalar products and note that: (i) The nonpositiveness of the first scalar product follows from (1) with $u = u^k - \alpha_k F(u^k), v = u^{k+1}$, and from the relation $\bar{u}^k = P_K(u^k - \alpha_k F(u^k))$; (ii) The second scalar product can be evaluated by the Schwarz inequality and (4). Namely, we have

$$\langle u^k - \bar{u}^k - \alpha_k F(\bar{u}^k), u^{k+1} - \bar{u}^k \rangle = \langle u^k - \alpha_k F(u^k) - \bar{u}^k, u^{k+1} - \bar{u}^k \rangle$$

$$+ \langle \alpha_k F(u^k) - \alpha_k F(\bar{u}^k), u^{k+1} - \bar{u}^k \rangle \leq \alpha_k \langle F(u^k) - F(\bar{u}^k), \bar{u}^{k+1} - \bar{u}^k \rangle \leq \alpha_k \|F(u^k) - F(\bar{u}^k)\| \|u^{k+1} - \bar{u}^k\| \leq \alpha_k L \|u^k - \bar{u}^k\| \|u^{k+1} - \bar{u}^k\|. \quad (12)$$

From (11) and (12) it follows that

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 - \|\bar{u}^k - u^{k+1}\|^2 + 2\alpha_k L \|u^k - \bar{u}^k\| \|u^{k+1} - \bar{u}^k\| - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2.$$

Since

$$2\alpha_k L \|u^k - \bar{u}^k\| \|u^{k+1} - \bar{u}^k\| \leq \alpha_k^2 L^2 \|u^k - \bar{u}^k\|^2 + \|u^{k+1} - \bar{u}^k\|^2,$$

we have

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \|u^k - \bar{u}^k\|^2 + \alpha_k^2 L^2 \|u^k - \bar{u}^k\|^2 - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2$$

$$\leq \|u^k - u^*\|^2 - (1 - \alpha_k^2 L^2) \|u^k - \bar{u}^k\|^2 - 2\gamma \alpha_k \|\bar{u}^k - u^*\|^2.$$

As $\alpha_k \to 0$, there exists $k_0 \in \mathbb{N}$ such that $2\gamma \alpha_k \leq 1 - \alpha_k^2 L^2$ for every $k \geq k_0$. Hence for $k \geq k_0$, we have $\gamma \alpha_k \leq \frac{1}{2} - \frac{1}{2} \alpha_k^2 L < 1$ and

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - 2\gamma \alpha_k \|u^k - \bar{u}^k\|^2 + \|\bar{u}^k - u^*\|^2 \leq \|u^k - u^*\|^2 - 2\gamma \alpha_k \|u^k - \bar{u}^k\| + \|\bar{u}^k - u^*\|^2 \leq \|u^k - u^*\|^2 - 2\gamma \alpha_k \|u^k - u^*\|^2. \quad (13)$$
Summing the inequalities \( \|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \gamma \alpha_k \|u^k - u^*\|^2 \) in (13) from \( k_0 \) to \( n \) over \( k \), we obtain

\[
\|u^{n+1} - u^*\|^2 \leq \|u^{k_0} - u^*\|^2 - \sum_{k=k_0}^{n} \gamma \alpha_k \|u^k - u^*\|^2.
\]

Thus

\[
\sum_{k=k_0}^{n} \gamma \alpha_k \|u^k - u^*\|^2 \leq \|u^{k_0} - u^*\|^2 - \|u^{n+1} - u^*\|^2.
\]

Since \( \|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 \) for \( k \geq k_0 \) by virtue of (13), we can assert that

\[
\gamma \left( \sum_{k=k_0}^{n} \alpha_k \right) \|u^n - u^*\|^2 \leq \sum_{k=k_0}^{n} \gamma \alpha_k \|u^k - u^*\|^2 \leq \|u^{k_0} - u^*\|^2 - \|u^{n+1} - u^*\|^2 \leq \|u^{k_0} - u^*\|^2.
\]

As \( \sum_{k=0}^{\infty} \alpha_k = +\infty \) by (5) and \( \gamma > 0 \), this implies that \( \lim_{n \to \infty} \|u^n - u^*\| = 0 \). Hence the sequence \( \{u^k\} \) converges in norm to \( u^* \).

Let us prove (7). For any \( k \geq k_0 \), from (13) it follows that

\[
\|u^{k+1} - u^*\|^2 \leq (1 - \gamma \alpha_k) \|u^k - u^*\|^2 \leq (1 - \gamma \alpha_k)(1 - \gamma \alpha_{k-1}) \|u^{k-1} - u^*\|^2 \leq \cdots \leq \prod_{j=k_0}^{k} (1 - \gamma \alpha_j) \|u^{k_0} - u^*\|^2.
\]

This implies

\[
\|u^{k+1} - u^*\| \leq \sqrt{\prod_{j=k_0}^{k} (1 - \gamma \alpha_j) \|u^{k_0} - u^*\|}
\]

and establishes (7).

To obtain (8), it suffices to observe that

\[
1 - \gamma \alpha_j \leq \frac{1}{1 + \gamma \alpha_j} \quad \forall j \geq k_0;
\]

therefore

\[
\prod_{j=k_0}^{k} (1 - \gamma \alpha_j) \leq \frac{1}{\prod_{j=k_0}^{k} (1 + \gamma \alpha_j)} \leq \frac{1}{1 + \gamma \sum_{j=k_0}^{k} \alpha_j} \to 0
\]
as $k \to \infty$. \hfill \Box

In the next section, we will show that the strong pseudomonotonicity assumption on $F$ and the two conditions in (5) are essential for the validity of the assertion of Theorem 3.2.

4 Further Analysis

Let us start with analyzing the condition $\sum_{k=0}^{\infty} \alpha_k = +\infty$ described in (5).

**Example 4.1** Put $K = \mathbb{R}$ and $F(u) = u$. It is clear that $F$ is Lipschitz continuous, strongly monotone on $K$, and $\text{Sol}(K, F) = \{0\}$. Choose $u^0 = 1 \in K$ and define the sequence $\{\alpha_k\}$ by setting

$$\alpha_k = \frac{1}{(k + 1)^2} \forall k \in \mathbb{N}. \quad (14)$$

Since $\sum_{k=0}^{\infty} \alpha_k < +\infty$, the first condition in (5) is violated. The iterative sequence $\{u^k\}$ produced by (6) is given by

$$u^{k+1} = P_K(u^k - \alpha_k F(P_K(u^k - \alpha_k F(u^k)))) = u^k - \alpha_k F(u^k - \alpha_k F(u^k)) = u^k - \alpha_k (u^k - \alpha_k u^k) = (1 - \alpha_k + \alpha_k^2) u^k.$$

Since $u^0 = 1$, (14) implies that

$$u^{k+1} = \prod_{j=0}^{k} (1 - \alpha_j + \alpha_j^2) = \prod_{j=0}^{k} \left(1 - \frac{1}{(j + 1)^2} + \frac{1}{(j + 1)^4}\right) \forall k \in \mathbb{N}.$$

Hence $\{u^k\}$ is a decreasing and bounded from below sequence; therefore $\{u^k\}$ is convergent.

Note that

$$\prod_{j=0}^{k} \left(1 - \frac{1}{(j + 1)^2} + \frac{1}{(j + 1)^4}\right) \geq \prod_{j=1}^{k} \left(1 - \frac{1}{(j + 1)^2}\right) \quad \prod_{j=1}^{k} \frac{j(j + 2)}{(j + 1)^2} = \frac{k + 2}{2(k + 1)}.$$

Letting $k \to \infty$, we obtain $\lim_{k \to \infty} u^k = u^*$ for some $u^* \geq \frac{1}{2}$. So the sequence $\{u^k\}$ does not converge to the unique solution of the problem $\text{VI}(K, F)$ under our consideration.
we have seen that the assumption $\sum_{k=0}^{\infty} \alpha_k = +\infty$ cannot be dropped in the formulation of Theorem 3.2.

Next, we will analyze the second condition in (5).

**Example 4.2** Let $K, F, u^0$ be the same as in Example 4.1 and let $\alpha_k = 1$ for all $k \in \mathbb{N}$. Here $\sum_{k=0}^{\infty} \alpha_k = +\infty$, but $\{\alpha_k\}$ does not converge to 0. By the calculations done in Example 4.1 we have that $u^k = 1$ for all $k \in \mathbb{N}$. So $\{u^k\}$ does not converge to the unique solution of VI($K, F$). We have seen that the condition $\lim_{k \to \infty} \alpha_k = 0$ cannot be omitted in the formulation of Theorem 3.2.

Finally, let us show that the assumed strong pseudomonotonicity of $F$ on $K$ is vital for the conclusion of Theorem 3.2. In the next example, $F$ is monotone (so it is pseudomonotone) on $K$, but the sequence $\{u^k\}$ produced by Algorithm 3.1 does not tend to the unique solution of the VI in question.

**Example 4.3** Put $K = \mathbb{R}^2$ and $F(u) = (-u_2, u_1)$ for all $u = (u_1, u_2) \in K$. It is clear that $F$ is Lipschitz continuous and monotone on $K$, and $\text{Sol}(K, F) = \{(0, 0)^T\}$. Let $u^0 = (u_1^0, u_2^0)^T$ be any point in $K \setminus \{(0, 0)^T\}$ and let $\{\alpha_k\}$ be defined by setting $\alpha_k = 1_{k+1} \forall k \in \mathbb{N}$. This sequence satisfies (5). To see that $F$ is not strongly pseudomonotone on $K$ with any constant $\gamma > 0$, it suffices to choose $u = (1, 0)^T$, $v = (2, 0)^T$ and note that $\langle F(u), v - u \rangle = 0$ and $\langle F(v), v - u \rangle = 0$. The iterative sequence $\{u^k\}$ in (6) is given by

\[
\begin{align*}
\begin{cases}
u^0 &= (u_1^0, u_2^0)^T \\
u_1^{k+1} &= \left(1 - \frac{1}{(k+1)^2}\right) u_1^k + \frac{1}{k+1} u_2^k \\
u_2^{k+1} &= \left(1 - \frac{1}{(k+1)^2}\right) u_2^k - \frac{1}{k+1} u_1^k
\end{cases}
\end{align*}
\tag{15}
\]

From (15) we have

\[
\|u^{k+1}\| = \sqrt{(u_1^{k+1})^2 + (u_2^{k+1})^2} = \sqrt{(u_1^k)^2 + (u_2^k)^2} \sqrt{1 - \frac{1}{(k+1)^2} + \frac{1}{(k+1)^4}}
\]

\[
= \|u^k\| \sqrt{1 - \frac{1}{(k+1)^2} + \frac{1}{(k+1)^4}}
\]

\[
= \|u^0\| \sqrt{\prod_{j=0}^{k} \left(1 - \frac{1}{(j+1)^2} + \frac{1}{(j+1)^4}\right)}.
\]

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Passing to the limit as $k \to \infty$, we obtain
\[
\mu = \lim_{k \to \infty} \sqrt[k]{\prod_{j=0}^{k} \left( 1 - \frac{1}{(j+1)^2} + \frac{1}{(j+1)^4} \right)} \geq \frac{\sqrt{2}}{2}.
\] (16)

(See the arguments given in Example 4.1.) Hence $\{u^k\}$ does not converge to the unique solution of $VI(K, F)$. We are going to prove that the set of the cluster points of the sequence $\{u^k\}$ is the circle
\[
S := \{ u \in \mathbb{R}^2 : \|u\| = \mu\|u^0\| \}.
\]
For each $k \in \mathbb{N}$, let $z^k = u_1^k + iu_2^k$ be the complex number generated by $u^k$. In order to obtain the above property, it suffices to show that the set of the cluster points of $\{z^k\}$ is the circle
\[
S := \{ z \in \mathbb{C} : |z| = \mu|z^0| \}
\]
in the complex plane. By (15) we have
\[
u_1^{k+1} + iu_2^{k+1} = \left( 1 - \frac{1}{(k+1)^2} - \frac{i}{k+1} \right) u_1^k + \left( \frac{1}{k+1} + i - \frac{i}{(k+1)^2} \right) u_2^k
\]
\[
= \left( 1 - \frac{1}{(k+1)^2} - \frac{i}{k+1} \right) u_1^k + i \left( -\frac{i}{k+1} + 1 - \frac{1}{(k+1)^2} \right) u_2^k
\]
\[
= \left( 1 - \frac{1}{(k+1)^2} - \frac{i}{k+1} \right) (u_1^k + iu_2^k).
\]
Setting $a_k = 1 - \frac{1}{(k+1)^2} - \frac{i}{k+1}$, we have $z^{k+1} = a_k z^k$. This yields $z^{k+1} = \left( \prod_{j=0}^{k} a_j \right) z^0$. For each $k \geq 1$ we write $a_k$ in the exponential form $a_k = r_ke^{i\theta_k}$, where
\[
r_k = |a_k| = \sqrt{1 - \frac{1}{(k+1)^2} + \frac{1}{(k+1)^4}}, \quad \theta_k = \arctan\left( \frac{-(k+1)^{-1}}{1 - (k+1)^{-2}} \right) \in (-\frac{\pi}{2}, 0).
\]
Then
\[
z^{k+1} = a_0 \left( \prod_{j=1}^{k} r_j \right) e^{\omega_k i} z^0 = a_0 \left( \prod_{j=1}^{k} r_j \right) |z^0| e^{\omega_k i} e^{\bar{\theta}i},
\] (17)
where $a_0 = -i$, $\omega_k = \sum_{j=1}^{k} \theta_j$ and $\bar{\theta} \in (-\pi, \pi]$ is the principal argument of $z^0$. Since
\[
\theta_k = \arctan\left( \frac{-(k+1)^{-1}}{1 - (k+1)^{-2}} \right) = -\frac{1}{k+2} + O\left( \frac{1}{k^2} \right),
\]
it follows that $\lim_{k \to \infty} \theta_k = 0$ and $\lim_{k \to \infty} \omega_k = -\infty$. 

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Let \( z = a_0 \mu |z^0| e^{i\theta} \) be an arbitrarily given point in \( S \). For every \( m \in \mathbb{N} \), there exists a unique \( k_m \in \mathbb{N} \) such that \( \omega_{k_m} \leq \theta - \bar{\theta} - m2\pi < \omega_{k_m+1} \). Hence

\[
|\omega_{k_m} - (\theta - \bar{\theta} - 2m\pi)| \leq |\omega_{k_m} - \omega_{k_m+1}| = |\theta_{k_m}|.
\]

Since \( \theta_{k_m} \to 0 \), we have \( \omega_{k_m} - (\theta - \bar{\theta} - 2m\pi) \to 0 \) as \( m \to \infty \). This means that \( \omega_{k_m} + 2m\pi \to \theta - \bar{\theta} \) as \( m \to \infty \). Therefore

\[
\lim_{m \to \infty} e^{\omega_{k_m}i} = \lim_{m \to \infty} e^{(\omega_{k_m}+2m\pi)i} = e^{(\theta-\bar{\theta})i}.
\]

By (16) we have

\[
\lim_{m \to \infty} \left( \prod_{j=1}^{k_m} r_j \right) = \lim_{m \to \infty} \sqrt{\prod_{j=0}^{k_m} \left( 1 - \frac{1}{(j+1)^2} + \frac{1}{(j+1)^4} \right)} = \mu.
\]

It follows from (17) and the choice of \( z \) that \( \lim_{m \to \infty} z^{k_m+1} = a_0 \mu e^{(\theta-\bar{\theta})i} e^{\bar{\theta}i} = z \). We have proved that the set of all the cluster points of \( \{z^k\} \) is the circle \( S \).

Example 4.3 shows that Algorithm 3.1, which works well for strongly pseudomonotone VIs, cannot serve as an adequate solution method for pseudomonotone VIs.

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