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Extended seminorms and extended topological vector spaces

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ABSTRACT

We introduce the notions of extended topological vector spaces and extended seminormed spaces, following the main ideas of extended normed spaces, which were introduced by G. Beer and J. Vanderwerff. We provide a topological study of such structures, giving a unifying theory with main applications in the study of spaces of continuous functions. We also generalize classical results of functional analysis, as open mapping theorem and closed graph theorem.

1. Introduction and problem formulation

In the papers [2,4,5], G. Beer and J. Vanderwerff introduced and studied the notion of *extended norm* on a vector space X over a field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), as a functional $\|\cdot\| : X \to [0, +\infty]$ which satisfies

1. $\|x\| = 0 \Leftrightarrow x = 0$;
2. For all $x \in X$ and $\alpha \in \mathbb{K}$, $\|\alpha x\| = |\alpha|\|x\|$ (with the convention $0 \cdot (+\infty) = 0$);
3. For all $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

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The space X endowed with the extended norm $\|\cdot\|$ was called the *extended normed space* $(X, \|\cdot\|)$. Of course, $\|\cdot\|$ induces a topology over X, which happens to fail to be compatible with the vectorial structure of X: it is compatible with the sum but not with the scalar multiplication.

In the same papers, many classical results concerning normed spaces are extended to this new framework, showing that this new object is in some sense "well-behaved". Motivated by these works, a natural generalization arises: Extended seminorms and Extended seminormed spaces.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$). A function $\rho : X \to [0, +\infty]$ is called an **extended seminorm** if, adopting the convention $0 \cdot (+\infty) = 0$, it satisfies

(a) $\rho(\lambda x) = |\lambda|\rho(x)$, for all $\lambda \in \mathbb{K}$ and $x \in X$.
(b) $\rho(x + y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$.

For a vector space X over $\mathbb{R}$ or $\mathbb{C}$ and a family of extended seminorms $\mathcal{P} := \{\rho_i \ : \ i \in I\}$ over X, the *induced topology of* $\mathcal{P}$ *on* X, denoted by $\mathfrak{T}(\mathcal{P})$, is the coarsest topology on X for which all functions

$$\begin{aligned} \rho_{i,x} : X &\to [0, +\infty] \\ x' &\mapsto \rho_i(x' - x) \end{aligned}$$

with $i \in I$ and $x \in X$ are continuous. It is easily verified that $\mathfrak{T}(\mathcal{P})$ is a group topology over X (see, e.g. [8, Ch. III, §1, Definition 1], or section 2 below), and so we are led to the following definition:

Definition 1.2. Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and τ be a group topology over X. We say that (X, τ) is an **extended seminormed space** (*esns*, for short) if there exists a family $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ of extended seminorms over X such that its induced topology $\mathfrak{T}(\mathcal{P})$ coincides with τ.

Extended seminormed spaces appear naturally in functional analysis when we study the convergences in functional spaces: Let (M, d) be a metric space and $\mathscr{B}$ be a *bornology* on M, namely, a family of subsets of M such that:

1. For each $B \in \mathscr{B}$, each subset of B also belongs to $\mathscr{B}$.
2. For each $B_1, B_2 \in \mathscr{B}$, $B_1 \cup B_2 \in \mathscr{B}$.
3. $\mathscr{B}$ forms a cover of M, namely, $M = \bigcup_{B \in \mathscr{B}} B$.

On the space of continuous functions over M with values in a normed space $(Y, \|\cdot\|)$, $\mathcal{C}(M, Y)$, the uniform convergence over the elements of $\mathscr{B}$ is posed as follows: a net $(f_\alpha) \subseteq \mathcal{C}(M, Y)$ converges $\mathscr{B}$-uniformly to a function $f \in \mathcal{C}(M, Y)$ if

$$\sup_{x \in B} \|f_\alpha(x) - f(x)\| \to 0, \quad \forall B \in \mathscr{B}.$$

This type of convergence induces a topology $\tau(\mathscr{B})$ which, unless $\mathscr{B}$ is a sub-bornology of the Hadamard bornology $\mathscr{H}$ (which is given by all relatively compact sets of (M, d)), fails to be a locally convex topology in the usual sense. Nevertheless, if we consider the family

$$\mathcal{P} = \{\rho_B \ : \ B \in \mathscr{B}\},$$

where $\rho_B(f) = \sup_{x \in B} \|f(x)\|$, then the functions ρ_B are extended seminorms, and so $(\mathcal{C}(M, Y), \tau(\mathscr{B}))$ is an extended seminormed space. For further information on bornologies, we refer the reader to [13].

Usual functional analysis doesn't allow us to fully understand this kind of structures, and therefore it is necessary to expand the theory. This is our main goal: We will provide a full picture of extended structures giving a unifying framework to work with, from both a functional analysis perspective as well as a topological one. It is worth mentioning that topologies induced by functionals that can take the value $+\infty$ have been previously studied by G. Beer and M. J. Hoffman in [1,3]. Toghether with [2,4,5], these papers contain the seminal ideas that motivated this work.

The paper is structured as follows: Section 2 gives all necessary preliminaries, specifically about linear projections, and fixes some notation. In Section 3, suitable notions of *extended topological vector spaces* and *extended locally convex spaces* are introduced and fully studied. This section provides the fundamental topological structure of extended seminormed spaces, since it will be proven that they coincide with the extended locally convex spaces.

Section 4 studies the special properties of esns. Also, in Subsection 4.2, the particular case of esns for which the topology is given by countable many extended seminorms is studied. The work ends with Subsection 4.3, where the main question of the paper is posed: When can an extended seminormed space be split into its finite space and a topological complement?

In what follows, we will assume that the reader is familiar with the basics in topological groups, including topological fields. We refer to [8, Ch. III] for further information on these topics. The notation and the definitions are based mainly on [8] and [15].

2. Preliminaries and notation

First, throughout this paper, for any topological space (T,τ) and any subset S of T, we will simply write (S,τ) to denote the topological subspace S endowed with the induced topology from τ. If the notation is ambiguous, we will write $\tau\big|_S$ to specify the induced topology. For a point $t_0 \in T$, we will denote the family of neighborhoods of t_0 as $\mathcal{N}_T(t_0,\tau)$. If there is no ambiguity, we will omit the space, the topology or both writing simply $\mathcal{N}_T(t_0), \mathcal{N}(t_0,\tau)$ or $\mathcal{N}(t_0)$, respectively.

For a family $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ of topologies on the same set T, we will denote by $\bigvee_{\alpha\in\mathscr{A}} \tau_\alpha$ the topology generated by $\bigcup_{\alpha\in\mathscr{A}} \tau_\alpha$, namely the coarsest topology on T containing each τ_α. We say that the family $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ is *directed by inclusion* if for every two elements $\alpha, \beta \in \mathscr{A}$ there exists a third one $\gamma \in \mathscr{A}$ such that $\tau_\alpha \cup \tau_\beta \subseteq \tau_\gamma$.

The following proposition is well-known in the literature (see, e.g., [8, Ch. II] and [10, Ch. III]) and we will use it many times during this work.

Proposition 2.1. *Let $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ be a family of topologies on X directed by inclusion. Denote $\tau := \bigvee_{\alpha\in\mathscr{A}} \tau_\alpha$. Then*

(a) *$\bigcup_{\alpha\in\mathscr{A}} \tau_\alpha$ is a basis of the topology τ; that is, every element of τ can be written as an union of elements in $\bigcup_{\alpha\in\mathscr{A}} \tau_\alpha$.*
(b) *For each $t_0 \in T$, $\mathfrak{B}_{t_0} = \bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(t_0,\tau_\alpha)$ is a fundamental system (of neighborhoods) of $\mathcal{N}(t_0,\tau)$; that is, for every $V \in \mathcal{N}_X(t_0,\tau)$, there exist $\alpha \in \mathscr{A}$ and $V_\alpha \in \mathcal{N}(t_0,\tau_\alpha)$ such that $V_\alpha \subseteq V$.*

Also, G and $\mathbb{K}$ will always denote an abelian group and a field, respectively. For G, we will always denote by $+$ its group law and by 0_G its identity element. Respectively, for $\mathbb{K}$ we will always denote by $+$ its first law, $\cdot$ its second law, $0_\mathbb{K}$ its zero and $1_\mathbb{K}$ its unit. We denote by $\mathrm{End}(G)$ the set of endomorphisms of G, and by $\mathrm{Aut}(G)$ the set of automorphisms of G. If two groups G and H are algebraically isomorphic (namely, there exists a bijective homomorphism $f : G \to H$), we will write $G \approx H$.

Recall that a topological space (G,τ) is said to be a *topological group* if G is a group and the mappings $(g,h) \in G\times G \mapsto g+h \in G$ and $g \in G \mapsto -g \in G$ are continuous. In such a case τ is said to be a *group* *topology* on G. Two topological groups (G,τ_G) and (H,τ_H) are *topologically isomorphic* if there exists a bijective homomorphism $f: G \to H$ which is bicontinuous. In such a case, we will write $(G,\tau_G) \cong (H,\tau_H)$ (or simply $G \cong H$ if there is no confusion).

Recall that a topological space (T,τ) is *connected* if T cannot be written as a disjoint union of two nonempty open sets. Also, a nonempty subset S of T is connected if it is connected as a topological subspace of (T,τ). The space (T,τ) is said to be *locally connected*, if each point $t \in T$ has a fundamental system of connected neighborhoods. For $t_0 \in T$ we will denote the *connected component* of t_0 by $C[t_0,\tau]$ (or simply $C[t_0]$ if there is no ambiguity). An easy observation is that a topological group (G,τ) is locally connected if and only if 0_G has a fundamental system of connected neighborhoods.

In the sequel, X will stand for a vector space over $\mathbb{K}$ and θ will be a field topology on $\mathbb{K}$. For $k \in \mathbb{K}$, we denote by φ_k the algebraic endomorphism induced by k, namely

$$\begin{aligned} \varphi_k : X &\to X \\ x &\mapsto kx. \end{aligned} \tag{1}$$

The following definition contains the primal structures that we will work with: Topological groups with operators and topological vector spaces. The second notion is well known and the first one can be found in [8, Ch. III, §6]. We present them as a definition since we will use them many times in the following sections.

Definition 2.2 *(Topological group with operators and topological vector space).* Given a group topology τ on X and a field topology θ on $\mathbb{K}$, we say that

1. (X,τ) is a **topological group with operators** in $\mathbb{K}$ if for each $k \in \mathbb{K}$, the function φ_k is τ-τ-continuous;
2. (X,τ) is a **topological vector space** (**tvs**, for short) over $\mathbb{K}$ if the scalar multiplication

$$\begin{aligned} \cdot : \mathbb{K} \times X &\to X \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

   is $(\theta \times \tau)$-τ-continuous.

For two subspaces M, N of X, and a group topology τ over X, we say that M and N are *algebraic complements* if $X = M \oplus N$. In such a case, for $x \in X$ we will denote by $P_{M,N}(x)$ and $P_{N,M}(x)$ the unique elements of M and N respectively such that $x = P_{M,N}(x) + P_{N,M}(x)$. The mappings $P_{M,N}, P_{N,M} : X \to X$, which are linear and idempotent, are called *parallel projections*. We will say that M and N are *τ-supplements* if they are algebraic complements and $P_{M,N}$ and $P_{N,M}$ are τ-τ-continuous. In such a case, we will write $X = M \oplus_\tau N$. In some contexts, it will be useful to denote the mapping from X to M which maps $x \mapsto P_{M,N}(x)$ also as $P_{M,N}$. If there is any confusion we will specify which of those functions we refer to.

Definition 2.3 *(Complement spaces w.r.t. a Hamel basis).* Let X be a vector space over a field $\mathbb{K}$ and let $\mathcal{H}$ be a Hamel basis of X. For a subspace Z of X we will say that $\mathcal{H}$ **generates** Z if $Z \cap \mathcal{H}$ is a Hamel basis of Z.

If Z is a subspace of X generated by $\mathcal{H}$ we will denote by $\mathscr{C}_{\mathcal{H}}(Z) := \operatorname{span}\{\mathcal{H} \setminus Z\}$, which is a complement subspace of Z in X, namely a subspace of X such that

$$X = Z \oplus \mathscr{C}_{\mathcal{H}}(Z). \tag{2}$$

In such a case, $P_{Z,\mathcal{H}}$ will stand for the parallel projection of X to Z (parallel to $\mathscr{C}_{\mathcal{H}}(Z)$).

Fixing M as a subspace of X, we will denote by X/M the quotient group, by π_M (or simply π, if there is no ambiguity) the canonical quotient map, and by $\pi_M(\tau)$ the quotient topology. The equivalence class of $x \in X$ is denoted by $[x]_M$, or simply $[x]$ is there is no confusion. We will say that $\ell : X/M \to X$ is a *lifting*, if $\pi_M \circ \ell = \mathrm{id}_{X/M}$.

Recall that a mapping $P : X \to X$ is called a *linear projection* if it is linear and idempotent. We end this section with some simple results on projections, many of which are contained in [8, Ch. III, §6] (written in different words) and are well known in the literature. Nevertheless we will sketch the arguments of some of them again in order to fix a common context and make the reading of this article easier.

First, since $P_{M,N} + P_{N,M} = \mathrm{id}_X$, we have the following proposition.

Proposition 2.4. *Let τ be a group topology over X and M, N be two complement subspaces in X. The following assertions are equivalent:*

(i) *$P_{M,N}$ is τ-τ-continuous.*
(ii) *$P_{N,M}$ is τ-τ-continuous.*
(iii) *M and N are τ-supplement subspaces in X.*
(iv) *The mapping*

$$\begin{aligned} + : \left(M \times N, \tau\big|_M \times \tau\big|_N\right) &\to (X, \tau) \\ (m, n) &\mapsto m + n \end{aligned}$$

is a topological isomorphism, where its inverse is given by the mapping $D : x \mapsto (P_{M,N}(x), P_{N,M}(x))$.

In the above proposition, taking $M = P(X)$ and $N = \mathrm{Ker}(P)$ for a linear projection $P : X \to X$, we obtain the following corollary:

Corollary 2.5. *Let $P : X \to X$ be a linear projection. We have that P is τ-τ-continuous if and only if $P(X)$ and $\mathrm{Ker}(P)$ are τ-supplements.*

Proposition 2.6. *Let (M_1, N_1) and (M_2, N_2) be two pairs of τ-supplement subspaces in X. Assume that there exists a Hamel basis $\mathcal{H}$ of X such that M_1, M_2, N_1 and N_2 are generated by subsets of $\mathcal{H}$. Then, $M = M_1 \cap M_2$ and $N = \mathrm{span}\{N_1 \cup N_2\}$ are also τ-supplement subspaces in X.*

Proof. Let $\mathcal{H} = \{b_j \ : \ j \in \Delta\}$ and define the sets $\Delta_{0,i}, \Delta_{1,i} \subset \Delta$ (with $i = 1, 2$) such that

$$\mathrm{span}\{b_j \ : \ j \in \Delta_{0,i}\} = M_i$$

and $\Delta_{1,i} = \Delta \setminus \Delta_{0,i}$ (thus, $\mathrm{span}\{b_j \ : \ j \in \Delta_{1,i}\} = N_i$). Defining $\Delta_0 = \Delta_{0,1} \cap \Delta_{0,2}$ and $\Delta_1 = \Delta \setminus \Delta_0$, it is easy to see that

$$M = \mathrm{span}\{b_j \ : \ j \in \Delta_0\} \quad \text{and} \quad N = \mathrm{span}\{b_j \ : \ j \in \Delta_{1,1} \cup \Delta_{1,2}\} = \mathrm{span}\{b_j \ : \ j \in \Delta_1\},$$

with the convention $\mathrm{span}\{\emptyset\} = \{0\}$. On the other hand, noting that $P_{M_i,N_i} = P_{M_i,\mathcal{H}}$ (with $i = 1, 2$) we have that

$$P_{M,N} = P_{M,\mathcal{H}} = P_{M_1,\mathcal{H}} \circ P_{M_2,\mathcal{H}},$$

and so, $P_{M,N}$ is τ-τ-continuous. The conclusion follows from Proposition 2.4. □

$$\begin{tikzcd}
M \times N \arrow[r, "+"'] \arrow[d, "P_N"'] & X \arrow[l, bend right, "D"'] \arrow[d, "\pi_M"] \arrow[dl, "P_{N,M}"] \\
N \arrow[r, "h=\pi_M \circ \hat{\iota}_N"] & X/M \arrow[l, bend left, "h^{-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times N \arrow[r, "+"'] & X \arrow[l, bend right, "D"'] \\
N \arrow[u, "\beta_N"] \arrow[ur, "\hat{\iota}_N"] \arrow[r, "h=\pi_M \circ \hat{\iota}_N"] & X/M \arrow[u, "\ell"'] \arrow[l, bend left, "h^{-1}"]
\end{tikzcd}$$

Fig. 1. Parallel diagram of (M, N).

When M and N are two complement spaces in X, it is well known that if we define $\hat{\iota}_N : N \to X$ as the canonical injection mapping, $P_N : M \times N \to N$ as the mapping given by $P_N(m, n) = n$, $D : X \to M \times N$ as the mapping given by $D(x) = (P_{M,N}(x), P_{N,M}(x))$ (which is the inverse of $+$) and $\beta : N \to M \times N$ as the mapping given by $\beta_N(n) = (0, n)$, the two diagrams in Fig. 1 commute, where $\ell : X/M \to X$ is the linear lifting such that for any $[x] \in X/M$, $\ell([x])$ is the unique element in $N \cap [x]$. In particular, $\pi_M\big|_N$ is an isomorphism, and its inverse is given by

$$\left(\pi_M\big|_N\right)^{-1} = P_N \circ D \circ \ell.$$

In the following, these two diagrams will be called the *parallel diagram of* (M, N) and the lifting ℓ just defined in it will be called the *parallel lifting* and will be denoted by $\ell_{M,N}$ or simply by ℓ_N if there is no ambiguity.

In these diagrams, if we endow X with a group topology τ, X/M with the quotient topology $\pi_M(\tau)$, $M \times N$ with the product topology $\tau\big|_M \times \tau\big|_N$ and N with the induced topology $\tau\big|_N$, then we get that the functions $+, P_N, \beta_N, \pi_M, \hat{\iota}_N$ and h are continuous.

Proposition 2.7. *Let M and N be two complement subspaces in X. The following assertions are equivalent:*

(i) *M and N are τ-supplements.*
(ii) *In the parallel diagram of (M, N), $h = \hat{\iota}_N \circ \pi_M$ is a topological isomorphism.*
(iii) *The parallel lifting $\ell_N : (X/M, \pi(\tau)) \to (X, \tau)$ is continuous.*

Proof.

(i)⇒(ii): If M and N are τ-supplements, then $+$ is a topological isomorphism in the parallel diagram of (M, N), and therefore D is continuous. We only need to prove that h is open. Let then $A \subset N$ be an open set. It is easy to see that

$$h(A) = \{[n] \ : \ n \in A\} = \pi_M(A + M).$$

On the other hand, denoting by $s : M \times N \to X$ the sum mapping $+$ in the parallel diagram, we see that $A + M = s \circ P_N^{-1}(A)$, and so, it is open. Since π_M is an open mapping, we conclude that $h(A)$ is open, which proves the implication.

(ii)⇒(iii): From the parallel diagram of (M, N), we know that

$$\ell_N = s \circ \beta_N \circ h^{-1},$$

and since h^{-1} is continuous, the conclusion follows.

(iii)⇒(i): It is easy to see that $P_{N,M} = \ell_N \circ \pi_M$. Since ℓ_N is continuous, the conclusion follows from Proposition 2.4. □

Corollary 2.8. *Let M be a subspace of X. Then, the following assertions are equivalent:*

(i) *There exists a subspace N of X such that M and N are τ-supplements.*
(ii) *There exists a continuous linear lifting $\ell : X/M \to X$.*
(iii) *There exists a continuous projection $P : X \to X$ such that $P(X) = M$.*

Proof. $(i) \Rightarrow (ii)$ follows from Proposition 2.7, and $(iii) \Rightarrow (i)$ is Corollary 2.5. Finally, to prove $(ii) \Rightarrow (iii)$ we take $P = \mathrm{id}_X - \ell \circ \pi_M$. Noting that

$$(\ell \circ \pi_M) \circ (\ell \circ \pi_M) = \ell \circ (\pi_M \circ \ell) \circ \pi_M = \ell \circ \pi_M,$$

we conclude that P is idempotent, and therefore it is a continuous projection with $P(X) = \mathrm{Ker}(\ell \circ \pi_M) = \mathrm{Ker}(\pi_M)$, where the last equality follows from the fact that every lifting is injective. Since $\mathrm{Ker}(\pi_M) = M$, the proof is finished. □

3. Extended topological vector spaces

A simple but central observation of Beer in [2] is that every extended normed space $(X, \|\cdot\|)$ can be decomposed algebraically as

$$X = X_{\mathrm{fin}} \oplus N,$$

where $X_{\mathrm{fin}} = \{x \in X \ : \|x\| < +\infty\}$. Clearly, $(X_{\mathrm{fin}}, \|\cdot\|)$ is a normed vector space and $(N, \|\cdot\|)$ is a discrete extended normed space, where the discrete extended norm $\|\cdot\|_d$ is given by

$$\|m\|_d = \begin{cases} 0 & \text{if } m = 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

In fact, provided by the simple fact that X_{fin} is an open subspace, it is easy to prove that such decomposition is also topological (see [2, Theorem 3.13]). In order to carry the notion of "extended structure" to the more general context, we will use this kind of decomposition.

3.1. Fundamental etvs

Definition 3.1. Let X be a vector space over a field $\mathbb{K}$, τ be a group topology over X and θ be a field topology over $\mathbb{K}$. We say that (X, τ) is a **fundamental extended topological vector space** (*fundamental etvs*, for short) if there exists an algebraic decomposition $X = M \oplus N$ satisfying

(A1) $\left(M, \tau|_M\right)$ is a tvs over $(\mathbb{K}, \theta)$;
(A2) $\left(N, \tau|_N\right)$ is a discrete space;
(A3) (X, τ) is topologically isomorphic to $\left(M, \tau|_M\right) \times \left(N, \tau|_N\right)$.

In such a case, we will say that (M, N) is a **$(\mathbb{K}, \theta)$-compatible decomposition** (or simply a compatible decomposition, if there is no confusion) of X.

In the above definition, if $(\mathbb{K},\theta)=(\mathbb{R},|\cdot|)$ or $(\mathbb{C},|\cdot|)$ and $\left(M,\tau\big|_M\right)$ is a locally convex space (lcs, for short) over $(\mathbb{K},\theta)$ we will rather say that (X,τ) is a **fundamental extended locally convex space** (*fundamental elcs*, for short) over $(\mathbb{K},\theta)$. This particular object will be further studied in Section 4.

The first natural question about fundamental etvs is whether there exists a unique compatible decomposition, in the sense that the space M in Definition 3.1 is unique (which clearly is the case of extended normed spaces since M must be X_{fin}). To answer this question, we will need some lemmas.

Lemma 3.2. *Let (X,τ) be a fundamental etvs over $(\mathbb{K},\theta)$. Then (X,τ) is a topological group with operators in $\mathbb{K}$.*

Proof. Let (M,N) be a compatible decomposition and let $\alpha\in\mathbb{K}$. From Definition 2.2, we need to prove that φ_α is continuous, where φ_α is the endomorphism given in equation (1). Since (M,τ) is a tvs over $(\mathbb{K},\theta)$, we get that $\varphi_\alpha\big|_M$ is continuous. Also, since (N,τ) is a discrete space, $\varphi_\alpha\big|_N$ is continuous. Finally, we can write

$$\varphi_\alpha=\left(\varphi_\alpha\big|_M\circ P_{M,N}\right)+\left(\varphi_\alpha\big|_N\circ P_{N,M}\right),$$

and since $P_{M,N}$ and $P_{N,M}$ are both continuous, φ_α is continuous, which finishes the proof. □

Lemma 3.3. *Let τ be a group topology over X and θ be a field topology over $\mathbb{K}$. Assume that (X,τ) is a topological group with operators in $\mathbb{K}$. Then, $C[0_X]$ (the connected component of 0_X) is a vector space.*

Moreover, if (X,τ) is a fundamental etvs over $(\mathbb{K},\theta)$, then for every compatible decomposition (M,N), $C[0_X]\subseteq M$.

Proof. For the first part, let $z\in C[0_X]$ and $\alpha\in\mathbb{K}$ with $\alpha\neq 0_{\mathbb{K}}$. Since $C[0_X]$ is already a subgroup of X (see e.g. [8, Ch. III, §2, Proposition 8]), we only need to prove that $\alpha z\in C[0_X]$. Let us consider then the set

$$C:=\alpha\cdot C[0_X].$$

Since the induced endomorphism φ_α (see equation (1)) is τ-continuous, the set C has to be connected. Also, since $0_X\in C$, we get that $C\subseteq C[0_X]$. Then,

$$\alpha z\in C\subseteq C[0_X],$$

which is what we wanted to prove. Now, let us prove that $C[0_X]\subseteq M$. Since M and N are τ-supplements, we have that $N\cong X/M$, and so X/M is discrete. Therefore, the connected component of $0_{X/M}$ in X/M for the quotient topology is the singleton $\{0_{X/M}\}$. Finally, since $\pi: X\to X/M$ is continuous, it maps connected sets into connected sets and so $\pi(C[0_X])\subseteq\{0_{X/M}\}$. Thus,

$$C[0_X]\subseteq\pi^{-1}(\{0_{X/M}\})=M,$$

which finishes the proof. □

Lemma 3.4. *Let (X,τ) be a topological group with operators in $\mathbb{K}$, Z be any open subspace of X and $\mathcal{H}$ be a Hamel basis which generates Z. Then,*

1. *$(\mathscr{C}_{\mathcal{H}}(Z),\tau)\cong(X/Z,\pi_Z(\tau))$ where the topological isomorphism is $\pi_Z\big|_{\mathscr{C}_{\mathcal{H}}(Z)}$.*
2. *$P_{\mathscr{C}_{\mathcal{H}}(Z),Z}=\ell_{\mathscr{C}_{\mathcal{H}}(Z)}\circ\pi_Z$, where $\ell_{\mathcal{C}_{\mathcal{H}}(Z)}$ is the parallel lifting of the decomposition $X=Z\oplus\mathcal{C}_{\mathcal{H}}(Z)$ (see Fig. 1).*

3. *Z and $\mathscr{C}_{\mathcal{H}}(Z)$ are τ-supplements.*

Moreover, if (X,τ) is a fundamental etvs over $(\mathbb{K},\theta)$ and (M,N) is a compatible decomposition, then, each subspace Z of X such that $M \subseteq Z$ is an open set of (X,τ).

In particular, for each subspace N' of X such that $X = M \oplus N'$, (M,N') is also a compatible decomposition (and therefore, $N' \cong N$).

Proof. For the first part, let Z be an open subspace of X. Applying again [8, Ch. III, §2, Proposition 18], X/Z is a discrete space. Let then $\mathcal{H}$ be a Hamel basis which generates Z. Since X/Z is a discrete space, the parallel lifting $\ell_{\mathscr{C}_{\mathcal{H}}(Z)} : X/Z \to X$ is continuous and therefore, by Proposition 2.7, the proof is finished.

For the second part, since M and N are τ-supplements, we have that $N \cong X/M$. Therefore, X/M endowed with the quotient topology is a discrete space. Applying [8, Ch. III, §2, Proposition 18], we get that M is an open set. Then, for each subspace $Z \supseteq M$ we can write

$$Z = \bigcup_{z\in Z} z + M,$$

and therefore, Z is an open set. □

Proposition 3.5. *Let (X,τ) be a fundamental etvs over $(\mathbb{K},\theta)$ and (M,N) be a compatible decomposition. Then:*

1. *If $(\mathbb{K},\theta)$ is non-discrete, then (M,N) is the unique compatible decomposition, in the sense that if (M',N') is another compatible decomposition, then $M = M'$ and $N \cong N'$.*
2. *If θ is the discrete topology, then for each subspace Z of X containing M and each Hamel basis $\mathcal{H}$ which generates Z, $(Z, \mathscr{C}_{\mathcal{H}}(Z))$ is a compatible decomposition. In particular, (X,τ) is a tvs over $(\mathbb{K},\theta)$.*

Proof.

1. Let be (M',N') be another compatible decomposition and $\mathcal{H}$ be a Hamel basis which generates the subspaces M, M' and $M'' = M \cap M'$ (it is not hard to see that such a basis exists). Provided by Lemma 3.4, we may replace N and N' by $\mathscr{C}_{\mathcal{H}}(M)$ and $\mathscr{C}_{\mathcal{H}}(M')$ respectively, and therefore we can assume without losing generality that $\mathcal{H}$ also generates N and N'. Thus, if we define $N'' = \operatorname{span}\{N \cup N'\}$, we can apply Proposition 2.6 to conclude that M'' and N'' are τ-supplements.
We will prove first that (N'',τ) is discrete. Observe that, since $\mathcal{H}$ generates $\operatorname{span}\{M \cup M'\}$, then it also generates the subspace $N \cap N'$. Thus, we can write
$$P_{N\cap N',\mathcal{H}} = P_{N,\mathcal{H}} \circ P_{N',\mathcal{H}},$$
and also, we have the equality
$$P_{N'',\mathcal{H}} = P_{N,\mathcal{H}} + P_{N',\mathcal{H}} - P_{N\cap N',\mathcal{H}}.$$
Now, let $(n_\lambda)_{\lambda\in\Lambda}$ be a net in N'' converging to $n \in N''$. Since (N,τ), (N',τ) and $(N\cap N',\tau)$ are discrete spaces and all three parallel projections with respect to $\mathcal{H}$ are continuous, we conclude that there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \geq \lambda_0$, $P_{N,\mathcal{H}}(n_\lambda) = P_{N,\mathcal{H}}(n)$, $P_{N',\mathcal{H}}(n_\lambda) = P_{N',\mathcal{H}}(n)$ and $P_{N\cap N',\mathcal{H}}(n_\lambda) = P_{N\cap N',\mathcal{H}}(n)$. Thus, for each $\lambda \geq \lambda_0$
$$\begin{aligned} n_\lambda &= P_{N'',\mathcal{H}}(n_\lambda) \\ &= P_{N,\mathcal{H}}(n_\lambda) + P_{N',\mathcal{H}}(n_\lambda) - P_{N\cap N',\mathcal{H}}(n_\lambda) \end{aligned}$$

$$
\begin{aligned}
&= P_{N,\mathcal{H}}(n) + P_{N',\mathcal{H}}(n) - P_{N\cap N',\mathcal{H}}(n) \\
&= P_{N'',\mathcal{H}}(n) = n.
\end{aligned}
$$

Then, all converging nets in (N'', τ) are stationary, and so the space is discrete. We conclude that (M'', N'') is also a compatible decomposition by just observing that (M'', τ) is a tvs since M'' is a subspace of M.

We will prove now that $M = M''$ which, by symmetry, will imply that $M = M'$, finishing the proof. Let $\mathcal{H}_0$ be the subset of $\mathcal{H}$ which generates M. Then, if we consider the topological group (M, τ), we have that (M'', N_0'') is a compatible decomposition of M which makes it a fundamental etvs over $(\mathbb{K}, \theta)$ (considering $N_0'' = \mathscr{C}_{\mathcal{H}_0}(M'') = N'' \cap M$). Let us suppose that there exists $n \in N_0''$ with $n \neq 0$. Since $(\mathbb{K}, \theta)$ is non-discrete, there exists $\alpha \in \mathbb{K} \setminus \{0\}$ such that $\{\alpha\}$ is not an open set. On the other hand, since N_0'' is a discrete space, $\{\alpha n\}$ is open in N_0''. Let (α_V) be a net in $\mathbb{K}$ indexed by the neighborhoods $V \in \mathcal{N}(\alpha, \theta)$ such that

$$\forall V \in \mathcal{N}(\alpha, \theta),\ \alpha_V \in V \setminus \{\alpha\}.$$

Since $\{\alpha\}$ is not open, such a net exists, and even more $\alpha_V \to \alpha$. Now, since $\alpha_V - \alpha \neq 0_{\mathbb{K}}$, we have that the map $m \mapsto (\alpha_V - \alpha)m$ is an automorphism of M, and therefore $\alpha_V n \neq \alpha n$. This implies that $\alpha_V n \not\to \alpha n$ in N_0''. But noting that $(\alpha_V, n) \to (\alpha, n)$ in $\mathbb{K} \times N_0''$, we conclude that (N_0'', τ) is not a tvs over $(\mathbb{K}, \theta)$. This last statement contradicts the fact that M is tvs, since N_0'' is a subspace of M and each subspace of a tvs over $(\mathbb{K}, \theta)$ endowed with the induced topology has to be also a tvs over $(\mathbb{K}, \theta)$. Therefore, $N_0'' = \{0_M\}$ and so, $M'' = M$, which is what we wanted to prove.

2. Assume that (M, N) is a compatible decomposition and that Z is a subspace of X with $M \subseteq Z$. Without losing generality we may assume that N is such that there exists a Hamel basis $\mathcal{H}$ which generates M, N and Z. Then, since $\mathscr{C}_{\mathcal{H}}(Z)$ is a subspace of N, we get that $(\mathscr{C}_{\mathcal{H}}(Z), \tau)$ is a discrete space. On the other hand, let $(\alpha_\lambda, z_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathbb{K} \times Z$ converging to $(\alpha, z) \in \mathbb{K} \times Z$. Since $(\mathbb{K}, \theta)$ is discrete, there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \geq \lambda_0$, $\alpha_\lambda = \alpha$. Then, applying Lemma 3.2, we get that for $\lambda \geq \lambda_0$

$$\alpha_\lambda z_\lambda = \alpha z_\lambda \to \alpha z,$$

and therefore, (Z, τ) is a tvs over $(\mathbb{K}, \theta)$. Finally, applying Lemma 3.4, we conclude that Z and $\mathscr{C}_{\mathcal{H}}(Z)$ are τ-supplements. Then, $(Z, \mathscr{C}_{\mathcal{H}}(Z))$ is a compatible decomposition, which finishes the proof. □

Remark 1. When a vector space X over a field $\mathbb{K}$ is endowed with a group topology τ such that (X, τ) is a topological group with operators in $\mathbb{K}$, it is easy to realize that (X, τ) is in fact a tvs over $(\mathbb{K}, \theta)$ when θ is the discrete topology. Such kinds of spaces have been already introduced in the literature by the name of *topological vector groups* (see [14]) and further developed during the 70's. Therefore, we won't work on those objects and so we will always assume that $(\mathbb{K}, \theta)$ is a non-discrete field.

Corollary 3.6. *Let (X, τ) be an etvs over a non-discrete field $(\mathbb{K}, \theta)$ and (M, N) be a compatible decomposition. Then*

$$M = \bigcap \{Z \ : \ Z \textit{ is a } \tau\textit{-open subspace of } X\}.$$

Proof. Let us denote $M' = \bigcap\{Z \ : \ Z \text{ is a } \tau\text{-open subspace of } X\}$. By Lemma 3.4, we get that $M \supseteq M'$. Suppose that the latter inclusion is strict. Then, there exists an open subspace Z such that $Z \not\supseteq M$. Replacing Z by $Z \cap M$, we may assume that $Z \subsetneq M$, and therefore, (Z, τ) is a tvs over $(\mathbb{K}, \theta)$. Applying again Lemma 3.4, we get that for any Hamel basis which generates Z, the pair $(Z, \mathscr{C}_{\mathcal{H}}(Z))$ is a compatible decomposition. Then, by Proposition 3.5, $M = Z$, which is a contradiction. We conclude that $M = M'$, finishing the proof. □

Observe that in the case of an extended normed space $(X, \|\cdot\|)$, the identification

$$X_{\text{fin}} = \bigcap\{Z \;:\; Z \text{ is a } \|\cdot\|\text{-open subspace of } X\},$$

is somehow direct, since X_{fin} is itself open and it doesn't contain any open proper subspaces provided that it is a connected subspace. Therefore, to adopt the notation of [2], for every vector space X endowed with a group topology τ, we will define its **finite space** as

$$X_{\text{fin}} = \bigcap\{Z \;:\; Z \text{ is a } \tau\text{-open subspace of } X\}. \tag{4}$$

Remark 2. Note that, provided by Corollary 3.6 and Lemma 3.4, we get that, if $(\mathbb{K}, \theta)$ is a non-discrete topological field, then (X, τ) is a fundamental etvs over $(\mathbb{K}, \theta)$ if and only if (X_{fin}, τ) is a tvs over $(\mathbb{K}, \theta)$ and X_{fin} is τ-open in X. Note also that, since every open subgroup of a topological group is also a closed subgroup (see [8, Ch. III, §2, Corollary of Proposition 4]), we have that X_{fin} is always τ-closed.

In the general case, the finite space of a fundamental etvs may not coincide with the connected component of zero. A counterexample arrives easily by just considering the vector space $X = \mathbb{Q}^2$ over the topological field $(\mathbb{Q}, |\cdot|)$ (where $\mathbb{Q}$ denotes the group of rational numbers) and the group topology τ over $\mathbb{Q}^2$ induced by the function

$$\begin{aligned} \rho : \mathbb{Q}^2 &\to [0, +\infty] \\ (q_1, q_2) &\mapsto \rho(q_1, q_2) = \begin{cases} |q_1| & \text{if } q_2 = 0 \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Here, the finite space is $X_{\text{fin}} = \mathbb{Q} \times \{0\}$, but $C[0_X] = \{(0, 0)\}$. This motivates the following definition.

Definition 3.7. Let (X, τ) be a fundamental etvs over a non-discrete topological field $(\mathbb{K}, \theta)$. We will say that (X, τ) is a **proper fundamental etvs** over $(\mathbb{K}, \theta)$ if $X_{\text{fin}} = C[0_X, \tau]$.

Since every tvs over $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$ has to be connected, we have the following corollary:

Corollary 3.8. *If (X, τ) is a fundamental etvs over $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$, then it is proper.*

3.2. Structure of etvs

Knowing the definition of fundamental etvs, we are ready to introduce what will be the most suitable notion of extended topological vector space. The main utility of such a definition will be fully appreciated in Theorem 4.3, which gives a topological characterization of esns in terms of what we will call "extended locally convex spaces". For the sake of deductive order of the paper, we will come back to this subject in Section 4 and in this section we will restrict our attention to the study of extended topological vector spaces.

Definition 3.9. Let X be a vector space over $\mathbb{K}$, τ be a group topology on X and θ be a non-discrete field topology on $\mathbb{K}$. We say that (X, τ) is an **extended topological vector space** (*etvs*, for short) if there exists a family $\{\tau_\alpha \;:\; \alpha \in \mathscr{A}\}$ of group topologies on X such that

1. $\tau = \bigvee_{\alpha \in \mathscr{A}} \tau_\alpha$.

2. For each $\alpha \in \mathscr{A}$, (X, τ_α) is a fundamental etvs over $(\mathbb{K}, \theta)$.

In such a case, the family $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ is called a **generating family** of τ. As before, the **finite space** X_{fin} of (X, τ) is defined as

$$X_{\text{fin}} = \bigcap \{M \ : \ M \ \tau\text{-open subspace of } X\}$$

and (X, τ) is said to be a **proper etvs** if $X_{\text{fin}} = C[0_X, \tau]$.

Observe first that, as in the case of fundamental etvs, if (X, τ) is an etvs then $C[0_X] \subseteq M$, whenever M is a τ-open subspace. Therefore, the notion of proper etvs is well-defined. Also, if in Definition 3.9 $(\mathbb{K}, \theta) = (\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$, and $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ is a *locally convex generating family* (i.e., (X, τ_α) is a fundamental elcs for each $\alpha \in \mathscr{A}$), then we will say that (X, τ) is an **extended locally convex space** (*elcs*, for short).

Note also that the generating family used in Definition 3.9 can be assumed to be directed by inclusion. Indeed, it is enough to consider instead of $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ the family

$$\left\{ \bigvee_{\alpha \in A} \tau_\alpha \ : \ A \in \mathcal{P}(\mathscr{A}) \text{ with } \operatorname{Card}(A) < \infty \right\},$$

which generates the same topology τ. In what follows, we will simply say that $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ is a *directed generating family*, whenever it is a generating family of τ and it is directed by inclusion.

Finally, recall that we will always assume that $(\mathbb{K}, \theta)$ is a non-discrete topological field.

Proposition 3.10. *Let (X, τ) be an etvs over $(\mathbb{K}, \theta)$. Then:*

1. *For every generating family $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ of τ, we have that*

$$X_{\text{fin}} = \bigcap_{\alpha \in \mathscr{A}} X^\alpha_{\text{fin}},$$

   where, for each $\alpha \in \mathscr{A}$, X^α_{fin} denotes the finite space of (X, τ_α).
2. *(X, τ) is a topological group with operators in $\mathbb{K}$.*
3. *X_{fin} is a tvs over $(\mathbb{K}, \theta)$.*

Proof.

1. Let $\mathscr{F}$ be the directed generating family induced by $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$, namely

$$\mathscr{F} = \left\{ \bigvee_{\alpha \in A} \tau_\alpha \ : \ A \in \mathcal{P}(\mathscr{A}) \text{ with } \operatorname{Card}(A) < \infty \right\}.$$

   Noting that for each $\sigma \in \mathscr{F}$ there exist $\alpha_1, \ldots, \alpha_n \in \mathscr{A}$ such that the finite space of (X, σ) is $X^\sigma_{\text{fin}} = \bigcap_{i=1}^n X^{\alpha_i}_{\text{fin}}$, we can write

$$\bigcap_{\sigma \in \mathscr{F}} X^\sigma_{\text{fin}} = \bigcap_{\alpha \in \mathscr{A}} X^\alpha_{\text{fin}}.$$

   Then, we may assume without loss of generality that $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ is directed by inclusion. Let us denote $Z = \bigcap \{X^\alpha_{\text{fin}} \ : \ \alpha \in \mathscr{A}\}$. It is direct that $Z \supseteq X_{\text{fin}}$. On the other hand let M be a τ-open subspace of X. By Proposition 2.1, there exist $\alpha \in \mathscr{A}$ and $V \in \mathcal{N}(0_X, \tau_\alpha)$ such that $V \subseteq M$. Then, M is τ_α-open by [8, Ch. III, §2, Corollary of Prop. 4]. Since $(\mathbb{K}, \theta)$ is non-discrete there is no τ_α-open proper subspace of X^α_{fin}, and therefore

$$Z \subseteq X^\alpha_{\text{fin}} \subseteq M.$$

By arbitrariness of M, we conclude that $Z \subseteq X_{\text{fin}}$.

2. Let $k \in \mathbb{K}$ and let $\{\tau_\alpha \; : \; \alpha \in \mathscr{A}\}$ be a directed generating family. Fix any $V \in \mathcal{N}(0_X, \tau)$. Since, by Proposition 2.1, $\mathfrak{B} = \bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ is a fundamental system of $\mathcal{N}(0_X, \tau)$, we get that there exist $\alpha \in \mathscr{A}$ and $V_\alpha \in \mathcal{N}(0_X, \tau_\alpha)$ such that $V_\alpha \subseteq V$. Then, since φ_k is τ_α-τ_α-continuous, we get that there exists $U_\alpha \in \mathcal{N}(0_X, \tau_\alpha)$ such that

$$\varphi_k(U_\alpha) \subseteq V_\alpha \subseteq V.$$

Finally, recalling that $U_\alpha \in \mathcal{N}(0_X, \tau)$, we conclude that φ_k is τ-τ-continuous, which finishes the proof.

3. Let $\{\tau_\alpha \; : \; \alpha \in \mathscr{A}\}$ be a directed generating family of τ. Since $\mathfrak{B} = \bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ is a fundamental system of $\mathcal{N}_X(0_X, \tau)$, we may consider

$$\mathfrak{B}_{\text{fin}} = \{V \cap X_{\text{fin}} \; : \; V \in \mathfrak{B}\}$$

as a fundamental system of $\mathcal{N}_{X_{\text{fin}}}(0_X, \tau)$. It is clear that (X_{fin}, τ) is a topological group and so, we only need to verify that the scalar multiplication $\cdot : \mathbb{K} \times X_{\text{fin}} \to X_{\text{fin}}$ is $(\theta \times \tau)$-τ-continuous, according to Definition 2.2. Choose then $V \cap X_{\text{fin}} \in \mathfrak{B}_{\text{fin}}$. There exists $\alpha \in \mathscr{A}$ such that $V \in \mathcal{N}(0_X, \tau_\alpha)$. Without losing generality, we may assume that $V \subseteq X^\alpha_{\text{fin}}$. Therefore, since $(X^\alpha_{\text{fin}}, \tau_\alpha)$ is a tvs over $(\mathbb{K}, \theta)$, there exist $O \in \mathcal{N}_{\mathbb{K}}(0, \theta)$ and $U \in \mathcal{N}(0_X, \tau_\alpha)$ such that $O \cdot U \subseteq V$. Then, since X_{fin} is trivially a vector space over $\mathbb{K}$, we have that $O \cdot (U \cap X_{\text{fin}}) \subseteq V \cap X_{\text{fin}}$ and so, provided by $U \cap X_{\text{fin}} \in \mathcal{N}_{X_{\text{fin}}}(0_X, \tau)$, the proof is complete. □

As a direct consequence of Lemma 3.4 and Proposition 3.10, we get the following corollary:

Corollary 3.11. *Let (X, τ) be an etvs over $(\mathbb{K}, \theta)$. Then, (X, τ) is a fundamental etvs if and only if X_{fin} is τ-open.*

Remark 3. Note that, as in Remark 2, X_{fin} is a τ-closed subspace of X, since every τ-open subspace of X is also τ-closed. Nevertheless, the above corollary shows that it is not necessarily τ-open.

Lemma 3.12. *Let Z be a subspace of X, (X, τ) be a topological group with operators in $\mathbb{K}$ and θ be a non-discrete field topology over $\mathbb{K}$. Then:*

1. *If (X, τ) is a fundamental etvs over $(\mathbb{K}, \theta)$, then (Z, τ) is also a fundamental etvs over $(\mathbb{K}, \theta)$ with $Z_{\text{fin}} = Z \cap X_{\text{fin}}$.*
2. *If (X, τ) is an etvs over $(\mathbb{K}, \theta)$, then (Z, τ) is also a etvs over $(\mathbb{K}, \theta)$ with $Z_{\text{fin}} = Z \cap X_{\text{fin}}$.*

Proof. Observe first that (Z, τ) is already a topological group with operators in $\mathbb{K}$ (see [8, Ch. III, §6]).

1. If (X, τ) is a fundamental etvs, then the space $Z' = X_{\text{fin}} \cap Z$ is a τ-open subspace of Z. Since Z' is also a subspace of X_{fin}, we get that (Z', τ) is a tvs over $(\mathbb{K}, \theta)$. Applying Lemma 3.4 we get that (Z, τ) is a fundamental etvs and (Z', N) is a compatible decomposition, where N is any algebraic complement of Z' in Z. By Proposition 3.5, we conclude that $Z_{\text{fin}} = Z'$, finishing the first part of the proof.

2. Assume now that (X,τ) is an etvs and let $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be a directed generating family of τ. By the first part of the proof, we have that (Z,τ_α) is a fundamental etvs with $Z^\alpha_{\text{fin}} = Z \cap X^\alpha_{\text{fin}}$. On the other hand, since by Proposition 2.1 the set $\bigcup_{\alpha\in\mathscr{A}} \tau_\alpha$ is a basis of the topology τ, it is direct that

$$\tau\big|_Z = \bigvee_{\alpha\in\mathscr{A}} \tau_\alpha\big|_Z,$$

and therefore (Z,τ) is an etvs. Finally, by Proposition 3.10, we have that

$$Z_{\text{fin}} = \bigcap_{\alpha\in\mathscr{A}} Z^\alpha_{\text{fin}} = \bigcap_{\alpha\in\mathscr{A}} (Z \cap X^\alpha_{\text{fin}}) = Z \cap X_{\text{fin}},$$

which finishes the proof. □

Lemma 3.13. *Let (X,τ) be an etvs over $(\mathbb{K},\theta)$. Then X_{fin} is the maximal subspace of X which is a tvs over $(\mathbb{K},\theta)$, namely, for each subspace Z of X such that $X_{\text{fin}} \cap Z \subsetneq Z$, we have that (Z,τ) fails to be a tvs over $(\mathbb{K},\theta)$.*

Proof. Let Z be subspace of X such that $X_{\text{fin}} \cap Z \subsetneq Z$. Suppose that (Z,τ) is a tvs over $(\mathbb{K},\theta)$. By definition of X_{fin}, there exists a τ-open subspace M such that $M' = Z \cap M \subsetneq Z$. Note that M' is an open subspace of (Z,τ). Then, by [8, Ch. III, §2, Proposition 18], we have $(Z/M', \pi(\tau))$ is a discrete space and therefore, since $(\mathbb{K},\theta)$ is not discrete, $(Z/M', \pi(\tau))$ is not a tvs. This last statement contradicts the assumption that (Z,τ) is a tvs, since the quotient space of a tvs endowed with the quotient topology is also a tvs. □

The following theorem will give us a more intuitive characterization of etvs: An extended topological vector space is a "limit" of fundamental etvs. To establish such a theorem we need to recall the suitable notion of limit that we will use.

Motivated by the known notion of projective limits for topological groups and topological vector spaces (see, e.g., [8] and [15]), we will apply the same notion for topological groups with operators in a field.

Recall that for a directed set of indexes $(\mathscr{A},\preceq)$, a *projective system* of topological groups with operators in $\mathbb{K}$, $(X_\alpha,\tau_\alpha,f_{\alpha\beta})_{\alpha\in\mathscr{A}}$, is a family $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\mathscr{A}}$ of topological groups with operators in the same field $\mathbb{K}$ which has an associated family of continuous linear mappings $\{f_{\alpha\beta} : X_\beta \to X_\alpha : \alpha,\beta \in \mathscr{A} \text{ and } \alpha \preceq \beta\}$ such that:

(i) Whenever $\alpha \preceq \beta \preceq \gamma$ in $\mathscr{A}$, $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$.
(ii) For each $\alpha \in \mathscr{A}$, $f_{\alpha\alpha} = \mathrm{id}_{X_\alpha}$.

For a projective system $(X_\alpha,\tau_\alpha,f_{\alpha\beta})_{\alpha\in\mathscr{A}}$, its *projective limit* $X = \varprojlim X_\alpha$ is the vector space

$$\varprojlim X_\alpha := \left\{ (x_\alpha)_{\alpha\in\mathscr{A}} \in \prod_{\alpha\in\mathscr{A}} X_\alpha \ :\ \forall \alpha \preceq \beta,\ f_{\alpha\beta}(x_\beta) = x_\alpha \right\}, \tag{5}$$

endowed with the induced topology of the product topology $\tau = \prod_{\alpha\in\mathscr{A}} \tau_\alpha$. This space is also a topological group with operators in $\mathbb{K}$, since any subspace of the product space is a topological group with operators in $\mathbb{K}$ (see [8, Ch. III, §6]).

Theorem 3.14. *Let $(\mathbb{K},\theta)$ be a non-discrete topological field. A topological group (X,τ) is an etvs over $(\mathbb{K},\theta)$ if and only if there exists a projective system $(X_\alpha,\tau_\alpha,f_{\alpha\beta})_{\alpha\in\mathscr{A}}$ such that*

(a) *for each* $\alpha \in \mathscr{A}$, (X_α, τ_α) *is a fundamental etvs over* $(\mathbb{K}, \theta)$*; and*
(b) (X, τ) *is topologically isomorphic to* $\varprojlim X_\alpha$.

In such a case, we have that X_{fin} *is topologically isomorphic to* $\varprojlim X^\alpha_{\text{fin}}$*, where* X^α_{fin} *is the finite space of* X_α.

Proof.

$\Rightarrow$) Assume that (X, τ) is an etvs over $(\mathbb{K}, \theta)$ and let $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be a directed generating family of τ. We will consider the set of indexes as $\mathscr{A}$, endowed with the order $\preceq$ given by

$$\alpha \preceq \beta \iff \tau_\alpha \subseteq \tau_\beta.$$

Clearly, $(\mathscr{A}, \preceq)$ is a directed set. Now, for each $\alpha \in \mathscr{A}$ we will consider the space $X_\alpha = X$, endowed with the topology τ_α, and for each $\alpha \preceq \beta \in \mathscr{A}$ we will consider the map $f_{\alpha\beta} = \text{id}_X : (X_\beta, \tau_\beta) \to (X_\alpha, \tau_\alpha)$, which is clearly continuous. Let us denote $\tilde{X} = \varprojlim X_\alpha$ and $\tilde{\tau} = \prod_{\alpha\in\mathscr{A}} \tau_\alpha$. It is easy to see that X is algebraically isomorphic to $\tilde{X}$, where the isomorphism is the function $h : X \to \varprojlim X_\alpha$ where $h(x)$ is the constant net (x_α) with $x_\alpha = x$. We will denote this latter net as $\tilde{x}$. We only need to verify that h is bicontinuous. Indeed, consider the fundamental system $\mathfrak{B} = \bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ of $\mathcal{N}_X(0_X, \tau)$ given by Proposition 2.1, and choose any $V \in \mathfrak{B}$. Let $\alpha \in \mathscr{A}$ be the index such that $V \in \mathcal{N}(0_X, \tau_\alpha)$. It is not hard to realize that

$$h(V) = \{\tilde{x} \ : \ x \in V\} = \tilde{X} \cap p_\alpha^{-1}(V),$$

where p_α is the αth canonical projection of the product space, given by $p_\alpha(x) = x_\alpha$. Since h is bijective and

$$\left\{\tilde{X} \cap p_\alpha^{-1}(V) \ : \ \alpha \in \mathscr{A},\ V \in \mathcal{N}(0_X, \tau_\alpha)\right\} \subseteq \mathcal{N}(0_{\tilde{X}}, \tilde{\tau}),$$

we conclude that h is bicontinuous, finishing this part of the proof.

$\Leftarrow$) Let $(X_\alpha, \tau_\alpha, f_{\alpha\beta})_{\alpha\in\mathscr{A}}$ be a projective system of fundamental etvs, such that X is topologically isomorphic to $\tilde{X} = \varprojlim X_\alpha$. We only need to show that $(\tilde{X}, \tilde{\tau})$ is an etvs, where $\tilde{\tau}$ is the product topology restricted to the projective limit. Define then $\tilde{\tau}_\alpha = p_\alpha^{-1}(\tau_\alpha)$, where p_α is the αth canonical projection. Clearly, $\tilde{\tau} = \bigvee_{\alpha\in\mathscr{A}} \tilde{\tau}_\alpha$, so, by Lemma 3.12, it is sufficient to show that $(X, \tilde{\tau}_\alpha)$ is a fundamental etvs, where $X = \prod_\alpha X_\alpha$. Let us consider the space $Z_\alpha = p_\alpha^{-1}(X^\alpha_{\text{fin}})$ as subspace of X. Since p_α is $\tilde{\tau}_\alpha$-τ_α-continuous, we have that Z_α is $\tilde{\tau}_\alpha$-open and therefore, by Lemma 3.4, we only need to show that $(Z_\alpha, \tilde{\tau}_\alpha)$ is a tvs over $(\mathbb{K}, \theta)$. Fix then $\tilde{V} \in \mathcal{N}(0_{Z_\alpha}, \tilde{\tau}_\alpha)$ which, by construction of $\tilde{\tau}_\alpha$, can be written as

$$\tilde{V} = V \times \prod_{\beta\neq\alpha} X_\beta,$$

where $V \in \mathcal{N}(0_{X^\alpha_{\text{fin}}}, \tau_\alpha)$. Since $(X^\alpha_{\text{fin}}, \tau_\alpha)$ is a tvs, there exist a neighborhood $O \in \mathcal{N}(0_\mathbb{K}, \theta)$ and a neighborhood $U \in \mathcal{N}(0_{X^\alpha_{\text{fin}}}, \tau_\alpha)$ such that

$$O \cdot U \subseteq V.$$

Then, is not hard to see that $O \cdot p_\alpha^{-1}(U) \subseteq \tilde{V}$, which finishes the proof since $p_\alpha^{-1}(U) \in \mathcal{N}(0_{Z_\alpha}, \tilde{\tau}_\alpha)$.

It only remains to prove that, whenever conditions (a) and (b) hold, $X_{\text{fin}} \cong \varprojlim X^{\alpha}_{\text{fin}}$. Since it is not hard to realize that $X_{\text{fin}} \cong \tilde{X}_{\text{fin}}$ (where $\tilde{X} = \varprojlim X_{\alpha}$, as before), it is enough to show that $\tilde{X}_{\text{fin}} = \varprojlim X^{\alpha}_{\text{fin}}$. Observe first that, if $x \in \tilde{X}_{\text{fin}}$, then for each $\alpha \in \mathscr{A}$,

$$x \in p_{\alpha}^{-1}(X^{\alpha}_{\text{fin}}) \cap \tilde{X}$$

since the latter is an open subspace of $\tilde{X}$. Therefore $p_{\alpha}(x) \in X^{\alpha}_{\text{fin}}$ which proves that

$$\tilde{X}_{\text{fin}} \subseteq \varprojlim X^{\alpha}_{\text{fin}}.$$

Note now that $\varprojlim X^{\alpha}_{\text{fin}}$ is a subspace of $\tilde{X}$ which, endowed with $\tilde{\tau}$, is a tvs over $(\mathbb{K}, \theta)$ since it is also the projective limit of a projective system of tvs over $(\mathbb{K}, \theta)$. Thus, by Lemma 3.13, $\varprojlim X^{\alpha}_{\text{fin}} \subseteq \tilde{X}_{\text{fin}}$, which finishes the proof. □

Theorem 3.15. *Let $(\mathbb{K}, |\cdot|)$ be a non-discrete Archimedean valuated field and τ be a group topology over X. Then, (X, τ) is an etvs over $(\mathbb{K}, |\cdot|)$ if and only if there exists a fundamental system $\mathfrak{B}$ of neighborhoods of 0_X such that:*

(i) *For each $V \in \mathfrak{B}$, $\mathbb{K} \cdot V$ is a subspace of X.*
(ii) *For each $V \in \mathfrak{B}$, there exists $U \in \mathfrak{B}$ such that $\mathbb{K} \cdot V = \mathbb{K} \cdot U$ and that $U + U \subseteq V$.*
(iii) *Each element $V \in \mathfrak{B}$ is* ***balanced****.*

Proof. First, we will prove the necessity. Assume then that (X, τ) is an etvs over $(\mathbb{K}, |\cdot|)$ and let $\{\tau_{\alpha} : \alpha \in \mathscr{A}\}$ be a directed generating family of τ. Fix $\alpha \in \mathscr{A}$. Since $(X^{\alpha}_{\text{fin}}, \tau_{\alpha})$ is a tvs over $(\mathbb{K}, |\cdot|)$, by [15, Ch. I, 1.2], there exists a fundamental system $\mathfrak{B}_{\alpha}$ of $\mathcal{N}_{X^{\alpha}_{\text{fin}}}(0_X, \tau_{\alpha})$ satisfying (i), (ii) and (iii). Also, since (X, τ_{α}) is a fundamental etvs, $\mathcal{N}_{X^{\alpha}_{\text{fin}}}(0_X, \tau_{\alpha})$ is a fundamental system of $\mathcal{N}(0_X, \tau_{\alpha})$, and therefore so is $\mathfrak{B}_{\alpha}$. Finally, considering the set $\mathfrak{B} = \bigcup_{\alpha \in \mathscr{A}} \mathfrak{B}_{\alpha}$, the conclusion follows, since $\bigcup_{\alpha \in \mathscr{A}} \mathcal{N}(0_X, \tau_{\alpha})$ is a fundamental system of $\mathcal{N}(0_X, \tau)$.

Now, to prove the sufficiency, assume the existence of a fundamental system $\mathfrak{B}$ of neighborhoods of 0_X satisfying (i), (ii) and (iii). We need to construct a generating family $\{\tau_{\alpha} : \alpha \in \mathscr{A}\}$ of τ. To do so, for each $V \in \mathfrak{B}$, we will define the subspace $M_V = \mathbb{K} \cdot V$. Then, we define

$$\mathfrak{B}_V = \{U \in \mathfrak{B} : \mathbb{K} \cdot U = M_V\}.$$

By (ii) and (iii), $\mathfrak{B}_V$ induces a group topology τ_V on X. It is not hard to see that in (M_V, τ_V), conditions (i) and (iii) imply that each element of $\mathfrak{B}_V$ is absorbing and balanced, and that condition (ii) implies that for each element $W \in \mathfrak{B}_V$, there exists $U \in \mathfrak{B}_V$ such that $U + U \subseteq W$. Therefore, applying [15, Ch. I, 1.2], we conclude that (M_V, τ_V) is a tvs over $(\mathbb{K}, |\cdot|)$, and therefore (X, τ_V) is a fundamental etvs over $(\mathbb{K}, |\cdot|)$. The proof is concluded noting that $\tau_V \subseteq \tau$ and the fact that $\tau \subseteq \bigvee_{V \in \mathfrak{B}} \tau_V$, provided by the equality $\mathfrak{B} = \bigcup_{V \in \mathfrak{B}} \mathfrak{B}_V$. □

In [2], many of the results obtained for extended normed spaces came from the fact that the finite space of these is always open or, in our terminology, that every normed space is a fundamental etvs. We would like then to have a similar splitting theorem for etvs. But we will show that even in the most elemental cases, we can find examples of spaces where the finite space is not topologically complementable.

Lemma 3.16. *Let (X, τ) be an etvs over $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$ such that (X_{fin}, τ) is Hausdorff and $\dim[X_{\text{fin}}] < \infty$. Suppose that X_{fin} has a τ-supplement space in X. Then, if $\{\tau_{\alpha} : \alpha \in \mathscr{A}\}$ is any directed generating family of τ, there exists $\alpha \in \mathscr{A}$ such that X_{fin} has a τ_{α}-supplement space in X^{α}_{fin}.*

Proof. Let $\dim[X_{\text{fin}}] = n$ with $\{e_1, \dots, e_n\}$ a Hamel basis of X_{fin}, and let $\{\tau_\alpha \;:\; \alpha \in \mathscr{A}\}$ be a directed generating family of τ. Since $X = X_{\text{fin}} \oplus_\tau N$ (for some subspace N of X) we have that the unique linear functionals $\delta_i : X \to \mathbb{R}$ defined by the relations $\delta_i(e_i) = 1$, $\delta_i(e_j) = 0$ for $j \neq i$ and $\delta_i\big|_N = 0$, are τ-continuous for each $i = 1, \dots, n$. Recalling that $\mathfrak{B} = \bigcup_{\alpha \in \mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ is a fundamental system of $\mathcal{N}(0_X, \tau)$ we get that for each $i \in \{1, \dots, n\}$ there exist $\alpha_i \in \mathscr{A}$ and $U_i \in \mathcal{N}(0_X, \tau_{\alpha_i})$ such that

$$\delta_i(U_i) \subseteq \{k \in \mathbb{K} \;:\; |k| < 1\},$$

(where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and so δ_i is τ_{α_i}-continuous (according to the fact that for each $\lambda > 0$, $\lambda U_i \in \mathcal{N}(0_X, \tau_{\alpha_i})$). Since the generating family is directed for the inclusion, there exists $\alpha \in \mathscr{A}$ such that δ_i is τ_α-continuous for each $i \in \{1, \dots, n\}$.

Define then the function $F : X^\alpha_{\text{fin}} \to X^\alpha_{\text{fin}}$ given by $F(x) = \sum_{i=1}^n \delta_i(x) e_i$. It is clear that F is a linear projection such that $F(X^\alpha_{\text{fin}}) = X_{\text{fin}}$. Therefore, it is sufficient to prove that F is τ_α-continuous. But this is direct since each δ_i restricted to X^α_{fin} is τ_α-continuous and $(X^\alpha_{\text{fin}}, \tau_\alpha)$ is a tvs. □

Example 3.17 *(An etvs with X_{fin} lacking of τ-supplements).* Consider $p \in (0, 1)$ and fix $X = L^p[0, 1]$. Let us denote by Θ the usual topology in $L^p[0, 1]$ and let $\mathcal{H}$ be a Hamel Basis of X such that $\mathbb{1} \in \mathcal{H}$, where $\mathbb{1}$ stands for the constant function $\mathbb{1}(t) = 1$. Let $\mathscr{A} = \{A \subset \mathcal{H} \setminus \{\mathbb{1}\} \;:\; \text{Card}(A) < +\infty\}$, and for each $A \in \mathscr{A}$ consider the subspace

$$X^A = \text{span}\{\mathcal{H} \setminus A\},$$

and define the topology Θ_A on X as the group topology induced by $\{O \cap X^A \;:\; O \in \Theta\}$, namely the unique topology on X such that the family

$$\mathfrak{B}_A = \{O \cap X^A \;:\; O \in \Theta \text{ and } 0_X \in O\}$$

is a fundamental system of neighborhoods of 0_X. It is easy to see that, with the latter construction, (X, Θ_A) is a proper fundamental etvs where X^A is its finite space; so we will write X^A_{fin} instead of X^A. Also, the family $\{\Theta_A \;:\; A \in \mathscr{A}\}$ is directed by inclusion: Indeed, let be $I, J \in \mathscr{A}$ and consider $A = I \cup J$ which, since both I and J are finite, is also finite and therefore $A \in \mathscr{A}$. We will show that $\Theta_I, \Theta_J \subset \Theta_A$. Let $O \in \Theta_I$. Without losing generality we assume that $0_X \in O$ and, since X^I_{fin} is open, $O \subseteq X^I_{\text{fin}}$. Then, by construction, there exists $O' \in \Theta$ such that $O = O' \cap X^I_{\text{fin}}$ and so, since $I \subseteq A$ we have that

$$O' \cap X^A_{\text{fin}} \subseteq O' \cap X^I_{\text{fin}} = O.$$

Finally, since $O' \cap X^A_{\text{fin}} \in \Theta_A$, we have that $O \in \mathcal{N}(0_X, \Theta_A)$. Therefore, $\mathcal{N}(0_X, \Theta_I) \subseteq \mathcal{N}(0_X, \Theta_A)$ which implies that $\Theta_I \subset \Theta_A$. Repeating the same reasoning for Θ_J instead of Θ_I we conclude the desired inclusions, proving that $\{\Theta_A \;:\; A \in \mathscr{A}\}$ is directed by inclusion as we claimed.

Define then $\tau = \bigvee_{A \in \mathscr{A}} \Theta_A$. We have then, that (X, τ) is an etvs over $(\mathbb{R}, |\cdot|)$ with

$$X_{\text{fin}} = \mathbb{R} \cdot \mathbb{1}.$$

We claim that X_{fin} doesn't have any τ-supplement space in X. Let us suppose the contrary. Applying Lemma 3.16, there exists $A \in \mathscr{A}$ such that X_{fin} has a Θ_A-supplement space in X^A_{fin}. Let us denote by N such a subspace. Now, we can write

$$X = Z \oplus N,$$

where $Z = \operatorname{span}\{A \cup \{\mathbb{1}\}\}$. Since Z is a finite dimensional subspace of $L^p[0,1]$ we get that N is Θ-dense in X (see [15, Ch. I, Exercises 6 and 7]) and therefore there exists a net $(n_\lambda) \subseteq N$ such that $n_\lambda \to^{\Theta} \mathbb{1}$. But since $N \subseteq X_{\text{fin}}^A$ we get that $n_\lambda \to^{\Theta_A} \mathbb{1}$, which contradicts the fact that X_{fin} and N are Θ_A-supplements in X_{fin}^A. Therefore, X_{fin} cannot have any τ-supplement space in X, as we claimed.

3.3. Subspaces, products and quotients of etvs

Proposition 3.18. *Let (X,τ) be an etvs over $(\mathbb{K},\theta)$, $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be a generating family of τ and Z be a subspace of X. The following assertions are equivalent:*

(i) *Z is τ-closed.*
(ii) *For each $\alpha \in \mathscr{A}$, $Z \cap X_{\text{fin}}^\alpha$ is τ-closed.*
(iii) *There exists $\alpha \in \mathscr{A}$ such that $Z \cap X_{\text{fin}}^\alpha$ is τ-closed.*

Proof. Noting that for each $\alpha \in \mathscr{A}$, X_{fin}^α is τ-closed (see Remark 2), $(i) \Rightarrow (ii)$ follows immediately, and $(ii) \Rightarrow (iii)$ is direct. To prove $(iii) \Rightarrow (i)$, let $\alpha \in \mathscr{A}$ such that $Z \cap X_{\text{fin}}^\alpha$ is τ-closed, and consider a Hamel basis $\mathcal{H}$ of X generating X_{fin}^α and Z. Then, denoting $N = \mathscr{C}_{\mathcal{H}}(X_{\text{fin}}^\alpha)$ and applying Lemma 3.4, we can write $X = X_{\text{fin}}^\alpha \oplus_\tau N$ and also

$$Z = P_{X_{\text{fin}}^\alpha, N}(Z) \oplus_\tau P_{N, X_{\text{fin}}^\alpha}(Z) = (Z \cap X_{\text{fin}}^\alpha) \oplus_\tau (Z \cap N).$$

Let (z_λ) be a net in Z converging to $z \in X$. Then, $P_{X_{\text{fin}}^\alpha, N}(z_\lambda) \to P_{X_{\text{fin}}^\alpha, N}(z) \in Z \cap X_{\text{fin}}^\alpha$ since $Z \cap X_{\text{fin}}^\alpha$ is τ-closed, and $P_{N, X_{\text{fin}}^\alpha}(z_\lambda) \to P_{N, X_{\text{fin}}^\alpha}(z) \in Z \cap N$ thanks to the discreteness of N (recall that (N, τ_α) is discrete by definition of fundamental etvs and by Lemma 3.4, and that τ_α is coarser than τ). Therefore, $z = P_{X_{\text{fin}}^\alpha, N}(z) + P_{N, X_{\text{fin}}^\alpha}(z) \in Z$, finishing the proof. □

Proposition 3.19. *Let (X,τ) be an etvs over $(\mathbb{K},\theta)$, $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be a directed generating family of τ and Z be a subspace of X. The following assertions are equivalent:*

(i) *Z is τ-open.*
(ii) *For each $\alpha \in \mathscr{A}$, $Z \cap X_{\text{fin}}^\alpha$ is τ-open.*
(iii) *There exists $\alpha \in \mathscr{A}$ such that $Z \cap X_{\text{fin}}^\alpha$ is τ-open.*
(iv) *There exists $\alpha \in \mathscr{A}$, such that $X_{\text{fin}}^\alpha \subseteq Z$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ are direct, and therefore we only need to prove $(iii) \Rightarrow (iv)$. Assume then that there exists $\alpha \in \mathscr{A}$ such that $Z \cap X_{\text{fin}}^\alpha$ is τ-open. Then, there exist $\beta \in \mathscr{A}$ and $V \in \mathcal{N}(0_X, \tau_\beta)$ such that $V \subseteq Z \cap X_{\text{fin}}^\alpha$. Without losing generality, we may assume that $\tau_\alpha \subseteq \tau_\beta$ and that $V \subseteq X_{\text{fin}}^\beta$. Since $(X_{\text{fin}}^\beta, \tau_\beta)$ is a tvs over $(\mathbb{K},\theta)$ and $(\mathbb{K},\theta)$ is non-discrete, V must generate X_{fin}^β and so, since $V \subseteq Z$ we conclude

$$X_{\text{fin}}^\beta = \operatorname{span}(V) \subseteq Z,$$

which finishes the proof. □

Proposition 3.20. *Let $(X_i, \tau_i)_{i \in I}$ be a family of etvs over $(\mathbb{K},\theta)$. The product space $X = \prod_i X_i$ endowed with the product topology $\tau = \prod \tau_i$ is also an etvs, with*

$$X_{\text{fin}} = \prod_i X_{\text{fin}}^i,$$

where X_{fin}^i stands for the finite space of (X_i, τ_i).

Proof. It is known (see [8, Ch. III, §6]) that (X, τ) is a topological group with operators in $(\mathbb{K}, \theta)$. For each $i \in I$, let us denote by $p_i : X \to X_i$ the canonical projection given by $p_i(x) = x_i$, and let $\{\tau_{i,\alpha} \ : \ \alpha \in \mathscr{A}_i\}$ be a generating family of τ_i. For $i \in I$ and $\alpha \in \mathscr{A}_i$, it is easy to see that $\tau(i, \alpha) := p_i^{-1}(\tau_{i,\alpha})$ is a group topology over X such that $(X, \tau(i, \alpha))$ is a fundamental etvs over $(\mathbb{K}, \theta)$ where its finite space is

$$X_{\text{fin}}^{\tau(i,\alpha)} = p_i^{-1}(X_{\text{fin}}^{\tau_{i,\alpha}}) = X_{\text{fin}}^{\tau_{i,\alpha}} \times \prod_{j \neq i} X_j.$$

Also, we have that $p_i^{-1}(\tau_i) = \bigvee_{\alpha \in \mathscr{A}_i} \tau(i, \alpha)$ and therefore

$$\tau = \bigvee_{i \in I} p_i^{-1}(\tau_i) = \bigvee \{\tau(i, \alpha) \ : \ i \in I, \ \alpha \in \mathscr{A}_i\},$$

which proves that (X, τ) is an etvs. Now, by Proposition 3.10, we have that

$$X_{\text{fin}} = \bigcap X_{\text{fin}}^{\tau(i,\alpha)}.$$

Let $x = (x_i) \in X_{\text{fin}}$. Then, for each $i \in I$ we have that

$$p_i(x) \in \bigcap_{\alpha \in \mathscr{A}} X_{\text{fin}}^{\tau_{i,\alpha}} = X_{\text{fin}}^i,$$

and so $X_{\text{fin}} \subseteq \prod X_{\text{fin}}^i$. Now, if $x \in \prod_i X_{\text{fin}}^i$, we have that for each $i \in I$ and each $\alpha \in \mathscr{A}_i$, $p_i(x) \in X_{\text{fin}}^{\tau_{i,\alpha}}$ and therefore $x \in X_{\text{fin}}^{\tau(i,\alpha)}$. Then, $x \in X_{\text{fin}}$, which finishes the proof. □

Since open subspaces of the product space $X = \prod_{i \in I} X_i$ are of the form $\prod_{j \in J} M_j \times \prod_{i \in I \setminus J} X_i$ where M_j is an open subspace of X_j and $\text{Card}(J) < \infty$, we get the following direct corollary:

Corollary 3.21. *Let $(X_i, \tau_i)_{i \in I}$ be a family of fundamental etvs over $(\mathbb{K}, \theta)$. The product space (X, τ) is a fundamental etvs if and only if the set of indexes*

$$\left\{i \in I \ : \ X_{\text{fin}}^i \subsetneq X_i\right\}$$

is finite.

Corollary 3.22. *Let (X_i, τ_i, f_{ij}) be a projective system of topological groups with operators in $\mathbb{K}$, and θ be a non-discrete field topology over $\mathbb{K}$. If (X_i, τ_i) is an etvs over $(\mathbb{K}, \theta)$ for each $i \in I$, then $X = \varprojlim X_i$ endowed with the product topology $\tau = \prod_i \tau_i$ is also an etvs over $(\mathbb{K}, \theta)$. Moreover,*

$$X_{\text{fin}} = \varprojlim X_{\text{fin}}^i,$$

where X_{fin}^i denotes the finite space of (X_i, τ_i).

Proof. Since $\varprojlim X_i$ is a subspace of $\prod_i X_i$, the first conclusion follows immediately from Proposition 3.20 and Lemma 3.12. Also, since $(X_{\text{fin}}^i, \tau_i)$ is a tvs over $(\mathbb{K}, \theta)$ for each $i \in I$, we have that $\left(\varprojlim X_{\text{fin}}^i, \tau\right)$ is also a tvs over $(\mathbb{K}, \theta)$. Therefore, $\varprojlim X_{\text{fin}}^i \subseteq X_{\text{fin}}$, by Lemma 3.13.

Now, let $x \in X_{\text{fin}}$ and fix $i \in I$. Let $\{\tau_\alpha : \alpha \in \mathscr{A}_i\}$ be a generating family of τ_i. For $\alpha \in \mathscr{A}_i$, we have that X^α_{fin} is a τ_i-open subspace of (X_i, τ_i) and therefore, $M_\alpha = X \cap p_i^{-1}(X^\alpha_{\text{fin}})$ is an open subspace of (X, τ). Then, $x \in M_\alpha$ and so, $p_i(x) \in X^\alpha_{\text{fin}}$. Then,

$$p_i(x) \in \bigcap_{\alpha \in \mathscr{A}_i} X^\alpha_{\text{fin}} = X^i_{\text{fin}},$$

where the last equality is provided by Proposition 3.10. Since the latter relation holds for every $i \in I$, $x \in \prod_i X^i_{\text{fin}}$ and, since

$$\varprojlim X^i_{\text{fin}} = \left(\prod_i X^i_{\text{fin}}\right) \cap X,$$

we conclude that $x \in \varprojlim X^i_{\text{fin}}$, which finishes the proof. □

Lemma 3.23. *Let (X, τ) be a fundamental etvs over $(\mathbb{K}, \theta)$ and Z be a subspace of X such that $Z \subseteq X_{\text{fin}}$. Then, $(X/Z, \pi(\tau))$ is a fundamental etvs over $(\mathbb{K}, \theta)$ and $(X/Z)_{\text{fin}} = X_{\text{fin}}/Z$.*

Proof. Let us note first that $(X/Z, \pi(\tau))$ is a topological group with operators in $\mathbb{K}$ (see [8, Ch. III, §6]). Also, since π is an open mapping, we have that $X_{\text{fin}}/Z = \pi(X_{\text{fin}})$ is open. By Lemma 3.4, it only remains to prove that X_{fin}/Z is a tvs over $(\mathbb{K}, \theta)$. This last statement is known to hold (see for example [7, Ch. I, §1, Part 3]) and therefore, the proof is complete. □

Lemma 3.24. *Let X, Y be two vector spaces over a field $\mathbb{K}$, τ_X, τ_Y be two group topologies over X and Y respectively and θ be a field topology over $\mathbb{K}$. Assume that there exists a linear mapping $h : X \to Y$ which is an isomorphism of topological groups. Then, (X, τ_X) is a topological group with operators in $\mathbb{K}$ if and only if (Y, τ_Y) is a topological group with operators in $\mathbb{K}$. Moreover,*

$$(X, \tau_X) \text{ is a tvs over } (\mathbb{K}, \theta) \iff (Y, \tau_Y) \text{ is a tvs over } (\mathbb{K}, \theta).$$

Proof. Assume first that (X, τ_X) is a topological group with operators in $\mathbb{K}$ and fix $k \in \mathbb{K}$. We will denote by φ_k both endomorphisms induced by k, the one defined over X and the other over Y (see equation (1)). It is not hard to see, according to the linearity of h, that we can write

$$\varphi_k = \varphi_k \circ h \circ h^{-1} = h \circ \varphi_k \circ h^{-1},$$

and so, φ_k is τ_Y-τ_Y-continuous. Thus (Y, τ_y) is a topological group with operators in $\mathbb{K}$ and, by symmetry, the first equivalence is proven.

Assume now that (X, τ_X) is a tvs over $(\mathbb{K}, \theta)$ and let (k_λ, y_λ) be a net in $\mathbb{K} \times Y$ converging to $(k, y) \in \mathbb{K} \times Y$. We have that

$$k_\lambda \cdot y_\lambda = k_\lambda \cdot h(h^{-1}(y_\lambda)) = h(k_\lambda \cdot h^{-1}(y_\lambda)) \to h(k \cdot h^{-1}(y)) = k \cdot y,$$

where the convergence is given by the continuity of h^{-1} and the fact that (X, τ_X) is a tvs over $(\mathbb{K}, \theta)$. We conclude that (Y, τ_Y) is a tvs over $(\mathbb{K}, \theta)$ and, by symmetry, the second equivalence is proven. □

Proposition 3.25. *Let (X, τ) be a fundamental etvs over $(\mathbb{K}, \theta)$ and Z be a subspace of X. Then, $(X/Z, \pi(\tau))$ is a fundamental etvs over $(\mathbb{K}, \theta)$ and $(X/Z)_{\text{fin}} = \pi(X_{\text{fin}})$.*

Proof. As in Lemma 3.23, $(X/Z, \pi(\tau))$ is a topological group with operators in $\mathbb{K}$ and $\pi(X_{\text{fin}})$ is an open subspace of X/Z. Therefore, we only need to prove that $\pi(X_{\text{fin}})$ is a tvs over $(\mathbb{K}, \theta)$.

Let us consider the space $Y = X_{\text{fin}} + Z$. By [8, Ch. III, §2, Proposition 20], we have that $\pi(X_{\text{fin}}) = \pi(Y)$ and Y/Z are isomorphic as topological groups, where the isomorphism is given by the mapping $\tilde{h}_1 : Y/\operatorname{Ker}(h_1) \to \pi(Y)$ induced by the mapping $h_1 : Y \to \pi(Y)$ defined as $h_1 = \pi \circ \hat{\imath}$, where $\hat{\imath}$ is the canonical injection from Y into X (see the comments before [8, Ch. III, §2, Proposition 20] and [6, Ch. I, §4, Theorem 3]). Observing that h_1 is a linear mapping, we have that $\tilde{h}_1$ meets the hypothesis of Lemma 3.24.

On the other hand, by Lemma 3.12, we know that (Y, τ) is a fundamental etvs over $(\mathbb{K}, \theta)$ with $Y_{\text{fin}} = Y \cap X_{\text{fin}} = X_{\text{fin}}$. We can consider then a Hamel basis $\mathcal{H}$ of Y which generates X_{fin} and Z and denote $N = \mathscr{C}_{\mathcal{H}}(X_{\text{fin}})$. By construction, we get that N is a subspace of Z and then, applying the corollary of [8, Ch. III, §7, Proposition 22], we get that

$$\pi(X_{\text{fin}}) \cong Y/Z \cong (Y/N)/(Z/N),$$

where the isomorphism between $\pi(X_{\text{fin}})$ and Y/Z is the mapping $\tilde{h}_1$ described before and the isomorphism between Y/Z and $(Y/N)/(Z/N)$ is given by the mapping $\tilde{h}_2 : Y/Z \to (Y/N)/(Z/N)$ induced by $h_2 : Y \to (Y/N)/(Z/N)$ defined as $h_2 = \pi_{Z/N} \circ \pi_N$. Observing that h_2 is also a linear mapping, the isomorphism $\tilde{h}_2$ meets the hypothesis of Lemma 3.24.

Noting that $Y/N \cong X_{\text{fin}}$ where the isomorphism is the parallel lifting $\ell_{N,X_{\text{fin}}}$ (which is linear) we conclude, thanks to Lemma 3.24 applied to $\ell_{N,X_{\text{fin}}}$, that Y/N is a tvs over $(\mathbb{K}, \theta)$ and therefore $(Y/N)/(Z/N)$ is also a tvs over $(\mathbb{K}, \theta)$. Finally, $\pi(X_{\text{fin}})$ also is a tvs over $(\mathbb{K}, \theta)$, by Lemma 3.24 applied to the isomorphism $\tilde{h} : (Y/N)/(Z/N) \to \pi(X_{\text{fin}})$ given by $\tilde{h} = \tilde{h}_1 \circ \tilde{h}_2^{-1}$. $\square$

Proposition 3.26. *Let (X, τ) be an etvs over $(\mathbb{K}, \theta)$ and Z be a subspace of X. Then, $(X/Z, \pi(\tau))$ is an etvs over $(\mathbb{K}, \theta)$ and we have*

$$\pi(X_{\text{fin}}) \subseteq \pi\left(\overline{X_{\text{fin}} + Z}^{\tau}\right) \subseteq \overline{\pi(X_{\text{fin}})}^{\pi(\tau)} \subseteq (X/Z)_{\text{fin}}.$$

Moreover, if $Z \subseteq X_{\text{fin}}$, then the latter inclusions hold with equality.

Proof. Let $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be a generating family of τ. By Proposition 3.25, we have that for each $\alpha \in \mathscr{A}$, $(X/Z, \pi(\tau_\alpha))$ is a fundamental etvs over $(\mathbb{K}, \theta)$. On one hand, applying [8, Ch. III, §2, Proposition 17] we have that

$$\pi(\tau) = \bigvee_{\alpha \in \mathscr{A}} \pi(\tau_\alpha).$$

Then, $(X/Z, \pi(\tau))$ is an etvs over $(\mathbb{K}, \theta)$. On the other hand, it is easy to see that $\pi(X_{\text{fin}}) \subseteq \pi\left(\overline{X_{\text{fin}} + Z}\right)$ and, by continuity of π, that $\pi\left(\overline{X_{\text{fin}} + Z}\right) \subseteq \overline{\pi(X_{\text{fin}})}$. Note also that for each $\alpha \in \mathscr{A}$ we have that $\pi(X^{\alpha}_{\text{fin}})$ is an open subspace of X/Z, therefore it is also closed (see Corollary of [8, Ch. III, §2, Proposition 4]), and it contains $\pi(X_{\text{fin}})$. Thus we can write

$$\overline{\pi(X_{\text{fin}})} \subseteq \bigcap_{\alpha \in \mathscr{A}} \pi(X^{\alpha}_{\text{fin}}) = \bigcap_{\alpha \in \mathscr{A}} (X/Z)^{\alpha}_{\text{fin}} = (X/Z)_{\text{fin}},$$

where the equality $\bigcap_{\alpha \in \mathscr{A}} \pi(X^{\alpha}_{\text{fin}}) = \bigcap_{\alpha \in \mathscr{A}} (X/Z)^{\alpha}_{\text{fin}}$ is given by Proposition 3.25. Thus, the inclusions in the statement hold.

Assume now that $Z \subseteq X_{\text{fin}}$ and let $[x] \in (X/Z)_{\text{fin}}$. Then,

$$[x] \subseteq \pi^{-1}\left(\bigcap_{\alpha \in \mathscr{A}} \pi(X^{\alpha}_{\text{fin}})\right) = \bigcap_{\alpha \in \mathscr{A}} (X^{\alpha}_{\text{fin}} + Z) = \bigcap_{\alpha \in \mathscr{A}} X^{\alpha}_{\text{fin}} = X_{\text{fin}}.$$

Then, $\pi^{-1}((X/Z)_{\text{fin}}) \subseteq X_{\text{fin}}$ and so, $(X/Z)_{\text{fin}} \subseteq \pi(X_{\text{fin}})$. Thus, the inclusions in the statement hold with equality, which was what we wanted to prove. □

3.4. Linear operators

Definition 3.27 *(Bounded sets).* Let (X, τ) be an etvs over a topological field $(\mathbb{K}, \theta)$ and A be a subset of X. We say that A is τ-bounded if for every neighborhood $V \in \mathcal{N}_X(0_X, \tau)$ there exist a finite set of centers $\{x_1, \ldots, x_n\} \subseteq A$ and a finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{K}$ such that

$$A \subseteq \sum_{j=1}^{n}(x_j + \alpha_j V).$$

Observe that this definition of bounded sets is exactly the same of [2, Definition 4.1] when (X, τ) is an extended normed space.

Proposition 3.28. *Let (X, τ) and (Y, σ) be two etvs over the same topological field $(\mathbb{K}, \theta)$, and $T : X \to Y$ be a linear operator. Then, T is continuous if and only if it is continuous at 0_X.*

In such a case, we have that:

1. *If (X, τ) is proper, then $T(X_{\text{fin}}) \subseteq Y_{\text{fin}}$.*
2. *T is bounded, namely, T maps τ-bounded sets into σ-bounded sets.*

Proof. The first part is clear, since for any net $(x_\lambda) \subseteq X$ converging to $x \in X$, we can write

$$T(x_\lambda) \to T(x) \iff T(x_\lambda - x) \to 0_Y.$$

Also, if T is continuous and (X, τ) is proper, we have that

$$T(X_{\text{fin}}) = T(C[0_X]) \subseteq C[0_Y] \subseteq Y_{\text{fin}}.$$

For the second part, let $A \subseteq X$ be a nonempty τ-bounded set and let $V \in \mathcal{N}_Y(0_Y, \sigma)$. Since T is continuous at 0_X and $T(0_X) = 0_Y$, there exists a neighborhood $U \in \mathcal{N}_X(0_X, \tau)$ such that $T(U) \subseteq V$. Since A is bounded, there exist sets $\{x_1, \ldots, x_n\} \subseteq A$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{K}$ such that

$$A \subseteq \sum_{i=1}^{n}(x_i + \alpha_i U).$$

Then, by linearity of T we have that

$$T(A) \subseteq \sum_{i=1}^{n}(T(x_i) + \alpha_i T(U)) \subseteq \sum_{i=1}^{n}(T(x_i) + \alpha_i V),$$

and therefore, noting that $\{T(x_1), \ldots, T(x_n)\} \subseteq T(A)$, we conclude that $T(A)$ is σ-bounded. □

Proposition 3.29. *Let $(\mathbb{K}, \theta) = (\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$, (X, τ) be an etvs over $(\mathbb{K}, \theta)$ and $f : X \to \mathbb{K}$ be a linear functional. We have that f is continuous if and only if $\operatorname{Ker}(f)$ is τ-closed.*

Proof. The necessity is trivial since $\{0_{\mathbb{K}}\}$ is closed. To prove the sufficiency, assume that $\operatorname{Ker}(f)$ is τ-closed. Then, applying Proposition 3.26, we have that the quotient space $X/\operatorname{Ker}(f)$ endowed with the quotient topology is an etvs over $(\mathbb{K}, \theta)$ of dimension 1. Also, since $\operatorname{Ker}(f)$ is τ-closed, we have that $(X/\operatorname{Ker}(f), \pi(\tau))$ is a Hausdorff space.

Thus, there are only two possibilities: either $\pi(\tau)$ is discrete or $(X/\operatorname{Ker}(f))_{\text{fin}} = X/\operatorname{Ker}(f)$. In both cases any linear function $g : X/\operatorname{Ker}(f) \to \mathbb{K}$ is continuous. In particular, the function $g : X/\operatorname{Ker}(f) \to \mathbb{K}$ given by $g([x]) = f(x)$ is continuous. The conclusion follows, since $f = g \circ \pi$. $\square$

Definition 3.30 *(Dual space).* Let (X, τ) be an etvs over $(\mathbb{R}, |\cdot|)$ (or $(\mathbb{C}, |\cdot|)$). We define the **dual space** $(X, \tau)^*$ (or simply X^* if there exists no confusion) as the vector space over $\mathbb{R}$ (or $\mathbb{C}$) of all linear functionals from X to $\mathbb{R}$ (or $\mathbb{C}$) which are continuous.

Proposition 3.31. *Let (X, τ) be a real (or complex) fundamental etvs. If N is a τ-supplement space of X_{fin} in X, then we have that*

$$X^* \approx X_{\text{fin}}^* \times N^{alg},$$

where N^{alg} denotes the algebraic dual of N.

Proof. Let $\mathcal{H}$ be a Hamel basis of X generating X_{fin} and N. We will consider the identification $\phi : X^* \to X_{\text{fin}}^* \times N^{alg}$ given by

$$\phi(x^*) = \left(x^*\big|_{X_{\text{fin}}}, x^*\big|_N \right).$$

Clearly, ϕ is well-defined and it is an algebraic homomorphism. Also, it is injective. Indeed, if $x^* \in X^*$ is such that $\phi(x^*) = (0, 0)$, then

$$\forall h \in \mathcal{H}, \langle x^*, h \rangle = 0,$$

and therefore $x^* = 0$. Finally, if we consider $(x_f^*, n^*) \in X_{\text{fin}}^* \times N^{alg}$, we can define

$$x^* = x_f^* \circ P_{X_{\text{fin}}, N} + n^* \circ P_{N, X_{\text{fin}}}.$$

Since the parallel projections are continuous, we get that $x^* \in X^*$ and that $\phi(x^*) = (x_f^*, n^*)$. Thus, ϕ is bijective, which finishes the proof. $\square$

Remark 4. Note that, following the notation of the Parallel Diagram described in Fig. 1, we get that ϕ corresponds to the mapping D, M to X_{fin}^* and N^{alg} to N. Furthermore, from the preceding proof, we get that $P_{X_{\text{fin}}^*, N^{alg}}$ is the restriction to X_{fin}, that is $P_{X_{\text{fin}}^*, N^{alg}}(x^*) = x^*\big|_{X_{\text{fin}}}$. Also, we would like to mention that the notation N^{alg} is not very common. Usually, the algebraic dual of a space N is noted by N' (or by N^*, when the topological dual is noted by N'). Nevertheless, to avoid confusions, we prefer to use this notation.

Recall that, for a directed set of indexes $(I, \preceq)$, an *inductive system* of vector spaces over a field $\mathbb{K}$, is a family $(X_i, f_{ji})_{i \in I}$ where X_i is a vector space over $\mathbb{K}$ and for each $i, j, k \in I$:

- Whenever $j \preceq i$, $f_{ji} : X_j \to X_i$ is a linear operator.
- $f_{ii} = \mathrm{id}_{X_i}$.
- Whenever $k \preceq j \preceq i$, $f_{ji} \circ f_{kj} = f_{ki}$.

For an inductive system $(X_i, f_{ji})_{i\in I}$, the *inductive limit* $X = \varinjlim(X_i, f_{ji})$ is given by

$$\varinjlim X_i = \left(\bigcup_{i\in I} X_i\right)/\mathcal{R},$$

where $\mathcal{R}$ is the equivalence relation given by

$$x_i\mathcal{R}y_j \Leftrightarrow \exists k \in I, i, j \preceq k \text{ and } f_{ik}(x) = f_{jk}(y).$$

The algebraic structure in $\varinjlim X_i$ (which is known to be a vectorial structure over the field $\mathbb{K}$) is the following: For $x_i \in X_i$, $y_j \in X_j$ and $\lambda \in \mathbb{K}$

- $[x_i]_{\mathcal{R}} + [y_j]_{\mathcal{R}} = [f_{ik}(x_i) + f_{jk}(y_j)]_{\mathcal{R}}$, for any $k \in I$ such that $i, j \preceq k$.
- $\lambda[x_i]_{\mathcal{R}} = [\lambda x_i]_{\mathcal{R}}$.

We want now to compute the dual space of an etvs (X, τ). The following proposition gives us a characterization of it in terms of a generating family $\{\tau_\alpha : \alpha \in \mathscr{A}\}$, as an inductive limit. Observe that, whenever τ_β, τ_α are two topologies on X with $\tau_\beta \subseteq \tau_\alpha$, then necessarily $(X, \tau_\beta)^* \subseteq (X, \tau_\alpha)^*$ and so, the canonical embedding $\hat{\imath}_{\beta\alpha} : (X, \tau_\beta)^* \to (X, \tau_\alpha)^*$ is well defined.

Proposition 3.32. *Let (X, τ) be a real (or complex) etvs, and $\{\tau_\alpha : \alpha \in \mathscr{A}\}$ be any directed generating family of τ. Then,*

$$X^* = \bigcup_{\alpha\in\mathscr{A}} (X, \tau_\alpha)^* \approx \varinjlim[(X^\alpha_{\text{fin}}, \tau_\alpha)^* \times N^{alg}_\alpha, f_{\beta\alpha}],$$

with, whenever $\tau_\beta \subseteq \tau_\alpha$, $f_{\beta\alpha} : (X^\beta_{\text{fin}}, \tau_\beta)^* \times N^{alg}_\beta \to (X^\alpha_{\text{fin}}, \tau_\alpha)^* \times N^{alg}_\alpha$ given by

$$f_{\beta\alpha} = \phi_\alpha \circ \hat{\imath}_{\beta\alpha} \circ \phi_\beta^{-1},$$

where ϕ_α is the isomorphism between $(X^\alpha_{\text{fin}}, \tau_\alpha)^* \times N^{alg}_\alpha$ and X^*_α given in Proposition 3.31, and $\hat{\imath}_{\beta\alpha}$ is the canonical embedding of $(X, \tau_\beta)^*$ into $(X, \tau_\alpha)^*$.

Proof. To simplify the proof, let us denote $X^*_1 = \bigcup_{\alpha\in\mathscr{A}} (X, \tau_\alpha)^*$ and $X^*_2 = \varinjlim[(X^\alpha_{\text{fin}}, \tau_\alpha)^* \times N^{alg}_\alpha, f_{\beta\alpha}]$. Let us also denote by $\mathbb{K}$ the field over which X is a vector space (which is $\mathbb{R}$ or $\mathbb{C}$). Observe that, since the generating family is directed, we have that X^*_1 is a subspace of the algebraic dual X^{alg}.

Now, we can define the mapping $\phi : X^*_1 \to X^*_2$ given by

$$\phi(x^*) = [\phi_\alpha(x^*)], \quad \text{whenever } x^* \in (X, \tau_\alpha)^*.$$

Note that whenever x^* belongs to $(X, \tau_\alpha)^*$ and to $(X, \tau_\beta)^*$, there exists $\gamma \in \mathscr{A}$ such that $\tau_\alpha \vee \tau_\beta \subseteq \tau_\gamma$, and therefore $x^* \in (X, \tau_\gamma)^*$. Thus, $[\phi_\alpha(x^*)] = [\phi_\gamma(x^*)] = [\phi_\beta(x^*)]$ and so, ϕ is well defined. Also, by definition, ϕ is an onto homomorphism. Finally, if $\phi(x^*) = 0$ for some $x^* \in X^*_1$, then there exist $\alpha, \beta, \gamma \in \mathscr{A}$ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$ such that $x^* \in (X, \tau_\alpha)^*$ and

$$f_{\alpha\gamma}(\phi_\alpha(x^*)) = f_{\beta\gamma}(\phi_\beta(0)).$$

Since ϕ_γ is an isomorphism, we have that $\hat{\imath}_{\alpha\gamma}(x^*) = \hat{\imath}_{\beta\gamma}(0) = 0$. Since $\hat{\imath}_{\alpha\gamma}$ is injective, we conclude that $x^* = 0$, proving that ϕ is injective. Thus, $X^*_1 \approx X^*_2$.

To prove that $X^* = X_1^*$, we only need to show that $X^* \subseteq X_1^*$ (the other inclusion is direct, since every τ_α-continuous functional is also τ-continuous, for each $\alpha \in \mathscr{A}$). Let $x^* \in X^*$. Since $\bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ is a fundamental system of neighborhoods of $\mathcal{N}(0_X, \tau)$, we have that there exist $\alpha \in \mathscr{A}$ and $V \in \mathcal{N}(0_X, \tau_\alpha)$ such that

$$x^*(V) \subseteq \{\lambda \in \mathbb{K} \ : \ |\lambda| < 1\}.$$

Since for each $r > 0$, $rV \in \mathcal{N}(0_X, \tau_\alpha)$, we have that the latter inclusion implies that x^* is τ_α-continuous, and therefore $x^* \in X_1^*$. So, $X^* \subseteq X_1^*$, finishing the proof. □

There is not much more that we can say about the dual space of a general etvs or about the linear operators defined over it. We will return to these subjects in the context of elcs, in Subsection 4.1.

Remark 5. The general strategy when we study extended topological vector spaces is to reduce them to their fundamental etvs. This approach may have applications in *Lie group theory* (see [11]). For example, if we have an abelian matrix Lie group (G, τ), induced by a commutative matrix Lie algebra $\mathfrak{g}$, it is known that

$$\exp(\mathfrak{g}) = C[0_G, \tau],$$

and even more, the exponential map is a local homeomorphism (see Proposition 2.3, Proposition 2.16, Corollary 2.29 and Corollary 2.31 of [11]). Therefore, $C[0_G, \tau]$ is an open subgroup of G and so, G can be decomposed as a topological sum of a *connected abelian Lie group* and a *discrete Lie group*. Thus, many of the techniques exposed in this section could be applied to the study of projective limits of Lie groups. This subject is outside of the scope of the paper.

4. Extended seminormed spaces

The notions of fundamental etvs and etvs gave us a global context to work with, but our attention, as in the classical theory, arrives quickly at the convex case. In this section we will study the properties of the extended locally convex spaces and the extended seminormed spaces, providing one of our principal results: (X, τ) is an esns if and only if it is an elcs.

In the following, $\mathbb{K}$ will be always $\mathbb{R}$ or $\mathbb{C}$, endowed with the usual absolute value. Recall that for a family $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ of extended seminorms over X, $\mathfrak{T}(\mathcal{P})$ denotes the induced topology of $\mathcal{P}$ on X, which was described in the comments above Definition 1.2.

Proposition 4.1. *Every fundamental elcs is an esns.*

Proof. Let (X, τ) be a fundamental elcs over $\mathbb{K}$. Then, (X_{fin}, τ) is a lcs over $\mathbb{K}$ and therefore there exists a family $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ of seminorms on X_{fin} such that

$$\tau\big|_{X_{\text{fin}}} = \mathfrak{T}(\mathcal{P}).$$

Let us write $X = X_{\text{fin}} \oplus_\tau N$ and, for each $i \in I$, set $\tilde{\rho}_i : X \to [0, +\infty]$ given by

$$\tilde{\rho}_i(x) = \rho_i(P_{X_{\text{fin}}}(x)) + \|P_N(x)\|_d,$$

where $\|\cdot\|_d$ is the discrete norm (see Equation (3) at the beginning of Section 3). We have that $\tilde{\mathcal{P}} = \{\tilde{\rho}_i \ : \ i \in I\}$ is a family of extended seminorms on X and

$$\tau = \mathfrak{T}(\tilde{\mathcal{P}}),$$

which proves that (X, τ) is an esns (see Definition 1.2). □

Lemma 4.2. *Let (X, τ) be a topological group and $\rho : X \to [0, +\infty]$ be a τ-continuous extended seminorm. Then, the sets $\rho^{-1}([0, +\infty))$ and $\rho^{-1}(\{+\infty\})$ are both open and closed.*

Proof. Clearly, $\rho^{-1}([0, +\infty))$ is open by continuity. On the other hand, choose $x_0 \in X$ with $\rho(x_0) = +\infty$. Since ρ is continuous and τ is a group topology, the set $U = \{x \in X \ : \ \rho(x_0 - x) < 1\}$ is open. Applying the triangle inequality of ρ we have that for each $x \in U$

$$\rho(x_0) \leq \rho(x_0 - x) + \rho(x),$$

and since $\rho(x_0 - x) < 1$, we conclude that $\rho(x) = +\infty$. Thus, $U \subseteq \rho^{-1}(\{+\infty\})$. Since x_0 is an arbitrary element of $\rho^{-1}(\{+\infty\})$, we conclude that $\rho^{-1}(\{+\infty\})$ is open, finishing the proof. □

Theorem 4.3. *(X, τ) is an esns if and only if it is an elcs.*

Proof. Assume first that (X, τ) is an esns and let $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ be a family of extended seminorms such that $\tau = \mathfrak{T}(\mathcal{P})$. Then, for each $i \in I$ let us consider the topology

$$\tau_i := \mathfrak{T}(\{\rho_i\}).$$

It is easy to see that (X, τ_i) is a fundamental elcs with $X^i_{\text{fin}} = \rho_i^{-1}(\mathbb{R})$. Noting that

$$\tau = \bigvee_{i \in I} \tau_i,$$

we conclude that (X, τ) is an elcs.

For the other direction, assume that (X, τ) is an elcs and let $\{\tau_\alpha \ : \ \alpha \in \mathscr{A}\}$ be a locally convex generating family (see comments after Definition 3.9) of τ. Since (X, τ_α) is a fundamental elcs, we can apply Proposition 4.1 to find a family $\mathcal{P}_\alpha = \{\rho_i \ : \ i \in I_\alpha\}$ of extended seminorms such that $\tau_\alpha = \mathfrak{T}(\mathcal{P}_\alpha)$. It is not hard to see that

$$\tau = \bigvee_{\alpha \in \mathscr{A}} \tau_\alpha = \bigvee_{\alpha \in \mathscr{A}} \mathfrak{T}(\mathcal{P}_\alpha) = \mathfrak{T}\left(\bigcup_{\alpha \in \mathscr{A}} \mathcal{P}_\alpha\right),$$

which finishes the proof. □

Remark 6. Observe that the study of projective limits of Section 3.2 and all stability results of Section 3.3 are still valid if we fix $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with their usual topology and replace fundamental etvs and etvs by fundamental elcs and elcs, respectively. The extended seminormed spaces inherit this structure.

4.1. Structure of esns

In the following, motivated by Theorem 4.3, we will denote by $\mathcal{S}(X, \tau)$ the set of all τ-continuous extended seminorms on X, and for each $\rho \in \mathcal{S}(X, \tau)$ we will denote $\tau_\rho := \mathfrak{T}(\{\rho\})$ and

$$X^\rho_{\text{fin}} := \{x \in X \ : \ \rho(x) < +\infty\}.$$

Note that (X, τ_ρ) is a fundamental elcs and its finite space coincides with X^ρ_{fin}. So this notation is not ambiguous.

Proposition 4.4. *Let (X,τ) be an esns, $\mathcal{P}$ be a family of extended seminorms with $\tau = \mathfrak{T}(\mathcal{P})$, and $\rho : X \to [0,+\infty]$ be an extended seminorm. The following assertions are equivalent:*

(i) $\rho \in \mathcal{S}(X,\tau)$.
(ii) *There exist $C > 0$ and $\{p_i\}_{i=1}^n \subseteq \mathcal{P}$ such that*

$$\rho \leq C \max\{p_i \ : \ i = 1,\dots,n\}.$$

(iii) *ρ is continuous at 0_X.*

Proof. Note first that $(ii) \Rightarrow (iii)$ is direct thanks to the positivity of ρ. Let us prove $(iii) \Rightarrow (i)$. Assume then that ρ is τ-continuous at 0_X and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X converging to $x \in X$. Then, $x_\lambda - x \to 0_X$ and so, there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \geq \lambda_0$, $\rho(x_\lambda - x) < 1$.

If $\rho(x) < +\infty$, we can apply the triangle inequality getting that for each $\lambda \geq \lambda_0$,

$$|\rho(x_\lambda) - \rho(x)| \leq \rho(x_\lambda - x) \to 0.$$

Then, in this case, $\rho(x_\lambda) \to \rho(x)$. On the other hand, if $\rho(x) = +\infty$, then since

$$\rho(x) \leq \rho(x_\lambda) + \rho(x_\lambda - x),$$

we can assure that, for each $\lambda \geq \lambda_0$, $\rho(x_\lambda) = +\infty$, and therefore $\rho(x_\lambda) = \rho(x)$. Then, $\rho(x_\lambda) \to \rho(x)$ and so the conclusion follows.

To prove $(i) \Rightarrow (ii)$, note that if for each $F \subseteq \mathcal{P}$ with $|F| < \infty$ we write $V(F) = \{x \in X \ : \ p(x) \leq 1, \forall p \in F\}$, then

$$\mathfrak{B} = \{C \cdot V(F) \ : \ C > 0,\ F \subseteq \mathcal{P} \text{ with } |F| < \infty\}$$

is a fundamental system of neighborhoods of $\mathcal{N}(0_X,\tau)$: the reasoning is the same as the one after Definition 3.9, namely, we replace $\{\tau_p \ : \ p \in \mathcal{P}\}$ by $\left\{\bigvee_{p\in F}\tau_p \ : \ F \subset \mathcal{P},\ \operatorname{Card}(F) < \infty\right\}$ as generating family of τ, which is directed by inclusion.

Therefore, since ρ is a continuous extended seminorm, we have that there exists an element $V \in \mathfrak{B}$ such that $V \subseteq \rho^{-1}((-1,1))$. Since $V = C \cdot V(F)$ for some $C > 0$ and some $F \subseteq \mathcal{P}$ finite, we conclude that

$$\rho \leq C^{-1} \cdot \max\{p \ : \ p \in F\},$$

finishing the proof. □

Theorem 4.5. *Let (X,τ) be an esns. Then, for each family of extended seminorms $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ such that $\tau = \mathfrak{T}(\mathcal{P})$ we have that*

$$X_{\text{fin}} = C[0_X] = \{x \in X \ : \ \rho_i(x) < +\infty,\ \forall i \in I\}.$$

Moreover, the following assertions are equivalent:

(i) *(X,τ) is a fundamental elcs.*
(ii) *(X,τ) is a locally connected esns.*
(iii) *(X,τ) is an esns and there exists $\rho \in \mathcal{S}(X,\tau)$ such that $X_{\text{fin}} = X_{\text{fin}}^{\rho}$.*

Proof. For the first part, let $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ be a family of extended seminorms such that $\tau = \mathfrak{T}(\mathcal{P})$. Since $\{\tau_\rho \ : \ \rho \in \mathcal{P}\}$ is a generating family of τ, we can apply Proposition 3.10 to get that

$$X_{\text{fin}} = \bigcap_{\rho\in\mathcal{P}} X^\rho_{\text{fin}} = \{x \in X \ : \ \rho_i(x) < +\infty, \ \forall i \in I\},$$

and that (X_{fin}, τ) is a locally convex space. Therefore, (X_{fin}, τ) is connected and so, by the inclusion $C[0_X] \subseteq X_{\text{fin}}$, we get that $C[0_X] = X_{\text{fin}}$.

For the second part, $(i) \Rightarrow (ii)$ follows directly from the fact that (X_{fin}, τ) is connected. For $(ii) \Rightarrow (iii)$, assume that there exists a connected neighborhood $V \in \mathcal{N}(0_X, \tau)$. Since

$$\mathfrak{B} = \{\rho^{-1}((-\varepsilon, \varepsilon)) \ : \ \rho \in \mathcal{S}(X, \tau), \ \varepsilon > 0\}$$

is a fundamental system of $\mathcal{N}(0_X, \tau)$, we get that there exist $\rho \in \mathcal{S}(X, \tau)$ and $\varepsilon > 0$ such that $\rho^{-1}((-\varepsilon, \varepsilon)) \subseteq V$. Then, we have that

$$X^\rho_{\text{fin}} = \bigcup_{n\in\mathbb{N}} \rho^{-1}((-n\varepsilon, n\varepsilon)) \subseteq \bigcup_{n\in\mathbb{N}} nV \subseteq C[0_X] = X_{\text{fin}} \subseteq X^\rho_{\text{fin}}.$$

It only remains to prove $(iii) \Rightarrow (i)$. Let $\rho \in \mathcal{S}(X, \tau)$ be such that $X_{\text{fin}} = X^\rho_{\text{fin}}$. In particular, by Lemma 4.2, we have that X_{fin} is τ-open and so the conclusion follows from Lemma 3.4 and the fact that (X_{fin}, τ) is a locally convex space (for example, by Proposition 3.10.3, adapted to elcs). □

Corollary 4.6. *Let X be a vector space over $\mathbb{K}$. The following assertions are equivalent:*

(i) *For any family of extended seminorms $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ on X, $(X, \mathfrak{T}(\mathcal{P}))$ is a fundamental elcs.*
(ii) *For any family of extended seminorms $\mathcal{P} = \{\rho_i \ : \ i \in I\}$, the finite space X_{fin} of $(X, \mathfrak{T}(\mathcal{P}))$ is open.*
(iii) *X is finite dimensional.*

Proof.

$(i) \Rightarrow (ii)$ Direct by Corollary 3.11.

$(ii) \Rightarrow (iii)$ Let us suppose that X has infinite dimension and let $\mathcal{H} = \{b_i \ : \ i \in \Delta\}$ be a Hamel basis of X. Consider the canonical embedding of X in $c_0(\Delta)$ given by

$$\hat{\imath} : x = \sum_{i\in\Delta} \alpha_i b_i \mapsto (\alpha_i)_{i\in\Delta} \in c_0(\Delta),$$

and for each $i \in \Delta$ define the extended seminorm $\rho_i : X \to [0, +\infty]$ given by

$$\rho_i(x) = \begin{cases} \|\hat{\imath}(x)\|_\infty & \text{if } \hat{\imath}(x)(i) = 0. \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to realize that, for $\tau = \mathfrak{T}(\{\rho_i \ : \ i \in \Delta\})$, $X_{\text{fin}} = \{0\}$ and therefore, for X_{fin} to be τ-open (X, τ) must be a discrete space. But this last statement doesn't hold, since for each infinite sequence (i_n) in Δ with no repeated elements, the non-stationary sequence $\left(\frac{1}{n} b_{i_n}\right)$ τ-converges to 0_X: Indeed, for each $j \in \Delta$ there exists $n_0 \in \mathbb{N}$ such that $i_n \neq j$ for each $n \geq n_0$ and therefore, starting from n_0, we have that

$$\rho_j\left(\tfrac{1}{n} b_{i_n}\right) = \tfrac{1}{n} \|\hat{\imath}(b_{i_n})\|_\infty = \frac{1}{n} \to 0.$$

$(iii) \Rightarrow (i)$ Let $\mathcal{P} = \{\rho_i \ : \ i \in I\}$ be a family of extended seminorms on X and let $\mathcal{H} = \{b_j\}_{j=1}^n$ be a Hamel basis of X generating X_{fin}. By reordering $\mathcal{H}$ if necessary, we may and do assume that $X_{\text{fin}} = \text{span}(\{b_1, \dots, b_k\})$ for some $k \in \{1, \dots, n\}$. If $k = n$ then $X = X_{\text{fin}}$ and the result follows directly. If $k < n$, then for each $j \in \{k+1, \dots, n\}$ there exists $\rho_{i_j} \in \mathcal{P}$ such that $\rho_{i_j}(b_j) = +\infty$. Considering $\rho = \max\{\rho_{i_{k+1}}, \dots, \rho_{i_n}\}$ we have that $\rho \in \mathcal{S}(X, \mathfrak{T}(\mathcal{P}))$ and

$$X_{\text{fin}}^{\rho} \subseteq X_{\text{fin}}.$$

Then, since the reverse inclusion always holds, $X_{\text{fin}} = X_{\text{fin}}^{\rho}$ and the conclusion follows from Theorem 4.5. $\square$

Proposition 4.7. *Let τ be a group topology over X. Then, (X, τ) is an esns over $\mathbb{K}$ if and only if there exists a neighborhood basis $\mathfrak{B}$ of 0_X such that each element $V \in \mathfrak{B}$ is* ***absolutely convex*** *(i.e. convex and balanced).*

Proof. For the necessity, assume that (X, τ) is an esns over $\mathbb{K}$ and consider the family

$$\mathfrak{B} = \left\{ V_{\rho,\varepsilon} = \rho^{-1}\left((-\varepsilon, \varepsilon)\right) \ : \ \rho \in \mathcal{S}(X, \tau), \ \varepsilon > 0 \right\}.$$

It is easy to see that for each $\rho \in \mathcal{S}(X, \tau)$ and each $\varepsilon > 0$ the set $V_{\rho,\varepsilon}$ is absolutely convex. Also, since $\tau = \mathfrak{T}(\mathcal{S}(X, \tau))$ and $\{\tau_\rho \ : \ \rho \in \mathcal{S}(X, \tau)\}$ is directed by inclusion, we have that for each $V \in \mathcal{N}(0_X, \tau)$ there exist $\rho \in \mathcal{S}(X, \tau)$ and $\varepsilon > 0$ such that

$$\rho^{-1}((-\varepsilon, \varepsilon)) \subseteq V,$$

and therefore $\mathfrak{B}$ is a fundamental system of $\mathcal{N}(0_X, \tau)$, which proves the necessity.

Assume now that there exists a fundamental system $\mathfrak{B}$ of $\mathcal{N}(0_X, \tau)$ such that each $V \in \mathfrak{B}$ is absolutely convex. Then, for each $V \in \mathfrak{B}$ consider ρ_V as the Minkowski functional of V, namely the function given by

$$\rho_V(x) = \inf\{\lambda > 0 \ : \ x \in \lambda V\},$$

with the convention $\inf \emptyset = +\infty$. Since V is absolutely convex, it is not hard to realize that ρ_V is an extended seminorm on X. Also, it is direct that $\mathfrak{T}(\{\rho_V \ : \ V \in \mathfrak{B}\}) \subseteq \tau$, since for each $V \in \mathfrak{B}$, ρ_V is τ-continuous. The other inclusion follows from the fact that $\mathcal{N}(0_X, \tau) = \mathcal{N}(0_X, \mathfrak{T}(\{\rho_V \ : \ V \in \mathfrak{B}\}))$ and that τ is already a group topology. $\square$

Proposition 4.8. *Let (X, τ) and (Y, σ) be two esns, $\mathcal{P} \subseteq \mathcal{S}(X, \tau)$ such that $\tau = \mathfrak{T}(\mathcal{P})$ and $T : X \to Y$ be a linear operator. Then, T is continuous if and only if for all $q \in \mathcal{S}(Y, \sigma)$ there exist $C > 0$ and a finite set $\{p_i\}_{i=1}^n \subseteq \mathcal{P}$ such that*

$$q(T(x)) \leq C \max\{p_i(x) \ : \ i = 1, \dots, n\}, \ \forall x \in X.$$

Proof. The necessity is direct, since, if T is continuous, then for each $q \in \mathcal{S}(Y, \sigma)$, we have that $q \circ T \in \mathcal{S}(X, \tau)$ and we can apply Proposition 4.4. For the sufficiency, fix $q \in \mathcal{S}(Y, \sigma)$ and let $C > 0$ and $\{p_i\}_{i=1}^n \subseteq \mathcal{P}$ be such that $q \circ T \leq C \max\{p_i \ : \ i = 1, \dots, n\}$. Then,

$$T^{-1}(\{y \in Y \ : \ q(y) \leq 1\}) \supseteq \bigcap_{i=1}^{n} \left\{x \in X \ : \ p_i(x) \leq \tfrac{1}{C}\right\},$$

where the latter set is known to be a neighborhood of 0_X. Since the family $\{q^{-1}([-1, 1]) \ : \ q \in \mathcal{S}(Y, \sigma)\}$ is a fundamental system of $\mathcal{N}(0_Y, \sigma)$, the conclusion follows. $\square$

We will end this section characterizing the dual space of X_{fin} when (X, τ) is an extended seminormed space. To do so, we will use a suitable adaptation of Hahn–Banach extension theorem (see Theorem 4.10 below) in order to understand what are the duals of the subspaces of X.

Recall first that, in a real vector space X, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be an *extended sublinear function* if $f(0) = 0$ and it satisfies

(a) for all $x \in X$ and $\lambda > 0$, $f(\lambda x) = \lambda f(x)$; and
(b) for all $x, y \in X$, $f(x + y) \leq f(x) + f(y)$.

Lemma 4.9. *Let (X, τ) be a real esns and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended sublinear function. Then, the following assertions are equivalent:*

(i) *f is τ-continuous.*
(ii) *f is τ-continuous at 0_X.*
(iii) *There exists $\rho \in \mathcal{S}(X, \tau)$ such that $f \leq \rho$.*

Proof.

(i) $\Rightarrow$ *(ii)*: Direct.

(ii) $\Rightarrow$ *(iii)*: Since f is continuous at 0_X, $f^{-1}((-1, 1)) \in \mathcal{N}(0_X, \tau)$. Then, by Proposition 4.7, there exists an absolutely convex neighborhood U of 0_X with $U \subseteq f^{-1}((-1, 1))$. Therefore, the Minkowski functional ρ_U is τ-continuous and, since f is positively homogeneous, $f \leq \rho_U$.

(iii) $\Rightarrow$ *(i)*: Let $\rho \in \mathcal{S}(X, \tau)$ such that $f \leq \rho$. By classical analysis we have that $f\big|_{X^\rho_{\text{fin}}}$ is continuous in $(X^\rho_{\text{fin}}, \tau_\rho)$, and therefore it is also continuous in $(X^\rho_{\text{fin}}, \tau)$. Now, let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X τ-converging to $x \in X$. Without loss of generality, we may assume that $(x_\lambda)_{\lambda\in\Lambda} \subset x + X^\rho_{\text{fin}}$. Noting that for each $\lambda \in \Lambda$ we have

$$f(x_\lambda) \leq f(x) + f(x_\lambda - x) \text{ and } f(x) \leq f(x_\lambda) + f(x - x_\lambda)$$

and recalling that $f(x - x_\lambda) \to 0$ and $f(x_\lambda - x) \to 0$, we have that

$$\limsup f(x_\lambda) \leq f(x) \leq \liminf f(x_\lambda),$$

and therefore, $f(x_\lambda) \to f(x)$, which finishes the proof. $\square$

Theorem 4.10. *Let (X, τ) be a real esns and let M be a subspace of X. Then, for any linear functional $\phi : M \to \mathbb{R}$ and any continuous extended sublinear function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $Z := f^{-1}(\mathbb{R})$ is a subspace of X and $\phi \leq f\big|_M$, there exists a continuous linear functional $\hat{\phi} : X \to \mathbb{R}$ such that*

$$\hat{\phi}\big|_M = \phi \qquad \textit{and} \qquad \hat{\phi} \leq f.$$

Proof. Since f is τ-continuous, we have that Z is a τ-open subspace. Choose a Hamel basis $\mathcal{H}$ generating Z and M and denote $N := \mathscr{C}_{\mathcal{H}}(Z)$. Clearly, by Lemma 3.4 we get that (N, τ) is a discrete space. Denote $\phi_1 = \phi\big|_{M\cap Z}$ and $\phi_2 = \phi\big|_{M\cap N}$. We will extend both functionals separately.

On one hand, note that $\phi_1 \leq f\big|_{M\cap Z}$ and so, by the classic Hahn–Banach extension theorem, there exists a linear functional $\hat{\phi}_1 : Z \to \mathbb{R}$ such that $\hat{\phi}_1\big|_{M\cap Z} = \phi_1$ and $\hat{\phi}_1 \leq f\big|_Z$.

On the other hand, since (N, τ) is discrete, we can extend continuously ϕ_2 to N directly: Just choose any subspace N_0 such that $N = (M \cap N) \oplus N_0$ and define the extension as $\hat{\phi}_2 = \phi_2 \circ P_{M\cap N, N_0}$, which clearly is a τ-continuous linear functional.

The proof is finished considering the extension of ϕ as $\hat{\phi} := \hat{\phi}_1 \circ P_{Z,N} + \hat{\phi}_2 \circ P_{N,Z}$. $\square$

Remark 7. Observe that the above result cannot be extended just erasing the hypothesis that $f^{-1}(\mathbb{R})$ is a subspace of X. Indeed, in [16, Counterexample to (3)], Simons provided a counterexample in $\mathbb{R}^2$ of an extended linear functional f dominating a linear functional ϕ in a subspace, for which there is no algebraic linear extension $\hat{\phi}$ preserving the inequality $\hat{\phi} \leq f$. Therefore, endowing $\mathbb{R}^2$ with the discrete topology, Simons' counterexample applies to our context as well.

Corollary 4.11. *Let X be a vector space over $\mathbb{K}$ and M be a subspace of X. Then, for any linear functional $\phi : M \to \mathbb{K}$ and any extended seminorm ρ on X (not necessarily continuous) such that $|\phi| \leq \rho\big|_M$, there exists a linear functional $\hat{\phi} : X \to \mathbb{K}$ such that*

$$\hat{\phi}\big|_M = \phi \qquad \text{and} \qquad |\hat{\phi}| \leq \rho.$$

In particular, if (X, τ) is an esns, then each element $\phi \in M^*$ admits a linear extension $\hat{\phi} \in X^*$.

Proof. If $\mathbb{K} = \mathbb{R}$, the proof follows directly from Theorem 4.10 endowing X with the topology τ_ρ and observing that, since ρ is an extended seminorm, $\hat{\phi} \leq \rho$ is equivalent to $|\hat{\phi}| \leq \rho$. Assume then that $\mathbb{K} = \mathbb{C}$ and let us denote by $\Re(\phi)$ the real part of ϕ. It is easy to see that $\Re(\phi) \leq |\phi| \leq \rho\big|_M$ and therefore, since any vector space over $\mathbb{C}$ is also a vector space over $\mathbb{R}$, there exists a real linear functional $\hat{\phi}_r : X \to \mathbb{R}$ such that $\hat{\phi}_r\big|_M = \Re(\phi)$ and $|\hat{\phi}_r| \leq \rho$. Then, considering the endomorphism φ_i induced by i given in equation (1) and applying [9, Lemma 6.3], we get that $\hat{\phi} := \hat{\phi}_r - i(\hat{\phi}_r \circ \varphi_i)$ is a complex linear functional on X satisfying $\hat{\phi}\big|_M = \phi$ and $|\hat{\phi}| \leq \rho$.

The second part of the corollary follows directly from Lemma 4.9. □

Corollary 4.12. *Let (X, τ) be an esns and M be a subspace of X. Then,*

$$M^* \approx X^*/M^\perp,$$

where $M^\perp$ stands for the annihilator of M.

Proof. Let us consider first the map $\phi : X^* \to M^*$ given by $\phi(x^*) = x^*\big|_M$. By Corollary 4.11, we have that ϕ is an onto homomorphism. Also, it is direct that $\mathrm{Ker}(\phi) = M^\perp$.

Therefore, taking any linear lifting $\ell : X^*/M^\perp \to X^*$, we have that $\phi \circ \ell$ is an isomorphism between $X^*/M^\perp$ and M^*, which finishes the proof. □

Proposition 4.13. *Let (X, τ) be an esns and $\{\tau_\alpha \;:\; \alpha \in \mathscr{A}\}$ be a directed locally convex generating family of τ. Then,*

$$(X_{\text{fin}}, \tau)^* \approx \varinjlim \left[(X^\alpha_{\text{fin}}, \tau_\alpha)^*, f_{\beta\alpha} \right],$$

where, whenever $\tau_\beta \subseteq \tau_\alpha$, $f_{\beta\alpha} : (X^\beta_{\text{fin}}, \tau_\beta)^* \to (X^\alpha_{\text{fin}}, \tau_\alpha)^*$ is given by $f_{\beta\alpha}(x^*_\beta) = x^*_\beta\big|_{X^\alpha_{\text{fin}}}$.

Proof. By Corollary 4.12, we know that $(X_{\text{fin}})^* \approx X^*/(X_{\text{fin}})^\perp$. Now, let $\ell : X^*/(X_{\text{fin}})^\perp \to X^*$ be a linear lifting. Since $\{\tau_\alpha \;:\; \alpha \in \mathscr{A}\}$ is directed we have, by Proposition 3.32, that $X^* = \bigcup_{\alpha \in \mathscr{A}} (X, \tau_\alpha)^*$. We define the assignation $\phi : X^* \to \varinjlim \left[(X^\alpha_{\text{fin}}, \tau_\alpha)^*, f_{\beta\alpha} \right]$ as follows: Whenever $x^* \in (X, \tau_\alpha)^*$,

$$\phi(x^*) = \left[x^*\big|_{X^\alpha_{\text{fin}}} \right].$$

To simplify notation, for $x^* \in X^*$ we will denote $x^*_\alpha = x^*\big|_{X^\alpha_{\text{fin}}}$, for each $\alpha \in \mathscr{A}$. Let us prove now that ϕ is well defined. Assume that $x^* \in X^*$ is τ_α-continuous and τ_β-continuous. Since the generating family is directed, there exists $\gamma \in \mathscr{A}$ such that $\tau_\alpha \subseteq \tau_\gamma$ and $\tau_\beta \subseteq \tau_\gamma$ and so, x^* is also τ_γ-continuous. Further, we have that $X_{\text{fin}}^\gamma \subseteq X_{\text{fin}}^\alpha$ and $X_{\text{fin}}^\gamma \subseteq X_{\text{fin}}^\beta$ and therefore

$$f_{\alpha\gamma}(x_\alpha^*) = x_\gamma^* = f_{\beta\gamma}(x_\beta^*).$$

So, $[x_\alpha^*] = [x_\beta^*]$, showing that ϕ is well defined. Also, it is not hard to see that ϕ is an onto homomorphism. To finish the proof, we will show that $\phi \circ \ell$ is bijective. To do so, it is sufficient to prove that $\mathrm{Ker}(\phi) = (X_{\text{fin}})^\perp$.

Fix first $x^* \in (X_{\text{fin}})^\perp \setminus \{0\}$. Then, since x^* is τ-continuous, we have that $\mathrm{Ker}(x^*)$ is a closed subspace of codimension 1. Let $x_0 \in X$ be any vector such that $\langle x^*, x_0\rangle = 1$. Therefore, since $X_{\text{fin}} \subseteq \mathrm{Ker}(x^*)$, then there exists $\rho \in \mathcal{S}(X,\tau)$ such that $\rho(x_0) = +\infty$. Noting that the projection $P : X \to \mathbb{K} \cdot x_0$ given by $P(x) = \langle x^*, x\rangle x_0$ is continuous, we have that $\tilde{\rho} = \rho \circ P \in \mathcal{S}(X,\tau)$ and then $X_{\text{fin}}^{\tilde{\rho}} \subseteq \mathrm{Ker}(x^*)$. Therefore, $\mathrm{Ker}(x^*)$ is τ-open. Now, since by Proposition 2.1 the set $\bigcup_{\alpha\in\mathscr{A}} \mathcal{N}(0_X, \tau_\alpha)$ is a fundamental system of $\mathcal{N}(0_X, \tau)$, we get that there exists $\alpha \in \mathscr{A}$ such that $\mathrm{Ker}(x^*)$ is τ_α-open (by [8, Ch. III, §2, Corollary of Proposition 4]). Therefore, $X_{\text{fin}}^\alpha \subseteq \mathrm{Ker}(x^*)$ and so, $x_\alpha^* = 0$. Finally, noting that $x^* \in (X, \tau_\alpha)^*$ (see Proposition 3.31) we can write

$$\phi(x^*) = [x_\alpha^*] = [0].$$

Thus, $(X_{\text{fin}})^\perp \subseteq \mathrm{Ker}(\phi)$. For the other inclusion, it is enough to note that if $x^* \in \mathrm{Ker}(\phi)$, then there exists $\alpha \in \mathscr{A}$ such that $x_\alpha^* = 0$. Then, since $X_{\text{fin}} \subseteq X_{\text{fin}}^\alpha$, we have that $x^* \in (X_{\text{fin}})^\perp$, finishing the proof. □

4.2. Countable esns

In this section we will focus on the special structure of extended seminormed spaces for which their topology can be induced by a countable family of extended seminorms.

Definition 4.14. Let (X,τ) be an esns over $\mathbb{K}$. We say that (X,τ) is a **countable extended seminormed space** if there exists a countable family $\mathcal{P} = \{\rho_n : X \to [0,+\infty] \ : \ n \in \mathbb{N}\}$ of extended seminorms such that $\tau = \mathfrak{T}(\mathcal{P})$.

If $\mathcal{P} = \{\rho_n \ : \ n \in \mathbb{N}\}$ is a family of extended seminorms, we can define the equivalent family $\tilde{\mathcal{P}} = \{\ \tilde{\rho}_n \ : \ n \in \mathbb{N}\}$ where

$$\tilde{\rho}_n = \max\{\rho_1, \dots, \rho_n\}.$$

Clearly $\mathfrak{T}(\mathcal{P}) = \mathfrak{T}(\tilde{\mathcal{P}})$, and therefore, when (X,τ) is a countable esns, we may assume that the countable family of extended seminorms is pointwise nondecreasing. In the following, when the countable family $\mathcal{P}$ has been fixed, we will use the notation

$$X_{\text{fin}}^n := X_{\text{fin}}^{\tilde{\rho}_n} = \{x \in X \ : \ \rho_1(x) < +\infty, \dots, \rho_n(x) < +\infty\}. \tag{6}$$

For any metric d on X, we will write $B_d(x,r)$ to denote the closed d-ball centered in $x \in X$ and of radius $r > 0$ and by τ_d the topology on X induced by d.

Also, for each extended seminorm ρ on X, each point $x \in X$ and each $r > 0$ we will write $B_{(X,\rho)}(x,r)$ to denote the closed ball centered in x of radius r induced by ρ, that is

$$B_{(X,\rho)}(x,r) := \{x' \in X \ : \ \rho(x'-x) \le r\}. \tag{7}$$

This notation will be natural after Lemma 4.15 below, on which we show that any countable esns is metrizable. Finally, recall that a metric d over X is said to be *translation-invariant* if

$$\forall x, y, z \in X,\ d(x, y) = d(x + z, y + z).$$

Lemma 4.15. *Let (X, τ) be a Hausdorff countable esns. Then, there exists a translation-invariant metric $d : X \times X \to \mathbb{R}_+$ such that $\tau = \tau_d$ and the balls*

$$B_d(0, r) := \{x \in X \ : \ d(0, x) \leq r\}$$

are absolutely convex.

Proof. We will proceed as in [15, Ch. I, Theorem 6.1]. Since (X, τ) is a countable esns, there exists a countable basis $\{V_n \ : \ n \in \mathbb{N}\}$ of $\mathcal{N}(0_X, \tau)$ such that each V_n is absolutely convex and, for all $n \in \mathbb{N}$, $V_{n+1} + V_{n+1} \subseteq V_n$.

Now, for each nonempty finite subset $F \subseteq \mathbb{N}$ we define $V_F = \sum_{n\in F} V_n$ and $p_F = \sum_{n\in F} 2^{-n}$. In [15, Ch. I, Theorem 6.1] it is proven that the translation-invariant function

$$d(x, y) = \min(1, \inf\{p_F \ : \ F \subseteq \mathbb{N} \text{ finite, } x - y \in V_F\})$$

is a metric on X with $\tau = \tau_d$ and with balanced balls. We will prove now that the balls induced by d are convex.

Let $r > 0$, $x, y \in B_d(0_X, r)$ and $\lambda \in (0, 1)$. Observing that for $r \geq 1$ the ball $B_d(0_X, r) = X$, we only need to prove the convexity for $r \in (0, 1)$. Let us denote by $\mathbb{D}$ the dyadic numbers in $[0, 1]$. Since the dyadic numbers are dense in $[0, 1]$, there exists a decreasing sequence $(r_n) \subseteq \mathbb{D}$ such that $r_n \searrow r$. Now, for each $n \in \mathbb{N}$ we have that $d(x, 0_X) < r_n$ and $d(0_X, y) < r_n$. Therefore there exist two finite subsets $F(n, x), F(n, y) \subseteq \mathbb{N}$ such that

$$x \in V_{F(n,x)}, \qquad y \in V_{F(y,n)}, \qquad \text{and} \qquad p_{F(n,x)} < r_n,\ p_{F(n,y)} < r_n.$$

Also, since r_n is dyadic, there exists a finite subset $F(n) \subseteq \mathbb{N}$ such that $p_{F(n)} = r_n$. The latter inequality shows that $V_{F(n,x)} \subseteq V_{F(n)}$ and $V_{F(n,y)} \subseteq V_{F(n)}$ and therefore $x, y \in V_{F(n)}$. Since $V_{F(n)}$ is absolutely convex, we obtain that $\lambda x + (1 - \lambda)y \in V_{F(n)}$ and therefore

$$d(0_X, \lambda x + (1 - \lambda)y) < r_n.$$

We conclude that $d(0_X, \lambda x + (1 - \lambda)y) \leq r$ and so $B_d(0_X, r)$ is convex. $\square$

Theorem 4.16. *Let (X, τ) be a Hausdorff countable esns, (Y, σ) be an esns and $T : X \to Y$ be a linear operator. The following assertions are equivalent:*

(i) *T is τ-σ-continuous.*
(ii) *T maps τ-bounded sets into σ-bounded sets.*
(iii) *For each sequence $(x_n) \subset X$ τ-converging to 0_X, the set $\{T(x_n) \ : \ n \in \mathbb{N}\}$ is σ-bounded.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are direct, and therefore we only need to prove $(iii) \Rightarrow (i)$. Reasoning by contradiction, suppose that (iii) holds but T is not continuous. Since (X, τ) is metrizable, the continuity of T is characterized by sequences and therefore, by Proposition 3.28, there exists a sequence $(x_n) \subseteq X$ converging to 0_X such that $T(x_n) \not\to 0_Y$. Without loss of generality, we may assume that there exists $q \in \mathcal{S}(Y, \sigma)$ such that for all $n \in \mathbb{N}$, $q(T(x_n)) > 1$.

Since $A := \{T(x_n) \ : n \in \mathbb{N}\}$ is bounded, by Definition 3.27 there exist a finite set $\{y_i\}_{i=1}^k \subseteq Y$ and a finite set $\{\alpha_i\}_{i=1}^k \subseteq \mathbb{R}$ with $\alpha_i > 0$ for each $i \in \{1, \ldots, k\}$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_{(Y,q)}(y_i, \alpha_i),$$

where, for $y \in Y$ and $\alpha > 0$, the set $B_{(Y,q)}(y, \alpha)$ is given as in equation (7). We have that at least for one $i \in \{1, \ldots, k\}$, the set of indexes $\{n \in \mathbb{N} \ : \ T(x_n) \in B_{(Y,q)}(y_i, \alpha_i)\}$ is infinite. Thus, up to subsequences, we may assume that $A \subseteq B_{(Y,q)}(y, \alpha)$ for some $y \in Y$ and some $\alpha > 0$.

Suppose first that $q(y) = +\infty$. Then, for each $n \in \mathbb{N}$, $q(T(x_n)) = +\infty$, but $A \subseteq y + X^q_{\text{fin}}$. Therefore, if we consider the new sequence $(\tilde{x}_n) \subseteq X$ given by $\tilde{x}_n = \frac{1}{n}x_n$ we will get that for each $n \in \mathbb{N}$, $T(\tilde{x}_n) \in \frac{1}{n}y + X^q_{\text{fin}}$ and so, the set $\{T(\tilde{x}_n) \ : \ n \in \mathbb{N}\}$ is unbounded (since $y \neq 0_Y$). This latter statement cannot hold, since $\tilde{x}_n \to 0$.

We conclude then that $q(y) < +\infty$, and therefore there exists $\beta > 0$ such that $A \subseteq B_{(Y,q)}(0_Y, \beta)$. Now, let $\mathcal{P} = \{\rho_k \ : \ k \in \mathbb{N}\}$ be a countable family of extended seminorms over X such that $\tau = \mathfrak{T}(\mathcal{P})$. Since $x_n \to 0_X$, we can build a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$

$$\rho_k(x_{n_j}) < \frac{1}{j^2}, \quad \forall j \geq k.$$

Therefore, the new sequence $(\tilde{x}_j) \subseteq \mathbb{N}$ given by $\tilde{x}_j = jx_{n_j}$ is still convergent to 0_X. But,

$$q(T(\tilde{x}_j)) = jq(T(x_{n_j})) > j,$$

and so $\{T(\tilde{x}_j) \ : \ j \in \mathbb{N}\}$ is unbounded, leading to a contradiction and finishing the proof. □

Lemma 4.17. *Let (X, τ) and (Y, σ) be two Hausdorff countable esns with (X, τ) complete, and let $T : X \to Y$ be a continuous map satisfying*

$$\forall r > 0, \ \exists p(r) > 0, \ \overline{T(B_X(0_X, r))} \supseteq B_Y(0_Y, p(r)).$$

Then, for all $\varepsilon > 0$, $T(B_X(0_X, r + \varepsilon)) \supseteq B_Y(0_Y, p(r))$.

Proof. The proof follows exactly as the Lemma previous to [15, Ch. III, Theorem 2.1]. □

Theorem 4.18 *(Open mapping theorem). Let (X, τ) and (Y, σ) be two complete Hausdorff countable esns, and let $T : X \to Y$ be a continuous linear operator such that*

$$\forall p \in S(X, \tau), \ \exists q \in S(Y, \sigma), \ T(X^p_{\text{fin}}) = Y^q_{\text{fin}}.$$

Then T maps τ-open sets of X to σ-open sets of Y.

Proof. Fix first $r > 0$ and let $V = B_X(0_X, \frac{r}{2})$. We have that V is absolutely convex and that the Minkowski functional $\rho_V \in \mathcal{S}(X, \tau)$. Therefore, there exists $q \in S(Y, \sigma)$ such that $T(X^{\rho_V}_{\text{fin}}) = Y^q_{\text{fin}}$. Also, since $X^{\rho_V}_{\text{fin}} = \bigcup_{n\in\mathbb{N}} nV$, we have that, by the linearity of T and the σ-closedness of Y^q_{fin},

$$Y^q_{\text{fin}} = \bigcup_{n\in\mathbb{N}} nT(V) = \bigcup_{n\in\mathbb{N}} \overline{nT(V)}.$$

Since Y^q_{fin} is a closed metric space of a complete metric space, we have that it is a Baire space, and so there exists $n \in \mathbb{N}$ such that $\overline{nT(V)}$ has nonempty $\sigma\big|_{Y^q_{\text{fin}}}$-interior. Thus, since Y^q_{fin} is also σ-open, we have that

$\overline{nT(V)}$ has nonempty σ-interior. Since φ_n (as given in equation (1)) is a topological isomorphism, we have that $\overline{T(V)}$ has nonempty σ-interior and therefore, there exists $p(r/2) > 0$ such that

$$\overline{T(V)} \supseteq B_Y(0_Y, p(r/2)).$$

We can apply Lemma 4.17 for $\varepsilon = \frac{r}{2}$, concluding that

$$T(B_X(0_X, r)) \supseteq B_Y(0_Y, p(r/2)),$$

which finishes the proof. □

Remark 8. In Theorem 4.18, the condition

$$\forall p \in \mathcal{S}(X,\tau), \exists q \in \mathcal{S}(Y,\sigma), T(X^p_{\text{fin}}) = Y^q_{\text{fin}},$$

is also necessary. Indeed, if $T : X \to Y$ is open, then for any $p \in \mathcal{S}(X,\tau)$, the set $T(X^p_{\text{fin}})$ is a σ-open subspace of Y. Therefore, the map $q : Y \to [0, +\infty]$ given by

$$q(y) = \begin{cases} 0 & \text{if } y \in T(X^p_{\text{fin}}) \\ +\infty & \text{otherwise,} \end{cases}$$

is a σ-continuous extended seminorm with $Y^q_{\text{fin}} = T(X^p_{\text{fin}})$.

Corollary 4.19 *(Closed graph theorem). Let (X,τ) and (Y,σ) be two Hausdorff complete countable esns and $u : X \to Y$ be a linear operator. Then, u is τ-σ-continuous if and only if*

(i) *for each $q \in \mathcal{S}(Y,\sigma)$, there exists $p \in \mathcal{S}(X,\tau)$ such that $u^{-1}(Y^q_{\text{fin}}) = X^p_{\text{fin}}$; and*
(ii) *the graph of u*

$$\text{gph}(u) = \{(x, u(x)) \;:\; x \in X\}$$

is $(\tau \times \sigma)$-closed in $X \times Y$.

Proof. The necessity is direct. For the sufficiency, let us suppose conditions (i) and (ii) hold. It is not hard to realize that $(X \times Y, \tau \times \sigma)$ is a Hausdorff complete countable esns and that $G = \text{gph}(u)$ is a closed subspace. Therefore, $(G, \tau \times \sigma)$ is also a Hausdorff complete countable esns. Let us consider now the map $T : G \to X$ given by $T(x, u(x)) = x$. Clearly T is bijective and, since it is the restriction of the parallel projection $P_{X,Y}$ to G, T is also continuous.

Now, let us consider an extended seminorm $\rho \in \mathcal{S}(G, \tau \times \sigma)$. The finite space G^ρ_{fin} is open, and therefore there exists a $(\tau \times \sigma)$-open set $V = V_X \times V_Y \subseteq X \times Y$ such that $G^\rho_{\text{fin}} = V \cap G$. Moreover, since G^ρ_{fin} is a vector space, we have that $\text{span}(V) \cap G = G^\rho_{\text{fin}}$. Let us write $\text{span}(V) = M_X \times M_Y$, where $M_X = \text{span}(V_X)$ and $M_Y = \text{span}(V_Y)$. We can compute

$$T(G^\rho_{\text{fin}}) = M_X \cap u^{-1}(M_Y).$$

Since M_X and M_Y are open subspaces, there exist $p_1 \in \mathcal{S}(X,\tau)$ and $q \in \mathcal{S}(Y,\sigma)$ such that $M_X = X^{p_1}_{\text{fin}}$ and $M_Y = Y^q_{\text{fin}}$: For example, we can define p_1 (resp. q) as $p_1(x) = 0$ (resp. $q(y) = 0$) if $x \in M_X$ (resp. $y \in M_Y$)

and $p_1(x) = +\infty$ (resp. $q(y) = +\infty$) otherwise. Therefore, by condition (ii), there exists $p_2 \in \mathcal{S}(X, \tau)$ such that $u^{-1}(M_Y) = X_{\text{fin}}^{p_2}$. Finally, we get that

$$T(G_{\text{fin}}^{\rho}) = X_{\text{fin}}^{\max(p_1,p_2)}.$$

Now, by Theorem 4.18, we get that T is also $(\tau\times\sigma)$-τ-open, which implies that the map $T^{-1} : x \mapsto (x, u(x))$ is τ-$(\tau \times \sigma)$-continuous. Noting that $u = P_{Y,X} \circ T^{-1}$, the proof is complete. $\square$

4.3. Splittable esns

The main structural principles of fundamental etvs, etvs and esns have been already established. We will end this work with what is perhaps the main structural remaining question: Can an etvs or an esns be split into two τ-supplement spaces where one of them is the finite space? The problem of splitting has been largely studied in the context of normed spaces and also has been treated for general topological groups (see, e.g., [12]).

Example 3.17 shows that in general etvs, such a decomposition cannot be always performed. Nevertheless, it is still an open question what happens in the case of esns. Theorem 4.10 gives us the first partial answer: Whenever we are able to write $X = X_{\text{fin}} \oplus_\tau N$, we also can perform extensions of linear functionals from $(X_{\text{fin}})^*$ to X^*, and so the extended seminormed spaces are the framework to search for an answer.

Definition 4.20. An esns (X, τ) is said to be a **splittable esns**, if there exists a subspace N of X such that

$$X = X_{\text{fin}} \oplus_\tau N.$$

Of course, every fundamental elcs is a splittable esns, since each algebraic complement of the finite space is also a topological supplement. Unfortunately, this situation doesn't hold in esns. Moreover, Proposition 4.21 shows that, in the context of countable esns, we always can find an algebraic complement of the finite space which fails to be a topological supplement.

Proposition 4.21. *Let (X, τ) be a countable esns of infinite dimension such that $X_{\text{fin}} \supsetneq \{0_X\}$. Then, one of the following holds:*

(i) *(X, τ) is a fundamental elcs.*
(ii) *There exists a subspace N of X such that X_{fin} and N are algebraic complements in X but not τ-supplements.*

Proof. Let $\mathcal{P} = \{\rho_n : n \in \mathbb{N}\}$ be a nondecreasing sequence of extended seminorms on X generating the topology τ. If we assume that (i) doesn't hold, then we can find a subsequence $\mathcal{P}_1 = \{\rho_{n_k} : k \in \mathbb{N}\}$ such that

$$X_{\text{fin}}^{n_k} \supsetneq X_{\text{fin}}^{n_{k+1}}, \ \forall k \in \mathbb{N}.$$

Note that, since the sequence $\mathcal{P}$ is nondecreasing, we have that $\mathfrak{T}(\mathcal{P}) = \mathfrak{T}(\mathcal{P}_1)$. Therefore, without loss of generality, we may assume that $\mathcal{P} = \mathcal{P}_1$. Now, we can construct inductively a sequence of subspaces $(N_n)_{n\in\mathbb{N}}$ such that, denoting $X_{\text{fin}}^0 = X$, we have that

$$\forall n \in \mathbb{N}, \ X_{\text{fin}}^{n-1} = X_{\text{fin}}^n \oplus_{\tau_n} N_n,$$

where $\tau_n = \mathfrak{T}(\rho_n)$. For each $n \in \mathbb{N}$ we can select $x_n \in N_{n+1} \setminus \{0_X\}$ such that $\rho_n(x_n) < \frac{1}{n}$. Therefore, since $\mathcal{P}$ is nondecreasing, we have that $x_n \to 0_X$. Also, we have that the set $\{x_n : n \in \mathbb{N}\}$ is linearly independent by construction.

Now, since $X_{\text{fin}} \neq \{0_X\}$, there exists $x_f \in X_{\text{fin}} \setminus \{0_X\}$ and we can consider the set $H = \{x_n + x_f : n \in \mathbb{N}\}$ which is still linearly independent. Since $H \cap X_{\text{fin}} = \emptyset$, there exists a Hamel basis $\mathcal{H}$ of X generating X_{fin} and with $H \subseteq \mathcal{H}$. Finally, if we fix $N = \mathscr{C}_{\mathcal{H}}(X_{\text{fin}})$, we have that N cannot be a τ-supplement with X_{fin}, since it is not closed: by construction, the point x_f belongs to $\overline{N} \setminus N$. □

Proposition 4.22. *Let (X, τ) be an esns such that (X_{fin}, τ) is Hausdorff and $\dim[X_{\text{fin}}] < \infty$. Then, there exists a subspace N of X such that $X = X_{\text{fin}} \oplus_\tau N$.*

Proof. Let $\dim[X_{\text{fin}}] = n$ and $\{b_j\}_{j=1}^n$ be a Hamel basis of X_{fin}. Since (X_{fin}, τ) is Hausdorff, for each $j \in \{1, \ldots, n\}$ there exists $\rho_j \in \mathcal{S}(X, \tau)$ such that $\rho_j(b_j) = 1$. Define $\rho = \max\{\rho_1, \ldots, \rho_n\}$. We then have that ρ is a norm on X_{fin} and therefore X_{fin} is τ_ρ-closed.

Since $(X^\rho_{\text{fin}}, \tau_\rho)$ is a locally convex space and X_{fin} is a τ_ρ-closed subspace of finite dimension, we have that there exists a subspace N_0 of X^ρ_{fin} such that $X^\rho_{\text{fin}} = X_{\text{fin}} \oplus_{\tau_\rho} N_0$. Then, since (X, τ_ρ) is an elcs, there exists a subspace N_1 of X such that $X = X^\rho_{\text{fin}} \oplus_{\tau_\rho} N_1$. If we denote $N = N_0 \oplus N_1$, we get that

$$X = X_{\text{fin}} \oplus_{\tau_p} N.$$

Now, since X_{fin} is of finite dimension and ρ is a norm on X_{fin}, we can conclude that $\tau_\rho\big|_{X_{\text{fin}}} = \tau\big|_{X_{\text{fin}}}$, according to the fact that there exists a unique Hausdorff topology compatible with the vector structure over a finite dimensional vector space. Then, since the parallel projection $P_{X_{\text{fin}},N}$ is τ_ρ-τ_ρ-continuous, we get that it is also τ-τ-continuous, and the conclusion follows from Corollary 2.8. □

5. Concluding remarks

Clearly, this work opens a new field for researchers in functional analysis and topology. There are still many questions to work through and we hope this to become a rich field in the future. There are some topics we are able to propose for further development:

1. There is still a necessity to study a suitable theory of duality. An interesting way to do this is, for an esns (X, τ), to study the "finest locally convex topology that is still coarser than τ" and apply classical duality theory to topologize the dual space.
2. In the context of extended normed and seminormed spaces, the subdifferential calculus and the optimization theory related to it remains undeveloped. Along this line, it is important to study Hahn–Banach-like separation theorems in this framework.
3. Further relations between the Bornology theory and the extended topologies presented in this work would be a very interesting research.

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