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PROBLEMAS INVERSOS Y CONTROLABILIDAD EN MODELOS DE LA MECÁNICA DE FLUIDOS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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PROBLEMAS INVERSOS Y CONTROLABILIDAD EN MODELOS DE LA MECÁNICA DE FLUIDOS

Esta tesis doctoral está dedicada al estudio de problemas inversos y de control en el área de la mecánica de fluidos. Nos centramos en las ecuaciones de Stokes y de Navier–Stokes, tanto sistemas estacionarios como evolutivos, los cuales son bien conocidos para el desarrollo matemático de los flujos viscosos incompresibles. En concreto, se analizaron tres temas principales:

- Realizamos la estimación del tamaño de una cavidad D inmersa en un dominio acotado $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, lleno de un fluido viscoso el cual se rige por el sistema de Stokes, por medio de la velocidad y las fuerzas de Cauchy en la frontera $\partial\Omega$. Más precisamente, establecemos una cota inferior y superior en términos de la diferencia entre las mediciones externas cuando el obstáculo está presente y cuando no lo está. La demostración del resultado se basa en los resultados de regularidad interior y estimaciones cuantitativas de continuación única para la solución del sistema de Stokes.
- Desarrollamos el estudio del fenómeno del turnpike que surge en el problema de control de seguimiento óptimo distribuido para las ecuaciones de Navier–Stokes. Obtenemos una respuesta positiva a esta propiedad en el caso de que los controles son funciones dependientes del tiempo, y también cuando son independientes del tiempo. En ambos casos se prueba una propiedad de turnpike exponencial, bajo el supuesto que el estado óptimo estacionario satisface ciertas propiedades de pequeñez.
- Consideramos las ecuaciones de Stokes evolutivas con viscosidad no constante. En primer lugar adaptamos la construcción de soluciones del tipo óptica geométrica complejas apropiadas para una ecuación de Stokes estacionaria modificada, con el fin de demostrar un resultado de identificabilidad siguiendo el enfoque dado por Uhlmann [110] y de Heck et al. [62]. Luego, se estudia la identificabilidad global para la función de viscosidad por medio de mediciones de contorno reduciendo el problema al caso estacionario, cuando consideramos el horizonte de tiempo suficientemente grande.

ABSTRACT OF MEMORY TO OBTAIN
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PROBLEMAS INVERSOS Y CONTROLABILIDAD EN MODELOS DE LA MECÁNICA DE FLUIDOS

This thesis is devoted to the study of inverse and control problems in the area of fluid mechanics. We focus on Stokes and Navier–Stokes equations, both stationary and evolutionary systems which are well known to the mathematical development of incompressible viscous flows. In particular, three main themes were analyzed:

- We are interested in estimating the size of a cavity D immersed in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, filled with a viscous fluid governed by the Stokes system, by means of velocity and Cauchy forces on the external boundary $\partial\Omega$. More precisely, we establish some lower and upper bounds in terms of the difference between the external measurements when the obstacle is present and without the object. The proof of the result is based on interior regularity results and quantitative estimates of unique continuation for the solution of the Stokes system.
- We study the turnpike phenomenon arising in the optimal distributed control tracking-type problem for the Navier-Stokes equations. We obtain a positive answer to this property in the case when the controls are time-dependent functions, and also when are independent of time. In both cases we prove an exponential turnpike property assuming that the stationary optimal state satisfy certain properties of smallness.
- We consider the evolutionary Stokes equations with non constant viscosity. Firstly we adapt the construction of an appropriate complex geometrical optics solutions for a modified stationary Stokes equation in order to prove an identifiability result, following the approach given by Uhlmann [110] and Heck et al.[62]. Later, we study a global identifiability for the viscosity function by boundary measurements using a stability result for the solutions of a non–steady Stokes equation when time tends to infinity.

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Chapter 1

Introduction

There is air around us, and there are rivers and seas near us. “The flow of a river never ceases to go past, nevertheless it is not the same water as before. Bubbles floating along on the stagnant water now vanish and then develop but have never remained.” So stated Chohmei Kamo, the famous thirteenth–century essayist of Japan, in the prologue of *Hohjohki*, his collection of essays. In this way, the air and the water of rivers and seas are always moving. Such a movement of gas or liquid (collectively called *fluid*) is called the *flow*, and the study of this is *fluid mechanics* (see [95]).

While the flow of the air and the water of rivers and seas are flows of our concern, so also are the flows of water, sewage and gas in pipes, in irrigation canals, and around rockets, aircraft, express trains, automobiles and boats. And so too is the resistance which acts on such flows.

Throwing baseballs and hitting golf balls are all acts of flow. Furthermore, the movement of people on the platform of a railway station or at the intersection of streets can be regarded as forms of flow. In a wider sense, the movement of social phenomena, information or history could be regarded as a flow, too. In this way, we are in so close a relationship to flow that the fluid mechanics which studies flow is really a very familiar thing to us.

Fluids are divided into liquids and gases. A liquid is hard to compress and as in the ancient saying “Water takes the shape of the vessel containing it”, it changes its shape according to the shape of its container with an upper free surface. Gas on the other hand is easy to compress, and fully expands to fill its container. There is thus no free surface.

Consequently, an important characteristic of a fluid from the viewpoint of fluid mechanics is its compressibility. Another characteristic is its viscosity. Whereas a solid shows its elasticity in tension, compression or shearing stress, a fluid does so only for compression. In other words, a fluid increases its pressure against compression, trying to retain its original volume. This characteristic is called compressibility. Furthermore, a fluid shows resistance whenever two layers slide over each other. This characteristic is called viscosity.

In general, liquids are called incompressible fluids and gases compressible fluids. Nevertheless, for liquids, compressibility must be taken into account whenever they are highly pressurized, and for gases compressibility may be disregarded whenever the change in pressure is small.

In this doctoral thesis, we are interested to study some mathematical properties of viscous incompressible fluid. For this it is necessary consider a mathematical model for the motion of the flow. Two of the more important approaches are known as **Stokes and Navier–Stokes equations**. Navier and Stokes were two scientists of the 19th century, which were the first who tried to derive equations of motion for fluids.

It is well known that the Navier–Stokes system is one of the pillars of fluid mechanics (the Stokes equations is a simplified version of the Navier–Stokes equations). These equations are useful because they describe the physics of many things of academic and economics interest. They may be used to model the weather behavior, ocean currents, water flow in a pipe and air flow around a wing. The Navier–Stokes equations in their full and simplified forms also help with the design of train, aircraft and cars, the study of blood flow, the design of power stations and pollution analysis.

Mathematically speaking, the viscous incompressible Navier–Stokes equations are given by

$$\left\{\begin{array}{rcll} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p &=& f & , \quad \text{in } Q, \\ \operatorname{div} u &=& 0 & , \quad \text{in } Q, \\ u &=& 0 & , \quad \text{on } \Sigma, \\ u(0) &=& u_0 & , \quad \text{in } \Omega. \end{array}\right. \tag{1.1}$$

Here, u denotes the velocity field of the fluid. The pressure of the fluid is denoted by p. The right hand side f is an given source term. The parameter $\mu > 0$ is the viscosity of the fluid. The initial profile of the flow is a given function u_0. The incompressible condition of the fluid is given by the assumption $\operatorname{div} u = 0$. The cylindrical domain is represented by $Q = \Omega \times (0, T)$ with boundary $\Sigma = \partial\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded region with boundary $\partial\Omega$.

The existence and uniqueness of classical solutions of the three–dimensional Navier-Stokes equations is still an open mathematical problem and is one of the Clay Institute's Millenium Problems[1]. In the two–dimensional case, existence and uniqueness of regular solutions for all time have been shown by Jean Leray in 1933. He also gave the theory for the existence of weak solutions in the three–dimensional case while uniqueness is still an open question. We refer the reader to [54, 38, 85, 86] and references therein, for an extensive literature of mathematical theory of these two models. We can also mention that in Chapter 2 of this thesis, we introduce and give the more important aspects of the Stokes and Navier–Stokes equations.

As mentioned above, we are interested to study some applications of these models. In this work we focus our attention in **inverse problems** for the Stokes equations and **control problem** for the Navier–Stokes system. But, what is an inverse and control problem? Let us briefly explain these two concepts.

[1]http://www.claymath.org/millennium/Navier-Stokes Equations

An inverse problem consists in formulate the inverse of a forward problem (causes implies results). Namely, it starts with the results and then calculates the causes. One of the most important aspects of inverse problem is that tell us about some parameters of a model that we cannot directly observe. This means that is a noninvasive methods to find some characteristics which describe some model.

The field of inverse problems was first discovered and introduced by Soviet–Armenian physicist, Viktor Ambartsumian. While still a student, Ambartsumian thoroughly studied the theory of atomic structure, the formation of energy levels, and the Schrödinger equation and its properties, and when he mastered the theory of eigenvalues of differential equations, he pointed out the apparent analogy between discrete energy levels and the eigenvalues of differential equations. He then asked: given a family of eigenvalues, is it possible to find the form of the equations whose eigenvalues they are? Essentially Ambartsumian was examining the inverse Sturm–Liouville problem, which dealt with determining the equations of a vibrating string. This paper was published in 1929 in the German physics journal Zeitschrift für Physik and remained in obscurity for a rather long time. Describing this situation after many decades, Ambartsumian said, "If an astronomer publishes an article with a mathematical content in a physics journal, then the most likely thing that will happen to it is oblivion." Nonetheless, toward the end of the Second World War, this article, written by the 20-year-old Ambartsumian, was found by Swedish mathematicians and formed the starting point for a whole area of research on inverse problems, becoming the foundation of an entire discipline. See for instance [18].

One of the most famous example in this field is the **Electrical Impedance Tomography (EIT)** inverse problem. The EIT is a non–invasive medical imaging technique in which an image of the conductivity or permittivity of a part of the body is inferred from surface electrode measurements. Mathematically, the problem of recovering the electrical conductivity from boundary measurements of current and voltage is an example of a non–linear inverse problem. The formulation of this mathematical problem is due to Alberto Calderón [32], and is also known as *Calderón's inverse problem* or the *Calderón problem.* For an extensive literature about this problem we can refer the reader the work of G. Uhlmann [110], and the references therein.

There are different aspects of an inverse problem that one could be interested in, for instance: uniqueness and stability with respect to the measurements, recovery of a parameter by means of reconstruction formula or estimate a few relevant parameters of an object immersed in a domain (geometric inverse problem). For Calderón's problem, we can refer the reader [105] for global uniqueness of the conductivity function and [77] in the case with partial data. In [14] the authors discuss the stability issue for Calderón's inverse conductivity problem. In the case of size estimates, we can refer [9] to an extensive survey.

Because it is impossible to quote all relevant contributions in this area, we refer the reader to the extensive survey [70, 39, 76, 78, 103] (and references therein),

A different kind of problem treated in this thesis is the called optimal control problems. Optimal control is closely related in its origins to the theory of calculus of variations. Some important contributors to the early theory of optimal control and calculus of variations include the books of Johann Bernoulli (1667–1748), Isaac Newton (1642–1727), Leonhard Euler

(1707–1793), Ludovico Lagrange (1736–1813), Andrien Legendre (1752–1833), Carl Jacobi (1804–1851), William Hamilton (1805–1865), Karl Weierstrass (1815–1897), Adolph Mayer (1839–1907), and Oskar Bolza (1857–1942). Some important milestones in the development of optimal control in the 20th century include the formulation dynamic programming by Richard Bellman (1920–1984) in the 1950s, the development of the minimum principle by Lev Pontryagin [97] (1908–1988) and co-workers also in the 1950s, and the formulation of the linear quadratic regulator and the Kalman filter by Rudolf Kalman (b. 1930) in the 1960s, see for instance [22, 40]. See the review papers Sussmann and Willems [104] and Bryson [31] for further historical details, the book of J.L. Lions [83] for the theory of optimal control problems in the context of partial differential equations, also the books [4, 111] for more theory of optimal control and the following books for details on control theory [42, 116, 87, 115].

Optimal control and its ramifications have found applications in many different fields, including aerospace, process control, robotics, bioengineering, economics, finance, and management science, and it continues to be an active research area within control theory. Before the arrival of the digital computer in the 1950s, only fairly simple optimal control problems could be solved. The arrival of the digital computer has enabled the application of optimal control theory and methods to many complex problems.

There are various types of optimal control problems, depending on the performance index, the type of time domain (continuous, discrete), the presence of different types of constraints, and what variables are free to be chosen. The formulation of an optimal control problem requires the following:

1. a mathematical model of the system to be controlled,
2. a specification of the performance index,
3. a specification of all boundary conditions on states, and constraints to be satisfied by states and controls,
4. a statement of what variables are free.

In order to apply mathematical approaches to the control of fluid flow these terminologies have to be translated into mathematical language. The first question to pose is how to model fluid flow mathematically. For this purpose firstly we have to the region of fluid flow has to be specified and to be made accessible to a mathematical description. This is, from the theoretical point of view, easy. Modelling fluid flow clearly depends on the physical properties of the fluid. Most commonly, the fluid is assumed to be incompressible, so that its state can approximately be described by the (instationary) Navier–Stokes equations. Once such a model is available, physical terms like drag and turbulent kinetic energy may be expressed in terms of the model variables and thus can be customized for mathematical performance indexes. The same holds true for control actions. Finally, blowing and suction or movement of walls may be regarded as boundary conditions for the flow variables, external forces applied in the domain of flow as inhomogeneities.

The choice of the control policy depends on the control target and the environment. If the system to be controlled is shielded against external influences, it can be desirable to provide a time dependent control function that, for example steers the system from a given state to a desired one. This would correspond to an optimal (open-loop) control problem (find the best time dependent control function for the specified environment). A more general form of control activity is (optimal) closed-loop or feedback control (find the best feedback control law). It allows for feeding back into the system control information obtained by currently available state information or estimation. This is a much more general concept than optimal control and, at least optimal feedback, also much more complicated to determine.

As mentioned earlier, our work focuses on inverse and control problems for the Stokes equation and Navier-Stokes system, respectively. In what follows, we describe the mathematical aspect of inverse and control problems, and the more recent and important results in the context of fluids mechanics.

1.1 Inverse problems

We start with the most important and exemplifying inverse problem, the Calderón's problem introduced in the previous section. The problem that A.P. Calderón proposed in [32] is whether it is possible to determine the conductivity of a body by making current and voltage measurements at the boundary.

We now describe more precisely the mathematical problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\mathrm{div}\,(\gamma \nabla u) = 0 \text{ in } \Omega \tag{1.2}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux.

Given a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{cases} \mathrm{div}\,(\gamma \nabla u) &= 0 \quad , \quad \text{in } \Omega, \\ u &= f \quad , \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$f \longmapsto \Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial n} \right) \Big|_{\partial\Omega} \tag{1.4}$$

where n denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine γ knowing Λ_γ. More precisely, we want to study properties of the map $\gamma \xrightarrow{\Lambda} \Lambda_\gamma$. We can divide this problem into several parts.

- Injectivity of Λ (identifiability).
- Continuity of Λ and its inverse if it exists (stability).
- What is the range of Λ? (characterization problem).

- Formula to recover γ from Λ_γ (reconstruction).
- Give an approximate numerical algorithm to find an approximation of the conductivity given a finite number os voltage and current measurements at the boundary (numerical reconstruction).

For example, the identifiability problem is formulated as: if we assume that

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f), \quad \forall f \in H^{1/2}(\partial\Omega),$$

implies that γ_1 and γ_2 are equal? The answers have been given in many cases, see for instance [94, 20, 105]. We refer the reader of [5] for stability result, [77] for identifiability result with partial data, [93, 92] for reconstruction of conductivity with measurements at the boundary and part of the boundary, respectively.

Here, we present the result of Sylvester and Uhlmann [105], where the authors resolved the problem in dimension three and for smooth conductivities.

Theorem 1.1 (Sylvester and Uhlmann [105]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boudary. Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

The proof of Theorem 1.1 is based on the construction of appropriate solutions to (1.3), called *exponentially growing solutions* or *complex geometrical optics solutions*. Their method has become a standard technique in treating inverse boundary value problems.

This result is a consequence of a more general result. Let $q \in L^\infty(\Omega)$. We define the Cauchy data as the set

$$C_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \right) \right\},$$

where $u \in H^1(\Omega)$ is a solution of

$$(\Delta - q)u = 0 \text{ in } \Omega.$$

We have that $C_q \subseteq H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$. If zero is not a Dirichlet eigenvalue of $\Delta - q$, then in fact C_q is a graph, namely

$$C_q = \{(f, \Lambda_q(f)) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)\}$$

where $\Lambda_q(f) = \frac{\partial u}{\partial n}\Big|_{\partial\Omega}$ with $u \in H^1(\Omega)$ the solution of

$$\left\{ \begin{array}{rcll} (\Delta - q)u &=& 0 \;, & \text{in } \Omega, \\ u &=& f \;, & \text{on } \partial\Omega. \end{array} \right. \tag{1.5}$$

Theorem 1.2 (Sylvester and Uhlmann [105]) *Let $q_i \in L^\infty(\Omega)$, $i = 1, 2$. Assume $C_{q_1} = C_{q_2}$, then $q_1 = q_2$.*

We have that, when the dimension is greater than three, Theorem 1.2 implies Theorem 1.1, see for instance [105].

Another interesting problem is the called geometrical inverse problem. Suppose that a given electrically conducting body Ω having some conductivity k, might contain an unknown inclusion D, having different conductivity. We can ask whether D can be determined by the knowledge of a prescribed current density φ on the boundary $\partial\Omega$ and of the corresponding voltage u measured on Ω.

Let us formulate analytically this problem. We suppose that the region containing the body is represented by a bounded open set $\Omega \subset \mathbb{R}^d$, with Lipschitz boundary, and let $\varphi \in H^{-1/2}(\partial\Omega)$, $\int_{\partial\Omega} \varphi = 0$, represent the prescribed current density on $\partial\Omega$. If the inclusion D is present, then the electrostatic potential $u = u(x)$ is determined as the $H^1(\Omega)$ solution to the Neumann problem

$$\begin{cases} \text{div }((1+\chi_D)\nabla u) &= 0 \quad , \quad \text{in } \Omega, \\ \nabla u \cdot n &= \varphi \quad , \quad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where χ_D denote the characteristic function of D.

The inverse problem, also known as *the inverse conductivity problem with one measurement*, consists of determining D when, for a prescribed nontrivial φ, the trace $u|_{\partial\Omega}$ is measured.

The uniqueness issue for this inverse problem remains a challenging unsolved problem. See [70, 7] for reviews of the known results and extensive bibliographical information. We can mention that, also if all boundary measurements are available, the inverse problem remains severely ill–posed. In fact Di Cristo and Rondi have shown by examples [44], that, under apriori assumptions on the C^m–regularity of D, the stability of this inverse problem is not better than logarithmic.

Therefore, from a practical point of view, the question of the identification of D should be reformulated in the following terms: *What parameters associated to D can be effectively evaluated from the available boundary measurements?*

The approach of *size estimates* is an attempt in this direction. The goal is to obtain upper and lower bounds on the size of the unknown enclosed object D in terms of some suitable number, extracteed from the boundary measurement. The issue of size estimates has been addressed in [102, 13, 75]. This type of approach has been extended also to the detection of inclusions in elastic bodies [10, 67] and to the detection of cavities or extreme inclusions in electric conductors [8] and in elastic bodies [88].

For the obtention of the size of the unknown object D, we define the following number

$$W = \int_{\partial\Omega} u\varphi$$

which can be easily calculated from the boundary measurement, in fact it has a clear physical interpretation as the power required to maintain the boundary current φ when the inclusion D is present. We shall compare the power W to the power

$$W_0 = \int_{\partial\Omega} u_0\varphi,$$

where u_0 denotes the potential corresponding to the current density φ when the inclusion is absent, namely the $H^1(\Omega)$ solution to the Neumann problem

$$\begin{cases} \Delta u_0 &= 0 \quad , \quad \text{in } \Omega, \\ \nabla u_0 \cdot n &= \varphi \quad , \quad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The reference power W_0 can be considered as a given quantity, since everything is known in the boundary value problem (1.7). Thus, in order to have a parameter which is suitable for the estimation of $|D|$, and is independent on the scaling of the boundary current, it is reasonable to consider the number

$$\frac{|W_0 - W|}{W_0}$$

which we shall call the normalized power gap, see for instance [9].

Now, we give the most relevant results of inverse problems in the context of fluids mechanics, in particular, for the Stokes equations. We start with an inverse identifiability result for the viscosity function when the fluid is described by the stationary Stokes equations in dimension three, using the idea given for the Calderón problem.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$. Assume that Ω is filled with an incompressible fluid. We consider the following stationary Stokes problem

$$\begin{cases} \text{div } \sigma_\mu(u,p) &= 0 \quad , \quad \text{in } \Omega, \\ \text{div } u &= 0 \quad , \quad \text{in } \Omega, \\ u &= \phi \quad , \quad \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where u represents the velocity of the fluid, the scalar function p represents the pressure, $\sigma_\mu = 2\mu\left(\frac{\nabla u + \nabla u^T}{2}\right) - pI$ is the stress tensor and I is the identity matriz of order $d \times d$, and $\mu = \mu(x) > 0$ is the viscosity function.

Given the boundary velocity $\phi \in (H^{1/2}(\partial\Omega))^3$ satisfying the compatibility condition

$$\int_{\partial\Omega} \phi \cdot n = 0,$$

where n denotes the unit outer normal to $\partial\Omega$, we consider the solution of problem (1.8), and measure the corresponding Cauchy force on $\partial\Omega$, $\psi = \sigma_\mu(u,p)|_{\partial\Omega} \subset H^{-1/2}(\partial\Omega)$, in order to recover the viscosity function μ.

Regarding the identifiability of the viscosity, under the following hypothesis

$$\begin{gathered} \mu_1, \mu_2 \in C^{n_0}(\overline{\Omega}), \ \forall n_0 \geq 8, \\ \mu_i \geq \mu_0 > 0, \ \forall i = 1, 2, \\ \partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x), \ \forall x \in \partial\Omega \ , \ |\alpha| \leq 1, \end{gathered} \tag{1.9}$$

Heck, Li and Wang [62] proved the injectivity of the S_μ given by

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)|_{\partial\Omega})\}.$$

Theorem 1.3 (Heck, Li and Wang [62]) *Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity functions satisfying* (1.9). *Let S_{μ_1} and S_{μ_2} be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1} = S_{\mu_2}$, then we obtain $\mu_1 = \mu_2$.*

The strategy for the proof of Theorem 1.3 follows from Sylvester and Uhlmann's seminal paper [105]. The problem here lies in the construction of exponentially growing solution for (1.8). The authors in [62] transformed (1.8) to the decoupled system which is a matrix–valued Schrödinger equation and then using the complex variable methods to construct exponentially growing solutions given by Eskin and Ralston in [48] and [47] for matrix–valued Schrödinger equations, they overcome this difficulty.

We can mention that in the case of Navier–Stokes equations, similar result was obtained by Fan et al. [52]. They proved the identification of the viscosity by observation data in a neighborhood of the boundary, and then obtain Lispchitz stability by the Carleman estimates in Sobolev spaces of negative order. Also, in [79] the authors obtained a global uniqueness of the viscosity for the stationary and evolutionary Stokes and Navier–Stokes equations in the two–dimensional case. Besides, Li and Wang [80] proved the identifiability result for the viscosity considering small boundary condition.

On the other hand, we discuss the following geometric inverse problem associated to the Stokes system. An inaccesible rigid body D is immersed in a viscous fluid, in such a way that D plays the role of an obstacle around which the fluid is flowing in a greater bounded domain Ω, and we wish to determine D via boundary measurements on the boundary $\partial\Omega$. Both for the stationary and the evolutionary problem, we show that under reasonable smoothness assumptions on Ω and D, one can identify D via the measurement of the velocity of the fluid and the Cauchy forces on some part of the boundary $\partial\Omega$.

Let Ω be a smooth bounded open set in $\mathbb{R}^d$, filled with an incompressible fluid, and let $D \subset\subset \Omega$ be an unknown rigid body immersed in it. Let $\varphi \in C^1([0,T];(H^{3/2}(\partial\Omega))^d)$ be non–homogeneous Dirichlet boundary data satisfying the compatibility condition

$$\int_{\partial\Omega} \varphi \cdot n = 0, \tag{1.10}$$

and let $(v,p) \in L^2(0,T;(H^1(\Omega \setminus \overline{D}))^d) \times L^2(0,T;L^2(\Omega \setminus \overline{D}))$ be the unique solution of the Stokes system of equations

$$\begin{cases} u_t - \mathrm{div}\,(\sigma(u,p)) &= 0 \quad, \quad \text{in } (\Omega \setminus \overline{D}) \times (0,T), \\ \mathrm{div}\, u &= 0 \quad, \quad \text{in } (\Omega \setminus \overline{D}) \times (0,T), \\ u &= 0 \quad, \quad \text{on } \partial D \times (0,T), \\ u &= \varphi \quad, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= 0 \quad, \quad \text{in } \Omega \setminus \overline{D}. \end{cases} \tag{1.11}$$

The problem is to obtain some information on the domain D through the observation of the Cauchy force $\sigma(u,p)n$ on some part of the boundary. Indeed the stationary version of the problem, that is,

$$\begin{cases} -\text{div } (\sigma(u,p)) &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ \text{div } u &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ u &= 0 \quad , \quad \text{on } \partial D, \\ u &= \varphi \quad , \quad \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

can also be considered and treated in the same way.

In this problem, we will look for the unknown domain D in the following class of admissible geometries

$$\mathcal{D}_{ad} := \{D \subset\subset \Omega \; : \; D \text{ is open, Lipschitz and } \Omega \setminus \overline{D} \text{ is connected}\}. \tag{1.13}$$

The corresponding Steklov–Poincaré map Λ_D is defined by

$$\Lambda_D(\varphi) := \sigma(u,p)n \quad \text{on} \quad \Gamma \times (0,T),$$

which maps $\varphi \in C^1([0,T];(H^{3/2}(\partial\Omega))^d)$ into the Cauchy forces $\sigma(u,p)n \in L^2(0,T;(H^{-1/2}(\Gamma))^d)$, where Γ is a relatively open subset of the boundary Ω. In the case of the stationary version of the problem, the operator Λ_D has to be considered as acting between the spaces $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\Gamma)$.

The result concerns the identifiability of D was proved by Alvarez et al. [16]. They proved that given fixed non–homogeneous Dirichlet boundary data φ, two different admissible geometries $D_0 \neq D_1 \in \mathcal{D}_{ad}$, yield two different Steklov–Poincaré operators $\Lambda_{D_0} \neq \Lambda_{D_1}$.

Theorem 1.4 (Alvarez et al. [16]) *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded $C^{1,1}$ domain, and Γ be a non–empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{ad}$ and $\varphi \in C^1([0,T];(H^{3/2}(\partial\Omega))^d)$ with $\varphi \neq 0$, satisfying the flux condition* (1.10)*. Let (u^j,p^j) for $j = 1,1$ be a solution of*

$$\begin{cases} u^j_t - \text{div } (\sigma(u^j,p^j)) &= 0 \quad , \quad \text{in } (\Omega \setminus D_j) \times (0,T), \\ \text{div } u^j &= 0 \quad , \quad \text{in } (\Omega \setminus D_j) \times (0,T), \\ u^j &= 0 \quad , \quad \text{on } \partial D_j \times (0,T), \\ u^j &= \varphi \quad , \quad \text{on } \partial\Omega \times (0,T), \\ u^j(0) &= 0 \quad , \quad \text{in } \Omega \setminus D_j. \end{cases}$$

Assume that (u^j,p^j) are such that $\sigma(u^0,p^0)n = \sigma(u^1,p^1)n$ on $\Gamma \times (0,T)$, then $D_0 \equiv D_1$.

The same identification result holds for the stationary problem:

Theorem 1.5 (Alvarez et al. [16]) *Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, be a bounded Lipschitz domain, and Γ be a non–empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{ad}$ and $\varphi \in (H^{3/2}(\partial\Omega))^d$ with $\varphi \neq 0$, satisfying the flux condition* (1.10)*. Let (u^j,p^j) for $j = 1,1$ be a solution of*

$$\begin{cases} -\text{div } (\sigma(u^j,p^j)) &= 0 \quad , \quad \text{in } \Omega \setminus D_j, \\ \text{div } u^j &= 0 \quad , \quad \text{in } \Omega \setminus D_j, \\ u^j &= 0 \quad , \quad \text{on } \partial D_j, \\ u^j &= \varphi \quad , \quad \text{on } \partial\Omega. \end{cases}$$

Assume that (u^j,p^j) are such that $\sigma(u^0,p^0)n = \sigma(u^1,p^1)n$ on Γ, then $D_0 \equiv D_1$.

The proof of these results are based by a suitable application of the unique continuation property for the Stokes equations due to Fabre and Lebeau [51]. Besides, we mention that the authors in [16] also proved the identification result in the case of Navier–Stokes equations, both stationary and evolutionary systems.

Now, we consider the inverse stability problem associated to the stationary Stokes problem (1.12), which we may roughly state as follows: *Given two solutions* (u^i, p^i) *to* (1.12) *for two different* D_i*, for* $i = 1, 2$*, with the same boundary data* φ*, if*

$$\|\sigma(u^1, p^1)n - \sigma(u^2, p^2)n\| \leq \varepsilon,$$

what is the rate of convergence of $d_{\mathcal{H}}(D_1, D_2)$ *as* $\varepsilon \to 0$*?* (here $d_{\mathcal{H}}$ denote the Hausdorff distance.)

We present the stability result proved by A. Ballerini [23], where the author prove a log–log type stability for the Hausdorff distance between the boundaries of the inclusions, assuming a $C^{2,\alpha}$ regularity bound. Such estimates have been established for various kinds of elliptic equations, for example, [6, 12] for the electric conductivity equation and [89, 90] for the elasticity system and the detection of cavities or rigid inclusions. The main tool used to prove stability is a quantitative estimate of continuation from boundary data, in the interior, in the form of a three spheres inequality and its main consequences. However, while in [6] the estimates are of log type for a scalar equations, here and in [89, 90], only an estimate of log–log type could be obtained for a system of equations. To improve that, one would need a doubling inequality at the boundary for systems of equations, which basically would allow to extend the reach of the unique continuation property up to the boundary. Unfortunately, to the present time, none are available.

Now, we state a priori hypotheses needed by Ballerini to prove the stability result. First, we will often make use of the following definition of regularity of a domain.

Definition 1.6 *Let* $\Omega \subset \mathbb{R}^d$ *be bounded domain. We say that* $\partial\Omega$ *is of class* $C^{k,\alpha}$*, with constants* ρ_0*,* $M_0 > 0$*, where* k *is a nonnegative integer and* $\alpha \in [0, 1)$*, if, for any* $x_0 \in \partial\Omega$*, there exists a rigid transformation of coordinates, in which* $x_0 = 0$ *and*

$$\Omega \cap B_{\rho_0}(0) = \{x = (x', x_n) \in B_{\rho_0}(0) : \ x_n > \varphi(x')\},$$

where φ *is a real valued function of class* $C^{k,\alpha}(B'_\rho(0))$*, such that*

$$\begin{aligned} \varphi(0) &= 0, \\ \nabla\varphi(0) &= 0, \ \text{if } k \geq 1, \\ \|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} &\leq M_0\rho_0. \end{aligned}$$

When $k = 0$ *and* $\alpha = 1$ *we will say that* $\partial\Omega$ *is of Lipschitz class with constants* ρ_0, M_0*.*

We assume $\Omega \subset \mathbb{R}^d$ to be a bounded domain such that $\partial\Omega$ is connected, and it has sufficiently smooth boundary, i.e., $\partial\Omega$ is of class $C^{2,\alpha}$ of constants ρ_0, M_0, where $\alpha \in (0, 1]$ is a real number, $M_0 > 0$, and $\rho_0 > 0$ is what we shall treat as our dimensional parameter. And

$$|\Omega| \leq M_1\rho_0^d, \tag{1.14}$$

where $M_1 > 0$.

We choose a special open and connected portion $\Gamma \subset \partial\Omega$ as being the accessible part of the boundary, where ideally all measurements are taken. We assume that there exists a point $P_0 \in \Gamma$ such that $\partial\Omega \cap B_{\rho_0}(P_0) \subset \Gamma$.

For the obstacle D, we requiere that $\Omega \setminus \overline{D}$ is connected, ∂D is connected. Also, we requiere the same regularity on D as we did for Ω, that is, ∂D is of class $C^{2,\alpha}$ with constants ρ, M_0. In addition, we suppose that the obstacle is well contained in Ω, meaning

$$d(D, \partial\Omega) \geq \rho_0.$$

Besides, for the Dirichlet data φ we assign on the accessible part of the boundary Γ, we assume that $\varphi \in H^{3/2}(\partial\Omega)^d$, $\varphi \neq 0$, $\text{supp}(\varphi) \subset\subset \Gamma$. We also ask that, for a given a constant $F > 0$, we have

$$\frac{\|\varphi\|_{H^{1/2}(\partial\Omega)}}{\|\varphi\|_{L^2(\partial\Omega)}} \leq F.$$

Let D_i, for $i = 1, 2$ satisfy the previous hypotheses. The following result, due to A. Ballerini [23], gives us the log–log type stability for the Stokes problem.

Theorem 1.7 (A. Bellerini [23]) *Let $\varphi \in (H^{3/2}(\Gamma))^d$ be the assigned boundary data satisfying the previous assumption. Let $u^i \in (H^1(\Omega))^d$ solve* (1.12) *for $D = D_i$, for $i = 1, 2$. If, for $\varepsilon > 0$, we have*

$$\rho\|\sigma(u^1, p^1)n - \sigma(u^2, p^2)n\|_{H^{-1/2}(\Gamma)} \leq \varepsilon,$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \rho\omega\left(\frac{\varepsilon}{\|\varphi\|_{H^{1/2}(\Gamma)}}\right),$$

where $\omega : (0, \infty) \to \mathbb{R}_+$ is an increasing function satisfying, for all $0 < t < \frac{1}{\varepsilon}$:

$$\omega(t) \leq C(\log|\log t|)^{-\beta}.$$

The constants $C > 0$ and $0 < \beta < 1$ only depends on d, M_0, M_1 and F.

The proof of Theorem 1.7 relies on a sequence of propositions. Here, we present only one of them. The result is the called *Lipschitz propagation of smallness.*

Proposition 1.8 (A. Ballerini [23]) *Let Ω be a bounded Lipschitz domain with constants ρ_0, M_0 satisfying* (1.14)*. Let (u, p) be a solution to the following problem*

$$\begin{cases} -\text{div}\,(\sigma(u,p)) &= 0 \quad , \quad \text{in } \Omega, \\ \text{div}\, u &= 0 \quad , \quad \text{in } \Omega, \\ u &= \varphi \quad , \quad \text{on } \partial\Omega, \end{cases}$$

where φ satisfies $\varphi \in (H^{3/2}(\partial\Omega))^d$, the compatibility condition, and $\frac{\|\varphi\|_{H^{1/2}(\partial\Omega)}}{\|\varphi\|_{L^2(\partial\Omega)}} \leq F$, for a given constant $F > 0$. Also suppose that there exists a point $P \in \partial\Omega$ such that $\varphi = 0$ on $\partial\Omega \cap B_{\rho_0}(P)$. Then there exists a constan $s > 1$, depending only on d and M_0 such that, for every $\rho > 0$ and for every $\overline{x} \in \Omega_{s\rho} = \{x \in \Omega \,:\, d(x, \partial\Omega) > s\rho\}$, we have

$$\int_{B_\rho(\overline{x})} |\nabla u|^2 \geq C_\rho \int_\Omega |\nabla u|^2,$$

where $C_\rho > 0$ is a constant depending only on d, M_0, M_1, F, ρ_0 and ρ.

1.2 Optimal control problems

The development of a theory of optimal control requires the following data:

1. A *control* u that we can handle according to our interests, which can be chosen among a family of feasible controls U_{ad} (the set of admissible controls).
2. The *state of the system* y to be controlled, which depends on the control. Some limitations can be imposed on the state, in mathematical terms $y \in \mathbb{C}$, which means that not every possible state of the system is satisfactory.
3. A *state equations* that establishes the dependence between the control and the state. In this thesis we consider this state equation will be a partial differential equation, y being the solution of the equations and u a function arising in the equations so that any change in the control u produces a change in the solution y. However the origin of control theory was connected with the control of systems governed by ordinary differential equations and there is a huge activity in this field, see for instance the classical book Prontyagin et al. [97].
4. A *function* to be minimized, called the objective function or the cost function, depending on the control and the state (y, u).

The objective is to determine an admissible control, called optimal control, that provides a satisfactory state for us and that minimizes the value of functional J. The basic questions to study are the existence of solution and its computations.

There are no many books devoted to all the questions. Firstly let me mention the book by Profesor J.L. Lions [83], which is an obliged reference in the study of the theory of optimal control problems of partial differential equations. In this text, that has left an indelible track, the reader will be able to find some of the methods used in the resolution of the two first questions above indicated. More recent books are X. Li and J. Yong [81], H.O. Fattorini [53] and F. Tröltzsch [109].

Let Ω be an open and bounded subset of $\mathbb{R}^d$, $d = 2, 3$, $\partial\Omega$ being its boundary that we will assume to be regular, $C^{1,1}$ is enough for us in this introduction. In Ω we will consider the linear operator A defined by

$$Ay := -\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x)\partial_{x_i} y(x) + a_0(x)y(x)),$$

where $a_{ij} \in C^{0,1}(\overline{\Omega})$ and $a_0 \in L^\infty(\Omega)$ satisfy:

1. $\exists m > 0$ such that $\displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2$, for all $\xi \in \mathbb{R}^d$ and for all $x \in \Omega$,
2. $a_0 \geq 0$ a.e. $x \in \Omega$.

Now let $\phi : \mathbb{R} \to \mathbb{R}$ be a non decreasing monotone function of class C^2, with $\phi(0) = 0$. For any $u \in L^2(\Omega)$, the Dirichlet problem

$$\begin{cases} Ay + \phi(y) = u \ , & \text{in } \Omega, \\ y = 0 \ , & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

has a unique solution $y_u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The control problem associated to this system is formulated as follows

$$(P) \begin{cases} \text{Minimize } J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx, \\ u \in U_{ad} = \{u \in L^\infty \ : \ \alpha \leq u(x) \leq \beta \text{ a.e. } x \in \Omega\}, \end{cases}$$

where $-\infty < \alpha, \beta < \infty$ and L satisfying some properties.

A typical functional in control theory is the *Linear Quadratic Control Problem*

$$J(u) = \int_\Omega [|y_u(x) - y_d(x)|^2 + Nu^2(x)]dx, \tag{1.16}$$

where $y_d \in L^2(\Omega)$ denotes the ideal state of the system, $N \geq 0$ and in this case the set of feasible controls is $U_{ad} = L^2(\Omega)$. The term $\int_\Omega Nu^2(x)dx$ can be considered as the cost term and it is said that the control is expensive if N is big, however the control is cheap if N is small or zero. From a mathematical point of view the presence of the term $\int_\Omega Nu^2(x)dx$, with $N > 0$, has a regularizing effect on the optimal control.

Now, (P) with the functional (1.16) is a convex control problem. In fact, the objective functional $J : L^2(\Omega) \to \mathbb{R}$ is well defined, continuos and strictly convex. Under these conditions, if U_{ad} is a convex and closed subset of $L^2(\Omega)$, we can prove the existence and uniqueness of an optimal control under one of the following assumptions:

1. U_{ad} is a bounded subset of $L^2(\Omega)$.
2. $N > 0$.

For the proof is enough to take a minimizing sequence and remark that the previous assumptions imply the boundedness of the sequence. Then it is possible to take a subsequence $(u_k)_k \subset U_{ad}$ converging weakly in $L^2(\Omega)$ to $\overline{u} \in U_{ad}$. Finally the convexity and continuity of J implies the weak lower semicontinuity of J, then

$$J(\overline{u}) \leq \liminf_{k\to\infty} J(u_k) = \inf(P).$$

The uniqueness of the solution is an immediate consequence of the strict convexity of J.

Now, let us consider the case of control of evolution equations, that is consider the following evolution state equation

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t}(x,t) + Ay(x,t) & = & u(x,t) & , \quad \text{in } Q_T = \Omega \times (0,T), \\ y(x,t) & = & 0 & , \quad \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ y(x,0) & = & y_0(x) & , \quad \text{in } \Omega, \end{array} \right. \tag{1.17}$$

where $y_0 \in C(\overline{\Omega})$ and $u \in L^2(Q_T)$. The existence and uniqueness are well now result for this type of equations, see for instance [49]. Thus we can formulate a control problem similar to the previous ones, taking as an objective function

$$J(u) = \int_{Q_T} L(x,t,y_u(x,t),u(x,t))dxdt.$$

To prove the existence of a solution of the control problem, we consider again the case of Linear Quadratic Control Problem, namely

$$J(u) = \int_{Q_T} [|y_u(x,t) - y_d(x)|^2 + Nu^2(x,t)]dxdt, \tag{1.18}$$

and using the same arguments that in the stationary case and compactness results for abstract spaces, it is possible to show the existence of an optimal solution.

Another property to study in optimal control problem is the first order conditions. This conditions are necessary conditions for local optimality, except in the case of convex problems, where they become also sufficient conditions for global optimality.

For problem (P) with functional (1.16), we have the following first order optimality conditions.

Theorem 1.9 (E. Casas [34]) *Let $\overline{u}$ be a local minimum of (P). Then there exists $\overline{y}, \overline{\varphi} \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ such that the following relationship hold*

$$\left\{ \begin{array}{rcll} A\overline{y} + \phi(\overline{y}) & = & \overline{u} & , \quad in\ \Omega, \\ \overline{y} & = & 0 & , \quad on\ \partial\Omega, \end{array} \right.$$

$$\left\{ \begin{array}{rcll} A^*\overline{\varphi} + \phi'(\overline{y})\overline{\varphi} & = & \overline{y} - y_d & , \quad in\ \Omega, \\ \overline{\varphi} & = & 0 & , \quad on\ \partial\Omega, \end{array} \right.$$

$$\int_\Omega (\overline{\varphi}(x) + N\overline{u}(x))(u(x) - \overline{u}(x))dx \geq 0 \quad \forall u \in U_{ad}.$$

In the case of evolutionary optimal problem (1.17) with functional (1.18), the expression of the optimality system is the following:

$$\left\{ \begin{array}{rcll} \dfrac{\partial \overline{y}}{\partial t} + A\overline{y} & = & u & , \quad \text{in } Q_T, \\ \overline{y} & = & 0 & , \quad \text{on } \Gamma_T, \\ \overline{y}(x,0) & = & y_0(x) & , \quad \text{in } \Omega, \end{array} \right.$$

$$\begin{cases} -\dfrac{\partial\overline{\varphi}}{\partial t} + A^*\overline{\varphi} &= \overline{y} - y_d \quad , \quad \text{in } Q_T, \\ \overline{\varphi} &= 0 \qquad\quad , \quad \text{on } \Gamma_T, \\ \overline{\varphi}(x,T) &= 0 \qquad\quad , \quad \text{in } \Omega, \end{cases}$$

$$\int_{Q_T} (\overline{\varphi} + N\overline{u})(u - \overline{u})dxdt \geq 0 \quad \forall u \in U_{ad}.$$

In the case of control problems with state constraints it is more difficult to derive the optimality conditions, mainly in the case of point–wise state constraints, for instance $|y(x)| \leq 1$ for every $x \in \Omega$. In fact this is an infinity number of constraints, one constraint for every point of Ω. The reader is referred to Bonnans and Casas [28, 27].

Difficulties also appear to obtain the optimality conditions in the case of state equations with more complicated linearities. A situation of this nature, especially interesting by the applications, is the one that arises in the control of the Navier–Stokes equations. See for instance Abergel and Casas [1].

Before giving the basic results on optimal control problems for the Navier-Stokes, we discuss the concept of *turnpike property*. More specifically, we analyze the convergence of the trajectories and controls which are optimal in $[0,T]$ toward the stationary state and control which are optimal for the corresponding stationary regime.

In general terms, the question could be set as follows, see for instance [98]. We are given a dynamics

$$\begin{cases} x_t &= f(x,u) \quad , \\ x(0) &= x_0 \qquad\quad , \end{cases} \tag{1.19}$$

and a corresponding control problem

$$\min_u J^T(u) := \int_0^T L(x,u)ds, \quad x \text{ solution of (1.19)},$$

under some conditions ensuring that this control problem as well as the stationary analogue, namely,

$$\min_u J_s(u) := L(x,u), \qquad \text{with the constraint } f(x,u) = 0,$$

are well posed. In other words, we assume that both J^T and J_s admit minimal controls and states, possibly unique.

The question is the following: to which extent do the long horizon optimal controls and states $u^T(t), x^T(t)$ approximate the stationary ones $\overline{u}, \overline{x}$ as $T \to \infty$?

Here, we present one of the result proved by Porretta and Zuazua [98], specifically the parabolic case. We consider the Dirichlet problem for a parabolic equation with internal control and observation. In a bounded domain $\Omega \subset \mathbb{R}^d$, let us consider the equation

$$\begin{cases} y_t - \text{div}\,(M(x)\nabla y) + c(x)y + B(x)\cdot\nabla y &= u\chi_\omega \quad , \quad \text{in } Q_T = \Omega \times (0,T), \\ y &= 0 \qquad , \quad \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ y(0) &= y_0 \qquad , \quad \text{in} L^2(\Omega), \end{cases} \tag{1.20}$$

where $M(x) \in L^\infty(\Omega; \mathbb{R}^d \times \mathbb{R}^d)$ satisfies $\lambda I \leq M(x) \leq \Lambda I$ for some $\lambda, \Lambda > 0$, and where $c(x) \in L^\infty(\Omega)$, $c(z) \geq 0$, and $B(x) \in L^\infty(\Omega)^d$ (though we could assume some more general integrability condition).

Then, consider the associated control problem

$$\min J^T(u) = \frac{1}{2} \int_0^T [|y(t) - z|^2_{L^2(\omega_0)} + |u(t)|^2_{L^2(\omega)}] dt, \tag{1.21}$$

where $u \in L^2(0, T; L^2(\omega))$ and y solves (1.20). Here ω and ω_0 are two open subsets of Ω.

Setting $X = H^1_0(\Omega)$, $X' = H^{-1}(\Omega)$, and $H = L^2(\Omega)$, and

$$A(y) = -\text{div}\ (M(x)\nabla y) + c(x)y + B(x) \cdot \nabla y,$$

we have $A \in \mathcal{L}(H^1_0(\Omega), H^{-1}(\Omega))$, and actually A is an isomorphism, since we have

$$\|y\|_X \leq K \|Ay\|_{X'}.$$

Moreover, observe that since A has positive first eigenvalue, both A and A^* are exponentially stable which implies some 'observability" inequality for the evolution equation (1.20) and also a similar 'observability" estimate for the adjoint state equations associated to (1.20), see the work [98] where the authors studied the turnpike phenomena in the abstract parabolic case. Then, we obtain the following stability result due to Porretta and Zuazua [98].

Theorem 1.10 (Porretta and Zuazua [98]) *Let us consider the control problem* (1.21) *and let* (u^T, y^T) *be the optimal control and state. Then, there exists* $\lambda > 0$ *such that*

$$\|y^T(t) - \overline{y}\|_{L^2(\Omega)} + \|u^T(t) - \overline{u}\|_{L^2(\Omega)} \leq K(e^{-\lambda t} + e^{-\lambda(T-t)}) \quad \forall t \in [0, T],$$

where $\overline{u}$ *and* $\overline{y}$ *are the optimal and state of the corresponding stationary control problem.*

Now, we present the most important results for the optimal control problem associated to the Navier–Stokes equations in the two dimensional case. Both stationary and evolutionary equations are considered. Because it is impossible to present all the bibliography related to this problem, we refer the reader the following literature [1, 2, 25, 33, 35, 45, 64, 65, 112].

We consider the mathematical formulation of an optimal control problem associated with the tracking of the velocity of a Navier–Stokes flow in a bounded two–dimensional domain through the adjustment of a distributed control. The existence of optimal solutions is showed and the first order necessary conditions for optimality are used to derive an optimality system of partial differential equations whose solutions provide optimal states and control.

The purpose of the velocity tracking problem is to steer, over time, a velocity field to a given target velocity field. We consider controls that act as a distributed body force; the state of the system, i.e., the velocity and pressure fields, is the solution of the Navier–Stokes system of partial differential equations. The cost or objective functional is a quadratic functional involving the state and the control variables; the functional measures, in an appropriate norm, the distance between the flow velocity and the target velocity fields, and through a penalty term (in the evolutionary case), also measures the cost of control. Thus, the minimization of the functional is used to both drive the flow towards the target flow and to limit the cost of control.

Firstly, we are concerned with the optimal control of the stationary Navier–Stokes equations, when the control is allowed to act as a point–wise constrained body force on a portion of the domain. We present the existence of the mentioned optimal solution and the first order necessary condition for the problem.

Let ω be an open connected subset of $\Omega \subset \mathbb{R}^d$. The mathematical setting of the problem is find $(\overline{y}, \overline{u}) \in H \times U_{ad}$, where $H = \{z \in (H^1(\Omega))^d \;:\; \text{div } z = 0\}$ and $U_{ad} = \{v \in (L^2(\omega))^d \;:\; a \leq v \leq b, \text{ a.e.}\}$, which solves

$$\min J(y,u) = \frac{1}{2}\int_\Omega |y - z_d|^2 dx + \frac{\alpha}{2}\int_\omega |u|^2 dx, \tag{1.22}$$

subject to

$$\begin{cases} -\mu\Delta y + (y\cdot\nabla)y + \nabla p &= Bu \quad , \quad \text{in } \Omega, \\ \text{div } y &= 0 \quad , \quad \text{in } \Omega, \\ y &= g \quad , \quad \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

where $z_d \in (L^2(\Omega))^d$, $a, b \in (L^2(\omega))^d$, $g \in (H^{1/2}(\partial\Omega))^d$ and $B \in \mathcal{L}(L^2(\omega)^d, L^2(\Omega)^d)$, with

$$Bu = \begin{cases} u \text{ in } \omega \\ 0 \text{ in } \Omega \setminus \omega. \end{cases}$$

Then, we have the following result of existence of minimum of the problem (1.22) due to De los Reyes [45].

Theorem 1.11 (De los Reyes [45]) *There exists an optimal solution* $(\overline{y}, \overline{u})$ *of problem* (1.22).

Besides, the author of [45] proved a sufficient requirement for the satisfaction of the regular point condition (see [114]). Thereafter the existence of appropriate Lagrange multipliers is justified and the optimality system for (1.22) is derived. Here, we present the optimality system associated to the problem (1.22).

We define the active sets for $(\overline{y}, \overline{u})$ by

$$\mathcal{A}^a := \{x \in \omega \;:\; \overline{u} = a(x) \text{ a.e.}\} \text{ and } \mathcal{A}^b := \{x \in \omega \;:\; \overline{u} = b(x) \text{ a.e.}\}$$

and the inactive set by

$$\mathcal{F} = \omega \setminus (\mathcal{A}^a \cup \mathcal{A}^b).$$

Theorem 1.12 (De los Reyes [45]) *Let* $(\overline{y}, \overline{u})$ *be an optimal solution for* (1.22), *such that* $\mu > \mathcal{M}(\overline{y}) = \sup_{v\in V} \dfrac{|((v\cdot\nabla)v, y)|}{\|v\|_V^2}$, *where* $V = \{v \in (H_0^1(\Omega))^d \;:\; \text{div } v = 0\}$. *Then it satisfies* *the following optimality system in variational sense:*

$$\begin{cases} -\mu\Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla p &= B\overline{u} \quad , \quad in\ \Omega, \\ \text{div}\ \overline{y} &= 0 \quad , \quad in\ \Omega, \\ \overline{y} &= g \quad , \quad on\ \partial\Omega, \\ -\mu\Delta\lambda - (\overline{y}\cdot\nabla)\lambda + (\nabla\overline{y})^T\lambda + \nabla\phi &= z_d - \overline{y} \quad , \quad in\ \Omega, \\ \text{div}\ \lambda &= 0 \quad , \quad in\ \Omega, \\ \lambda &= 0 \quad , \quad on\ \partial\Omega, \\ \alpha\overline{u} &= B^*\lambda - \nu \quad , \quad in\ \Omega, \\ a \leq & \overline{u} \leq b \quad , \quad a.e., \\ \nu &\geq 0 \quad , \quad on\ \mathcal{A}^b, \\ \nu &\leq 0 \quad , \quad on\ \mathcal{A}^a, \\ \nu &= 0 \quad , \quad on\ \mathcal{F}. \end{cases}$$

Beside, the Lagrange multiplier $\lambda \in V$ satisfies the following estimate:

$$\|\lambda\|_V \leq k\sigma(\overline{y})\|\overline{y} - z_d\|,$$

where $\sigma(y) := \dfrac{1}{\mu - \mathcal{M}(y)}$ and k is the Poincaré inequality constant.

For the instationary Navier–Stokes system we consider the following initial–boundary value problem

$$\begin{cases} y_t - \mu\Delta y + (y\cdot\nabla)y + \nabla p &= u \quad , \quad \text{in } Q, \\ \text{div}\ y &= 0 \quad , \quad \text{in } Q, \\ y &= 0 \quad , \quad \text{on } \Sigma, \\ y(0) &= u_0 \quad , \quad \text{in } \Omega, \end{cases} \tag{1.24}$$

where the cylindrical domain is represented by $Q = \Omega\times(0,T)$ with boundary $\Sigma = \partial\Omega\times(0,T)$, where $\Omega \subset \mathbb{R}^2$ is a bounded region with boundary $\partial\Omega$.

Let $u \in L^2(0,T;L^2(\Omega)^2)$ be the distributed control. Given $T > 0$, we define the functional

$$J(y,u) := \frac{\alpha}{2}\int_0^T \|y(t) - z_d(t)\|^2_{L^2(\Omega)}dt + \frac{\beta}{2}\int_0^T \|u(t)\|^2_{L^2(\Omega)}dt + \frac{\gamma}{2}\int_\Omega |y(T) - z_d(T)|^2 dx, \tag{1.25}$$

where $z_d \in C([0,T];H^2(\Omega)^2 \cap V)$.

The minimization of the first term in this functional is the real goal of the velocity tracking problem, which is to keep the solution y close to z_d over Q. The second term is introduced in order to bound the control function and to prove the existence of an optimal control. And the last term is necessary in order to keep the solution y close to z_d near the time T.

As in the stationary case, the existence of minimum for the functional (1.25) was proved by Gunzburger and Manservisi [60].

Theorem 1.13 (Gunzburger and Manservisi [60]) *There exists an optimal solution $(\overline{y},\overline{u})$ of problem* (1.25).

After testing that the functional (1.25) is Gâteaux differentiable, the authors proved the following first order necessary optimality condition.

Theorem 1.14 (Gunzburger and Manservisi [60]) *If $(\overline{y}, \overline{u})$ is an optimal pair, then we have*

$$\overline{u} = -\frac{1}{\beta}\varphi,$$

where φ is the solution of the adjoint problem

$$\begin{cases} -\varphi_t - \mu\Delta\varphi - (\overline{y}\cdot\nabla)\varphi + (\nabla\overline{y})^T\varphi + \nabla q &= \alpha(y - z_d) \quad , \quad \textit{in } Q, \\ \operatorname{div}\overline{y} &= 0 \quad , \quad \textit{in } Q, \\ \overline{y} &= 0 \quad , \quad \textit{on } \Sigma, \\ \overline{y}(T) &= \gamma(\overline{y}(T) - z_{ad}(T)) \quad , \quad \textit{in } \Omega. \end{cases}$$

1.3 Contents of the thesis

In this section we briefly introduce the problems studied in this thesis. Three main subjects are covered:

1. **Size estimates for the stationary Stokes equations**: In Chapter 3, we analyze the inverse problem of establish a quantitative estimate of the size of an obstacle immersed in a domain, which is filled with a viscous fluid. We prove that the size of the obstacle can be estimated in terms of the boundary measurements of the Cauchy forces at the boundary of the domain. The results of this chapter are based on the articles [8, 88], in collaboration with E. Beretta, C. Cavaterra, and J.H. Ortega.

2. **Turnpike property of Navier–Stokes equations:** In Chapter 4, we consider an optimal control tracking–type problem associated to the Navier–Stokes equations. We study the relationship between the optimal solution of the evolutionary and stationary problem, when the time–horizon goes to infinity. We prove in a local sense that, when the time is large enough, the optimal forward and backward trajectories converge towards the stationary optimal solution and adjoint state, with an exponential rate. In this case we need to ensure some smallness condition of the solution for the stationary Navier–Stokes equations. Besides, we then consider the simpler problem of time–independent controls showing that the accumulation point of a sequence of controls for the evolutionary optimal control problem is an optimal control for the stationary problem. The results of this chapter are based on the articles [99, 98], in collaboration with E. Zuazua.

3. **Inverse problem for the evolutionary Stokes equations:** In Chapter 5, we present our work about the identifiability of the viscosity function, when the fluid is modeled by the nonstationary Stokes equations. The main idea is to prove that, as the time goes to infinity, thus the determination of the viscosity is reduced to prove the result for the stationary case. We formulate our initial results and describe the associated

future work. The chapter is based on the articles [47, 48, 62, 69], in collaboration with R. Lecaros and J.H. Ortega.

In what follows, we describe the most important aspects of each of these topics, the obtained results and the methods we have developed.

1.3.1 Chapter 3: Size Estimates of an Obstacle in a Stationary Stokes Fluid.

Chapter 3 is devoted to the study of an inverse problem for the Stokes equations. We prove, under a list of suitable hypothesis, that if we consider an obstacle D immersed in a region which is filled with a viscous fluid, the size of this inclusion D can be estimated for above and below in terms of the normalized power gap (concept introduced by Alessandrini et al [8, 10, 13, 102]).

Mathematically, consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega$ of Lipschitz class with constants ρ_0, M_0 and an obstacle $D \subset \Omega$ with boundary ∂D of Lipschitz class too with constants ρ, L. So, we consider the pair $(u, p) \in (H^1(\Omega \setminus \overline{D}))^d \times L^2(\Omega \setminus \overline{D})$ as the weak solution to the following Stokes problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\sigma(u,p)) &=& 0 &, \quad \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div}\ u &=& 0 &, \quad \text{in } \Omega \setminus \overline{D}, \\ u &=& g &, \quad \text{on } \partial\Omega, \\ u &=& 0 &, \quad \text{on } \partial D. \end{array} \right. \tag{1.26}$$

On the other hand, when the obstacle is absent, we shall denote by $(u_0, p_0) \in (H^1(\Omega))^d \times L^2(\Omega)$ the unique weak solution to the Dirichlet problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\sigma(u_0,p_0)) &=& 0 &, \quad \text{in } \Omega, \\ \mathrm{div}\ u_0 &=& 0 &, \quad \text{in } \Omega, \\ u_0 &=& g &, \quad \text{on } \partial\Omega. \end{array} \right. \tag{1.27}$$

In both cases (1.26) and (1.27) we consider $g \in (H^{1/2}(\partial\Omega))^d, \quad g \not\equiv 0, \quad \dfrac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}} \leq c_0.$

As we want to estimate the size of D in terms of the pair of Cauchy boundary data $(g, \sigma(u,p)n)$ on $\partial\Omega$, we introduce the following number

$$W = \int_{\partial\Omega} (\sigma(u,p)n) \cdot u \quad , \quad W_0 = \int_{\partial\Omega} (\sigma(u_0,p_0)n) \cdot u_0,$$

and the normalized power gap

$$\frac{W - W_0}{W_0}.$$

Therefore, our goal is to derive estimates of the size of D in terms of $\frac{W-W_0}{W_0}$.

In order to give the two main results of this Chapter, we introduce the most important assumption for the obtention of the estimate. For the lower estimate, we assume that there exists a positive constants d_0 and h_1, such that

$$d(D, \partial\Omega) \geq d_0 > 0, \quad |D_{h_1}| \geq \frac{1}{2}|D|, \quad \int_{\partial\Omega} \sigma(u,p)n = 0. \tag{1.28}$$

The first assumption tell us that the obstacle is strictly contained in Ω, the second means that D satisfies the fatness condition and finally, the third is a technical condition on the solution of (1.26).

Then, we obtain the first main result of this Chapter, specifically the upper estimate of the size of the inclusion D.

Theorem 1.15 *Assume the conditions* (1.28)*. Then, we obtain*

$$|D| \leq K \left(\frac{W - W_0}{W_0} \right),$$

where the constant $K > 0$ *depends on* $\Omega, d, d_0, h_1, M_0, M_1,$ *and* $\|g\|_{H^{1/2}(\partial\Omega)} / \|g\|_{L^2(\partial\Omega)}$.

Let us give some comments about the second assumption on D, the *fatness condition.* In [8] the authors proved a similar Theorem for the estimates of cavities for conductivity problem. They used similar assumption on the cavity, but without the fatness condition and obtain the following upper estimate

$$|D| \leq K \left(\frac{W - W_0}{W_0} \right)^{1/p},$$

where $p > 1$. The proof of this inequality is based on the fact that the gradient of the solution of the background conductivity problem is a Muckenhoupt weight, see [58]. This type of weight is a consequence on the fact that the conductivity problem satisfy the doubling inequality. However, in the case of the Stokes problem, as far as we know, the doubling inequality has not been proved. For this reason, in this Chapter we need to assume the fatness condition on D.

On the other hand, for the upper estimate we need the following scale–invariant fatness assumption with constant $Q > 0$ on the obstacle:

$$\mathrm{diam}(D) \leq Q\rho. \tag{1.29}$$

Therefore, we obtain the second main result of this Chapter, the lower estimate of the volume of D.

Theorem 1.16 *Assume the condition* (1.29)*. Then, it holds*

$$C \frac{(W - W_0)^2}{W W_0} \leq |D|, \tag{1.30}$$

where $C > 0$ *depends on* $|\Omega|, d, d_0, L,$ *and* Q.

The proof of this inequality is based on interior regularity, the Rellich's identity (see [30, 50]) to the solution of the Stokes problem, among others.

1.3.2 Chapter 4: Turnpike Property for Two–Dimensional Navier–Stokes Equations.

In Chapter 4 we analyze the turnpike property, introduced in the previous section, in the case when we consider the Navier–Stokes equations. The motivation of this type of property in models of the fluids mechanics arises naturally. For example, in aeronautics and automotive industry the optimal design problem is based on stationary models, see for instance [66], where is assumed or understood that the optimal stationary shape is close enough to the respective evolutionary shape.

This Chapter gives a first approach to this situation, considering the case of optimal control problem. Due to the nonlinearity of the problem, we prove the turnpike property for the optimal forward and backward state in a local sense. Specifically, we show that the solution of the optimality system associated to the optimal control problem satisfies the turnpike.

The main idea is to develop a local analysis around a given steady optimal control, and then, by fixed point argument, prove the local nature of the turnpike. Namely, we have the following optimality system for the nonstationary and stationary case

$$\left\{\begin{array}{rcll} y^T_t-\mu\Delta y^T+(y^T\cdot\nabla)y^T+\nabla p^T &=& -q^T & ,\ \text{in } Q_T,\\ \text{div }\ y^T &=& 0 & ,\ \text{in } Q_T,\\ y^T &=& 0 & ,\ \text{on } \Gamma_T,\\ y^T(x,0) &=& y_0(x) & ,\ x\in\Omega,\\ -q^T_t-\mu\Delta q^T-(y^T\cdot\nabla)q^T+(\nabla y^T)^Tq^T+\nabla\pi^T &=& y^T-x^d & ,\ \text{in } Q_T,\\ \text{div }\ q^T &=& 0 & ,\ \text{in } Q_T,\\ q^T &=& 0 & ,\ \text{on } \Gamma_T,\\ q^T(x,T) &=& q_0 & ,\ x\in\Omega. \end{array}\right. \tag{1.31}$$

and

$$\left\{\begin{array}{rcll} -\mu\Delta\overline{y}+(\overline{y}\cdot\nabla)\overline{y}+\nabla\overline{p} &=& -\overline{q} & ,\ \text{in } \Omega,\\ \text{div }\ \overline{y} &=& 0 & ,\ \text{in } \Omega,\\ \overline{y} &=& 0 & ,\ \text{on } \partial\Omega,\\ -\mu\Delta\overline{q}-(\overline{y}\cdot\nabla)\overline{q}+(\nabla\overline{y})^T\overline{q}+\nabla\overline{\pi} &=& \overline{y}-x^d & ,\ \text{in } \Omega\\ \text{div }\ \overline{q} &=& 0 & ,\ \text{in } \Omega,\\ \overline{q} &=& 0 & ,\ \text{on } \partial\Omega. \end{array}\right. \tag{1.32}$$

Considering $y=\overline{y}+z$, $p=\overline{p}+\eta$, $q=\overline{q}+\varphi$, and $\pi=\overline{\pi}+\nu$, the optimality system (1.31) linearized around this solutions takes the form

$$\left\{\begin{array}{rcll} z_t-\mu\Delta z+(\overline{y}\cdot\nabla)z+(z\cdot\nabla)\overline{y}+\nabla\eta &=& -\varphi & ,\ \text{in } Q_T,\\ \text{div }\ z &=& 0 & ,\ \text{in } Q_T,\\ z &=& 0 & ,\ \text{on } \Gamma_T,\\ z(x,0) &=& z_0 & ,\ \text{in } \Omega,\\ -\varphi_t-\mu\Delta\varphi-(\overline{y}\cdot\nabla)\varphi+(\nabla\overline{y})^T\varphi+\nabla\nu &=& z-(\nabla z)^T\overline{q}+(z\cdot\nabla)\overline{q} & ,\ \text{in } Q_T,\\ \text{div }\ \varphi &=& 0 & ,\ \text{in } Q_T,\\ \varphi &=& 0 & ,\ \text{on } \Gamma_T,\\ \varphi(x,T) &=& \varphi_0 & ,\ \text{in } \Omega, \end{array}\right. \tag{1.33}$$

where $z_0 = y_0 - \overline{y}$ and $\varphi_0 = q_0 - \overline{q}$.

The system (1.33) can be seen as an optimal control problem for the Oseen equation, and using the result for the linear case given in [98], we obtain the following theorem.

Theorem 1.17 *Under suitable smallness condition, there exists some $\varepsilon > 0$ such that for every y_0, q_0 with*

$$\|y_0 - \overline{y}\|_{L^2(\Omega)} + \|q_0 - \overline{q}\|_{L^2(\Omega)} \leq \varepsilon,$$

there exists a solution of the optimality system (1.31) *such that*

$$\|y^T(t) - \overline{y}\|_{L^2(\Omega)} + \|q^T(t) - \overline{q}\|_{L^2(\Omega} \leq C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad \forall t < T, \tag{1.34}$$

where $\gamma > 0$ is the stabilizing rate of the linearized optimality system (1.33).

In the case when the controls are independent of time, $u = u(x)$, using the classical Γ–convergence and suitable exponential stabilization of the nonstationary solutions of the Navier–Stokes problem, we prove the following theorem.

Theorem 1.18 *Let (y^{T_n}, u^{T_n}) be an optimal state and control for $T = T_n$. Then any accumulation point (y_∞, u_∞), as $n \to \infty$, is an optimal steady state and control.*

1.3.3 Chapter 5: Inverse Viscosity Boundary Value Problem for the Evolutionary Stokes Equation.

In Chapter 5 we study an inverse identification problem associated to the Stokes equations in three dimensions. We focus on the following boundary value problem

$$\left\{ \begin{array}{rcll} -\text{div }(\sigma_\mu(u,p)) + \lambda u & = & 0 & , \quad \text{in } \Omega, \\ \text{div } u & = & 0 & , \quad \text{in } \Omega, \\ u & = & g^0 & , \quad \text{on } \partial\Omega, \end{array} \right. \tag{1.35}$$

where $\sigma_\mu(u,p) = 2\mu e(u) - pI$ is the stress tensor, $e(u) = ((\nabla u) + (\nabla u)^T)/2$ is the strain tensor and $\lambda > 0$. Here $\mu(x) > \mu_0 > 0$ is the kinematic viscosity function.

We are interested in the inverse problem of determining μ from the knowledge of boundary measurements. Mathematically, the boundary measurements are encoded in the Cauchy data of all solutions satisfying (1.35). Precisely, we define the set

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)n|_{\partial\Omega})\} \subset (H^{3/2}(\partial\Omega))^3 \times (H^{1/2}(\partial\Omega))^3, \tag{1.36}$$

where (u,p) is the solution of (1.35). Then, our inverse problem is to determine μ from the knowledge of the operator S_μ.

In order to obtain the identification result, we need to consider a suitable regularity for the viscosity, given by the following hypothesis.

(H1) Let μ_1 and μ_2 be two viscosity functions. We assume that

$$\mu_1, \mu_2 \in C^{n_0}(\overline{\Omega}), \ \forall n_0 \geq 8, \tag{1.37}$$

$$\mu_i \geq \mu_0 > 0, \ \forall i = 1, 2, \tag{1.38}$$

$$\mu_1(x) = \mu_2(x), \ \forall x \in \partial\Omega. \tag{1.39}$$

Using the construction of exponentially growing solutions given by Eskin and Ralston [47], we have the following result.

Theorem 1.19 *Let* (u, p) *be the solution to the stationary Stokes problem* (1.35)*. Assume that* $\mu_1(x)$ *and* $\mu_2(x)$ *are two viscosity function satisfying* (**H1**)*. Let* $S^E_{\mu_1}$ *and* $S^E_{\mu_2}$ *be the Cauchy data associated with* μ_1 *and* μ_2*, respectively. If* $S^E_{\mu_1} = S^E_{\mu_2}$*, then* $\mu_1 = \mu_2$*.*

Now, we consider the following boundary value problem

$$\begin{cases} u_t - \text{div}\,(\sigma_\mu(u, p)) &= 0 \quad , \quad \text{in } \Omega \times (0, T), \\ \text{div}\ u &= 0 \quad , \quad \text{in } \Omega \times (0, T), \\ u &= g \quad , \quad \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x) \quad , \quad \text{in } \Omega, \end{cases} \tag{1.40}$$

and the boundary measurements encoded in the Cauchy data

$$S_\mu = \{(u|_{\partial\Omega\times(0,T)}, \sigma_\mu(u, p)n|_{\partial\Omega\times(0,T)})\}, \tag{1.41}$$

where (u, p) is the solution of (1.40).

The idea is to obtain an identifiability result of the viscosity, in the evolutionary case (1.40), using S_μ above defined, based on the uniqueness given in the stationary case, see Theorem 1.19.

The question we address is as follows: *can we get identifiability of the viscosity function of the evolutionary Stokes equation from the stationary problem as the horizon time goes to infinity?*

Chapter 2

Mathematical Theory of Incompressible Stokes and Navier–Stokes Equations

In this chapter we introduce and present the basics results concerning with the theory of the incompressible Stokes and Navier–Stokes equations, both evolutionary and stationary. We provide some functional analytic background to study this equations. We rely on the books by Boyer and Fabrie [29], Galdi [57], Girault and Raviart [59], and Temam [107].

The existence, uniqueness and regularity for the Stokes systems is completely understood in the two and three dimensional case. The Navier–Stokes systems is a little different, because in the two–dimensional case we know all the previous properties; however in the three–dimensional case there are many open problems connected with smoothness and uniqueness of solutions.

The outline of the chapter is a follows. In Section 2.1 we introduce the notations, basic definitions and functionals spaces that we will use through the thesis. Section 2.2 and 2.3 are devoted to the study of the Stokes equations, both stationary and evolutionary, respectively. We analyzed existence and uniqueness of solutions, also regularity results and spectral properties of the Stokes operator. Analogously, in Sections 2.4 and 2.5 we analyze the Navier–Stokes system for the stationary and nonstationary case, respectively. Finally, in Section 2.6 we introduce the linearized equations from the Navier–Stokes system. This model is called the Oseen equation or the Stokes–Oseen equation.

2.1 Function spaces and auxiliary results

In the whole chapter we assume that $\Omega \subset \mathbb{R}^d$, with $d = 2$ o $d = 3$, is a open bounded domain with boundary $\Gamma = \partial\Omega$. We denoted by n the outward normal vector of the boundary. In general, we will need some smoothness condition of Ω and Γ. In many situations, it suffices to assume

$$\Omega \text{ is locally Lipschitz.} \tag{2.1}$$

This means that for each point $x \in \Gamma$, there is a neighborhood $\mathcal{O}$ such that $x \in \mathcal{O}$ and $\Gamma \cap \mathcal{O}$ is the graph of a Lipschitz continuous function, see [3]. However, this assumption can be insufficient in some other cases. For this reason, sometime we will need that

$$\Omega \text{ is of class } C^k, \tag{2.2}$$

for some $k \geq 1$ integer. This mean that the boundary Γ is a $n-1$–dimensional manifold of class C^k and Ω is locally located on one side of Γ, see [3]. We observe that if Γ is of class C^1, then the boundary Γ is locally Lipschitz. Both smoothness assumption imply the so–called cone property.

In the Chapter 3 we need another type of regularity of Ω. Let $x \in \mathbb{R}^d$, we denote by $B_r(x)$ the ball in $\mathbb{R}^d$ centered in x of radius r. We will indicate by $\cdot$ the scalar product between vectors or matrices. We set $x = (x_1, \ldots, x_d)$ as $x = (x', x_d)$, where $x' = (x_1, \ldots, x_{d-1})$.

Definition 2.1 (Def. 2.1 [8]) *Let $\Omega \subset \mathbb{R}^d$ be bounded domain. We say that $\partial\Omega$ is of class $C^{k,\alpha}$, with constants ρ_0, $M_0 > 0$, where k is a nonnegative integer and $\alpha \in [0,1)$, if, for any $x_0 \in \partial\Omega$, there exists a rigid transformation of coordinates, in which $x_0 = 0$ and*

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) : \; x_n > \varphi(x')\},$$

where φ is a function of class $C^{k,\alpha}(B'_\rho(0))$, such that

$$\begin{aligned} \varphi(0) &= 0, \\ \nabla\varphi(0) &= 0, \; \textit{if } k \geq 1, \\ \|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} &\leq M_0\rho_0. \end{aligned}$$

When $k = 0$ and $\alpha = 1$ we will say that $\partial\Omega$ is of Lipschitz class with constants ρ_0, M_0.

Observation We normalize all norms in such a way that they are dimensionally equivalent to their argument, and coincide with the usual norms when $\rho_0 = 1$. In this setup, the norm taken in the previous definition is intended as follows:

$$\|\phi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} = \sum_{i=0}^{k} \rho_0^i \|D^i\phi\|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+\alpha}|D^k\phi|_{\alpha, B'_{\rho_0}(0)},$$

where $|\cdot|$ represents the α-Hölder seminorm

$$|D^k\phi|_{\alpha, B'_{\rho_0}(0)} = \sup_{x',y' \in B'_{\rho_0}(0), x' \neq y'} \frac{|D^k\phi(x') - D^k\phi(y')|}{|x'-y'|^\alpha},$$

and $D^k\phi = \{D^\beta\phi\}_{|\beta|=k}$ is the set of derivatives of order k. Similarly we set the norms

$$\|u\|^2_{L^2(\Omega)} = \frac{1}{\rho_0^d}\int_\Omega |u|^2 \quad \text{and} \quad \|u\|^2_{H^1(\Omega)} = \frac{1}{\rho_0^d}\left(\int_\Omega |u|^2 + \rho_0^2 \int_\Omega |\nabla u|^2\right).$$

We denote by $L^p(\Omega)$, with $1 \le p \le \infty$, the space of real functions defined on Ω with the p-th power absolutely integrable for the Lebesgue measure $dx = dx_1 \dots dx_d$ when $p \in [1, \infty)$, and essentially bounded real functions in the case $p = \infty$. The $L^p(\Omega)$ space is a Banach space with the norm

$$\|u\|_p := \|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)^p| dx \right)^{1/p}, \ p \in [1, \infty),$$
$$\|u\|_\infty := \|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{\Omega} |u(x)|.$$

In the particular case when $p = 2$, we have that $L^2(\Omega)$ is a Hilbert space with scalar product

$$(u, v)_2 = \int_\Omega u(x) v(x) dx.$$

Further, we consider the quotient space

$$L^2_0(\Omega) = L^2(\Omega)/\mathbb{R} = \left\{ q \in L^2(\Omega) \ : \ \int_\Omega q(x) dx = 0 \right\},$$

represented by the class of functions of $L^2(\Omega)$ which differ by an additive constant. We equip this space with the quotient norm

$$\|v\|_{L^2_0(\Omega)} = \inf_{\alpha \in \mathbb{R}} \|v + \alpha\|_2.$$

Let $W^{m,p}(\Omega)$ be the space of functions in $L^p(\Omega)$ whose weak derivatives up to order m are functions in $L^p(\Omega)$, where m is an integer and $1 \le p \le \infty$. Equipped with the norm

$$\|u\|_{m,p} := \|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|j| \le m} \|D^j u\|_p^p \right)^{1/p}$$

it is a Banach space. When $p = 2$, the space $H^m(\Omega) := W^{m,2}(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_{m,2} = \sum_{|j| \le m} (D^j u, D^j v)_2.$$

Let $\mathcal{D}(\Omega)$ be the space of all C^∞–functions with compact support in Ω. The closure of $\mathcal{D}(\Omega)$ in the norm of $W^{m,p}(\Omega)$ is denoted by $W^{m,p}_0(\Omega)$, respectively $H^m_0(\Omega)$ when $p = 2$.

For a comprehensive introduction of Solobev spaces we refer the reader to [3, 49].

In this thesis we shall work with d–component vector–valued functions, whose components are elements of some of the above spaces. We shall use the following notation

$$(\mathcal{D}(\Omega))^d = \{(u_1, \dots, u_d) \ : \ u_i \in C^\infty(\Omega), \ i = 1, \dots, d\},$$
$$(L^p(\Omega))^d = \{(u_1, \dots, u_d) \ : \ u_i \in L^p(\Omega), \ i = 1, \dots, d\},$$
$$(W^{m,p}(\Omega))^d = \{(u_1, \dots, u_d) \ : \ u_i \in W^{m,p}(\Omega), \ i = 1, \dots, d\},$$

which is analogously employed for all other kinds of spaces. These product spaces will be equipped with the usual product and norm or an equivalent norm, except the case $(\mathcal{D}(\Omega))^d$, that is not a normed space at all.

By X^* we denote the *dual* of a normed linear space X. The symbol $\langle \cdot, \cdot \rangle$ denotes the duality: if $f \in X^*$, $\varphi \in X$, then $\langle f, \varphi \rangle = f(\varphi)$.

We recall that if Ω is bounded or bounded in some direction, then the Poincaré inequality holds [49]

$$\|u\|_2 \leq c(\Omega)\|Du\|_2 \ , \ \forall u \in H_0^1(\Omega), \tag{2.3}$$

where D is the derivative in the sense of the distributions and $c(\Omega)$ is a constant depending only on Ω. We know that, see [49], in this case the norm on $H_0^1(\Omega)$ is equivalent to the norm

$$\|u\|_{1,0} := \|u\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^{d} \|D^i u\|_2 \right)^{1/2} .$$

The space $H_0^1(\Omega)$ is also a Hilbert space with the scalar product

$$(u, v)_{1,0} := \sum_{i=1}^{d} (D^i u, D^i(v))_2.$$

Further, let us recall that $H^{1/2}(\Gamma)$ is the subspace of $L^2(\Gamma)$ formed by the traces on Γ of all functions from $H^1(\Omega)$. We use the notation $u|_\Gamma = (u_1|_\Gamma, \dots, u_d|_\Gamma)$ for the trace of a vector function $u = (u_1, \dots, u_d) \in (H^1(\Omega))^d$ on Γ.

Also, a very useful tool when dealing with Sobolev spaces is the following theorem. We refer the reader to [3, 49] for the proof and further details.

Theorem 2.2 (Sobolev imbedding theorem) *Let $\Omega \subset \mathbb{R}^d$ be a domain having the cone property. Suppose that $mp < d$, then the imbedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for $q < \frac{dp}{d-mp}$. If $mp > d$, then we hace continuity of the imbedding

$$W^{m,p}(\Omega) \hookrightarrow C(\overline{\Omega}).$$

In the theory of Stokes and Navier–Stokes equations we shall work with spaces of *solenoidal functions* satisfying the constraint div $u = 0$. We put

$$\mathcal{V} = \{u \in (\mathcal{D}(\Omega))^d \ : \ \text{div } u = 0\}.$$

The closures of $\mathcal{V}$ in $(L^2(\Omega))^d$ and $(H_0^1(\Omega))^d$ are the basic spaces used in the theory of incompressible Stokes and Navier–Stokes systems

$$H := \overline{\mathcal{V}}^{(L^2(\Omega))^d},$$
$$V := \overline{\mathcal{V}}^{(H_0^1(\Omega))^d}.$$

Their norms are the usual $(L^2(\Omega))^d$ and $(H_0^1(\Omega))^d$–norms and will be denoted by $\|\cdot\|_H$ and $\|\cdot\|_V$, respectively. Although these spaces have the same scalar product as L^2 and H_0^1, we will use $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ instead.

The following result is essential for the study, later, of the variational formulation of Stokes and Navier–Stokes and related equations. This property can be proved by using a result of De Rham [101].

Proposition 2.3 *Let $\Omega \subset \mathbb{R}^d$ be an open set and $f = \{f_1, \ldots, f_d\}$, $f_i \in \mathcal{D}'(\Omega)$, $i = 1, \ldots, d$. A necessary and sufficient condition that*

$$f = \nabla p,$$

for some $p \in \mathcal{D}'(\Omega)$, is that

$$\langle f, v\rangle = 0 \ , \ \forall v \in \mathcal{V}.$$

Proposition 2.4 ([107], Chapter I, Section 1, Proposition 1.2) *Let Ω be a bounded Lispchitz open set in $\mathbb{R}^d$. If a distribution p has all its first derivatives $D_i p$, $1 \le i \le d$, in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and*

$$\|p\|_{L^2(\Omega)\setminus\mathbb{R}} \le c(\Omega)\|\nabla p\|_{H^{-1}(\Omega)}.$$

We can now give the following characterizations of the spaces H, $H^\perp$ (the orthogonal complement of H in $(L^2(\Omega))^d$), and V.

Theorem 2.5 ([107], Theorems I.1.4 and I.1.6) *Let Ω be a Lipschitz open bounded set in $\mathbb{R}^d$. Then*

$$\begin{aligned} H &= \{u \in (L^2(\Omega))^d \ : \ \operatorname{div} u = 0, \ \gamma_n u = 0\}, \\ H^\perp &= \{u \in (L^2(\Omega))^d \ : \ u = \nabla p, \ p \in H^1(\Omega)\}, \\ V &= \{u \in (H_0^1(\Omega))^d \ : \ \operatorname{div} u = 0\}, \end{aligned}$$

where γ_n is the normal trace operator $\gamma_n \ : \ u \longmapsto n \cdot u|_\Gamma$.

In H, we can define a norm by

$$\|u\|_* := \sup_{v \in V\setminus\{0\}} \frac{(u,v)_H}{\|v\|_V}.$$

The closure of H with respect to the $\|\cdot\|_*$–norm is equal to V', which is the usual dual space of V. In V' we will use the above defined norm: $\|\cdot\|_{V'} := \|\cdot\|_*$. The spaces V, H, and V' satisfies

$$V \subset H = H' \subset V'$$

with dense and continuous imbedding. The duality pairing of V' and V is then the continuation of the H–scalar product to $V' \times V$. It is denoted by $\langle \cdot,\cdot\rangle_{V',V}$. Obviously, for $u \in H$ and $v \in V$ we have the following property

$$\langle u, v\rangle_{V',V} = \int_\Omega u(x)v(x)dx.$$

We will use once in a while the following spaces, which are the L^p–counterparts of H and V,

$$H_p := \overline{\mathcal{V}}^{(L^p(\Omega))^d},$$
$$V_p := \overline{\mathcal{V}}^{(W_0^{1,p}(\Omega))^d}.$$

2.2 The stationary Stokes equations

2.2.1 Stokes equations with homogeneous boundary conditions

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with boundary Γ, and let $f \in (L^2(\Omega))^d$ be a given vector function in Ω. We are looking vector function $u = (u_1, \ldots, u_d)$ representing the velocity of the fluid, and a scalar function p representing the pressure, which are defined in Ω and satisfy the following equations and boundary conditions

$$\begin{cases} -\mu\Delta u + \nabla p &= f \quad , \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad , \quad \text{in } \Omega, \\ u &= 0 \quad , \quad \text{on } \Gamma, \end{cases} \tag{2.4}$$

where μ is the coefficient of kinematic viscosity, a positive constant.

Definition 2.6 *A couple (u, p) is called the classical solution of the Stokes problem with homogeneous boundary conditions, if $u \in (C^2(\overline{\Omega}))^d$ and $p \in C^1(\overline{\Omega})$ satisfy equation* (2.4).

Let us assume there is a classical solution (u, p) satisfying (2.4). If v is an element of V, multiplying the first equation in (2.4) by v and integrating over Ω, we obtain

$$-\mu \int_\Omega \Delta u \cdot v dx + \int_\Omega v \cdot \nabla p dx = \int_\Omega f \cdot v dx. \tag{2.5}$$

Using the Green's theorem, the boundary condition $u|_\Gamma = 0$, and the incompressible assumption $\operatorname{div} v = 0$ on Ω, the equation (2.5) can be rewritten in the form

$$\mu(u, v)_V = \langle f, v \rangle_{V', V} \;, \; \forall v \in V. \tag{2.6}$$

Since u is divergence–free, the pressure disappears in the weak formulation due to $(u, \nabla p)_2 = 0$. Equation (2.6) suggests the following generalization of the concept of the solution of the Stokes problem.

Definition 2.7 *Let $\mu > 0$, $f \in (L^2(\Omega))^d$. We say that a vector function $u : \Omega \to \mathbb{R}^d$ is the weak solution of the Stokes problem with homogeneous boundary conditions, if*

$$u \in V \text{ and } \mu(u, v)_V = \langle f, v \rangle_{V', V} \; \forall v \in V. \tag{2.7}$$

Observation Let us notice that conditions div $u = 0$ and $u|_\Gamma = 0$ are already hidden in the assumption $u \in V$. Conditions (2.7) form the *weak formulation* of the Stokes problem.

We are now in a position to guarantee the existence and uniqueness of solutions to (2.7). For this we need the classical projection Lemma, known as Lax–Milgram Lemma.

Lemma 2.8 *Let W be a separable real Hilbert space with norm $\|\cdot\|_W$ and let $a(u, v)$ be a bilinear continuous form on $W \times W$, which is coercive. Then for each $l \in W'$, the dual space of W, there exists one and only one $u \in W$ such that*

$$a(u, v) = \langle l, v\rangle \ , \ \forall v \in W.$$

Then, we obtain the following result about the existence and uniqueness of the weak formulation of the Stokes equations.

Theorem 2.9 (Existence and Uniqueness) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain. Given $f \in (L^2(\Omega))^d$, the problem (2.7) has a unique solution u.*

Proof. The proof is an immediate consequence of the preceding lemma. Indeed, we take $W = V$ equipped with the norm associated $\|\cdot\|_V$ which is a separable space as a closed subspace of the separable space $(H_0^1(\Omega))^d$, $a(u, v) = \mu(u, v)_V$, and the form $v \longmapsto \langle f, v\rangle_{V',V}$. Then, we need to prove that the form $a(u, v)$ and $\langle f, \cdot\rangle_{V',V}$ are linear and continuous. Its linearity is obvious.

The continuity of the functional $\langle f, \cdot\rangle_{V',V}$ is a consequence of the inequalities

$$|(f, v)| \leq \|f\|_2\|v\|_2 \leq \|f\|_2\|v\|_{1,2} \leq c\|f\|_2\|v\|_V \ , \ \forall v \in V.$$

The properties of the form $a(\cdot, \cdot)$ follow from the fact that $a(\cdot, \cdot)$ is a positive multiple of a scalar product in the space V.

□

Observation It is easily observed that the velocity field u obtained above through the Lax–Milgram Lemma is the unique minimizer on V of the energy functional

$$J(w) = \frac{1}{2}\mu \int_\Omega |\nabla w|^2 dx - \int_\Omega f \cdot w dx \ , \ \forall w \in V.$$

This is a general property of the solution of variational problems in the symmetric case.

The term $-\mu\Delta u - f \in V'$ is nothing but the gradient of J considered as a functional defined on $(H_0^1(\Omega)^d$. Since u minimizes J on the subspace V of divergence–free vector fields, the Lagrange multiplier theorem show that, at the optimum point u, the gradient of J has to be proportional to the gradient of the constraint in some sense. More precisely, the constraint here is linear and defined by the operator div $: (H_0^1(\Omega))^d \to L_0^2(\Omega)$ and thus there must exist a $\Pi \in (L_0^2(\Omega))'$ such that

$$(\nabla J)(u) = \Pi \circ \text{div} \ \in V'.$$

Since $(L^2_0(\Omega))'$ can be identified with $L^2_0(\Omega)$ itself, there exists a unique $p \in L^2_0(\Omega)$ such that

$$\langle \Pi, q\rangle_{(L^2_0)', L^2_0} = \int_\Omega pq dx \ , \ \forall q \in L^2_0(\Omega),$$

so that

$$\langle \Pi \circ \mathrm{div} \ , w\rangle_{V',V} = \int_\Omega p(\mathrm{div}\ w)dx = -\langle \nabla p, w\rangle_{V',V} \ , \ \forall w \in V,$$

that is $\Pi \circ \mathrm{div} \ = -\nabla p$. It follows from these computations that, for the Stokes problem, the pressure can be understood as the Lagrange multiplier related to the divergence–free constraint.

There is a question how to introduce the pressure to the velocity satisfying the weak formulation (2.7). The following Theorem guarantee the existence of the pressure.

Theorem 2.10 *Let u be a weak solution of the Stokes problem with homogeneous boundary conditions. Then u belongs to $(H^1_0(\Omega))^d$ and there exists $p \in L^2_0(\Omega)$ such that the couple (u, p) satisfies the problem* (2.4) *in the distribution sense in Ω.*

PROOF. Let us suppose that u satisfies (2.7). Then, $u \in (H^1_0(\Omega))^d$ implies that the traces $\gamma_0 u$ of its components are zero in $H^{1/2}(\Gamma)$. Besides, $u \in V$ implies that div $u = 0$ in the distribution sense. And using (2.7) we obtain

$$\langle -\mu\Delta u - f, v\rangle_{V',V} = 0 \ , \ \forall v \in V.$$

Then, by Proposition 2.3, there exists some distribution $p \in L^2_0(\Omega)$ such that

$$-\mu\Delta u - f = -\nabla p$$

in the distribution sense in Ω.

□

2.2.2 Stokes equations with non homogeneous boundary conditions

For a given constant $\mu > 0$ and given vectors functions $f : \Omega \to \mathbb{R}^d$ and $g : \Gamma \to \mathbb{R}^d$, we consider the boundary value problem

$$\begin{cases} -\mu\Delta u + \nabla p &= f \ , & \text{in } \Omega, \\ \mathrm{div}\ u &= 0 \ , & \text{in } \Omega, \\ u &= g \ , & \text{on } \Gamma. \end{cases} \tag{2.8}$$

The classical solution of this problem is defined analogously as in Definition 2.6. Let us assume that the source term $f \in (L^2(\Omega))^d$, the boundary data $g \in (H^{1/2}(\Gamma))^d$ and

$$\int_\Gamma g \cdot n dS = 0. \tag{2.9}$$

Observation We note that provided a function $u \in (H^1(\Omega))^d$ satisfies the first equation to (2.8) and the third equation (in the sense of traces), then relation (2.9) is fulfilled. It means that (2.9) is a necessary condition for the solvability of problem (2.8).

Lemma 2.11 *Let the function $g \in (H^{1/2}(\Gamma))^d$ satisfy* (2.9)*. Then there exists $\varphi \in (H^1(\Omega))^d$ such that*

$$\begin{aligned} \operatorname{div} \varphi = 0 \ \textit{in}\ \Omega, \\ \varphi|_\Gamma = g \ (\textit{ in the sense of traces}). \end{aligned} \tag{2.10}$$

Proof. By the traces theorem, we have that there exists $g_1 \in (H^1(\Omega))^d$ such that $g_1|_\Gamma = g$. Besides,

$$\int_\Omega \operatorname{div}\, g_1 dx = \int_\Gamma g_1 \cdot n dS = \int_\Gamma g \cdot n dS = 0.$$

This implies that div $g_1 \in L_0^2(\Omega)$. We know that the operator div is an isomorphism of the ortogonal complement

$$V^\perp = \{u \in (H_0^1(\Omega))^d \ : \ (u, v)_{1,0} = 0 \ \forall v \in V\}$$

of the subspace $V \subset (H_0^1(\Omega))^d$ onto $L_0^2(\Omega)$, see Chapter I, Corollary 2.4 of [59]. Then, there exists $g_2 \in (H_0^1(\Omega))^d$ such that div $g_1 =$ div g_2. Now, we put $\varphi = g_1 - g_2$. It is obvious that $\varphi \in (H^1(\Omega))^d$, div $\varphi = 0$, and $\varphi = g$.

□

The weak formulation of the Stokes equations with non homogeneous boundary conditions can be obtained similarly as in the previous case with the use of Green's Theorem. Now, we introduce the concept of a weak solution.

Definition 2.12 *Let $f \in (L^2(\Omega))^d$, $g \in (H^{1/2}(\Gamma))^d$, and assume that the condition* (2.9) *hold. Supposing that φ is a function from Lemma* 2.11*, we call u a weak solution of the Stokes problem* (2.8) *if*

$$\begin{aligned} &a)\ u \in (H^1(\Omega))^d \\ &b)\ u - \varphi \in V \\ &c)\ \mu(u, v)_V = \langle f, v\rangle_{V',V} \ , \ \forall v \in V. \end{aligned} \tag{2.11}$$

Theorem 2.13 *The problem* (2.11) *$a) - c)$ has a unique solution which does not depend on the choice of the function φ from Lemma* 2.11.

Proof. The condition b) on (2.11) can be sought in the form $u = \varphi + z$, where $z \in V$ is a solution of the problem

$$\mu(z, v)_V = \langle f, v\rangle_{V',V} - \mu(\varphi, v)_V \ , \ \forall v \in V. \tag{2.12}$$

The Lax–Milgram Lemma implies that problem (2.12) has a unique solution $z \in V$. It is obvious that $u = \varphi + z$ is a weak solution of the Stokes problem. Now we show that u does not depend on the choice of the function φ. Let g_1 and g_2 be two functions associated with the given by Lemma 2.11 and let u_1 and u_2 be the corresponding weak solutions. Then we obtain

$$\mu(u_i, v)_V = \langle f, v\rangle_{V',V} \ , \ \forall v \in V, \ i = 1, 2.$$

Tacking $v := u_1 - u_2 \in V$, we have

$$0 = (u_1 - u_2, v)_V = (u_1 - u_2, u_1 - u_2)_V = \|u_1 - u_2\|_V^2,$$

which immediately implies that $u_1 = u_2$.

□

Observation The existence of a pressure function $p \in L_0^2(\Omega)$ to a weak solution of the Stokes problem with non homogeneous boundary conditions can be obtained in a similar way as in Theorem 2.8.

2.2.3 The Stokes operator

The bilinear form $a(\cdot, \cdot)$ defined in the proof of the Theorem 2.9 being continuous on $V \times V$, then we can define an operator A from V to V' by

$$\langle Au, v\rangle_{V',V} = \int_\Omega \nabla v : \nabla u dx \ , \ \forall u, v \in V. \tag{2.13}$$

The operator A is called the *Stokes operator*. From the Lax–Milgram Lemma this operator is an isomorphism from V onto V'.

We can now consider A as an unbounded operator in H with domain

$$D(A) = \{u \in V \ : \ Au \in H\}.$$

The Stokes operator defined above satisfies the following properties.

Lemma 2.14 *The operator $(A, D(A))$ has a closed graph in $H \times H$.*

Proof. Let $(u_n)_n$ be a sequence of elements of $D(A)$ which converges in H towards an element $u \in H$ and such thta $(Au_n)_n$ converges in H towards a certain f. Since $f \in H$, there exists a unique $v \in D(A)$ such that $Av = f$. We then need to prove that $v = u$.

The convergence in H implies that in V', therefore we deduce that $(Au_n)_n$ converges towards Av in V'. However, A is an isomorphism from V onto V' and hence $(u_n)_n$ converges towards v in V. The convergence in V implies the convergence in H, thus we have shown that $u = v$.

□

As we know in the general theory of unbounded operators this property implies that $D(A)$, equipped with the scalar product

$$(u,v)_{D(A)} = (u,v)_H + (Au, Av)_H \ ,; \forall u,v \in D(A),$$

is a Hilbert space and that the operator A is an isomorphism from $D(A)$ onto H. Furthermore, for all $u \in D(A)$ we have

$$\|u\|_V \leq C\|Au\|_{V'} \leq \tilde{C}\|Au\|_H \leq \tilde{C}\|u\|_{D(A)},$$

which implies that the canonical embedding from $D(A)$ to V is continuous. Finally, since the embedding from V to H is compact, we have that the embedding from $D(A)$ to H is compact.

Lemma 2.15 *The Stokes operator $(A, D(A))$ is self–adjoint in H.*

PROOF. First, by definition, for all $u, v \in D(A)$ we have

$$(Au, v)_H = \langle Au, v\rangle_{V',V} = \int_\Omega \nabla u \cdot \nabla v dx,$$

and therefore it is clear that

$$(Au, v)_H = (u, Av)_H \ , \ \forall u, v \in D(A). \tag{2.14}$$

This implies in particular that $D(A) \subset D(A^*)$ and that for all $u \in D(A)$, $A^*u = Au$.

We now need to prove that $D(A^*) = D(A)$. Let $u \in D(A^*)$. By definition there exists $f = A^*u \in H$ such that for all $v \in D(A)$ we have $(Av, u)_H = (v, f)_H$. However, A is bijective from $D(A)$ onto H, and thus there exists $\tilde{u} \in D(A)$ such that $f = A\tilde{u}$. Then, we need to prove that $u = \tilde{u}$. To do this, let $w \in H$. Then there exists $\tilde{w} \in D(A)$ such that $A\tilde{w} = w$. Then we have

$$(w, u - \tilde{u})_H = (A\tilde{w}, u)_H - (A\tilde{w}, \tilde{u})_H. \tag{2.15}$$

From the definition of the adjoint we obtain $(A\tilde{w}, u)_H = (\tilde{w}, A^*u)_H$, and from (2.14), since $\tilde{u} \in D(A)$, we have $(A\tilde{w}, \tilde{u})_H = (\tilde{w}, A\tilde{u})_H$. By definition of $\tilde{u}$, we have $A\tilde{u} = A^*u$ and hence, (2.15) becomes

$$(w, u - \tilde{u})_H = 0,$$

and this for all $w \in H$, which proves that $u = \tilde{u}$, and in particular, that $u \in D(A)$.

□

In consequence of all the preceding results, we can therefore construct a spectral decomposition of the Stokes operator. More precisely, we have the following proposition.

Proposition 2.16 ([29], Chapter IV, Section 5, Theorem 5.5) *Let Ω be a bounded, connected, Lipschitz domain of $\mathbb{R}^d$. There exists an increasing sequence of positive real numbers $(\lambda_k)_k$, which tends to $+\infty$, and a sequence of functions $(u_k)_k$, which is orthonormal in H, orthogonal in V and in $D(A)$, forming a complete family in H, in V, and in $D(A)$, and a sequence of functions $(p_k)_k$ in $L_0^2(\Omega)$ satisfying*

$$\left\{\begin{array}{rcll} -\mu\Delta u_k + \nabla p_k &=& \lambda_k u_k &, \quad \text{in } \Omega, \\ \operatorname{div} u_k &=& 0 &, \quad \text{in } \Omega, \\ u_k &=& 0 &, \quad \text{on } \Gamma. \end{array}\right.$$

We kwon that the solution $u \in H_0^1(\Omega)$ of the Dirichlet problem $-\Delta u + u = f$ belongs to $H^{m+2}(\Omega)$ when $f \in H^m(\Omega)$ and Ω is sufficiently smooth. The Stokes operator satisfies elliptic regularity properties which are similar to those of the Laplace operator, for instance. More precisely, we can state the following result.

Proposition 2.17 ([107], Chapter I, Section 2, Proposition 2.2) *Let Ω be an open bounded set of class C^r, $r = \max(m+2, 2)$, m positive integer. Let us suppose that*

$$u \in W^{1,\alpha}(\Omega)\ ,\ p \in L^\alpha\ ,\ 1 < \alpha < \infty,$$

are solutions of the following Stokes problem

$$\left\{\begin{array}{rcll} -\mu\Delta u + \nabla p &=& f &, \quad \text{in } \Omega, \\ \operatorname{div} u &=& 0 &, \quad \text{in } \Omega, \\ u &=& g &, \quad \text{on } \Gamma. \end{array}\right. \tag{2.16}$$

If $f \in W^{m,\alpha}(\Omega)$, $g \in W^{m+2-1/\alpha,\alpha}(\Gamma)$, then

$$u \in W^{m+2,\alpha}(\Omega)\ ,\ p \in W^{m+1,\alpha}. \tag{2.17}$$

Besides, there exists a constant $c = c(\alpha, \mu, m, \Omega)$ such that

$$\|u\|_{m+2,\alpha} + \|p\|_{m+1,\alpha} \leq c(\|f\|_{m,\alpha} + \|g\|_{m+2-1/\alpha,\alpha} + d_\alpha \|u\|_\alpha), \tag{2.18}$$

where $d_\alpha = 0$ for $\alpha \geq 2$, $d_\alpha = 1$ for $1 < \alpha < 2$.

We observe that the Proposition 2.17 does not assert the existence of u, p satisfying the Stokes equations (2.16), but gives only a result on the regularity of an eventual solution. The following proposition gives a general existence result for $N = 2$ or $N = 3$.

Proposition 2.18 ([107], Chapter I, Section 2, Proposition 2.3) *Let Ω be an open set of class C^r, with $r = \max(m+2, 2)$, m integer bigger than -1, and let $f \in W^{m,\alpha}(\Omega)$, $g \in W^{m+2-\alpha,\alpha}(\Omega)$ be given satisfying the compatibility condition*

$$\int_\Gamma g \cdot n dS = 0. \tag{2.19}$$

Then there exists unique functions u and p (p is unique up to a constant) which are solutions of (2.16) and satisfy (2.17) and (2.18) with $d_\alpha = 0$ for any α, $1 < \alpha < \infty$. In particular, *for any* $f \in (L^2(\Omega))^d$ *and* $g \in (H^{3/2}(\partial\Omega))^d$ *satisfying* (2.19), *the unique solution to* (2.16) *is such that, see [19],*

$$(u, p) \in (H^2(\Omega))^d \times H^1(\Omega). \tag{2.20}$$

Moreover, we have

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)}), \tag{2.21}$$

where C *is a positive constant depending only on* Ω.

Let us finish this section with a new sight to the Poincaré inequality.

Proposition 2.19 ([29], Chapter IV, Section 5, Proposition 5.12) *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Then, we have the following Poincaré inequalities*

$$\|u\|_H^2 \leq \frac{1}{\lambda_1}\|\nabla u\|_2^2 \ , \ \forall v \in V, \tag{2.22}$$

$$\|\nabla u\|_2^2 \leq \frac{1}{\lambda_1}\|Au\|_2^2 \ , \ \forall u \in D(A),$$

where λ_1 *is the smallest eigenvalue of the Stokes operator.*

2.3 The non stationary Stokes equations

In this part of the thesis, we use the elements introduced before concerning the Stokes operator in order to solve the evolutionionary incompressible Stokes problem.

First, we introduce some notations and auxiliary spaces. We denote by $Q := \Omega \times (0, T)$ the space–time cylinder. Here, $T > 0$ is a given final time. Further, we set $\Sigma := \Gamma \times (0, T)$.

We shall work in the standard space of abstract functions from $[0, T]$ to a real Banach space X, $L^p(0, T; X)$, endowed with its natural norm

$$\|u\|_{L^p(X)} := \|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} \ , \ 1 \leq p < \infty,$$

The space $L^\infty(0, T; X)$ is the space of essentially bounded functions from $[0, T]$ into X, and is equipped with the Banach norm

$$\|u\|_{L^\infty(X)} := \text{ess sup}_{t\in[0,T]}\|u(t)\|_X.$$

When no confusion can arise, we will use the following notations

$$L^p(X) := L^p(0, T; X) \ , \ 1 \leq p \leq \infty$$

For our proposes, we need the following technical Lemma. The proof can be found in [107].

Lemma 2.20 ([107], Chapter III, Section 1, Lemma 1.1) *Let X be a given Banach space with dual X' and let u and g be two functions belonging to $L^1(X)$. Then ,the following three conditions are equivalent:*

(i) u is a.e. equal to a primitive function of g,

$$u(t) = \xi + \int_0^t g(s)ds \ , \ \xi \in X, \ \text{a.e. } t \in [0,T].$$

(ii) For each test function $\varphi \in \mathcal{D}((0,T))$,

$$\int_0^T u(t)\varphi'(t)dt = -\int_0^T g(t)\varphi(t)dt \quad \left(\varphi' = \frac{d\varphi}{dt}\right).$$

(iii) For each $\eta \in X'$,

$$\frac{d}{dt}\langle u, \eta\rangle = \langle g, \eta\rangle,$$

in the scalar distribution sense on $(0,T)$.

If $(i)-(iii)$ are satisfied u, in particular, is a.e. equal to a continuous function from $[0,T]$ into X.

Let Ω be a Lipschitz open bounded domain in $\mathbb{R}^d$ and let $T > 0$ fixed. We consider the evolutionionary incompressible Stokes problem

$$\begin{cases} u_t - \mu\Delta u + \nabla p &= f \quad , \quad \text{in } Q, \\ \text{div } u &= 0 \quad , \quad \text{in } Q, \\ u &= 0 \quad , \quad \text{on } \Sigma, \\ u(0) &= u_0 \quad , \quad \text{in } \Omega, \end{cases} \tag{2.23}$$

where the vector function $u : Q \to \mathbb{R}^d$ represents the velocity of the fluid, the scalar function $p : Q \to \mathbb{R}$ represents the pressure, the vector functions f and u_0 are given, f defined on Q, u_0 defined on Ω.

As in the previous section, we will formulate the weak formulation of the problem (2.23). Let us suppose that $u \in C^2(\overline{Q})$ and $p \in C^1(\overline{Q})$ are classical solutions of (2.23). Let v be any element of V, then we have that

$$(u_t, v)_2 + \mu(u, v)_V = \langle f, v\rangle_{V',V}, \tag{2.24}$$

holds for all $t \in (0,T)$. Since u is divergence–free, the pressure disappears in the weak formulation. Equation (2.24) suggests the following formulation of the problem (2.23).

For given $f \in L^2(0,T;V')$ and $u_0 \in H$, find a solution $u \in L^2(0,T;V)$ such that

$$\begin{cases} \dfrac{d}{dt}(u, v)_2 + \mu(u, v)_V &= \langle f, v\rangle_{V',V} \quad , \quad \forall v \in V. \\ u(0) &= u_0. \end{cases} \tag{2.25}$$

We observe that if $u \in L^2(0,T;V)$, then the condition $u(0) = u_0$ does not make sense in general. For this reason we need to give an alternative formulation of the weak problem.

Using the results above the Stokes operator, we can write the first equation in (2.25) as

$$\frac{d}{dt}\langle u, v\rangle_{V',V} = \langle f - \mu A u, v\rangle_{V',V} \ , \ \forall v \in V. \tag{2.26}$$

Since A is linear and continuos from V into V' and $u \in L^2(0,T;V)$, we obtain that the function Au belongs to $L^2(0,T;V')$. Hence $f - \mu Au$ belongs to $L^2(0,T;V)$ and by Lemma 2.20, we have

$$u' \in L^2(0,T;V'), \tag{2.27}$$

and the equation (2.26) can be written as

$$u' + \mu A u = f.$$

Therefore, an alternative weak formulation of the evolutionary Stokes problem is the following:

Given $f \in L^2(0,T;V')$ *and* $u_0 \in H$, *to find* u *such that*

$$u \in L^2(0,T;V) \ , \quad u' \in L^2(0,T;V'), \tag{2.28}$$
$$u' + \mu A u = f \ , \ \text{on } (0,T), \tag{2.29}$$
$$u(0) = u_0. \tag{2.30}$$

Then, we can prove the following result about the existence and uniqueness of solution of these problems.

Theorem 2.21 ([107], Chapter III, Section 1, Theorem 1.1) *For given* $f \in L^2(0,T;V')$ *and* $u_0 \in H$, *there exists a unique function* u *which satisfies* (2.28)–(2.30).

Observation As in the stationary case, we need to ensure the existence of the pressure p satisfying the problem (2.23) in the distribution sense in Q, with u defined by Theorem 2.21.

Lemma 2.22 *Under the assumptions of Theorem 2.21, there exists a distribution* p *on* Q, *such that the function* u *defined by Theorem 2.21 and* p *satisfy* (2.23).

Proof. We consider the functions

$$U(t) = \int_0^t u(s)ds \ , \quad F(t) = \int_0^t f(s)ds. \tag{2.31}$$

Note that, at least, the functions satisfying

$$U \in C([0,T];V) \ , \quad F \in C([0,T];V').$$

Integrating over time the first equation in (2.25), we obtain

$$(u(t) - u(0), v) + \mu(U(t), v)_V = \langle F(t), v \rangle \ , \ \forall t \in [0, T], \ \forall v \in V.$$

Equivalently, we can write the previous equation as

$$\langle u(t) - u_0 - \mu \Delta U(t) - F(t), v \rangle = 0 \ , \ \forall t \in [0, T], \ \forall v \in V.$$

By Proposition 2.3, for each $t \in [0, T]$ there exists a function $P(t) \in L^2(\Omega)$, such that

$$u(t) - u_0 - \mu \Delta U(t) + \nabla P(t) = F(t). \tag{2.32}$$

From Proposition 2.4 we have that the gradient operator is an isomorphism from $L^2(\Omega) \backslash \mathbb{R}$ into $H^{-1}(\Omega)$. From (2.32), we have that $\nabla P = F + \mu \Delta U - u + u_0$, which implies that $P \in C([0, T]; H^{-1}(\Omega))$. Then, $P \in C([0, T]; L^2(\Omega))$.

Finally, taking the derivative with respect to the t–variable, we obtain

$$u'(t) - \mu \Delta u(t) + \nabla p(t) = f(t),$$

where $p = \frac{\partial P}{\partial t}$. □

We finished this section with a simple result of regularity.

Proposition 2.23 ([107], Chapter III, Section 1, Proposition 1.2) *Let us assume that Ω is of class C^2, that $f \in L^2(0, T; H)$ and $u_0 \in V$. Then,*

$$u \in L^2(0, T; H^2(\Omega)) \ , \quad u' \in L^2(0, T; H) \ , \quad p \in L^2(0, T; H^1(\Omega)). \tag{2.33}$$

2.4 The stationary Navier–Stokes equations

The present section we will be concerned with the study of the stationary Navier–Stokes equations. We will show the basic properties of this equations, existence and uniqueness of the solution. But, unlike the Stokes systems, there exists important differences between these two models:

- The Navier–Stokes systems is a nonlinear partial differential equations. Then, we need to ensure some strong convergence results, and this properties can be obtained by compactness arguments.
- The strong convergence, Sobolev inequalities, Sobolev embedding, among others, depend according to the dimension of the space. Namely, unlike the Stokes systems, the dimension can be consider a parameter.
- The uniqueness of solutions occurs only when the data are small enough, or the viscosity term is large enough.

Let us consider the boundary value problem for the stationary nonlinear Navier–Stokes equations with homogeneous boundary data. Namely, let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$, be an open bounded Lipschitz set with boundary $\partial\Omega = \Gamma$. We are seek vector function $u = (u_1, \ldots, u_d)$ and scalar function p, such that

$$\begin{cases} -\mu\Delta u + (u \cdot \nabla)u + \nabla p &= f \quad , \quad \text{in } \Omega, \\ \text{div } u &= 0 \quad , \quad \text{in } \Omega, \\ u &= 0 \quad , \quad \text{on } \Gamma, \end{cases} \tag{2.34}$$

where $f \in (L^2(\Omega))^d$ is a given vector function and $\mu > 0$ the viscosity coefficient.

The nonlinear term $(u \cdot \nabla)u$ can be written in the following form

$$(u \cdot \nabla)u = \sum_{i=1}^{d} u_i \frac{\partial u}{\partial x_i}.$$

As in the linear case, let (u, p) be its classical solution of (2.34), namely, $u \in (C^2(\overline{\Omega}))^d$ and $p \in C^1(\overline{\Omega})$ satisfy (2.34). Then, multiplying the first equation of (2.34) by an arbitrary vector function $v \in \mathcal{V}$, integrating over Ω ans using the Green's Theorem, we obtain

$$\mu(u, v)_V + b(u, u, v) = (f, v)\ ,\ \forall v \in \mathcal{V}, \tag{2.35}$$

where

$$b(u, v, w) = \sum_{i,j=1}^{d} \int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j dx. \tag{2.36}$$

Using continuity argument, we can show that equation (2.35) is satisfied by any $v \in V$.

The variational formulation of the problem (2.34) will be slightly different with respect the Stokes equations, and this will be stated after studying some properties of the form $b(u, v, w)$.

Lemma 2.24 *The mapping b is defined and trilinear continuos form on $(H^1(\Omega))^d \times (H^1(\Omega))^d \times (H^1(\Omega))^d$.*

Proof. Let $u, v, w \in (H^1(\Omega))^d$. Then u_i, v_i, w_i belongs to $H^1(\Omega)$. Recalling that we are working in dimension two or three and in the bounded case, in virtue of the continuos embedding of $H^1(\Omega)$ into $L^4(\Omega)$, we have $u_i, w_j \in L^4(\Omega)$ and $\frac{\partial v_i}{\partial x_j} \in L^2(\Omega)$. By the Hölder inequality we obtain

$$\begin{aligned} |\int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j dx| &\leq \left(\int_\Omega |u_i w_j|^2\right)^{1/2} \left(\int_\Omega |\frac{\partial v_i}{\partial x_j}|^2\right)^{1/2} \\ &\leq \left(\int_\Omega |u_i|^4\right)^{1/4} \left(\int_\Omega |w_j|^4\right)^{1/4} \left(\int_\Omega |\frac{\partial v_i}{\partial x_j}|^2\right)^{1/2}, \end{aligned} \tag{2.37}$$

which implies that $u_i \frac{\partial v_i}{\partial x_j} w_j$ belongs to $L^1(\Omega)$. It means that the integral in (2.36) exists and is finite. The form b is thus defined on $(H^1(\Omega))^d \times (H^1(\Omega))^d \times (H^1(\Omega))^d$. Its linearity with respect the arguments u, v, w is obvious.

Let us prove the continuity of b. Due to the continuos embedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a constant $c > 0$ such that

$$\|u\|_{L^4(\Omega)} \leq c\|u\|_{H^1(\Omega)} \ , \ \forall u \in (H^1(\Omega))^d.$$

Then, from (2.37) we obtain

$$|\int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j dx| \leq c^2 \|u_i\|_{H^1(\Omega)} \|w_j\|_{H^1(\Omega)} \|v_i\|_{H^1(\Omega)}. \tag{2.38}$$

Summing these inequalities for $i, j = 1, \ldots, d$, (2.38) ensures the continuity of b. □

In particular, we have the following result.

Corolary 2.25 *The function b is a continuos trilinear from on $V \times V \times V$. Besides, there exists a constant $c > 0$ such that*

$$|b(u, v, w)| \leq c\|u\|_V \|v\|_V \|w\|_V \ , \ \forall u, v, w \in V. \tag{2.39}$$

The following property of b is fundamental for the study of the Navier–Stokes equations, and will be used throughout the thesis.

Lemma 2.26 *Let $u \in (H^1(\Omega))^d$, $\text{div } u = 0$, and $v, w \in (H_0^1(\Omega))^d$. Then, we obtain*

$$b(u, v, v) = 0, \tag{2.40}$$

$$b(u, v, w) = -b(u, w, v), \tag{2.41}$$

$$b(u, v, w) = ((\nabla v)^T w, u), \ \forall u, v, w \in (H^1(\Omega))^d. \tag{2.42}$$

A proof can be found for instance in [107], Chapter III, Section 1, Lemma 1.3.

The above considerations leads us to the following variational formulation of the Navier–Stokes system.

Definition 2.27 *Let $\mu > 0$ and $f \in (L^2(\Omega))^d$ be given. We say that u is a weak solutions of the Navier–Stokes problem with homogeneous boundary data, if*

$$u \in V \quad \textit{and} \quad \mu(u, v)_V + b(u, u, v) = (f, v) \ , \ \forall v \in V. \tag{2.43}$$

It is obvious that any classical solution is a weak one. Similarly as in Stokes equations, we can prove that to the weak solution u there exists the pressure $p \in L_0^2(\Omega)$ satisfying the identity

$$\mu(u, v)_V + b(u, u, v) - (p, \text{div } u) = (f, v) \ , \ \forall v \in (H_0^1(\Omega))^d.$$

Then, the couple (u, p) satisfies the equation (2.34) in the sense of distributions.

As we said before, we need some results of strong converge to ensure some convergence of the nonlinear term. The following result will be helpful in the proof of the existence of a weak solution of the Navier–Stokes problem.

Lemma 2.28 *Let $(u_\alpha)_\alpha$ be a sequence in V such that converges strongly in $(L^2(\Omega))^d$ to some $u \in V$, as $\alpha \to \infty$. Then*

$$b(u_\alpha, u_\alpha, v) \to b(u, u, v) \ , \ \forall v \in \mathcal{V}. \tag{2.44}$$

Proof. From (2.41) we have

$$b(u_\alpha, u_\alpha, v) = -b(u_\alpha, v, u_\alpha) = -\sum_{i,j=1}^{d} \int_\Omega u_\alpha^i u_\alpha^j \frac{\partial v_i}{\partial x_j} dx.$$

By the assumption that $u_\alpha \to u$ in $(L^2(\Omega))^d$, imply

$$\int_\Omega |u_\alpha^i u_\alpha^j - u^i u^j| \to 0 \ , \ \forall i, j = 1, \ldots, d. \tag{2.45}$$

Since $v \in \mathcal{V}$, there exists $c > 0$ such that

$$\left| \frac{\partial v_i}{\partial x_j} \right| \le c \ , \ \forall x \in \overline{\Omega}, \ \forall i, j = 1, \ldots, d. \tag{2.46}$$

From (2.45) and (2.46), we have

$$\left| \int_\Omega \left(u_\alpha^i u_\alpha^j \frac{\partial v_i}{\partial x_j} - u^i u^j \frac{\partial v_i}{\partial x_j} \right) \right| \le c \int_\Omega |u_\alpha^i u_\alpha^j - u^i u^j| \to 0,$$

and thus

$$\int_\Omega u_\alpha^i u_\alpha^j \frac{\partial v_i}{\partial x_j} \to \int_\Omega u^i u^j \frac{\partial v_i}{\partial x_j}. \tag{2.47}$$

Summing (2.47) we obtain

$$b(u_\alpha, u_\alpha, v) = -b(u_\alpha, v, u_\alpha) = -\sum_{i,j=1}^{d} \int_\Omega u_\alpha^i u_\alpha^j \frac{\partial v_i}{\partial x_j} \to -\sum_{i,j=1}^{d} \int_\Omega u^i u^j \frac{\partial v_i}{\partial x_j} = b(u, u, v).$$

□

Theorem 2.29 (Existence of solution) *Let Ω be a bounded set in $\mathbb{R}^d$ and let f be given in $(H^{-1}(\Omega))^d$. Then problem* (2.43) *has at least one solution $u \in V$.*

Proof. For the proof of the existence, which is carried out using a Galerkin approximation, we refer to Temam [107], . □

In the following we shall investigate the uniqueness of a weak solution of the Navier–Stokes problem.

Theorem 2.30 ([107], Chapter II, Section 1, Theorem 1.3) *Suppose that $d \le 4$ and if μ is sufficiently large or f sufficiently small, so that*

$$\mu^2 > c(d)\|f\|_{V'} \tag{2.48}$$

then there exists a unique solution u of (2.43).

2.5 The non stationary Navier–Stokes equations

In this section we are dealing with the mathematical setting of the non stationary and incompressible Navier–Stokes equations in the **two–dimensional case**.

As in the case of the evolutionary Stokes problem, we denote by $Q = \Omega \times (0,T)$ the space–time cylinder. Here, Ω is a Lispchitz open bounded domain in $\mathbb{R}^2$ and $T < \infty$ is a given final time. Further, we set $\Sigma = \Gamma \times (0,T)$.

To deal with the time derivative in the state equation, we introduce the common spaces of functions y whose time derivative y_t exist as abstract functions,

$$\begin{aligned} X^k(0,T;V) &:= \{y \in L^2(0,T;V) \ : y_t \in L^k(0,T;V')\}, \\ X(0,T) &:= X^2(0,T;V), \end{aligned}$$

where $k \in [1,2]$. Endowed with the norm

$$\begin{aligned} \|y\|_{X^k} := \|y\|_{X^k(0,T;V)} &= \|y\|_{L^2(V)} + \|y_t\|_{L^k(V')}, \\ \|y\|_X &:= \|y\|_{X^2}, \end{aligned}$$

these spaces are Banach spaces. Every function of $X(0,T)$ is equivalent to a function of $C([0,T],H)$, and the imbedding $X(0,T) \rightharpoonup C([0,T],H)$ is continuos, see [3, 84].

An interesting property of the space $X(0,T)$ is the following Lemma.

Lemma 2.31 *Let $y \in X(0,T)$ be given. Then it holds $y \in (L^4(Q))^2$ with $\|y\|_4 \leq c\|y\|_X$.*

Proof. We known the following interpolation inequality proven in [107], for every $y \in H^1(\Omega)$, $\Omega \in \mathbb{R}^d$, $d \leq 4$, it holds

$$\|y\|_4 \leq 2^{1/4}\|y\|_2^{1/2}\|y\|_{1,2}^{1/2}. \tag{2.49}$$

Then, we obtain

$$\|y\|_4^4 = \int_0^T \int_\Omega |y(x,t)|^4 dxdt \leq 2\int_0^T |y(t)|_H^2 |y(t)|_V^2 dt \leq 2\|y\|_{L^\infty}^2 \|y\|_{L^2(V)}^2.$$

We observe that the claim remains true for $y \in L^2(0,T;V) \cap L^\infty(0,T;H)$, since we did not use any regularity of the time derivative. □

We introduce the following spaces of abstract functions in the L^p–context:

$$X_p^{2,1} := \{y \in L^p(0,T;(W^{2,p}(\Omega))^2) \cap V_p \ : \ y_t \in L^p(0,T;(L^p(\Omega))^2)\}. \tag{2.50}$$

We abbreviate $H^{2,1} = X_2^{2,1}$, for $p = 2$. The space $H^{2,1}$ is continuously imbedded in $C([0,T];V)$, see Lions and Magenes [84].

Now, we are ready to investigate the non stationary Navier–Stokes equations. Given $f \in L^2(0,T;(L^2(\Omega))^2)$ and $u_0 \in H$, we are looking for solutions of the system

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \nabla p &= f \quad , \quad \text{in } Q, \\ \text{div } u &= 0 \quad , \quad \text{in } Q, \\ u &= 0 \quad , \quad \text{on } \Sigma, \\ u(0) &= u_0 \quad , \quad \text{in } \Omega. \end{cases} \tag{2.51}$$

As in the previous section, let us assume there is a classical solution, say $y \in C^2([0,T];(C^2(\overline{\Omega}))^2)$ and $p \in C^1([0,T];C^1(\overline{\Omega}))$ satisfying system (2.51). If v belongs to V, we obtain that

$$(u_t, v)_2 + \mu(u,v)_V + b(u,u,v) = \langle f, v\rangle_{V',V}\ ,\ \forall t \in (0,T). \tag{2.52}$$

Equation (2.52) suggests the following weak formulation of the problem (2.51) which is due to Leray.

For given $f \in L^2(0,T;V')$ and $u_0 \in H$, find a solution $u \in L^2(0,T;V)$ with $u_t \in L^2(0,T;V')$ that fulfills

$$\langle u_t(t), v\rangle_{V',V} + \mu(u(t),v)_V + b(u(t),u(t),v) = \langle f, v\rangle\ ,\ \forall v \in V,\ \text{a.e. on } (0,T), \tag{2.53}$$

$$u(0) = u_0. \tag{2.54}$$

We observe that the condition (2.54) makes sense, because it is well known the embedding $X(0,T) \hookrightarrow C([0,T];H)$. For our proposes, we want to give an equivalent formulation as an equation in function space. To this aim, we introduce the linear continuous operator $A: L^2(0,T;V) \to L^2(0,T;V')$ for $u, v \in L^2(0,T;V)$ by

$$\begin{aligned} \langle Au, v\rangle_{L^2(V'),L^2(V)} &= \int_0^T \langle (Au)(t), v(t)\rangle_{V',V} dt \\ &= \int_0^T (u(t),v(t))_V dt = \int_0^T \nabla u(t)\cdot\nabla v(t) dt \end{aligned}$$

and a nonlinear operator $B: X(0,T) \to L^2(0,T;V')$ for $u \in X(0,T)$, $w \in L^2(0,T;V')$ by

$$\langle B(u), w\rangle_{L^2(V'),L^2(V)} = \int_0^T \langle (B(u))(t), w(t)\rangle_{V',V} dt = \int_0^T b(u(t),u(t),w(t))dt.$$

The operator B is a bounded mapping from $X(0,T)$ into $L^2(0,T;V')$.

Lemma 2.32 ([112], Chapter I) *For all $y \in L^2(0,T;V)\cap L^\infty(0,T;H)$ it holds $B(y) \in L^2(0,T;V')$.*

Then, we can transformed the system (2.53)–(2.54) to an operator equation.

Definition 2.33 *Let $f \in L^2(0,T;V')$ and $u_0 \in H$ be given. A function $u \in L^2(0,T;V)$ with $u_t \in L^2(0,T;V')$ is called weak solution of (2.51) if it fulfills*

$$u_t + \mu Au + B(u) = f \quad \text{in } L^2(0,T;V'), \tag{2.55}$$

$$u(0) = u_0 \quad \text{in } H. \tag{2.56}$$

In [112], the author prove that the weak formulation (2.53)–(2.54) and (2.55)–(2.56) are equivalent problems. Using this fact, in the sequel, we will work with the second and more handy definition of a weak solution (2.55)–(2.56). We will restrict our considerations to the spatial two–dimensional case. Here, results concerning the solvability of (2.55)–(2.56) are standars, for example see [59, 107] for proofs and further details. The next result concerning the existence and uniqueness of solution to (2.55)–(2.56) is only valid for two–dimensional domains.

Theorem 2.34 ([107], Chapter III, Section 2) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then, for every $f \in L^2(0,T;V')$ and $u_0 \in H$, the equation (2.55)–(2.56) has a unique solution $u \in X(0,T)$. Moreover, the mapping $(u_0, f) \mapsto u$ is locally Lispchitz from $L^2(0,T;V') \times H$ into $X(0,T)$.*

With respect to the more regular solutions, we have the following result.

Theorem 2.35 *Let $u_0 \in V$ and $f \in L^2(0,T;(L^2(\Omega))^2)$ be given. Then the weak solution of (2.55)–(2.56) fulfills*

$$u \in L^2(0,T;(H^2(\Omega))^2) \cap L^\infty(0,T;V)\ ,\ u_t \in L^2(0,T;H).$$

2.6 Linearized Navier–Stokes equations

We will need in the following some results about the linearized equations. In the literature, this problem is so–caled *Oseen equation*, and in the Barbu's book [25] is called *Stokes–Oseen equation*. We refer the reader to the extensive survey [25, 57] and references therein.

Given a state $\overline{u} \in (H^2(\Omega))^2 \cap V$, solution of the steady Navier–Stokes problem (2.43), we consider the linearized Navier–Stokes equation around the state $\overline{u}$. Namely, we consider $u = \overline{u} + w$, where u is the solution of the evolutionary Navier–Stokes equation (2.55)-(2.56). Then, we obtain

$$w_t + \mu A w + B'(\overline{u})w = f \quad \text{in } L^2(0,T;V'), \tag{2.57}$$

$$w(0) = w_0 \quad \text{in } H, \tag{2.58}$$

where $w_0 = u_0 - \overline{u}$. Here, $B'(\overline{u})w$ denotes the Fréchet derivative of B with respect to the state $\overline{u}$. It is itself a functional of $L^2(0,T;V')$, which for $v \in L^2(0,T;V)$ is given by

$$\langle B'(\overline{u})w, v\rangle_{L^2(V'),L^2(V)} = \int_0^T (b(\overline{u}, w(t), v(t)) + b(w(t), \overline{u}, v(t)))dt. \tag{2.59}$$

Since B is of quadratic nature, its differentiability is a simple conclusion of the above considerations.

Lemma 2.36 *The operator $B:\ X(0,T) \to L^2(0,T;V')$ is twice Fréchet differentiable. All derivatives of third or higher order vanish. The first derivative is given by (2.59). It can be* *estimated as*

$$\|B'(\overline{u})w\|_{L^2(V')} \leq c\|\overline{u}\|\|w\|_X. \tag{2.60}$$

Besides, let $B'(y)^*$ denote the adjoint of $B'(y)$ for the duality between V and V', then we have

$$\langle B'(\overline{u})^* v, w\rangle = \int_0^T [b(w(t), \overline{u}, v(t)) - b(\overline{u}, v(t), w(t))]dt.$$

As for quadratic functions, the second derivative is independent of $\overline{u}$:

$$\langle B''(\overline{u})[w_1, w_2], v\rangle_{L^2(V'),L^2(V)} = \int_0^T (b(w_1(t), w_2(t), v(t)) + b(w_2(t), w_1(t), v(t)))dt. \tag{2.61}$$

The proofs of the following existence and regularity results of solutions of the linearized system are proven in [65].

Theorem 2.37 *Let $w_0 \in H$, $f \in L^2(0,T;V')$, and $\overline{u} \in V$ be given. Then there exists a unique weak solution $w \in X(0,T)$ of* (2.57)–(2.58)*. Moreover, if $w_0 \in V$, $f \in L^2(0,T;(L^2(0,T))^2)$, and $\overline{u} \in (H^2(\Omega))^2 \cap V$ be given. Then the weak solution of* (2.57)–(2.58) *satisfies also $w \in H^{2,1}$.*

Now, we give some properties for this equation. We define the Oseen operator $\mathcal{A}$ as

$$\mathcal{A}v := -\mu P(\Delta v) + P[(\overline{y}\cdot\nabla)v + (v\cdot\nabla)\overline{y}], \tag{2.62}$$

where P is the Leray projector.

This operator is closed and has the domain $D(\mathcal{A}) = D(A) = (H^2(\Omega))^2 \cap V$, where A is the Stokes operator defined at the beginning.

Assuming that the spaces are complex, we denote by $\rho(\mathcal{A})$ the resolvent set of operator $\mathcal{A}$, namely, the set of $\lambda \in \mathbb{C}$ such that the resolvent operator

$$R(\lambda, \mathcal{A}) \equiv (\lambda I - \mathcal{A})^{-1}$$

is defined and continuous. Here I is the identity operator. The complement of $\rho(\mathcal{A})$ is called the spectrum of the operator $\mathcal{A}$ and is denoted by $\Sigma(\mathcal{A})$.

It is well known that for $\lambda \in \rho(\mathcal{A})$, the resolvent of Oseen operator (2.62) is a compact operator, and the spectrum $\Sigma(\mathcal{A})$ consists of a discrete set of points. Moreover, Oseen operator is sectorial.

Now, let us consider the adjoint operator $\mathcal{A}^*$ to Oseen operator

$$\mathcal{A}^*v := -\mu P(\Delta v) - P[(\overline{y}\cdot\nabla)v - (\nabla\overline{y})^T v], \tag{2.63}$$

where T denote the transpose of $\nabla\overline{y}$.

Evidently, $\mathcal{A}^*$ are the same properties than $\mathcal{A}$. Namely, is closed with domain $D(\mathcal{A}^*) = (H^2(\Omega))^2 \cap V$. Moreover, $\mathcal{A}^*$ is sectorial with a compact resolvent. Besides, we assume that $\overline{y} \in (H^2(\Omega))^2 \cap V$, then $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$.

Let $\sigma > 0$ be a constant satisfying

$$\Sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C} \ : \ \text{Re}\lambda = \sigma\} = \emptyset. \tag{2.64}$$

Denote by $X_\sigma^+(\mathcal{A})$ the subspace of H generated by all eigenfunctions and associated functions of operator $\mathcal{A}$ corresponding to all eigenvalues of $\mathcal{A}$ placed in the set $\{\lambda \in \mathbb{C} \ : \ \text{Re}\lambda < \sigma\}$. By $X_\sigma^+(\mathcal{A}^*)$ we denote analogous subspace corresponding to adjoint operator $\mathcal{A}^*$. We denote the orthogonal complement to $X_\sigma^*(\mathcal{A}^*)$ in H by X_σ. Then, we have the following result of Fursikov [56].

Theorem 2.38 (see [56]) *Suppose that $\mathcal{A}$ is the operator* (2.62) *and $\sigma > 0$ satisfies* (2.64)*. Then for each $w_0 \in X_\sigma$ we have*

$$\|w(t,\cdot)\|_V \leq c\|w_0\|_V \, e^{-\sigma t}, \quad \textit{for } t \geq 0. \tag{2.65}$$

Chapter 3

Size Estimates of an Obstacle in a Stationary Stokes Fluid

3.1 Introduction

In this chapter we consider an obstacle D immersed in a region $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is filled with a viscous fluid. Then, the velocity vector u and the scalar pressure p of the fluid in the presence of the obstacle D fulfill the following boundary value problem for the Stokes system:

$$\begin{cases} -\text{div}\,(\sigma(u,p)) &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ \text{div}\ u &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ u &= g \quad , \quad \text{on } \partial\Omega, \\ u &= 0 \quad , \quad \text{on } \partial D, \end{cases} \tag{3.1}$$

where $\sigma(u,p) = 2\mu e(u) - pI$ is the stress tensor, $e(u) = \frac{(\nabla u + \nabla u^T)}{2}$ is the strain tensor, I is the identity matrix of order $d \times d$, n denotes the exterior unit normal to $\partial\Omega$ and $\mu > 0$ is the kinematic viscosity. The condition $u|_{\partial D} = 0$ is the so called *no-slip* condition.

Given the boundary velocity $g \in (H^{1/2}(\partial\Omega))^d$ satisfying the compatibility condition

$$\int_{\partial\Omega} g \cdot n = 0,$$

we consider the solution of problem (3.1), $(u,p) \in (H^1(\Omega \backslash \overline{D}))^d \times L^2(\Omega \backslash \overline{D})$, and measure the corresponding Cauchy force on $\partial\Omega$, $\psi = \sigma(u,p)n|_{\partial\Omega}$, in order to recover the obstacle D. Then, it is well known that this inverse problem has a unique solution. In fact, in [16], the authors prove uniqueness in the case of the steady-state and evolutionary Stokes system using unique continuation property of solutions. By uniqueness we mean the following fact: if u_1 and u_2 are two solutions of (3.1) corresponding to a given boundary data g, for obstacles D_1 and D_2 respectively, and we consider that the Cauchy forces satisfy $\sigma(u_1,p_1)n = \sigma(u_2,p_2)n$ on an open subset $\Gamma_0 \subset \partial\Omega$, then $D_1 = D_2$. Moreover, in [23], $\log - \log$ type stability estimates for the Hausdorff distance between the boundaries of two cavities in terms of the Cauchy forces have been derived. Reconstruction algorithms for the detection of the obstacle have been proposed in [17], [36] and in [63]. The method used in [63] relies on the construction of special complex geometrical optics solutions for the stationary Stokes equation with a variable viscosity. In [17], the reconstruction algorithm released in a nonconvex optimization algorithm (simulating annealing) for the reconstruction of parametric objects. In [36], the detection algorithm is based on topological sensitivity and shape derivatives of a suitable functional. We would like to mention that there hold log type stability estimates for the Hausdorff distance between the boundaries of two cavities in terms of boundary data, also in the case of conducting cavities and elastic cavities (see [6], [37] and [89]). These very weak stability estimates reveal that the problem is severely ill posed limiting the possibility of efficient reconstruction of the unknown object. The above problem motives the study or the identification of partial information on the unknown obstacle D like, for example, the size.

In literature we can find several results concerning the determination of inclusions or cavities and the estimate of their sizes related to different kind of models. Without being exhaustive, we quote some of them. For example in [73] and [74] the problem of estimating the volume of inclusions is analyzed using a finite number of boundary measurements in electrical impedance tomography. In [46], the authors prove uniqueness, stability and reconstruction of an immersed obstacle in a system modeled by a linear wave equation. These results are obtained applying the unique continuation property for the wave equation and in the two dimensional case the inverse problem is transformed in a well-posed problem for a suitable cost functional. We can also mention [63], in which it is analyzed the problem of reconstructing obstacles inside a bounded domain filled with an incompressible fluid by means of special complex geometrical optics solutions for the stationary Stokes equation.

Here we follow the approach introduced by Alessandrini et al. in [8] and in [88] and we establish a quantitative estimate of the size of the obstacle D, i.e. $|D|$, in terms of suitable boundary measurements. More precisely, let us denote by $(u_0, p_0) \in (H^1(\Omega))^d \times L^2(\Omega)$ the velocity vector of the fluid and the pressure in the absence of the obstacle D, namely the solution to the Dirichlet problem

$$\begin{cases} -\text{div } (\sigma(u_0, p_0)) &= 0 \quad , \quad \text{in } \Omega, \\ \text{div } u_0 &= 0 \quad , \quad \text{in } \Omega, \\ u_0 &= g \quad , \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

and let $\psi_0 = \sigma(u_0, p_0)n|_{\partial\Omega}$. We consider now the following quantities

$$W_0 = \int_{\partial\Omega} g \cdot \psi_0 \qquad \text{and} \qquad W = \int_{\partial\Omega} g \cdot \psi,$$

representing the measurements at our disposal. Observe that the following identities hold true

$$W_0 = 2\int_{\Omega} |e(u_0)|^2 \qquad \text{and} \qquad W = 2\int_{\Omega\setminus\overline{D}} |e(u)|^2,$$

giving us the information on the total deformation of the fluid in the corresponding domains, Ω and $\Omega\setminus\overline{D}$. We will establish a quantitative estimate of the size of the obstacle D, $|D|$, in terms of the difference $W - W_0$. In order to accomplish this goal, we will follow the main track of [8] and [88] applying fine interior regularity results, Poincaré type inequalities and quantitative estimates of unique continuation for solutions of the stationary Stokes system. The plan of the chapter is as follows. In Section 3.2 we provide the rigorous formulations of the direct problem and state the main results, Theorems 3.2-3.3. Section 3.3 is devoted to some auxiliar results and to give the proofs of Theorems 3.2-3.3. In Section 3.4 we prove Proposition 3.8 which deals with some estimates for the trace of the Cauchy force on the boundary of the cavity D. Finally, in Section 3.5 we show some computational examples of the behavior of the rate with respect to the shape and the size of the interior obstacle.

3.2 Main results

In this section we introduce some preliminary results we will use through the chapter and we will state our main theorems. Let $x \in \mathbb{R}^d$, we denote by $B_r(x)$ the ball in $\mathbb{R}^d$ centered in x of radius r and $B'_r(0)$ the ball in $\mathbb{R}^{d-1}$. In what follows we will consider the notation $\cdot$ for the scalar product between vectors in $\mathbb{R}^d$, $:$ for the inner product between matrices, and $\otimes$ for the tensorial product between vectors. We set $x = (x_1, \dots, x_d)$ as $x = (x', x_d)$, where $x' = (x_1, \dots, x_{d-1})$.

3.2.1 Preliminaries

In order to prove our main results we need the following a-priori assumptions on Ω, D and the boundary data g.

(H1) $\Omega \subset \mathbb{R}^d$ is a bounded domain with a connected boundary $\partial\Omega$ of Lipschitz class with constants ρ_0, M_0. Further, there exists $M_1 > 0$ such that

$$|\Omega| \leq M_1 \rho_0^d. \tag{3.3}$$

(H2) $D \subset \Omega$ is such that $\Omega \setminus \overline{D}$ is connected and it is strictly contained in Ω, that is there exists a positive constant d_0 such that

$$d(D, \partial\Omega) \geq d_0 > 0. \tag{3.4}$$

Moreover, D has a connected boundary ∂D of Lipschitz class with constants ρ, L.

(H3) D satisfies (**H2**) and the scale-invariant fatness condition with constant $Q > 0$, that is

$$\operatorname{diam}(D) \leq Q\rho. \tag{3.5}$$

(H4) The boundary condition g is such that

$$g \in (H^{1/2}(\partial\Omega))^d, \quad g \not\equiv 0, \quad \frac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}} \leq c_0,$$

for a given constant $c_0 > 0$, and satisfies the compatibility condition

$$\int_{\partial\Omega} g \cdot n = 0.$$

Also suppose that there exists a point $P \in \partial\Omega$, such that,

$$g = 0 \text{ on } \partial\Omega \cap B_{\rho_0}(P).$$

(H5) Since one measurement g is enough in order to detect the size of D, we choose g in such a way that the corresponding solution u satisfies the following condition

$$\int_{\partial\Omega} \sigma(u,p)n = 0. \tag{3.6}$$

(H6) There exists a constant $h_1 > 0$, such that the *fatness condition* holds, namely

$$|D_{h_1}| \geq \frac{1}{2}|D|. \tag{3.7}$$

Concerning assumption **(H5)**, the following result holds.

Proposition 3.1 *There exists at least one function g satisfying* (**H4**) *and* (**H5**).

PROOF. Consider $(d+1)$ linearly independent functions g_i satisfying (**H4**), $i = 1, \dots, d+1$.

Let

$$\int_{\partial\Omega} \sigma(u_i, p_i)n = v_i \in \mathbb{R}^d,$$

where (u_i, p_i) is the corresponding solution of (3.1) associated to g_i, $i = 1, \dots, d+1$.

If, for some i, we have that $v_i = 0$, then the result follows. So, assume that all the v_i are different from the null vector. Then, there exist some constants λ_i, with $i = 1, \dots, d+1$, not all zero, such that

$$\sum_{i=1}^{d+1} \lambda_i v_i = 0$$

and we can choose our Dirichlet boundary data as

$$g = \sum_{i=1}^{d+1} \lambda_i g_i.$$

Therefore, g satisfies (**H4**) and since the Cauchy force is linear with respect to the Dirichlet boundary condition we have

$$\int_{\partial\Omega} \sigma(u,p)n = 0,$$

where (u,p) is the corresponding solution to (3.1), associated to g. □

With respect to these hypotheses, we make some remarks.

Observation Integrating the first equation of (3.1) on $\Omega \setminus \overline{D}$, applying the Divergence Theorem and using (3.6), we obtain

$$\int_{\partial D} \sigma(u,p)n = 0. \tag{3.8}$$

Observation Notice that the constant ρ in (**H2**) already incorporates information on the size of D. In fact, an easy computation shows that if D has a Lipschitz boundary class, with positive constants ρ and L, then we have

$$|D| \geq C(L)\rho^d.$$

Moreover, if also condition (**H3**) is satisfied, then it holds

$$|D| \leq C(Q)\rho^d.$$

Then, it will be necessary to consider ρ as an unknown parameter while the constants L and Q will be assumed as given pieces of a priori information on the unknown inclusion D.

Observation The fatness condition assumption (**H6**) is classic in the context of the size estimates (see [10, 91]), and is satisfied when mild a priori regularity assumptions are made on D. For instance, if D has a boundary of class $C^{1,\alpha}$, then there exists a constant $h_1 > 0$, such that (see [102])

$$|D_{h_1}| \geq \frac{1}{2}|D|. \tag{3.9}$$

where we set, for any $A \subset \mathbb{R}^d$ and $h > 0$,

$$A_h = \{x \in A : \ d(x, \partial A) > h\}.al$$

Observation The non-slip condition for viscous fluids establishes that, on the boundary of the solid, the fluid has zero speed. The fluid velocity in any liquid-solid boundary is the same as that of the solid surface. Conceptually, we can think that the molecules of the fluid closest to the surface of the solid "stick" to the molecules of the solid on which it flows. For that reason, the condition $g = 0$, on $\partial\Omega \cap B_{\rho_0}(P)$, in the assumption (**H4**) is a congruent hypothesis with the non-slip condition on the boundary data. On the other hand, in our case this condition is also a technical assumption. This can be seen in the proof of the main theorems (Section 3), where we need to use the classical Poincaré inequality and one result of Ballerini [23] about the Lipschitz propagation of smallness.

Observation Condition (**H5**) is merely technic and it is used in the proof of Theorem 3.2. We can see that in the case where there is no obstacle in the interior, the condition holds directly. Moreover, we mention that replacing the Dirichlet boundary condition by $\sigma(u,p)n = g$, then assumption (**H5**) is straightforward, due to the compatibility condition.

3.2.2 Main results

Under the previous assumptions we consider the following boundary value problems. When the obstacle $D \subset \Omega$ is present, the pair given by the velocity and the pressure of the fluid in $\Omega \setminus \overline{D}$ is the weak solution $(u,p) \in (H^1(\Omega \setminus \overline{D}))^d \times L^2(\Omega \setminus \overline{D})$ to

$$\begin{cases} -\text{div}\,(\sigma(u,p)) &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ \text{div}\ u &= 0 \quad , \quad \text{in } \Omega \setminus \overline{D}, \\ u &= g \quad , \quad \text{on } \partial\Omega, \\ u &= 0 \quad , \quad \text{on } \partial D. \end{cases} \tag{3.10}$$

Then we can define the function ψ by

$$\psi = \sigma(u,p)n|_{\partial\Omega} \in (H^{-1/2}(\partial\Omega))^d \tag{3.11}$$

and the quantity

$$W = \int_{\partial\Omega} (\sigma(u,p)n) \cdot u = \int_{\partial\Omega} \psi \cdot g.$$

When the obstacle D is absent, we shall denote by $(u_0, p_0) \in (H^1(\Omega))^d \times L^2(\Omega)$ the unique weak solution to the Dirichlet problem

$$\begin{cases} -\text{div}\ (\sigma(u_0,p_0)) &= 0 \ , & \text{in } \Omega, \\ \text{div}\ \ u_0 &= 0 \ , & \text{in } \Omega, \\ u_0 &= g \ , & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Let us define

$$\psi_0 = \sigma(u_0,p_0)n|_{\partial\Omega} \in (H^{-1/2}(\partial\Omega))^d, \tag{3.13}$$

and

$$W_0 = \int_{\partial\Omega} (\sigma(u_0,p_0)n) \cdot u_0 = \int_{\partial\Omega} \psi_0 \cdot g.$$

Our goal is to derive estimates of the size of D, $|D|$, in terms of W and W_0.

Theorem 3.2 *Assume* (**H1**)*,* (**H4**)*,* (**H5**)*,* (**H6**)*, and* (3.4)*. Then, we obtain*

$$|D| \leq K \left(\frac{W - W_0}{W_0} \right), \tag{3.14}$$

where the constant $K > 0$ depends on $\Omega, d, d_0, h_1, M_0, M_1$, and $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$.

Theorem 3.3 *Assume* (**H1**)*,* (**H2**)*,* (**H3**) *and* (**H4**)*. Then, it holds*

$$C\frac{(W - W_0)^2}{WW_0} \leq |D|, \tag{3.15}$$

where $C > 0$ depends on $|\Omega|, d, d_0, L$, and Q.

Observation We expect that a similar result to the one obtained in Theorem 3.2 and 3.3 can be derived when we replace the Dirichet boundary data with

$$\sigma(u,p)n = g, \quad \text{on } \partial\Omega,$$

g satisfying suitable regularity assumptions and the compatibility condition

$$\int_{\partial\Omega} g = 0.$$

Observation In the work [13], the authors showed that the upper bound without assuming a priori information on D, has the form

$$|D| \leq K \left(\frac{W - W_0}{W_0} \right)^{1/p},$$

where $p > 1$. The proof of this inequality is strongly based on the fact that the gradient of the solution of the background conductivity problem, namely u_0, is a Muckenhoupt weight, [58]. Namely, for any $\tilde{r} > 0$ there exists $B > 0$ and $p > 1$ such that

$$\left(\frac{1}{|B_r|}\int_{B_r} |\nabla u_0|^2\right)\left(\frac{1}{|B_r|}\int_{B_r} |\nabla u_0|^{-\frac{2}{p-1}}\right)^{p-1} \leq B,$$

for any ball B_r such that $B_{4r} \subset \Omega_{\tilde{r}}$. This estimate is based on the Caccioppoli inequality, Poincaré–Sobolev inequality, and the called Doubling inequality. It is known that the Doubling inequality holds for some classes of elliptic systems [11]. Unfortunately, as far as we know, for the Stokes system the doubling inequality has not been proved. For instance, see the paper by Lin, Uhlmann and Wang [82] where the authors explain that they were not able to prove a doubling inequality for the Stokes systems, but only to derive a certain optimal three spheres inequality, which is also a strong unique continuation property.

3.3 Proofs of the main theorems

The main idea of the proof of Theorem 3.2 is an application of a three spheres inequality. In particular, we apply a result contained in [82] concerning the solutions to the following Stokes systems

$$\begin{cases} -\Delta u + A(x)\cdot\nabla u + B(x)u + \nabla p &= 0 \quad , \quad \text{in } \Omega, \\ \text{div } u &= 0 \quad , \quad \text{in } \Omega. \end{cases} \tag{3.16}$$

Indeed it holds:

Theorem 3.4 (Theorem 1.1 [82]) *Consider* $0 \leq R_0 \leq 1$ *satisfying* $B_{R_0}(0) \subset \Omega \subset \mathbb{R}^d$. *Then, there exists a positive number* $\tilde{R} < 1$, *depending only on d, such that, if* $0 < R_1 < R_2 < R_3 \leq R_0$ *and* $R_1/R_3 < R_2/R_3 < \tilde{R}$, *we have*

$$\int_{|x|<R_2} |u|^2 dx \leq C\left(\int_{|x|<R_1} |u|^2 dx\right)^{\tau}\left(\int_{|x|<R_3} |u|^2 dx\right)^{1-\tau},$$

for $(u,p) \in (H^1(B_{R_0}(0)))^d \times H^1(B_{R_0}(0))$ *solution to* (3.16). *Here* $C > 0$ *depends on* R_2/R_3, *d, and* $\tau \in (0,1)$ *depends on* R_1/R_3, R_2/R_3, *d. Moreover, for fixed* R_2 *and* R_3, *the exponent* τ *behaves like* $1/(-\log R_1)$, *when* R_1 *is sufficiently small.*

Based on this result, the following proposition holds:

Proposition 3.5 (Lipschitz propagation of smallness, Proposition 3.1 [23]) *Let* Ω *satisfy* (**H1**) *and g satisfies* (**H4**). *Let u be a solution to the problem*

$$\begin{cases} -\text{div}\,(\sigma(u_0,p)) &= 0 \quad , \quad \textit{in } \Omega, \\ \text{div } u_0 &= 0 \quad , \quad \textit{in } \Omega, \\ u_0 &= g \quad , \quad \textit{on } \partial\Omega. \end{cases} \tag{3.17}$$

Then, there exists a constant $s > 1$, depending only on d and M_0, such that for every $r > 0$ there exists a constant $C_r > 0$, such that for every $x \in \Omega_{sr}$, we have

$$\int_{B_r(x)} |\nabla u_0|^2 dx \geq C_r \int_\Omega |\nabla u_0|^2 dx, \tag{3.18}$$

where the constant $C_r > 0$ depends only on $d, M_0, M_1, \rho_0, r, \dfrac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}}$.

Following the ideas developed in [8], we establish a key variational inequality relating the boundary data $W - W_0$ with the L^2 norm of the gradient of u_0 inside the cavity D.

Lemma 3.6 *Let $u_0 \in (H^1(\Omega))^d$ be the solution to problem (3.12) and $u \in (H^1(\Omega \setminus \overline{D}))^d$ be the solution to problem (3.10). Then, there exists a positive constant $C = C(\Omega)$ such that*

$$\int_D |\nabla u_0|^2 \leq C(W - W_0) = C \int_{\partial D} u_0 \cdot \sigma(u,p)n, \tag{3.19}$$

where n denotes the exterior unit normal to ∂D.

Proof. Let (u, p) and (u_0, p_0) be the solutions to problems (3.10) and (3.12), respectively. We multiply the first equation of (3.10) by u_0 and after integrating by parts, we have

$$\int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} (\sigma(u,p)n) \cdot u_0 + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 = 0, \tag{3.20}$$

where n denotes either the exterior unit normal to $\partial\Omega$ or to ∂D.

In a similar way, multiplying the first equation of (3.12) by u_0, we obtain

$$\int_\Omega \sigma(u_0,p_0) : \nabla u_0 - \int_{\partial\Omega} (\sigma(u_0,p_0)n) \cdot u_0 = 0. \tag{3.21}$$

Now, replacing $\psi = \sigma(u,p)n|_{\partial\Omega}$ and $\psi_0 = \sigma(u_0,p_0)n|_{\partial\Omega}$ into the equations (3.20)-(3.21), we get

$$\left\{ \begin{array}{r} \displaystyle\int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 = 0, \\ \displaystyle\int_\Omega \sigma(u_0,p_0) : \nabla u_0 - \int_{\partial\Omega} \psi_0 \cdot g = 0. \end{array} \right. \tag{3.22}$$

Let us define

$$\tilde{u}(x) = \left\{ \begin{array}{ll} u & \text{if } x \in \Omega \setminus \overline{D}, \\ 0 & \text{if } x \in \overline{D}. \end{array} \right.$$

Since $u = 0$ on ∂D, we have $\tilde{u} \in (H^1(\Omega))^d$. So, multiplying (3.10) and (3.12) by $\tilde{u}$, we obtain

$$\left\{ \begin{array}{r} \displaystyle\int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla\tilde{u} - \int_{\partial\Omega} \psi \cdot g + \underbrace{\int_{\partial D} (\sigma(u,p)n) \cdot \tilde{u}}_{=0} = 0, \\ \displaystyle\int_{\Omega\setminus\overline{D}} \sigma(u_0,p_0) : \nabla\tilde{u} - \int_{\partial\Omega} \psi_0 \cdot g = 0. \end{array} \right. \tag{3.23}$$

Using that $\sigma(u,p) = 2e(u) - pI$, where $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, in the first equation of (3.22), we have

$$\begin{aligned}
0 &= \int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\
&= \int_{\Omega\setminus\overline{D}} (2e(u) - pI) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\
&= \int_{\Omega\setminus\overline{D}} 2e(u) : \nabla u_0 - \int_{\Omega\setminus\overline{D}} p(\text{div } \ u_0) - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\
&= \int_{\Omega\setminus\overline{D}} 2e(u) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0,
\end{aligned}$$

where we use the fact that div $\ u_0 = 0$. For the next step, we need a different expression for the term $e(u) : \nabla u_0$. We claim that, for every $v \in (H^1(\Omega))^d$ such that div $\ v = 0$, we have $e(u) : \nabla v = e(u) : e(v)$. Indeed,

$$\begin{aligned}
2e(u) : \nabla v &= \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial v_i}{\partial x_j} \\
&= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial v_i}{\partial x_j} + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial v_j}{\partial x_i} \\
&= e(u) : \nabla v + e(u) : \nabla v^T = 2e(u) : e(v).
\end{aligned}$$

Therefore, equalities (3.22) and (3.23) can be rewritten as

$$2 \int_{\Omega\setminus\overline{D}} e(u) : e(u_0) - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} u_0 \cdot (\sigma(u,p)n) = 0, \tag{3.24}$$

$$2 \int_{\Omega} |e(u_0)|^2 - \int_{\partial\Omega} \psi_0 \cdot g = 0, \tag{3.25}$$

$$2 \int_{\Omega\setminus\overline{D}} |e(u)|^2 - \int_{\partial\Omega} \psi \cdot g = 0, \tag{3.26}$$

$$2 \int_{\Omega\setminus\overline{D}} e(u_0) : e(u) - \int_{\partial\Omega} \psi_0 \cdot g = 0. \tag{3.27}$$

We note that if we subtract (3.27) from (3.24) we get

$$\int_{\partial\Omega} (\psi - \psi_0) \cdot g = \int_{\partial D} u_0 \cdot (\sigma(u,p)n). \tag{3.28}$$

Now, let us consider the quadratic form

$$\begin{aligned}
\int_{\Omega} e(\tilde{u} - u_0) : e(\tilde{u} - u_0) &= \int_{\Omega} |e(u_0)|^2 + \int_{\Omega\setminus\overline{D}} |e(u)|^2 - 2 \int_{\Omega\setminus\overline{D}} e(u) : e(u_0) \\
&= \frac{1}{2} \int_{\partial\Omega} \psi_0 \cdot g + \frac{1}{2} \int_{\partial\Omega} \psi \cdot g - \int_{\partial\Omega} \psi_0 \cdot g \\
&= \frac{1}{2} \int_{\partial\Omega} (\psi - \psi_0) \cdot g.
\end{aligned}$$

By Korn's inequality there exists a constant $C = C(\Omega) > 0$, such that

$$\int_{\Omega} |\nabla(\tilde{u} - u_0)|^2 \leq C \int_{\Omega} |e(\tilde{u} - u_0)|^2.$$

Finally, by the chain of inequalities

$$\int_D |\nabla u_0|^2 = \int_D |\nabla(\tilde{u} - u_0)|^2 \leq \int_\Omega |\nabla(\tilde{u} - u_0)|^2$$
$$\leq C \int_\Omega |e(\tilde{u} - u_0)|^2 = C \int_{\partial\Omega} (\psi - \psi_0) \cdot g = C(W - W_0),$$

and (3.28) the claim follows. □

Now, using the previous results, we are able to prove Theorem 3.2.

Proof. The proof is based on arguments similar to those used in [8] and [10]. Let us consider the intermediate domain $\Omega_{d_0/2}$. Recalling that $d(D, \partial\Omega) \geq d_0$, we have $d(D, \partial\Omega_{d_0/2}) \geq \frac{d_0}{2}$. Let $\varepsilon = \min\left(\frac{d_0}{2}, \frac{h_1}{\sqrt{d}}\right) > 0$. Let us cover the domain D_{h_1} with cubes Q_l of side ε, for $l = 1, \ldots, N$. By the choice of ε, the cubes Q_l are contained in D. Then,

$$\int_D |\nabla u_0|^2 \geq \int_{\cup_{l=1}^N Q_l} |\nabla u_0|^2 \geq \frac{|D_{h_1}|}{\varepsilon^d} \int_{Q_{\bar{l}}} |\nabla u_0|^2, \tag{3.29}$$

where $\bar{l}$ is chosen in such way that

$$\int_{Q_{\bar{l}}} |\nabla u_0|^2 = \min_l \int_{Q_l} |\nabla u_0|^2 > 0.$$

We observe that the previous minimum is strictly positive, in fact, if the minimum is zero, then u_0 would be constant in $Q_{\bar{l}}$. Thus, from the unique continuation property, u_0 would be constant in Ω and since there exists a point $P \in \partial\Omega$, such that,

$$g = 0 \text{ on } \partial\Omega \cap B_{\rho_0}(P),$$

we would have that $u_0 \equiv 0$ in Ω, contradicting the fact that g is different from zero. Then, the minimum is strictly positive.

Let $\overline{x}$ be the center of $Q_{\bar{l}}$. From the estimate (3.18) in Proposition 3.5 with $x = \overline{x}$, $r = \frac{\varepsilon}{2}$, we deduce that

$$\int_{Q_{\bar{l}}} |\nabla u_0|^2 \geq C \int_\Omega |\nabla u_0|^2. \tag{3.30}$$

On account of Remark 3.2.1, we obtain

$$\int_D |\nabla u_0|^2 \geq \frac{\frac{1}{2}|D|}{\varepsilon^d} C \int_\Omega |\nabla u_0|^2 = |D|C' \int_\Omega |\nabla u_0|^2. \tag{3.31}$$

We estimate the right hand side of (3.31). First, using (3.25) we have

$$\int_{\partial\Omega} \psi_0 \cdot g \quad = 2 \int_\Omega |e(u_0)|^2 = 2 \int_\Omega \frac{|\nabla u_0 + \nabla u_0^T|^2}{4}$$
$$= 2 \left(\int_\Omega \frac{|\nabla u_0|^2 + |\nabla u_0^T|^2 + 2\nabla u_0 : \nabla u_0^T}{4} \right)$$

Now, Hölder's inequality implies

$$\int_{\partial\Omega} \psi_0 \cdot g \leq 2 \int_{\Omega} |\nabla u_0|^2. \tag{3.32}$$

Then, coming back to (3.31), we obtain that there exists a constant K, depending on $\Omega, d, d_0, h_1, \rho_0, M_0, M_1$, and $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$ such that

$$\int_{D} |\nabla u_0|^2 \geq |D| K \int_{\partial\Omega} \psi_0 \cdot g. \tag{3.33}$$

Combining (3.33) and Lemma 3.6 we have

$$C \int_{\partial\Omega} (\psi - \psi_0) \cdot g \geq \int_{D} |\nabla u_0|^2 \geq \left(K \int_{\partial\Omega} \psi_0 \cdot g \right) |D|. \tag{3.34}$$

Therefore, we can conclude that

$$|D| \leq K \frac{W - W_0}{W_0},$$

where $\tilde{K}$ is a positive constant depending on $\Omega, d, d_0, h_1, \rho_0, M_0, M_1$, and $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$.
□

In order to prove Theorem 3.3, we make use of the following two propositions. The first Proposition can be found in [8] and the second Proposition will be shown in the next section.

Proposition 3.7 (Poincaré type inequality, Proposition 3.2 [8]) *Let D be a bounded domain in $\mathbb{R}^d$ of Lipschitz class with constants ρ, L and such that* (3.5) *holds. Then, for every $u \in (H^1(D))^d$ we have*

$$\int_{\partial D} |u - u_{\partial D}|^2 \leq \overline{C_1} \rho \int_{D} |\nabla u|^2, \tag{3.35}$$

$$\int_{D} |u - u_{D}|^2 \leq \overline{C_2} \rho^2 \int_{D} |\nabla u|^2, \tag{3.36}$$

where

$$u_{\partial D} = \frac{1}{|\partial D|} \int_{\partial D} u \qquad \text{and} \qquad u_D = \frac{1}{|D|} \int_{D} u,$$

and the constants $\overline{C_1}, \overline{C_2} > 0$ depend only on L, Q.

Proposition 3.8 *Assume* (**H1**)*,* (**H2**)*,* (**H3**) *and* (**H4**)*. The Cauchy force $\sigma(u,p)n$ on ∂D belongs to $L^2(\partial D)$ and the following estimate holds:*

$$\int_{\partial D} |\sigma(u,p)n|^2 \leq \frac{C}{\min\{\rho, 1\}} \int_{\Omega\setminus\overline{D}} |\nabla u|^2, \tag{3.37}$$

where $C > 0$ only depends on $|\Omega|$, L, Q and d_0.

Using this results and Lemma 3.6, we can prove now Theorem 3.3.

Proof. Let $\overline{u}_0$ be the following number

$$\overline{u}_0 = \frac{1}{|\partial D|} \int_{\partial D} u_0. \tag{3.38}$$

Then, we deduce that

$$\int_{\partial D} (\sigma(u,p)n) \cdot u_0 = \int_{\partial D} (\sigma(u,p)n) \cdot u_0 - \int_{\partial D} (\sigma(u,p)n) \overline{u}_0, \tag{3.39}$$

because $\int_{\partial D} \sigma(u,p)n = 0$. From equality (3.28) in Lemma 3.6, we have

$$W - W_0 = \int_{\partial D} (\sigma(u,p)n) \cdot u_0 = \int_{\partial D} (\sigma(u,p)n) \cdot (u_0 - \overline{u}_0). \tag{3.40}$$

Applying Hölder inequality in the right hand side of (3.40) we obtain

$$W - W_0 \leq \left(\int_{\partial D} |u_0 - \overline{u_0}|^2 \right)^{1/2} \left(\int_{\partial D} |\sigma(u,p)n|^2 \right)^{1/2}. \tag{3.41}$$

Now, using Poincaré inequality (3.35) and inequality (3.37) on the right hand side of (3.41), we get

$$W - W_0 \leq C \left(\int_D |\nabla u_0|^2 \right)^{1/2} \left(\int_{\Omega \setminus \overline{D}} |\nabla u|^2 \right)^{1/2}, \tag{3.42}$$

where $C > 0$ depends on $|\Omega|$, Q, L, and d_0. The first integral on the right hand side of (3.42) can be estimated as

$$\int_D |\nabla u_0|^2 \leq |D| \sup_D |\nabla u_0|. \tag{3.43}$$

Now, we need to give an interior estimate for the gradient of u_0. We known that the pressure is an harmonic function. This implies that each component of u_0 is a biharmonic function. Then, using interior regularity estimates for fourth order equations, we deduce that

$$\sup_D |\nabla u_0| \leq C \|u_0\|_{L^2(\Omega)}, \tag{3.44}$$

where the constant C depends on Q, $|\Omega|$ and d_0. Estimate (5.30) can be obtained considering the following results. We know that the embedding from $H^4(\Omega)$ to $C^k(\Omega)$ is continuous for $0 \leq k < 4 - \frac{d}{2}$, with $d = 2, 3$. Then, in particular,

$$\|u_0\|_{C^1(D)} \leq C \|u_0\|_{H^4(D)}.$$

Moreover, from the interior regularity of fourth order equations, see [91, Th. 8.3], we obtain

$$\|u_0\|_{H^4(D)} \leq C \|u_0\|_{H^2(\Omega_{d_0/2})}.$$

Finally, considering the estimates in [21] and [29], we have

$$\|u_0\|_{H^2(\Omega_{d_0/2})} \leq C \|u_0\|_{L^2(\Omega_{d_0/4})} \leq C \|u_0\|_{L^2(\Omega)},$$

and (3.42) holds. We refer to [21, 26, 41], and references therein, for more details on interior estimates for elliptic operators.

As the boundary data g satisfies (**H4**), we use the classical Poincaré inequality and obtain

$$\|u_0\|_{L^2(\Omega)} \leq C\|\nabla u_0\|_{L^2(\Omega)}. \tag{3.45}$$

Therefore, by means of the inequality $\displaystyle\int_\Omega |\nabla u_0|^2 \leq C \int_{\partial\Omega} \psi_0 \cdot g$, we deduce

$$\left(\int_D |\nabla u_0|^2\right)^{1/2} \leq C|D|^{1/2} W_0^{1/2}.$$

Now, concerning the second integral in (3.42), by (3.26), we get

$$\int_{\Omega\setminus\overline{D}} |\nabla u|^2 \leq C \int_{\Omega\setminus\overline{D}} |e(u)|^2 \leq CW. \tag{3.46}$$

Therefore, it holds

$$C\frac{(W-W_0)^2}{WW_0} \leq |D|,$$

where C depends on $|\Omega|, d, L, d_0$ and Q. This completes the proof. □

Observation We note that the last inequality can be rewritten in the form

$$C\phi\left(\frac{W-W_0}{W_0}\right) \leq |D|,$$

where the function ϕ is given by

$$\phi(t) = \frac{t^2}{1-t},\ \forall t \in [0,1].$$

The previous expression is identical to the one obtained in [8].

3.4 Proof of Proposition 3.8

The proof closely follows the arguments of [8]. For technical reason, we introduce the following notation. Given $\tau, L > 0$, and a Lipschitz function $\varphi : B_{2\tau}(0) \subset \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\varphi(0) = 0$, $\|\varphi\|_{C^{0,1}(B_{2\tau}(0))} \leq 2\tau L$. We define for every t, with $0 < t \leq 2\tau$, the following sets

$$\begin{aligned} C_t^+ &:= \{x = (x', x_d) \in \mathbb{R}^d \ :\ |x'| < t,\ \varphi(x') < x_d < Lt\}, \\ \Delta_t &:= \{x = (x', x_d) \in \mathbb{R}^d \ :\ |x'| < t,\ x_d = \varphi(x')\}. \end{aligned}$$

Before proving Proposition 3.8, we need some auxiliary result.

We start by some algebraic formalisms associated with the Stokes system. Let us consider the following family of coefficients, with δ_{jk} denoting the Kronecker symbol,

$$a_{jk}^{\alpha\beta} := \delta_{jk}\delta_{\alpha\beta} + \delta_{j\beta}\delta_{k\alpha},\ 1 \leq j,k,\alpha,\beta \leq d.$$

We denote by A the fourth order tensor associated to the family of coefficients $a_{jk}^{\alpha\beta}$, namely $A = (a_{jk}^{\alpha\beta})$. Let B be any matrix in $\mathbb{R}^d$. Adopting the summation convention over repeated indices, we obtain that this tensor A applied on matrix B, component–wise, is

$$(AB)_{\alpha j} = a_{jk}^{\alpha\beta} b_{k\beta} = \sum_{k,\beta} \delta_{jk}\delta_{\alpha\beta} b_{k\beta} + \sum_{k,\beta} \delta_{j\beta}\delta_{k\alpha} b_{k\beta} = b_{j\alpha} + b_{\alpha j},\ 1 \le \alpha, j \le d.$$

From the previous considerations, we can write the strain tensor $e(u) = \frac{\nabla u + \nabla u^T}{2}$, for $u = (u_\beta)_{1\le\beta\le d}$, component–wise, as

$$(e(u))_{\alpha j} = \frac{a_{jk}^{\alpha\beta}\partial_k u_\beta}{2} = \frac{\partial_\alpha u_k + \partial_k u_\alpha}{2},\ 1 \le \alpha, j \le d,$$

and in matrix form as

$$2e(u) = A\nabla u,$$

where ∇u is the Jacobian matrix associated to u. Then, the Stokes system in a domain $\Omega \subset \mathbb{R}^d$ can be written as

$$\text{div } (A\nabla u - qI) = 0, \quad \text{div } \ u = 0, \text{ in } \Omega. \tag{3.47}$$

From the previous computations, it follows that the α–th component of the normal derivative is

$$[(\nabla u)n]_\alpha = \sum_l (\partial_l u_\alpha) n_l,\ 1 \le \alpha \le d.$$

Then, the tangential component of the gradient of u, $\nabla_T u$, can be expressed by

$$(\nabla_T u)_{\alpha j} = \partial_j u_\alpha - \sum_l (\partial_l u_\alpha) n_l n_j,\ 1 \le \alpha, j \le d. \tag{3.48}$$

Lemma 3.9 *Let* $(u,q) \in (H^{3/2}(C_{2\tau}^+))^d \times L^2(C_{2\tau}^+)$ *such that* $\text{div } (\sigma(u,q)) \in L^2(C_{2\tau}^+)$ *and* $\text{div } \ u = 0$ *in* $C_{2\tau}^+$. *Then, there exists* $C > 0$ *such that*

$$\int_{\partial C_{2\tau}^+} (|\nabla u|^2 + q^2) \le C \int_{\partial C_{2\tau}^+ \setminus \Delta_{2\tau}} |\sigma(u,q)n|^2 + C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}^+} |\text{div } (\sigma(u,q))||\nabla u|, \tag{3.49}$$

where we indicate by $\nabla_T u$ *the tangential gradient of* u *(see* (3.48)*).*

Proof. The proof is based on the Rellich's identity for the Stokes system [30, 50] and elliptic system [96] from which it holds, for any vector valued field $f \in C^\infty(\mathbb{R}^n)$,

$$\int_{C_{2\tau}^+} \text{div } (\sigma(u,q)) \cdot ((\nabla u) f) = \int_{\partial C_{2\tau}^+} \sigma(u,q)n \cdot ((\nabla u)f) - \frac{1}{2}\int_{\partial C_{2\tau}^+} (f\cdot n)|\nabla u|^2$$
$$+ \frac{1}{2}\int_{C_{2\tau}^+} (\text{div } \ f)|\nabla u|^2 + \int_{C_{2\tau}^+} q\partial_i u_k \partial_k f_i - \partial_i u_k(\partial_j u_k + \partial_k u_j)\partial_j f_i. \tag{3.50}$$

More precisely, in Theorem 4.1 and Corollary 4.2 of [30] the authors studied a mixed problem for the Stokes system and they established a technical estimate. This estimate (where we have taken $r = 1$) implies in our particular case

$$\int_{\partial C_{2\tau}^+} (|\nabla u|^2 + q^2) \leq C \int_{\partial C_{2\tau}^+ \setminus \Delta_{2\tau}} |\sigma(u, q)n|^2 + C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}^+} (|\nabla u|^2 + q^2) + C \int_{C_{2\tau}^+} |\text{div } (\sigma(u, q))||\nabla u|.$$

Then, following the proof of Theorem 4.1 in [30] and choosing $f = e_d$ in the Rellich's identity (3.50), we obtain that any terms involving derivatives of f vanish. So that, as in Corollary 4.2 in [30], we have

$$\int_{\partial C_{2\tau}^+} (|\nabla u|^2 + q^2) \leq C \int_{\partial C_{2\tau}^+ \setminus \Delta_{2\tau}} |\sigma(u, q)n|^2 + C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}^+} |\text{div } (\sigma(u, q))||\nabla u|.$$

□

Proposition 3.10 *Let* $(u, q) \in (H^{3/2}(C_{2\tau}^+))^d \times L^2(C_{2\tau}^+)$ *such that* $\text{div } (\sigma(u, q)) \in L^2(C_{2\tau}^+)$, $q = u = |\nabla u| = 0$ *on* $\partial C_{2\tau}^+ \setminus \Delta_{2\tau}$, *and* $\text{div } \ u = 0$ *in* $C_{2\tau}^+$. *Then, we have*

$$\int_{\Delta_\tau} |\sigma(u, q)n|^2 \leq C \left(\int_{\Delta_{2\tau}} |\nabla_T u|^2 + \int_{C_{2\tau}^+} \big(|\nabla u|^2 + |\nabla u||\text{div } (\sigma(u, q))|\big) \right), \tag{3.51}$$

where the constant $C > 0$ *only depends on* L.

Proof. Using again the Rellich identity (3.50) with $f = e_d$ and recalling that $q = u = |\nabla u| = 0$ on $\partial C_{2\tau}^+ \setminus \Delta_{2\tau}$, $u = (u_1, \ldots, u_d)$ and $n = (n_1, \ldots, n_d)$, then we obtain

$$\int_{\Delta_{2\tau}} \left(\sigma(u, q)n \cdot (\partial_d u) - \frac{1}{2} n_d |\nabla u|^2 \right) = \int_{C_{2\tau}^+} \text{div } (\sigma(u, q)) \cdot (\partial_d u). \tag{3.52}$$

Now, we express the matrix ∇u in terms on its tangential component $\nabla_T u$ and the Cauchy forces $\sigma(u, q)n$. From (3.48), we have

$$(\nabla u)_{\alpha j} = \sum_l (\partial_l u_\alpha) n_l n_j + \sum_l (\partial_\alpha u_l) n_l n_j + (\nabla_T u)_{\alpha j} - \sum_l (\partial_\alpha u_l) n_l n_j.$$

Recalling that the tensorial product is denoted by $\otimes$, we obtain

$$\nabla u = (A\nabla u - qI)n \otimes n + \nabla_T u - (\nabla u - qI)^T n \otimes n.$$

Using the above expression we can write the scalar terms $\sigma(u, q)n \cdot (\partial_d u)$ and $\frac{1}{2} n_d |\nabla u|^2$ as

$$\begin{aligned} \sigma(u, q)n \cdot (\partial_d u) &= n_d |\sigma(u, q)n|^2 + \sum_j (\sigma(u, q)n)_j (\nabla_T u)_{jd} - \sum_j (\sigma(u, q)n)_j ((\nabla u - qI)^T n : n)_{jd} \\ &= n_d |\sigma(u, q)n|^2 + \sigma(u, q)n \cdot (\nabla_T u)_{\cdot d} - \sigma(u, q)n \cdot ((\nabla u - qI)^T n)_{\cdot d}, \end{aligned} \tag{3.53}$$

and

$$\frac{1}{2}n_d|\nabla u|^2 = \frac{1}{2}n_d|\sigma(u,q)n|^2 + \frac{1}{2}n_d|\nabla_T u|^2 + \frac{1}{2}n_d|(\nabla u - qI)^T n|^2 +$$
$$n_d\Big(|[\sigma(u,q)n\otimes n] : \nabla_T u| + |[\sigma(u,q)n\otimes n] : [(\nabla u - qI)^T n\otimes n]| + |\nabla_T u : [(\nabla u - qI)^T n\otimes n]|\Big) \tag{3.54}$$

Replacing (3.53) and (3.54) in (3.52), we obtain

$$\frac{1}{2}\int_{\Delta_{2\tau}} n_d|\sigma(u,q)n|^2 = \int_{\Delta_{2\tau}} \Big(\frac{1}{2}n_d|\nabla_T u|^2 + \frac{1}{2}n_d|(\nabla u - qI)^T n|^2 + \frac{1}{2}n_d|[\sigma(u,q)n\otimes n] : \nabla_T u|$$
$$+ \frac{1}{2}n_d|[\sigma(u,q)n\otimes n] : [(\nabla u - qI)^T n\otimes n]| + \frac{1}{2}n_d|\nabla_T u : [(\nabla u - qI)^T n\otimes n]| +$$
$$\sigma(u,q)n\cdot(\nabla_T u)_{\cdot d} - \sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d}\Big) + \int_{C^+_{2\tau}} \mathrm{div}\,(\sigma(u,q))\cdot(\partial_d u). \tag{3.55}$$

Now, we apply the Young inequality with weight $\varepsilon > 0$, namely $ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{\varepsilon}b^2$, to the last five terms on $\Delta_{2\tau}$ on the right hand side in (3.55). We obtain that

$$|[\sigma(u,q)n\otimes n] : \nabla_T u| \le \frac{\varepsilon}{2}|\sigma(u,q)n|^2 + \frac{1}{2\varepsilon}|\nabla_T u|^2 = \frac{\varepsilon}{2}|\sigma(u,q)n|^2 + C_\varepsilon|\nabla_T u|^2,$$

$$|[\sigma(u,q)n\otimes n] : [(\nabla u - qI)^T n\otimes n]| \le \frac{\varepsilon}{2}|\sigma(u,q)n|^2 + C_\varepsilon|(\nabla u - qI)^T n|^2,$$

$$|\nabla_T u : [(\nabla u - qI)^T n\otimes n]| \le \frac{\varepsilon}{2}|(\nabla u - qI)^T n|^2 + C_\varepsilon|\nabla_T u|^2,$$

$$\sigma(u,q)n\cdot(\nabla_T u)_{\cdot d} \le \frac{\varepsilon}{2}|\sigma(u,q)n|^2 + C_\varepsilon|\nabla_T u|^2,$$

$$-\sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d} \le |\sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d}| + \frac{\varepsilon}{2}|\sigma(u,q)n|^2 + C_\varepsilon|(\nabla u - qI)^T n|^2.$$

This implies

$$\frac{1}{2}\int_{\Delta_{2\tau}} n_d|\sigma(u,q)n|^2 \le C_\varepsilon\int_{\Delta_{2\tau}} \left(|\nabla_T u|^2 + |\nabla u|^2 + q^2\right) + \varepsilon\left(C\int_{\Delta_{2\tau}} \left(|\sigma(u,q)n|^2 + |\nabla u|^2 + q^2\right)\right)$$
$$+ \int_{C^+_{2\tau}} \mathrm{div}\,(\sigma(u,q))\cdot(\partial_d u), \tag{3.56}$$

where $C > 0$. From Lemma 3.9 and using the assumptions $q = u = |\nabla u| = 0$ on $\partial C^+_{2\tau}\setminus\Delta_{2\tau}$, we obtain

$$\int_{\Delta_{2\tau}} |\nabla u|^2 + q^2 \le C\int_{\Delta_{2\tau}} |\nabla_T u|^2 + C\int_{C_{2\tau}} \mathrm{div}\,(\sigma(u,q))\cdot(\partial_d u). \tag{3.57}$$

Combining (3.56) and (3.57), and using the inequality $|n_d| \geq \frac{1}{\sqrt{1+L^2}}$, then we derive

$$\frac{1}{2\sqrt{1+L^2}} \int_{\Delta_{2\tau}} |\sigma(u,q)n|^2 - \varepsilon \int_{\Delta_{2\tau}} |\sigma(u,q)n|^2 \leq C_\varepsilon \left(\int_{\Delta_{2\tau}} |\nabla_T u|^2 + \int_{C_{2\tau}} \text{div } (\sigma(u,q)) \cdot (\partial_d u) \right)$$
$$+ \varepsilon \left(C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}} \text{div } (\sigma(u,q)) \cdot (\partial_d u) \right) + \int_{C_{2\tau}} \text{div } (\sigma(u,q)) \cdot (\partial_d u).$$

Choosing $\varepsilon > 0$ small enough, we have

$$\int_{\Delta_\tau} |\sigma(u,q)n|^2 \leq C \left(\int_{\Delta_{2\tau}} |\nabla_T(u)|^2 + \int_{C^+_{2\tau}} \left(|\nabla u|^2 + |\nabla u| |\text{div } (\sigma(u,q))| \right) \right),$$

where the constant $C > 0$ depends on L, and the proof is finished.

□

Proposition 3.11 *Let $(v,p) \in (H^1(C^+_{2\tau}))^d \times L^2(C^+_{2\tau})$ be the solution of the problem*

$$\begin{cases} -\text{div } (\sigma(v,p)) = 0, & \text{in } C^+_{2\tau}, \\ \text{div } v = 0, & \text{in } C^+_{2\tau}. \end{cases} \tag{3.58}$$

If $v|_{\Delta_{2\tau}} \in H^1(\Delta_{2\tau})$, then $\sigma(v,p)n \in L^2(\Delta_\tau)$ and

$$\int_{\Delta_\tau} |\sigma(v,p)n|^2 \leq C \left[\int_{\Delta_{2\tau}} |\nabla_T v|^2 + \left(1 + \frac{1}{\tau}\right) \int_{C^+_{2\tau}} |\nabla v|^2 \right], \tag{3.59}$$

where the constant $C > 0$ only depends on L.

Proof. First, we assume that the function v is more regular, namely $v \in H^{3/2}(C^+_{2\tau})$. We consider the following vector field cut–off function $\eta = (\eta_1, \ldots, \eta_d)$ in $\mathbb{R}^d$

$$\eta_i(x', x_d) = \varphi_i(x')\psi_i(x_d), \ \forall i = 1, \ldots, d, \ \text{div } \eta = 0, \tag{3.60}$$

where

$$\varphi_i \in C_0^\infty(\mathbb{R}^{d-1}), \quad \varphi_i(x') = 1 \text{ if } |x'| \leq \tau, \quad \varphi_i(x') = 0 \text{ if } |x'| \geq \frac{3}{2}\tau, \tag{3.61}$$
$$\|\nabla \varphi_i\|_\infty \leq C_1 \tau^{-1}, \qquad \|\nabla^2 \varphi_i\|_\infty \leq C_1 \tau^{-2}, \tag{3.62}$$
$$\psi_i \in C_0^\infty(\mathbb{R}), \quad \psi_i(x_d) = 1 \text{ if } |x_d| \leq \tau L, \quad \psi_i(x_d) = 0 \text{ if } |x_d| \geq \frac{3}{2}\tau L, \tag{3.63}$$
$$\|\nabla \psi_i\|_\infty \leq C_2 \tau^{-1}, \qquad \|\nabla^2 \psi_i\|_\infty \leq C_2 \tau^{-2}. \tag{3.64}$$

Here C_1 is an absolute constant and C_2 is a constant only depending on L. For $u = (u_1, \ldots, u_d)$ and $c \in \mathbb{R}$, we consider the function

$$u_i = \eta_i(v_i - c), \ q = \eta_j p, \ i = 1, \ldots, d, \ \text{for some } j \in \{1, \ldots, d\}. \tag{3.65}$$

We note that if we take $\tau = t$ in Proposition 3.10, for every $\frac{3}{4}\tau < t < \tau$, we obtain that $|x'| \in (\frac{3}{2}\tau, 2\tau)$. This implies $\varphi_i = 0$, for every $i = 1, \dots, d$. Then, the pair (u, q) satisfies the hypotheses of Proposition 3.10, with $\tau = t$, for every $\frac{3}{4}\tau < t < \tau$. Namely,

$$\int_{\Delta_\tau} |\sigma(u,q)n|^2 \leq C \left(\int_{\Delta_{2\tau}} |\nabla_T u|^2 + \int_{C^+_{2\tau}} \left(|\nabla u|^2 + |\nabla u| |\mathrm{div}\,(\sigma(u,q))| \right) \right).$$

Recalling that (v, p) satisfies equation (3.58) and the definition of the cut–off function (3.61)–(3.64), we obtain

$$\int_{\Delta_t} |\sigma(v,p)n|^2 \leq C \left(\int_{\Delta_{2t}} |\nabla_T v|^2 + \left(1 + \frac{1}{t}\right) \int_{C^+_{2t}} \left[\frac{(v-c)^2}{t^2} + |\nabla v|^2 \right] \right),$$

for every $\frac{3}{4}\tau < t < \tau$. Choosing the constant c such that

$$c = \frac{1}{|C^+_{2t}|} \int_{C^+_{2t}} v,$$

and applying the Poincaré inequality (3.36), we obtain

$$\int_{\Delta_t} |\sigma(v,p)n|^2 \leq C \left(\int_{\Delta_{2t}} |\nabla_T v|^2 + \left(1 + \frac{1}{t}\right) \int_{C^+_{2t}} |\nabla v|^2 \right).$$

Then passing to the limit for $t \to \tau$, we deduce (3.59). We observe that the assumption of the regularity on v is satisfied when $\varphi \in C^\infty$, by the regularity of the Stokes problem.

Now, given a Lipschitz function φ, let $\{\varphi_m\}_m$ be a sequence of C^∞ equi–Lipschitz functions with constant L, such that

$$\varphi_m(0) = 0, \quad \varphi_m \to \varphi \text{ uniformly,}$$
$$\nabla \varphi_m \to \nabla \varphi \text{ in } L^p, \ \forall p < \infty, \text{ as } m \to \infty.$$

Therefore, we have that (3.59) is valid when φ is replaced by φ_m, for every m.

For every m and for every t, with $0 < t \leq 2\tau$, let us consider the following sets

$$\begin{aligned} C^+_{t,m} &:= \{x = (x', x_d) \in \mathbb{R}^d \ : \ |x'| < t, \ \varphi_m(x') < x_d < Lt\}, \\ \Delta_{t,m} &:= \{x = (x', x_d) \in \mathbb{R}^d \ : \ |x'| < t, \ x_d = \varphi_m(x')\}. \end{aligned}$$

Let $(u_m, p_m) \in (H^1(C^+_{2\tau}))^d \times L^2 C^+_{2\tau}$ be the solution to the following Stokes problem

$$\begin{cases} -\mathrm{div}\,(\sigma(v_m, p_m)) = 0 \ , & \text{in } C^+_{2\tau}, \\ \mathrm{div}\ v_m = 0 \ , & \text{in } C^+_{2\tau}, \\ v_m = v \ , & \text{on } \partial C^+_{2\tau}. \end{cases} \tag{3.66}$$

Then, multiplying the first equation in (3.66) by v_m and integrating by parts, we obtain

$$\int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla v_m = \int_{\partial C^+_{2\tau}} [\sigma(v_m, p_m)n] v_m = \int_{\partial C^+_{2\tau}} [\sigma(v_m, p_m)n] v = \int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla v.$$

Equivalently, from (3.24) and the fact that div $v_m = 0$, we deduce

$$\int_{C_{2\tau}^+} e(v_m) : e(v_m) = \int_{C_{2\tau}^+} e(v_m) : e(v).$$

Therefore, $e(v_m)$ is a bounded sequence in $L^2(C_{2\tau}^+)$ and

$$\int_{C_{2\tau}^+} |e(v_m)|^2 \le C \int_{C_{2\tau}^+} |e(v)|^2, \tag{3.67}$$

where $C > 0$. We note that $(v_m - v)|_{\partial C_{2\tau}^+} = 0$, then applying the Poincaré inequality to $v_m - v$ and the Korn's inequality, we deduce

$$\int_{C_{2\tau}^+} |v_m - v|^2 \le C \int_{C_{2\tau}^+} |\nabla v_m|^2 \le C \int_{C_{2\tau}^+} |e(v_m)|^2 \le C \int_{C_{2\tau}^+} |e(v)|^2.$$

Namely, $\{v_m\}_m$ is a bounded sequence in $(H^1(C_{2\tau}^+))^d$. Then, there exists a subsequence, still denoted by v_m, such that $\{v_m\}_m$ converges weakly in $(H^1(C_{2\tau}^+))^d$ to some function $u \in (H^1(C_{2\tau}^+))^d$. From estimate (2.21), we obtain that the sequence $\{p_m\}_m$ also converges weakly in $L^2(C_{2\tau}^+)$ to some $q \in L^2(C_{2\tau}^+)$. Besides, div $v_m = 0$ in $C_{2\tau}^+$, for every m, and then we have div $u = 0$ in $C_{2\tau}^+$. Recalling (3.66), for every $\xi \in V = \{f \in (H_0^1(C_{2\tau}^+))^d \ : \ \text{div} \ f = 0\}$, and using again (3.24), we obtain

$$-\int_{C_{2\tau}^+} \text{div } (\sigma(u,q)) \cdot \xi = \int_{C_{2\tau}^+} \sigma(u,q) : \nabla \xi - \int_{\partial C_{2\tau}^+} \sigma(u,q) n \cdot \xi = \int_{C_{2\tau}^+} e(u) : \nabla \xi, \tag{3.68}$$

and

$$0 = -\int_{C_{2\tau}^+} \text{div } (\sigma(v_m, p_m)) \cdot \xi = \int_{C_{2\tau}^+} \sigma(v_m, p_m) : \nabla \xi - \int_{\partial C_{2\tau}^+} \sigma(v_m, p_m) n \cdot \xi$$
$$= \int_{C_{2\tau}^+} e(v_m) : \nabla \xi. \tag{3.69}$$

This implies

$$-\int_{C_{2\tau}^+} \text{div } (\sigma(u,q)) \cdot \xi = \int_{C_{2\tau}^+} e(u) : \nabla \xi - \int_{C_{2\tau}^+} e(v_m) : \nabla \xi = \int_{C_{2\tau}^+} e(u - v_m) : \nabla \xi. \tag{3.70}$$

From the weak convergence in $(H^1(C_{2\tau}^+))^d$ of $\{v_m\}m$ to u, we obtain that the right hand side of (3.70) converges to zero, as m tends to infinity, for every $\xi \in V$. Then, (u, q) is a weak solution to

$$\begin{cases} -\text{div } (\sigma(u,q)) &= 0 \ , & \text{in } C_{2\tau}^+, \\ \text{div } \ u &= 0 \ , & \text{in } C_{2\tau}^+. \end{cases}$$

On the other hand, on account of the trace Theorem we have $v_m \rightharpoonup u$ weakly in $(H^{1/2}(\partial C_{2\tau}^+))^d$. So that $v_m = v$ on $\partial C_{2\tau}^+$ implies $u = v$ on $\partial C_{2\tau}^+$. From the uniqueness of the solution to the Stokes problem we obtain that $u = v$ and $q = p$ in $C_{2\tau}^+$. Therefore, we get

$$v_m \rightharpoonup v \text{ weakly in } (H^1(C_{2\tau}^+))^d,$$

and, by compactness,

$$v_m \to v \text{ in } (L^2(C^+_{2\tau}))^d.$$

Now, as noticed before, the equation (3.59) holds for $v = v_m$ and $p = p_m$, then

$$\int_{\Delta_\tau} |\sigma(v_m, p_m)n|^2 \leq C \left(\int_{\Delta_{2\tau}} |\nabla_T v_m|^2 + \left(1 + \frac{1}{\tau}\right) \int_{C^+_{2\tau}} |\nabla v_m|^2 \right), \tag{3.71}$$

where $C > 0$ is independent of m. We observe that $v = v_m$ on $\Delta_{2\tau}$, $\nabla_T v \in (L^2(\Delta_{2\tau}))^d$ by hypotheses. So that, using the equation (3.67) we deduce

$$\int_{\Delta_\tau} |\sigma(v_m, p_m)n|^2 \leq C,$$

where $C > 0$ is independent of m. Hence, up to asubsequence, $\sigma(v_m, p_m)n$ converges weakly in $(L^2(\Delta_\tau))^d$ to some $h \in (L^2(\Delta_\tau))^d$. On the other hand, let us take any $\xi \in \tilde{V} = \{f \in (H^1(C_{2\tau}))^d \ : \ \text{div} \ \ f = 0\}$. Using (3.24), it follows

$$0 = \int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla\xi - \int_{\partial C^+_{2\tau}} \sigma(v_m, p_m)n \cdot \xi, 0 = \int_{C^+_{2\tau}} \sigma(v, p) : \nabla\xi - \int_{\partial C^+_{2\tau}} \sigma(v, p)n \cdot \xi.$$

Therefore,

$$\int_{\partial C^+_{2\tau}} \sigma(v_m, p_m)n \cdot \xi - \int_{\partial C^+_{2\tau}} \sigma(v, p)n \cdot \xi = \int_{C^+_{2\tau}} e(v_m - v) : \nabla\xi,$$

and the last integral converges to zero, as m tends to infinity. Namely,

$$\sigma(v_m, p_m)n \rightharpoonup \sigma(v, p)n \text{ weakly in } (L^2(\partial C^+_{2\tau}))^d. \tag{3.72}$$

Finally, we obtain $\sigma(v, p)n = h \in (L^2(\Delta_\tau))$. Then, by definition

$$\|\sigma(v, p)n\|_{L^2(\Delta_\tau)} \leq \liminf_{m\to\infty} \|\sigma(v_m, p_m)n\|_{L^2(\Delta_\tau)}.$$

On account of (3.67) and (3.71), we deduce (3.37).

□

Using the previous result, we are able to prove the Proposition 3.8.

Proof. First, assume that $\rho < d_0$. Let us cover ∂D with internally nonoverlapping closed cubes Q_j, $j = 1, \dots, J$, with side $\tilde{\rho} = \gamma(L)\rho$, where $\gamma(L) = \frac{\min\{1,L\}}{2\sqrt{d}\sqrt{1+L^2}}$. From the result of [8], we have

$$J \leq C \frac{|D|}{\rho^d} \leq CQ^d, \tag{3.73}$$

where $C > 0$ only depends on L. For every $j = 1, \dots, J$ there exists $x_0 \in \partial D \cap Q_j$ such that $Q_j \cap (\Omega \setminus \overline{D}) \subset C^+_{\overline{\rho}}$, where $\overline{\rho} = \frac{\rho}{2\sqrt{1+L^2}}$ and $C^+_t = \{y = (y', y_d) \in \mathbb{R}^d \ : \ |y'| < t \ \varphi(y') < y_d <$

$tL\}$, for every t, with $0 < t \leq 2\overline{\rho}$. In this case, φ is a Lipschitz function in $B_{2\overline{\rho}}(0) \subset \mathbb{R}^{d-1}$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{0,1}(B_{2\overline{\rho}}(0))} \leq 2\overline{\rho}L$, representing locally the boundary of D in a suitable coordinate system $y = (y_1, \dots, y_d)$, $y = Rx$, with R an orthogonal transformation and $x = (x_1, \dots, x_d)$ the reference coordinate system. We note that from (3.47), the functions $u \in (H^1(\Omega \setminus \overline{D}))^d$, $p \in L^2(\Omega \setminus \overline{D})$ satisfies

$$-\text{div}\,(\tilde{\sigma}(u,p)) = 0, \quad \text{in } C^+_{2\overline{\rho}},$$

where $\tilde{\sigma}(u,p) = (RA(R^T\nabla u)R^T - RpIR^T)$. We have that $u = 0$ on ∂D, then applying equation (3.59) with $\tau = \overline{\rho}$, we obtain

$$\int_{\partial D \cap Q_j} |\sigma(u,p)|^2 \leq C\left(1 + \frac{1}{\rho}\right)\int_{C^+_{2\overline{\rho}}} |\nabla u|^2,$$

where $C > 0$ only depends on L. Following the same arguments as in the proof of Proposition 3.3 in [8], we deduce (3.37). □

3.5 Computational examples

In this section we will perform some numerical experiments to compute $|\frac{W-W_0}{W_0}|$ for classes of cavities for which our result holds. In particular, we expect to collect numerical evidence that the ratio between $\frac{|D|}{|\Omega|}$ and $|\frac{W-W_0}{W_0}|$ is bounded from below and above by two constants, representing the ones appearing in our estimates.

Moreover, we are interested in studying the dependence of this ratio on d_0, which bounds from below the distance of D from $\partial\Omega$, and the size of the inclusions.

A more systematic analysis would require the knowledge of explicit solutions u and u_0. This would allow to compute analytically the constants in the upper and lower bounds, at least for some particular geometries. On the contrary to the case in [8], for the Stokes system it is difficult to find explicit solutions.

For the experiments we use the free software *FreeFem++* (see [61]). Moreover, in all numerical tests we consider a square domain Ω, discretized with a mesh of 100×100 elements, and with boundary condition $u|_{\partial\Omega} = g$ as in Figure 3.1. The datum g satisfies the assumptions (**H4**) and (**H5**).

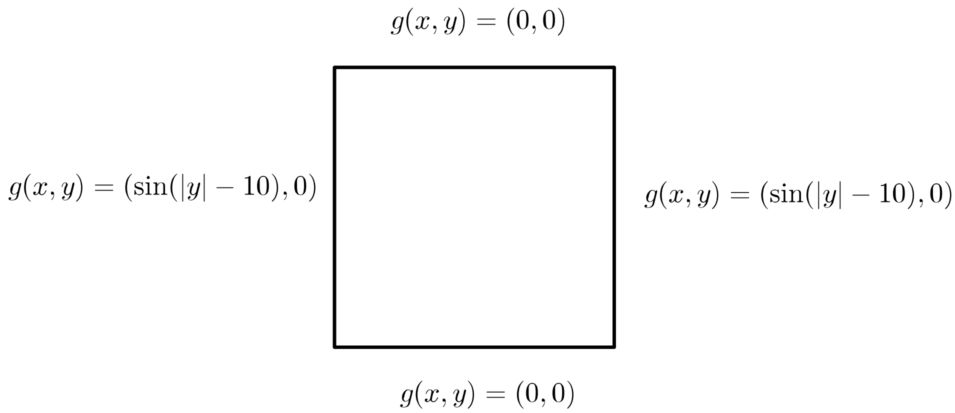


Figure 3.1: Square domain in 2-D with boundary condition g.

The first series of numerical tests has been performed by varying the position and the size of a circle inclusion D with volume up to 8% of the total size of the domain. In particular, we consider a circle inclusion with volume 0.2%, 3.1% and 7.1% with respect to $|\Omega|$. We have placed these circles in eight different positions, see Figure 3.2. The results are collected in Figure 3.3, 3.4 and 3.5, for different values of the distance d_0 between the object D and the boundary of Ω. Also, the averages of all this simulations are collected in Figure 3.6.

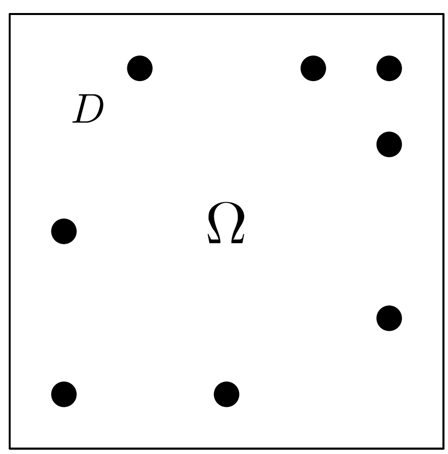
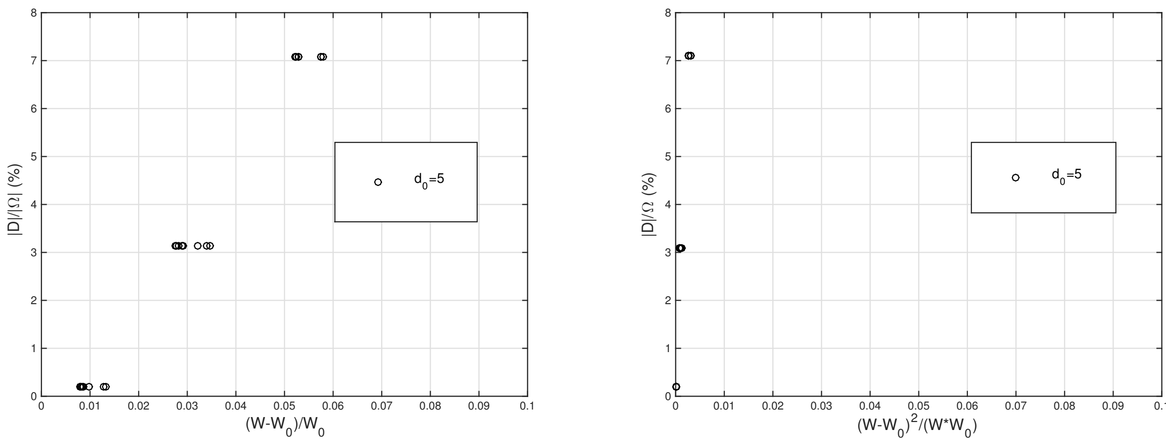


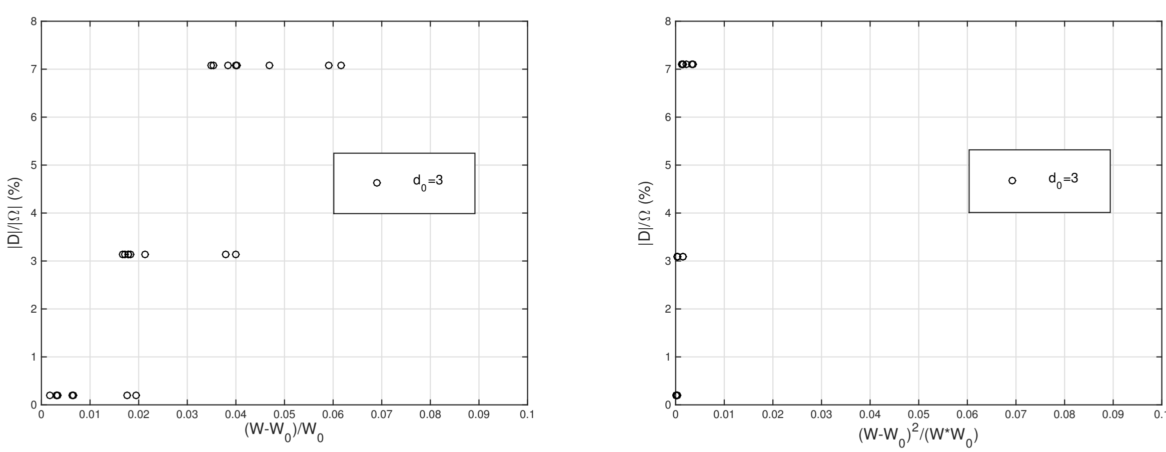
Figure 3.2: The eight positions of the circle inclusion D.

In order to compare our numerical results with the theoretical upper and lower bounds (3.14) and (3.15), it is interesting to study the relationship between $\frac{|D|}{|\Omega|}$ and $|\frac{W-W_0}{W_0}|$. As we expected from the theory, the points $(\frac{W-W_0}{W_0}, \frac{|D|}{|\Omega|})$ are confined inside an angular sector delimited by two straight lines.



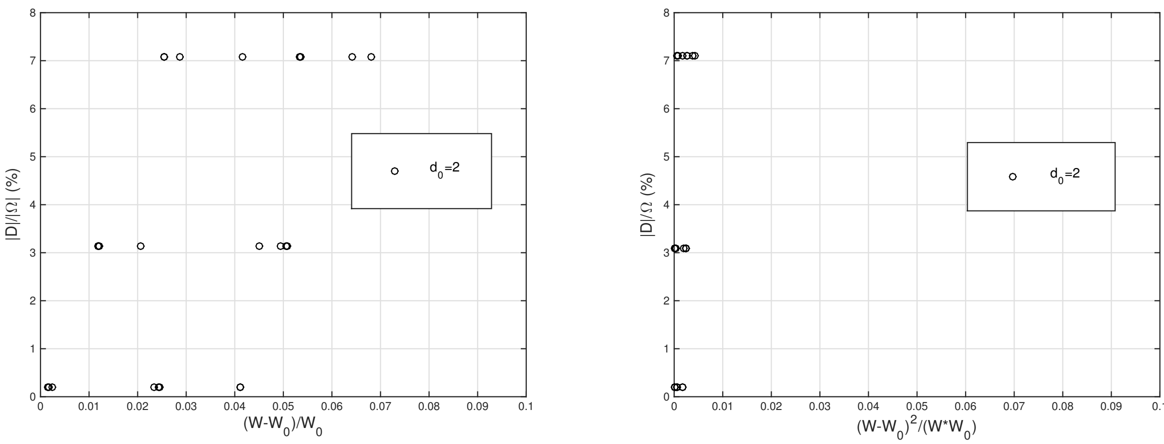
(a) Upper Estimate (b) Lower Estimate

Figure 3.3: Case $d_0 = 5$ for circle inclusion.



(a) Upper Estimate (b) Lower Estimate

Figure 3.4: Case $d_0 = 3$ for circle inclusion.



(a) Upper Estimate (b) Lower Estimate

Figure 3.5: Case $d_0 = 2$ for circle inclusion.

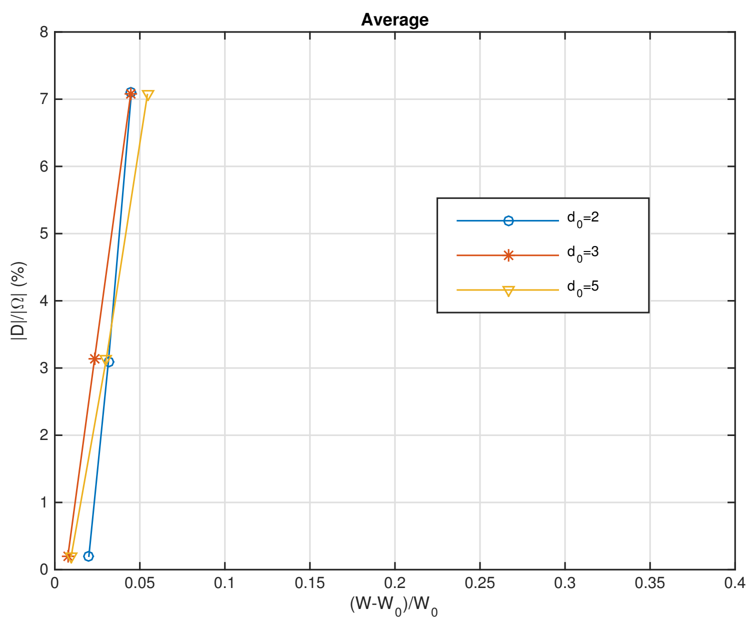
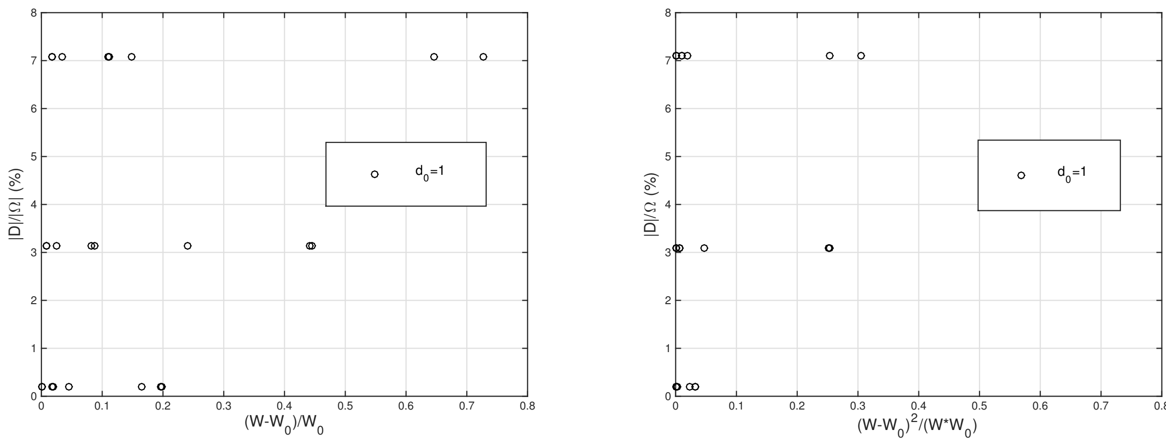


Figure 3.6: Averages of the ratio $\frac{W-W_0}{W_0}$ with different d_0 for circle inclusion.

However, it is quite clear that when d_0 decreases, then the lower bound becomes worse. To illustrate this situation, we simulate also the case when the distance is $d_0 = 1$, see Figure 3.7.



(a) Upper Estimate

(b) Lower Estimate

Figure 3.7: Case $d_0 = 1$ for circle inclusion.

As a second class of experiments, we consider what happens when the size of the circle increases. In this case we can observe that the number $|\frac{W-W_0}{W_0}|$ grows rapidly when the volume occupies almost the entire domain. The result is collected in Figure 3.8.

Again it is observed the relationship between the volume of the object with the quotient $(W - W_0)/W_0$. This gives us an indication that the estimates found in Theorems 3.2 and 3.3 involve constants that do not depend on the inclusion.

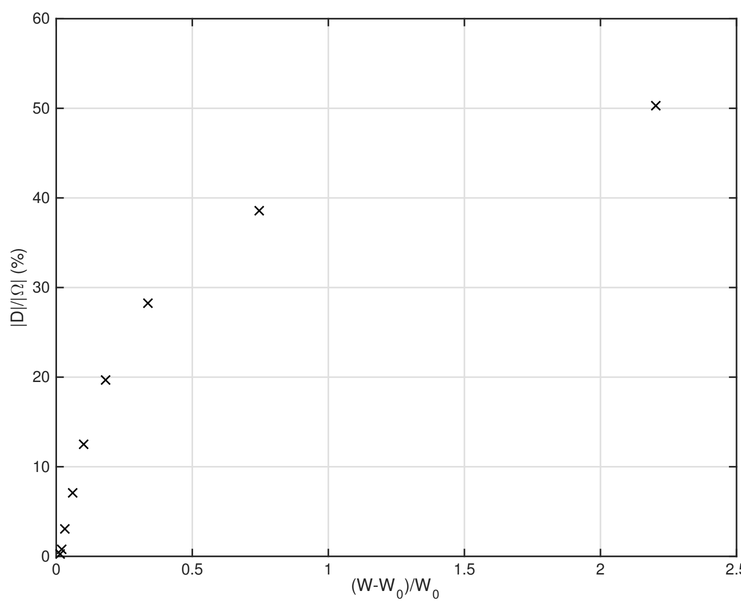


Figure 3.8: Influence of the size of the circle.

Observation From the previous analysis an interesting problem would be to find optimal lower and upper bounds for this model. Another interesting issue would be to weaken the a-priori assumptions imposed on the obstacle, as for example the fatness condition (see, for instance, [13, 43], where this restriction is removed in the case of the conductivity and shallow shell equations, respectively).

Chapter 4

Turnpike Property for Two–Dimensional Navier–Stokes Equations

4.1 Introduction

In this chapter, we are dealing with optimal control problem of the incompressible Navier-Stokes equations in two dimensions, both evolutionary and stationary problem. We try to understand what is the relationship between the optimal solution of the nonstationary and the stationary problem, when the times goes to infinity. Specifically, we want to know the conditions under which the nonstationary optimal control and state converges to the stationary optimal control and state, respectively.

We consider two cases, when the controls are dependent on time and the case where are independent on time. In both cases the optimal control problem consists in minimizing a functional involving both the control and the measure of the difference between the state and a desired stationary state, and terminal constraint. Then, the main idea of this chapter is to prove that the optimal controls achieve to get the target state and remains on this situation most of the time.

In the first case, we establish a result of exponential convergence of the optimality systems associated to the Navier-Stokes equations. We prove, see Theorem 4.9 in Section 3.2, under some appropriate smallness conditions of the optimal solutions for the stationary problem, that both optimal evolutionary state and control converge to the respective optimal stationary control and state in a local sense with an exponential rate.

For the second case, as we consider time-independent controls, using the Γ-convergence we prove that the accumulation point of a sequence of controls for the evolutionary optimal control problem is an optimal control for the stationary problem, see Theorem 4.11 in Section 3.3. In this case, we need to ensure the exponential stabilization of the solution of the nonstationary Navier-Stokes problem to the solution of the stationary Navier-Stokes equation, under some smallness condition.

The smallness condition for the optimal state of the stationary equations is because it is well known that the solution of the stationary Navier-Stokes system is unique when the viscosity is large enough with respect to the right hand side [107]. If we remove this condition we need to work with solutions for which the equation is locally unique. These solutions are called *nonsingular solutions*, see, for instance, Casas et al. [35].

The study of this type of relationship is commonly used in many models of the fluid mechanics, where the stationary model is considered instead of the evolutionary system. Namely, the underlying idea is that when the time horizon is large enough, the evolutionary optimal control are sufficiently close to the stationary optimal control.

For example, in aeronautics most of the techniques to solve shape optimization are based on stationary models. In that case, is assumed or understood that the optimal shape is close enough to the evolutionary optimal shape, see, for instance, [66]. There are no results justifying such assumptions, specially for models from the fluids mechanics, as the Navier-Stokes or Euler equations (see [72]).

A recent answer to this problem is given in [98]. The authors examined such questions in the context of linear control problems both in the finite dimensional case as infinite dimensional systems, including the linear heat and wave equations. They proved, under suitable observability and controllability assumptions, that optimal controls and state converge exponentially when the time is sufficiently large, to the corresponding stationary case. Porretta and Zuazua in [98] mentioned that this type of property in the economy field, specifically in econometry, is known as the *turnpike property*, concept introduced by P. Samuelson. In [108], the authors proved the turnpike property in the case of nonlinear optimal control problem in the finite-dimensional case.

Also, this type of approach can be observed in optimal design. We mention [15], where the autohrs proved that when the time tends to infinity, the optimal design of coefficients of parabolic dynamics converge to those of the elliptic steady state problem. This approach use the classical Γ-convergence, because they consider coefficients which are independent of time.

In this work, we consider the Navier-Stokes equations in two dimensions. Navier-Stokes equations are useful because they describe the physics of many things of scientific and engineering interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier-Stokes system in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, among others.

It is well known that in the three-dimensional cases there are many open problems connected with smoothness and uniqueness of weak solutions, both nonstationary as stationary models. Hence, in this chapter we restrict our attention to the two-dimensional case. In particular, we consider incompressible and newtonian fluids. Namely, the density remains constant within a parcel of fluid that moves with the flow velocity and constant viscosity, respectively. Obviously, the next step is consider both Euler as Navier-Stokes equations for compressible and viscous fluids. This type of fluids are more realistic in the field of aeronautic (see [71]), but this models could be much more complex.

The literature of optimal control problem for the Navier-Stokes equations are very extensive. We mention the work of [2] for evolution optimal control problems in fluids mechanics in the case of two-dimensional flows. Also, the PhD thesis [64] and [112] for Navier-Stokes equations. In the stationary case, we refer [1] and [45], and the references therein.

Since the main result of this chapter is in a local sense, for technical reasons, we need some properties about the linearized Navier-Stokes equations. This equation is known as *Oseen equation* or *Stokes-Oseen equation*. The importance of this equation for the study of the Navier-Stokes system is fundamental, specially for the feedback stabilization of the Navier-Stokes problem around an unstable stationary solution, see [24, 55, 56, 100]. In our case, the Oseen equation is fundamental to obtain a positive response on the turnpike property for the Navier-Stokes problem.

The outline of the chapter is a follows. In Section 4.2 we formulate the optimal control problem for both nonstationary problem and stationary, and present existence results, first order necessary and second order conditions. In Section 4.3, we state and prove the main result of the chapter, see Theorem 4.9. Finally, in Section 4.4, we prove a turnpike property in the special case when the controls are independent of time.

4.2 Optimal control problem and existence of solutions

In this section we introduce the optimal control problem for the evolutionary and stationary Navier-Stokes problem in two dimensions. We show the existence of optimal solution and state the theorems about the first-order optimality conditions. Besides, we prove that, in the case when the tracking term is sufficiently small, the second derivative of the functional to minimize is positive definite.

4.2.1 Evolutionary optimal control for Navier-Stokes equations

We recall that our analysis is in two dimension. In this case, the relation between the control and the state is differentiable, which simplifies the analysis for the optimality conditions (see below the Lemmas 4.2 and 4.3). For the three-dimensional case is more complicated to derive some optimality conditions. A possibility, as in [33], is to work with the so-called strong solutions of the Navier-Stokes problem. This type of solution is well known, see, for instance, [29] Chapter V.2. The advantage of these solutions is that the uniqueness is known, but the existence is still an open problem.

Firstly, we remember the nonstationary Navier-Stokes equations in two–dimension. Given $T > 0$, we denote $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \partial\Omega \times (0, T)$. Under the framework introduced in the Chapter 1, we consider the incompressible Navier-Stokes problem

$$\begin{cases} y_t - \mu\Delta y + (y\cdot\nabla)y + \nabla p &= u \quad , \text{ in } Q_T,\\ \text{div } y &= 0 \quad , \text{ in } Q_T,\\ y &= 0 \quad , \text{ on } \Gamma_T,\\ y(x,0) &= y_0(x) \quad , \ x\in\Omega, \end{cases} \tag{4.1}$$

where the forcing term (control) u belongs to $L^2(0,T;H)$, the initial data y_0 belongs to H, and the kinematic viscosity $\mu > 0$.

The Navier-Stokes equations (4.1) in Q_T can be written under the following form, following the ideas in Chapter 2.

$$\begin{cases} \dfrac{dy(t)}{dt} + \mu Ay(t) + By(t) &= u(t) \quad , \quad t \leq 0,\\ y(0) &= 0 \quad . \end{cases} \tag{4.2}$$

Let us introduce the optimal control tracking-type problem of the evolutionary Navier-Stokes equations:

find $u^T \in L^2(0,T;H)$, y^T is the solution of (4.2) associated to u^T, minimizing the functional

$$J^T(u) = \frac{1}{2}\int_0^T \|y(t) - x^d\|^2_{L^2(\Omega)}dt + \frac{k}{2}\int_0^T \|u(t)\|^2_{L^2(\Omega)}dt + q_0\cdot y(T), \tag{4.3}$$

where $x^d \in (L^2(\Omega))^2$ is desired state, $q_0 \in (L^2(\Omega))^2$ and $k > 0$ is a constant.

Let us remark that the controls u can act on all domain Q_T or on a subset of Q_T.

We observe that the problem (4.3) is a nonconvex optimization problem because the solution mapping $u \mapsto y_u$ is nonlinear. But we show that if the tracking term, $\|y(t)-x^d\|_{L^2(\Omega)}$, is sufficiently small, then the Hessian of J^T is positive definite.

Theorem 4.1 *Let $y_0 \in V$. There exists at least an element $u^T \in L^2(0,T;H)$, and $y^T \in C([0,T];V) \cap L^2(0,T;(H^2(\Omega))^2)$ such that the functional $J^T(u)$ attains its minimum at u^T, and y^T is the solution of (4.1) associated to u^T.*

Proof. The functional J^T is bounded from below. Hence, there exists the infimum of J^T. Moreover, let us take a minimizing sequence (y_n, u_n). Since

$$\frac{k}{2}\int_0^T \|u_n\|_{L^2(\Omega)}dt \leq J^T(u_n) < \infty,$$

we deduce that (u_n) is bounded in $L^2(0,T;H)$ and, consequently, (y_n) is bounded in $C([0,T];V) \cap L^2(0,T;(H^2(\Omega))^2)$ as well. Therefore, we can extract a subsequence, denoted in the same way, converging weakly in $L^2(0,T;(H^2(\Omega))^2) \times L^2(0,T;H)$ to (y^*, u^*).

Now, we need to prove that the pair (y^*, u^*) satisfies the equation (4.1). The only problem is to pass to the limit in the nonlinear term $(y_n\cdot\nabla)y_n$. By the result in Chapter III in [107], we obtain a compactness property, this implies that $y_n \to y^*$ strongly in $L^2(0,T;H)$. By Lemma 3.2 Chapter III in [107], we obtain that

$$b(y_n, y_n, v) \to b(y^* y^*, v), \text{ as } n \to \infty.$$

Then, taking into account the linearity and continuity of the other terms involved, the limit (y^*, u^*) satisfies the state equations.

Finally, the objective functional consists of several norms, thus it is weakly semicontinuous which implies

$$J^T(u^*) \le \liminf J^T(u_n) = \inf J^T(u).$$

Therefore, u^* is an optimal solution, with y^* the solution of (4.1) associated to u^*. □

First-Order necessary optimality conditions

We now proceed to derive the first-order optimality conditions associated with the problem (4.3). This is done by studying the Gâteaux derivative of the functional $J^T(u)$.

We will need, in the following, some results about the so-called control-to-state mapping. The next two lemmas can be found in [2].

Lemma 4.2 *Let y_0 be in V. The mapping $u \mapsto y_u$, from $L^2(0,T;H)$ into $L^2(0,T;V)$, has a Gâteaux derivative $((\frac{Dy_u}{Du}) \cdot h)$ in every direction h_1 in $L^2(0,T;H)$. Furthermore, $(\frac{Dy_u}{Du}) \cdot h_1 = w(h_1)$ is the solution of the linearized problem*

$$\begin{cases} \dfrac{dw}{dt} + \mu A w + B'(y_u) \cdot w &= h_1 \quad , \quad t \le 0, \\ w(0) &= 0 \quad . \end{cases} \tag{4.4}$$

Finally, w is in $L^\infty(0,T;V) \cap L^2(0,T;(H^2(\Omega))^2)$ and $\|B'(y_u)w\|_{L^2(V')} \le c\|y_u\|\|w\|$.

Lemma 4.3 *Let h_1 be given in $L^2(0,T;H)$, and let $w(h_1)$ be defined as above. Then, for every h_2 in $L^2(0,T;H)$ we have*

$$\iint_{Q_T} (h_2 \cdot w(h_1))(x,t) dx dt = \iint_{Q_T} (\tilde{w}(h_2) \cdot h_1)(x,t) dx dt,$$

where $\tilde{w}(h_2)$ is the solution of the adjoint linearized problem

$$\begin{cases} -\dfrac{d\tilde{w}}{dt} + \mu A \tilde{w} + B'(y_u)^* \cdot \tilde{w} &= h_2 \quad , \quad t \le 0, \\ \tilde{w}(T) &= 0 \quad . \end{cases} \tag{4.5}$$

Observation Writing systems (4.4) and (4.5) in a extended way, it is possible to express w and $\tilde{w}$ as the respective solutions of the following equations:

$$\left\{\begin{array}{rcll} w_t - \mu\Delta w + (y_u\cdot\nabla)w + (w\cdot\nabla)y_u + \nabla p &=& h_1 &, \text{ in } Q_T,\\ \text{div } w &=& 0 &, \text{ in } Q_T,\\ w &=& 0 &, \text{ on } \Gamma_T,\\ w(x,0) &=& 0 &, x\in\Omega, \end{array}\right. \tag{4.6}$$

and

$$\left\{\begin{array}{rcll} -\tilde{w}_t - \mu\Delta\tilde{w} + (\nabla y_u)^T\tilde{w} - (y_u\cdot\nabla)\tilde{w} + \nabla\tilde{p} &=& h_2 &, \text{ in } Q_T,\\ \text{div } \tilde{w} &=& 0 &, \text{ in } Q_T,\\ \tilde{w} &=& 0 &, \text{ on } \Gamma_T,\\ \tilde{w}(x,T) &=& 0 &, x\in\Omega. \end{array}\right. \tag{4.7}$$

Using the last two Lemmas, Abergel and Temam [2] prove the following first-order optimality condition for the optimal control problem (4.3). The proof can be obtained by the usual approach.

Theorem 4.4 (see [2]) *Let (y^T, u^T) be an optimal pair for problem (4.3). The following equality holds*

$$u^T + q = 0,$$

where q is the adjoint state that is the solution of the linearized adjoint problem

$$\left\{\begin{array}{rcll} -q_t - \mu\Delta q + (\nabla y^T)^T q - (y^T\cdot\nabla)q + \nabla\tilde{p} &=& y^T - x^d &, \text{ in } Q_T,\\ \text{div } q &=& 0 &, \text{ in } Q_T,\\ q &=& 0 &, \text{ on } \Gamma_T,\\ q(x,T) &=& q_0 &, x\in\Omega. \end{array}\right. \tag{4.8}$$

Moreover, u^T is in $L^\infty(0,T;V)\cap L^2(0,T;(H^2(\Omega))^2)$.

Second order conditions

In the following result we assert positive definiteness of the Hessian provided that $\|y - x^d\|_{L^2(0,T;V)}$ is sufficiently small, a condition which is applicable to tracking type problems.

We observe that in [99] the authors proved that the functional to minimize is also positive definite at least when the target and the initial data are small enough.

Theorem 4.5 *If $\|y - x^d\|_{L^2(0,T;V)}$ is sufficiently small, then the Hessian $J^T(u)''$ is positive definite.*

Proof. By Chapter 2 of [64], we have that the second Gâteaux derivative of J^T is given by

$$J^T(u)''v^2 = \iint_{Q_T} |y_v|^2 dxdt + \iint_{Q_T} |v|^2 dxdt - 2\iint_{Q_T} (y_v\cdot\nabla)y_v\cdot q_u dxdt, \tag{4.9}$$

where y_v is the solution of the linearized equation

$$\left\{ \begin{array}{rcll} \dfrac{dy_v}{dt} + \mu A y_v + B'(y_u) \cdot y_v & = & v & , \quad t \leq 0, \\ v(0) & = & 0 & , \end{array} \right. \tag{4.10}$$

in the direction v, and q_u the solution of the adjoint linearized problem

$$\left\{ \begin{array}{rcll} -\dfrac{dq_u}{dt} + \mu A q_u + B'(y_u)^* \cdot q_u & = & y_u - x^d & , \quad t \leq 0, \\ q_u(T) & = & q_0 & . \end{array} \right. \tag{4.11}$$

Since B is of quadratic nature, we have that the second derivative of B is given by

$$B''(y)y_v^2 = B'(y_v)y_v = 2B(y_v).$$

Besides, we known that

$$\|B'(y_v)y_v\|_{L^2(0,T;V')} \leq C\|y_v\|^2_{L^2(0,T;V)}.$$

Moreover, the solution of the linearized equation (4.19) satisfy

$$\|y_v\|_{L^2(0,T;V)} \leq C\|v\|_{L^2(0,T;V)}.$$

For the adjoint linearized problem (4.11) we obtain that

$$\|q_u\|_{L^2(0,T;V)} \leq C\|y_u - x^d\|_{L^2(0,T;V)}.$$

Then, we conclude that the second derivative of J^T can be estimated as

$$J^T(u)''v^2 \geq \iint_{Q_T} |y_v|^2 dxdt + (1 - C\|y_u - x^d\|_{L^2(0,T;V)}) \iint_{Q_T} v^2 dxdt,$$

which gives the assertion. □

4.2.2 Stationary optimal control problem for Navier-Stokes equations

We recall the Navier–Stokes equations in the stationary case. We consider the following problem

$$\left\{ \begin{array}{rcll} -\mu\Delta y + (y \cdot \nabla)y + \nabla p & = & u & , \quad \text{in } \Omega, \\ \text{div } y & = & 0 & , \quad \text{in } \Omega, \\ y & = & 0 & , \quad \text{on } \partial\Omega, \end{array} \right. \tag{4.12}$$

where $u \in (L^2(\Omega))^2$.

As for the nonstationary Navier-Stokes equations, our optimal control problem is to find $\overline{u}$, $\overline{y}$ being the solution of (4.12) associated to $\overline{u}$, minimizing the functional

$$J(u) = \frac{1}{2}\|y - x^d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}, \tag{4.13}$$

where $x^d \in (L^2(\Omega))^2$ is a target and $\alpha > 0$ is a constant.

We are going to show that the optimal control problem (4.13) has a solution.

Theorem 4.6 *There exists at least an element $\overline{u} \in L^2(\Omega)$, and $\overline{y} \in H^2(\Omega) \cap V$ solution of* (4.12) *associated to $\overline{u}$, such that the functional $J(u)$ attains its minimum at $\overline{u}$.*

Proof. The functional J is bounded below by zero. Then we can take a minimizing sequence (y_n, u_n). Is easy to see that $\frac{\alpha}{2}\|u_n\|^2 \leq J(u_n) < \infty$, which implies that the sequence (u_n) is uniformly bounded in $L^2(\Omega)$.

From the regularity of the Navier-Stokes problem we obtain that the sequence (y_n) is uniformly bounded in $H^2(\Omega) \cap V$, and then implies that we can extract a weakly convergent subsequence, denoted in the same way (y_n, u_n), such that

$$y_n \rightharpoonup y^* \text{ in } H^2(\Omega) \cap V, \quad u_n \rightharpoonup u^* \text{ in } L^2(\Omega).$$

Now, we need to ensure that (y^*, u^*) is a solution of the Navier-Stokes problem. For this steep we use the trilinear continuous form b. Thanks to the compact embedding $H^2(\Omega) \cap V \hookrightarrow V$ and the continuity of b, we obtain that $b(y_n, y_n, v) \to b(y^*, y^*, v)$, as $n \to \infty$. Then, we have that (y^*, u^*) satisfies the Navier-Stokes problem.

Therefore, as J is weakly lower semicontinuous, the result is proved.

$\square$

First-Order necessary optimality conditions

The following result of J. De los Reyes [45], shows the first-order optimality conditions in the case of the stationary Navier-Stokes equations. This theorem is more general, since De los Reyes consider the constrained optimal control problem. He proved the result based on a result of Lagrange multipliers.

Theorem 4.7 (see [45]) *Let $(\overline{u}, \overline{y})$ be an optimal solution for* (4.13)*, such that $\mu > \mathcal{M}(\overline{y})$, where $\mathcal{M}(y) = \sup_{v \in V} \dfrac{|b(v, v, y)|}{\|v\|^2_V}$. Then there exists $q \in V$ such that satisfies the following* *optimality system in variational sense*

$$\left\{\begin{array}{rcll} -\mu\Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} &=& -q &, \ \textit{in } \Omega,\\ \text{div } \overline{y} &=& 0 &, \ \textit{in } \Omega,\\ \overline{y} &=& 0 &, \ \textit{on } \partial\Omega,\\ -\mu\Delta q - (\overline{y}\cdot\nabla)q + (\nabla\overline{y})^T q + \nabla\pi &=& \overline{y} - x^d &, \ \textit{in } \Omega\\ \text{div } q &=& 0 &, \ \textit{in } \Omega,\\ q &=& 0 &, \ \textit{on } \partial\Omega. \end{array}\right. \tag{4.14}$$

Moreover, $(q, \pi) \in (H^2(\Omega))^2 \times H^1(\Omega)$ *and satisfies the estimate*

$$\|q\|_V \leq \frac{c}{\mu - \mathcal{M}(\overline{y})}\|\overline{y} - x^d\|_{L^2(\Omega)}. \tag{4.15}$$

Observation The assumption $\mu > \mathcal{M}(\overline{y})$ is a sufficient requirement for the satisfaction of the regular point condition, see [114].

Second order conditions

The next result is relevant for our purposes. In the next section we use this result to prove the turnpike property for a particular system, the Oseen equation.

Theorem 4.8 *Assume that* $\|y - x^d\|_V$ *is sufficiently small and* $\mu > \mathcal{M}(\overline{y})$*. Then, the Hessian* $J(u)''$ *is positive definite.*

Proof. By Theorem 3.3 in [35], we have that the second Gâteaux derivative of J is given by

$$J''(u)v^2 = \int_\Omega |y_v|^2 dx + \int_\Omega |v|^2 dx - 2\int_\Omega (y_v \cdot \nabla)y_v \cdot q_v dx, \tag{4.16}$$

where y_v is the solution of the linearized problem

$$\left\{\begin{array}{rcll} -\mu\Delta y_v + (y\cdot\nabla)y_v + (y_v\cdot\nabla)y + \nabla p_v &=& v &, \ \text{in } \Omega,\\ \text{div } y_v &=& 0 &, \ \text{in } \Omega,\\ y_v &=& 0 &, \ \text{on } \partial\Omega, \end{array}\right. \tag{4.17}$$

in the direction v, and q_u the solution of the adjoint linearized problem

$$\left\{\begin{array}{rcll} -\mu\Delta q_u + (\nabla y)^T q_u - (y\cdot\nabla)q_u + \nabla\tilde{p} &=& y - x^d &, \ \text{in } \Omega,\\ \text{div } q_u &=& 0 &, \ \text{in } \Omega,\\ q_u &=& 0 &, \ \text{on } \partial\Omega, \end{array}\right. \tag{4.18}$$

Reasoning as in Theorem 4.5, using the Theorem 4.8, we deduce that

$$J''(u)v^2 \geq \int_\Omega |y_v|^2 dx + (1 - C\|y - x^d\|_V)\int_\Omega |v|^2 dx,$$

which implies the claim. $\square$

4.3 Turnpike property for the two-dimensional Navier-Stokes problem with time-dependent control

In this section we prove a turnpike result for the optimality system of Navier-Stokes problem, under the condition that the initial and final states are close enough to the stationary primal and dual state, respectively. Also, we need some assumption of smallness for the solution of the stationary adjoint equation.

As in the paper of Porretta and Zuazua [99], the smallness condition is to ensure the exponential turnpike property of the linearized optimality system. In [99], the authors prove under the smallness of the target and the initial condition that the linearized optimality system satisfies the turnpike property. However, by the quadratic nature of the nonlinear term B, in this chapter we only assume the smallness of the tracking term.

From the results of Section 3.1, we have the following optimality system for the nonstationary Navier-Stokes equations (see Theorem 4.4)

$$\begin{cases} y_t^T - \mu\Delta y^T + (y^T\cdot\nabla)y^T + \nabla p^T &= -q^T \quad , \text{ in } Q_T,\\ \text{div } y^T &= 0 \quad , \text{ in } Q_T,\\ y^T &= 0 \quad , \text{ on } \Gamma_T,\\ y^T(x,0) &= y_0(x) \quad , x\in\Omega,\\ -q_t^T - \mu\Delta q^T - (y^T\cdot\nabla)q^T + (\nabla y^T)^T q^T + \nabla\pi^T &= y^T - x^d \quad , \text{ in } Q_T,\\ \text{div } q^T &= 0 \quad , \text{ in } Q_T,\\ q^T &= 0 \quad , \text{ on } \Gamma_T,\\ q^T(x,T) &= q_0 \quad , x\in\Omega. \end{cases} \tag{4.19}$$

And, for the stationary Navier-Stokes problem, see Theorem 4.7, we obtain

$$\begin{cases} -\mu\Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} &= -\overline{q} \quad , \text{ in } \Omega,\\ \text{div } \overline{y} &= 0 \quad , \text{ in } \Omega,\\ \overline{y} &= 0 \quad , \text{ on } \partial\Omega,\\ -\mu\Delta\overline{q} - (\overline{y}\cdot\nabla)\overline{q} + (\nabla\overline{y})^T\overline{q} + \nabla\overline{\pi} &= \overline{y} - x^d \quad , \text{ in } \Omega\\ \text{div } \overline{q} &= 0 \quad , \text{ in } \Omega,\\ \overline{q} &= 0 \quad , \text{ on } \partial\Omega. \end{cases} \tag{4.20}$$

Now, we develop a local analysis around a given steady state optimal control $(\overline{y},\overline{u})$.

We consider $y=\overline{y}+z$, $p=\overline{p}+\eta$, $q=\overline{q}+\varphi$, and $\pi=\overline{\pi}+\nu$. Then, the optimality system linearized around the stationary solutions takes the form

$$\begin{cases} z_t - \mu\Delta z + (\overline{y}\cdot\nabla)z + (z\cdot\nabla)\overline{y} + \nabla\eta &= -\varphi \quad , \text{ in } Q_T,\\ \text{div } z &= 0 \quad , \text{ in } Q_T,\\ z &= 0 \quad , \text{ on } \Gamma_T,\\ z(x,0) &= z_0 \quad , \text{ in } \Omega,\\ -\varphi_t - \mu\Delta\varphi - (\overline{y}\cdot\nabla)\varphi + (\nabla\overline{y})^T\varphi + \nabla\nu &= z - (\nabla z)^T\overline{q} + (z\cdot\nabla)\overline{q} \quad , \text{ in } Q_T,\\ \text{div } \varphi &= 0 \quad , \text{ in } Q_T,\\ \varphi &= 0 \quad , \text{ on } \Gamma_T,\\ \varphi(x,T) &= \varphi_0 \quad , \text{ in } \Omega, \end{cases} \tag{4.21}$$

where $z_0 = y_0 - \overline{y}$ and $\varphi_0 = q_0 - \overline{q}$.

We observe that the right hand side of the equation satisfied by φ in (4.21), can be written using the definition of B as

$$(\nabla z)^T \overline{q} - (z \cdot \nabla)\overline{q} = B'(z)^* \overline{q}.$$

Since the nonlinear function B is of quadratic nature, we deduce that the derivative of $B'(z)^*\overline{q}$ with respect to z is the same function $B'(z)^*\overline{q}$. Then, the optimality system (4.21), in the references case when $\varphi_0 = 0$, can be expressed as a linear quadratic optimal control problem, minimizing the functional

$$L(u) = \frac{1}{2}\int_{Q_T} |z|^2 dxdt - \int_{Q_T} [(\nabla z)^T \overline{q} - (z \cdot \nabla)\overline{q}] dxdt + \frac{1}{2}\int_0^T \|v(t)\|^2_{L^2(\Omega)} dt, \tag{4.22}$$

such that (z, φ) is the unique solution of

$$\begin{cases} z_t - \mu \Delta z + (\overline{y} \cdot \nabla) z + (z \cdot \nabla)\overline{y} + \nabla \eta &= v \quad , \quad \text{in } Q_T, \\ \text{div } z &= 0 \quad , \quad \text{in } Q_T, \\ z &= 0 \quad , \quad \text{on } \Gamma_T, \\ z(x,0) &= z_0 \quad , \quad \text{in } \Omega. \end{cases}$$

For our purposes, we need to give the basic hypothesis such that the optimal control problem for Oseen equation (4.22) satisfies the turnpike property. To ensure this, we will use the result of Porretta and Zuazua [98]. In this paper the authors prove the turnpike property for linear problems.

Consider the control problem for Oseen equation

$$\begin{cases} z_t + \mathcal{A} z &= v \quad , \quad \text{in } (0,T), \\ z(0) &= z_0 \quad , \end{cases} \tag{4.23}$$

where $\mathcal{A}$ is the Oseen operator defined by (2.62) and the control v is in $L^2(0,T;H)$.

It is easy to prove that the Oseen operator satisfies

$$\exists \gamma, \xi > 0 \ : \quad \langle \mathcal{A} z, z \rangle_{V',V} + \gamma \|z\|^2_H \geq \xi \|z\|^2_V \ , \forall x \in V. \tag{4.24}$$

Also, if we assume that the initial data z_0 is in X_σ and $\sigma > 0$ satisfies (2.64), we obtain by Theorem 2.38 that the semigroup associated to the Oseen equation decays exponentially.

Then, there exists $C > 0$ such that for every solution z of (4.23) and $z_0 \in X_\sigma$, we have

$$\|z(T)\|_H \leq C \left(\|z_0\|_H + \int_0^T \|v(s)\|_{V'} ds \right). \tag{4.25}$$

Besides, from the paper of Fursikov [56] we know that there exists a linear bounded operator $L : V \to V$ such that the control $v(t, \cdot)$ can be expressed by

$$v(t, \cdot) = L z(t, \cdot)$$

with the solution of (4.23) satisfying

$$\|z(t,\cdot)\|_V \leq c\|z_0\|_V\, e^{-\sigma t}, \quad \text{for } t \geq 0. \tag{4.26}$$

Assuming that the tracking term $\|\overline{y} - x^d\|_V$ is sufficiently small, the viscosity function satisfies $\mu > \mathcal{M}(\overline{y})$, and $z_0 \in X_\sigma$, reasoning as in the proof of Theorem 4.8, we deduce that the functional L is coercive. This implies, by Theorem 3.10 in [98], that the optimality system (4.21) satisfies the turnpike property. Namely,

$$\|z^T(t)\|_{L^2(\Omega)} + \|\varphi^T(t)\|_{L^2(\Omega)} \leq C(e^{-\gamma t} + e^{-\gamma(T-t)})\,, \quad \forall t \in (0,T).$$

Then, as in [98], we can define a linear bounded operator in $(L^2(\Omega))^2$ as

$$P(T)z_0 = \varphi(0)$$

such that

$$\|P(t) - \hat{P}\|_{\mathcal{L}((L^2(\Omega))^2,(L^2(\Omega))^2)} \leq Ce^{-2\gamma t}, \tag{4.27}$$

for some constant $C > 0$ and $\gamma > 0$. $\hat{P}$ being the corresponding operator for the infinite horizon control problem.

Using the previous turnpike property for Oseen equation, we can state and prove the main theorem of this chapter.

Theorem 4.9 *We assume that the tracking term $\|\overline{y} - x^d\|_V$ is sufficiently small, $\mu > \mathcal{M}(\overline{y})$, and $z_0 = y_0 - \overline{y} \in X_\sigma$. Then, there exists some $\varepsilon > 0$ such that for every y_0, q_0 with*

$$\|y_0 - \overline{y}\|_{L^2(\Omega)} + \|q_0 - \overline{q}\|_{L^2(\Omega)} \leq \varepsilon,$$

there exists a solution of the optimality system (4.19) *such that*

$$\|y^T(t) - \overline{y}\|_{L^2(\Omega)} + \|q^T(t) - \overline{q}\|_{L^2(\Omega} \leq C(e^{-\gamma t} + e^{-\gamma(T-t)}), \quad \forall t < T, \tag{4.28}$$

where $\gamma > 0$ is the stabilizing rate of the linearized optimality system (4.21).

Proof. The proof follows the arguments of [98, 99].

The main idea of the proof is to consider a perturbed problem of (4.21) and then to implement a fixed point argument, which gives the solutions of the optimality system (4.19).

Let X be the set

$$X = \{(z,\varphi) \;:\; \|z\|_V + \|\varphi\|_V \leq M(e^{-\gamma t} + e^{-\gamma(T-t)}),\ \forall t \in [0,T]\},$$

for some $M \leq 1$. For $(\hat{z},\hat{\varphi}) \in X$, we consider

$$R_1(\hat{z}) = -(\hat{z}\cdot\nabla)\hat{z}$$

and

$$R_2(\hat{z},\hat{\varphi}) = (\hat{z}\cdot\nabla)\hat{\varphi} - (\hat{\varphi}\cdot\nabla)\hat{z}.$$

Note that the terms R_1 and R_2 can be expressed in an abstract way, namely

$$R_1(\hat{z}) = -B(\hat{z}) \quad , \quad R_2(\hat{z},\hat{\varphi}) = -B'(\hat{z})^*\hat{\varphi}.$$

Then, using the properties for the nonlinear form B, we obtain that

$$\begin{aligned} \|R_1(\hat{z})(t)\|_{L^2(\Omega)} \le \|R_1(\hat{z})(t)\|_V \le c_0 M^2(e^{-2\gamma t} + e^{-2\gamma(T-t)}), \\ \|R_2(\hat{z},\hat{\varphi})(t)\|_{L^2(\Omega)} \le \|R_2(\hat{z},\hat{\varphi})(t)\|_V \le c_1 M^2(e^{-2\gamma t} + e^{-2\gamma(T-t)}), \end{aligned} \tag{4.29}$$

where c_1 depend on $\|\overline{y}\|_V$.

Besides, we define the operator

$$R(\hat{z},\hat{\varphi}) = (z,\varphi), \tag{4.30}$$

where (z,φ) solve the problem

$$\left\{ \begin{array}{rcll} z_t - \mu\Delta z + (\overline{y}\cdot\nabla)z + (z\cdot\nabla)\overline{y} + \nabla\eta & = & -\varphi + R_1(\hat{z}) & , \text{ in } Q_T, \\ \text{div } z & = & 0 & , \text{ in } Q_T, \\ z & = & 0 & , \text{ on } \Gamma_T, \\ z(x,0) & = & z_0 & , \text{ in } \Omega, \\ -\varphi_t - \mu\Delta\varphi - (\overline{y}\cdot\nabla)\varphi + (\nabla\overline{y})^T\varphi + \nabla\nu & = & z - (\nabla z)^T\overline{q} + (z\cdot\nabla)\overline{q} & \\ & & +R_2(\hat{z},\hat{\varphi}) & , \text{ in } Q_T, \\ \text{div } \varphi & = & 0 & , \text{ in } Q_T, \\ \varphi & = & 0 & , \text{ on } \Gamma_T, \\ \varphi(x,T) & = & \varphi_0 & , \text{ in } \Omega, \end{array} \right. \tag{4.31}$$

Then, we need to prove that the operator R has a fixed point which is a solution of (4.19) and satisfies the estimate (4.28).

Define h as a solution of the equation

$$\left\{ \begin{array}{rcll} -h_t - \mu\Delta h - (\overline{y}\cdot\nabla)h + (\nabla\overline{y})^T h + \nabla\nu + P(T-t)h & = & P(T-t)R_1(\hat{z}) & \\ & & +R_2(\hat{z},\hat{\varphi}) & , \text{ in } Q_T, \\ \text{div } h & = & 0 & , \text{ in } Q_T, \\ h & = & 0 & , \text{ on } \Gamma_T, \\ h(x,T) & = & \varphi_0 & , \text{ in } \Omega. \end{array} \right. \tag{4.32}$$

Then, it is easy to prove that h satisfies

$$h = \varphi - P(T-t)z \tag{4.33}$$

in a weak sense, namely for all test function ϕ

$$\int_\Omega h(t)\phi = \int_\Omega \varphi(t)\phi dx - \int_\Omega z(t)[P(T-t)\phi]dx.$$

We observe that h can be estimated as

$$h(t) = e^{-\mathcal{M}(T-t)}\varphi_0 + \int_t^T e^{\mathcal{M}(t-s)})[P(T-s)R_1(\hat{z})(s) + R_2(\hat{z},\hat{\varphi})(s)]ds \\ - \int_t^T e^{\mathcal{M}(t-s)}[\hat{P} - P(T-s)]h(s)ds,$$

where $\mathcal{M}v = -\mu\Delta v - (\overline{y}\cdot\nabla)v + (\nabla\overline{y})^T v + \hat{P}$. We observe that $\mathcal{M}$ is exponentially stable with rate γ. Using the estimates (4.27) and (4.29), we obtain

$$\begin{aligned} \|h(t)\|_{L^2(\Omega)} &\leq e^{-\gamma(T-t)}\|\varphi_0\|_{L^2(\Omega)} + cM^2\int_t^T e^{\gamma(t-s)}(e^{-2\gamma s} + e^{-2\gamma(T-s)})ds \\ &\quad + \int_t^T e^{\gamma(t-s)}e^{-2\gamma(T-s)}\|h(s)\|_{L^2(\Omega)}ds \\ &\leq e^{-\gamma(T-t)}\|\varphi_0\|_{L^2(\Omega)} + cM^2[e^{-2\gamma t} + e^{-\gamma(T-t)}] \\ &\quad + \int_t^T e^{-2\gamma T+\gamma t+\gamma s}\|h(s)\|_{L^2(\Omega)}ds. \end{aligned}$$

By the Gronwall inequality

$$\|h(t)\|_{L^2(\Omega)} \leq e^{-\gamma(T-t)}\|\varphi_0\|_{L^2(\Omega)} + cM^2[e^{-2\gamma t} + e^{-\gamma(T-t)}]\exp\left(\int_t^T e^{-2\gamma T+\gamma t+\gamma s}ds\right).$$

The last integral can be estimated easily by $\frac{1}{\gamma}$. Therefore

$$\|h(t)\|_{L^2(\Omega)} \leq e^{-\gamma(T-t)}[\|\varphi_0\|_{L^2(\Omega)} + cM^2] + cM^2e^{-2\gamma t}. \tag{4.34}$$

From the estimate for h we can find a similar estimate for z and φ. Indeed, observe that z satisfies the following equation

$$z_t - \mu\Delta z + (\overline{y}\cdot\nabla)z + (z\cdot\nabla)\overline{y} + \hat{P} + \nabla\eta = (\hat{P} - P(T-t))z - h + R_1(\hat{z}).$$

Therefore, we obtain that

$$z(t) = e^{-\mathcal{N}t}z_0 + \int_0^t e^{-\mathcal{N}(t-s)}[\hat{P} - P(T-s)]z(s)ds + \int_0^t e^{-\mathcal{N}(t-s)}(R_1(\hat{z})(s) - h(s))ds,$$

where $\mathcal{N}v = -\mu\Delta v + (\overline{y}\cdot\nabla)v + (v\cdot\nabla)\overline{y} + \hat{P}$. We note that $\mathcal{N}$ satisfies the exponentially decay with rate γ. Again, using the estimate (4.29), (4.27), and (4.34) we get

$$\begin{aligned}
\|z(t)\|_{L^2(\Omega)} \leq e^{-\gamma t}\|z_0\|_{L^2(\Omega)} &+ \int_0^t e^{-\gamma(t-s)}e^{-2\gamma(T-s)}\|z(s)\|_{L^2(\Omega)}ds \\
&+ cM^2\int_0^t e^{-\gamma(t-s)}(e^{-2\gamma(T-s)} + e^{-2\gamma s})ds \\
&+ \int_0^t e^{-\gamma(t-s)}[e^{-\gamma(T-s)}(\|\varphi_0\|_{L^2(\Omega)} + cM^2) + cM^2e^{-2\gamma s}]ds \\
&\leq e^{-\gamma t}\|z_0\|_{L^2(\Omega)} + cM^2[e^{-2\gamma(T-t)} + e^{-\gamma t}] + [\|\varphi_0\|_{L^2(\Omega)} + cM^2]e^{-\gamma(T-t)} \\
&+ \int_0^t e^{-2\gamma T-\gamma t+3\gamma s}\|z(s)\|_{L^2(\Omega)}ds.
\end{aligned}$$

Applying again the Gronwall inequality, we obtain

$$\|z(t)\|_{L^2(\Omega)} \leq [\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}). \tag{4.35}$$

Using now that $\varphi = h + P(T-t)z$, we get an estimate for φ

$$\|\varphi(t)\|_{L^2(\Omega)} \leq [\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}). \tag{4.36}$$

Now, we go back on the first equation of (4.31). Observe that

$$\|-\varphi(t) + R_1(\hat{z})(t)\|_{L^2} \leq [\|z_0\|_{L^2} + \|\varphi_0\|_{L^2} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}). \tag{4.37}$$

Then, by the regularity of the solution of the linearized problem, see Lemma 4.2, we have that

$$\|z(t)\|_{H^2(\Omega)} \leq [\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}). \tag{4.38}$$

And, we can conclude that

$$\|z(t)\|_V \leq [\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}).$$

Analogously, we obtain the same estimate for φ, namely

$$\|\varphi(t)\|_V \leq [\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + cM^2](e^{-\gamma t} + e^{-\gamma(T-t)}).$$

Finally, we choose $M \leq 1$ such that $cM^2 \leq \frac{M}{2}$. Then, if we assume that the initial and final state are close enough to the stationary primal and dual state, respectively, we obtain that

$$c[\|z_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)} + M^2] \leq M.$$

So, we deduce that the space X becomes an invariant convex subset of $L^2(0,T;(L^2(\Omega))^2)$. Besides, we observe that operator R is continuous and compact, then we conclude the existence of a fixed point (z,φ) of R. It is easy to see that (z,φ) is a solution of the optimality system (4.19). Then the proof is complete. ☐

Observation Since we develop a local analysis around the optimal solution for the stationary problem, the turnpike property for Oseen equation is fundamental in our work. In this point is fundamental the smallness assumption on the tracking term. If we remove the last condition, we need to suppose that the optimality system (5.3) satisfy the turnpike property to ensure our result.

An interesting problem is to prove the necessary and sufficient conditions to obtain the turnpike property for the linearized optimality systems.

4.4 Turnpike property for the two-dimensional Navier-Stokes problem with time-independent control

In this section we prove a turnpike property for the two-dimensional Navier-Stokes problem in the particular case when the controls are independent on time.

The proof is different from that given in the previous section when the control function depends on time. In this case, we obtain the result using the classical Γ-convergence, and a standard stability property of the Navier-Stokes equation, see Theorem 4.10, under suitable conditions of smallness of the data.

That technique is a general principle proved by Porretta and Zuazua [99] for the semilinear heat equation. Of course, can also be employed for a larger class of semilinear problems enjoying standard exponentially stability.

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain, with boundary $\partial\Omega$ of class C^2. We consider the Navier-Stokes control problem

$$\left\{ \begin{array}{rcll} y_t - \mu\Delta y + (y\cdot\nabla)y + \nabla p &=& u(x) &, \text{ in } Q_T,\\ \text{div } y &=& 0 &, \text{ in } Q_T,\\ y &=& 0 &, \text{ on } \Gamma_T,\\ y(x,0) &=& y_0(x) &, x\in\Omega, \end{array}\right. \tag{4.39}$$

with controls $u = u(x)$ independent of time.

We consider the optimal control problem

$$\left\{ \begin{array}{ll} \min J^T(u) = \dfrac{1}{2}\displaystyle\int_0^T \|y(t)-z\|^2_{L^2(\Omega)}dt + \dfrac{T}{2}\|u\|^2_{L^2(\Omega)}, & \\ \text{s. a.} \qquad y \text{ solution of (4.39) and } u\in C, & \end{array}\right. \tag{4.40}$$

where C is a closed convex subset of $(L^2(\Omega))^2$ and $z \in (L^2(\Omega))^2$ denotes the desired state.

In addition, we consider the analogous stationary optimal control problem

$$\left\{ \begin{array}{rcll} -\mu\Delta y + (y\cdot\nabla)y + \nabla p &=& u(x) &, \text{ in } \Omega,\\ \text{div } y &=& 0 &, \text{ in } \Omega,\\ y &=& 0 &, \text{ on } \partial\Omega, \end{array}\right. \tag{4.41}$$

together with the corresponding functional

$$\begin{cases} \min J(u) = \dfrac{1}{2}\left(\|y-z\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right), \\ \text{s. a.} \qquad y \text{ solution of (4.41) and } u \in C. \end{cases} \tag{4.42}$$

In both cases, we consider that C has the following form

$$C \equiv U_{ad} := \{u \in (L^2(\Omega))^2 \ : \ \|u\| \leq c(\Omega)\mu^2, \ \forall x \in \Omega\}.$$

Observation In view of the Theorems 4.1 and 4.6, we note that in both cases the optima are achieved. The only difference is that in this case we consider constrains on the controls. However, since U_{ad} is a convex closed subset of $(L^2(\Omega))^2$, we assert the result using the classical results of convex analysis.

For a given source term u which does not depend on time, we consider a steady solution $(y_\infty, p_\infty) \in ((H^2(\Omega))^2 \cap V) \times (H^1(\Omega) \cap L^2_0(\Omega))$ to the stationary Navier-Stokes problem. Then, the solution (y,p) to (4.39) converge to (y_∞, p_∞) as $t \to \infty$, under suitable assumptions.

Theorem 4.10 *There exists $C > 0$ and $\alpha > 0$ depending only on Ω such that, under the condition*

$$\|\nabla y_\infty\|_{L^2(\Omega)} \leq C\mu,$$

there exists a unique weak solution (y,p) of (4.39) which satisfies

$$\|y(t) - y_\infty\|_{L^2(\Omega)} \leq \|y_0 - y_\infty\|_{L^2(\Omega)} e^{-\alpha t}, \quad \forall t \geq 0. \tag{4.43}$$

Proof. Let (y,p) and (y_∞, p_∞) be the solution of the evolutionary and stationary Navier-Stokes problem, respectively. Let $y = y_\infty + w$ and $p = p_\infty + q$, where (w,q) solves the problem

$$\begin{cases} w_t - \mu\Delta w + (w\cdot\nabla)w + (y_\infty\cdot\nabla)w + (w\cdot\nabla)y_\infty + \nabla q = 0, & \text{in } Q_T, \\ \text{div} \ \ w = 0, & \text{in } Q_T, \\ w = 0, & \text{on } \Gamma_T, \\ w(x,0) = y_0(x) - y_\infty, & x \in \Omega. \end{cases} \tag{4.44}$$

Multiplying the equation (4.44) by w and using the definition of b, we obtain

$$\int_\Omega w_t \, w dx - \mu \int_\Omega \nabla w \, w + b(w,w,w) + b(y_\infty, w, w) + b(w, y_\infty, w) = 0$$

and by Lemma 2.26, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \mu\|\nabla w(t)\|^2_{L^2(\Omega)} \leq C\|\nabla y_\infty\|_{L^2(\Omega)}\|w(t)\|_{L^2(\Omega)}\|\nabla w(t)\|_{L^2(\Omega)}.$$

We remind the following Young inequality

$$x_1 \cdot \ldots \cdot x_n \le e_1 x_1^{p_1} + \ldots + e_{n-1} x_{n-1}^{p_{n-1}} + C(e_1, \ldots, e_{n-1}) x_n^{p_n},$$

where $p_1^{-1} + \ldots + p_n^{-1} = 1$ and $e_1, \ldots, e_{n-1}, x_1, \ldots, x_n$. are positive real numbers.

Then, using the Young inequality for $x_1 = \|\nabla y_\infty\|_{L^2(\Omega)} \|w(t)\|_{L^2(\Omega)}$, $x_2 = \|\nabla w(t)\|_{L^2(\Omega)}$, $e_1 = \frac{1}{2\mu}$, $e_2 = \frac{\mu}{2}$ and $p_1 = p_2 = 2$, we obtain

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \mu\|\nabla w(t)\|^2_{L^2(\Omega)} \le C\frac{1}{\mu}\|\nabla y_\infty\|^2_{L^2(\Omega)}\|w(t)\|^2_{L^2(\Omega)}.$$

From the Poincaré inequality for the Stokes operator, see Proposition 2.19, we have that

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \left(C_1\mu - \frac{C}{\mu}\|\nabla y_\infty\|^2_{L^2(\Omega)}\right)\|w(t)\|^2_{L^2(\Omega)} \le 0.$$

Provided that $\dfrac{C}{\mu^2}\|\nabla y_\infty\|^2_{L^2(\Omega)} \le C_1$, we have

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + 2\alpha\|w(t)\|^2_{L^2(\Omega)} \le 0,$$

which finally gives

$$\|y(t) - y_\infty\|^2_{L^2(\Omega)} \le \|y_0 - y_\infty\|^2_{L^2(\Omega)} e^{-2\alpha t}, \quad \forall t \ge 0.$$

□

Now, we can prove the following turnpike result for controls independent of time.

Theorem 4.11 *Let (y^{T_n}, u^{T_n}) be an optimal solution of (4.40) for $T = T_n$. Then any accumulation point (y_∞, u_∞), as $n \to \infty$, is an optimal solution of (4.42).*

Proof. The main idea of the proof is to use the Γ-convergence, since we consider the control function independent of time.

Then, let $(T_n)_{n\in\mathbb{N}}$ be an increasing sequence of times converging to infinity. For each $n \in \mathbb{N}$, by Theorem 4.1 the optimal control problem (4.40) has at least a minimizer $(y^{T_n}, u^{T_n}) \in U_{ad}$. In particular, (u^{T_n}) is uniformly bounded in $(L^2(\Omega))^2$, so we can extract a subsequence, still labeled by n, such that

$$u^{T_n} \rightharpoonup u_\infty, \text{ weakly in } L^2(\Omega), \text{ as } n \to \infty.$$

We claim that

$$\lim_{n\to\infty} \frac{1}{T_n}\left(\frac{1}{2}\int_0^{T_n} \|y^{T_n}(t) - z\|^2_{L^2(\Omega)} dt + \frac{T_n}{2}\|u^{T_n}\|^2_{L^2(\Omega)}\right) = \frac{1}{2}\left(\|y_\infty - z\|^2_{L^2(\Omega)} + \|u_\infty\|^2_{L^2(\Omega)}\right), \quad (4.45)$$

where y_∞ solves

$$\begin{cases} -\mu\Delta y_\infty + (y_\infty\cdot\nabla)y_\infty + \nabla p_\infty &= u_\infty(x) \quad , \quad \text{in } \Omega, \\ \text{div } y_\infty &= 0 \quad , \quad \text{in } \Omega, \\ y_\infty &= 0 \quad , \quad \text{on } \partial\Omega. \end{cases} \tag{4.46}$$

Observe that the previous limit is equivalent to saying that if we consider I^{T_n} and I the values of the minimizers for the time dependent problem in $[0, T_n]$ and the steady state, respectively, then

$$\lim_{n\to\infty} \frac{I^{T_n}}{T_n} = I.$$

Indeed, if $\|\nabla \overline{y}^{T_n}\|_{L^2(\Omega)} \le C\mu$ by Theorem 4.10 we have that

$$\|y^{T_n} - \overline{y}^{T_n}\|_{L^2(\Omega)} \le \|y_\infty - \overline{y}^{T_n}\|_{L^2(\Omega)} e^{-\alpha t}, \ \forall t > 0, \tag{4.47}$$

where $\nabla\overline{y}^{T_n}$ satisfies

$$\begin{cases} -\mu\Delta \overline{y}^{T_n} + (\overline{y}^{T_n}\cdot\nabla)\overline{y}^{T_n} + \nabla \overline{p}^{T_n} &= u^{T_n}(x) \quad , \quad \text{in } \Omega, \\ \text{div } \overline{y}^{T_n} &= 0 \quad , \quad \text{in } \Omega, \\ \overline{y}^{T_n} &= 0 \quad , \quad \text{on } \partial\Omega. \end{cases} \tag{4.48}$$

We observe that from the regularity of the stationary Navier-Stokes problem, we have that

$$\|\overline{y}^{T_n}\|_{H^2(\Omega)} + \|\overline{p}^{T_n}\|_{H^1(\Omega)} \le C(1 + \|u^{T_n}\|^3_{L^2(\Omega)}),$$

and in particular

$$\|\nabla \overline{y}^{T_n}\|_{L^2(\Omega)} \le C\|u^{T_n}\|_{L^2(\Omega)}.$$

Then, $\|\nabla \overline{y}^{T_n}\|_{L^2(\Omega)} \le C\mu$. By Theorem 4.10, we obtain the estimate (4.47).

Now, we decompose

$$\frac{J^{T_n}(u^{T_n})}{T_n} - J(u_\infty) = J_1^n + J_2^n,$$

where

$$J_1^n = \frac{J^{T_n}(u^{T_n})}{T_n} - J(u^{T_n}), \tag{4.49}$$

and

$$J_2^n = J(u^{T_n}) - J(u_\infty). \tag{4.50}$$

We study the convergence of J_1^n and J_2^n as $n \to \infty$. First, we analyze J_1^n:

$$\begin{aligned} J_1^n =& \frac{1}{T_n}\left(\frac{1}{2}\int_0^{T_n} \|y^{T_n}(t) - z\|^2_{L^2(\Omega)} dt + \frac{T_n}{2}\|u^{T_n}\|^2_{L^2(\Omega)}\right) \\ &- \frac{1}{2}\|\overline{y}^{T_n} - z\|^2_{L^2(\Omega)} - \frac{1}{2}\|u^{T_n}\|^2_{L^2(\Omega)} \\ =& \frac{1}{2T_n}\int_0^{T_n} \|y^{T_n}(t) - z\|^2_{L^2(\Omega)} dt - \frac{1}{2}\|\overline{y}^{T_n} - z\|^2_{L^2(\Omega)}. \end{aligned}$$

Since u^{T_n} is uniformly bounded in $(L^2(\Omega))^2$, from the regularity of the Navier-Stokes problem, we obtain that $\overline{y}^{T_n}$ is uniformly bounded in $(H^2(\Omega))^2 \cap V$, in particular, in $(L^2(\Omega))^2$. Then, using again the exponential stability property (4.47), we deduce that

$$I_1^n \to 0 \quad \text{as } n \to \infty.$$

For I_2^n we have

$$\begin{aligned} I_2^n &= J(u^{T_n}) - J(u_\infty) \\ &= \frac{1}{2}\|\overline{y}^{T_n} - z\|^2_{L^2(\Omega)} + \frac{1}{2}\|u^{T_n}\|^2_{L^2(\Omega)} - \frac{1}{2}\|y_\infty - z\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_\infty\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $\overline{y}^{T_n}$ is bounded in $(H^2(\Omega))^2 \cap V$, there exists some $y^* \in (H^2(\Omega))^2 \cap V$ and a subsequence of $\overline{y}^{T_n}$ such that

$$\overline{y}^{T_n} \rightharpoonup y^* \text{ weakly in } H^2(\Omega) \cap V.$$

We know that the injection of V into $(L^2(\Omega))^2$ is compact, so we have also

$$\overline{y}^{T_n} \to y^* \text{ in the norm of } L^2(\Omega).$$

Besides, the trilinear function b is continuous and by the Lemma 2.28, we obtain that $b(\overline{y}^{T_n}, \overline{y}^{T_n}, v) \to b(y^*, y^*, v)$, for all $v \in V$. Finally, since u^{T_n} converge to $u_\infty \in U_{ad}$, by the uniqueness of the Navier-Stokes problem, we obtain that $y^* = y_\infty$. Therefore, we conclude that

$$I_2^n \to 0, \quad \text{as } n \to \infty.$$

This completes the proof of the claim.

Now, we need to prove that u_∞ is an optimal solution of (4.41). Indeed, by the weak convergence of u^{T_n} we obtain that

$$\|u_\infty\|_{L^2(\Omega)} \le \liminf_{n\to\infty} \|u^{T_n}\|_{L^2(\Omega)}.$$

Also, we have that $\overline{y}^{T_n}$ weakly converges to y_∞, as $n \to \infty$. Then,

$$\left\| \frac{\int_0^{T_n} y^{T_n}(t)dt}{T_n} - y_\infty \right\|_{L^2(\Omega)} \le \left\| \frac{\int_0^{T_n} y^{T_n}(t)dt}{T_n} - \overline{y}^{T_n} \right\|_{L^2(\Omega)} + \|\overline{y}^{T_n} - y_\infty\|_{L^2(\Omega)},$$

and we obtain that

$$\frac{\int_0^{T_n} y^{T_n}(t)dt}{T_n} \to y_\infty \text{ in } L^2(\Omega), \text{ as } n \to \infty.$$

Then, necessarily we have

$$J(u_\infty) \le \liminf_{n\to\infty} \frac{J^{T_n}(u^{T_n})}{T_n}.$$

Therefore, using the claim (4.45), we obtain

$$J(u_\infty) \leq I$$

and finally, u_∞ is a minimizer for the steady state problem, with y_∞ the associated state. $\square$

Observation We observe that the proof of the Theorem 4.11 uses the exponential stabilization result (Theorem 4.10) in many times. We know that in the three-dimensional case this property is also true for strong solutions, but under more smallness condition of the stationary solutions. This implies that the three-dimensional case is more complex that the two-dimensional problem.

Observation Note that we considered the L^2-norm in the tracking term on the functional to minimize. However, in the three-dimensional case, this choice is not correct because there is no way to assure the optimal state to be a strong solution of the evolutionary Navier-Stokes problem. The good choice would be, for instance [33],

$$J(u) = \frac{1}{8}\int_0^T \left(\int_\Omega |y - x^d|^4 dx \right)^2 dt + \frac{T}{2}\|u\|_{L^2(\Omega)},$$

with $x^d \in L^8([0,T];(L^4(\Omega))^3)$.

Chapter 5

Inverse Viscosity Boundary Value Problem for the Evolutionary Stokes Equation

5.1 Introduction

In this chapter, we address the question of the unique determination of the viscosity function in an incompressible fluid described by the evolutionary Stokes equations in three dimensions, proving that, as the time–horizon tends to infinity, thus the determination of the viscosity is reduced to prove the identification of the viscosity for the stationary case. This strategy is close to the work of Isakov [69]. Isakov studied the inverse parabolic identifiability problem of a coefficient from the knowledge of the Dirichlet–to–Neumann map. His method make use of the stabilization of solutions of parabolic problems when the time–horizon tends to infinity, reducing the inverse parabolic problems to inverse elliptic problems with parameters.

Namely, one can simply consider the steady–state Stokes problem, forgetting the time dependence, and take the corresponding steady–state inverse problem as an approximation of this time evolution ones. This principle of replacing the time–dependent inverse problem by the steady–state one is not very common in the literature of inverse problem, the usual approaches are to construct the appropriate complex geometrical optics solutions and prove some result of density of the solutions or the gradient of the solutions, see [76] and the reference therein, and the other way is the use of suitable Carleman estimates, see [106].

Nevertheless, this kind of approach is related to the called *turnpike property*, in the context of optimal control problem. This property, which establishes that, under certain conditions, the optimal solution of a control problem remains exponentially close to the optimal solution of the steady–state problem, for most of the time. For instance, in [98] Porreta and Zuazua studied this phenomenon in the context of linear ODE and PDE (heat and wave equation), in that case they prove, under suitable controllability and observability assumptions, that the evolutionary optimal control and state, exponentially converge as the time–horizon tends to infinity to the ones of the corresponding steady state model. Besides, in [108] Trelat and

Zuazua shows the similar property in the context of nonlinear ODE, we refer to [113] and references therein for an extensive literature for the turnpike property. This tool has been studied in other contexts, for instance, in the context of identification of optimal materials for heat equation. In [15] it was proved that for large time optimization horizons, such processes can be approximated by the optimal steady–state ones, using the Γ–convergence ensuring the convergence of optimizers from parabolic towards elliptic.

On the other hand, the inverse problem for fluids mechanics models was studied by several authors. In the context of the inverse Stokes equations, Heck, Li and Wang [62] proved a global identifiability of the viscosity parameter in the stationary three–dimensional case, by boundary measurements reducing the stationary Stokes system, which contains two equations with different orders, into a second order decoupled systems (matrix–valued Schrödinger equations) and then constructed exponentially growing solutions for matrix–valued Schrödinger equations using the result of Eskin and Ralston [47, 48] for the linear isotropic elasticity problem. Moreover in [52] Fan et al. consider the inverse problem of determining a viscosity coefficient in the Navier–Stokes equation by observation data in a neighborhood of the boundary, and they obtain Lipschitz stability by the Carleman estimates in Sobolev spaces of negative order.

In [68] the authors prove the global uniqueness for the viscosity in two cases, stationary Navier–Stokes equations and Lamé system in two dimensions. They used the classical Dirichlet–to–Neumann map for the solution of the stationary Navier–Stokes system, namely

$$\Lambda_\mu(g) = \left(\frac{\partial u}{\partial n}, p\right)|_{\partial\Omega},$$

where g is the Dirichlet boundary condition and (u, p) is the solution of the Navier–Stokes problem. However, this boundary measurements are unnatural, as indicated in [79]. The authors in [79] consider the more natural boundary measurements, the Cauchy forces than the Dirichlet–to–Neumann map. Besides, they obtain also a global uniqueness of the viscosity for the stationary Stokes and Navier–Stokes system in the two–dimensional case. In three dimension, Li and Wang [80] proved the global uniqueness of the viscosity in an incompressible fluid described by the stationary Navier–Stokes equations using the Cauchy forces instead of the Dirichlet–to–Neumann map, and they considering small boundary condition to ensure the existence and uniqueness of solution. They proved the global determination reducing the problem to the Stokes system.

In this chapter, we analyze the parameter identification of the viscosity for the three–dimensional evolutionary Stokes equations. Mathematically, let $\Omega \subset \mathbb{R}^3$ be an open bounded connected domain with boundary $\partial\Omega \in C^2$, assume also that Ω is filled with an incompressible fluid. Given $T > 0$, let $u = (u^1, u^2, u^3)$ be the velocity vector field and the scalar function p, representing the pressure, satisfying the initial boundary value problem

$$\left\{\begin{array}{rcll} u_t - \text{div}\ (\sigma_\mu(u,p)) &=& 0 & , \quad \text{in } \Omega \times (0,T), \\ \text{div}\ u &=& 0 & , \quad \text{in } \Omega \times (0,T), \\ u &=& g & , \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &=& u_0(x) & , \quad \text{in } \Omega, \end{array}\right. \tag{5.1}$$

where $\sigma_\mu(u,p) = 2\mu e(u) - pI$ is the stress tensor and $e(u) = ((\nabla u) + (\nabla u)^T)/2$ is the strain tensor. Here $\mu(x) > \mu_0 > 0$ is the kinematic viscosity function. The exact regularity of μ

will be specified as we go along. Let $g \in C^1(0,T,(H^{3/2}(\partial\Omega))^3)$ satisfying the compatibility condition

$$\int_{\partial\Omega} g \cdot n ds = 0,$$

where n is the unit outer normal of $\partial\Omega$ and $u_0 \in H$, then there exists $(u,p) \in L^2(0,T;(H^2(\Omega))^3) \times L^2(0,T;H^1(\Omega))$ the unique solution of (5.1), see Chapter 1.

We are interested in the inverse problem of determining μ from the knowledge of boundary measurements. Mathematically, the boundary measurements are encoded in the Cauchy data of all solutions satisfying (5.1). Precisely, we define the set

$$S_\mu = \{(u|_{\partial\Omega}, \sigma_\mu(u,p)n|_{\partial\Omega})\} \subset (H^{3/2}(\partial\Omega))^3 \times (H^{1/2}(\partial\Omega))^3, \tag{5.2}$$

where (u,p) is the solution of (5.1). Then, our inverse problem is to determine μ from the knowledge of the operator S_μ.

As we mention before, the idea is to obtain an identifiability result of the viscosity using S_μ above defined, based on the uniqueness given in the stationary case. Namely, we consider the stationary Stokes problem

$$\left\{ \begin{array}{rcll} -\text{div}\ (\sigma_\mu(u,p)) + \lambda u & = & 0 & , \quad \text{in } \Omega, \\ \text{div}\ \ u & = & 0 & , \quad \text{in } \Omega, \\ u & = & g^0 & , \quad \text{on } \partial\Omega, \end{array} \right. \tag{5.3}$$

where $g^0 \in (H^{3/2}(\partial\Omega))^3$ satisfying the compatibility condition, $\lambda > 0$ is a constant and $\sigma_\mu(v,q)$ as before, and let S_μ^E the Cauchy data of all solutions satisfying (5.3).

The question we address is as follows: *can we get identifiability of the viscosity function of the evolutionary Stokes equation from the stationary problem as the horizon time goes to infinity?*

The first step to prove the uniqueness of the viscosity, is to show the identifiability in the stationary case, namely for the system (5.3). In order to satisfy the hypotheses for the construction of exponentially growing solutions given by Eskin and Ralston [47, 48], we need to consider a suitable regularity for the viscosity, given by the following hypothesis.

(H1) Let μ_1 and μ_2 be two viscosity functions. We assume that

$$\begin{aligned} \mu_1, \mu_2 \in C^{n_0}(\overline{\Omega}),\ \forall n_0 \geq 8, \\ \mu_i \geq \mu_0 > 0,\ \forall i = 1,2, \\ \mu_1(x) = \mu_2(x),\ \forall x \in \partial\Omega. \end{aligned}$$

Based on [62] and [47, 48], we have the following result, which will be shown in Section 2.

Theorem 5.1 *Let (u,p) be the solution to the stationary Stokes problem (5.3). Assume that $\mu_1(x)$ and $\mu_2(x)$ are two viscosity function satisfying (**H2**). Let $S_{\mu_1}^E$ and $S_{\mu_2}^E$ be the Cauchy data associated with μ_1 and μ_2, respectively. If $S_{\mu_1}^E = S_{\mu_2}^E$, then $\mu_1 = \mu_2$.*

We remark that Theorem 5.1 is a generalization of work by Heck, Li, and Wang [62]. In this paper the authors prove a similar result for the stationary Stokes equation (Theorem 1.1, [62]), but they consider the Stokes equation without perturbation, namely, $-\text{div }(\sigma_\mu(u,p)) = 0$, in Ω, and the additional condition on the viscosity function

$$\partial^\alpha \mu_1(x) = \partial^\alpha \mu_2(x),\ \forall x \in \partial\Omega,\ |\alpha| \leq 1. \tag{5.4}$$

We note that our result relax the hypothesis (5.4) for the viscosity function on the boundary, because we need only that $\mu_1 = \mu_2$ on $\partial\Omega$, unlike the work of Heck.

The rest of the chapter is organized as follows. In Section 5.2 we prove the global identifiability of the viscosity function when the fluid is described by the stationary Stokes equations, Theorem 5.1. Finally, in Section 5.3 we state our future work.

5.2 Proof of Theorem 5.1

The main idea for the proof of Theorem 5.1 is the construction of the classical exponentially growing solutions for the Stokes system. The difficulties in the such construction is that we are dealing with a systems of PDEs. Following the approach given by Uhlmann [110], we reduced the system into a second order decoupled matrix–valued Schrödinger equations. Using the results given by [47, 48], we can construct this special solutions.

The proof of the Theorem 5.1 is a slightly different with respect to that one proved by Heck et al. [62]. Our proof have the difficulties that in each step we need to add more terms of the viscosity. This changes all matrices that appears in our calculations. At the end, we can reduced the assumptions on the viscosity on the boundary, because these terms help, in some way, to the conclusion of the proof.

Proof. Theorem 5.1.

The proof closely follows the arguments of [62, 48]. For simplicity and without loss of generality, we consider the case $\lambda = 1$. Let $S^E_{\mu_1}$ and $S^E_{\mu_2}$ be the Cauchy data associated with μ_1 and μ_2, respectively. Denote by (u_1, p_1) and (u_2, p_2) the solutions. The duality pair between $H^{3/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$. Using the Green's formula, we obtain

$$\begin{aligned}\langle u_2, \sigma_{\mu_1}(u_1,p_1)\rangle &= \int_\Omega (\mu_1(x)e(u_2)\cdot e(u_1) - \text{div }(u_2)p_2)dx - \int_\Omega u_1\cdot u_2 dx\\ &= \int_\Omega \mu_1(x)e(u_2)\cdot e(u_1)dx - \int_\Omega u_1\cdot u_2 dx.\end{aligned}$$

and

$$\begin{aligned}\langle u_1, \sigma_{\mu_2}(u_2,p_2)\rangle &= \int_\Omega (\mu_2(x)e(u_2)\cdot e(u_1) - p_2\text{div }(u_1))dx - \int_\Omega u_2\cdot u_1 dx\\ &= \int_\Omega \mu_2(x)e(u_2)\cdot e(u_1)dx - \int_\Omega u_2\cdot u_1 dx.\end{aligned}$$

Then, if $S^E_{\mu_1} = S^E_{\mu_2}$, we conclude that

$$\int_\Omega (\mu_1 - \mu_2) e(u_2) \cdot e(u_1) dx = 0. \tag{5.5}$$

We observe that if we denote by $V_1(\mu_1, \mu_2) = \text{span}\{e(u_1) \cdot e(u_2) : \ u_i \in (H^2(\Omega))^3, \ u_i$ solution of (5.3)$\}$. Then, note that a practical way to prove the identifiability of the viscosity is to show that the space V_1 is dense in $L^1(\Omega)$, see [76].

However, for our purposes is better to use another systems of equations in place of (5.3). Inspired by the isotropic elasticity system, see [48, 110], thus we consider the new functions (w, f) such that

$$u = \mu^{-1/2} w + \mu^{-1} \nabla f - f \nabla \mu^{-1},$$

and we must check which equation must fulfill (w, f) such that (u, p) is solution of (5.3). We need that u satisfies the divergence condition, namely,

$$0 = \text{div} \ \ u = \mu^{-1} \Delta f + \mu^{-1/2} \text{div } w - \Delta \mu^{-1} f + \nabla \mu^{-1/2} \cdot w.$$

Now, we compute the Cauchy tensor componentwise

$$\begin{aligned}
2\mu e(u)_{jk} &= \mu(\partial_k u^j + \partial_j u^k) \\
&= \mu \partial_k (\mu^{-1/2} w^j + \mu^{-1} \partial_j f - f \partial_j \mu^{-1}) + \mu \partial_j (\mu^{-1/2} w^k + \mu^{-1} \partial_k f - f \partial_k \mu^{-1}) \\
&= \mu(\partial_k \mu^{-1/2} w^j + \mu^{-1/2} \partial_k w^j + \partial_k \mu^{-1} \partial_j f + \mu^{-1} \partial^2_{jk} f \\
&\quad - \partial_j \mu^{-1} \partial_k f - f \partial^2_{jk} \mu^{-1} + \partial_j \mu^{-1/2} w^k + \mu^{-1/2} \partial_j w^k \\
&\quad + \partial_j \mu^{-1} \partial_k f + \mu^{-1} \partial^2_{jk} f - \partial_k \mu^{-1} \partial_j f - f \partial^2_{jk} \mu^{-1}) \\
&= -\partial_k \mu^{1/2} w^j - \partial_j \mu^{1/2} w^k + \mu^{1/2} (\partial_j w^k + \partial_k w^j) + 2\partial^2_{jk} f - 2\mu \partial^2_{jk} \mu^{-1} f.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\sum_j \partial_j (2\mu e(u)_{jk}) &= \mu^{1/2} \Delta w^k + \partial_k (\nabla \mu^{1/2} \cdot w + \mu^{1/2} \text{div } w + 2\Delta f) - 2\partial_k \mu^{1/2} \text{div } w \\
&\quad - \sum_j 2\mu \partial^2_{jk} \mu^{-1} \partial_j f - 2 \sum_j \partial^2_{jk} \mu^{1/2} w^j - \Delta \mu^{1/2} w^k - 2 \sum_j (\partial_j \mu \partial^2_{jk} \mu^{-1} + \mu \partial^3_{jjk} \mu^{-1}) f.
\end{aligned}$$

From [110], we know that functions (w, f) are related to functions (u, p). In fact, for different (u, p) we have different (w, f) and obviously a different equation which is satisfied by (w, f). As in the equation (5.5) does not shown the pressure, we hence some freedom in choosing it such that (w, f) satisfy an equation which is easy to attack. In fact, we can consider

$$p \ = \ \nabla \mu^{1/2} \cdot w + \mu^{1/2} \text{div } w + 2\Delta f \ = \ \text{div } (\mu^{1/2} w) + 2\Delta f,$$

then

$$\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \mu^{-1/2} w + \mu^{-1} \nabla f - f \nabla \mu^{-1} \\ \text{div } (\mu^{1/2} w) + 2\Delta f \end{pmatrix}$$

is a solution of (5.3) and (w,f) satisfies

$$\Delta \begin{pmatrix} w \\ f \end{pmatrix} + A_1(x) \begin{pmatrix} \nabla f \\ \text{div } w \end{pmatrix} + A_0(x) \begin{pmatrix} w \\ f \end{pmatrix} = 0, \tag{5.6}$$

where

$$A_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2\mu^{-1} + \mu^{-1} & -\mu^{-1}\nabla\mu \\ 0 & \mu^{1/2} \end{pmatrix}$$

and

$$A_0(x) = \begin{pmatrix} -2\mu^{-1/2}\nabla^2\mu^{1/2} - \mu^{-1/2}\Delta\mu^{1/2} + \mu^{-1/2} & -4\nabla\mu^{1/2}\nabla^2\mu^{-1} - 2\mu^{1/2}\nabla^3\mu^{-1} - \nabla\mu^{-1} \\ \mu(\nabla\mu^{-1/2})^T & -\mu\Delta\mu^{-1} \end{pmatrix}.$$

We note that (5.6) is a matrix–valued Schrödinger equation. Moreover, if $\mu \in C^{n_0}(\overline{\Omega})$ and $\mu > 0$, then $A_1 \in C^{n_0-2}(\overline{\Omega})$ and $A_0 \in C^{n_0-3}(\overline{\Omega})$. Now, we shall construct exponentially growing solutions with large parameter for (5.6) and (5.3).

Since it is convenient to work in the whole space $\mathbb{R}^3$ instead of Ω, we pick up a large domain $\tilde{\Omega} \supset \Omega$, and extend μ to $\mathbb{R}^3$ by preserving its smoothness.

Let α, β and l be pairwise orthogonal vectors in $\mathbb{R}^3$ with $|\alpha| = |\beta| = 1$. We set $\theta = \alpha + i\beta$ and for $\tau \gg 0$ we set $\rho_1(\tau) = \frac{l}{2} + \sqrt{\tau^2 - \frac{|l|^2}{4}}\alpha$ and $\zeta_1(\tau) = \rho_1 + i\tau\beta$. Note that $\lim_{\tau\to\infty} \tau^{-1}\zeta_1(\tau) = \theta$.

Then, we can construct the exponentially growing solutions using the following result, which was proved in [48].

Lemma 5.2 (Section 2, [48]) *Consider the Schrödinger equation with external Yang–Mills potential*

$$Lu := -\Delta u - 2iA(x) \cdot \frac{\partial}{\partial x} u + B(x)u = 0, \;\; x \in \tilde{\Omega} \subset \mathbb{R}^3,$$

where $A(x) \in C^{n_1}(\tilde{\Omega})$ and $B(x) \in C^{n_1-1}(\tilde{\Omega})$, $n_1 \geq 6$. One can construct solutions of the form

$$u = e^{ix\cdot\zeta_1}v = e^{ix\cdot\zeta_1}(C_0(x,\theta)p(\theta \cdot x) + O(\frac{1}{\tau}))$$

where $C_0 \in C^{n_1}(\tilde{\Omega})$ is solution of

$$i\theta \cdot \frac{\partial}{\partial x} C_0(x,\theta) = \theta \cdot A(x)C_0(x,\theta)$$

with $\det C_0 \neq 0$, *$p(z)$ is an arbitrary polynomial in complex variable z, and $O(\frac{1}{\tau})$ is the term bounded by $C\frac{1}{\tau}$ in $H^k(\tilde{\Omega})$, $0 \leq k \leq n_1 - 2$.*

From this Lemma we can construct the exponentially growing solutions to (5.6) for $\mu = \mu_1$, that is

$$\begin{pmatrix} w_1 \\ f_1 \end{pmatrix} = e^{ix\cdot\zeta_1}v_1$$

with

$$v_1 =: \begin{pmatrix} r_1(x,\zeta_1) \\ s_1(x,\zeta_1) \end{pmatrix} = C_1(x,\theta)g_1(\theta\cdot x) + O(\tau^{-1})$$

where C_1 is an invertible matrix satisfying

$$-2\theta\cdot\nabla C_1(x,\theta) = \begin{pmatrix} -2\mu_1^{1/2}\nabla^2\mu_1^{-1}+\mu_1^{-1} & -\mu_1^{-1}\nabla\mu_1 \\ 0 & \mu_1^{1/2} \end{pmatrix}\begin{pmatrix} 0_{3\times 3} & \theta \\ \theta^T & 0 \end{pmatrix} C_1(x,\theta) \tag{5.7}$$

in $\tilde{\Omega}$ and $g_1(z)$ is an arbitrary polynomial in complex variable z. Similarly, let $\rho_2(\tau) = \frac{l}{2} + \sqrt{\tau^2 - \frac{|l|^2}{4}}\alpha$ and $\zeta_2(\tau) = \rho_2 - i\tau\beta$. Note that $\lim_{\tau\to\infty}\tau^{-1}\zeta_2(\tau) = \overline{\theta}$. We construct the exponentially growing solutions to (5.6) for $\mu = \mu_2$

$$\begin{pmatrix} w_2 \\ f_2 \end{pmatrix} = e^{ix\cdot\zeta_2}v_2$$

with

$$v_2 =: \begin{pmatrix} r_2(x,\zeta_1) \\ s_2(x,\zeta_1) \end{pmatrix} = C_2(x,\overline{\theta})g_2(\overline{\theta}\cdot x) + O(\tau^{-1})$$

where C_2 is an invertible matrix satisfying

$$-2\overline{\theta}\cdot\nabla C_2(x,\overline{\theta}) = \begin{pmatrix} -2\mu_2^{1/2}\nabla^2\mu_2^{-1}+\mu_2^{-1} & -\mu_2^{-1}\nabla\mu_2 \\ 0 & \mu_2^{1/2} \end{pmatrix}\begin{pmatrix} 0_{3\times 3} & \overline{\theta} \\ \overline{\theta}^T & 0 \end{pmatrix} C_2(x,\overline{\theta})$$

in $\tilde{\Omega}$ and $g_2(z)$ is an arbitrary polynomial in complex variable z. To simplify, we denotes $C_3(x,\theta) = \overline{C_2(x,\overline{\theta})}$. Then we obtain

$$-2\theta\cdot\nabla C_3(x,\theta) = \begin{pmatrix} -2\mu_2^{1/2}\nabla^2\mu_2^{-1}+\mu_2^{-1} & -\mu_2^{-1}\nabla\mu_2 \\ 0 & \mu_2^{1/2} \end{pmatrix}\begin{pmatrix} 0_{3\times 3} & \theta \\ \theta^T & 0 \end{pmatrix} C_3(x,\theta). \tag{5.8}$$

We substitute the previous solutions into (5.5). We denote by

$$H(u_1,u_2) = \int_\Omega (\mu_1-\mu_2)e(u_2)\cdot\overline{e(u_1)}dx.$$

Then we prove the indentifiability of $\mu(x)$ from

$$\lim_{\tau\to\infty}\tau^{-2}H(u_1,u_2) = 0.$$

To express the result in more compact form, denotes

$$\begin{pmatrix} r_1^{(0)}(x,\theta) \\ s_1^{(0)}(x,\theta) \end{pmatrix} = C_1(x,\theta)g_1(\theta\cdot x), \begin{pmatrix} r_3^{(0)}(x,\theta) \\ s_3^{(0)}(x,\theta) \end{pmatrix} = \begin{pmatrix} r_2^{(0)}(x,\overline{\theta}) \\ s_2^{(0)}(x,\overline{\theta}) \end{pmatrix} = C_2(x,\overline{\theta})g_2(\overline{\theta}\cdot x).$$

Then

$$\begin{pmatrix} r_3^{(0)}(x,\theta) \\ s_3^{(0)}(x,\theta) \end{pmatrix} = C_3(x,\theta) g_3(\theta\cdot x),$$

where we take $g_3(z) = g_2(z)$. We define

$$\begin{pmatrix} R_1^{(0)}(x,\theta) \\ s_1^{(0)}(x,\theta) \end{pmatrix} = \begin{pmatrix} \mu_1^{-1/2}\theta\cdot r_1^{(0)} \\ s_1^{(0)} \end{pmatrix}, \begin{pmatrix} R_3^{(0)}(x,\theta) \\ s_3^{(0)}(x,\theta) \end{pmatrix} = \begin{pmatrix} \mu_2^{-1/2}\theta\cdot r_3^{(0)} \\ s_3^{(0)} \end{pmatrix}.$$

With this notations, we obtain from (5.7) and (5.8)

$$\theta\cdot\nabla \begin{pmatrix} R_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 0 & (\theta\cdot\nabla)^2\mu_1^{-1} + \frac{1}{2}\mu_1^{-3/2} \\ -\frac{1}{2}\mu_1 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ s_1 \end{pmatrix} \tag{5.9}$$

and

$$\theta\cdot\nabla \begin{pmatrix} R_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 & (\theta\cdot\nabla)^2\mu_2^{-1} + \frac{1}{2}\mu_2^{-3/2} \\ -\frac{1}{2}\mu_2 & 0 \end{pmatrix} \begin{pmatrix} R_3 \\ s_3 \end{pmatrix}. \tag{5.10}$$

From Lemma 2.1 in [48], we get

$$0 = \lim_{\tau\to\infty} \tau^{-2} H(u_1,u_2) = \int_\Omega e^{ix\cdot l}(R_3,s_3)V(x,\theta) \begin{pmatrix} R_1 \\ s_1 \end{pmatrix} dx, \tag{5.11}$$

where

$$V(x,\theta) = \frac{1}{2}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right) \begin{pmatrix} 0 & \theta\cdot\nabla\mu_2 \\ \theta\cdot\nabla\mu_1 & \mu_1(\theta\cdot\nabla)^2\mu_1^{-1} + \mu_2(\theta\cdot\nabla)^2\mu_2^{-1} + \frac{1}{4}\mu_1^{-1/2} + \frac{1}{4}\mu_2^{-1/2} \end{pmatrix}. \tag{5.12}$$

Arguying as in [48], we can take

$$\begin{pmatrix} R_1 \\ s_1 \end{pmatrix} = \tilde{C}_1(x,\theta)\tilde{g}_1(\theta\cdot x)\ ,\ \begin{pmatrix} R_3 \\ s_3 \end{pmatrix} = \tilde{C}_3(x,\theta)\tilde{g}_3(\theta\cdot x)$$

where $\tilde{C}_1$ and $\tilde{C}_3$ are invertible 2×2 matrix solutions of (5.9) and (5.10), respectively, and $\tilde{g}_1$ and $\tilde{g}_3$ are two arbitrary vector of polynomials of z. Thus, (5.11) can be written as

$$\int_\Omega e^{ix\cdot l}\tilde{g}_3^T(\theta\cdot x)\tilde{C}_3^T(x,\theta)V(x,\theta)\tilde{C}_1(x,\theta)\tilde{g}_1(\theta\cdot x)dx = 0. \tag{5.13}$$

As in [48], we take advantage of the freedom of choosing θ, using the next parameterization. Denote by $\xi(s)$

$$\xi(s) = \left(\frac{1}{2}(s-s^{-1}), \frac{i}{2}(s+s^{-1}), 1\right)^T\ ,\ s\in\mathbb{C}\setminus\{0\}.$$

Note that $\xi(s) \cdot \xi(s) = 0$ and $Re(\xi(s)) \neq 0$. We have that $\theta = |Re(\xi(s))|^{-1}\xi(s)$.

Let $(x_1', x_2', x_\perp)$ be coordinates in $\mathbb{R}^3$, where $x_1' = x \cdot \alpha$, $x_2' = x \cdot \beta$ and $x_\perp = x - x_1'\alpha - x_2'\beta$.

Following the ideas of [48], we define the matrix

$$B(x, \xi(s)) = P(\tilde{C}_3^T(x, \xi(s))V(x, \xi(s))\tilde{C}_1(x, \xi(s)))$$

where $P(s)$ denotes the operator

$$P(s)f = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{(\mathcal{F}f)(\eta)e^{ix\cdot\eta}}{i\eta \cdot \xi(s)} d\eta.$$

Note that

$$\xi(s) \cdot \nabla Pf = f \ , \ \forall f \in L_c^2(\mathbb{R}^3). \tag{5.14}$$

Next, we define $f_1(x_1', x_2', x_\perp)$ the function $f(x)$ in the new coordinates. Then

$$(Pf)_1(x_1', x_2', x_\perp) = |Re(\xi(s))|^{-1}\Pi f_1, \tag{5.15}$$

where Π is the Cauchy operator

$$(\Pi f_1)(z, x_\perp) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f_1(y, x_\perp)}{z - y} dy_1' dy_2', \tag{5.16}$$

with $z = x_1' + ix_2'$ and $y = y_1' + iy_2'$. Since V is supported in the ball of radio R, B_R, using the representation of $P(s)$ to compute the Laurent series for $B(z, x_\perp, \xi(s))$ in $\{|z| > R\}$, one sees that each term in the Laurent series vanishes for $|x| > R$. Then Lemma 6.1 in [48], implies that

$$B'(x, \xi(s)) = (\tilde{C}_3^T)^{-1}(x, \xi(s))B(x, \xi(s))\tilde{C}_1^{-1}(x, \xi(s)) \tag{5.17}$$

is analytic in s on $\mathbb{C} \setminus \{0\}$.

Next we study the homogeneity of $B'(x, \xi(s))$ in θ. By (5.9) and (5.10), we obtain that

$$\tilde{C}_1(x, \lambda\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tilde{C}_1(x, \theta) \ , \ \tilde{C}_3(x, \lambda\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tilde{C}_3(x, \theta)$$

and (5.12) implies

$$V(x, \lambda\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} V(x, \theta) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

From the definition of P, we have

$$B'(x, \lambda\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \frac{B'(x, \theta)}{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then if

$$B'(x,\theta) = \begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix}$$

we have $B'_{jk}(x,\lambda\theta) = \lambda^{j+k-3} B'_{jk}(x,\theta)$ for $j,k = 1,2$. Thus Liouville's theorem implies that $B'_{11} = 0$, B'_{12} and B'_{21} are functions of x alone, and

$$B'_{22}(x,\xi(s)) = \sum_{|k|\leq 1} s^k B'_{22k}(x).$$

In view of the homogeneity of the entries of B' in z we conclude

$$B'(x,\theta) = \begin{pmatrix} 0 & b_{12}(x) \\ b_{21}(x) & b_{22}(x)\cdot\theta \end{pmatrix}.$$

From (5.9),(5.10), and (5.14), one gets

$$\theta\cdot\nabla B' = -\begin{pmatrix} 0 & -\frac{1}{2}\mu_2 \\ (\theta\cdot\nabla)^2\mu_2^{-1} & \end{pmatrix} B' + V - B' \begin{pmatrix} 0 & (\theta\cdot\nabla)^2\mu_1^{-1} \\ -\frac{1}{2}\mu_1 & 0 \end{pmatrix}.$$

Denote by

$$V(x,\theta) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

and following the method in [48], we conclude that

$$-\frac{1}{2}\mu_2 b_{21} - \frac{1}{2}\mu_1 b_{12} = v_{11} = 0, \tag{5.18}$$

$$\theta\cdot\nabla b_{12} - \frac{1}{2}\mu_2 b_{22}\cdot\theta = v_{12}, \tag{5.19}$$

$$\theta\cdot\nabla b_{21} - \frac{1}{2}\mu_1 b_{22}\cdot\theta = v_{21}, \tag{5.20}$$

$$\theta\cdot\nabla(b_{22}\cdot\theta) + (\theta\cdot\nabla)^2\mu_1^{-1}b_{21} + (\theta\cdot\nabla)^2\mu_2^{-1}b_{12} + \frac{1}{2}\mu_1^{-3/2}b_{21} + \frac{1}{2}\mu_2^{-3/2}b_{12} = v_{22}. \tag{5.21}$$

From (5.18)-(5.21) we prove that $\mu_1 = \mu_2$. Applying $\theta\cdot\nabla$ to (5.18) we obtain

$$\theta\cdot\nabla\mu_1 b_{12} + \mu_1\theta\cdot\nabla b_{12} + \theta\cdot\nabla\mu_2 b_{21} + \mu_2\theta\cdot\nabla b_{21} = 0. \tag{5.22}$$

Now, using (5.19) and (5.20) to cancel $b_{22}\cdot\theta$, implies

$$\mu_1\theta\cdot\nabla b_{12} - \mu_2\theta\cdot\nabla b_{21} = \mu_1 v_{12} - \mu_2 v_{21}. \tag{5.23}$$

Putting together (5.22) and (5.23) we have

$$\theta\cdot\nabla\mu_1 b_{12} + \theta\cdot\nabla\mu_2 b_{21} + 2\mu_1\theta\cdot\nabla b_{12} = \mu_1 v_{12} - \mu_2 v_{21}. \tag{5.24}$$

Solving (5.18) gives

$$b_{21} = -\frac{\mu_1}{\mu_2} b_{12}.$$

Putting the previous expression, the expressions v_{12} and v_{21} of (5.12), in (5.24) we obtain

$$2\sqrt{\mu_1\mu_2}\theta \cdot \nabla \left(\sqrt{\frac{\mu_1}{\mu_2}} b_{12} \right) = -\sqrt{\mu_1\mu_2}\theta \cdot \nabla \left(\sqrt{\frac{\mu_1}{\mu_2}} \right) - \sqrt{\mu_1\mu_2}\theta \cdot \nabla \left(\sqrt{\frac{\mu_2}{\mu_1}} \right).$$

Since $b_{12} = 0$ and $\mu_1 = \mu_2$ on $\partial\Omega$, by the uniqueness of the Cauchy problem we have

$$2\sqrt{\frac{\mu_1}{\mu_2}} b_{12} = -\sqrt{\frac{\mu_1}{\mu_2}} - \sqrt{\frac{\mu_2}{\mu_1}} + 2,$$

that is,

$$b_{12} = -\frac{1}{2}\left(1 + \frac{\mu_2}{\mu_1} - 2\sqrt{\frac{\mu_2}{\mu_1}}\right). \tag{5.25}$$

Similarly, we have

$$b_{21} = \frac{1}{2}\left(1 + \frac{\mu_1}{\mu_2} - 2\sqrt{\frac{\mu_1}{\mu_2}}\right). \tag{5.26}$$

Thus, replacing (5.25) and (5.26) in (5.19), implies that

$$b_{22} \cdot \theta = \frac{2}{\mu_2}\theta \cdot \nabla \left(\frac{\mu_2}{\mu_1} \right) + \theta \cdot \nabla \mu_2^{-1} - \theta \cdot \nabla \mu_1^{-1}. \tag{5.27}$$

On the other hand, from v_{22} of (5.12), using (5.25) and (5.26), in (5.21), implies

$$\theta \cdot \nabla(b_{22} \cdot \theta) + \left(1 - \sqrt{\frac{\mu_1}{\mu_2}}\right)(\theta \cdot \nabla)^2 \mu_1^{-1} - \left(1 - \sqrt{\frac{\mu_2}{\mu_1}}\right)(\theta \cdot \nabla)^2 \mu_2^{-1} + \frac{3}{8}\left(\frac{1}{\mu_1^{3/2}} - \frac{1}{\mu_2^{3/2}}\right) + \frac{5}{8}\left(\frac{1}{\mu_1^{1/2}\mu_2} - \frac{1}{\mu_2^{1/2}\mu_1}\right) = 0. \tag{5.28}$$

Putting together (5.26) and (5.27) we obtain

$$\theta \cdot \nabla \left(\frac{2}{\mu_2}\theta \cdot \nabla \left(\frac{\mu_1}{\mu_2} \right) \right) = \sqrt{\frac{\mu_1}{\mu_2}}(\theta \cdot \nabla)^2 \mu_1^{-1} - \sqrt{\frac{\mu_2}{\mu_1}}(\theta \cdot \nabla)^2 \mu_2^{-1} - \frac{3}{8}\left(\frac{1}{\mu_1^{3/2}} - \frac{1}{\mu_2^{3/2}}\right) - \frac{5}{8}\left(\frac{1}{\mu_1^{1/2}\mu_2} - \frac{1}{\mu_2^{1/2}\mu_1}\right).$$

Note that

$$\frac{2}{\mu_2}\theta\cdot\nabla\left(\frac{\mu_1}{\mu_2}\right)=\sqrt{\frac{\mu_1}{\mu_2}}\theta\cdot\nabla\mu_1^{-1}-\sqrt{\frac{\mu_2}{\mu_1}}\theta\cdot\nabla\mu_2^{-1},$$

then

$$\theta\cdot\nabla\left(\frac{2}{\mu_2}\theta\cdot\nabla\left(\frac{\mu_1}{\mu_2}\right)\right)=\theta\cdot\nabla\left(\sqrt{\frac{\mu_1}{\mu_2}}\right)\theta\cdot\nabla\mu_1^{-1}+\sqrt{\frac{\mu_1}{\mu_2}}(\theta\cdot\nabla)^2\mu_1^{-1}$$
$$-\theta\cdot\nabla\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\theta\cdot\nabla\mu_2^{-1}-\sqrt{\frac{\mu_2}{\mu_1}}(\theta\cdot\nabla)^2\mu_2^{-1}.$$

Therefore

$$\theta\cdot\nabla\left(\sqrt{\frac{\mu_1}{\mu_2}}\right)\theta\cdot\nabla\mu_1^{-1}-\theta\cdot\nabla\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\theta\cdot\nabla\mu_2^{-1}+\frac{3}{8}\left(\frac{1}{\mu_1^{3/2}}-\frac{1}{\mu_2^{3/2}}\right)+$$
$$\frac{5}{8}\left(\frac{1}{\mu_1^{1/2}\mu_2}-\frac{1}{\mu_2^{1/2}\mu_1}\right)=0. \qquad (5.29)$$

It is easy to prove that

$$\theta\cdot\nabla\left(\sqrt{\frac{\mu_1}{\mu_2}}\right)\theta\cdot\nabla\mu_1^{-1}=\frac{1}{2\mu_1^2\mu_2\sqrt{\mu_1\mu_2}}(\mu_2\theta\cdot\nabla\mu_1-\mu_1\theta\cdot\nabla\mu_2)\theta\cdot\nabla\mu_1$$

and

$$\theta\cdot\nabla\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\theta\cdot\nabla\mu_2^{-1}=\frac{1}{2\mu_2^2\mu_1\sqrt{\mu_1\mu_2}}(\mu_2\theta\cdot\nabla\mu_1-\mu_1\theta\cdot\nabla\mu_2)\theta\cdot\nabla\mu_2.$$

From (5.29) we obtain

$$4[(\mu_2\theta\cdot\nabla\mu_1)^2-(\mu_1\theta\cdot\nabla\mu_2)^2]+3\mu_1\mu_2(\mu_2^{3/2}-\mu_1^{3/2})+5(\sqrt{\mu_1}-\sqrt{\mu_2})\mu_1^{3/2}\mu_2^{3/2}=0. \qquad (5.30)$$

Note that the equation (5.30) is true for all θ, in particular, for all $\varepsilon\theta$, with $\varepsilon>0$. Then,

$$4\varepsilon^2(\mu_2\theta\cdot\nabla\mu_1)^2-(\mu_1\theta\cdot\nabla\mu_2)^2]+3\mu_1\mu_2(\mu_2^{3/2}-\mu_1^{3/2})+5(\sqrt{\mu_1}-\sqrt{\mu_2})\mu_1^{3/2}\mu_2^{3/2}=0.$$

Taking limit $\varepsilon\to 0$, we obtain

$$3\mu_1\mu_2(\mu_2^{3/2}-\mu_1^{3/2})+5(\sqrt{\mu_1}-\sqrt{\mu_2})\mu_1^{3/2}\mu_2^{3/2}=0.$$

Let $\phi=\sqrt{\mu_1}$ and $\psi=\sqrt{\mu_2}$. Then

$$3\phi^2\psi^2(\psi^3-\phi^3)+5(\phi-\psi)\phi^3\psi^3=0,$$

thus

$$(\psi - \phi)\phi^2\psi^2[3(\phi^2 + \psi\phi + \psi^2) - 5\phi\psi] = 0,$$

therefore

$$(\psi - \phi)\phi^2\psi^2[3\phi^2 + 3\psi^2 - 2\phi\psi] = 0.$$

Suppose that $\mu_1 \neq \mu_2$ in Ω. Then

$$3(\phi^2 + \psi\phi + \psi^2) - 5\phi\psi = 0,$$

thus

$$2(\psi^2 + \phi^2) + (\phi^2 + \psi^2 - 2\psi\phi) = 0.$$

Since $\phi^2 + \psi^2 - 2\psi\phi \geq 0$, we obtain that

$$\phi^2 + \psi^2 \leq 0,$$

which is a contradiction. Therefore, $\phi = \psi$, that is,

$$\mu_1 = \mu_2 \ , \ \text{in } \Omega.$$

□

5.3 Perspective

As mentioned in the introduction, we are interested in the identification of the viscosity function for the evolutionary case. One idea for this result is to use the approach given by Isakov [69]. Namely, we want to obtain the identifiability of viscosity when the time horizon tends to infinity, reducing the inverse evolutionary problem to inverse stationary problem. For this, its necessary to obtain a result of stabilization of solutions, but the big problem is that the viscosity function play a important role in the convergence of solutions when time tends to infinity. In particular, we lose that the pressure is a harmonic function.

Another approach is to formulate the problem from the point of view of the optimal control theory, specifically use the approach of the turnpike property. In this case, the problem arises to ensure the uniqueness of the optimal viscosity function for the evolutionary case.

Chapter 6

Conclusions

In this Thesis we have applied theoretical results about the inverse and control problems on some models in the fluid mechanics field. In particular, we have focused on the Stokes and Navier–Stokes equations. We conclude this Thesis with some final remarks and perspectives related to these subjects.

- In Chapter 3 we studied the geometric inverse problem related to the stationary Stokes equations. Specifically, we analyzed the size estimates, lower and upper bounds, problem for an inclusion immersed in a bounded domain fulfilled with a incompressible viscous fluid. Our results rely in a series of theoretical inequalities. In fact, the lower bound are based on the application of Rellich's identity for the Stokes system, an appropriate Poincaré type inequality and interior estimate for the gradient of the solution. On the other hand, the upper estimates are based on a variational technical inequality associated to the Stokes equations, Lipschitz propagation of smallness (consequence of a three–spheres inequality) and Korn's inequality.

  As we mentioned in the Chapter 3, the upper bound on our volume estimates is different with respect to the result proved by Alessandrini et al. [13]. The authors show that the upper estimate can be realized in a better way, without using a priori information on the obstacle. However, this improvement is a consequence of the doubling inequality for the conductivity problem, which, as far as we know, is not yet proven to the Stokes equations.

  Besides, we have studied some numerical experiments about the size estimates, but only experiments. In a future work (in progress) following the guidelines drawn up in this Chapter, an extended numerical investigation will be performed in order to prove the effectiveness of this approach. We will analyze the sensitivity with respect to various relevant parameters.

- In Chapter 4 we have analyzed the large–time behavior of some optimal control problem for the incompressible viscous evolutionary Navier–Stokes equations in the two–dimensional case. We proved, under certain smallness condition, that the optimal solution of the nonsteady optimality system satisfies the turnpike property in a local sense.

Our result are based strongly on an appropriate definition of the functional to minimize, that is, consider a tracking–type optimal control problem with terminal constraint. This implies that when linearize the optimality system, we can translate the system as a optimal control problem for the Oseen equations. Besides, using that the nonlinear term is of quadratic order, we need only assume the smallness condition on the tracking term of the analogous stationary optimal problem.

A future work (in progress) is to consider a similar problem, but for optimal design approach. This will help us to understand, mathematically speaking, why in the aeronautic field the optimal aircraft design are based on stationary models. We will analyze different models before to the treatment to the Navier–Stokes equations, for instance, we will studied the optimal design of coefficient in the finite dimensional case and optimal design in the context of parabolic equations.

- In Chapter 5 we have studied the parameter identification question for the evolutionary Stokes system. First, we analyzed the problem of the viscosity identifiability problem in the stationary case using the technique given by Uhlmann [110], that is the construction of special solutions for the problem. Then, in a work in progress, we will studied the identification of the viscosity for the evolutionary equation when we consider the time tends to infinity. We hope to have a result of this property using the idea of Isakov [69] for the parabolic case.

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