SLOWLY DECAYING RADIAL SOLUTIONS OF AN ELLIPTIC EQUATION WITH SUBCRITICAL AND SUPERCRITICAL EXPONENTS

By

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Abstract. We study radial solutions of the problem

$$\Delta u + u^p + u^q = 0, \ u > 0 \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$ and

$$\frac{N}{N-2} < p < \frac{N+2}{N-2} < q.$$

We show that if p is close to $N/(N-2)$, q is close to $(N+2)/(N-2)$, and a certain relation holds between them, then the problem has slowly decaying solutions.

1 Introduction

Let $N \geq 3$. We are interested in finding radially symmetric solutions $u(r)$, $r = |x|$, of

$$\Delta u + u^p + u^q = 0, \ u > 0, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$p^s < p < p^* < q, \tag{1.2}$$

where here and throughout the paper $p^* = (N+2)/(N-2)$ and $p^s = N/(N-2)$.

Solutions u of (1.1) such that $\lim_{|x|\to+\infty} u(x) = 0$ are called **ground states**. A ground state u such that $\lim_{|x|\to+\infty} |x|^{N-2}u(x)$ exists and is positive is said to have **rapid decay**, and a ground state u such that $\lim_{|x|\to+\infty} |x|^{2/(p-1)}u(x) = \ell > 0$, is said to have **slow decay**. For a solution with slow decay, the constant ℓ depends

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on p and N only and is given by

$$K_p := \left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{1/(p-1)}.$$

When $p = q$ in equation (1.1), the existence of ground states is well understood. In the **subcritical** case $p < p^*$, there are no solutions [8], whereas ground states exist in the **supercritical case** $p \geq p^*$. In the **critical** case $p = p^*$, all solutions are radial around some point [7]. Radial ground states in the critical or supercritical case are parametrized by $u(0)$ and are unique up to the natural scaling of the equation. In the critical case, the ground state is explicit and has rapid decay; whereas in the supercritical case, the radial ground state has slow decay.

Lin and Ni [9] considered equation (1.1) to provide a counterexample to the Nodal Domain Conjecture and found slowly decaying ground states when $q = 2p - 1$ given explicitly by

$$u(x) = a(b + |x|^2)^{-\frac{2}{q-1}} = a(b + |x|^2)^{-1/p-1} \tag{1.3}$$

with $a = K_p$ and $b = \frac{1}{p}\big((N-2) - 2/(p-1)\big)^2$.

Ni then asked whether there exist radial ground states under condition (1.2). Bamón, Flores, and del Pino [1] addressed this question and discovered a complex picture of solutions. First they found an increasing number of rapidly decaying ground states if one of the exponents is fixed and the other one is sufficiently close to p^*. More precisely, they proved that for $p^s < p < p^*$ and an integer $k \geq 1$, (1.1) has at least k radial ground states with rapid decay if $q > p^*$ is close enough to p^*. They also showed that for fixed $q > p^*$ and $k \geq 1$ integral, (1.1) has at least k radial ground states with rapid decay if $p < p^*$ is sufficiently close to p^* Furthermore, for $q > p^*$ fixed, there exists $p_0 > p^s$ such that there are no radial ground states if $1 < p < p_0$. They obtained their results using dynamical systems arguments. Recently, Campos [2] gave a different proof of the same main result.

Our main interest is the existence of slowly decaying radial solutions. Such solutions, if they exist, are unique. Indeed, an Emden-Fowler change of variables, following [1, p. 555] (see also [6]), transforms (1.1) into a first order 3-dimensional system of ODEs. Slowly decaying solutions correspond to trajectories contained in the 1-dimensional stable manifold of a stationary point, which implies uniqueness. But regular slowly decaying solutions also lie in the 2-dimensional unstable manifold of another stationary point, which suggests that their existence is nongeneric in the parameters p, q.

The only indication of existence of regular slowly decaying radial solutions is a result in [1], which states that for $p^s < p < p^*$, there exists a sequence of exponents

$\{q_j\}$, such that $q_j > p^*$, $q_j \to p^*$, for which there exists a radial solution with slow decay; but it is unknown whether these solutions are regular or singular.

We conjecture that slowly decaying *singular* solutions either do not exist, or exist at most for a finite number of pairs (p, q). The reason is that such solutions must satisfy the two constraints

$$\lim_{|x|\to 0} |x|^{2/(q-1)} u(x) = K_q \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{2/(p-1)} u(x) = K_p. \tag{1.4}$$

If one could discard the existence of slowly decaying singular solutions, the result of [1] would imply the existence of slowly decaying radial regular solutions associated to the sequence of exponents $\{q_j\}$.

In this work, we prove the existence of slowly decaying radial regular solutions if p is close to p^s, q is close to p^*, and p and q are related by some equation. More precisely, we prove the following theorem.

Theorem 1.1. *For each integer $k \geq 2$, there exist $\delta_0(k) > 0$ and a function $\varepsilon_k(\delta) > 0$,*

$$\varepsilon_k(\delta) = \frac{k}{2}\delta + o(\delta) \quad \text{as } \delta \to 0,$$

such that for $0 < \delta \leq \delta_0(k)$ and exponents given by

$$q = p^* + \delta \quad p = p^s + \varepsilon_k(\delta), \tag{1.5}$$

there exists a radial slowly decaying solution u of (1.1). *Moreover, there exist constants $\alpha_1, \dots \alpha_k$, depending on N and k, such that*

$$u(x) = \gamma_N \sum_{j=1}^{k} \left(\frac{1}{1 + (\alpha_j \delta^{-(j+\frac{N-4}{2})})^{4/N-2} |x|^2} \right)^{1/(p-1)} \delta^{-(j+\frac{N-4}{2})} \alpha_j \left(1 + o(1)\right),$$

where $\gamma_N = (N(N-2))^{(N-2)/4}$ and $o(1) \to 0$ uniformly on $\mathbb{R}^N$ as $\delta \to 0$.

The constants $\alpha_1, \dots, \alpha_k$ have explicit formulas in terms of numbers Λ_j^* given in (2.8) below, from which it follows that $\alpha_1 = \lim_{\delta\to 0} \frac{\gamma_N}{K_p} \delta^{1/(p-1)}$. This is consistent with the second constraint in (1.4)

Solutions of (1.1) corresponding to $k = 1$ are the explicit ones found by Lin and Ni, given in (1.3). In this case, $q = 2p - 1$, which corresponds to the relation $\delta = 2\varepsilon$. It is likely that when p is close to p^s and q is close to p^*, the solutions we construct in Theorem 1.1 are the same as those detected in [1].

The existence of slowly decaying solutions is interesting because of the following result of Flores [6]. If for some p, q in the range (1.2) there exists a radial

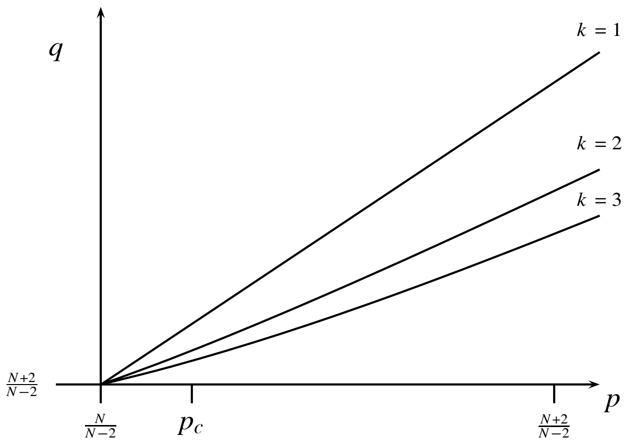

Figure 1. Bifurcation diagram for (1.1) based on numerical computations with $N = 5$. Points on the curves are pairs (p, q) for which we have found regular slowly decaying solutions.

ground state with slow decay, and $p > p_c := \frac{N+2\sqrt{N-1}}{N+2\sqrt{N-1}-4}$, then there exist infinitely many radial ground states with rapid decay.

Figure 1 shows a bifurcation diagram for (1.1) based on numerical computations. It is likely that for p close to p^s and q close to p^*, these solutions are those constructed in Theorem 1.1 for curves $q = q_k(p)$, $k = 1, 2, 3, \ldots$. The curve shown for $k = 1$ is the line $q = 2p - 1$, and the curves for $k = 2$ and $k = 3$ start at $p = p^s$ and have derivative consistent with Theorem 1.1. They bend slightly upwards. Our numerical computations show that these curves can be continued even for $p > p^*$. Hence, we see, at least numerically, that solutions with slow decay exist for $p > p_c$ and $q = q_k(p)$, and therefore the result of Flores [6] applies.

A dual phenomenon to the existence of bounded solutions with slow decay is the existence of singular solutions with rapid decay. Bamón, Flores, and del Pino [1] showed that for $q > p^*$, there exists a sequence of exponents $\{p_j\}$ such that $p_j < p^*$, $p_j \to p^*$, for which there is either a rapidly decaying singular solution or a slowly decaying singular solution.

Numerically, we have found a family of curves relating $p \in (p^s, p^*)$ and $q > p^*$ for which singular rapidly decaying solutions exist; see Figure 2. These curves are asymptotic to the line $p = p^*$ as $q \to \infty$.

Noting that singular solutions satisfy $\lim_{|x|\to 0} |x|^{2/(q-1)} u(x) = K_q$, and using formal asymptotic expansions, we arrive at the following conjecture.

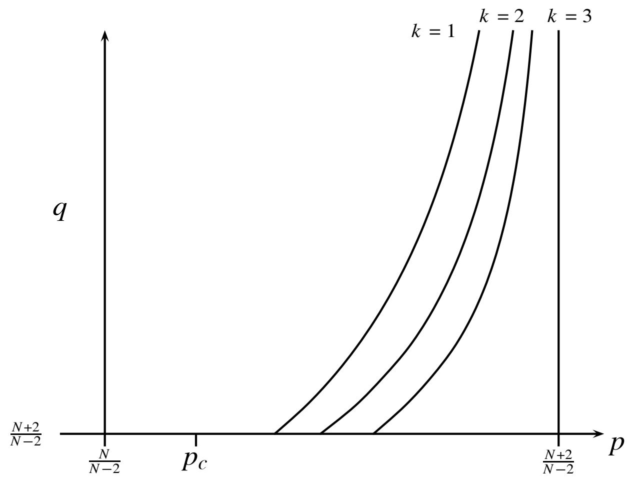

Figure 2. Bifurcation diagram for (1.1) showing singular rapidly decaying solutions for $N = 5$.

Conjecture 1.2. Let $k \geq 1$ be an integer and $p = p^* - \varepsilon$. Then there exist $\varepsilon_0 > 0$ and a function $q_k(\varepsilon) > 0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists a radial singular rapidly decaying solution u of (1.1). Moreover, there exist positive constants $\beta_1, \ldots \beta_k$, depending on N and k, such that

$$u(x) = K_q|x|^{-\frac{2}{q-1}} \left[\gamma_N \sum_{j=1}^{k} \left(\frac{1}{|x|^2 + (\beta_j \varepsilon^{(j-1)})^{\frac{4}{N-2}}} \right)^{\frac{N-2}{2} - \frac{1}{q-1}} \varepsilon^{(j-1)} \beta_j \, (1 + o(1)) \right], \tag{1.6}$$

and $q_k(\varepsilon)$ satisfies

$$\left(\frac{1}{q_k(\varepsilon) - 1} \right)^{N/2} = c_N k \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

where

$$\gamma_N = \big(N(N-2)\big)^{(N-2)/4}, \quad c_N = \frac{1}{2} \left(\frac{N-2}{2} \right)^{(N+2)/2} \frac{\Gamma(N/2)}{\Gamma(N)}$$

and $o(1) \to 0$ uniformly on $\mathbb{R}^N$ as $\delta \to 0$. Here, $\beta_1 = \gamma_N^{-1}$.

2 Scheme of the proof of Theorem 1.1

The **Emden-Fowler change of variables** $r = e^t$, $v(t) = r^\alpha u(r)$, where $\alpha = (N-2)/2$, shows that the equation $\Delta u + u^p + u^q = 0$ in $\mathbb{R}^N$ is equivalent to

$$v'' - \alpha^2 v + e^{\sigma_p t} v^p + e^{\sigma_q t} v^q = 0 \quad \text{in } \mathbb{R}, \tag{2.1}$$

where $\sigma_p = \alpha + 2 - \alpha p$ and $\sigma_q = \alpha + 2 - \alpha q$. Equation (2.1) is the Euler-Lagrange equation of the functional

$$I(v) = \int_{-\infty}^{\infty} \left(\frac{1}{2}(v')^2 + \frac{\alpha^2}{2} v^2 - e^{\sigma_p t} \frac{|v|^{p+1}}{p+1} - e^{\sigma_q t} \frac{|v|^{q+1}}{q+1} \right) dt. \tag{2.2}$$

When p and q are given by (1.5), $\sigma_p = 1 - \alpha\varepsilon$ and $\sigma_q = -\alpha\delta$. In the sequel, for $\delta > 0$, we always work with ε in the range $\delta/C \le \varepsilon \le C\delta$, for some fixed $C > 1$.

For small $\varepsilon, \delta > 0$ a first approximation to a solution of (2.1) is given by

$$U_0(t) = \gamma_N 2^{-\alpha} \cosh(t)^{-\alpha}, \quad t \in \mathbb{R}, \tag{2.3}$$

where

$$\gamma_N = (N(N-2))^{(N-2)/4}; \tag{2.4}$$

U_0 satisfies

$$U_0'' - \alpha^2 U_0 + U_0^{\frac{N+2}{N-2}} = 0.$$

In the original variables, this function is the **standard bubble**

$$u_0(x) = \gamma_N \frac{1}{(1+|x|^2)^{(N-2)/2}}$$

and satisfies

$$\Delta u_0 + u_0^{\frac{N+2}{N-2}} = 0, \; u_0 > 0 \text{ in } \mathbb{R}^N.$$

Thus U_0 corresponds to a function with rapid decay. The translate $U_0(t-\xi)$ becomes a good approximation of (2.1) as $\xi \to -\infty$ and $\varepsilon, \delta \to 0$. To achieve an approximation with slow decay, we set $\beta = 1/(p-1)$ and define

$$U(t) = \gamma_N \frac{e^{\alpha t}}{(1+e^{2t})^{\beta}}, \text{ in } \mathbb{R}.$$

If $q = 2p - 1$, U is solution of (2.1) with one bump. As in [3] and [2], one can find a multibump solution of (2.1) starting with

$$V(t) = \sum_{j=1}^{k} U(t - \xi_j), \tag{2.5}$$

where $\xi_j \in \mathbb{R}$ are parameters to be adjusted. After a change of variables, V is, at main order, the solution in the statement of Theorem 1.1.

The location of the points ξ_j, $j = 1, \ldots k$, can be determined by an expansion of $I(V)$. Indeed, assuming they are sufficiently separated, we have

$$I(V) = -c_1 \sum_{j=1}^{k} e^{\xi_j} - c_2 \sum_{j=1}^{k-1} e^{\alpha(\xi_{j+1} - \xi_j)} + c_3 \delta \sum_{j=1}^{k} \xi_j + k c_0 + A\delta + o(\delta),$$

where c_1, c_2, c_3, A, c_0 are constants; see Proposition 3.1, where the values of these constants are given. Note that $c_1, c_2, c_3 > 0$. To yield a solution, $\xi_1, \dots, \xi_k$ must be close to a critical point of the above functional. To see this more clearly, write

$$\begin{aligned} \xi_1 &= \log\delta - \log\Lambda_1, \\ \xi_{j+1} &= \xi_j + \frac{1}{\alpha}\log\delta - \log\Lambda_{j+1} \quad \text{for } j = 1, \dots, k-1, \end{aligned} \tag{2.6}$$

where

$$\frac{1}{M} \le \Lambda_j \le M \text{ for all } j = 1, \dots, k, \tag{2.7}$$

and $M > 1$ is a constant to be fixed later. Note that

$$\xi_j = \left(1 + \frac{j-1}{\alpha}\right)\log\delta - \sum_{i=1}^{j}\log\Lambda_i \quad \text{for } j = 1, \dots, k.$$

With this choice of the points ξ_j, $I(V)$ takes the form

$$\begin{aligned} &- c_1\delta\Lambda_1^{-1} - c_2\delta\sum_{j=2}^{k}\Lambda_j^{-\alpha} \\ &\quad - c_3\delta\sum_{j=1}^{k}(k-j+1)\log\Lambda_j + c_3k\left(1 + \frac{k-1}{\alpha}\right)\delta\log\delta + kc_0 + A\delta + o(\delta). \end{aligned}$$

Note that

$$\varphi(\Lambda_1, \dots, \Lambda_k) = \frac{c_1}{\Lambda_1} + c_3k\log\Lambda_1 + \sum_{j=2}^{k}\left(c_2\Lambda_j^{-\alpha} + (k-j+1)c_3\log\Lambda_j\right)$$

has a unique critical point $\Lambda^* = (\Lambda_1^*, \dots, \Lambda_k^*)$, given by

$$\Lambda_1^* = \frac{c_1}{kc_3}, \quad \Lambda_j^* = \left(\frac{c_2\alpha}{c_3(k-j+1)}\right)^{1/\alpha}, \ j = 2, \dots, k, \tag{2.8}$$

and that this critical point is a nondegenerate minimum. In the sequel, we fix the number M in (2.7) so that $\Lambda_i^* \in (1/2M, 2M)$, $i = 1, \dots k$.

To find a solution v of (2.1) close to V, we perform a Lyapunov-Schmidt reduction, i.e., we look for a solution v of the form $v = V + \phi$, where ϕ is a lower order correction. We find the following equation for ϕ:

$$L\phi + E + N(\phi) = 0 \text{ in } \mathbb{R}, \tag{2.9}$$

where

$$
\begin{aligned}
(2.10)\qquad L\phi &= \phi'' - \alpha^2\phi + (pe^{\sigma_p t}V^{p-1} + qe^{\sigma_q t}V^{q-1})\phi,\\
N(\phi) &= e^{\sigma_p t}\big((V+\phi)^p - V^p - pV^{p-1}\phi\big) + e^{\sigma_q t}\big((V+\phi)^q - V^q - qV^{q-1}\phi\big),\\
(2.11)\qquad E &= V'' - \alpha^2 V + e^{\sigma_p t}V^p + e^{\sigma_q t}V^q.
\end{aligned}
$$

The perturbation $\phi : \mathbb{R} \to \mathbb{R}$ is small in an appropriate norm, defined by $\|\phi\|_* = \sup_{t\in\mathbb{R}} |\phi(t)|/w(t)$, where

$$
w(t) = \begin{cases} e^{-(\alpha+\tau)(t-\xi_1)} & \text{if } t \ge \xi_1,\\ \sum_{i=1}^k e^{-\nu|t-\xi_i|} & \text{if } t \le \xi_1 \end{cases} \tag{2.12}
$$

for small $\tau > 0$ and $0 < \nu < \min(2, \alpha)$. To motivate this choice of norm, we remark that the exponential decay of ϕ between the points ξ_j is expected because away from these points, the dominant terms in (2.1) are the linear ones $\phi'' - \alpha^2\phi$. Bounded solutions then have exponential decay away from the ξ_j of the form $e^{-\nu|t-\xi_i|}$ with $0 < \nu < \alpha$. In general, for $t \ge \xi_1$, one can expect the same behavior. However, the solution we are looking for has slow decay as $t \to +\infty$ in the sense that it behaves like $e^{(\alpha-2\beta)t}$ as $t \to +\infty$, where $\beta = 1/(p-1)$.

We want to use the contraction mapping principle to solve our nonlinear problem. For this, we need for ϕ to decay more rapidly than $e^{-\alpha t}$ as $t \to +\infty$. To see this, observe that

$$
e^{\sigma_p t}\big((V+\phi)^p - V^p - pV^{p-1}\phi\big) \sim e^{\sigma_p t}V^{p-2}\phi^2 \sim e^{\sigma_p t + (\alpha-2\beta)(p-2)t}\phi^2,
$$

where $\beta = 1/(p-1)$. Also $\sigma_p + (\alpha - 2\beta)(p-2) = 2\beta - \alpha = \alpha + O(\varepsilon)$. Hence, if there exists a constant A such that ϕ satisfies $|\phi(t)| \le Ae^{-m|t-\xi_1|}$ for $t \ge \xi_1$, the first term in $N(\phi)$ is of the form

$$
CA^2 e^{(\alpha+O(\epsilon))\xi_1} e^{(\alpha+O(\epsilon)-2m)(t-\xi_1)} \quad t \ge \xi_1.
$$

The contraction principle then applies if m satisfies $\alpha + O(\epsilon) - 2m \le -m$ for small ϵ, which leads to the choice $m = \alpha + \tau$ for some $\tau > 0$.

Having introduced a suitable norm for application of the contraction mapping principle, let us look at the error E defined by (2.11). Note that E contains a term of the form $Se^{(\alpha-2\beta)t}$, where S is a function of $\varepsilon, \delta, \Lambda_1, \dots, \Lambda_k$, which we call the slowly decaying part, and other terms, which decay more rapidly. Since $\beta = \alpha + O(\varepsilon)$, $\|E\|_* = +\infty$ unless $S = 0$. In Proposition 4.1, we prove that there exists a function $\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k) > 0$ such that $S = 0$ if $\varepsilon = \varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k)$, and then also $\|E\|_* \le C\delta^\theta$ for some $\theta > 1/2$ and all $\delta > 0$ small. The function $\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k)$ is at main order of the form $C\delta/\Lambda_1$ for some constant C.

Using the contraction mapping principle and a suitable right inverse of L, constructed in Section 5, which preserves the norm $\| \ \|_*$, we prove in Section 6 that for small enough $\delta > 0$ and $\varepsilon = \varepsilon_k(\Lambda, \delta)$, there exists a solution ϕ of the nonlinear projected problem

$$L\phi + E + N(\phi) = \sum_{i=1}^{k} c_i \tilde{Z}_i$$

which satisfies $\|\phi\|_* \leq A\delta^\theta$ for a suitable constant $A > 0$. Here, the $\tilde{Z}_i$ are defined in (5.2). Finally, to find a solution of (2.9), it remains to verify that there exists $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ such that the constants c_i all vanish. We do this in Section 6.

3 Expansion of the energy

Proposition 3.1. *Let $M > 1$, and $k > 2$ be an integer. Let $\xi_1, \dots, \xi_k$ be given by* (2.6)*, $\Lambda = (\Lambda_1, \dots, \Lambda_k) \in [1/M, M]^k$, and V be given by* (2.5)*. If $0 < \varepsilon = O(\delta)$ as $\delta \to 0$, the functional I of* (2.2) *satisfies*

$$I(V) = -\delta\varphi(\Lambda) + kc_0 + A\delta + B\delta\log\delta + \delta\Theta_\delta(\Lambda),$$

where

$$\varphi(\Lambda_1, \dots, \Lambda_k) = \frac{c_1}{\Lambda_1} + c_3 k \log\Lambda_1 + \sum_{j=2}^{k} \left(c_2 \Lambda_j^{-\alpha} + (k - j + 1)c_3 \log\Lambda_j \right),$$

and $\Theta_\delta \to 0$ in C^1 norm on $[1/M, M]^k$ as $\delta \to 0$. The constants are given by

$$c_0 = \frac{1}{2}\int_{-\infty}^{\infty} \left((U_0')^2 + \alpha^2 U_0^2\right) - \frac{1}{2^*}\int_{-\infty}^{\infty} U_0^{2^*},$$

$$c_1 = \frac{N-2}{2N-2}\int_{-\infty}^{\infty} e^t U_0^{\frac{2N-2}{N-2}}\,dt \tag{3.1}$$

$$c_2 = \frac{\gamma_N}{2}\int_{-\infty}^{\infty} U_0(t)^{2^*-1} e^{\alpha t}\,dt \tag{3.2}$$

$$c_3 = \frac{\alpha}{2^*}\int_{-\infty}^{\infty} U_0^{2^*}\,dt, \tag{3.3}$$

$$A = \frac{k}{(2^*)^2}\int_{-\infty}^{\infty} U_0^{2^*} - \frac{k}{2^*}\int_{-\infty}^{\infty} U_0^{2^*}\log U_0,$$

$$B = \frac{\alpha}{2^*}k\left(1 + \frac{k-1}{2\alpha}\right)\int_{-\infty}^{\infty} U_0^{2^*},$$

where $2^* = 2N/(N-2)$.

These constants can be explicitly computed using the identity

$$\int_{-\infty}^{\infty} \cosh(s)^{-q}\, e^{-\mu s}\,ds = 2^{q-1}\frac{\Gamma(\frac{q-\mu}{2})\Gamma(\frac{q+\mu}{2})}{\Gamma(q)}$$

for all $\mu \in \mathbb{R}$ and $q > \max\{\mu, -\mu\}$. Note that $c_1, c_2, c_3 > 0$.

Proof. We write $I = I_1 + I_2 + I_3 + I_4 + I_5$, where

$$I_1(v) = \int_{-\infty}^{\infty} \left(\frac{1}{2}(v')^2 + \frac{\alpha^2}{2}v^2 - \frac{|v|^{2^*}}{2^*}\right),$$

$$I_2(v) = \frac{1}{2^*}\int_{-\infty}^{\infty}(|v|^{2^*} - |v|^{q+1}) + \left(\frac{1}{2^*} - \frac{1}{q+1}\right)\int_{-\infty}^{\infty} e^{\sigma_q t}|v|^{q+1},$$

$$I_3(v) = \int_{-\infty}^{\infty} \left(1 - e^{\sigma_q t}\right)\frac{|v|^{q+1}}{2^*},$$

$$I_4(v) = -\int_{-\infty}^{\infty} e^{\sigma_p t}\frac{|v|^{p+1}}{p+1},$$

following the computation in [3, Lemma 1.3]. Let us start with the computation of $I_2(V)$. Since $q = p^* + \delta$, recalling that U depends on ε and $\varepsilon = O(\delta)$, we have

$$\begin{aligned}\frac{1}{2^*}\int_{-\infty}^{\infty}(V^{2^*} - V^{q+1}) &= -\frac{\delta}{2^*}\int_{-\infty}^{\infty} V^{2^*}\log V + o(\delta) = -\frac{k\delta}{2^*}\int_{-\infty}^{\infty} U^{2^*}\log U + o(\delta)\\ &= -\frac{k\delta}{2^*}\int_{-\infty}^{\infty} U_0^{2^*}\log U_0 + o(\delta),\end{aligned}$$

The second term in $I_2(V)$ is

$$\frac{\delta}{(2^*)^2}\int_{-\infty}^{\infty} e^{\sigma_q t}V^{q+1} + o(\delta) = \frac{\delta}{(2^*)^2}\int_{-\infty}^{\infty} e^{\sigma_q t}V^{2^*} + o(\delta) = \frac{\delta k}{(2^*)^2}\int_{-\infty}^{\infty} U_0^{2^*} + o(\delta).$$

Therefore,

$$I_2(V) = A\delta + o(\delta). \tag{3.4}$$

Regarding $I_3(V)$, we have

$$\begin{aligned}I_3(V) &= \frac{\alpha\delta}{2^*}\int_{-\infty}^{\infty} tV(t)^{q+1}dt + o(\delta) = \frac{\alpha\delta}{2^*}\sum_{i=1}^{k}\int_{-\infty}^{\infty} tU(t-\xi_i)^{q+1}\,dt + o(\delta)\\ &= \frac{\alpha\delta}{2^*}\sum_{i=1}^{k}\xi_i\int_{\infty}^{\infty} U_0^{2^*} + o(\delta).\end{aligned}$$

Since ξ_i are given by (2.6),

$$I_3(V) = \frac{\alpha\delta}{2^*}\int_{\infty}^{\infty} U_0^{2^*}\left[k\left(1 + \frac{k-1}{2\alpha}\right)\log\delta - \sum_{i=1}^{k}(k-i+1)\log\Lambda_i\right] + o(\delta). \tag{3.5}$$

For $I_4(V)$, we see that

$$\begin{aligned} I_4(V) &= -\frac{1}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} V^{p+1} = -\frac{1}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t-\xi_1)^{p+1} + o(\delta) \\ &= -\frac{e^{\sigma_p \xi_1}}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t)^{p+1} + o(\delta) = -\frac{\delta\Lambda_1^{-1}}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t)^{p+1} + o(\delta) \\ &= -\frac{\delta}{\Lambda_1}\frac{N-2}{2N-2}\int_{-\infty}^{\infty} e^t U_0(t)^{\frac{2N-2}{N-2}}\, dt + o(\delta). \end{aligned} \tag{3.6}$$

Finally, we compute $I_1(V)$. Writing $U_i(t) = U(t-\xi_i)$, we have

$$\begin{aligned} I_1(V) &= \int_{-\infty}^{\infty} \left(\frac{1}{2}\Big(\sum_i U_i'\Big)^2 + \frac{\alpha^2}{2}(\sum_i U_i)^2 - \frac{1}{2^*}\Big(\sum_i U_i\Big)^{2^*} \right) \\ &= kI_U + \frac{1}{2}\sum_{i\neq j}\int_{-\infty}^{\infty}(-U_i'' + \alpha^2 U_i)U_j - \frac{1}{2^*}\int_{-\infty}^{\infty}\left[\Big(\sum_i U_i\Big)^{2^*} - \sum_i U_i^{2^*}\right], \end{aligned}$$

where we have set

$$I_U = \frac{1}{2}\int_{-\infty}^{\infty}\big((U')^2 + \alpha^2 U^2\big) - \frac{1}{2^*}\int_{-\infty}^{\infty} U^{2^*}.$$

Note that

$$I_U = c_0 + o(\delta) \quad \text{as } \delta \to 0. \tag{3.7}$$

Indeed, I_U is a function of ε, and

$$\frac{d}{d\varepsilon} I_U = \int_{-\infty}^{\infty}(-U'' + \alpha^2 U - U^{2^*-1})\frac{\partial U}{\partial \varepsilon},$$

so that $\frac{d}{d\varepsilon} I_U = 0$ at $\varepsilon = 0$. Let $F_i(t) = F(t-\xi_i)$, where $F = -U'' + \alpha^2 U - U^{2^*-1}$. Then, by (3.7),

$$I_1(V) = \frac{1}{2}\sum_{i\neq j}\int_{-\infty}^{\infty}(U_i^{2^*-1} + F_i)U_j - \frac{1}{2^*}\int_{-\infty}^{\infty}\left[\Big(\sum_i U_i\Big)^{2^*} - \sum_i U_i^{2^*}\right] + kc_0 + o(\delta).$$

Letting

$$t_1 = 0, \quad t_j = \left(1 + \frac{j-1/2}{\alpha}\right)\log\delta, \quad j = 2, \dots, k-1, \quad t_k = -\infty,$$

we can write

$$I_1(V) = -\frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} U_i^{2^*-1} U_j + kc_0 + R,$$

where R is given by

$$R = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} F_i U_j + \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{\mathbb{R}\setminus[t_i,t_{i-1}]} U_i^{2^*-1}U_j + \frac{1}{2^*}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} U_j^{2^*}$$
$$-\frac{1}{2^*}\sum_{i=1}^{k}\int_{t_i}^{t_{i-1}}\Big[\Big(U_i+\sum_{j\neq i}U_j\Big)^{2^*} - U_i^{2^*} - 2^* U_i^{2^*-1}\sum_{j\neq i}U_j\Big] + o(\delta).$$

If $|i-j|\geq 2$, then $\int_{t_i}^{t_{i-1}} U_i^{2^*-1}U_j = o(\delta)$. We have the expansions

$$U(t) = \gamma_N e^{\alpha t}(1+O(e^{2t})) \quad \text{as } t\to-\infty,$$
$$U(t) = \gamma_N e^{(\alpha-2\beta)t}(1+O(e^{-2t})) \quad \text{as } t\to+\infty.$$

Therefore, for $i=1,\dots,k-1$ and $j=i+1$,

$$\begin{aligned}\int_{t_i}^{t_{i-1}} U_i^{2^*-1}U_j &= \int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}}U(t-(\xi_{i+1}-\xi_i))\,dt\\ &= \gamma_N\int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}}e^{(\alpha-2\beta)(t-(\xi_{i+1}-\xi_i))}(1+O(e^{2(t-(\xi_{i+1}-\xi_i))}))dt\\ &= \gamma_N\delta^{\frac{2\beta-\alpha}{\alpha}}\Lambda_{i+1}^{\alpha-2\beta}\int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}}e^{(\alpha-2\beta)t}\big(1+O(e^{2(t-(\xi_{i+1}-\xi_i))})\big)dt\\ &= \gamma_N\delta\Lambda_{i+1}^{-\alpha}\int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}}e^{-\alpha t}dt + o(\delta).\end{aligned}$$

A similar calculation shows that if $i=2,\dots,k$ and $j=i-1$, then

$$\int_{t_i}^{t_{i-1}} U_i^{2^*-1}U_j = \gamma_N\delta\Lambda_i^{-\alpha}\int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}}e^{\alpha t}\,dt + o(\delta).$$

Now $R=o(\delta)$ as $\delta\to 0$. Indeed,

$$\begin{aligned}\int_{t_i}^{t_{i-1}}\Big[(U_i+\sum_{j\neq i}U_j)^{2^*} - U_i^{2^*} - 2^*U_i^{2^*-1}\sum_{j\neq i}U_j\Big] &\leq C\int_{t_i}^{t_{i-1}} U_i^{2^*-2}\sum_{j\neq i}U_j^2\\ &\leq C\int_0^{\frac{1}{2\alpha}|\log\delta|} e^{-\alpha\frac{4}{N-2}t}e^{-2\alpha(\frac{1}{\alpha}|\log\delta|-t)}\,dt\\ &\leq C\delta^2\int_0^{\frac{1}{2\alpha}|\log\delta|} e^{2\alpha\frac{N-4}{N-2}t}dt,\end{aligned}$$

which is $O(\delta^{1+\frac{2}{N-2}})$ if $N\geq 5$, $O(\delta^2|\log\delta|)$ if $N=4$, and $O(\delta^2)$ if $N=3$. The other terms in R can be handled similarly. Therefore,

$$I_1(V) = -\delta\gamma_N\int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}}e^{\alpha t}\,dt\sum_{j=2}^{k}\Lambda_j^{-\alpha} + kc_0 + o(\delta). \tag{3.8}$$

Combining (3.4), (3.5), (3.6), and (3.8), we arrive at

$$I(V) = -\delta\varphi(\Lambda) + kc_0 + A\delta + B\delta\log\delta + o(\delta) \quad \text{as } \delta \to 0$$

for some constants A, B, with $o(\delta)$ uniformly in the region $\Lambda_i \in [1/M, M]$, $i = 1, \dots, k$. A similar calculation shows that this expansion is also valid in the C^1 norm with respect to $\Lambda = (\Lambda_1, \dots, \Lambda_k)$. □

4 Error estimate

Proposition 4.1. *Let $k \geq 2$ be an integer and $M > 1$. Let $\xi_1, \dots, \xi_k$ be given by* (2.6), *$\Lambda = (\Lambda_1, \dots, \Lambda_k) \in [1/M, M]^k$, and E be given by* (2.11). *For sufficiently small $\nu > 0$ and $\tau > 0$ in* (2.12), *there exist $\delta_0 > 0$, $\theta > 1/2$, and a function $\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k) > 0$ such that*

$$\|E\|_* \leq C\delta^\theta$$

for $0 < \delta \leq \delta_0$ and $\varepsilon = \varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k)$, with the constant C independent of δ. The function ε_k is C^1 and satisfies

$$\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k) = \frac{\gamma_N^{p-1}}{4\alpha^3\Lambda_1}\delta + o(\delta) \quad \text{as } \delta \to 0, \tag{4.1}$$

$$\frac{\partial\varepsilon_k}{\partial\Lambda_i}(\delta, \Lambda_1, \dots, \Lambda_k) = O(\delta) \quad \text{as } \delta \to 0, \tag{4.2}$$

where $o(\delta)$, $O(\delta)$ are uniform in the region $\Lambda \in [1/M, M]^k$.

Proof. Let us write $U_j(t) = U(t-\xi_j)$, $V = \sum_{j=1}^k U_j$, and $E = \sum_{j=1}^k E_j + A + B$, where

$$E_j = U_j'' - \alpha^2 U_j + e^{\sigma_p t}U_j^p + e^{\sigma_q t}U_j^q,$$

$$A = e^{\sigma_p t}\left(V^p - \sum_{j=1}^k U_j^p\right), \text{ and}$$

$$B = e^{\sigma_q t}\left(V^q - \sum_{j=1}^k U_j^q\right).$$

Let $\beta = 1/(p-1) = \alpha - \varepsilon\alpha^2 + O(\varepsilon^2)$. Then a computation shows that

$$U_j'' - \alpha^2 U_j + e^{\sigma_p t}U_j^p = \left[e^{\sigma_p\xi_j}\gamma_N^p + 4\gamma_N\beta(\beta-\alpha)\right]\frac{e^{(\alpha+2)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+1}} - 4\gamma_N\beta(\beta+1)\frac{e^{(\alpha+2)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+2}}.$$

Note that the terms $U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p$ of E_j have slow decay, i.e.,

$$U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p \sim e^{(\alpha-2\beta)t} \quad \text{as } t \to \infty.$$

Define the slowly decaying part of E_j as

$$S_j = \chi_{[t \ge \xi_1/2]} \left(4\gamma_N \beta(\beta - \alpha) e^{-(\alpha-2\beta)\xi_j} + \gamma_N^p e^{-(\alpha-2\beta)p\xi_j} \right) e^{(\alpha-2\beta)t},$$

where $\chi_{[t\ge\xi_1/2]}$ is the indicator function of the set $[\xi_1/2, +\infty)$ and $\tilde{E}_j = E_j - S_j$. The term A also has a slowly decaying part,

$$S_A = \chi_{[t\ge\xi_1/2]} \gamma_N^p \left[\left(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} \right)^p - \sum_{j=1}^k e^{-(\alpha-2\beta)p\xi_j} \right] e^{(\alpha-2\beta)t}.$$

Set $\tilde{A} = A - S_A$. Given small $\delta > 0$ and $\xi_1, \dots, \xi_k$ satisfying (2.6) and (2.7), choose $\varepsilon > 0$ such that $\sum_{j=1}^k S_j + S_A = 0$, which is equivalent to

$$0 = 4\gamma_N \beta(\beta-\alpha) \sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} + \gamma_N^p \left(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} \right)^p. \tag{4.3}$$

By (2.6), at main order (in ε and δ), this equation has the form

$$4\gamma_N \beta(\beta-\alpha) e^{-(\alpha-2\beta)\xi_1} + \gamma_N^p e^{-(\alpha-2\beta)p\xi_1} = 0,$$

so that we have the asymptotic expansion (4.1). The estimate (4.2) also follows from (4.3).

We claim that

$$\|\tilde{E}_j\|_* \le C\delta^{1-2\tau} \tag{4.4}$$

for small $\delta > 0$. To prove this, consider separately the regions $t \ge \xi_1/2$ and $t \le \xi_1/2$. Using the formula for S_j, we have

$$\tilde{E}_j = O(e^{(\alpha-2\beta-2)(t-\xi_j)}) + O(e^{\sigma_p t} e^{((\alpha-2\beta)p-2)(t-\xi_j)}) + O(e^{\sigma_q t} e^{(\alpha-2\beta)q(t-\xi_j)})$$

for $t \ge \xi_1/2$, from which we see that

$$\sup_{t\ge\xi_1/2} |\tilde{E}_j| e^{(\alpha+\tau)(t-\xi_1)} \le C\delta^{1-\tau/2+O(\delta)} \le C\delta^{1-\tau}$$

for small $\delta > 0$.

We now estimate in the interval $t \le \xi_1/2$, in which

$$\begin{aligned} \tilde{E}_j &= -4\gamma_N \beta(\alpha+1) \frac{e^{(\alpha+2)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+1}} + 4\gamma_N \beta(\beta+1) \frac{e^{(\alpha+4)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+2}} \\ &\quad + e^{\sigma_p t} \gamma_N^p \frac{e^{\alpha p(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta p}} + e^{\sigma_q t} \gamma_N^q \frac{e^{\alpha q(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta q}} \\ &= O(\delta)(1+|t|+|\xi_j|) e^{-(\alpha+O(\delta))|t-\xi_j|}, \end{aligned} \tag{4.5}$$

where $O(\delta)$ here designates a quantity bounded by a constant times δ. From (4.5), we have

$$\sup_{t\le\xi_1} e^{\nu|t-\xi_i|}|\tilde{E}_j| \le C\delta \quad \text{for } i = 1,\dots,k, \tag{4.6}$$

and

$$\sup_{\xi_1\le t\le\xi_1/2} e^{(\alpha+\tau)|t-\xi_1|}|\tilde{E}_j| \le C\delta^{1-2\tau} \tag{4.7}$$

for small $\delta > 0$. Using (4.6), (4.7), we deduce (4.4).

Similarly, we estimate $\tilde{A}$ first in the interval $t \ge \xi_1/2$. In this interval, $\tilde{A} = A - S_A = A_1 + A_2$, where

$$A_1 = \gamma_N^p e^{\sigma_p t} e^{(\alpha-2\beta)pt}\left[\left(\sum_{j=1}^k \frac{e^{-(\alpha-2\beta)\xi_j}}{(1+e^{2(\xi_j-t)})^\beta}\right)^p - \left(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j}\right)^p\right]$$

and

$$A_2 = -\gamma_N^p e^{\sigma_p t} e^{(\alpha-2\beta)pt}\left[\sum_{j=1}^k \frac{e^{-(\alpha-2\beta)p\xi_j}}{(1+e^{2(\xi_j-t)})^{\beta p}} - \sum_{j=1}^k e^{-(\alpha-2\beta)p\xi_j}\right].$$

Thus, $(1+se^{2(\xi_j-t)})^{-\beta-1} = O(1)$; so, by the Mean Value Theorem,

$$|A_1| \le Ce^{\sigma_p t}e^{(\alpha-2\beta)pt}\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j p}e^{2(\xi_j-t)} \quad \text{for } t \ge \xi_1/2.$$

We then compute

$$\sup_{t\ge\xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A_1| \le C\delta^{2-\tau/2+O(\delta)}.$$

Similarly,

$$\sup_{t\ge\xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A_2| \le C\delta^{2-\tau/2+O(\delta)},$$

and we deduce

$$\sup_{t\ge\xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|\tilde{A}| \le C\delta^{2-\tau/2+O(\delta)}. \tag{4.8}$$

In the region $\xi_1 \le t \le \xi_1/2$, we have

$$|A| \le Ce^{\sigma_p t}\left[\sum_{j=2}^k U(t-\xi_j)U(t-\xi_1)^{p-1} + \sum_{j=2}^k U(t-\xi_j)^p\right],$$

which gives

$$\sup_{\xi_1 \le t \le \xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A| \le C\delta^{2-\tau/2+O(\delta)}. \tag{4.9}$$

We now estimate the term A for $t \le \xi_1$. Using the fact that $e^{\sigma_p t} \le C\delta^{1+O(\delta)}$ in this interval, we see that

$$\sup_{t\le\xi_1} e^{\nu|t-\xi_i|}|A| \le C\delta^{1+O(\delta)} \quad \text{for } i = 1, \dots, k. \tag{4.10}$$

Hence, by (4.8), (4.9) and (4.10), we find that $\|\tilde{A}\|_* \le C\delta^{1+O(\delta)}$.

Finally, we estimate $\|B\|_*$. We claim that there exists $\theta > 1/2$ such that

$$\|B\|_* \le C\delta^{\theta} \tag{4.11}$$

for $\delta > 0$ sufficiently small. Indeed, let $i = 1, \dots, k-1$ and estimate

$$\sup_{\xi_{i+1} \le t \le \xi_i} (e^{\nu|t-\xi_{i+1}|} + e^{\nu|t-\xi_i|})|B|.$$

Let $\lambda \in (0, 1/2)$, to be fixed later. Consider the three intervals

$$\begin{aligned} I_1 &= [\xi_{i+1}, (1-\lambda)\xi_{i+1} + \lambda\xi_i], \\ I_2 &= [(1-\lambda)\xi_{i+1} + \lambda\xi_i, \lambda\xi_{i+1} + (1-\lambda)\xi_i], \\ I_3 &= [\lambda\xi_{i+1} + (1-\lambda)\xi_i, \xi_i]. \end{aligned}$$

The worst term in each sum of B is either $U(t-\xi_{i+1})^q$ or $U(t-\xi_i)^q$. We estimate

$$\begin{aligned} \sup_{t\in I_2} e^{\nu|t-\xi_i|} U(t-\xi_i)^q &\le C \sup_{t\in I_2} e^{\nu(\xi_i - t)} e^{-\alpha q(\xi_i - t)} \le Ce^{(\nu-\alpha q)\xi_i} \sup_{t\in I_2} e^{(\alpha q - \nu)t} \\ &= C\delta^{\lambda(q-\nu/\alpha)}. \end{aligned}$$

Since $q > 1$, we may choose $\nu > 0$ small so that $q - \nu/\alpha > 1$. Then take $\lambda \in (0, 1/2)$ such that

$$\lambda(q - \nu/\alpha) > \frac{1}{2}. \tag{4.12}$$

We also have

$$\sup_{t\in I_2} e^{\nu|t-\xi_{i+1}|} U(t-\xi_{i+1})^q \le C\delta^{\lambda(q-\nu/\alpha)}.$$

This gives

$$\sup_{t\in I_2} \frac{|B|}{w(t)} \le C\delta^{\lambda(q-\nu/\alpha)+O(\delta)}.$$

We now compute

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}\left[\left(\sum_{j=1}^{k} U(t-\xi_j)\right)^q - \sum_{j=1}^{k} U(t-\xi_j)^q\right].$$

In this interval, $U(t-\xi_i)$ is dominant, so

$$\left(\sum_{j=1}^{k} U(t-\xi_j)\right)^q = U(t-\xi_i)^q\left(1+\sum_{j\neq i}^{k}\frac{U(t-\xi_j)}{U(t-\xi_i)}\right)^q$$
$$= U(t-\xi_i)^q + \sum_{j\neq i}^{k} O(U(t-\xi_j)U(t-\xi_i)^{q-1}).$$

Hence

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}\left|\left(\sum_{j=1}^{k} U(t-\xi_j)\right)^q - \sum_{j=1}^{k} U(t-\xi_j)^q\right|$$
$$\leq C\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}\left[\sum_{j\neq i}^{k} U(t-\xi_j)U(t-\xi_i)^{q-1} + \sum_{j=1}^{k} U(t-\xi_j)^q\right].$$

The worst case is $j = i+1$ in the first sum

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}U(t-\xi_{i+1})U(t-\xi_i)^{q-1}$$
$$\leq Ce^{\nu\xi_i}e^{-(\alpha-2\beta)\xi_{i+1}}e^{-\alpha\xi_i(q-1)}\sup_{t\in I_3} e^{-\nu t}e^{\sigma_q t}e^{(\alpha-2\beta)t}e^{\alpha(q-1)t}$$

If the sup is attained at $t = \xi_i$,

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}U(t-\xi_{i+1})U(t-\xi_i)^{q-1} = Ce^{\alpha(\xi_{i+1}-\xi_i)+O(\delta|\log\delta|)} \leq C\delta.$$

If the sup is attained at $t = \lambda\xi_{i+1} + (1-\lambda)\xi_i$,

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|}e^{\sigma_q t}U(t-\xi_{i+1})U(t-\xi_i)^{q-1} \leq C\delta^{(q-1-\nu/\alpha)\lambda}\delta^{(2\beta-\alpha)/\alpha(1-\lambda)}e^{\delta|\log\delta|}$$
$$\leq C\delta^{(q-\nu/\alpha)\lambda+1-2\lambda}.$$

Since $\lambda \in (0, 1/2)$, $(q-\nu/\alpha)\lambda + 1 - 2\lambda > 1/2$, by (4.12).

By similar estimates in the remaining intervals, we obtain the validity of (4.11) with $\theta = \lambda(q-\nu/\alpha) > 1/2$. □

5 The linearized equation

In this section, given $\xi_1, \dots, \xi_k \in \mathbb{R}$ satisfying (2.6) and (2.7) for some fixed $M > 1$, we study the linear problem

$$\begin{cases} L(\phi) = h + \sum_{i=1}^{k} c_i \tilde{Z}_i & \text{in } \mathbb{R}, \\ \lim_{t \to \pm\infty} \phi(t) = 0, \end{cases} \tag{5.1}$$

where L is the operator defined in (2.10) and $\tilde{Z}_i$ is defined by

$$\tilde{Z}_i(t) = U_0'(t - \xi_i)\eta(t - \xi_i), \tag{5.2}$$

where $\eta \in C^\infty(\mathbb{R})$ is an even cut-off function, $\eta \geq 0$, such that $\operatorname{supp}(\eta) = [-R, R]$ and $R > 0$ is a fixed constant. We also write $Z_i(t) = U_0'(t - \xi_i)$.

The main result in this section is the following.

Proposition 5.1. *Let $M > 1$ and $k > 2$ be an integer. Let $\xi_1, \dots, \xi_k \in \mathbb{R}$ satisfy* (2.6) *and* (2.7)*. Then there exist $\delta_0, C > 0$ such that for $0 < \delta \leq \delta_0$ and $0 < \varepsilon \leq C\delta$, there exists a linear operator T from $\|\cdot\|_*$ to $(\|\cdot\|_*, \mathbb{R}^k)$ such that $T(h) = (\phi, c_1, \dots, c_k)$ solves* (5.1) *for all h with $\|h\|_* < \infty$. Moreover,*

$$t\|\phi\|_* \leq C\|h\|_* \quad \text{and} \quad |c_i| \leq C\|h\|_*, \quad i = 1, \dots, k.$$

For $\tau > 0$, $0 < \nu < \alpha$, and $\phi : \mathbb{R} \to \mathbb{R}$, define

$$\|\phi\|_1 = \sup_{t \geq \xi_1}(e^{(\alpha+\tau)(t-\xi_1)}|\phi(t)|) + \sup_{t \leq \xi_1}(e^{\nu(\xi_1 - t)}|\phi(t)|). \tag{5.3}$$

Lemma 5.2. *Let $\tau > 0$ and $0 < \nu < \alpha$. For each h such that $\|h\|_1 < +\infty$, there exist a unique ϕ with $\|\phi\|_1 < +\infty$ and $c_1 \in \mathbb{R}$ such that*

$$\phi'' - \alpha^2\phi + 2^* U_0(t - \xi_1)^{2^*-1}\phi = h + c_1\tilde{Z}_1 \quad \text{in } \mathbb{R}. \tag{5.4}$$

Moreover, there exists $C > 0$ such that

$$\|\phi\|_1 \leq C\|h\|_1 \quad \text{and} \quad |c_1| \leq C\|h\|_1. \tag{5.5}$$

Proof. Translating, if necessary, we may assume that $\xi_1 = 0$.

Let U_0 be the function defined by (2.3). Then $z_1 = U_0'$ satisfies

$$z_1(t) = -\gamma_N 2^{-\alpha}\alpha \cosh(t)^{-N/2}\sinh(t) \tag{5.6}$$

and the initial value problem

$$z'' - \alpha^2 z + 2^* U_0^{2^*-1} z = 0 \quad \text{in } \mathbb{R}, \tag{5.7}$$

$$z(0) = 0 \quad \text{and} \quad z'(0) = -2^{-\alpha}\alpha\gamma_N.$$

Let z_2 be the solution of (5.7) satisfying the initial conditions

$$z(0) = 1 \quad \text{and} \quad z'(0) = 0.$$

To prove uniqueness, observe that if $h = 0$, multiplication of (5.4) by z_1 gives $c_1 = 0$. Then ϕ must be a linear combination of z_1 and z_2; and, since $\|\phi\|_1 < +\infty$, $\phi = cz_1$ for some c. But again, because $\|\phi\|_1 < +\infty$, $\phi = 0$.

To prove existence, suppose that $\|h\|_1 < \infty$ and $\int_{-\infty}^{\infty} hz_1 = 0$. The function

$$\phi(t) = \frac{2^\alpha}{\alpha\gamma_N}\left(z_1(t)\int_t^\infty z_2(s)h(s)ds - z_2(t)\int_t^\infty z_1(s)h(s)ds\right) \tag{5.8}$$

is a solution of the linear problem

$$\phi'' - \alpha^2\phi + 2^* U_0^{2^*-1}\phi = h \quad \text{in } \mathbb{R}.$$

Moreover,

$$\|\phi\|_1 \le C\|h\|_1. \tag{5.9}$$

Indeed, from (5.6), we have $z_1(t) = ce^{-\alpha|t|} + o(e^{-\alpha|t|})$ as $t \to \pm\infty$ for some constant c . Furthermore, one can also prove that $z_2(t) = c'e^{\alpha|t|} + o(e^{\alpha|t|})$ as $t \to \pm\infty$ for some constant $c' \neq 0$. Then (5.9) follows from (5.8) and the behaviors of z_1, z_2 at $\pm\infty$.

In the general case, when h is not necessarily orthogonal to z_1, define

$$c_1 = -\frac{\int_{-\infty}^{\infty} hz_1}{\int_{-\infty}^{\infty} \tilde{Z}_1 z_1}$$

and apply the previous construction to $h + c_1\tilde{Z}_1$. □

For $\phi : \mathbb{R} \to \mathbb{R}$, define the norm

$$\|\phi\|_2 = \sup_{t\in\mathbb{R}} \left(\sum_{i=2}^{k} e^{-\nu|t-\xi_i|}\right)^{-1} |\phi(t)|. \tag{5.10}$$

Lemma 5.3. *Let* $0 < \nu < \alpha$ *in* (5.10). *Then there exist* $\delta_0, C > 0$ *such that for* $0 < \delta \le \delta_0$ *and* $\|h\|_2 < \infty$, *there exist* $c_2, \ldots, c_k \in \mathbb{R}$ *and a unique solution* ϕ

with $\|\phi\|_2 < +\infty$ *of*

$$\begin{cases} \phi'' - \alpha^2\phi + 2^* \sum_{i=2}^{k} U_0(t-\xi_i)^{2^*-1}\phi = h + \sum_{i=2}^{k} c_i \tilde{Z}_i \text{ in } \mathbb{R} \\ \int_{\mathbb{R}} \phi \tilde{Z}_i = 0, \quad i = 2, \dots, k. \end{cases} \tag{5.11}$$

Moreover,

$$\|\phi\|_2 \le C\|h\|_2, \quad |c_i| \le C\|h\|_2, \quad i = 2, \dots, k. \tag{5.12}$$

The proof is similar to that of [2, Proposition 1].

Lemma 5.4. *Let* $0 < \nu < \min(2, \alpha)$ *and* $\tau > 0$. *Then there are* $\delta_0, C > 0$ *such that for* $0 < \delta \le \delta_0$, *there exists a linear operator* T_0 *from* $\|\cdot\|_*$ *to* $(\|\cdot\|_*, \mathbb{R}^k)$ *such that* $T_0(h) = (\phi, c_1, \dots, c_k)$ *solves*

$$\phi'' - \alpha^2\phi + 2^* \sum_{i=1}^{k} U_0(t-\xi_i)^{2^*-1}\phi = h + \sum_{i=1}^{k} c_i \tilde{Z}_i \text{ in } \mathbb{R} \tag{5.13}$$

for each h *with* $\|h\|_* < \infty$, *Moreover,*

$$\|\phi\|_* \le C\|h\|_* \quad \text{and} \quad |c_i| \le C\|h\|_*, \quad i = 1, \dots, k. \tag{5.14}$$

Proof. Define

$$W_i(t) = 2^* U_0(t-\xi_i)^{2^*-1}. \tag{5.15}$$

Let $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ be such that $0 \le \eta_1, \eta_2 \le 1$, and

$$\eta_1 \equiv 1 \text{ in } (-\infty, (1+\tfrac{1}{2\alpha})\log\delta], \quad \eta_1 \equiv 0 \text{ in } [(1+\tfrac{1}{4\alpha})\log\delta, \infty)$$
$$\eta_2 \equiv 1 \text{ in } (-\infty, (1+\tfrac{3}{4\alpha})\log\delta], \quad \eta_2 \equiv 0 \text{ in } [(1+\tfrac{1}{2\alpha})\log\delta, \infty).$$

We look for a solution of (5.13) of the form $\phi = \phi_1 + \phi_2\eta_2$. It suffices for ϕ_1, ϕ_2 to satisfy the system

$$\phi_1'' - \alpha^2\phi_1 + W_1\phi_1 = (1-\eta_2)h + c_1\tilde{Z}_1 - (1-\eta_2)\sum_{i=2}^{k} W_i\phi_1, \tag{5.16}$$
$$- 2\phi_2'\eta_2' - \phi_2\eta_2''$$

$$\phi_2'' - \alpha^2\phi_2 + \sum_{i=2}^{k} W_i\phi_2 = \eta_1 h + \sum_{i=2}^{k} c_i\tilde{Z}_i - \eta_1 W_1\phi_2 - \eta_1 \sum_{i=2}^{k} W_i\phi_1 \text{ in } \mathbb{R}. \tag{5.17}$$

Define $\phi = T_1(h)$ to be the solution of (5.4) (Lemma 5.2) and $\phi = T_2(h)$ to be the solution of (5.11) obtained in Lemma 5.3. To find a solution of (5.16), (5.17) with the correct bounds, we are then led to the system

$$\phi_1 = T_1\left[(1-\eta_2)h - (1-\eta_2)\sum_{i=2}^{k} W_i\phi_1 - 2\phi_2'\eta_2' - \phi_2\eta_2''\right], \tag{5.18}$$

$$\phi_2 = T_2\left[\eta_1 h - \eta_1 W_1\phi_2 - \eta_1\sum_{i=2}^{k} W_i\phi_1\right]. \tag{5.19}$$

We solve this system in the Banach space E consisting of pairs (ϕ_1, ϕ_2) of functions $\phi_j : \mathbb{R} \to \mathbb{R}$ such that ϕ_2 is Lipschitz continuous and the norm

$$\|(\phi_1, \phi_2)\|_E = \|\phi_1\|_1 + \|\phi_2\|_2 + \|\phi_2'\|_2$$

is finite, where $\| \ \|_1$ is defined by (5.3) and $\| \ \|_2$ by (5.10). We verify that the operator $\tilde{T} : E \to E$ defined by the right hand side of (5.18), (5.19) is a contraction on E. For this we use (5.5) to obtain the estimate

$$\begin{aligned}
&\|T_1[-(1-\eta_2)\sum_{i=2}^{k} W_i\phi_1 - 2\phi_2'\eta_2' - \phi_2\eta_2'']\|_1 \\
&\qquad \le C(\|(1-\eta_2)\sum_{i=2}^{k} W_i\phi_1\|_1 + \|\phi_2'\eta_2'\|_1 + \|\phi_2\eta_2''\|_1).
\end{aligned}$$

Computation shows that

$$\begin{aligned}
\|(1-\eta_2)\sum_{i=2}^{k} W_i\phi_1\|_1 &\le C\delta^{\frac{1}{\alpha}}\|\phi_1\|_1, \\
\|\phi_2'\eta_2'\|_1 &\le \frac{C}{|\log\delta|}\|\phi_2'\|_2, \\
\|\phi_2\eta_2''\|_1 &\le \frac{C}{|\log\delta|^2}\|\phi_2\|_2.
\end{aligned}$$

Using (5.12), we have

$$\|T_2[-\eta_1 W_1\phi_2 - \eta_1\sum_{i=2}^{k} W_i\phi_1]\|_2 \le C\left(\|\eta_1 W_1\phi_2\|_2 + \|\eta_1\sum_{i=2}^{k} W_i\phi_1\|_2\right).$$

Another computation yields

$$\|\eta_1 W_1\phi_2\|_2 \le C\delta^{1/2\alpha}\|\phi_2\|_2,$$

and

$$\Big\| \eta_1 \sum_{i=2}^k W_i \phi_1 \Big\|_2 \le C \delta^{(3-\nu)/2\alpha} \|\phi_1\|_1$$

if $\nu \ge 1$, while

$$\Big\| \eta_1 \sum_{i=2}^k W_i \phi_1 \Big\|_2 \le C \delta^{\nu/\alpha} \|\phi_1\|_1$$

if $\nu < 1$. It follow that if $\nu < 3$, $\tilde{T}$ is a contraction in E. □

Proof of Proposition 5.1. First, let us prove existence of a solution. Let W_i be defined by (5.15), and rewrite equation (5.1) in the form

$$\phi = T_0 \Big[h + \Big(\sum_{i=1}^k W_i - p e^{\sigma_p t} V^{p-1} - q e^{\sigma_q t} V^{q-1} \Big) \phi \Big], \tag{5.20}$$

where T_0 is the operator defined in Lemma 5.4. Let X the Banach space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\|\phi\|_* < \infty$, equipped with the norm $\| \ \|_*$. By (5.14),

$$\begin{aligned} \Big\| T_0 \Big[\Big(\sum_{i=1}^k W_i - p e^{\sigma_p t} V^{p-1} - q e^{\sigma_q t} V^{q-1} \Big) \phi \Big] \|_* \\ \le C \Big\| \sum_{i=1}^k W_i - p e^{\sigma_p t} V^{p-1} - q e^{\sigma_q t} V^{q-1} \Big\|_{L^\infty(\mathbb{R})} \|\phi\|_* . \end{aligned}$$

A computation shows that

$$\Big\| \sum_{i=1}^k W_i - p e^{\sigma_p t} V^{p-1} - q e^{\sigma_q t} V^{q-1} \Big\|_{L^\infty(\mathbb{R})} = o(1) \quad \text{as } \delta \to 0. \tag{5.21}$$

Indeed, let us estimate $\|e^{\sigma_p t} V^{p-1}\|_{L^\infty(\mathbb{R})}$. We have

$$\begin{aligned} e^{\sigma_p t} V^{p-1} &= e^{\sigma_p t} \Big(\sum_{j=1}^k U(t - \xi_j) \Big)^{p-1} \le C e^{\sigma_p t} \Big(\sum_{j=1}^k e^{\alpha(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-\beta} \Big)^{p-1} \\ &\le C e^{\sigma_p t} \sum_{j=1}^k e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1}. \end{aligned}$$

For $t \ge \xi_j$,

$$e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1} \le C e^{(\alpha - 2\beta)(p-1)(-\xi_j)} \le C e^{(1-\alpha\varepsilon)\xi_j},$$

since $\sigma_p + (\alpha - 2\beta)(p-1) = 0$. Since the ξ satisfy (2.6), (2.7) for some $M > 0$,

$$e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1} \le C \delta^{(1-\alpha\varepsilon)((j-1)/\alpha + 1)} \le C \delta^{1-\alpha\varepsilon}$$

for $t \ge \xi_j$. For $t \le \xi_j$,

$$e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)}(1+e^{2(t-\xi_j)})^{-1} \le C e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} \le C e^{\sigma_p t}$$
$$\le C e^{\sigma_p \xi_j} \le C\delta^{\sigma_p((j-1)/\alpha+1)} \le C\delta^{1-\alpha\varepsilon}.$$

Therefore,

$$\|pe^{\sigma_p t} V^{p-1}\|_{L^\infty(\mathbb{R})} \le C\delta^{1-\alpha\varepsilon}.$$

The difference $\sum_{i=1}^k W_i - qe^{\sigma_q t} V^{q-1}$ in (5.21) can be handled similarly. Thus, if $\|h\|_* < \infty$ and $\varepsilon, \delta > 0$ are suitably small, (5.20) has a unique solution in X. □

6 Proof of Theorem 1.1

Let us fix an integer $k \ge 2$. By Proposition 4.1, there exist $\theta > 1/2$ and a function $\varepsilon_k(\Lambda, \delta) > 0$ such that if $\varepsilon = \varepsilon_k(\Lambda, \delta)$ and δ is sufficiently small, then $\|E\|_* \le C\delta^\theta$. We claim that for small enough $\delta > 0$ and $\varepsilon = \varepsilon_k(\Lambda, \delta)$, there exists a solution ϕ of the nonlinear projected problem

$$L\phi + E + N(\phi) = \sum_{i=1}^k c_i \tilde{Z}_i \tag{6.1}$$

such that $\|\phi\|_* \le A\delta^\theta$, for a suitable constant $A > 0$. Here, $\tilde{Z}_i$ are the functions defined in (5.2). Indeed, let T be the operator defined in Proposition 5.1. Then we obtain a solution of (6.1) by solving the fixed point problem

$$\phi + T(E - N(\phi)) = 0. \tag{6.2}$$

Consider the Banach space X of all continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\|\phi\|_* < +\infty$ with norm $\|\ \|_*$. Let $A > 0$. One checks easily that for $\phi_1, \phi_2 \in E$ with $\|\phi_i\|_* \le A\delta^\theta$, $i = 1, 2$,

$$\|N(\phi_1) - N(\phi_2)\|_* \le C_A \delta^a \|\phi_1 - \phi_2\|_*$$

for some $a > 0$. We conclude from this estimate and the boundedness of the operator T that the fixed point problem (6.2) has a unique solution ϕ in the region $\|\phi\|_* \le A\delta^\theta$ for some suitably chosen A. We write this solution as $\phi(\Lambda)$.

To find a solution of (2.9), it remains to verify that for some $\Lambda = (\Lambda_1, \dots, \Lambda_k)$, the constants c_i in (6.1) all vanish. Testing equation (6.1) against $Z_j(t) = U_0'(t - \xi_j)$ for $i = 1, \dots, k$, we obtain

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j + \int_{-\infty}^{\infty} E Z_j = c_j \int_{-\infty}^{\infty} \tilde{Z}_j Z_j.$$

Thus $c_i = 0$, $i = 1, \dots k$, is equivalent to

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j + \int_{-\infty}^{\infty} E Z_j = 0 \tag{6.3}$$

for all j. A calculation shows that

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j = o(\delta)$$

as $\delta \to 0$, where $o(\delta)$ a continuous function of Λ that tends to 0 is uniformly in the region considered as $\delta \to 0$ (for this, it is important that $\|\phi\|_* \leq C\delta^\theta$ with $\theta > 1/2$). Write $\mathcal{E}(v) = v'' - \alpha^2 v + e^{-\sigma_p t} v^p + e^{-\sigma_q t} v^q$. Since $E = \mathcal{E}(V)$ and $Z_i = \partial_{\xi_i} V$,

$$\int_{-\infty}^{\infty} E Z_i = \int_{-\infty}^{\infty} \mathcal{E}(V) \partial_{\xi_i} V = \partial_{\xi_i} I(V)$$

From the expansion for $I(V)$ in Proposition 3.1 and the relations (2.6), we see that the system (6.3) is equivalent to

$$\nabla \varphi(\Lambda) + o(1) = 0,$$

where the quantity $o(1)$ tends to 0 uniformly in the region considered for the parameters Λ_i and depends continuously on them. Recall that the functional φ possesses a unique critical point Λ^*, which is nondegenerate. Therefore, the above equation has a solution that is close to Λ^* for $\delta > 0$ small.

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