MATHEMATICAL MODELLING AND ECONOMIC EVALUATION

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Abstract. In small developing countries many significant agroindustrial projects are not marginal. Therefore a model is proposed that includes all the main components of the sector. This model can be used to evaluate new projects, policies, to test the prevailing degree of market imperfection in the sector and in general to provide a better understanding of the sector so that the economic agents in it can increase their efficiency. This approach is based on Samuelson's proof that a spatial partial equilibrium system can be converted into a maximization problem. The approach is exemplified by a model of the chilean wheat sector in 1978.

Keywords. Economics; modelling; optimisation; agriculture; developing countries.

INTRODUCTION

A marginalist approach to project evaluation might be inappropiate in a small developing country where many agroindustrial projects are not marginal. A method is proposed that consists of building a model which contains all the relevant components of the sector and the relationships between them. This approach is based on Samuelson's proof that a spatial partial equilibrium system can be converted into a maximization problem. In fact, Samuelson (1952) proved that a spatial equilibrium system consisting of a number of locations, each having a supply function and a demand function, can be converted into a maximization problem, the maximization of the Net Social Payoff (NSP). We extend this approach to include processing plants in a multiperiod framework, but the idea is still the same: the equilibrium of the system can be obtained by maximizing the surplus of the economy.

A limitation of this kind of model is that it assumes perfect competition. But on the other hand this limitation can be one of its uses, to test for competitiveness in the sector being studied.

The kind of model we propose is designed to take into account all the main effects a new project or policy might have in the sector. The solution of the proposed model, besides evaluating the new project or policy, gives information about changes in input supplies, in processing levels in plants, in product consumption, and in shadow prices.

Other authors have already extended Samuelson's model to include both storage and intermediate processing. The most extensive treatment can be found in Takayama and Judge (1971). Several applications of this type of model are reported in Judge and Takayama (1973). However, our approach is somewhat different to that of these authors. Their models are intended to help economic planning, ours to evaluate projects (and policies) in a free-market context.

THE MODEL

In the rest of this paper I present the application we did for the wheat agroindustrial sector in Chile.

In Chile, the wheat producing regions are different from those where population (and flour demand) is concentrated. Since transport costs are far from being negligible the spatial dimension has to be taken into account, and so the country is divided into regions. Furthermore a special region is added, 'the rest of the world', or, in other words, import and export activities are considered.

Samuelson's simple spatial model is complicated by the introduction of intermediate processing plants. Wheat before consumption is processed in mills. The model has to include processing costs and new restrictions: milling capacities.

The model is further expanded by the introduction of the time dimension. Wheat is harvested in a short span of time, but consumption is spread throughout the year. To handle this problem the year is divided into periods and a new activity has to be introduced: wheat storage, with its own costs and capacities.

For expository reasons, the model we present here is a simplification of the original one. The wheat sector can be described in the following way: the country is divided into n regions, each of which has a demand for flour, a supply of wheat, an aggregate storage capacity and an aggregate mill capacity. Wheat flows from farms into silos, which are also supplied from the rest of the world, from silos the wheat flows into mills where it is processed into flour, then flour is distributed for consumption.

The model is a set of mathematical expressions representing the different components of the sector and their interrelations.

$$\text{Max} \sum_t \sum_\ell F_{\ell t}(y_{\ell t}) - \sum_t \sum_i G_{it}(x_{it}) - TC - PC - IC \quad (1)$$

s.t:

$$x_{it} = \sum_j z_{ijt}, \qquad i = 1,\ldots,n;\ t = 1,\ldots,T \quad (2)$$

$$s_{jt} = s_{j,t-1} + m_{jt} + \sum_i z_{ijt} - \sum_k v_{jkt}, \qquad j = 1,\ldots,n;\ t = 1,\ldots,T \quad (3)$$

$$s_{j0} = 0, \qquad j = 1,\ldots,n \quad (4)$$

$$(1/r_k)\sum_\ell w_{k\ell t} = \sum_j v_{jkt}, \qquad k = 1,\ldots,n;\ t = 1,\ldots,T \quad (5)$$

$$\sum_k w_{k\ell t} = y_{\ell t}, \qquad \ell = 1,\ldots,n;\ t = 1,\ldots,T \quad (6)$$

$$s_{jt} \le K_j^1, \qquad j = 1,\ldots,n;\ t = 1,\ldots,T \quad (7)$$

$$\sum_j v_{jkt} \le K_k^2, \qquad k = 1,\ldots,n;\ t = 1,\ldots,T \quad (8)$$

where $F_{\ell t}(y_{\ell t}) = \int_0^{y_{\ell t}} f_{\ell t}(q)dq$

and $G_{it}(x_{it}) = \int_0^{x_{it}} g_{it}(q)dq.$

$f_{\ell t}(\ )$ denoting the flour demand of region ℓ in period t, and $g_{it}(\ )$ denoting the wheat supply of region ℓ in period t (in both cases the price is a function of the quantity).

TC, PC and IC are respectively the aggregate transport cost, the aggregate processing cost, and the total value of imported wheat.

$$TC = \sum_{i,j,t} t^1_{ij} z_{ijt} + \sum_{j,k,t} t^2_{jk} v_{jkt} + \sum_{k,\ell,t} t^3_{k\ell} w_{k\ell t} + \sum_{j,t} t^4_j m_{it} \quad (9)$$

$$PC = \sum_{j,t} c^1_j s_{jt} + \sum_{k,\ell,t} c^2_k w_{k\ell t} \quad (10)$$

$$IC = \sum_{j,t} e_t m_{jt} \quad (11)$$

The subindices are:

$i,j,k,\ell = 1,\ldots,n$; regions

$t = 1,\ldots,T$; periods

The variables are:

x_{it} : wheat supply of region i, period t

z_{ijt} : amount of wheat shipped from farms located in the i^{th} region to silos located in the j^{th} region, in period t

v_{jkt} : amount of wheat received in mills of the k^{th} region coming from silos in the j^{th} region, in period t

$w_{k\ell t}$: flour sent from mills in the k^{th} region to consumers living in the ℓ^{th} region, in period t

$y_{\ell t}$: flour demand in the ℓ^{th} region, in period t

m_{jt} : amount of imported wheat received by silos in the j^{th} region, in period t

and the parameters are:

r_k : mill yield in the k^{th} region

t^1_{ij}, t^2_{jk}, $t^3_{k\ell}$, t^4_j : transport cost per unit

c^1_j : storage cost (per period) in the j^{th} zone

c^2_k : processing cost per unit in the k^{th} zone

e_t : cost of one unit of imported wheat in period t.

c_j^1 and c_k^2 are variable costs. They will differ from the total per unit cost if and only if the relevant capacity is restrictive. For instance in some region and in some period the processing capacity might be restrictive. In this case the per unit processing cost of this region at this time will include a contribution to the fixed costs of the plant.

The interpretation of the restrictions is straightforward. Restrictions (2) - (6) are balance equations. Equation (2) says that for each region wheat shipments are equal to the total supply of the period. Equation (3) says that for each region and period, the stock at the end of the period is equal to the stock at the beginning of the period plus wheat inflows and minus wheat outflows. Equation (4) says that stocks are zero at the beginning of the first period. Equation (5) says that for each region and period the total flour production is a fixed proportion of the wheat being processed. Finally equation (6) says that total consumption in each region and period is equal to the flour supply. Expressions (7) and (8) are capacity restrictions. In each region and period, the wheat stored cannot exceed the storage capacity, and the wheat processed cannot exceed the mill processing capacity.

The model as presented has various implicit assumptions and simplifications.

i) It is implicit in the model that wheat producers have perfect foresight. When farmers are sowing they know the price wheat will have at the time of the harvest. This assumption is in accordance with some empirical evidence. Furthermore it is logically plausible. In Chile, most of the sowing is done when the harvest results of the major world wheat producers are known, and there is a good forecast of the international wheat price that will prevail during the time of the domestic harvest.

ii) For each unit of wheat entering a mill, the flour yield is about 76.5%, 21.5% are byproducts and the remaining 2% are impurities. A major part of the byproducts' output is exported, and its demand is highly elastic. This fact allowed us to consider a fixed price for byproducts, and instead of including a demand for byproducts, we substracted the revenue obtained by the sale of byproducts from the mill processing cost.

iii) A significant part of the wheat production remains in the rural sector, and does not go through industrial processing. So, by wheat supply we mean the net supply of the rural sector. The model should improve if a rural demand for wheat were included and the model considered the gross supply of wheat.

iv) The model assumes that per unit transport costs are constant. It further assumes that the per unit storage and processing variable costs are constant. The last assumption seems unrealistic, especially if we take into account the fact that the aggregate processing capacity of a region is obtained by adding up capacity of several mills. A way of solving this problem is by dividing the mills of a region into groups. Than, each of these groups is included in the model with its own capacity and processing costs. An equivalent division can be done with the silos.

v) A major component of stockholding costs is the financial cost. The financial cost depends on the price of wheat, i.e. it is an endogenous variable of the model. In order to simplify the model we used an *a priori* estimation of the price of the wheat to calculate the financial cost.

vi) At the time the model was built there was a specific tariff that effectively precluded flour imports, and for this reason we did not include a flour import activity. Neither did we include wheat or flour exports because they were completely unthinkable.

THE OPTIMAL SOLUTION

The dual variable associated with each restriction of the system has an economic interpretation. The dual variable associated with restriction (2), pf_{it}, is the market price of one unit of wheat in a farm of the i^{th} region in period t. The dual variable associated with restriction (3), ps_{jt}, is the market price of one unit of wheat in a silo of the j^{th} region in period t. The dual variable associated with restriction (5), pm_{kt}, is the market price of one unit of wheat in a mill of region k in period t. The dual variable associated with restriction (6), $pc_{\ell t}$, is the market price of one unit of flour in region ℓ, in period t.

The dual variables associated with restrictions (7) and (8) have a different interpretation. The dual variable associated with restriction (7), π_{jt}^1, is the per unit contribution to fixed costs. The parameter c_{jt}^1 used in the model only accounts for variable costs. If a region has an excess storage capacity in a period, then the dual variable π_{jt}^1 is zero, and the operating plants only recover their variable costs. If capacity is restrictive then the dual variable is strictly positive and it gives the per unit contribution to the payment of the fixed costs.

The dual variable associated with restriction (8), π_{kt}^2, has a similar interpretation.

If the supply and demand functions have normal slope, then expression (1) is strictly concave, and the maximization problem has a unique solution. The solution is characterized by the Kuhn-Tucker conditions

$$f_{\ell t}(y_{\ell t}) - pc_{\ell t} + \alpha_{\ell t} = 0,$$
$$y_{\ell t} \cdot \alpha_{\ell t} = 0; \ \alpha_{\ell t} \geq 0 \qquad (12)$$

$$pf_{it} - g_{it}(x_{it}) + \beta_{it} = 0,$$
$$x_{it} \cdot \beta_{it} = 0; \ \beta_{it} \geq 0 \qquad (13)$$

$$ps_{jt} - pf_{it} - t^1_{ij} + \gamma_{ijt} = 0$$
$$z_{ijt} \cdot \gamma_{ijt} = 0; \ \gamma_{ijt} \geq 0 \qquad (14)$$

$$ps_{jt} - e_{jt} - t^4_j + \delta_{jt} = 0,$$
$$m_{jt} \cdot \delta_{jt} = 0; \ \delta_{jt} \geq 0 \qquad (15)$$

$$ps_{jt+1} - ps_{jt} - c^1_{jt} - \pi^1_{jt} + \eta_{jt} = 0,$$
$$s_{jt} \cdot \eta_{jt} = 0; \ \eta_{jt} = 0 \qquad (16)$$

$$pm_{kt} - ps_{jt} - t^2_{jk} + \tau_{jkt} = 0,$$
$$v_{jkt} \cdot \tau_{jkt} = 0; \ \tau_{jkt} = 0 \qquad (17)$$

$$pc_{\ell t} - 1/r_k pm_{kt} - c^2_{kt} - \pi^2_{kt} - t^3_{k\ell} + \theta_{k\ell t} = 0,$$
$$w_{k\ell t} \cdot \theta_{k\ell t} = 0; \ \theta_{k\ell t} \geq 0 \qquad (18)$$

The meaning of these equations is as follows: equation (12) says that in the case of nonzero flour consumption its market price is equal to the price at which the amount supplied is demanded, i.e. the flour market is in equilibrium, in zone ℓ period t. If the market price is too high there is no flour consumption. Equation (13) says that in case of nonzero production, the market price of wheat is equal to the price at which the amount demanded is supplied by the producers, i.e. the wheat market is in equilibrium. If the market price of wheat is too low there is no wheat consumption. Equation (14) says that the wheat price differential between the i^{th} region and the j^{th} region must be less than or equal to the transport cost between the two regions, and it must be an equality if there are shipments.

Equation (15) says that the wheat price in a silo must be less than or equal to the cost of importing wheat plus the cost of transporting the wheat from the port to the silo. If some wheat is imported then both values must be equal.

Equation (16) says that the wheat price differential between two consecutive periods in the same region must be less than or equal to the stockholding cost. It must be an equality if there is storage.

Equation (17) says that if silos in the i^{th} region send wheat to mills in the j^{th} region, then the price differential between the two places is equal to the transport cost, if no wheat is sent the per unit transport cost exceeds the price differential.

Finally, equation (18) says that if mills in the k^{th} region supply consumers in the ℓ^{th} region, then the price consumers pay is equal to the price at which mills sell flour plus the per unit transport cost. Otherwise the per unit transport cost exceeds the price differential.

Then,the solution represents a market equilibrium, specifically the competitive equilibrium solution. The solution we obtain will be close to the actual equilibrium if the forecast of the stochastic variables is not far from the real values.

PROJECT EVALUATION

The model assumes perfect competition, so it can only be used to evaluate situations in which perfect competition operates. But this limitation is one of the uses this kind of model has: the detection of market imperfections, which otherwise is a difficult task in a multiperiod spatial system. Before using this model to do any kind of evaluation, a first run must be used to detect market imperfections.

I will consider first the private evaluation of a new mill. Then I will look at the social evaluation. Assume that a new mill with the capacity for processing K tons of wheat per period can be built in one year at a total cost of U.S. \$ Co. Assume also that the mill has a useful life of S years, and that the yearly fixed costs are F_s, s=1,..,S.

To evaluate the project the model has to be run for each of the S years. After the model has been run for the s^{th} year we can calculate this year's pretax benefit as:

$$B_s = \sum_{i=1}^{T} \pi^1_{it} K - F_s,$$
$$s = 1,\ldots,s \qquad (19)$$

This kind of evaluation requires an extraordinary amount of data, in particular it seems cumbersome to have to estimate supply and demand curves for future years. But any other type of evaluation includes, in one manner or other, the same information. Furthermore we think it is more reliable to make assumptions on the basic data than on derived data.

These values can be used to calculate the standard economic and financial project evaluation indicators.

The social evaluation is quite easy. Let us assume that the costs used in the model, including the price of foreign reserves, are equal to the social opportunity costs. If this is the case, we first run our model including the new project, for each year, and we obtain the economy's surplus ES_s^p, $s = 1,\dots,S$. Then for each year we run the model without including the new project, obtaining the surplus of each year ES_s, $s = 1,\dots,S$. The social benefit is obtained simply, for each year.

$$SB_s = ES_s^p - ES_s - F_s, \quad s = 1,\dots,S \tag{20}$$

If the economy has some sales taxes, such as the 20% value added tax Chile has, the objective function of the model (ES) no longer represents the surplus of the economy, and formula (20) has to be modified to include tax collection.

CONCLUSIONS

The main justification for proposing the use of this kind of model in project evaluation is that it can easily handle the indirect effects of a new project. For instance using the model we could calculate the impact a new mill would have on wheat producers, consumers, silos and other mills. Of course the model cannot be large enough to take into account all the indirect effects explicitly, but it should handle all those that are not negligible. We will clarify this point with two examples.

Silos can be used to store other grains. In Chile, the only other grain that is stored in significant amounts is corn, so, we built a joint model for wheat and corn. In this presentation we have excluded the corn sector to simplify the exposition.

Grains, especially wheat and corn, represent a large share of the total cargo handled by ports. As a consequence, in those years with large grain imports ports are overwhelmed, increasing waiting time for ships and shipping costs for grains as well as for other commodities. The inclusion of port activities should increase the accuracy of the results of the model.

This kind of model is also suited to policy evaluation. I would be quite easy to evaluate the impact of a new tariff, or a producers' subsidy, or a consumers' subsidy, or other policies. Besides evaluating the policy the solution of the model would give information about the effects the new policy would have in every component of the sector.

REFERENCES

Judge, G.G. and T. Takayama (eds.)(1973). Studies in Economic Planning over Space and Time. North-Holland, Amsterdam.

Samuelson, P.A. (1952). Spatial Price Equilibrium and Linear Programming. American Economic Review, 42, 283-303.

Takayama, T., and G.G. Judge (1971). Spatial and Temporal Price and Allocation Models. North-Holland, Amsterdam.