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ON THE KLEE–SAINT RAYMOND'S CHARACTERIZATION OF CONVEXITY*

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Abstract. Using techniques of convex analysis, we provide a direct proof of a recent characterization of convexity given in the setting of Banach spaces in [J. Saint Raymond, *J. Nonlinear Convex Anal.*, 14 (2013), pp. 253–262]. Our results also extend this characterization to locally convex spaces under weaker conditions.

Key words. convexity, epi-pointed functions, conjugate and biconjugate functions

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1. Introduction.

Saint Raymond observes [18] that for a given nonconvex continuous function $f : \mathbb{R} \to \mathbb{R}$, which satisfies $\lim_{|x|\to+\infty} \frac{f(x)}{|x|} = +\infty$, there exists an affine function h that minorizes f, such that $f - h$ vanishes on a nonconvex set. This fact characterizes the convexity of a function. More generally, the following holds.

THEOREM 1 (see [18, Theorem 10]). *Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a weakly lower semicontinuous (lsc) proper function such that $f - x^*$ is weakly inf-compact for all $x^* \in X^*$. If the argmin set of the function $f - x^*$ is convex for all x^* in a convex dense subset of X^*, then f is convex.*

Observe that in the original statement of [18, Theorem 10] the hypothesis of weak inf-compactness used above is replaced by the equivalent fact that the function $f - x^*$ attains its minimum for every $x^* \in X^*$. This equivalence, being a functional counterpart of James's theorem [11, Theorem 3.130], has been established in [19, Theorem 2.4]. We call this work Klee–Saint Raymond characterization of convexity because in the framework of Hilbert spaces, Theorem 1 is equivalent to the famous characterization given by Klee [12] for the convexity of weakly closed sets. See [4] for a recent review of this problem related to Chebychev sets.

To prove Theorem 1 the author uses classical deep tools of Banach space theory, like James's theorem and Brouwer's fixed-point theorem for multifunctions, among others. More recently, another proof has been given in [17], under the assumption that X is a reflexive Banach space, by using techniques of operator theory.

In this work we use techniques of convex analysis to give a direct proof for a generalization of Theorem 1 for functions defined on locally convex spaces. This generalization, given in Corollary 11, is an immediate consequence of the main result of this work that provides an explicit expression of the closed convex hull of a function; see Theorem 8. Our hypotheses are weaker than those used in Theorem 1 and rely on

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the epi-pointedness property that has been successfully utilized recently ([2], [5], [6], [7], [8], [15]) with the purpose of extending results, which were known exclusively for Banach spaces or convex functions, to locally convex spaces and nonconvex functions.

2. Notation and preliminary results. In the following, X and Y will be two (separated) locally convex spaces in duality by the bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$. The space X will be endowed with the weak topology $w(X, Y)$, while on Y we use the Mackey topology $\tau(Y, X)$. We will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.

For a given function $f : X \to \overline{\mathbb{R}}$, the (effective) *domain* of f is $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$. We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f > -\infty$, and *inf-compact* if for every $\lambda \in \mathbb{R}$ the set $[f \leq \lambda] := \{x \in X \mid f(x) \leq \lambda\}$ is compact. The *conjugate* of f is the function $f^* : Y \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\},$$

and the *biconjugate* of f is $f^{**} := (f^*)^* : X \to \overline{\mathbb{R}}$. The *subdifferential* of f at a point $x \in X$ where it is finite is the set

$$\partial f(x) := \{x^* \in Y \mid \langle x^*, y - x \rangle \leq f(y) - f(x) \ \forall y \in X\};$$

if $f(x)$ is not finite, we set $\partial f(x) := \emptyset$.

The *indicator* and the *support* functions of a set A ($\subseteq X, Y$) are, respectively,

$$I_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A, \end{cases} \qquad \sigma_A := I_A^*.$$

The *inf-convolution* of $f, g : X \to \overline{\mathbb{R}}$ is the function $f \Box g := \inf_{z \in X}\{f(z) + g(\cdot - z)\}$; it is said to be exact at x if there exists z such that$f \Box g(x) = f(z) + g(x - z)$. For a set $A \subseteq X$, we denote by $\operatorname{int}(A)$, $\overline{A}$, $\operatorname{co}(A)$, $\overline{\operatorname{co}}(A)$, and $\operatorname{aff}(C)$, the interior, the closure, the *convex hull*, the *convex closed hull*, and the affine subspace generated by A, respectively. By $\operatorname{ri}(A)$ we denote the interior of A with respect to $\overline{\operatorname{aff}(C)}$.

The *polar* of A is the set

$$A^o := \{x^* \in Y \mid \langle x^*, x \rangle \leq 1, \forall x \in A\},$$

while the normal cone of A at x is $N_A(x) := \partial I_A(x)$.

We introduce in the following definition the family of functions that will play a key role in our analysis.

Definition 2. *A function $f : X \to \overline{\mathbb{R}}$ is said to be epi-pointed if f^* is $\tau(Y, X)$-continuous at some point of its domain.*

This class of epi-pointed functions was introduced by Benoist and Hiriart-Urruty [3] in the nineties when X is finite-dimensional, but the original definition goes back to Debreu in the fifties [9].

The following lemma is a compilation of classical results in convex analysis that we will use in the proof of our main results that correspond to Proposition 7 and Theorem 8. They can be found in the pioneer reference of convex analysis [14] and also in [13, Chapter 6].

Lemma 3.

(a) *Given two lsc proper convex functions $g, h : X \to \overline{\mathbb{R}}$ such that g^* is continuous at some point of* $\operatorname{dom} h^*$, *for all $x \in X$ there exist $x_1, x_2 \in X$ such that $x_1 + x_2 = x$ and*

$$\partial g(x_1) \cap \partial h(x_2) = \partial(g \Box h)(x_1 + x_2).$$

(b) *Given two functions $g, h : X \to \overline{\mathbb{R}}$, we have for all $x_1, x_2 \in X$*

$$\partial g(x_1) \cap \partial h(x_2) \subseteq \partial(g \square h)(x_1 + x_2).$$

(c) *Given two functions $g, h : X \to \overline{\mathbb{R}}$, we have $(f \square g)^* = f^* + g^*$.*

(d) *Let $g, h : Y \to \overline{\mathbb{R}}$ be proper convex functions such that h is continuous at some point in $\operatorname{dom} g \cap \operatorname{dom} h$. Then*

$$(g + h)^*(x) = (g^* \square h^*)(x) \;\; \forall x \in X,$$

and the inf-convolution is exact.

(e) *Given a function $g : X \to \overline{\mathbb{R}}$, if $\partial g(x) \neq \emptyset$, then $g(x) = g^{**}(x)$.*

(f) *An lsc proper convex function $g : Y \to \overline{\mathbb{R}}$ is $\tau(Y, X)$-continuous at $x^* \in \operatorname{dom} g$ if and only if $g^* - x^*$ is $w(X, Y)$-inf-compact.*

The following lemma is a slight extension of [1, Theorem 2.40] to the case of nets of functions. For completeness we give a proof.

LEMMA 4. *Let X be a topological space and let $(f_\alpha)_{\alpha \in D}$ be a net of lsc proper functions defined on X such that*

$$\alpha, \beta \in D,\; \alpha \leq \beta \Rightarrow f_\alpha \leq f_\beta. \tag{1}$$

For $\varepsilon_\alpha \underset{\alpha \in D}{\to} 0^+$, let $(x_\alpha)_{\alpha \in D}$ be a relatively compact net such that $x_\alpha \in \varepsilon_\alpha\text{-}\operatorname{argmin} f_\alpha$ for each α. Then

$$\inf_{x \in X} \sup_{\alpha \in D} f_\alpha(x) = \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x),$$

and every accumulation point of (x_α) is a minimizer of the function $\sup_{\alpha \in D} f_\alpha$.

Proof. It is easy to see that every subnet of (f_α) has a subnet that preserves property (1); so, without loss of generality, we may assume that $x_\alpha \to \bar{x} \in X$. We start by showing that for every $V \in \mathcal{N}_{\bar{x}}$, the neighborhood system of $\bar{x}$, we have

$$\sup_{\alpha \in D} \inf_{v \in V} f_\alpha(v) \leq \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x). \tag{2}$$

Given $\delta > 0$ and $V \in \mathcal{N}_{\bar{x}}$, we choose $\alpha_0 \in D$ such that $x_\alpha \in V$ and $\varepsilon_\alpha < \delta$ for all $\alpha \geq \alpha_0$. Then for any $\beta \in D$ we get

$$\inf_{v \in V} f_\beta(v) \leq \sup_{\alpha \geq \alpha_0} f_\alpha(x_\alpha) \leq \sup_{\alpha \geq \alpha_0} \left\{ \inf_{x \in X} f_\alpha(x) + \varepsilon_\alpha \right\} \leq \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x) + \delta.$$

This yields (2) by taking the supremum on $\beta \in D$ and the limit as $\delta \to 0^+$.

Now, (2) leads us to

$$\begin{aligned} \inf_{x \in X} \sup_{\alpha \in D} f_\alpha(x) \leq \sup_{\alpha \in D} f_\alpha(\bar{x}) &= \sup_{\alpha \in D} \sup_{V \in \mathcal{N}_{\bar{x}}} \inf_{v \in V} f_\alpha(v) \\ &= \sup_{V \in \mathcal{N}_{\bar{x}}} \sup_{\alpha \in D} \inf_{v \in V} f_\alpha(v) \leq \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x), \end{aligned}$$

which yields

$$\inf_{x \in X} \sup_{\alpha \in D} f_\alpha(x) \leq \sup_{\alpha \in D} f_\alpha(\bar{x}) \leq \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x) \leq \inf_{x \in X} \sup_{\alpha \in D} f_\alpha(x),$$

and, so, the proof is completed. ∎

We close this section by recalling the key tool in the proof of our main result, which is an extension of the classical Fenchel formula $\partial f^* = (\partial f)^{-1}$ for lsc proper convex functions, to weakly lsc epi-pointed (nonnecessarily convex) functions.

PROPOSITION 5 (see [6, Corollary 6]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a weakly lsc epi-pointed function. Then for every $x^* \in Y$ we have that*

$$\partial f^*(x^*) = \overline{\text{co}}\left[(\partial f)^{-1}(x^*)\right] + N_{\text{dom}\, f^*}(x^*).$$

3. The characterization of convexity. We start with a comparison between the subdifferentials of an epi-pointed function and its biconjungate.

PROPOSITION 6. *Let $f : X \to \overline{\mathbb{R}}$ be a weakly lsc epi-pointed function and denote*

$$M_f := \{x^* \in Y : \text{argmin}\{f - x^*\} \textit{ is convex}\}.$$

Then for every $x \in X$ we have that

$$\text{int}(\text{dom}\, f^*) \cap M_f \cap \partial f^{**}(x) \subseteq \partial f(x).$$

Proof. We choose $x^* \in \text{int}(\text{dom}\, f^*) \cap M_f \cap \partial f^{**}(x)$ and $x \in X$. Since $(\partial f)^{-1}(x^*) = \text{argmin}\{f - x^*\}$ is convex and weakly closed (f is weakly lsc), according to Proposition 5 we have that

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*).$$

Hence, since $x^* \in \partial f^{**}(x)$ we have that $x \in \partial f^*(x^*) = (\partial f)^{-1}(x^*)$, which is equivalent to $x^* \in \partial f(x)$. ∎

We give now a first relation between an epi-pointed function and its biconjugate.

PROPOSITION 7. *Let $f : X \to \overline{\mathbb{R}}$ be a weakly lsc epi-pointed function and M_f as in proposition above. Then, for all nonempty, convex, and compact sets $C \subset \text{int}(\text{dom}\, f^*) \cap M_f$, we have that*

$$\sigma_C \Box f^{**} = \sigma_C \Box f.$$

Proof. We fix $x \in X$. By Lemma 3(a), applied with $g := \sigma_C$ and $h := f^{**}$, since $h^* = f^*$ is continuous at any point of $\text{dom}\, g^* = C$, there are $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and

$$\partial\sigma_C(x_1) \cap \partial f^{**}(x_2) = \partial(\sigma_C \Box f^{**})(x). \tag{3}$$

Since $\partial\sigma_C(x_1) \subseteq C \subseteq \text{int}(\text{dom}\, f^*) \cap M_f$, Proposition 6 gives us the relation

$$\partial\sigma_C(x_1) \cap \partial f^{**}(x_2) \subseteq \partial\sigma_C(x_1) \cap \partial f(x_2).$$

So, invoking Lemma 3(b), from (3) we infer that

$$\partial(\sigma_C \Box f^{**})(x) \subseteq \partial(\sigma_C \Box f)(x). \tag{4}$$

On the one hand, by Lemma 3(c) we have $(\sigma_C \Box f)^* = f^* + I_C$ and so, invoking Lemma 3(d), we get

$$(\sigma_C \Box f)^{**} = \sigma_C \Box f^{**}. \tag{5}$$

On the other hand, since

$$(\sigma_C \square f)^{**}(x) = (I_C + f^*)^*(x) = \sup_{x^* \in C} \{\langle x^*, x\rangle - f^*(x^*)\},$$

the continuity of f^* and the compacity of C yield the existence of some $\overline{x}^* \in C$ such that

$$(\sigma_C \square f)^{**}(x) = \langle \overline{x}^*, x\rangle - f^*(\overline{x}^*) = \langle \overline{x}^*, x\rangle - (f^* + I_C)(\overline{x}^*) = \langle \overline{x}^*, x\rangle - (\sigma_C \square f)^*(\overline{x}^*).$$

In other words, in view of (5) and (4), respectively,

$$\overline{x}^* \in \partial(\sigma_C \square f)^{**}(x) = \partial(\sigma_C \square f^{**})(x) \subseteq \partial(\sigma_C \square f)(x)$$

and, so, $\partial(\sigma_C \square f)(x) \neq \emptyset$. Due to Lemma 3(e), and using (5) again, this implies that

$$\sigma_C \square f(x) = (\sigma_C \square f)^{**}(x) = \sigma_C \square f^{**}(x).$$

This finishes the proof, since x was arbitrarily chosen. ∎

We are now able to prove the main result of this work, which has as a consequence the required characterization of convexity.

THEOREM 8. *Let $f : X \to \overline{\mathbb{R}}$ be a weakly lsc epi-pointed function such that*

$$\operatorname{argmin}\{f - x^*\} \text{ is convex } \forall x^* \in D,$$

where D is a convex dense subset of $\operatorname{dom} f^*$*. Then we have that*

$$f^{**} = \sigma_{\operatorname{dom} f^*} \square f.$$

Proof. Since $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$, without loss of generality, we assume that $D \subseteq \operatorname{int}(\operatorname{dom} f^*)$. Define $\mathfrak{C} := \{\operatorname{co}(F) : F \text{ is a finite subset of } D\}$. Clearly $(\mathfrak{C}, \supseteq)$ is a directed set.

It is easy to check, using Proposition 7, that

$$\sup_{C \in \mathfrak{C}} \sigma_C \square f = \sup_{C \in \mathfrak{C}} \sigma_C \square f^{**} \leq \sigma_{\operatorname{dom} f^*} \square f^{**} \leq \sigma_{\operatorname{dom} f^*} \square f.$$

Now we will prove that

$$\sup_{C \in \mathfrak{C}} \sigma_C \square f = \sigma_{\operatorname{dom} f^*} \square f, \tag{6}$$

and the conclusion will follow from Lemma 3(d), which shows that $\sigma_{\operatorname{dom} f^*} \square f^{**} = (I_{\operatorname{dom} f^*} + f^*)^* = f^{**}$.

We fix $x \in X$ and for any $C \in \mathfrak{C}$ define $g_C(y) := f(y) + \sigma_C(x - y)$ and $g(y) := f(y) + \sigma_{\operatorname{dom} f^*}(x - y)$. Clearly, g_C and g are weakly lsc functions and $g_C \nearrow g$ pointwise.

If there exists $C \in \mathfrak{C}$ such that $g_C \equiv +\infty$, (6) is trivially verified.

Assume that for every $C \in \mathfrak{C}$, $g_C \not\equiv +\infty$. From Lemma 3(f) applied to f^* at $x^* \in C (\subset \operatorname{int}(\operatorname{dom} f^*))$ we conclude that $f^{**} - x^*$ is weakly inf-compact, and from the fact that

$$g_C(\cdot) = f(\cdot) + \sigma_C(x - \cdot) \geq f^{**}(\cdot) - \langle x^*, \cdot\rangle + \langle x^*, x\rangle,$$

we see that g_C is weakly inf-compact.

Now, for every $C \in \mathfrak{C}$ we take $x_C \in \operatorname{argmin}\{g_C\}$ and fix some $C_0 \in \mathfrak{C}$. Then for every $K \in \mathfrak{C}$ such that $C_0 \subseteq K$ we have $x_K \in \Gamma := \{y \in X : g_{C_0}(y) \leq \sup_{C \in \mathfrak{C}}\{\sigma_C \square f(x)\}\}$.

Finally, if $\sup_{C \in \mathfrak{C}}\{\sigma_C \square f(x)\} = +\infty$, equality (6) is trivial. On the other hand, if $\sup_{C \in \mathfrak{C}}\{\sigma_C \square f(x)\} < +\infty$, then Γ is compact and we apply Lemma 4 to the family g_C, x_C indexed by the directed set $\{C \in \mathfrak{C} : C_0 \subseteq C\}$ to obtain equality (6). ∎

The following example illustrates the necessity of considering the support function of $\operatorname{dom} f^*$ in the formula for the biconjugate given in Theorem 8.

Example 9. Let $f : \mathbb{R} \to \mathbb{R}$ be the lsc nonconvex function defined by

$$f(x) = \begin{cases} |x| & \text{if} \quad x \in [-1, 1], \\ |x| + e^{-|x|} & \text{if} \quad x \in \mathbb{R}\backslash[-1, 1]. \end{cases}$$

Then it is easy to prove that $f^* = I_{[-1,1]}$, which shows that f is epi-pointed, and

$$\operatorname{argmin}\{f - \alpha\} = \begin{cases} \{0\} & \text{if} \quad \alpha \in (-1, 1), \\ [0, 1] & \text{if} \quad \alpha = 1, \\ [-1, 0] & \text{if} \quad \alpha = -1. \end{cases}$$

Hence, the hypothesis of Theorem 8 holds, but not the equality $f^{**} = f$. However, we easily check that

$$f^{**} = |\cdot| = \sigma_{\operatorname{dom} f^*} \Box f.$$

The following remark gives a geometrical interpretation of the conclusion of Theorem 8 in terms of the epigraph of involved functions.

Remark 10. It is well known that when the inf-convolution is exact, then its epigraph is the sum of the epigraphs of the two functions. Since it can be shown that convolution in the equality of Theorem 8 is exact, we see that Theorem 8 corresponds to the set equality

$$\overline{\text{co}}(\operatorname{epi} f) = \operatorname{epi} f + \operatorname{epi} \sigma_{\operatorname{dom} f^*}. \tag{7}$$

On the other hand, if we consider the asymptotic cone of epi f given by (see [10])

$$(\operatorname{epi} f)_\infty := \bigcap_{\varepsilon > 0} \overline{]0, \varepsilon] \operatorname{epi} f},$$

which is the epigraph of the asymptotic function f_∞, since $\overline{\text{co}}(f_\infty) = \sigma_{\operatorname{dom} f^*}$ when f is weakly lsc and epi-pointed (see [8, Theorem 7]), we can rewrite Theorem 8 as

$$f^{**} = f \Box \overline{\text{co}}(f_\infty)$$

and (7) as

$$\overline{\text{co}}(\operatorname{epi} f) = \overline{\text{co}}((\operatorname{epi} f)_\infty) + \operatorname{epi} f.$$

It is worth noting that this characterization does not involve dual objects.

The next corollary corresponds to the announced result of this work: The extension of Theorem 1 to the setting of locally convex spaces.

COROLLARY 11. *Let $f : X \to \overline{\mathbb{R}}$ be a weakly lsc epi-pointed function such that* $\operatorname{dom} f^* = Y$. *If there exists a convex dense set $D \subset Y$ such that* $\operatorname{argmin}\{f - x^*\}$ *is convex for all $x^* \in D$, then f is convex.*

The next remark compares the hypotheses of Theorem 1 and Corollary 11.

Remark 12. It is worth observing that in Theorem 1 the density assumption on D is with respect to the norm topology in X^*. This is clearly stronger than the condition used in Corollary 11 asking that D is dense only with respect to the Mackey topology. On the other hand, according to Lemma 3(f) we see that the hypothesis of weakly infcompactness of $f - x^*$ is equivalent to the continuity of f^* over all X^* with respect to the Mackey topology.

The following example shows the necessity of the convexity assumption of D in Corollary 11.

Example 13. Let h be a proper lsc convex function defined on a reflexive Banach space Z such that $\operatorname{dom} h^* = Z^*$; hence, h is epi-pointed (with respect to the norm-topology in Z^*). Choose any nonconvex positive and weakly lsc function g such that the function $f := h + g$ is not convex. Then f defines a weakly lsc epi-pointed function, due to the relation $h^* \geq f^*$, which implies that $\operatorname{dom} f^* = Z^*$ and f^* is norm-continuous on Z^*. Since f^* is Fréchet-differentiable in a (G_δ)-dense subset of Z^* (see, e.g., [16]), $D \subset Z^*$, by [5, Proposition 6] we deduce that, for all $x^* \in D$,

$$\operatorname{argmin}\{f - x^*\} = (\partial f)^{-1}(x^*) = \{\nabla f^*(x^*)\}.$$

The next corollary shows that the epi-pointedness assumption in Corollary 11 can be replaced by the assumption of convexity of $\varepsilon\text{-}\operatorname{argmin}\{f - x^*\}$ when ε is sufficiently small and the density of D in the norm topology.

COROLLARY 14. *Suppose that X is a normed space, and let $f : X \to \overline{\mathbb{R}}$ be a weakly lsc function such that* $\operatorname{dom} f^* = X^*$. *Assume that there exist a convex (norm-)dense set $D \subset X^*$ such that for all $x^* \in D$ there is some $\delta > 0$ such that*

$$\varepsilon\text{-}\operatorname{argmin}\{f - x^*\} \textit{ is convex } \forall \varepsilon \in (0, \delta).$$

Then f is convex.

Proof. We consider the duality pair $((X^{**}, w^*), (X^*, \|\cdot\|_*))$ together with the function $\overline{f}^{w^*} : X^{**} \to \overline{\mathbb{R}}$ given by

$$\overline{f}^{w^*}(x^{**}) := \liminf_{\substack{x_\alpha \xrightarrow{w^*} x^{**} \\ x_\alpha \in X}} f(x_\alpha).$$

It is clear that $\overline{f}^{w^*}$ is weakly lsc and epi-pointed. Moreover, since for every $x^* \in D$ we have

$$\operatorname{argmin}\{\overline{f}^{w^*} - x^*\} = \bigcap_{\varepsilon > 0} \overline{\varepsilon\text{-}\operatorname{argmin}\{f - x^*\}}^{w^*},$$

we deduce that $\operatorname{argmin}\{\overline{f}^{w^*} - x^*\}$ is convex. Hence, by applying Corollary 11 we conclude that $\overline{f}^{w^*}$ is convex, and so is the function f. ∎

In the final part, keeping in mind a possible application to the finite-dimensional case, we consider the relative interior within the definition of epi-pointed functions. More generally, we have the following.

THEOREM 15. *Let $f : X \to \overline{\mathbb{R}}$ be a function with a proper conjugate, and denote $F := \overline{\operatorname{aff}}(\operatorname{dom} f^*)$. We suppose that the following conditions hold:*

(a) *The restriction of f^* to F, ${f^*}_{|F}$, is continuous on* $\operatorname{ri}(\operatorname{dom} f^*)$.

(b) *There exists $x_0^* \in \operatorname{ri}(\operatorname{dom} f^*)$ such that $f - x_0^*$ is weakly lsc and weakly inf-compact.*

(c) *There exists a convex set $D \subseteq \operatorname{ri}(\operatorname{dom} f^*)$ with $\operatorname{ri}(\operatorname{dom} f^*) \subset \overline{D}$, such that* $\operatorname{argmin}\{f - x^*\}$ *is convex for all $x^* \in D$.*

Then we have

$$\sigma_{\operatorname{dom} f^*} \square f = f^{**}. \tag{8}$$

Moreover, if $\overline{D} = F$, *then*

$$\sigma_{\operatorname{dom} f^*} \Box f = f^{**} = f_F, \tag{9}$$

where $f_F(x) := \inf\{f(w) : w \in x + (F - x_0^*)^\perp\}$.

Proof. We may suppose without loss of generality that $x_0^* = 0$ so that the function f_F defined above is written as

$$f_F(x) := \inf\{f(w) : w \in x + F^\perp\}.$$

We also denote by $Z := X/F^\perp$ the quotient space of X by the orthogonal space of F and introduce the function $h : Z \to \overline{\mathbb{R}}$ defined as

$$h([x]) := f_F(x).$$

Let us consider the dual pair $(Z, \sigma(Z, F), F, \tau(Y, X))$ endowed with the bilinear form $\langle x^*, [x]\rangle = \langle x^*, x\rangle$. Then, from the relation

$$\{z \in Z : h(z) \leq \lambda\} = \Pi(\{x \in X : f(x) \leq \alpha\}) \;\; \forall \lambda \in \mathbb{R},$$

where $\Pi : X \to X/F^\perp$ is the canonical projection, i.e., $\Pi(x) = [x]$, it follows that h is weakly lsc. Also, since f is weakly inf-compact, the relation above together with the fact that $h^* = {f^*}_{|F}$ also implies that

$$\operatorname{int}(\operatorname{dom} h^*) = \operatorname{ri}(\operatorname{dom} f^*),$$

and h is epi-pointed.

Next, because f is weakly inf-compact, we get

$$\operatorname{argmin}\{h - x^*\} = \Pi(\operatorname{argmin}\{f - x^*\}),$$

which shows that $\operatorname{argmin}\{h - x^*\}$ is convex.

Now, we are able to apply Theorem 8 to get, for every $x \in X$,

$$\sigma_{\operatorname{dom} h^*} \Box h = \sup_{x^* \in F} \{\langle \cdot, x^*\rangle - h^*(x^*)\}.$$

Moreover, using the fact that $h^* = {f^*}_{|F}$ and $\sigma_{\operatorname{dom} f^*}(x - y - z) = \sigma_{\operatorname{dom} f^*}(x - y)$, for every $z \in F^\perp$, we see that for all $x \in X$

$$\sigma_{\operatorname{dom} h^*} \Box h([x]) = \sigma_{\operatorname{dom} f^*} \Box f(x)$$

and

$$\sup_{x^* \in F} \{\langle x, x^*\rangle - h^*(x^*)\} = f^{**}(x).$$

Finally, if $F \subseteq \overline{\operatorname{dom} f^*}$, then $x - y \notin F^\perp$, and so

$$\sigma_{\operatorname{dom} f^*}(x - y) = \sigma_F(x - y) = +\infty.$$

Therefore, $\sigma_{\operatorname{dom} f^*} \Box f = f_F$. ∎

Remark 16. With the hypothesis of Theorem 15, it is not possible to get the equality $f = f^{**}$ instead of $f_{F-x_0^*} = f^{**}$. Indeed, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{1}{2}x^2 + \ln(|y| + 1).$$

Then the conjugate is given by

$$f^*(\alpha, \beta) = \begin{cases} \frac{\alpha^2}{2} & \text{if} \quad \beta = 0, \\ +\infty & \text{if} \quad \beta \neq 0 \end{cases}$$

so that $\operatorname{dom} f^* = \mathbb{R} \times \{0\}$ and for every $(\alpha, 0)$,

$$\operatorname{argmin}\{f - (\alpha, 0)\} = \{(\alpha, 0)\};$$

that is, $\operatorname{argmin}\{f - (\alpha, 0)\}$ is convex, while f is not convex.

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