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SCHOOL CHOICE WITH RANDOM ASSIGNMENTS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN ECONOMÍA APLICADA
MEMORIA PARA OPTAR AL TITULO DE INGENIERO CIVIL MATEMÁTICO

MARTÍN PABLO CASTILLO QUINTANA

PROFESOR GUÍA:
JUAN ESCOBAR CASTRO

MIEMBROS DE LA COMISIÓN:
JOSÉ SOTO SAN MARTÍN
MATTEO TRIOSSI VERONDINI

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El objetivo de este trabajo es estudiar el problema de asignación escolar como uno de asignación probabilística y poder entender como diversos mecanismos de asignación escolar se desempeñan en términos de las probabilidades que le asignan a los alumnos de poder acceder a los colegios. Para éste fin se asume que el planificador central determina una función que les permite generar preferencias sobre loterías desde preferencias ordinales por los colegios, estás funciones se denominan *extensiones*.

Se elabora una nueva noción de equidad (estabilidad) la cual generaliza nociones previas tanto en la literatura de asignación escolar como en la de asignación probabilística. El resultado principal de éste trabajo corresponde a la caracterización, bajo supuestos razonables en las preferencias, del conjunto de asignaciones probabilísticas estables. También se desarrollan nuevos resultados de existencia de asignaciones probabilísticas estables y eficientes, se presentan resultados de mecanismos probabilísticos compatibles en incentivos y se evalúan los mecanismos de asignación escolar *Boston*, *Deferred Acceptance*, *Top Trading Cycles* y *Fraction Deferred Acceptance* en términos de eficiencia, estabilidad e incentivos.

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The aim of this work is to study the school choice problem as a random assignment problem and understand how real school choice programs perform in terms of the lotteries that they assign to students. For this purpose it is assumed that the central planner is endowed with a mapping that allows her to build preferences over lotteries from ordinal preferences, this mappings are called *extensions*.

I provide a new concept of fairness (stability) which generalizes previous notions of stability in school choice and fairness in the random assignment problem. The main result of this work is that the set of stable random matchings, under reasonable conditions on the preferences, is the same. An existence result for efficient and stable random matchings are provided, as well as new results of incentives for random mechanisms, and different school choice random mechanisms such as *Boston*, *Deferred Acceptance*, *Top Trading Cycles* and *Fraction Deferred Acceptance*, are evaluated in terms of stability, efficiency and strategy-proofness.

A los Quintana-Peña, Castillo-Astudillo y Warnken-Lihn...

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Contents

Chapter 1

Introduction

Nowadays price-based markets allocate and distribute large fraction of resources and goods in many societies around the world. Nevertheless there are things that social norms or laws prohibits to be assigned under price-based markets. Still societies need to find ways to assign them, and this is were *market design* and *matching* arises. Some examples of many important matching markets are dating, kidney exchange, college admissions, the assignment of doctors to hospitals and the assignment of students to schools.

While the theoretical scope of this work allows it to be applicable to numerous matching markets, the development has been inspired by the assignment of students to schools. This is because I had been working, as a programmer, on the team that is implementing the new centralized school choice admissions for Chile. Before the implementation of the new centralized admissions the Chilean secondary education system consisted on a hybrid system of state school managed by local governments and charter schools, this system generated unprecedented levels of socio-economic levels of segregation (Valenzuela, Bellei, and De Los Ríos, 2010) and thus the government decided to implement a reform.

1.1. School choice

In *School Choice* is the topic that studies, analyzes and designs the mechanisms under which the seats of schools are assigned to students. The main conflict in school choice is that almost everywhere in the world there are over-demanded public or charter schools, thus a lot of attention is put on the procedure that assign students to school in an efficient and fair manner.

Different societies might have different ideas of what is efficient and what is fair, and thus the policy makers should have a robust set of tools to design different assignment rules under different targets or constraints. For example in some societies the seats at over-demanded schools should be given to students that have higher scores on standardized tests, in others this might not be reasonable and believe that all students should have the same access opportunities to over-demanded schools regardless of the student's record. For the first case the problem is solved by the seminal works, for the other it is not crystal clear what is the best to do, and this work aims to contribute to the answer.

Abdulkadiroglu and Sönmez (2003) were the first to formulate the school choice problem as a *mechanism design* problem. Under this approach each student has a strict-preference list, each school has a a strict-priority list and fixed number of seats. There are three main criteria that would be ideal for a school choice mechanism to satisfy:

- *Efficiency*: No student can be better off without hurting other student.
- *Stability*: No student can better off by getting to a school where there is a student with worse priority at that school.
- *Strategy-proofness*: All students reporting their true preferences can be sustained as *Nash equilibrium* under the game that the mechanism induces.

Since efficiency and stability may not be compatible (Roth, 1982) a weaker version of efficiency for stable matchings was introduced:

- *Constrained efficiency*: No student can be better off without hurting other student, or breaking stability.

Two decades ago the *Boston Mechanism* and its variants were widely used in several districts of the United States. Its main flaw was on the incentives that fostered (Abdulkadiroglu and Sönmez, 2003; Glazerman and Meyer, 1994). The problem with this was that students whose parents were naive ended up in worse schools than students with strategic parents, and due to the fact that reports were not representative of the true preferences, they are not reliable for the welfare analysis.

Abdulkadiroglu and Sönmez (2003) proposed two compelling mechanisms that overcame this flaw on the incentives: the *Gale-Shapley Student Optimal Stable Mechanism*[1] (SOSM) which is constrained efficient (David Gale and L. S. Shapley, 1962) and strategy-proofness (Dubins and Freedman, 1981; Roth, 1982); and the *Top Trading Cycles Mechanism*[2] (TCC) which is efficient and strategy-proofness (Abdulkadiroglu and Sönmez, 2003).

Roth (1982) also showed that stability might be incompatible with efficiency and thus central planners, when selecting the allocation rule in matching markets, would have to choose one.

Some effort has been made to amend the tension between efficiency and stability. In particular Onur Kesten (2010) proposes the *Efficiency-Adjusted Deferred Acceptance Mechanism* (EADAM) that allow students to waive certain priorities and obtain a more efficient

[1] Which is a modified version of the mechanism that David Gale and L. S. Shapley (1962) proposed for the College admissions.

[2] Which is a modified version of the mechanism that L. Shapley and Scarf (1974) proposed for the housing market.

outcome, Dur, Gitmez, and O. Yılmaz (2015) generalize this idea and propose the *Student Exchange under Partial Fairness* class of algorithms which can reach all constrained efficient matchings that Pareto dominates SOSM, Morrill (2016) defines a weaker version of stability called *legal assignments* which states that matchings can only be blocked with legal assignments and thus justified envy is not enough to block a matching, Hakimov, Onur Kesten, et al. (2014) proposes the *Equitable Top Trading Cycles Mechanism* (ETTC) that modifies the way in which the trading power is distributed among students and eliminates justified envy due to pairwise exchanges.

In real school choice problems the priorities are not strict and many students have the same priority at a given school. Under this environment SOSM and TTC cannot operate. The way that this problem has been addressed is by randomly breaking the ties before running an algorithm. Erdil and Ergin (2008) showed that there is a welfare loss when breaking ties and then running SOSM, so constrained efficiency is lost. They also show that under the weak priority environment there is no constrained efficient and strategy-proof mechanism. Ehlers and Westkamp (2011) characterize the priority structure under which constrained efficiency and strategy-proofness are compatible, and it turns out to be too restrictive for the school choice framework.

The two most discussed ways of tie-breaking in the literature are *single* (STB) and *multiple tie-breaking* (MTB). Abdulkadiroğlu, Pathak, and Roth (2009) investigated the welfare loss of SOSM under STB and MTB. Using field data from New York City they found out that under STB more students are assigned to their top preferences, but MTB leaves less students unassigned. They also showed that there is no strategy-proof mechanism (stable or not) that Pareto dominates SOSM with STB. Pathak and Sethuraman (2011) proved that TTC with STB and MTB are equivalent in the distribution over the matchings that both mechanism induce. Ashlagi, Nikzad, and Romm (2015) compare mechanisms using an overall rank distribution and find out that SOSM under STB assign a constant fraction of students to one of their top choices, while SOSM under MTB a vanishing fraction of students are matched to one of their top choices. Ashlagi and Nikzad (2016) compare STB and MTB in a random preference environment with over-demanded schools and conclude that STB outperforms MTB. They propose to run STB only in popular schools and MTB in the rest. Ashlagi and Shi (2014) propose a new way of implementing the tie-breaking by allowing correlation in such a way that community cohesion is preserved. Shi (2016) uses discrete choice models to find optimal tie-breaking (among other things) that improve on welfare by maintaining stability.

As counter-intuitive it may seem uniformly randomizing over the weak-priority structure and then running a matching algorithm such as SOSM or TTC does not imply uniformly randomizing over the desired support of matchings, thus an analysis that focus on random matchings is justified.

On this work I do not take the approach of finding particular ways of tie-breaking. Instead I focus on finding *random matchings* and *random mechanisms* that would satisfy efficiency, stability and strategy-proofness requirements without breaking the ties. This introduces a new fairness constraint: envy among equal priority students at each school. I interpret two students having the same priority at a school as both having the same right to get to that school, this interpretation combined with the fact that schools are over-demanded yields an impossibility result in which there is no ex-post stable mechanism that eliminates justified envy among equal priority students. This impossibility result forces to work with random matchings such that elimination of justified envy among equal priority students could be feasible, thus justifying the approach of school choice as a *random assignment* problem.

If we are assigning lotteries to students then issues of efficiency, fairness and incentives should be taken in account as well as it is done in the deterministic case. Here is when school choice meets the random assignment to provide new solutions.

1.2. Random assignment

In matching theory random assignment is a field that studies how to assign lotteries over objects to agents embedded with preferences over the objects.

This literature was inaugurated by Hylland and Zeckhauser (1979), in their model every agent has cardinal utility over objects, and no priority structure over the objects. An implicit market procedure is developed which is efficient. An algorithm was provided to conduct the lotteries in such a way that original probabilities are respected and an ex-post efficient matching is obtained. Satterthwaite and Sonnenschein (1981) under the cardinal approach showed that every non-bossy and strategy-proof mechanism is a *Serial Dictatorship*[3]. Zhou (1990) proved[4] that under the cardinal utility environment there is no mechanism that is Pareto efficient, strategy-proof and has equal treatment of equals[5].

Abdulkadiroğlu and Sönmez (1998) were the first to adopt an ordinal structure of the preferences, and showed that a mechanism is Pareto efficient if and only if it is a *Serial Dictatorship.* They proposed the *Random Priority Mechanism*[6] (RP) and showed that it can be obtained as the TTC with random initial endowment, and as a consequence RP is ex-post Pareto efficient, anonymous and strategy-proof. Crès and Moulin (2001) introduced the *Probabilistic Serial Mechanism* (PS) which first order stochastically dominates RP. Bogomolnaia and Moulin (2001, 2002) introduced first order stochastic dominance as a notion of ex-ante efficiency: SD-efficiency[7], which is stronger than ex-post efficiency. They showed that RP is ex-post but not SD-efficient, not SD-envy-free and strategy-proof; PS is SD-efficient, SD-envy-free, but not strategy-proof. They also proved that there is no SD-efficient and strategy-proof mechanism with equal treatment of equals.

Abdulkadiroğlu and Sönmez (2003) characterized SD-efficiency as lotteries with sup-

[3]In fact their result is more general since they allow the set of objects to be non-finite, but require the utility functions to be linear and the mechanisms to be continuously differentiable on the reports.

[4]In response to the conjecture proposed by D Gale (1987) which asked for the existence of a Pareto efficient, envy-free and strategy-proof mechanism under the cardinal utility framework.

[5]On his work this property was called *Symmetry*, and it is weaker than the concept of symmetry proposed by D Gale (ibid.).

[6]They called it *Random Serial Dictatorship.*

[7]On their work it was called *Ordinally Efficiency.*

port on a particular class of matching called *undominated sets of assignments* which is a refinement of the set of efficient matchings. O Kesten (2006) showed that TTC with equal division and PS are equivalent. Katta and Sethuraman (2006) extended PS to the case with weak-preferences and concluded under this domain that SD-strategy-proofness is incompatible with SD-efficiency and SD-envy-free. Che and Kojima (2010) showed that when the market becomes too large PS and RP converge. Saban and Sethuraman (2014) and Schulman and Vazirani (2012) study the random assignment problem when agents have lexicographic preferences. WJ Cho (2012, 2014), Wonki Cho (2013), and W. J. Cho and Dogan (2016) introduced the *extensions* which are mapping from preferences over objects to preferences over lotteries, and evaluate assignment rules under different extensions, this approach is one of the cornerstones of this work.

1.3. Preferences over lotteries

In economics expected utility has been the most common approach for building preferences over lotteries. This work takes some distance from the expected utility theory and considers different ways in which preferences over lotteries can be built.

There might be different sources of critiques to expected utility, one of the oldest was made by Knight (1921) arguing that expected utility is a theory of risk rather than one of uncertainty. Later Allais (1955) and Ellsberg (1961) gave experimental evidence that contradicted expected utility theory. Another reason to depart from expected utility, even in the case at which we believe that real preferences of students and their parents are as expected utility tell, is that central planners may not have the means for obtaining the cardinal preferences of students, thus there may be lotteries that the central planner would not know how to compare, and then allowing for the preferences to be incomplete may be a reasonable feature of a model.

1.4. Related literature

There is few literature approaching stability in probabilistic environments. Vate (1989) in the marriage problem characterized stable matchings as extreme points of a polytope, later this result was simplified and extended by Rothblum (1992) to the College admissions problem. Roth, Rothblum, and Vande Vate (1993) exploited this linear programming formulation and explored stability requirements over random matchings[8]. Their stability concept for random matchings is equivalent to the ex-post stability and their strong stability notion to the vertically SD-stability concept, both defined on this work.

Manjunath (2013) explores the logical connections between the core and the stable set when lotteries are assigned in the marriage problem. When all schools have quota one and

[8] In their work were referred as *Fractional Matchings*.

priorities are strict his strongly stability concept is equivalent to the SD-vertically stability presented on this work.

WJ Cho (2012, 2014) and Wonki Cho (2013) developed one of the main tools used on this work: the extension approach. The e-efficiency and e-strategy-proofness defined on his work are the same concepts that I use to evaluate efficiency and strategy-proofness for the random environment in school choice. His e-weak no-envy definitions is a weaker form of the horizontally e-stability notion when all students have the same priority at all schools that is presented on this work.

Onur Kesten and Ünver (2015) developed two stability notions in the school choice setting with weak-priorities. Their ex-ante stability concept is equivalent to my vertically SD-stability notion; and their strongly ex-ante stability notion is equivalent to my strongly SD-stability notion. They also provide a new random mechanism: the *Fractional Deferred Acceptance* which always finds the unique strongly SD-stable random matching that SD-Pareto dominates all other SD-stable random matching. I regard this article as the second cornerstone of this work, the e-stability notions presented here are inspired by the ex-ante notions of stability presented on their work.

1.5. Thesis organization

The rest of this Thesis is organized as follows: on Section 2 is presented the model and all the related definitions, on Section 3 are presented all the results and Section 4 concludes with final remarks. Proofs of propositions, lemmas and corollaries are presented at the Appendix.

Chapter 2

The model

First I introduce some basic notation in order to avoid notational problems through this work.

2.1. Notation

Zero is not included in the natural numbers: $\mathbb{N} \doteq \{1, 2, 3, \ldots\}$.

Given two sets $\mathcal{X}$ and $\mathcal{Y}$, denote $\mathcal{Y}^{\mathcal{X}}$ as the set of all functions with domain $\mathcal{X}$ and co-domain $\mathcal{Y}$.

Given set $\mathcal{X}$ and $n, m \in \mathbb{N}$ denote $\mathcal{X}^{n \times m}$ as the set of all $n \times m$ matrices with entries in $\mathcal{X}$.

A *lottery* over the set $\mathcal{X}$ is a probability distribution over $\mathcal{X}$. Given $\mathcal{X}$ denote $\Delta\mathcal{X}$ as the set of all lotteries over $\mathcal{X}$.

With abuse of notation I will denote by $\mathcal{X}$ to the set of degenerate lotteries of $\Delta\mathcal{X}$.

A *preference relation* R over the set $\mathcal{X}$ is a subset of $\mathcal{X} \times \mathcal{X}$.Throughout this work $(x, x') \in R$ is interpreted as: x is at least as preferred as x'. In what follows xRx' means $(x, x') \in R$, and $x\not{R}x'$ means $(x, x') \notin R$.

A preference relation R over X is said to be:

- *reflexive* if xRx for all $x \in X$.
- *anti-symmetric* if xRx' and $x'Rx$ implies $x = x'$.
- *transitive* if xRx' and $x'Rx''$ implies xRx''.
- *complete* if $x'Rx$ or xRx' holds.

- *upper semi-continuous* if $Rx \doteq \{x' \in X : x'Rx\}$ is closed[1].
- *lower semi-continuous* if $xR \doteq \{x' \in X : xRx'\}$ is closed.

Denote $\mathcal{R}(\mathcal{X})$ to the set of all complete, transitive and anti-symmetric preference relations over $\mathcal{X}$.

Denote $\mathcal{T}(\mathcal{X})$ to the set of all complete and transitive preference relations over $\mathcal{X}$.

Denote $\mathcal{R}(\Delta\mathcal{X})$ to the set of all reflexive preference relations over $\Delta\mathcal{X}$.

2.2. Setup

A *school choice economy* is a tuple $\Gamma = \left[I, S, \vec{q}, \vec{P}, \vec{\succsim}\right]$ where:

- I is a finite set of students.
- S is a finite set of schools.
- $\vec{q} \in \mathbb{N}^{|S|}$ is a maximum quota vector for schools such that there is as many seats as students $\sum_{s \in S} q_s = |I|$[2].
- $\vec{P} = (P_{\mathrm{i}})_{\mathrm{i} \in I} \in \mathcal{R}(S)^{|I|}$ is a strict preference profile for each student over schools.
- $\vec{\succsim} = (\succsim_s)_{s \in S} \in \mathcal{T}(I)^{|S|}$ is a weak priority structure for schools over students.

Denote by Σ to the set of all school choice economies.

For each $P \in \mathcal{R}(S)$ I denote by R to its reflexivization: sRs' if sPs' or $s = s'$.

Define the *rank function* κ that given a student preference list and a school, returns the position of the school at the preference list of the student.

$$\begin{aligned} \kappa : \mathcal{R}(S) \times S &\longrightarrow \{1, \ldots, |S|\}, \\ (P, s) &\longmapsto \kappa(P, s) \doteq |\{t \in S : tRs\}|. \end{aligned}$$

Given school choice economies $\Gamma = \left[I, S, \vec{q}, \vec{P}, \vec{\succsim}\right]$ and $\Gamma' = \left[I, S, \vec{q}, \vec{P}, \vec{\succsim}'\right]$, $\vec{\succsim}'$ has *stronger priority structure*[3] than $\vec{\succsim}$ if for all $\mathrm{i}, j \in I$ and $s \in S$, then $\mathrm{i} \succsim_s j$ implies $\mathrm{i} \succsim'_s j$, and denote it by $\vec{\succsim} \subseteq \vec{\succsim}'$.

[1]On this work I will work on finite dimensional vector spaces only so no particular topology needs to be specified because they are all equivalent.

[2]There is no loss of generality in assuming that there are as much seats as students. If there are more seat than students it is possible to add fictitious students to the model that have the worst priority at all schools with any preference profiles and such that there would be as many seats as students, on the opposite if there are more students than seats it is possible to add a fictitious school that is the less preferred school for all students and with any priority structure such that the number of seats and students are equal.

[3]Here I follow the notation introduced by Kojima, 2012. Intuitively speaking at $\succsim'$ there are less indifference than at $\succsim$.

For $\vec{\succsim}, \vec{\succsim}' \in \mathcal{T}(I)^{|S|}$, $\vec{\succsim}'$ is a tie-break of $\vec{\succsim}$ if:

(i) $\vec{\succsim} \subseteq \vec{\succsim}'$, and

(ii) $\vec{\succsim}'$ is anti-symmetric.

Let $\vec{\succsim}$ be a weak priority structure, denote by $\Lambda(\vec{\succsim})$ to the set of all possible tie-breaking of $\vec{\succsim}$.

$$\Lambda(\vec{\succsim}) \doteq \left\{ \vec{\succsim}' \in \mathcal{T}(I)^{|S|} : \ \vec{\succsim} \subseteq \vec{\succsim}' \text{ and } \vec{\succsim}' \text{ is anti-symmetric} \right\}.$$

2.3. Extensions

All centralized school choice admissions around the world work with students ordinal preferences only, and reports on the intensities of their preferences are not allowed in these mechanisms. For this reason tools that enable us to construct preferences over lotteries from ordinal preferences over schools are needed.

The framework of extensions that is going to be presented allow us to work with arbitrary preferences over lotteries using ordinal preferences as input. The concept of extension was introduced by WJ Cho, 2012 in the context of the random assignment problem in an attempt to formalize and axiomatize the procedure of evaluating random assignments with ordinal preferences.

An *extension* is a mapping from preferences over schools to preference over random assignments.

$$\begin{aligned} \mathrm{e} : \mathcal{R}(S) &\longrightarrow \mathcal{R}(\Delta S), \\ P &\rightrightarrows \mathrm{e}(P). \end{aligned}$$

2.3.1. Properties on extensions

Given the ordinal preferences, an extension shapes the preferences over lotteries, different extensions induce different preferences over lotteries, and later different preferences over lotteries may shape different sets of stable and efficient random matchings, and different sets of strategy-proof random mechanisms.

Assigning properties to extensions is an indirect way of putting properties to the preferences over lotteries. Next are presented the different properties that I will consider for the analysis.

Given a property over $\mathcal{R}(\Delta S)$, an extension e satisfies such property if $\mathrm{e}(P)$ satisfies the property for all $P \in \mathcal{R}(S)$[4].

Next are presented some properties that can be seen as weaker forms of stochastic dominance. By doing this it is possible to isolate which features of stochastic dominance are needed for different purposes, and disentangle what makes stochastic dominance so powerful approach when evaluating lotteries, and what are the effects in the sets of stable and efficient random matchings.

For random assignments $\rho, \rho' \in \Delta S$ we say that ρ is a *monotone improvement* of ρ' for $P \in \mathcal{R}(S)$ if there exist schools $s, s' \in S$ such that:

(i) $s^* P s_*$,

(ii) $\rho_t = \rho'_t$ holds for all schools $t \in S \setminus \{s^*, s_*\}$, and

(iiii) $\rho_{s^*} > \rho'_{s^*}$.

On this work it will be important for extensions not to contradict the monotone improvement relation.

An extension e is *monotone* if for random assignments $\rho, \rho' \in \Delta S$ such that ρ is a monotone improvement of ρ' for $P \in \mathcal{R}(S)$, then $\rho \mathrm{e}(P) \rho'$, and $\rho' \cancel{\mathrm{e}(P)} \rho$.

A monotone extension is an extension that when compares lotteries that differ at only two coordinates always strictly prefer the one that weights higher the most preferred school.

The following two properties (CMPO and CLPO) will also be important for extensions to fulfill.

An extension e is said to *compensate for more preferred objects* (CMPO) if for each $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$: $\rho \mathrm{e}(P) \rho'$ and $\rho' \cancel{\mathrm{e}(P)} \rho$, implies that for each school $s \in S$ such that $\rho_s < \rho'_s$ there is a more preferred school $s^* \in S$ (i.e. $s^* P s$) such that $\rho_{s^*} > \rho'_{s^*}$.

An extension e is said to *compensate for less preferred objects* (CLPO) if for each $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$: $\rho \mathrm{e}(P) \rho'$ and $\rho' \cancel{\mathrm{e}(P)} \rho$, implies that for each school $s \in S$ such that $\rho_s > \rho'_s$ there is a less preferred school $s_* \in S$ (i.e. $s P s_*$) such that $\rho_{s_*} < \rho'_{s_*}$.

In a way CMPO and CLPO imply that if under a particular extension a lottery is at least as preferred as another lottery, then the former cannot be stochastically dominated by the last.

Since I do not require extensions to be complete, extensions can be related in the same way as sets can be related, and we can have bigger extensions in the sense that if two elements are compared at both extensions they are compared in the same way and one of the extensions compares more elements than the other.

[4]Since $\mathcal{R}(\Delta S)$ is the set of all reflexive preference relations over ΔS, by construction all extensions on this work are reflexive.

Given extensions e_1 and e_2, $e_1 \subseteq e_2$ holds if for all $P \in \mathcal{R}(S)$ and $\rho, \rho' \in S$: $\rho e_1(P)\rho'$ implies $\rho e_2(P)\rho'$.

Bigger or smaller extensions are not necessarily better. Some extensions may be smaller for good reasons such as when the central planners do not know how students compare lotteries or extensions could be bigger if the central planner is embedded with more information on how students prefer the lotteries. In some sense extensions can be bigger or smaller depending on the taste.

2.3.2. A toolkit of extensions

Next I present various extensions. Many of which were introduced by WJ Cho, 2012. This particular selection of extensions are intended to later support and illustrate different ideas and limitations of the framework presented here.

Identity extension: ID

Given schools $s, s' \in S$ and $P \in \mathcal{R}(S)$, then $sID(P)s'$[5] if sPs'.

$$ID(P) \cong P, \ \forall P \in \mathcal{R}(S).$$

This is the smaller extension that only compares degenerate lotteries the same way that schools are compared.

Given extension e, denote as $e|_{S\times S}$ to the extension that compares only degenerate lotteries, and compares them the same way that e does.

$$e|_{S\times S}(P) \doteq e(P) \cap (S \times S), \ \forall P \in \mathcal{R}(S).$$

An extension e is *consistent* if $e|_{S\times S} = ID$[6]. Denote C to the set of all consistent extensions.

$$\mathsf{C} \doteq \{e \in \mathcal{R}(\Delta S)^{\mathcal{R}(S)} : e|_{S\times S} = ID\}.$$

Throughout this work I will focus on consistent extensions only, because it makes no sense that an extension flips around the explicit preferences declared by the students.

[5]Here s and s' are compared as degenerated lotteries.

[6]Reader may notice that $ID|_{S\times S} = ID$.

Monotone extension: MO

Given $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho MO(P)\rho'$ if ρ is a monotone improvement of ρ' for P; or $\rho = \rho'$.

Given two lotteries which differ at only two schools probabilities, the monotone extension always strictly prefers the lottery that weights higher the most preferred school. This is the smaller extension that respects monotone improvements and as a consequence this extension is very incomplete, we will see later that this is not a deterrence for carrying the stability analysis.

Expected utility extension: EU

Given a utility vector $\vec{u} = (u_1, \ldots, u_{|S|})$ in descending order (i.e. $u_{\mathrm{i}} > u_j$ for all $\mathrm{i} < j$); $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho EU(P, \vec{u})\rho'$ if $\sum_{s \in S} \rho_s \cdot u_{\kappa(P,s)} \geq \sum_{s \in S} \rho'_s \cdot u_{\kappa(P,s)}$.

Even though expected utility is as old as economic theory itself, the work of Von Neumann and Morgenstern, 1947 made this extension the mainstream for evaluating lotteries in economics and social sciences. Due to its simplicity and nice properties (completeness, transitivity, independence and continuity) makes the welfare analysis tractable.

Denote by $\mathcal{U}$ to the set of utility vectors compatible with the expected utility extensions.

$$\mathcal{U} \doteq \{\vec{u} \in \mathbb{R}^{|S|} : \ u_1 > u_2 > \ldots > u_{|S|}\}.$$

(First order) Stochastic dominance extension: SD

For $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho SD(P)\rho'$ if $\sum_{tRs} \rho_t \geq \sum_{tRs} \rho'_t$ for all $s \in S$.

Since Quirk and Saposnik (1962) this extensions was the first departure from expected utility theory, because it allows to compare lotteries in a way that respects all cardinal preferences compatible with the original ordinal preferences. Thus intensities of preferences are not needed to compare lotteries, which ends up in a lot of lotteries that cannot be compared under first order stochastic dominance.

Downward lexicographic extension: DL

For $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho DL(P)\rho'$ if either exists $s \in S$ such that $\rho_t = \rho'_t$ for all tPs and $\rho_s > \rho'_s$; or $\rho = \rho'$.

Under this extension when comparing the lotteries a students always prefers the one that has higher probability at his most preferred school, if the probabilities are equal at that school then the analysis proceeds sequentially to the second most preferred school and so on.

Upward lexicographic extension: UL

For $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho UL(P)\rho'$ if either exists $s \in S$ such that $\rho_t = \rho'_t$ for all sPt and $\rho_s < \rho'_s$; or $\rho = \rho'$.

Under this extension when comparing the lotteries a students always prefers the one that has lower probability at his less preferred school, if the probabilities are equal at that school then the analysis proceeds sequentially to the second less preferred school and so on.

The use of lexicographic preferences dates back from Hausner (1952). This particular way of preferences over lotteries is complete but cannot be represented by an utility function and it is not continuous.

Downward naive extension: DN

For $\rho, \rho' \in \Delta S$ and $P \in \mathcal{R}(S)$, then $\rho DN(P)\rho'$ if $N(\rho, P) \doteq \min_{\{s:\nu_s>0\}} \kappa(P, s) \leq N(\rho', P)$.

Under this extension when comparing the lotteries a students always prefers the one that has positive probability at his most preferred school while the other has probability zero at that school, if both lotteries have positive or zero probability at that school, then the analysis proceeds sequentially to the second more preferred school and so on.

This extension has the same spirit as the DL extension except that students only care of having a positive probability of getting their most preferred object, which of course that is a naive way of building preferences over lotteries.

DN is consistent, complete, transitive and satisfies CMPO property, but it's neither monotone nor anti-symmetric. Since this extension will not be compatible with the monotone extension it will be useful for some counterexamples.

A more complete set of extensions can be found at WJ Cho (2012, 2014) and Wonki Cho (2013).

Until this point all the introduction of the framework has been about how the preferences of the agents (students) are shaped. Next section uses this framework to define desirable (and non-desirable) properties over matchings of students to schools.

2.4. Random matchings

A *random matching* is a real stochastic matrix $\pi = (\pi_{\mathrm{i}s})_{(\mathrm{i},s)\in I\times S}$ such that $\sum_{\mathrm{i}\in I}\pi_{\mathrm{i},s} = q_s$ for all $s \in S$. Given a school choice economy Γ denote the set of random matchings as $\Pi(\Gamma)$[7]
.

$$\Pi \doteq \left\{\pi \in [0,1]^{|I|\times|S|} : \sum_{s\in S}\pi_{\mathrm{i}s} = 1 \ \forall \mathrm{i} \in I, \ \sum_{\mathrm{i}\in I}\pi_{\mathrm{i}s} = q_s \ \forall s \in S\right\}.$$

A *matching* is a random matching that is zero or one at every coordinate. The set of matchings is denoted by $\mathcal{M}$.

$$\mathcal{M} \doteq \Pi \cap \{0,1\}^{|I|\times|S|}.$$

Each random matching is interpreted as follows:

$$\pi_{\mathrm{i}s} \doteq \mathbb{P}(\text{student i is assigned at school } s), \ (\mathrm{i}, s) \in I \times S.$$

This interpretation is a key feature because it implies that the central planners should have a way of implementing them, and while students may only care about their particular lottery being respected, central planners have to find a way of implementing lotteries over matchings in such a way does not violates every students requirement while also respecting some ex-post properties in some cases. So part of this work is also devoted on how and when this requirements are compatible in an ex-post implementation sense.

For the sake of the previous discussion, and for future simplicity the classic expected operator from probability theory is introduced.

For a convex sub-set contained on a vector space over $\mathbb{R}$, define the *expectancy function* $\mathbb{E}$ as a function that receives a lottery and returns the weighted sum induced by it.

$$\begin{aligned}\mathbb{E} : \Delta\mathcal{X} &\longrightarrow \mathcal{X}, \\ \sigma &\longmapsto \mathbb{E}(\sigma) \doteq \sum_{\chi\in\mathcal{X}} \chi \cdot \sigma_\chi.\end{aligned}$$

It is presented in this general way because in some parts of this work $\mathcal{X}$ will represent sets of matchings, and in others will be functions over sets of matchings (mechanisms).

[7] In most parts of this work I will omit the school choice economy dependence of the sets of random matchings, because it will induce no ambiguity. For example when I say $\pi \in \Pi$ I will assume that there is an underlying school choice economy Γ at which π belongs. Moreover when I state that a inclusion among random matchings sets holds, it will mean for all school choice economies:

$$\mathcal{X}_1 \subseteq \mathcal{X}_1 \text{ if } \mathcal{X}_1(\Gamma) \subseteq \mathcal{X}_1(\Gamma), \ \forall \Gamma \in \Sigma,$$

for any $\mathcal{X}_1$ and $\mathcal{X}_2$ sets of random matchings depending on the school choice economy structure.

2.4.1. Stability

In what comes old and new notions of stability are presented, both for matchings and random matchings. Recall that on this setting stability is understood as a way of fulfilling fairness of the assignments, dropping the traditional interpretation of unstability as the situation where agents of opposite sides of the market (student and school) would make an arrangement by themselves violating the arrangement made by the central planner.

Stability in matchings

Matching $\mu \in \mathcal{M}$ induces *justified envy* of student $\mathrm{i} \in I$ towards student $j \in I$ at school $s \in S$ if:

(i) $sP_{\mathrm{i}}s'$, for $s' \in S$ being i's assignment under μ: $\mu_{\mathrm{i}s'} = 1$,
(ii) $\mu_{js} = 1$, and
(iii) $\mathrm{i} \succ_s j$.

A matching is *stable* if it does not induces justified envy. Denote the set of stable matchings as $\mathcal{S}$.

$$\mathcal{S} \doteq \{\mu \in \mathcal{M} : \ \mu \text{ does not induces justified envy}\}.$$

This concept of stability was presented by David Gale and L. S. Shapley (1962) in the context of college admissions, in a model at which the priorities were strict and provided the deferred acceptance algorithm to show the existence of stable matchings.

Strong stability in matchings

Matching $\mu \in \mathcal{M}$ induces *discrimination*[8] of student $\mathrm{i} \in I$ towards student $j \in I$ at school $s \in S$ if:

(i) $sP_{\mathrm{i}}s'$, for $s' \in S$ being i's assignment under μ: $\mu_{\mathrm{i}s'} = 1$,
(ii) $\mu_{js} = 1$, and
(iii) $\mathrm{i} \succsim_s j$.

A matching is *strongly stable* if it does not induces discrimination. Denote the set of strongly stable matchings as $\mathcal{SS}$.

$$\mathcal{SS} \doteq \{\mu \in \mathcal{M} : \ \mu \text{ does not induces discrimination}\}.$$

[8]Here I follow the terminology introduced by Onur Kesten and Ünver (2015).

Despite its intuitive extension I did not found it on the literature, perhaps because the existence of strongly stable matchings, as I will show later, is not guaranteed.

Ex-post stability

A random matching is *ex-post stable* if it can be induced by a lottery whose support includes stable matchings only. Denote the set of ex-post stable random matchings as $\mathcal{S}^{xp}$.

$$\mathcal{S}^{xp} \doteq \mathbb{E}\left(\Delta\mathcal{S}\right)$$

The first allusion to this kind of stability was made by Roth, Rothblum, and Vande Vate (1993) in the context of one-to-one two sided matching market. This kind of stability only cares that the lotteries could be done without violating ex-post stability. Later it will be shown that this condition does not eliminates the most reasonable and intuitive notion of ex-ante stability.

e-Weak no-envy

Given an extension e, random matching π satisfies e-*weak no-envy*[9] if for all students $\mathrm{i}, j \in I$: $\pi_j \mathrm{e}(P_\mathrm{i})\pi_\mathrm{i}$ implies $\pi_\mathrm{i}\mathrm{e}(P_\mathrm{i})\pi_j$. Denote by $\mathcal{S}^{\mathrm{e}}_W$ to the set of all random matchings satisfying e-weak no-envy.

$$\mathcal{S}^{\mathrm{e}}_W \doteq \left\{\pi \in \Pi : \pi \text{ satisfies e-weak no-envy}\right\}.$$

Under this approach envy takes places on the complete lotteries and does not care about the priority structure, which makes it harder to connect it to previous notions of stability in matchings.

This notion of stability was formalized and introduced by WJ Cho (2012), but it has been widely used under the expected utility (Hylland and Zeckhauser, 1979; Manjunath, 2013; Zhou, 1990), stochastic dominance (Athanassoglou and Sethuraman, 2011; Bogomolnaia, 2015; Bogomolnaia and Moulin, 2001; Katta and Sethuraman, 2006; Kojima, 2009; Ö. Yılmaz, 2009, 2010) and lexicographic preference extensions (Saban and Sethuraman, 2014; Schulman and Vazirani, 2012) in the random assignment literature.

(Strongly) Ex-ante stability

A random matching π causes *ex-ante justified envy* of student i towards (a lower priority student) j at school s if:

[9] This definition is weak in the sense that allows preferences over lotteries to be incomplete, and some agents may not envy another lottery, not because the prefer the one they have, but because they cannot compare them.

(i) $\mathrm{i} \succ_s j$,

(ii) $\pi_{\mathrm{i}s'} > 0$ for some school s' such that $sP_{\mathrm{i}}s'$, and

(iii) $\pi_{js} > 0$.

A random matching π causes *ex-ante discrimination* of student i towards (an equal priority student) j at school s if:

(i) $\mathrm{i} \sim_s j$,

(ii) $\pi_{\mathrm{i}s'} > 0$ for some school s' such that $sP_{\mathrm{i}}s'$, and

(iii) $\pi_{js} > \pi_{\mathrm{i}s}$.

A random matching is *ex-ante stable* if it eliminates ex-ante justified envy. Denote by $\mathcal{S}^{xa}$ to the set of ex-ante stable random matchings.

$$\mathcal{S}^{xa} \doteq \{\pi \in \Pi : \pi \text{ does not induces ex-ante justified envy}\}.$$

A random matching is *strongly ex-ante stable* if it eliminates ex-ante discrimination and ex-ante justified envy. Denote by $\mathcal{S}^{sxa}$ to the set of strongly ex-ante stable random matchings.

$$\mathcal{S}^{sxa} \doteq \mathcal{S}_{xa} \cap \{\pi \in \Pi : \pi \text{ does not induces ex-ante discrimination}\}.$$

These stability notions were presented by Onur Kesten and Ünver (2015), and it has the feature of that envy arises not from the random assignment but just over particular coordinates of them. While not explicitly stated the following interpretation arises: when looking at the lotteries, agents may want to get other's probability weight to build another lottery that would dominate in the first order stochastic dominance sense.

Another interesting feature of these stability notions is that two different sources of envyness arises: the one from different priority students, and the one from same priority students at a particular school.Thus a new restriction is imposed and this may imply a welfare cost when avoiding ex-ante discrimination.

Vertically, horizontally and strongly e-stability

Given an extension e, random matching π induces e-*envy* from student $\mathrm{i} \in I$ toward student $j \in I$ if there exists a *blocking random assignment* for student i, that I call ρ_{i}[10], and a non-empty set of *blocking schools* S' such that:

[10]Here the index i can be eliminated, but I introduce the index i to make allusion that is a blocking assignment for student i.

(i) $\pi_{\mathrm{i}s} < \rho_{\mathrm{i}s} \leq \pi_{\mathrm{i}s} + \pi_{js}$ for all $s \in S'$,
(ii) $\rho_{\mathrm{i}s} \leq \pi_{\mathrm{i}s}$ for all $s \in S \setminus S'$,
(iii) $\rho_{\mathrm{i}}\mathrm{e}(P_{\mathrm{i}})\pi_{\mathrm{i}}$, and
(iv) $\pi_{\mathrm{i}}\cancel{\mathrm{e}(P_{\mathrm{i}})}\rho_{\mathrm{i}}$.

Given an extension e, random matching π induces *vertically justified* e-*envy* from student $\mathrm{i} \in I$ towards student $j \in I$ in the subset of blocking schools $S' \subseteq S$ if:

(i) student i has e-envy towards student j at S', and
(ii) for the whole set of blocking schools S': $\mathrm{i} \succ_s j$, $s \in S'$.

Given an extension e, random matching π induces *horizontally justified* e-*envy* from student $\mathrm{i} \in I$ towards student $j \in I$ in the subset of blocking schools $S' \subseteq S$ if:

(i) student i has e-envy towards student j with blocking random assignment ρ_{i} at S',
(ii) ρ_{i} satisfies $\rho_{\mathrm{i}s} \leq \frac{\pi_{\mathrm{i}s}+\pi_{js}}{2}$[11], and
(iii) for the whole set of blocking schools S': $\mathrm{i} \sim_s j$, $s \in S'$.

A random matching is *horizontally* e-*stable* if there is no horizontally justified e-envy among students. Denote by $\mathcal{S}_H^{\mathrm{e}}$ to the set of horizontally e-stable random matchings.

$$\mathcal{S}_H^{\mathrm{e}} \doteq \{\pi \in \Pi : \pi \text{ does not induces horizontally justified e-envy}\}$$

A random matching is *vertically* e-*stable* if there is no vertically justified e-envy among students. Denote by $\mathcal{S}_V^{\mathrm{e}}$ to the set of vertically e-stable random matchings.

$$\mathcal{S}_V^{\mathrm{e}} \doteq \{\pi \in \Pi : \pi \text{ does not induces vertically justified e-envy}\}$$

A random matching is *strongly* e-*stable* if it is vertically and horizontally e-stable. Denote by $\mathcal{S}^{\mathrm{e}}$ to the set of strongly e-stable random matchings.

$$\mathcal{S}^{\mathrm{e}} \doteq \mathcal{S}_H^{\mathrm{e}} \cap \mathcal{S}_V^{\mathrm{e}}$$

I propose this new concepts of stability inspired by WJ Cho (2012) and Onur Kesten and Ünver (2015) so that a concept of stability could account to the weak-priority structure and different preferences over lotteries.

[11] Which implies $\pi_{\mathrm{i}s} < \pi_{js}$.

In trying to find the appropriate notions there were some things that were not obvious in how should be defined, for example under this notion random matchings are blocked by a student justifiably envying another student at a set of schools. What if we allow random matchings to be blocked by a set of students at a set of schools? What happens if we restrict the blocking notion so that only a student at a school could block? The equivalence of these different ways of defining blocking rules depends on whether the extension considered is monotone or not.

2.4.2. Efficiency

On this subsection I present previous and new notions of efficiency. The new ones have direct relation to the new concepts of stability that I introduce in the previous subsection.

Efficiency on matchings

A matching is *efficient* if it is not Pareto dominated by any other matching. Denote by $\mathcal{E}$ to the set of efficient matchings.

$$\mathcal{E} \doteq \{\mu \in \mathcal{M} : \ \mu \text{ is not Pareto dominated by any other matching}\}.$$

Constrained efficiency on matchings

A stable matching is *constrained efficient* if it is not Pareto dominated by any other stable matching. Denote by $\mathcal{E}_C$ to the set of efficient matchings.

$$\mathcal{E}_C \doteq \{\mu \in \mathcal{S} : \ \mu \text{ is not Pareto dominated by any other stable matching}\}.$$

Ex-post efficiency

A random matching is *ex-post efficient* if it can be induced by a lottery with support on efficient matchings only. Denote the set of ex-post efficient random matchings as $\mathcal{E}^{xp}$.

$$\mathcal{E}^{xp} \doteq \mathbb{E}\left(\Delta\mathcal{E}\right).$$

e-Efficiency

Given an extension e, for $\pi, \pi' \in \Pi$, π e-*Pareto dominates* π' if for all students $\mathrm{i} \in I$: $\pi_{\mathrm{i}}\mathrm{e}(P_{\mathrm{i}})\pi'_{\mathrm{i}}$, and there exist a student $j \in I$ such that $\pi'_{\mathrm{i}}\cancel{\mathrm{e}(P_{\mathrm{i}})}\pi_{\mathrm{i}}$. Denote by $<_{\mathrm{e}}$ to the e-Pareto dominance relation among random matchings, and by $\leq_{\mathrm{e}}$ to its reflexivization.

A random matching is e-*efficient* if there is no other random matching that e-Pareto dominates it. Denote by $\mathcal{E}^{\mathrm{e}}$ to the set of all e-efficient random matchings.

$$\mathcal{E}^{\mathrm{e}} \doteq \{\pi \in \Pi : \ \pi \text{ is not e-Pareto dominated}\}.$$

Constrained e-efficiency

Given extension e, a vertically e-stable random matching is *constrained* e-*efficient* if it is not e-Pareto dominated by any other vertically e-stable random matching. Denote by $\mathcal{E}^{\mathrm{e}}_{C}$ to the set of all constrained e-efficient random matchings.

$$\mathcal{E}^{\mathrm{e}}_{C} \doteq \{\pi \in \mathcal{S}^{\mathrm{e}}_{V} : \ \pi \text{ is not e-Pareto dominated at } \mathcal{S}^{\mathrm{e}}_{V}\}.$$

Strongly constrained e-efficiency

Given extension e, a strongly e-stable random matching is *strongly constrained* e-*efficient* if it is not e-Pareto dominated by any other strongly e-stable random matching. Denote by $\mathcal{E}^{\mathrm{e}}_{SC}$ to the set of all strongly constrained e-efficient random matchings.

$$\mathcal{E}^{\mathrm{e}}_{SC} \doteq \{\pi \in \mathcal{S}^{\mathrm{e}} : \ \pi \text{ is not e-Pareto dominated at } \mathcal{S}^{\mathrm{e}}\}.$$

In general all these efficiency concepts share the same structure: the Pareto dominance builds up an incomplete order, and given a set of feasible random matchings we want to restrict our attention only to the ones that are not Pareto dominated by any other feasible random matching.

2.5. Algorithms, mechanisms and random mechanisms

Next is presented a framework that is intended to capture the fact that central planners do not know the preferences of the agents, and thus at some point students have to report their preferences. It makes sense that the assignment rule depends on the reported information of the agents, but central planners must be careful because depending on the assignment rule some students may face incentives to lie on their true preferences in order to get a better assignment as it was the case in Boston.

A *mechanism* is a function that maps school choice economies to matchings of adequate size. Denote by $\mathfrak{M}$ to the set of mechanisms.

$$\mathfrak{M} \doteq \bigcup_{\Gamma \in \Sigma} \mathcal{M}(\Gamma)^{\mathcal{R}(S)^{|I|}}.$$

A *random mechanism* is a function that maps school choice economies to random matchings of adequate size. Denote by $\mathfrak{RM}$ to the set of random mechanisms.

$$\mathfrak{RM} \doteq \bigcup_{\Gamma \in \Sigma} \Pi(\Gamma)^{\mathcal{R}(S)^{|I|}}.$$

Given an algorithm $\mathcal{A}$ with Γ as input and output in Π, then the random mechanism that such algorithm induces is denoted as $\phi^{\mathcal{A}}$. For simplicity when the co-domain of a random mechanism is a subset of $\mathcal{M}$ it will be denoted by φ, and $\varphi^{\mathcal{A}}$ when it is induced by an algorithm with output in $\mathcal{M}$.

With abuse of notation, when evaluating a random mechanism, I will show the preferences of students only[12] (i.e. $\phi[\vec{P}]$ instead of $\phi[\Gamma]$).

Given a random mechanism ϕ and a student i, then $\phi_{\mathrm{i}}[\vec{P}]$ denotes the random assignment of i under $\phi[\vec{P}]$ (i.e. the restriction to the ith row).

2.5.1. Properties on mechanisms

Given a property over Π, we say that a random mechanism ϕ satisfies such property if $\phi\left[\vec{P}\right]$ satisfies the property for all $\vec{P} \in \mathcal{R}(S)^{|I|}$.

Denote by: $\mathfrak{S}$ to the set of all stable mechanisms, $\mathfrak{E}$ to the set of all Pareto efficient mechanisms and $\mathfrak{E}_C$ to the set of all constrained efficient mechanisms.

$$\begin{aligned}
\mathfrak{S} &\doteq \{\varphi \in \mathfrak{M} : \varphi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{S}\},\\
\mathfrak{E} &\doteq \{\varphi \in \mathfrak{M} : \varphi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{E}\},\\
\mathfrak{E}_C &\doteq \{\varphi \in \mathfrak{M} : \varphi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{E}_C\}.
\end{aligned}$$

[12] The justification for this is that students are the only ones who can misrepresent their real preferences, and in this way the sight is not overburdened without paying cost in ambiguity.

Denote by: $\mathfrak{S}^{xp}$ to the set of all ex-post stable random mechanisms, $\mathfrak{E}^{xp}$ to the set of all ex-post efficient random mechanisms and $\mathfrak{E}^{xp}_C$ to the set of all ex-post constrained efficient random mechanisms.

$$\begin{aligned}\mathfrak{S}^{xp} \doteq& \mathbb{E}(\Delta\mathfrak{S}),\\ \mathfrak{E}^{xp} \doteq& \mathbb{E}(\Delta\mathfrak{E}),\\ \mathfrak{E}^{xp}_C \doteq& \mathbb{E}(\Delta\mathfrak{E}_C).\end{aligned}$$

Let e be an extension. Denote by: $\mathfrak{S}^{\mathrm{e}}_V$ to the set of all vertically e-stable random mechanisms, $\mathfrak{S}^{\mathrm{e}}_H$ to the set of all horizontally e-stable random mechanisms, $\mathfrak{S}^{\mathrm{e}}_V$ to the set of all strongly e-stable random mechanisms, $\mathfrak{E}^{\mathrm{e}}$ to the set of all e-efficient random mechanisms, $\mathfrak{E}^{\mathrm{e}}_C$ to the set of all constrained e-efficient random mechanisms and $\mathfrak{E}^{\mathrm{e}}_{SC}$ to the set of all strongly constrained e-efficient random mechanisms.

$$\begin{aligned}\mathfrak{S}^{\mathrm{e}}_V \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{S}^{\mathrm{e}}_V\},\\ \mathfrak{S}^{\mathrm{e}}_H \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{S}^{\mathrm{e}}_H\},\\ \mathfrak{S}^{\mathrm{e}} \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{S}^{\mathrm{e}}\},\\ \mathfrak{E}^{\mathrm{e}} \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{E}^{\mathrm{e}}\},\\ \mathfrak{E}^{\mathrm{e}}_C \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{E}^{\mathrm{e}}_C\},\\ \mathfrak{E}^{\mathrm{e}}_{SC} \doteq& \{\phi \in \mathfrak{RM} : \phi\left[\mathcal{R}(S)^{|I|}\right] \subseteq \mathcal{E}^{\mathrm{e}}_{SC}\}.\end{aligned}$$

Strategy-proofness

A mechanism $\varphi \in \mathfrak{M}$ is *strategy-proof* if $\varphi_{\mathrm{i}}[\vec{P}]P_{\mathrm{i}}\varphi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]$ for all $\vec{P} \in \mathcal{R}(S)^{|I|}$, $P'_{\mathrm{i}} \in \mathcal{R}(S)$ and $\mathrm{i} \in I$. Denote by $\mathfrak{SP}$ to the set of strategy-proof mechanisms.

$$\mathfrak{SP} \doteq \left\{\varphi \in \mathfrak{M} : \varphi_{\mathrm{i}}[\vec{P}]P_{\mathrm{i}}\varphi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}],\ \forall(\mathrm{i}, \vec{P}, P'_{\mathrm{i}}) \in I \times \mathcal{R}(S)^{|I|} \times \mathcal{R}(S)\right\}.$$

Strategy-proofness of a mechanism implies that truth-telling can be sustained as a Nash equilibrium. This does not necessarily imply that is the only Nash equilibrium induced by the mechanism, for that a mechanism would need to be group strategy-proof.

e-Strategy-proofness

Let e be an extension. A random mechanism $\phi \in \mathfrak{RM}$ is e-*strategy-proof* if $\phi_{\mathrm{i}}[\vec{P}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]$ for all $\vec{P} \in \mathcal{R}(S)^{|I|}$, $P'_{\mathrm{i}} \in \mathcal{R}(S)$ and $\mathrm{i} \in I$. Denote by $\mathfrak{SP}^{\mathrm{e}}$ to the set of e-strategy-proof random mechanisms.

$$\mathfrak{SP}^{\mathrm{e}} \doteq \left\{\phi \in \mathfrak{RM} : \phi_{\mathrm{i}}[\vec{P}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}],\ \forall(\mathrm{i}, \vec{P}, P'_{\mathrm{i}}) \in I \times \mathcal{R}(S)^{|I|} \times \mathcal{R}(S)\right\}.$$

This is the most intuitive extension of strategy-proofness to the probabilistic environment, but it turns out to be too strong, because for a mechanism to be strategy-proof the preference over lotteries that an extension induces would need to compare all possible lotteries obtained by unilaterally deviation of all agents, and since we are allowing extensions to induce non-complete order the comparison could not be done. This is the main reason for defining a weaker version of the strategy-proofness of random mechanisms.

e-Weak strategy-proofness

Let e be an extension. A random mechanism $\phi \in \mathfrak{RM}$ is e-*weak strategy-proof* if $\phi_{\mathrm{i}}[\vec{P}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]$ is implied by $\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[\vec{P}]$ for all $\vec{P} \in \mathcal{R}(S)^{|I|}$, $P'_{\mathrm{i}} \in \mathcal{R}(S)$ and $\mathrm{i} \in I$. Denote by $\mathfrak{SP}^{\mathrm{e}}_{W}$ to the set of e-weak strategy-proof random mechanisms.

$$\mathfrak{SP}^{\mathrm{e}}_{W} \doteq \left\{\phi \in \mathfrak{RM} : \forall(\mathrm{i}, \vec{P}, P'_{\mathrm{i}}) \in I \times \mathcal{R}(S)^{|I|} \times \mathcal{R}(S) : \ \phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[\vec{P}] \Rightarrow \phi_{\mathrm{i}}[\vec{P}]\mathrm{e}(P_{\mathrm{i}})\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]\right\}.$$

This definition allows for a richer set of strategy-proof mechanisms, and allows to draw a thiner separation of mechanisms in terms of their incentives. It is expected that weaker extensions induce bigger sets of strategy-proof mechanisms.

2.5.2. A toolkit of matching algorithms

Next I present different mechanisms and random mechanisms that are actually being used, had been used or are proposed to be used in the context of school choice. Most of them formalize intuitive notions of how seats of schools should be assigned to students. I will introduce the mechanisms by provided the algorithms that operate on the school choice economy, which fully characterize the mechanism.

Boston

In the early centralized school choice programs of the U.S. were in the 90's this was the main mechanism.

Input: A school choice economy with strict priorities.

Output: A matching.

Step n: Every student who is not assigned to a school proposes to her most preferred school that has not rejected her. Then every school consider the students who applied at this step, if there are no seats left it rejects all applicants, if has available seats it proceeds to assign them one by one following the priority until no seats or applicants are left, those applicant become accepted and the rest are rejected. If any applicant was rejected at this step go to next step, if not stop the algorithm.

This mechanism was widely used in the United States for assigning public seats. His flaws on the incentives triggered the research and the application of mechanism design to school choice.

Deferred Acceptance

This algorithm is now-a-days the most used for centralized school choice programs around the world.

Input: A school choice economy with strict priorities.

Output: A matching.

Step n: Every student who is not assigned to a school proposes to her most preferred school that has not rejected her. Then every school consider the students who applied at this step and those who had not been rejected in any previous step, then every school rejects them one by one following the priority until there are enough seats for the non-rejected applicants[13]. If any applicant was rejected at this step go to next step, if not stop the algorithm.

This mechanism was first introduced for the college admission problem and its main feature is that tries to improve welfare of students while maintaining stability.

Top Trading Cycles

Input: A school choice economy with strict priorities.

Output: A matching.

Step n: Any student who is not assigned to a school points to his most preferred school that has available seats, each school that has available seats points to its higher priority student that is not assigned to a school. Chose any cycle[14] and assign each student to the school that is pointing to. If there are still students and seats unassigned go to the next step, stop otherwise.

This mechanism was first introduced for the housing market problem and its main feature is that tries to replicate what rational agents with complete information would have done in a decentralized exchange market. Its application to centralized school choice programs

[13] This implies that if at any step a school has more seats than applicants, then it doesn't reject anyone.

[14] There always is at least one cycle and the order selection doesn't affect the final outcome.

has been scarce, because of its potential stability violations.

Fractional Deferred Acceptance

This algorithm is a generalization of the DA algorithm in which schools are allowed to accept/reject fractions of student, thus when a student applies, the school may accept 1/2 of the student and the other 1/2 of student is rejected and has to find a school. Later the fractions are interpreted as probabilities.

Input: A school choice problem.

Output: A random matching.

Step n: Fix some student $\mathrm{i} \in I$ who has an unassigned fraction. He applies to the next best school that has not yet rejected any fraction of him. Let s_1 be this school. Two cases are possible:

1. If the student i_1 induces a rejection cycle

$$\mathrm{i}_1 \hookrightarrow_{s_1} \mathrm{i}_2 \hookrightarrow_{s_2} \cdots \hookrightarrow_{s_{m-1}} \mathrm{i}_m \hookrightarrow_{s_m} \mathrm{i}_1,$$

   then we resolve it as follows: For $\mathrm{i}_{m+1} \equiv \mathrm{i}_1$ and $s_0 \equiv s_m$, s_1 tentatively accepts the maximum possible fraction of i_1 such that each school s tentatively accepts:

   - all fractions of applicants tentatively accepted in the previous step except the ones belonging to the lowest-priority level,
   - the total rejected fraction of student i from school s_1 , and
   - an equal fraction (if possible) among the lowest-priority applicants tentatively accepted in the previous step (including student $\mathrm{i}+1$) so that it does not exceed its quota q_s.

2. If i_1 does not induce a rejection cycle, school s_1 considers its tentatively assigned applicants from the previous step together with the new fraction of i_1. It tentatively accepts these fractions starting from the highest priority. In case its quota is filled in this process, it tentatively accepts an equal fraction (if possible) of all applicants belonging to the lowest accepted priority level. It rejects all outstanding fractions.

Continue until no fraction of a student remains unassigned. At this point, we terminate the algorithm by making all tentative random assignments permanent.

This mechanism is the first effort in the school choice literature to provide a random mechanism capable of operate on the weak priority structure without explicitly relying on the tie-breaking.

2.5.3. Randomizing mechanisms: the tie-breaking approach

In real world applications I know no random mechanism that is not defined through randomization of mechanisms. By this way mechanisms are generalized to the weak priority structure.

Let $\mathcal{A}$ be an algorithm with with a school choice economy with strict priorities as input, denote by $\phi_{TB}^{\mathcal{A}}$ to the random mechanism induced by randomly breaking the ties and then running algorithm $\mathcal{A}$.

$$\phi_{TB}^{\mathcal{A}} \doteq \sum_{\vec{\succeq}' \in \Lambda\left(\vec{\succeq}\right)} \varphi_{\vec{\succeq}'}^{\mathcal{A}} \cdot \frac{1}{\left|\Lambda\left(\vec{\succeq}\right)\right|}.$$

Here I define it to the MTB case, but generalizations to other probability distributions over mechanisms are obvious. In fact some recent literature tries to find optimal ways of tie-breaking the priorities.

Chapter 3

Results

On this chapter are gathered all the results obtained throughout this research. First are presented results regarding extensions and preferences over lotteries, then an alternative proof of the school choice Birkhoff-Von Neumann theorem, next a sequence of examples are presented to convince the reader that new notions of stability for the random environment are necessary and finally results regarding efficiency, stability and strategy-proofness are presented both random matchings and random mechanisms.

3.1. Extensions

On this section some logical connections among properties of extensions are presented. This results are not so important on their own, but are very useful for demonstrations on next sections. Since stability, efficiency and incentive notions depend directly on how preferences of students are shaped, then it is important to understand what are the implications of different approaches and how they relate to each other when building preferences over lotteries.

Proposition 1 (A folk result) Lottery ρ^1 stochastically dominates ρ^2 if and only if the expected utility of ρ^1 is higher that ρ_2 for any cardinal preferences respecting the ordinal preference.

$$SD(P) = \bigcap_{\vec{u} \in \mathcal{U}} EU(P, \vec{u}), \ \forall P \in \mathcal{R}(S).$$

This result illustrates the trade-off between SD and EU: on one hand reporting cardinal utilities $\vec{u}$ might be very complicated or even not feasible, but without that information we have to take a very big intersection which ends up in a very incomplete preference relation for lotteries.

Next propositions help to understand how properties on extensions and commonly known extensions such as lexicographic and stochastic dominance interact.

Proposition 2 (WJ Cho 2012)

1. If extension e is monotone and transitive, then $SD \subseteq$ e.
2. If extension e is anti-symmetric and has the CMPO, then e $\subseteq DL$.
3. If extension e is anti-symmetric and has the CLPO, then e $\subseteq UL$.

If one considers monotonicity and transitivity as a natural property for an extension to have, then SD arises again as a minimal departure to start from. Less expected is the intrinsic relation between compensation properties and lexicographic preferences.

Proposition 3

1. Let e_1, e_2 be two extensions such that $e_1 \subseteq e_2$ and e_2 being anti-symmetric, then e_1 is anti-symmetric.
2. If e is an extension such that e $\subseteq SD$, then e has both CMPO and CLPO property.

The structure of two extensions with one being contained in the other and the stronger being anti-symmetric appears frequently on this work, thus is important to have in mind so that we do not overvalue some results, that in that case we are implying the anti-symmetry of the smaller one. It is also important to have in mind that probability compensation properties are a weaker form of stochastic dominance.

3.2. An alternative proof of the school choice Birkhoff-Von Neumann theorem

On this section I provide an alternative proof which I believe that is simpler and clearer than the one provided by Kojima and Manea (2010). We both use the procedure presented by Hylland and Zeckhauser (1979), this result only relies on one step of the aforementioned algorithm that provides a possible constructive decomposition of a random matching.

Lemma 1 $\mathcal{M}$ characterizes the whole set of extreme points of Π.

For the purpose of the following result the expectancy function is defined as follows:

$$\mathbb{E} : \Delta\mathcal{M} \longrightarrow \Pi.$$

Theorem 1 For every random matching there is a lottery over matchings that induces it.

$$\Pi \subseteq \mathbb{E}\left(\Delta\mathcal{M}\right).$$

Proof. Since $\mathcal{M}$ are all the extreme points of Π, which is compact and convex subset of $\mathbb{R}^{|I|+|S|}$. Then by Carathéodory (1907), Krein and Milman (1940), and Minkowski (1911) every random matching is a convex combination of matchings. $\square$

Corollary 1 (Kojima and Manea 2010) $\mathbb{E}$ is surjective.

$$\mathbb{E}\left(\Delta\mathcal{M}\right) = \Pi.$$

Build new proofs of old results help us to view those results from a new perspective. In this case understanding matchings as extreme points of the polotype of random matchings explains how this literature is related to linear programming. On the other hand the surjectivity of the expectancy function, while obvious, has not been exploited. An interesting question is that if for some generalizations of this model the surjectivity on the expectancy function could be dropped off.

Next is presented an example that negates the inyectivity of $\mathbb{E}$.

Example 1 ($\mathbb{E}$ is not inyective) Consider $I = \{\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3\}$, $S = \{s_1, s_2, s_3\}$ and the following matchings:

$$\begin{array}{c|ccc} \mu_1 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 1 & 0 & 0 \\ \mathrm{i}_2 & 0 & 1 & 0 \\ \mathrm{i}_3 & 0 & 0 & 1 \end{array}, \quad \begin{array}{c|ccc} \mu_2 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 0 & 1 & 0 \\ \mathrm{i}_2 & 0 & 0 & 1 \\ \mathrm{i}_3 & 1 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} \mu_3 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 0 & 0 & 1 \\ \mathrm{i}_2 & 1 & 0 & 0 \\ \mathrm{i}_3 & 0 & 1 & 0 \end{array},$$

$$\begin{array}{c|ccc} \mu'_1 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 0 & 0 & 1 \\ \mathrm{i}_2 & 0 & 1 & 0 \\ \mathrm{i}_3 & 1 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} \mu'_2 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 1 & 0 & 0 \\ \mathrm{i}_2 & 0 & 0 & 1 \\ \mathrm{i}_3 & 0 & 1 & 0 \end{array}, \quad \begin{array}{c|ccc} \mu'_3 & \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \hline \mathrm{i}_1 & 0 & 1 & 0 \\ \mathrm{i}_2 & 1 & 0 & 0 \\ \mathrm{i}_3 & 0 & 0 & 1 \end{array}.$$

Now consider lotteries $\sigma, \sigma' \in \Delta\mathcal{M}$ such that

$$\sigma_{\mu_\mathrm{i}} = \sigma'_{\mu'_\mathrm{i}} = 1/3, \ \forall \mathrm{i} \in \{1, 2, 3\}.$$

Then $\mathbb{E}(\sigma) = \mathbb{E}(\sigma')$, but $\sigma \neq \sigma'$.

The previous example reveals a risk for the designers of random mechanisms: there are many ways in which a random matching can be decomposed in a lottery over matchings, and some of those decompositions might not be desirable. Onur Kesten and Ünver (2015) show an example of this in which an ex-post random matching can be expressed as lottery over unstable matchings. Related to this result is the work of Abdulkadiroğlu and Sönmez (2003) in which they show that SD-efficiency can only be decomposed under a particular and restrictive class of efficient matchings, and thus restricting the possibility of an SD-efficient random matching to have 'too many' decompositions.

3.3. On the necessity of a new stability concept

This section is advocated to convince the reader that previous notions of stability are insufficient. In general I use four main criteria to evaluate a stability concept:

- Elimination of ex-post envy among different priority students: there should be a decomposition over stable matchings only.

- Elimination of ex-ante envy among equal and different priority students: students should not want to take probability share from other's student lottery at a particular school at which they have better or equal priority.
- Accounts for explicit preferences over lotteries: different ways of comparing lotteries should have the potential of generating different sets of stable random matchings.
- Should always exist: the set of stable random matchings should not be empty.

Next I will show how previous notions of stability fail to accomplish some of this criteria.

Example 2 (Insufficiency of stability concepts over matchings.) Consider the following school choice economy with $q_a = q_b = 1$.

$$\begin{array}{c|c} P_1 & P_2 \\ \hline \text{a} & \text{a} \\ \text{b} & \text{b} \end{array}, \qquad \begin{array}{c|c} \succeq_a & \succeq_b \\ \hline 1,2 & \vdots \\ & \end{array}.$$

There are two stable matchings:

$$\begin{array}{c|cc} & \text{a} & \text{b} \\ \hline \text{i}_1 & 1 & 0 \\ \text{i}_2 & 0 & 1 \end{array}, \qquad \begin{array}{c|cc} & \text{a} & \text{b} \\ \hline \text{i}_1 & 0 & 1 \\ \text{i}_2 & 1 & 0 \end{array}.$$

None of them eliminates discrimination of equal priority students.

On the other hand the set of strongly stable matchings is empty[1].

$$\mathcal{SS} = \emptyset.$$

Then the intuitive step is to randomize over stable matchings, but as next example shows, it does not eliminates ex-ante justified envy.

Example 3 (Insufficiency of ex-post stability.) Consider the following school choice economy with $q_a = q_b = 2$.

$$\begin{array}{c|c|c|c} P_1 & P_2 & P_3 & P_4 \\ \hline \text{b} & \text{b} & \text{a} & \text{a} \\ \text{a} & \text{a} & \text{b} & \text{b} \end{array}, \qquad \begin{array}{c|c} \succeq_a & \succeq_b \\ \hline 1 & 3,4 \\ 2 & 1 \\ 3,4 & 2 \end{array}.$$

Consider the lottery $\sigma \in \Delta\mathcal{M}$ with its support on the following stable matchings:

[1]In what follows every time that a negative existence or contained result appears it means that there exist a school choice economy such that the result holds. $x \notin \mathcal{X}$ if there exist a school choice economy Γ such that $x \notin \mathcal{X}(\Gamma)$.

μ_1	a	b
i_1	1	0
i_2	1	0
i_3	0	1
i_4	0	1

,

μ_2	a	b
i_1	0	1
i_2	0	1
i_3	1	0
i_4	1	0

Then σ induces is the following random matching:

$\mathbb{E}(\sigma)$	a	b
i_1	σ_{μ_1}	σ_{μ_2}
i_2	σ_{μ_1}	σ_{μ_2}
i_3	σ_{μ_2}	σ_{μ_1}
i_4	σ_{μ_2}	σ_{μ_1}

But i_1 has ex-ante justified envy towards i_2 for school a.

The problem illustrated in the previous example gives place to a way in which ex-ante justified envy is eliminated, but the problem now is that this definition is not flexible to different ways in which agents or the central planner extend preferences over lotteries.

Example 4 (Insufficiency of strongly ex-ante stability.) Consider the school choice economy as in example 5, and suppose agents extend their preferences as DN.

The only strongly ex-ante stable random matching is the following.

π	a	b
i_1	1	0
i_2	0	1

But, since agents extend their preferences as DN, it might be too restrictive. For any $\varepsilon \in (0, 1)$ consider the following random matching.

	a	b
i_1	1-ε	ε
i_2	ε	1-ε

Which DN-Pareto dominates π and does not induce any ex-ante stability violation.

Now let' see that the stability criterion that comes from the random assignment literature has trouble because it does not consider the possibility of agents having different priorities at different objects (schools), thus making sometimes ex-post stability incompatible.

Example 5 (Insufficiency of e-weak no-envy.) Consider the following school choice economy with $q_a = q_b = 1$.

P_1	P_2
a	a
b	b

$\succeq_a$	$\succeq_b$
1	$\vdots$
2	

The following random matching is SD-weak no-envy.

	a	b
i_1	1/2	1/2
i_2	1/2	1/2

But it is not ex-post stable.

This set of examples justify the introduction of a new concept of stability in random environments. In some way all necessary ingredients are on the different notions presented here, the main contribution of this work is on how to mix them and come up with a new one: the e-stability notion.

3.4. Stability

In this section are all the results concerning relations and existence of different concepts of stable matchings are presented.

3.4.1. Conciliation between different concepts of stability

Next I show how previous notions of stability interact with e-stability concepts. This is important because we want to understand how intuitive and non-intuitive properties of different definitions carry on when refinements of definitions are made, and what properties are fundamental preserve intuitive results.

The e-weak no-envy and e-stability conciliation

In the previous work that used e-weak no-envy as a fairness requirement there was no priority structure over the objects. Which meant that all agents had the same priority over all objects. In light of this interpretation I will show what happens to the new concepts of stability on the particular case when all students have the same priority at all schools.

Proposition 4 Let e be a monotone extension. When all students have the same priority at all schools, horizontally e-stability implies e-weak no-envy.

$$\mathcal{S}_H^{\mathrm{e}} \subseteq \mathcal{S}_W^{\mathrm{e}}.$$

This implies that horizontally e-stability is a stronger notion of stability compared to the e-weak no-envy.

The reciprocal inclusion is not true (neither it is wanted to be true) because e-envyness operates at particular coordinates of students random assignments. Since students can envy some coordinates and not others, and have different preferences there are no reasons for students to desire other' students random assignments when they have e-envy.

Ex-ante stability and SD-stability conciliation

Even though ex-ante and strongly ex-ante stability makes no explicit allusion to the way in which students rank lotteries over schools, it is most likely that this definition was inspired by the SD notion, since it is the most intuitive and commonly used way to compare lotteries.

The next proposition shows in fact that when restricting attention to SD extension, these stability notions are equivalent.

Proposition 5 A random matching is:

(i) ex-ante stable if and only if is vertically SD-stable; and
(ii) strongly ex-ante stable if and only if is strongly SD-stable.

$$\mathcal{S}^{xa} = \mathcal{S}_V^{SD},$$
$$\mathcal{S}^{sxa} = \mathcal{S}^{SD}.$$

This results supports the idea that the new concepts of stability are reasonable, in the sense that satisfies previous notions of stability on the environments under they were introduced.

Ex-post and e-stability conciliation

Next I explore under which conditions of the extensions considered the new concepts of stability can be implemented as a lottery over stable matchings only, it turns out to hold a stronger result: it is only needed monotonicity and vertically e-stability to guarantee that the random matching can only be decomposed with support over stable matchings.

Theorem 2 Let e be a monotone extension. If a lottery over matchings induces a vertically e-stable random matching, then it has support over stable matchings only.

$$\mathbb{E}^{-1}(\mathcal{S}_V^{\mathrm{e}}) \subseteq \Delta\mathcal{S}.$$

Proof. By contrapositive if a lottery over matching σ has not support over stable matching only, then denoting μ to this matching, and $\sigma_\mu > 0$ its support, there exist students $\mathrm{i}, j \in I$ and schools $s^*, s' \in S$ such that: (i) $\mu_{js^*} = \mu_{\mathrm{i}s'} = 1$, (ii) $\mathrm{i} \succ_{s^*} j$, and (iii) $s^* P_\mathrm{i} s'$. Then by the monotonicity of e:

$$\left[(\pi_{\mathrm{i}s^*} + \sigma_\mu)s^* + (\pi_{\mathrm{i}s'} - \sigma_\mu)s' + \sum_{s \in S \setminus \{s^*, s'\}} \pi_{\mathrm{i}s} s\right] \mathrm{e}(P_\mathrm{i})\pi_\mathrm{i}, \text{ and}$$

$$\pi_\mathrm{i} \cancel{\mathrm{e}(P_\mathrm{i})} \left[(\pi_{\mathrm{i}s^*} + \sigma_\mu)s^* + (\pi_{\mathrm{i}s'} - \sigma_\mu)s' + \sum_{s \in S \setminus \{s^*, s'\}} \pi_{\mathrm{i}s} s\right].$$

hence π is not vertically e-stable. □

As a consequence the initial desired and more intuitive result.

Corollary 2 If e is a monotone extension, then vertically e-stability implies ex-post stability.

$$\mathcal{S}_V^{\mathrm{e}} \subseteq \mathcal{S}^{xp}.$$

As a consequence a recent result in the literature is generalized.

Corollary 3 (Onur Kesten and Ünver 2015) Ex-ante stability implies ex-post stability. Moreover any lottery that induces ex-post stability has support over stable matchings only.

$$\mathcal{S}_V^{SD} = \mathcal{S}^{xa} \subseteq \mathcal{S}^{xp},$$
$$\mathbb{E}^{-1}(\mathcal{S}_V^{SD}) = \mathbb{E}^{-1}(\mathcal{S}^{xa}) \subseteq \Delta\mathcal{S}.$$

Next an example is shown that proves the necessity of the monotonicity of the extension considered if we want the e-stable random matchings to be ex-post stable.

Example 6 (Why monotonicity?) Consider the following school choice economy:

P_1	P_2
a	b
b	a

$\succeq_a$	$\succeq_b$
1	2
2	1

$q_a = q_b = 1$ and $\mathrm{e} = DN$. Consider the following random matching π:

π	a	b
i_1	1/2	1/2
i_2	1/2	1/2

Is clear that π is DN-stable but not ex-post stable.

$$\mathcal{S}_V^{DN} \not\subseteq \mathcal{S}^{xp}.$$

This result is crucial because we are looking for random matchings that could be converted to matchings respecting the probabilities, and that stability and efficiency may hold ex-post. So if we want to maintain stability when drawing a matching we should restrict attention to monotone extensions only.

3.4.2. On the existence of e-stable random matching

Next we want to understand under which conditions horizontally, vertically and strongly e-stable random matching exist. To assure that the set of strongly e-stable matchings is not empty, structure over the extension is needed. But for the SD extension Onur Kesten and Ünver (ibid.) provide a constructive proof, through their novel algorithm: Fractional Deferred Acceptance, that there always exist a strongly SD-stable random matching.

Proposition 6 (ibid.) The set of strongly SD-stable random matchings is non-empty.

$$\mathcal{S}^{SD} \neq \emptyset.$$

To show that the set of strongly e-stable random matchings is not empty for a general extension e, some intermediate results are needed. It will turn out that these intermediate results will be useful for the characterization of e-stable random matchings when e is monotone, anti-symmetric and has either the CMPO or CLPO.

Next lemma formalizes the idea that if an extension is stronger, then there is more space to envy and smaller is the set of stable random matchings.

Lemma 2 Given e_1, e_2 extensions such that $e_1 \subseteq e_2$ and e_2 is anti-symmetric, then the following inclusions hold:

$$\mathcal{S}_H^{e_2} \subseteq \mathcal{S}_H^{e_1},$$
$$\mathcal{S}_V^{e_2} \subseteq \mathcal{S}_V^{e_1},$$
$$\mathcal{S}^{e_2} \subseteq \mathcal{S}^{e_1}.$$

Next lemma is less intuitive, since monotonicity is not implied neither implies CMPO or CLPO, but at the e-envy level they interact in a particular way: envy at an extension with the CMPO or CLPO property allows to construct explicitly a blocking assignment that generates justified envy at any monotone extension.

Lemma 3 Consider extensions $\mathrm{e}_1, \mathrm{e}_2 \in \mathsf{C}$ such that e_1 is monotone and e_2 having the CMPO or CLPO property, then the following inclusions hold:

$$\begin{aligned} \mathcal{S}_H^{\mathrm{e}_1} &\subseteq \mathcal{S}_H^{\mathrm{e}_2}, \\ \mathcal{S}_V^{\mathrm{e}_1} &\subseteq \mathcal{S}_V^{\mathrm{e}_2}, \\ \mathcal{S}^{\mathrm{e}_1} &\subseteq \mathcal{S}^{\mathrm{e}_2}. \end{aligned}$$

Now we are at hand to show one of the main results of this section, and by the way provide a characterization of e-stable random matchings to extensions that may be unfamiliar.

Theorem 3 Let e be a monotone, anti-symmetric and has either CMPO or CLPO property, then

$$\begin{aligned} \mathcal{S}_H^{\mathrm{e}} &= \mathcal{S}_H^{MO} = \mathcal{S}_H^{SD} = \mathcal{S}_H^{DL} = \mathcal{S}_H^{UL} \neq \emptyset, \\ \mathcal{S}_V^{\mathrm{e}} &= \mathcal{S}_V^{MO} = \mathcal{S}_V^{SD} = \mathcal{S}_V^{DL} = \mathcal{S}_V^{UL} \neq \emptyset, \\ \mathcal{S}^{\mathrm{e}} &= \mathcal{S}^{MO} = \mathcal{S}^{SD} = \mathcal{S}^{DL} = \mathcal{S}^{UL} \neq \emptyset. \end{aligned}$$

Proof. First note that by lemmas 2 and 3, for any pair of extensions e_1 and e_2 such that $\mathrm{e}_1 \subseteq \mathrm{e}_2$, e_1 monotone, e_2 being anti-symmetric and having either CMPO or CLPO, then

$$\begin{aligned} \mathcal{S}_H^{\mathrm{e}_1} &= \mathcal{S}_H^{\mathrm{e}_2}, \\ \mathcal{S}_V^{\mathrm{e}_1} &= \mathcal{S}_V^{\mathrm{e}_2}, \\ \mathcal{S}^{\mathrm{e}_1} &= \mathcal{S}^{\mathrm{e}_2}. \end{aligned}$$

The following pairs of extensions suffice previous conditions: $MO - \mathrm{e}$, $\mathrm{e} - DL$, $\mathrm{e} - UL$. Besides SD is monotone, anti-symmetric, CMPO and CLPO. This proves all equalities of sets provided by the stated theorem.

Using Onur Kesten and Ünver (2015), since $\mathcal{S}^{SD} \neq \emptyset$, I conclude that all previous sets are non-empty. □

One of the most curious results is that for the existence of strongly e-stable random matchings transitivity is not needed.

Another important consequence of last theorem is that the set of e-stable random matchings is very robust to various degrees of completeness of the extension that is taken for account.

The following corollary provides the minimum requirement for a an extension so that strongly e-stable random matchings exist.

Corollary 4 Let e with CMPO or CLPO, then the set of strongly e-stable random matchings is non-empty.

$$\mathcal{S}^{\mathrm{e}} \neq \emptyset.$$

The following example shows that CMPO or CLPO are necessary conditions.

Example 7 (Why CMPO or CLPO?) Consider the following school choice economy with $q_a = q_b = q_c = 1$.

P_1	P_2	P_3
a	a	a
b	b	b
c	c	c

$\succsim_a$	$\succsim_b$	$\succsim_c$
1,2,3	1,2,3	1
		2,3

Consider an extension e monotone and anti-symmetric such that[2]:

$$\left(\frac{1}{3}-\frac{\varepsilon}{2},\frac{1}{3}-\frac{\varepsilon}{2},\frac{1}{3}+\varepsilon\right)\mathrm{e}(P_{\mathrm{i}})\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\ \forall\varepsilon\in(0,2/3).$$

Lets see that there is no horizontally e-stable random matching:

Set π as a random matching:

If $\pi_{\mathrm{i}a} < \pi_{ja}$ for any $\mathrm{i}, j \in \{1,2,3\}$ then student i has horizontally justified e-envy towards student j at school a.

Now fix $\pi_{\mathrm{i}a} = \pi_{ja}$ for all $\mathrm{i}, j \in \{1,2,3\}$.

If $\pi_{\mathrm{i}b} < \pi_{jb}$ for any $\mathrm{i}, j \in \{1,2,3\}$ then student i has horizontally justified e-envy towards student j at school b.

Now fix $\pi_{\mathrm{i}b} = \pi_{jb}$ for all $\mathrm{i}, j \in \{1,2,3\}$, this implies (since π is bi-stochastic) that $\pi_{\mathrm{i}s} = 1/3$ for all $\mathrm{i} \in \{1,2,3\}$ and $s \in \{a,b,c\}$, and in this random matching student 1 has vertically justified e-envy towards students 2 and 3 at school c.

Then there is no strongly e-stable random matching.

$$\mathcal{S}^{\mathrm{e}} = \mathcal{S}^{\mathrm{e}}_V \cap \mathcal{S}^{\mathrm{e}}_H = \emptyset.$$

[2]On this example expressing a lottery as a vector is interpreted as each school represented as the canonical base in the alphabetical order (i.e. p_1 is the probability of object a, p_2 of b, etc...).

3.5. Efficiency

On this section analogous results as the ones obtained in the stability section are shown, some of them were previously proved. The contribution of this section relies on conditions on the extensions so that ex-post properties hold, and an existence non-constructive result for e-efficient random matchings that relies on the axiom of choice.

3.5.1. Relation of efficiency concepts

Just like I did in the section of stability, here various logical relations of different notions of efficiency are presented.

Theorem 4 (WJ Cho 2012) Let e be a monotone, anti-symmetric and has the CMPO or CLPO, then the follow equalities between efficient sets hold.

$$\mathcal{E}^{\mathrm{e}} = \mathcal{E}^{MO} = \mathcal{E}^{SD} = \mathcal{E}^{UL} = \mathcal{E}^{DL} \neq \emptyset.$$

Theorem 5 Let e be a monotone, anti-symmetric and CMPO or CLPO extension, then following equalities between constrained efficient sets hold:

$$\mathcal{E}_C^{\mathrm{e}} = \mathcal{E}_C^{MO} = \mathcal{E}_C^{SD} = \mathcal{E}_C^{UL} = \mathcal{E}_C^{DL} \neq \emptyset,$$
$$\mathcal{E}_{SC}^{\mathrm{e}} = \mathcal{E}_{SC}^{MO} = \mathcal{E}_{SC}^{SD} = \mathcal{E}_{SC}^{UL} = \mathcal{E}_{SC}^{DL} \neq \emptyset.$$

Proof. Since theorem 3 characterizes the stable sets, the proof is the same as theorem 4 but with stable improvement cycles instead of improvement cycles.

The non-emptiness result are consequences of the algorithms fractional deferred acceptance and fractional deferred acceptance with trade presented by Onur Kesten and Ünver (2015). □

It is interesting that the hypothesis for this propositions are the same that required the characterization of e-stable random matchings and existence of e-stable random matchings. In both cases the non-emptiness is shown constructively using the fractional deferred acceptance algorithm for the e-stability and a generalized version of the eating algorithm for the e-efficiency.

Next the relation between e-efficiency and ex-post efficiency is explored.

Theorem 6 If e is a monotone extension, then any lottery that induces an e-efficient random matching has support over efficient matching only.

$$\mathbb{E}^{-1}(\mathcal{E}^{\mathrm{e}}) \subseteq \Delta\mathcal{E}.$$

Proof. By contrapositive, if a lottery σ has not support over efficient matchings only, there exist $\mu^*, \mu' \in \mathcal{M}$ such that is μ' Pareto dominates μ^* and $\sigma_\mu > 0$. Then by e's monotonicity $0 \cdot \mu^* + (\sigma_{\mu'} + \sigma_{\mu^*}) \cdot \mu' + \sum_{\mu \in \mathcal{M} \setminus \{\mu', \mu^*\}} \sigma_\mu \cdot \mu$ e-Pareto dominates $\mathbb{E}(\sigma)$. □

Just as in the case of e-stability, an intuitive result can be obtained.

Corollary 5 If e is monotone, then e-efficiency implies ex-post efficiency.

$$\mathcal{E}^{\mathrm{e}} \subseteq \mathcal{E}^{xp}.$$

This theorem also recovers a well known result from the early literature on random matchings.

Corollary 6 (Bogomolnaia and Moulin 2001) If a random matching is SD-efficient[3], then it is also ex-post efficient.

$$\mathcal{E}^{SD} \subseteq \mathcal{E}^{xp}.$$

Even though e-stability and e-efficiency are defined in very different ways, for these notions to be compatible with ex-post implementation the same condition over the extension (monotonicity) has to be made.

Next is an example in which monotonicity is violated and e-efficiency is not compatible with ex-post efficiency.

Example 8 (Why monotonicity?) Consider example 4, DN-efficiency does not implies ex-post efficiency.

$$\mathcal{E}^{DN} \not\subseteq \mathcal{E}^{xp}.$$

The same kind of results can be obtained for constrained e-efficiency.

Theorem 7 If e is a monotone extension, then any lottery that induces a constrained e-efficient random matching has support over constrained efficient matchings only.

$$\mathbb{E}^{-1}(\mathcal{E}_C^{\mathrm{e}}) \subseteq \Delta\mathcal{E}_C.$$

Proof. The proof follows both procedures as in theorems 2 and 6, but supposing that there is a stable matching that is not constrained efficient. □

Corollary 7 If e is monotone, then constrained e-efficiency implies ex-post constrained efficiency.

$$\mathcal{E}_C^{\mathrm{e}} \subseteq \mathcal{E}_C^{xp}.$$

[3]In their work they called it *Ordinally Efficiency*

Which suggest again that a modeler should not get away from the monotonicity of an extension when allowing agents to have preferences over lotteries.

3.5.2. Existence of e-efficient random matchings

Next it is proposed a proof of existence for e-efficient random matchings. This new approach uses the fact that if an extension e induces a partial order over random assignments, then $\leq_{\mathrm{e}}$ induces a partial order over the set of random matchings. Then *Kuratowski-Zorn lemma* is applied to show the existence of a maximal element, which in this case correspond to an e-efficient random matching.

Lemma 4 When extension e is transitive, then $(\Pi, \leq_{\mathrm{e}})$ is a partially ordered set.

Lemma 5 Let e be transitive and continuous, then every chain in $(\Pi, \leq_{\mathrm{e}})$ has an upper bound at Π.

Theorem 8 If e is a transitive and continuous extension, then there exist an e-efficient random matching.

$$\mathcal{E}^{\mathrm{e}} \neq \emptyset.$$

Proof. By lemma 4 $(\Pi, \leq_{\mathrm{e}})$ is a partially ordered set, and by lemma 5 every chain has an upper bound in Π. Thus by *Kuratowski-Zorn's lemma* (Kuratowski, 1922; Zorn, 1935) there exist an e-Pareto efficient random matching. □

Next I want to know under which conditions this existence result can be applied over constrained e-efficient random matchings, to this end more hypothesis are needed so that we can assure that the sets of vertically and horizontally e-stable matchings is closed.

Lemma 6 Suppose e is a monotone extension with CMPO or CLPO property, then $\mathcal{S}_V^{\mathrm{e}}$, $\mathcal{S}_H^{\mathrm{e}}$ and $\mathcal{S}^{\mathrm{e}}$ are closed.

The proof idea of this lemma is quite simple: if a random matching is not e-stable, then for any random matching not different enough there would remain the same kind of envyness as in the original non e-stable random matching.

Theorem 9 If e is monotone, anti-symmetric, CMPO or CLPO, transitive and continuous then there always exist constrained e-efficient and strongly constrained e-efficient random matchings.

$$\mathcal{E}_C^{\mathrm{e}} \neq \emptyset,$$
$$\mathcal{E}_{SC}^{\mathrm{e}} \neq \emptyset.$$

Proof. Using lemma 6 it follows the same proof of theorem 8. □

Previous techniques for showing the existence of extremal matchings relied heavily on the fact that the universe for possible matchings was finite, and then using iterative procedures would end by finitude. On this case the set of random matchings is infinite and thus a new procedure was necessary.

It is also quite interesting that the result uses Zorn lemma, which is equivalent to the axiom of choice. Around this axiom there is an endless debate of mathematicians on whether should be regarded as true or not.

3.6. Random mechanisms

This section is devoted to analyze commonly used and proposed random mechanisms under the scope of the framework developed in previous sections. Most of the results presented here are old results interpreted under the framework presented on this work.

3.6.1. Random mechanisms as lotteries of mechanisms

Next theorem is the cornerstone for the extension of ex-ante/ex-post results over random matchings to random mechanisms.

Theorem 10 (Pycia and Ünver 2015) Every random mechanism is a convex combination of mechanisms.

$$\mathbb{E}(\mathfrak{M}) = \mathfrak{R}\mathfrak{M}.$$

With this theorem at hand the following can be shown.

Proposition 7 Let e be a monotone extension. Then

$$\begin{aligned} \mathbb{E}^{-1}(\mathfrak{S}_V^{\mathrm{e}}) &\subseteq \Delta\mathfrak{S}, \\ \mathbb{E}^{-1}(\mathfrak{E}^{\mathrm{e}}) &\subseteq \Delta\mathfrak{E}, \\ \mathbb{E}^{-1}(\mathfrak{E}_C^{\mathrm{e}}) &\subseteq \Delta\mathfrak{E}_C. \end{aligned}$$

Corollary 8 Let e be a monotone extension, then

$$\begin{aligned} \mathfrak{S}_V^{\mathrm{e}} &\subseteq \mathfrak{S}^{xp}, \\ \mathfrak{E}^{\mathrm{e}} &\subseteq \mathfrak{E}^{xp}, \\ \mathfrak{E}_C^{\mathrm{e}} &\subseteq \mathfrak{E}_C^{xp}. \end{aligned}$$

This states on the same line that the results from previous sections that for a e-stable-efficient-constrained efficient random mechanisms, then its decompositions ca only have support over stable-efficient-constrained efficient mechanisms.

3.6.2. Relation of strategy-proof concepts

Next relation between strategy-proof concepts are explored. The most important results were shown by Cho's work.

Proposition 8 (WJ Cho 2012)

1. For any extension e, e-strategy-proofness implies e-weak strategy-proofness.

$$\mathfrak{SP}^{\mathrm{e}} \subseteq \mathfrak{SP}_W^{\mathrm{e}}.$$

2. Let e_1, e_2 extensions such that $\mathrm{e}_1 \subseteq \mathrm{e}_2$, then e_1-strategy-proofness implies e_2-strategy-proofness.

$$\mathfrak{SP}^{\mathrm{e}_1} \subseteq \mathfrak{SP}^{\mathrm{e}_2}.$$

3. A random mechanism is SD-strategy-proof if and only if is DL-strategy-proof and UL-strategy-proof.

$$\mathfrak{SP}^{SD} = \mathfrak{SP}^{DL} \cap \mathfrak{SP}^{UL}.$$

The first two results are quite intuitive, but the third is a non-trivial characterization that is useful to show SD-strategy-proofness.

Proposition 9 Let e_1 and e_2 extensions such that $\mathrm{e}_1 \subseteq \mathrm{e}_2$ with e_2 anti-symmetric, then e_2-weak strategy-proofness implies e_1-weak strategy-proofness.

$$\mathfrak{SP}_W^{\mathrm{e}_2} \subseteq \mathfrak{SP}_W^{\mathrm{e}_1}.$$

It is worth noting that when weakening an extension the set of strategy-proof random mechanism squeezes, while the set of weak strategy-proof random mechanism widens.

Proposition 10 (Another folk result) Any random mechanism that can be written as convex combination of strategy-proof mechanisms, is SD-strategy-proof[4].

$$\mathfrak{SP}^{xp} \subseteq \mathfrak{SP}^{SD}.$$

[4]Pycia and Ünver (2015) show that the inclusion is strict.

Perhaps this is one of the strongest argument to stick with randomization over strategy-proof mechanisms, because it will maintain the strategy-proofness in a context were such a property is very demanding, as I will show later this is not for free and some other desirable ex-ante properties are lost.

With this result the existence of some strategy-proof random mechanisms come very easy.

Corollary 9 1. For any extension e such that e $\subseteq SD$, there exist a e-weak strategy-proof random mechanism.

$$\mathfrak{SP}_W^{\mathrm{e}} \neq \emptyset.$$

2. For any extension e such that $SD \subseteq$ e, there exist a e-strategy-proof random mechanism.

$$\mathfrak{SP}^{\mathrm{e}} \neq \emptyset.$$

Thus very weak extensions guarantee existence of weakly strategy-proof random mechanism, while stronger extensions guarantee existence of strategy-proof mechanism. The existence result comes directly from the randomization of strategy-proof mechanisms.

3.6.3. Evaluating random mechanisms

Now we are ready to evaluate random mechanisms in the context of school choice with this robust framework.

Boston with tie-breaking

Recall that Boston mechanism was discarded because of the mess in terms of the incentives that raised during its application. This mess in terms of the incentives endures to the random framework.

Proposition 11 (Abdulkadiroglu and Sönmez 2003; Glazerman and Meyer 1994) Boston mechanism is not strategy-proof.

$$\varphi^B \notin \mathfrak{SP}.$$

Corollary 10 Boston with tie-breaking random mechanism is not e-weak strategy-proof for any e $\in \mathsf{C}$ and anti-symmetric.

$$\phi_{TB}^B \notin \mathfrak{SP}_W^{\mathrm{e}}.$$

The interpretation of this result is that even for very weak structure on the extension the flaws of incentives persist (which is what was expected).

Deferred Acceptance with tie-breaking

Recall that this random mechanism is the most used in the context of school choice, and while it has good ex-post stability properties it does not in an ex-ante stability approach.

Proposition 12 (Onur Kesten and Ünver 2015) 1. Deferred acceptance with tie-breaking is not vertically SD-stable.

$$\phi_{TB}^{DA} \notin \mathfrak{S}_V^{SD}.$$

2. Deferred acceptance with tie-breaking is not horizontally SD-stable.

$$\phi_{TB}^{DA} \notin \mathfrak{S}_H^{SD}.$$

Corollary 11 For any monotone, anti-symmetric and CMPO or CLPO extension e:

- Deferred acceptance with tie-breaking is not vertically e-stable.

$$\phi_{TB}^{DA} \notin \mathfrak{S}_V^{\mathrm{e}}.$$

- Deferred acceptance with tie-breaking is not horizontally e-stable.

$$\phi_{TB}^{DA} \notin \mathfrak{S}_H^{\mathrm{e}}.$$

- Deferred acceptance with tie-breaking is not strongly e-stable.

$$\phi_{TB}^{DA} \notin \mathfrak{S}^{\mathrm{e}}.$$

The interpretation of this result is still not totally clear, but at least there are two approaches to this: (i) if central planners care about the fair distribution of probabilities they should change the mechanism, and (ii) if at some point parents start asking for the lotteries, they would have good reasons to complain that the system is not being fair to his student.

Next the same previous results on efficiency and strategy-proofness on the random environment.

Proposition 13 (Roth 1982) There is no stable and efficient mechanism.

$$\mathfrak{E} \cap \mathfrak{S} = \emptyset.$$

Corollary 12 For any monotone extension e, Deferred Acceptance with tie-breaking is not e-efficient.

$$\phi_{TB}^{DA} \notin \mathfrak{E}^{\mathrm{e}}.$$

Proposition 14 (Dubins and Freedman 1981; Roth 1982) Deferred acceptance is strategy-proof.

$$\varphi^{DA} \in \mathfrak{S}\mathfrak{P}.$$

Corollary 13 1. For any extension e such that $SD \subseteq$ e, deferred acceptance with tie-breaking is e-strategy-proof.

$$\phi_{TB}^{DA} \in \mathfrak{SP}^{\mathrm{e}}.$$

2. For any extension e such that e $\subseteq SD$, deferred acceptance with tie-breaking is e-weak strategy-proof.

$$\phi_{TB}^{DA} \in \mathfrak{SP}_{W}^{\mathrm{e}}.$$

Top Trading Cycles with tie-breaking

While the use of TTC in school choice has been scarce it has been widely applied to other markets. Its main problem is that for the sake of efficiency violates stability and justified envy arises, of course that these features extend to the probabilistic environment.

Corollary 14 For any monotone extension e, top trading cycles with tie-breaking is not vertically e-stable.

$$\phi_{TB}^{TTC} \notin \mathfrak{S}_{V}^{\mathrm{e}}.$$

Proposition 15 (O Kesten 2006) Top trading cycles with multiple tie-breaking is not SD-efficient.

$$\phi_{TB}^{TTC} \notin \mathfrak{E}^{SD}.$$

Corollary 15 For any monotone, anti-symmetric and CMPO or CLPO extension e, top trading cycles with multiple tie-breaking is not e-efficient.

$$\phi_{TB}^{TTC} \notin \mathfrak{E}^{\mathrm{e}}.$$

Corollary 16 1. For any extension e such that $SD \subseteq$ e, top trading cycles with tie-breaking is e-strategy-proof.

$$\phi_{TB}^{TTC} \in \mathfrak{SP}^{\mathrm{e}}.$$

2. For any extension e such that e $\subseteq SD$, top trading cycles with tie-breaking is e-weak strategy-proof.

$$\phi_{TB}^{TTC} \in \mathfrak{SP}_{W}^{\mathrm{e}}.$$

At this point the only good property of TTC with TB are the ex-post efficiency and the SD-strategy-proofness. So if SD-efficiency is important a market procedure that allows agents to exchange probability shares should be considered.

Fractional Deferred Acceptance

To my knowledge this random mechanism is the only proposed alternative that does not rely on randomization of mechanisms for the school choice problem with weak priority. It will be shown that its good ex-ante properties extend to a wider class of extensions at the cost of loosing the SD-strategy-proofness.

Proposition 16 (Onur Kesten and Ünver 2015) Fractional deferred acceptance is strongly constrained SD-efficient.

$$\phi^{FDA} \in \mathfrak{E}_C^{SD}.$$

Corollary 17 For any monotone, anti-symmetric and CMPO or CLPO extension e, fractional deferred acceptance is strongly constrained e-efficient.

$$\phi^{FDS} \in \mathfrak{E}_C^{\mathrm{e}}.$$

Proposition 17 Fractional Deferred Acceptance is not DL-strategy-proof.

Proof. The proof is by means of an example. Consider the following school choice economy with $q_a = q_b = q_c = 1$.

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

,

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	1	$\vdots$
1,2	$\vdots$	

.

Under truth-telling student 1 obtains lottery $(0,1,0)$, but if she reports P_1' : a-c-b, she obtains the more preferred lottery $(1/2, 0, 1/2)$. $\square$

Corollary 18 Fractional deferred acceptance is not SD-strategy-proof.

This negative result might still be revisited, because recent literature suggest that common people do not know how to do complicated manipulations of complicated mechanisms[5] thus there might be a milder interpretation of incentives that could vanish this negative result.

[5]In fact highly trained economists might not know how to manipulate the mechanism.

Chapter 4

Concluding remarks

This work provides several contributions to solve the problem of fair assignment of students to schools when fairness is interpreted as equal priority students having the same right of getting a seat. This task is accomplished including a generalization that allows to have different preferences over lotteries.

It is shown that previous notions of stability and fairness were insufficient: the (deterministic) notion of stability is intrinsically incompatible with the elimination of envy for equal priority students, the ex-post stability notion does not eliminates SD-envy for different and equal priority students, the ex-ante stability and strongly ex-ante stability is not flexible enough to account for different preferences over the lotteries that students and their parents may have and the e-weak no-envy does not account for the priority structure and thus they do not fulfill the minimum requirement of being implementable with support on stable matchings.

A new concept of stability is presented for the school choice problem with random assignments: the vertically e-stability, horizontally e-stability and the (strongly) e-stability notions. This new definition respects the previous definitions of stability and fairness that were presented in the previous literature of school choice and random assignment such as the e-weak no-envy, ex-ante stability and strongly ex-ante stability.

Monotonicity on the extension is necessary and sufficient condition for a horizontally e-stable random matching to have support over stable matchings only. The same result holds for e-efficient and constrained e-efficient random matchings.

When the extension e compensates for either more or less preferred objects (schools) the existence of e-stable random matchings is guaranteed as a consequence of the Fractional Deferred Acceptance algorithm, and an example shows in fact that this condition is also necessary. If e is also monotone and anti-symmetric then it defines the same set of e-stable random matchings that monotone, stochastic dominance, upward and downward lexicographic extensions induce, yielding an interesting characterization result that provides a novel justification of the widely use of stochastic dominance as the main way of comparing lotteries.

It is surprising that transitivity is not necessary for any result regarding e-stability.For e-efficiency it matters.

A novel existing result is provided for the existence of e-efficient random matchings when e es transitive and continuous. The most interesting thing about this result is that relies on the axiom of choice, I have tried to prove the same result with weaker hypothesis (using transfinite induction) but nothing has come out.

Well known random mechanisms are evaluated under this framework: Boston with tie-breaking' bad incentive properties persist, Deferred Acceptance with tie-breaking does not eliminates ex-ante sources of justified envy but is still strategy-proof, Top Trading Cycles with tie-breaking is not SD-efficient but is still strategy-proof, and the Fractional Deferred Acceptance Algorithm eliminates both sources of horizontally and vertically justified envy, it is constrained efficient but it is not strategy-proof. An open question is whether Fractional Deferred Acceptance fulfills some weaker form of strategy-proofness.

Previous results suggest new trade-off dimensions in random matching: there is an intrinsic trade-off between ex-ante properties regarding stability, fairness and efficiency; and incentive properties.

Appendix of Proofs

Proof of proposition 3.

1. Suppose $\rho^1 \mathrm{e}_1(P)\rho^2$ and $\rho^2 \mathrm{e}_1(P)\rho^2$, then by the inclusion $\rho^1 \mathrm{e}_2(P)\rho^2$ and $\rho^2 \mathrm{e}_2(P)\rho^2$ hold. By $\mathrm{e}_2(P)$ anti-symmetry $\rho^1 = \rho^2$.
2. Suppose CMPO or CLPO do not hold at extension e. Then, in either case, there exist lotteries ρ^1, ρ^2 such that $\rho^1 \mathrm{e}(P)\rho^2$, $\rho^2 \cancel{\mathrm{e}(P)}\rho^1$, and the following inequality holds for some $s^* \in S$:

$$\sum_{sRs^*} \rho^1_s < \sum_{sRs^*} \rho^2_s.$$

Thus $\rho^1 \cancel{SD(P)} \rho^2$.

$\square$

Proof of lemma 1. It is straightforward that every matching is an extreme point of Π.

For the converse suppose that μ is an extreme point of Π and for some $\mathrm{i} \in I$ and $s \in S$: $\mu_{\mathrm{i}s} \in (0,1)$.

Then operating a-la Hylland and Zeckhauser (1979) μ can be decomposed as the convex combination of two different random matchings ($\rightharpoonup\!\leftharpoondown$). $\square$

Proof of proposition 4. Suppose π is not e-weakly no-envy because i envies j's random assignment.

Since $\pi_j \mathrm{e}(P_\mathrm{i})\pi_\mathrm{i}$ and $\pi_\mathrm{i} \cancel{\mathrm{e}(P_\mathrm{i})} \pi_j$. Let $\underline{s} = \arg\max_{s \in S} \kappa(P_\mathrm{i}, s)$ be the worst school for i, then setting $S' = \arg\min_{\{s \in S : \pi_{\mathrm{i}s} < \pi_{js}\}} \kappa(P_\mathrm{i}, s)$ as the blocking set of schools ($S' = \{\overline{s}\}$), and the blocking assignment:

$$\rho_{\mathrm{i}s} = \begin{cases} \pi_{\mathrm{i}s}, & s \in S \setminus \{\overline{s}, \underline{s}\} \\ \pi_{\mathrm{i}s} + \min\left\{\pi_{\mathrm{i}s}, \frac{\pi_{js} - \pi_{\mathrm{i}s}}{2}\right\}, & s = \overline{s} \\ \pi_{\mathrm{i}s} - \min\left\{\pi_{\mathrm{i}s}, \frac{\pi_{js} - \pi_{\mathrm{i}s}}{2}\right\}, & s = \underline{s} \end{cases}$$

Thus π is not horizontally e-stable. $\square$

Proof of proposition 5. Given π to be the random matching:

(i) $\underline{\Rightarrow}$: Suppose that π is not ex-ante stable.

Since $\pi_{is'} > 0$, sP_is' and $\pi_{js} > 0$ then

$$\rho_i = (\pi_{is} + \min\{\pi_{is'}, \pi_{js}\})s + (\pi_{is'} - \min\{\pi_{is'}, \pi_{js}\})s' + \sum_{t \in S \setminus \{s,s'\}} \pi_{st} t,$$

generates vertically justified SD-envy, and thus π is not vertically SD-stable.

$\underline{\Leftarrow}$: Suppose that π is not vertically SD-stable.

Since there exist blocking random assignment ρ_i and non-empty set of schools S' such that $\pi_{is} < \rho_{is} \leq \pi_{is} + \pi_{js}$, $\forall s \in S'$; $\rho_i SD(P_i)\pi_i$ and $\pi_i \cancel{SD(P_i)} \rho_i$. Then, by contradiction, it must be that there exists school s' such that $\pi_{is'} > 0$ and sP_is', if not then ρ_i could not stochastically dominate π_i ($\rightharpoonup\!\leftharpoondown$).

Then i would have ex-ante justified envy towards j at school s', and thus π would not be ex-ante stable.

(ii) $\underline{\Rightarrow}$: Suppose that π is ex-ante stable but is not strongly ex-ante stable. Then π induces ex-ante discrimination.

Since $\pi_{is'} > 0$, sP_is' and $\pi_{js} > \pi_{is}$ then

$$\rho_i = \left(\pi_{is} + \min\left\{\pi_{is'}, \frac{\pi_{js} - \pi_{is}}{2}\right\}\right) s + \left(\pi_{is'} - \min\left\{\pi_{is'}, \frac{\pi_{js} - \pi_{is}}{2}\right\}\right) s' + \sum_{t \in S \setminus \{s,s'\}} \pi_{st} t,$$

generates horizontally justified SD-envy, and thus π is not horizontally neither strongly SD-stable.

$\underline{\Leftarrow}$: Suppose that π is not horizontally SD-stable.

Since there exist blocking random assignment ρ_i and non-empty set of schools S' such that $\pi_{is} < \rho_{is} \leq \frac{\pi_{is} + \pi_{js}}{2}$, $\forall s \in S'$; $\rho_i SD(P_i)\pi_i$ and $\pi_i \cancel{SD(P_i)} \rho_i$. Then, by contradiction, it must be that there exists school s' such that $\pi_{is'} > 0$ and sP_is', if not then ρ_i could not stochastically dominate π_i ($\rightharpoonup\!\leftharpoondown$).

Then i would be ex-ante discriminated towards j at school s', and thus π would not be strongly ex-ante stable.

$\square$

Proof of corollary 2. Since $\mathbb{E}$ is surjective[1], then

$$\mathbb{E} \circ \mathbb{E}^{-1}(\mathcal{S}_V^{e}) = \mathcal{S}_V^{e} \subseteq \mathbb{E}(\Delta \mathcal{S}) = \mathcal{S}^{xp}.$$

$\square$

Proof of lemma 2. If there is a student i that has e_1-envy at π then there exist ρ_i such that $\rho_i e_1(P_i)\pi_i$, which implies $\rho_i e_2(P_i)\pi_i$. By e_2 anti-symmetry $\pi_i \cancel{e_2(P_i)} \rho_i$ then student i has e_2 envy at π.

$$\rho_i e_1(P_i)\pi_i \Rightarrow \rho_i e_2(P_i)\pi_i \Rightarrow \pi_i \cancel{e_2(P_i)} \rho_i$$

$\square$

[1]For any function $f \circ f^{-1}(B) \subseteq B$, and if f is surjective then $f \circ f^{-1}(B) = B$.

Proof of lemma 3. Suppose that at random matching π student i e_2-envies student j at the set of schools S' because of random assignment ρ_{i}. So

(i) $\pi_{\mathrm{i}s} < \rho_{\mathrm{i}s} \leq \pi_{\mathrm{i}s} + \pi_{js}$ for all $s \in S'$,
(ii) $\rho_{\mathrm{i}s} \leq \pi_{\mathrm{i}s}$ for all $s \in S \setminus S'$,
(iii) $\rho_{\mathrm{i}} \mathrm{e}_2(P_{\mathrm{i}})\pi_{\mathrm{i}}$, and
(iv) $\pi_{\mathrm{i}} \cancel{\mathrm{e}_2(P_{\mathrm{i}})} \rho_{\mathrm{i}}$.

By e_2's CMPO or CLPO there must be schools $s^* \in S \setminus S'$ and $s' \in S'$ such that $s' P_{\mathrm{i}} s^*$, and $\pi_{\mathrm{i},s^*} > 0$; $\pi_{js'} > 0$.

Define random assignment ν_{i} as

$$\begin{cases} (\pi_{\mathrm{i}s'} + \min\{\pi_{\mathrm{i}s^*}, \pi_{js'}\})s' + (\pi_{\mathrm{i}s^*} - \min\{\pi_{\mathrm{i}s^*}, \pi_{js'}\})s^* + \sum_{s\in S\setminus\{s^*,s'\}} \pi_{\mathrm{i}s} s & \text{, if } \pi_{\mathrm{i}s'} \geq \pi_{js'} \\ \left(\pi_{\mathrm{i}s'} + \min\left\{\pi_{\mathrm{i}s^*}, \frac{\pi_{js'}-\pi_{\mathrm{i}s'}}{2}\right\}\right) s' + \left(\pi_{\mathrm{i}s^*} - \min\left\{\pi_{\mathrm{i}s^*}, \frac{\pi_{js'}-\pi_{\mathrm{i}s'}}{2}\right\}\right) s^* + \sum_{s\in S\setminus\{s^*,s'\}} \pi_{\mathrm{i}s} s & \text{, if } \pi_{\mathrm{i}s'} < \pi_{js'} \end{cases}$$

which induces e_1-envy at π. And it would be justified e_1-envy if originally existed justified e_2-envy. □

Proof of corollary 4. By lemma 3 $\mathcal{S}^{SD} \subseteq \mathcal{S}^{\mathrm{e}}$, and by theorem 3 is non-empty. □

Proof of corollary 5. Since $\mathbb{E}$ is surjective, then

$$\mathbb{E} \circ \mathbb{E}^{-1}(\mathcal{E}^{\mathrm{e}}) = \mathcal{E}^{\mathrm{e}} \subseteq \mathbb{E}(\Delta\mathcal{E}) = \mathcal{E}_{xp}.$$

□

Proof of lemma 4 . Lets prove that reflexivity, anti-symmetry and transitivity are met.

- Reflexive: by construction $\pi \leq_{\mathrm{e}} \pi \ \forall \pi \in \Pi$.
- Anti-symmetry: Let $\pi, \pi' \in \Pi$ such that $\pi \leq_{\mathrm{e}} \pi'$ and $\pi' \leq_{\mathrm{e}} \pi$. By contradiction suppose $\pi <_{\mathrm{e}} \pi'$ which means that there exist $j \in I$ such that $\pi_j \cancel{\mathrm{e}(P_j)} \pi'_j$ which contradicts $\pi' \leq_{\mathrm{e}} \pi$ ($\rightharpoonup\leftharpoondown$), thus it must be that $\pi = \pi'$.
- Transitive: let $\pi, \pi', \pi'' \in \Pi$ such that $\pi'' \leq_{\mathrm{e}} \pi' \leq_{\mathrm{e}} \pi$. If $\pi'' = \pi'$ or $\pi' = \pi$ hold, then transitivity is concluded from the anti-symmetry of $\leq_{\mathrm{e}}$.

  Let's see the case where $\pi'' <_{\mathrm{e}} \pi' <_{\mathrm{e}} \pi$. By e's transitivity $\pi''_{\mathrm{i}} \mathrm{e}(P_{\mathrm{i}})\pi_{\mathrm{i}}$ for all $\mathrm{i} \in I$. Let $j \in I$ be a student such that $\pi'_j \cancel{\mathrm{e}(P_j)} \pi''_j$, by contradiction if $\pi_j \mathrm{e}(P_j)\pi''_j$ then $\pi'_j \mathrm{e}(P_j)\pi''_j$ ($\rightharpoonup\leftharpoondown$).

  Then it must be that $\pi_j \cancel{\mathrm{e}(P_j)} \pi''_j$. Thus $\leq_{\mathrm{e}}$ is transitive.

□

Proof of lemma 5. Let C be a chain at $(\Pi, \leq_{\mathrm{e}})$. From now on denote $F(\pi) \doteq \pi \leq_{\mathrm{e}}$, I will show that $\cap_{\pi \in C} F(\pi)$ is non-empty.

Since C is a chain and e is transitive, for every pair $\pi^1, \pi^2 \in C$:

$$\begin{aligned}\pi^1 \leq_{\mathrm{e}} \pi^2 &\Leftrightarrow F(\pi^2) \subseteq F(\pi^1), \\ &\Leftrightarrow \mathrm{di}am(F(\pi^2)) \leq \mathrm{di}am(F(\pi^1)).\end{aligned}$$

Now consider the following infimization succession $(\omega_n)_{n \in \mathbb{N}} \subseteq C$ such that

$$\mathrm{di}am(F(\omega_n)) \searrow \inf_{\pi \in C} \mathrm{di}am(F(\pi)).$$

Since ω is an infimization succession for every $\pi \in C$ there exist $N \in \mathbb{N}$ such that $\mathrm{di}am(F(\omega_n)) \leq \mathrm{di}am(F(\pi))$ for all $n \geq N$.

Thus there is a function $\eta \in \mathbb{N}^C$ such that $F(\omega_{\eta(\pi)}) \subseteq F(\pi)$ for all $\pi \in C$.

The following inclusions hold:

$$\begin{aligned}\bigcap_{\pi \in C} F(\pi) &\subseteq \bigcap_{n=1}^{\infty} F(\omega_n), \\ \bigcap_{\pi \in C} F(\omega_{\eta(\pi)}) &\subseteq \bigcap_{\pi \in C} F(\pi), \\ \bigcap_{n=1}^{\infty} F(\omega_n) &\subseteq \bigcap_{\pi \in C} F(\omega_{\eta(\pi)}).\end{aligned}$$

Thus the following equality hold:

$$\bigcap_{\pi \in C} F(\pi) = \bigcap_{n=1}^{\infty} F(\omega_n).$$

Because e is continuous the collection $\{F(\omega_n)\}_n$ contains only closed sets, and it is decreasing. It concludes by *Cantor's intersection theorem*:

$$\bigcap_{n=1}^{\infty} F(\omega_n) \neq \emptyset.$$

The proof concludes by the fact that Π is closed. $\square$

Proof of lemma 6. For $\mathcal{S}_V^{\mathrm{e}}$ and $\mathcal{S}_H^{\mathrm{e}}$ I will show that the complement is open:

- Suppose π is not vertically e-stable. Let i be a student with vertically justified e-envy towards student j. By CMPO or CLPO there exist $s' \in S'$ and $s^* \in S \setminus S'$ such that $s' P_{\mathrm{i}} s^*$ and $\pi_{\mathrm{i}s^*} > 0$. Let $\hat{\pi}$ be a random matching such that

$$\|\pi - \hat{\pi}\|_{\infty} < \min\{\pi_{js'}, \pi_{\mathrm{i}s^*}\}.$$

  By construction $\hat{\pi}_{js'} > 0$ and $\hat{\pi}_{\mathrm{i}s^*} > 0$. Thus is possible to build a monotone improvement of $\hat{\pi}_{\mathrm{i}}$ for i by raising the probability at s and lowering it at s', and would imply vertically justified e-envy at π^*.

- Suppose π is not vertically e-stable. Let i be a student with vertically justified e-envy towards student j. By CMPO or CLPO there exist $s' \in S'$ and $s^* \in S \setminus S'$ such that $s' P_{\mathrm{i}} s^*$ and $\pi_{\mathrm{i}s^*} > 0$. Let $\hat{\pi}$ be a random matching such that
$$\|\pi - \hat{\pi}\|_\infty < \min\left\{\frac{\pi_{js'} - \pi_{\mathrm{i}s'}}{2}, \pi_{\mathrm{i}s^*}\right\}.$$
By construction $\hat{\pi}_{js'} > \hat{\pi}_{\mathrm{i}s'}$ and $\hat{\pi}_{\mathrm{i}s^*} > 0$. Thus is possible to build a monotone improvement of $\hat{\pi}_{\mathrm{i}}$ for i by raising the probability at s and lowering it at s', and would imply vertically justified e-envy at π^*.
- Since the intersection of closed sets is closed $\mathcal{S}^{\mathrm{e}}$ is closed.

□

Proof of proposition 7. In general for property $P \in$ {stability,efficiency,constrained efficiency}, if a random mechanism has positive support over a mechanism that does not satisfies that property, then by theorems 2, 6, and 7 it won't satisfy e-property $P' \in$ {vertically e-stability, e-efficiency, constrained e-efficiency}. □

Proof of proposition 9. Suppose $\phi \in \mathfrak{S}\mathfrak{P}^{\mathrm{e}_2}_W$. If $\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]\mathrm{e}_1(P_{\mathrm{i}})\phi_{\mathrm{i}}\left[\vec{P}\right]$, then $\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]\mathrm{e}_2(P_{\mathrm{i}})\phi_{\mathrm{i}}\left[\vec{P}\right]$, and by e_2-weak strategy-proof and e_2 anti-symmetry: $\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}] = \phi_{\mathrm{i}}\left[\vec{P}\right]$.

Thus $\phi_{\mathrm{i}}\left[\vec{P}\right]\mathrm{e}_1(P_{\mathrm{i}})\phi_{\mathrm{i}}[P'_{\mathrm{i}}, P_{-\mathrm{i}}]$. □

Proof of corollary 9. By Abdulkadiroglu and Sönmez (2003) $\varphi^{DA}, \varphi^{TTC} \in \mathfrak{S}\mathfrak{P} \neq \emptyset$.

1. By proposition 8.1: $\mathfrak{S}\mathfrak{P}^{SD} \subseteq \mathfrak{S}\mathfrak{P}^{SD}_W$, and by proposition 9 $\mathfrak{S}\mathfrak{P}^{SD}_W \subseteq \mathfrak{S}\mathfrak{P}^{\mathrm{e}}_W$.
2. By proposition 8.2: $\mathfrak{S}\mathfrak{P}^{SD} \subseteq \mathfrak{S}\mathfrak{P}^{\mathrm{e}}$.

□

Proof of corollary 10. Using the same counterexample than proposition 11, then $\phi^B_{TB} \notin \mathfrak{S}\mathfrak{P}^{ID}_W$. By the anti-symmetry of e and proposition 9, then $\mathfrak{S}\mathfrak{P}^{\mathrm{e}}_W \subseteq \mathfrak{S}\mathfrak{P}^{ID}_W$. □

Proof of corollary 11. Direct result from the characterization given by theorem 3. □

Proof of corollary 12. Since $\mathfrak{E}^{\mathrm{e}} \subseteq \mathfrak{E}^{xp}$, and $\phi^{DA}_{TB} \in \mathfrak{S}^{xp}$. By proposition 13: $\mathfrak{S}^{xp} \cap \mathfrak{E}^{xp} = \emptyset$. □

Proof of corollary 13. Straightforward from propositions 8 and 9. □

Proof of corollary 14. Since $\phi^T_{TB}TC \in \mathfrak{E}^{xp}$ and $\mathfrak{S}^{\mathrm{e}}_V \subseteq \mathfrak{S}^{xp}$, it concludes with proposition 13. □

Proof of corollary 15. Is a consequence of the characterization provided by proposition 4. □

Proof of corollary 16 . Using the fact that $\varphi^{TTC} \in \mathfrak{SP}$ (Abdulkadiroglu and Sönmez, 2003) it follows the same as corollary 13. □

Proof of corollary 18. This is a direct consequence of 8.3. □

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