UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FíSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

CONTRIBUTIONS TO THE PRINCIPAL-AGENT THEORY AND APPLICATIONS IN ECONOMICS

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS
DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
EN COTUTELA CON LA UNIVERSIDAD PARIS-DAUPHINE

NICOLÁS IVÁN HERNÁNDEZ SANTIBÁÑEZ

PROFESORES GUÍAS:
ALEJANDRO JOFRÉ CÁCERES
DYLAN POSSAMAÏ

MIEMBROS DE LA COMISIÓN:
BRUNO BOUCHARD
JEAN-FRANÇOIS CHASSAGNEUX
ULRICH HORST
HUYÊN PHAM
HÉCTOR RAMÍREZ CABRERA
NIZAR TOUZI

Este trabajo ha sido parcialmente financiado por CONICYT-Beca Doctorado Nacional 2013

SANTIAGO DE CHILE
2017

Resumen: Contribuciones a la teoría de Agente-Principal y aplicaciones en economía

En esta tésis se estudian aspectos teóricos del modelo de Agente-Principal y se presentan algunas aplicaciones en economía.

En la primera parte de la tésis se presentan dos aplicaciones del modelo. En la primera, un proveedor de electricidad determina la tarifa óptima para cobrar a los clientes por su consumo. La población es heterogénea y el proveedor observa perfectamente el consumo de cada cliente. Esto conlleva a una situación de selección adversa sin riesgo moral. El problema del Principal se escribe como un problema variacional no estándar que se resuelve para formas particulares de la utilidad de reserva de la población. El contrato óptimo resulta ser o bien lineal o polinomial con respecto al consumo y el proveedor contrata solo a aquellos consumidores que presentan una alta o una baja necesidad de electricidad.

En la segunda aplicación, un banco monitorea un conjunto de préstamos idénticos sujetos a contagio Markoviano. El banco obtiene fondos de un inversor, que no puede observar las acciones del banco y tampoco conoce su competencia para el trabajo. Este trabajo es una extensión del modelo de Pagès and Possamaï [84] al caso de incluye tanto riesgo moral como selección adversa. Siguiendo el enfoque de Cvitanić, Wan and Yang [31] para este tipo de problemas, el conjunto creíble dinámico es calculado explícitamente y la función valor del inversor se obtiene a través de un sistema recursivo de inecuaciones variacionales. Las propiedades del contrato óptimo se discuten en detalle.

En la segunda parte de la tesis se estudia el problema de un Agente que controla el retorno esperado de un proceso de difusión bajo incerteza de la volatilidad. Se asume que tanto el Principal como el Agente tiene un enfoque pesimista al problema y actúan como si un tercer jugador, la Naturaleza, escogiera la peor volatilidad posible. Este trabajo es una extensión de Mastrolia y Possamaï [64] y de Sung [125] a un marco más general. Se demuestra que la función valor del Agente puede ser representada como la solución de una Ecuación Diferencial Estocástica Retrógrada de segundo orden, y también que la función valor del Principal corresponde a la única solución viscosa de la ecuación de Hamilton-Jacobi-Bellman-Isaacs asociada, asumiendo que esta última satisface un principio de comparación.

Key words: problema de agente-principal, riesgo moral, selección adversa, tarificación de electricidad, monitoreo de bancos, incerteza en la volatilidad, EDERs de segundo orden, análisis variacional, análisis $u-$convexo, juegos diferenciales estocásticos, ecuación HJB, ecuación HJBI.

Résumé: Contributions à la théorie d'agent principal et applications en économie

Dans cette thèse, les aspects théoriques et les applications en économie du modèle Principal-Agent sont étudiés.

La première partie de la thèse présente deux applications du modèle. Dans la première, un fournisseur d'électricité détermine le tarif de consommation optimal pour ses clients. La population est hétérogène et le fournisseur observe parfaitement la consommation des clients. Cela conduit à un problème d'anti-sélection sans aléa moral. Le problème du Principal s'écrit comme un problème variationel non standard, qui peut être résolu sous certaines formes particulières de l'utilité de réservation de la population. Les contrats optimaux obtenus sont linéaires ou polynomiaux par rapport à la consommation et le fournisseur d'électricité ne propose un contrat que pour les consommateurs avec une faible ou une forte appétence pour l'électricité.

Dans la deuxième application, une banque surveille un ensemble de prêts identiques soumis à une contagion Markovienne. La banque collecte des fonds auprès d'un investisseur, qui ne peut pas observer les actions de la banque et n'a pas accès à sa capacité à faire son travail. Ce travail étend le modèle de Pagès et Possamaï [84] au cas de l'aléa moral avec anti-sélection. Suivant l'approche de Cvitanić, Wan et Yang [31] pour traiter ce genre de problèmes, l'ensemble crédible est calculé explicitement et la fonction valeur de l'investisseur est obtenue à l'aide d'un système récursif d'inégalités variationnelles. Les propriétés des contrats optimaux sont discutées en détail.

Dans la deuxième partie de la thèse, le problème d'un Agent contrôlant le drift d'un processus de diffusion sous volatilité incertaine est étudié. On suppose que le Principal et l'Agent ont une approche pessimiste du problème et agissent comme si un troisième joueur, la Nature, choisissait la pire volatilité possible pour leurs problèmes d'optimisation respectifs. Ce travail est une extension de l'étude de Mastrolia et Possamaï [64] et Sung [125] à un cadre plus général. On montre ainsi que la fonction valeur de l'Agent peut être représentée à l'aide de la solution à un EDSR du second ordre, et que la fonction valeur du Principal correspond à l'unique solution de viscosité de l'équation d'Hamilton-Jacobi-Bellman-Isaacs associée, sous réserve d'existence d'un résultat de comparaison.

Mots clés : problème principal-agent, aléa moral, anti-sélection, tarification de l'électricité, surveillance des banques, volatilité incertaine, EDSR du seconde ordre, analyse variationnelle, analyse $u-$convexe, jeux différentiels stochastiques, équation de HJB, équation de HJBI.

Abstract: Contributions to the Principal-Agent theory and applications in economics

In this thesis, theoretical aspects and applications in economics of the Principal-Agent model are studied.

The first part of the thesis presents two applications of the model. In the first one, an electricity provider determines the optimal tariff of consumption for its clients. Population is heterogeneous and the provider observes perfectly the consumption of the clients. This leads to a setting of adverse selection without moral hazard. The problem of the Principal writes as a non-standard variational problem, which can be solved under certain particular forms of the reservation utility of the population. The optimal contracts obtained are either linear or polynomial with respect to the consumption and the electricity provider contracts only consumers with either low or high appetite for electricity.

In the second application, a bank monitors a pool of identical loans subject to Markovian contagion. The bank raises funds from an investor, who cannot observe the actions of the bank and neither knows his ability to do the job. This is an extension of the model of Pagès and Possamaï [84] to the case of both moral hazard and adverse selection. Following the approach of Cvitanić, Wan and Yang [31] to these problems, the dynamic credible set is computed explicitly and the value function of the investor is obtained through a recursive system of variational inequalities. The properties of the optimal contracts are discussed in detail.

In the second part of the thesis, the problem of an Agent controlling the drift of a diffusion process under volatility uncertainty is studied. It is assumed that the Principal and the Agent have a worst–case approach to the problem and they act as if a third player, the Nature, was choosing the worst possible volatility. This work is an extension to Mastrolia and Possamaï [64] and Sung [125] to a more general framework. It is proved that the value function of the agent can be represented as the solution to a second–order BSDE, and also that the value function of the Principal corresponds to the unique viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation, given that the latter satisfies a comparison result.

Key words: principal-agent problem, moral hazard, adverse selection, power tarification, bank monitoring, volatility uncertainty, second–order BSDEs, variational analysis, $u-$convex analysis, stochastic differential games, HJB equation, HJBI equation.

A Juan Carlos y Lucy.
A Jocelyn.
A Sofía.

Agradecimientos

Quisiera comenzar agradeciendo a Erhan Bayraktar, Jakša Cvitanić y Huyên Pham por aceptar revisar esta tésis. Sin duda han dedicado una gran cantidad de tiempo y trabajo en su lectura y en la emisión de los reportes. También agradezco a quienes aceptaron ser parte de mi jurado, y cuya presencia en mi disertación me honra: René Aïd, Bruno Bouchard, Jean-François Chassagneux, Ulrich Horst, Héctor Ramírez y Nizar Touzi.

Muchas cosas han pasado durante estos 4 años y medio de doctorado, y muchas personas han sido importantes para mí en este periodo. La realización de este sueño se la debo en gran parte a Alejandro, quien fue fundamental para tomar la decisión de comenzar mi doctorado. Gracias por hacerme ver posible lo que para mí parecía inalcanzable y por motivarme a iniciar esta aventura. Gracias también por brindarme tu apoyo y consejo en todo momento durante estos años.

Al viajar por primera vez a París, no esperaba encontrarme con un tutor como Dylan. Ha sido genial descubrir nuestros intereses en común (sin duda nos hemos beneficiado de la poca diferencia de edad!) y poder trabajar de la forma en la que lo hemos hecho. A día de hoy te considero un gran amigo y me alegra que hayas forjado lazos tan fuertes con Chile [1], que te traerán frecuentemente a estas tierras.

He tenido la suerte de trabajar con varias personas durante estos años. He compartido grandes momentos, más allá de lo profesional, con Chao, Nabil y Thibaut, a quienes considero mis amigos. También agradezco a Clémence, Ivar y Romuald, a quienes respeto mucho, la oportunidad de haber trabajado juntos. Quisiera mencionar a mis compañeros de doctorado en París: Roméo, Shouquing, Marco, Raphael, Fang, Camille; y en Santiago: Vale, Sébastien, Pedro, Bao y Tich. Además de amigos a quienes veía frecuentemente y con quienes compartí en ambos establecimientos: Fili, Donoso, Obando, Miquel, Caro.

Agradezco también a los funcionarios del DIM y CMM por toda la ayuda brindada en innumerables ocasiones, destacando María Inés, Eterin, Natacha, Óscar, Luis, Silvia y Karen.

Muchas gracias a mis amigos de la vida, que me han acompañado durante estos años: Orly, Pancho, Z, Mati, Emilio, Pipe, Juanelo, Wilson, Pablo y Pancha. A los amigos con quienes estreché lazos durante mi estadía en París e hicieron que ésta fuera genial: Víctog, Riffo, Niko, Gery, Javi, Cristopher, Marce, Octavio. Y a mis amigos *de vacaciones* con quienes viajé y/o me dieron techo en Europa: Johan, Dani, Marrano, Jajo. En esta parte debo

[1] Un saludo para Cel!

separar a mis tíos adoptivos, Verónica y Luis, que junto a Alexandre me abrieron las puertas de su hogar en múltiples ocasiones y me hicieron sentir realmente en familia.

A Jimmy y Fer, mis nuevos convivientes. Ha sido un agrado compartir el día a día con ustedes, han sido una nueva fuente de alegría y diversión para nuestro hogar. Además estoy muy agradecido por toda la ayuda que me brindaron mientras terminaba esta tésis.

A mis padres, Juan Carlos y Lucy, gracias a quienes soy la persona que soy. Han sido un ejemplo para mí y me siento afortunado de poder contar con ustedes siempre que lo he necesitado.

A Jocelyn, por estar a mi lado todo este tiempo. Nuestra historia se sigue escribiendo y nos sigue preparando sorpresas. Como ya te he dicho, siento que juntos podemos lograr cualquier cosa que nos propongamos. Gracias por darme fuerzas, quererme y aguantarme. Gracias también a Sara, Jasmín y Joaquín, que los considero parte de mi familia.

A Sofía, mi fuente ilimitada de energía. Llegaste a nuestras vidas para darles felicidad e intento día a día devolverte el favor. Me fascina verte crecer y desarrollarte como persona. Deseo siempre estar cuando me necesites, para lo que sea.

Finalmente, quisiera agradecer a CONICYT por financiar esta tesis mediante la Beca Doctorado Nacional 2013, e igualmente al Centro de Modelamiento Matemático, a la Fondation Sciences Mathématiques de Paris y a EDF, France.

Contents

List of Figures

Chapter 1

Introduction

The Principal-Agent problem arises when an individual or entity (she, the Principal) wants to hire another one (he, the Agent) to do some work or action on her behalf. When working, the Agent performs one of several possible actions. By doing so, he incurs into a personal cost and an outcome which benefits the Principal is generated (possibly in a non-deterministic way). Since both parties are selfish and have opposite interests, the Principal offers a contract to the Agent which specifies a compensation depending on the result of the work. The Agent can accept or reject the contract, depending on the benefits he expects to obtain from it. In a competitive setting, it is considered that the Agent has an outside option so he will not accept any contract which reports him less benefits than what he could get by making use of this option. The problem of the Principal consists in designing a contract which will be accepted by the Agent and under which his work is expected to produce the best outcome for her.

The situation just described fits into a great number of economic interactions that take place everyday in real life. Therefore the Principal-Agent model can be applied to formally study such interactions. Just to mention some of them, the model has natural applications into agency problems, delegated portfolio management and project selection. Apart from the situation of a boss hiring an employee, the model can also be applied into the study of insurance contracts, derivatives design, electricity tarification, pollution regulators and even into non-economic interactions such as the one between the voters and a candidate in a election. [1]

An important feature of the Principal-Agent model is the asymmetry of information between the two parties. There are three main cases which are studied in the literature. In the risk-sharing or first-best problem, the Principal can observe and contract the action performed by the Agent. The Principal can freely choose what action will be performed by the Agent and she only takes care of remunerating the Agent in a way which provides him more benefits than his outside option. In the moral hazard problem, the action performed by the Agent is

[1] See for instance the works of De Marzo and Sannikov [35], Cvitanić, Possamaï and Touzi [26], Cadenillas, Cvitanić and Zapatero [21], Zeckhauser [134], Carlier, Ekeland and Touzi [24], Aïd, Possamaï and Touzi [1], section 2.5 of this introduction and Banks and Sundaram [5] respectively.

not observable or non-contractible. In this case the Principal can only observe the outcome of the work and she cannot deduce merely from this information what action the Agent did. If the Principal wants the Agent to perform a specific action she must provide incentives through the contract in order to align both their interests. Compared to the risk-sharing problem, the Principal faces an additional constrain which results in a loss of utility for her. For this reason the moral hazard problem is also referred to as the second-best problem. In the adverse selection case, the Agent possesses private characteristics (such as his ability for the work) which can be resumed on a type. The Principal does not know the type of the Agent, but only the distribution of types in the population. In this case the Principal offers a menu of contracts to the Agent, one for each type, and then the Agent reveals his type and accepts or rejects the corresponding contract. Since the Agent is not forced to reveal his true type, the Principal has to provide incentives through the menu of contracts to make sure the Agent will not lie (this could happen if a contract designed for a certain type is much better than the contract designed for the real type of the Agent). This constrain supposes another loss of utility for the Principal and for this reason the problem including both moral hazard and adverse selection is also called the third-best problem.

If the work of the Agent consists in performing several actions, each one of them producing an outcome, then time plays a crucial role in the model. Two classes of Principal-Agent models can be distinguished, the one with models in which time is continuous and the one with models in which time is considered as a discrete variable. The latter class include also the static models in which the Agent acts only once and a single outcome is generated.

Discrete–time models are the result of a game theoretic analysis of the relationship between the Principal and the Agent. They have a simple and natural formulation but they suffer the disadvantage of being quite hard to solve in general. In non-static problems, the optimality conditions obtained are recursive systems of equations which require advanced computational algorithms to be solved. This fact makes difficult to study the dynamic properties of the contract as well as its dependence on the contractual environment. On the other side, the continuous–time framework allows to use classic tools from stochastic control and calculus of variation which render the continuous–time models more tractable. Different approaches can be used to solve the problems, based mainly on the martingale representation theorem, the stochastic maximum principle, and the dynamic programming principle. As soon as the problem of the Principal is related to a Hamilton-Jacobi-Bellman equation, the optimal contract can be fully studied. In some cases by solving explicitly the equation and in others by approximating it numerically, where a deep literature on the subject is available.

In the rest of the introduction, a detailed description of the moral hazard problem and the adverse selection problem is given. For the moral hazard problem, the discrete–time and continuous–time settings are considered separately. The standard models and main results in the literature are presented. Also, a contribution from this Thesis to the continuous–time setting is introduced. In the adverse selection problem, two particular models and approaches in the literature are presented, with the purpose of introducing related applications in economics which are part of this Thesis.

1 Moral hazard in the Principal-Agent problem

1.1 Discrete-time model

The Principal-Agent problem with moral hazard caught the attention of economists in the 1970s, in the context of equilibrium theory under uncertainty. Arrow [4] and Pauly [88] were the first who pointed out that unobservable behaviour by insured persons could reduce the efficiency of the economy. However, they did not provide any mathematical model for the economic behaviour of insured persons. Early models for insurance contracts under moral hazard were those of Zeckhauser [134] and Spence and Zeckhauser [120], where solutions were computed using first-order optimality conditions.

Mirrlees [71, 72] introduced a satisfactory formulation of the general moral hazard problem, extended later by Hölmstrom [52], which constituted the starting point of the rigorous study of the Principal-Agent problem by the economical community. This basic model was referred to as the single time model, since it considered the situation where the Agent acts only once and a single outcome is generated. In game theory terms, the single time model corresponds to a Stackelberg game in which the Principal is the leader and the Agent is the follower. There is a vast literature on the single time model including, among others, the works of Shavell [113], Grossman and Hart [47], Rogerson [105] and Jewitt [55].

When the single time model was well understood, a dynamic version of it was necessary. In many real life situations in which the Principal-Agent model is suitable, the interaction between both parties continues over time. Based on the repeated games theory, the natural extension was then a repeated version of the single time model. There is a large literature of the repeated Principal-Agent problem, including Chiappori, Macho, Rey and Salanié [25], Fudenberg, Hölmstrom and Milgrom [45], Hölmstrom and Milgrom [53], Lambert [60], Malcomson and Spinnewyn [62], Radner [96], Rey [99], Rey and Salanie [100], Rogerson [104] and Spear and Srivastava [118] among others. In the dynamic problem new questions were of interest, such as the role memory, savings and the commitment of the Agent in the optimal contract. A question of practical importance was if the optimal contract of the repeated model could be replicated by a sequence of single time contracts. This question is related to the concept of perfect equilibria in repeated games. Chiappori, Macho, Rey and Salanié [25] collected the main results on these aspects in the literature and presented some extensions.

1.1.1 Single time model

In the single time model, the Principal hires the Agent to perform a single action which will generate an outcome. The model for the moral hazard problem, introduced by Mirrless [71, 72] and extended by Hölmstrom [52], is presented next. Denote by $A \subset \mathbb{R}$ the set of actions of the Agent. For every action $a' \in A$ chosen by the Agent, the outcome x is a random variable taking values in $X \subset \mathbb{R}$, with distribution function $F(\cdot, a')$ and density $f(\cdot, a')$. The Principal does not observe what action is chosen by the Agent but only the outcome. The Principal will offer to the Agent a payment rule w, depending on the outcome of his work, and she will also recommend the Agent to perform a certain action a. The Principal benefits from the outcome of the project through his utility function U_P, which depends also on the payments she provides to the Agent. The Agent possesses utility function U_A, which depends on the payments he receives and the action he performs. The Agent is not forced to accept the contract and he has an outside option which provides him utility R_0, this quantity is called the reservation utility of the Agent.

Since the interaction between both parts is sequential (first the Principal offers the contract to the Agent and then the Agent accepts it or rejects it and eventually works), the action chosen by the Agent depends directly on the contract offered by the Principal. From the game theory point of view, this means that the Principal and the Agent play a Stackelberg game. Therefore, the Principal can anticipate the reaction of the Agent and take it into account when he is designing the contract he will offer.

Mathematically, the problem of the Principal consists in finding the most convenient payment rule and recommended action, solutions to the following optimization problem.

$$(P)\left\{\begin{array}{lll} \underset{w(\cdot),a}{\text{maximize}} & \int_X U_P(x,w(x))f(x|a)\mathrm{d}x & \\ \text{s.t.} & \int_X U_A(w(x),a)f(x|a)\mathrm{d}x \geq R_0, & \text{(IR)} \\ & a \in \underset{a'\in A}{\text{argmax}} \int_X U_A(w(x),a')f(x|a')\mathrm{d}x. & \text{(IC)} \end{array}\right.$$

The above formulation corresponds to the moral hazard problem faced by the Principal in a single time setting. Constrain (IR) is called the individual rationality constrain and represents the fact that the Agent will accept only contracts which provide him more utility than his reservation utility R_0. Constrain (IC) is referred to as the incentive compatibility constrain and it states the fact that the Agent acts according to his own benefit. Given any contract offered by the Principal, since the action performed by the Agent is not observable, he will simply choose the one which provides him the highest expected utility. Consequently, when the Principal recommends an action, she must make sure that the Agent will not find another one under which he can obtain more utility.

Moral hazard is present because the Principal can not observe the action of the Agent. In case she could do so, she would design a forcing contract guaranteeing that the Agent selects the most convenient action for her, even if such an action does not maximize the Agent's utility. In that case the Principal would face the following first-best problem, in which (IC) constrain is omitted.

$$(P_{FB})\left\{\begin{array}{lll} \underset{w(\cdot),a}{\text{maximize}} & \int_X U_P(x,w(x))f(x|a)\mathrm{d}x & \\ \text{s.t.} & \int_X U_A(w(x),a)f(x|a)\mathrm{d}x \geq R_0. & \text{(IR)} \end{array}\right.$$

In the described single time model, three main aspects arise and are subject of study in the literature.

- Existence of solutions to (P).
- Characterization of solutions to (P).
- Comparison between the value of (P) and the value of (P_{FB}).

While the importance of the first two points is evident in a rigorous study of the moral hazard problem, the last aspect can be understood as the main purpose of the model, since it seeks to quantify the loss of utility suffered by the Principal due to her inability to observe the Agent's actions. The difference between the value of the first-best problem and the value of the moral hazard problem is the resulting loss of efficiency considered by Arrow [4] and Pauly [88].

It will be seen that the mentioned aspects are very connected and they are frequently studied under similar assumptions in the literature. Whether problem (P) is well–posed or not, *i.e.* it possesses a solution which can be characterized, will generally determine if its value is strictly inferior to the value of problem (P_{FB}). In the next sections, the main results for each one of these points are detailed.

1.1.1.1 Characterization of solutions: First-order approach

The main tool for the characterization of the solutions to problem (P) is the so–called first-order approach, which will be explained in this section. For the sake of simplicity, we assume that outcome is monetary, the utility function of the Principal depends only on his wealth and the utility function of the Agent is additively separable in consumption (earned money) and effort (chosen action). These assumptions are common in the literature and are present in most of the applications of the single time model. Thus, by abusing the notation, the utility functions have the following form

$$U_P(x, w(x)) = U_P(x - w(x)),\ U_A(w(x), a) = U_A(w(x)) - c(a),$$

where $c : A \longrightarrow \mathbb{R}_+$ is the cost function of the Agent for performing actions. In this case the moral hazard problem corresponds to

$$(P) \left\{ \begin{array}{cl} \underset{w(\cdot),a}{\text{maximize}} & \displaystyle\int_X U_P(x - w(x)) f(x|a) \mathrm{d}x \\ \text{s.t.} & \displaystyle\int_X U_A(w(x)) f(x|a) \mathrm{d}x - c(a) \geq R_0, \qquad \text{(IR)} \\ & a \in \underset{a' \in A}{\text{argmax}} \displaystyle\int_X U_A(w(x)) f(x|a') \mathrm{d}x - c(a'). \quad \text{(IC)} \end{array} \right.$$

Problem (P) is a non-standard optimization problem, the main difficulty of it being the variational inequality constrain (IC). For non-trivial sets A of admissible actions, the (IC) constrain represents a continuum of inequalities and therefore problem (P) cannot be solved by means of the Karush-Kuhn-Tucker multipliers. A common way of avoiding this difficulty is the so–called first-order approach, in which the utility maximizing requirement for the recommended action is relaxed and instead only a stationary condition is asked. More precisely, the following relaxed problem is solved instead of (P)

$$(RP)\begin{cases} \underset{w(\cdot),a}{\text{maximize}} & \int_X U_P(x-w(x))f(x|a)\mathrm{d}x \\ \text{s.t.} & \int_X U_A(w(x))f(x|a)\mathrm{d}x - c(a) \geq R_0, \quad \text{(IR)} \\ & \int_X U_A(w(x))f_a(x|a)\mathrm{d}x - c'(a) = 0. \quad \text{(RIC)} \end{cases}$$

where the relaxed incentive compatibility condition (RIC) replaces the original constrain (IC). If the set A is open, the solutions to problem (P) are feasible points in problem (RP). However, both problems have different values in general and their solutions do not coincide. In such a case, the first-order approach is not valid. Mirrlees [71] was the first to give importance to this fact, by showing examples where the solution to problem (RP) is not even a feasible point in (P). Nevertheless, a substitute of the first-order approach was not proposed and effort was put into finding classes of problems for which the solutions to problems (P) and (RP) are the same. Mirrlees [73] introduced two conditions under which the first-order approach should be valid. Assume that for every $a \in A$ that the density $f(\cdot, a)$ is of class $\mathcal{C}^1(X)$, we say the monotone likelihood ratio condition (MLRC) is satisfied if it holds

$$x \longmapsto \frac{f_a(x|a)}{f(x|a)} \text{ is strictly increasing, } \forall a \in A. \tag{MLRC}$$

The convexity of the distribution function condition (CDFC) is defined as follows

$$F_{aa}(x|a) \geq 0, \ \forall x \in X, \forall a \in A. \tag{CDFC}$$

Rogerson [105] proved that if (MLRC) and (CDFC) are satisfied, the solutions of the relaxed problem (RP) and the moral hazard problem (P) do coincide. The advantage of the relaxed problem is that it can be easily solved. Point-wise optimization of the associated Lagrangian provides the following characterization of the optimal pair (w, a)

$$\frac{U_P'(x-w(x))}{U_A'(w(x))} = \lambda + \mu\frac{f_a(x,a)}{f(x,a)}, \tag{1.1}$$

where $\lambda \geq 0$ and $\mu \in \mathbb{R}$ are the associated KKT multipliers. Moreover, second order optimality conditions imply that $\mu \geq 0$.[2] Consequently, under the natural assumption that U_P and U_A are concave functions, condition (CDFC) implies that the optimal sharing rule is increasing in output, a pleasing result which follows the intuition.

1.1.1.2 Existence of solutions

Mirrlees [71] constructed an example of a moral hazard problem with no solution, in which no contract attains the value of problem (P). This construction is based on the unboundedness of the likelihood ratio $\frac{f_a}{f}$ and the utility function of the Agent U_A.

[2] Rogerson [105] works with a different doubly-relaxed problem in which the relaxation of the (IC) constrain is an inequality and the multiplier associated to it is immediately non-negative.

To deal with Mirrlees's example it is generally assumed that the set of outcomes X is compact and for every action $a \in A$ the density $f(\cdot, a)$ is a strictly positive function of class $\mathcal{C}^2(X)$. If, in addition, proper integrability conditions are imposed on the utility function of the Agent U_A and the class of feasible contracts is restricted by imposing point-wise boundedness, it can be proved that problem (P) admits a solution. (See Hölmstrom [52] and Jewitt, Kadan and Swinkels [56]).

If no constrains on the set of feasible contracts are desired, then the existence of solutions to (P) depends on the utility function of the Agent. If U_A is bounded below, it was proved by Jewitt, Kadan and Swinkels [56] that a solution to the moral hazard problem exists. Moroni and Swinkels [76] addressed the existence of solutions when U_A is unbounded below. They identify two cases: if the utility of the Agent diverges when the consumption of the Agent goes to $-\infty$ then a solution exists if and only if the following (ARA) condition holds

$$\lim_{w \downarrow -\infty} \frac{-U_A(w)}{U_A'(w)} + w = -\infty. \tag{ARA}$$

In case U_A diverges at a finite level of consumption, it is possible to construct examples where problem (P) does not have a solution, by choosing different cost functions c and values of R_0.

1.1.1.3 Cost of moral hazard

The risk-sharing problem has been widely studied in economics. Early works on this subject can be found, among others, in Arrow [3], Borch [17] and Wilson [131]. With the introduction of the moral hazard problem, it arose naturally the question of quantifying the loss of utility suffered by the Principal, due to the unobservable behaviour of the Agent. It was shown by Mirrlees [71] that in some cases the Principal is not significantly affected by the self-interested behaviour of the Agent, since she can offer incentive compatible contracts to the Agent which provide her utility as close as desired to the value of the first-best problem. In such examples, the moral hazard problem does not have a solution and the value of the second-best problem is equal to the value of the first-best problem. However, these examples are based on the unboundedness of the likelihood ratio and this situation is in general ruled out, as explained in the previous section, precisely for avoiding the non-existence of solutions to the moral hazard problem.

In the single time setting, under the assumptions that outcome is monetary and the utility function of the Agent is additively separable, problem (P_{FB}) takes the form

$$(P_{FB}) \begin{cases} \underset{w(\cdot),a}{\text{maximize}} & \displaystyle\int_X U_P(x - w(x)) f(x|a) \mathrm{d}x \\ \text{s.t.} & \displaystyle\int_X U_A(w(x) f(x|a) \mathrm{d}x - c(a) \geq R_0. \quad \text{(IR)} \end{cases}$$

The above problem is a very simple standard optimization problem with an inequality constrain. The optimality condition associated to it is the following Borch's rule

$$\frac{U_P'(x - w(x))}{U_A'(w(x))} = \lambda, \tag{1.2}$$

where $\lambda \geq 0$ is the associated Lagrange multiplier. Comparing this equation with (1.1), the difference lies in the multiplier $\mu \geq 0$ which can make the right-hand side of (1.1) non-constant. If $\mu > 0$ then necessarily problem (P) and (P_{FB}) have different solutions and therefore the value of the latter problem is strictly greater than the value of the former one. Different approaches can be used for proving that μ is indeed positive, depending on the assumptions of the model. For instance, Hölmstrom [52] concluded this fact from the validity of the first–order approach, while Rogerson [105] argued that (MLRC) is sufficient in the case where the Principal is risk-neutral. Jewitt [55] gave a direct argument which proves the result in the case of a risk-neutral Principal neither assuming (MLRC) nor (CDFC).

1.1.2 Repeated model

From the game theory perspective, the natural extension of the single time model is the repeated version of the game played by the Principal and the Agent.

In the repeated moral hazard problem there are T periods of time, indexed by $t \in \{1, \dots, T\}$. The Principal hires the Agent for performing T actions, one during each period. Every action a_t generates an outcome x_t which is a real random variable that depends exclusively on the action performed during the current period. At this point, the Principal can observe only the outcomes $X_t = (x_1, \dots, x_t)$ of the project produced so far, but not the actions $(a_1, \dots, a_t)$ chosen by the Agent. After observing X_t the Principal remunerates the Agent with some payment rule w_t and the Agent consumes. If the Agent has access to a credit line he may decide to save an amount of money s_t at every period and his consumption will be given by $c_t = w_t + s_{t-1} - s_t$. If the Agent does not have access to a credit line his consumption is simply $c_t = w_t$ at every t.

For simplicity it will be assumed that the utility functions of the Principal and the Agent are separable in time and they are equal to the undiscounted sum of some within-period utility functions. The problem faced by the Principal in this case is

$$(P_T)\begin{cases} \underset{\{(w_t, a_t, s_t, \widetilde{s}_t)(\cdot)\}}{\text{maximize}} & \mathbb{E}\left[\sum_{t=1}^{T} U_P\left(x_t - w_t(X_t) + \widetilde{s}_{t-1}(X_{t-1}) - \widetilde{s}_t(X_t)\right)\right] \\ \text{s.a} & \mathbb{E}\left[\sum_{t=1}^{T} U_A\left(w_t(X_t) + s_{t-1}(X_{t-1}) - s_t(X_t), a_t(X_{t-1})\right)\right] \geq R_0, \quad (IR) \\ & \{(a_t, s_t)\} \text{ maximizes} \\ & \qquad \mathbb{E}\left[\sum_{t=1}^{T} U_A\left(w_t(X_t) + s_{t-1}(X_{t-1}) - s_t(X_t), a_t(X_{t-1})\right)\right], \quad (IC) \end{cases}$$

where $\widetilde{s}_t$ is the amount of money saved by the Principal at time t. A solution to this problem will be referred to as an optimal long–term contract.

The introduction of time in the moral hazard problem arises new questions which are inherent to repeated games. One of them is the presence of memory in the optimal long–term contract, *i.e.* whether the payments w_t, actions a_t and consumptions c_t depend actually on the previous outcomes X_{t-1} or not. A second interesting question is whether the optimal long–term contract is implementable via single time contracts, *i.e.* there exists a perfect Bayesian equilibrium of the repeated single time game, the outcome of which replicates the long–term contract. The importance of this point is related to the commitment of the Agent to the contract. Indeed, it may be too costly (illegal) to design (enforce) a contract which covers the full length of the relationship and cannot be broken by the Agent at any time. It is important therefore to study in which situations the Principal and the Agent obtain more benefits by committing themselves to a long–term contract than by simply negotiating a single time contract at the beginning of every period.

Chiappori, Macho, Rey and Salanié [25] was a unifying paper which presented several results from the literature in repeated moral hazard and also extended some of them. In this work it was shown that the answer to the previous questions depends mainly on what access to credit has the Agent. They studied three different cases, the one with free access to credit (as in (P_T)), the case in which the Agent does not have access to credit ($s_t = 0$) and the case in which his savings are monitored by the Principal (s_t is not a maximizer in equation (IC)).

1.1.2.1 Presence of memory

Lambert [60] and Rogerson [104] proved that when the Agent has no access to credit, the optimal long–term contract exhibits memory of consumption (equivalently memory of payments). If during certain period the outcome of the project affects the payment received by the Agent, then in the next period his consumption and salary will also do. As shown by Malcomson and Spinnewyn [62], the presence of memory is just a consequence of intertemporal smoothing between consecutive periods.

To illustrate this, consider $T = 2$ and there are N possible outcomes $\{x_1, \dots, x_N\}$ with probabilities $\{p_k(\cdot)\}_{k=1}^N$ induced by the Agent's actions. Denote by a_0 the action of the Agent in period 1 and by a_j the action in period 2, given that the outcome in period 1 is x_j. Call w_i the payment in period 1 if outcome x_i occurs and w_{ij} the payment in period 2 if x_i occurs in period 1 and x_j occurs in period 2. Then, if U_A is separable in money and effort the following optimality condition is satisfied

$$\frac{U_P'(x_j - w_j)}{U_A'(w_j)} = \sum_{k=1}^{N} p_k(a_j) \frac{U_P'(x_k - w_{jk})}{U_A'(w_{jk})}, \ j \in \{1, \dots, N\}. \tag{1.3}$$

Chiappori, Macho, Rey and Salanié [25] studied the case in which the Agent has free access to credit and the case when his savings can be monitored by the Principal. They showed that in both cases memory of consumption is also present due to intertemporal smoothing. Concerning the memory of payments, it depends on the presence of wealth effects in the utility function of the Agent. For instance, in the particular case of CARA functions it disappears.

1.1.2.2 Implementability of the long–term contract

The conclusions of Chiappori, Macho, Rey and Salanié [25] were disappointing because in the most interesting cases of access to credit, the long–term optimal contract cannot be implemented by single time contracts.

Consider first the case when the Agent has access to credit, but he can be monitored by the Principal. This setting can be reduced to an equivalent situation in which the Agent has no access to credit and only the Principal does. Rey [99] extended a result of Rogerson [104], for the case of no access to credit, and showed that if the Principal is risk-neutral and can monitor the Agent's savings, then under the optimal long–term contract the Agent will always want to save more money than what he is allowed to. This result is problematic, since a situation of constrained savings is not appropriated for many applications. Therefore, it renders the case of constrained savings very unattractive.

It is shown in Chiappori, Macho, Rey and Salanié [25] that the optimal long–term contract cannot be implemented by single time contracts if the Agent does not have access to credit or if he has free access which cannot be monitored by the Principal. On the other side, if the Agent's access to credit can be monitored, then the implementation is possible. These results were distressing, since implementability was possible only in the least realistic models. To quote Chiappori, Macho, Rey and Salanié [25] "The analysis of repeated moral hazard therefore seems to lead to a dead-end in the general case". More complex analysis of the problem, or a new model, was needed.

1.2 Continuous-time model

The main drawback of discrete–time models was that the resulting problems were in general quite hard to solve. One of the motivations to move to the continuous–time setting was to construct tractable models, by making use of the technical advantages of stochastic calculus and stochastic control.

The continuous–time Principal–Agent literature started with the work of Hölmstrom and Milgrom [53]. In their model, the Agent is hired by the Principal to control the drift of a diffusion process, the output, which may be understood as the value of a firm. The Principal observes only the output process, so the actions of the Agent are hidden. They studied the case in which the Principal and the Agent have exponential utilities, where the main result is that the optimal contract turns out to be a linear function of the output. This simple setting was generalized by Schättler and Sung [111, 112], Sung [122, 123] and Müller [77, 78] among others.

There are different approaches to the moral hazard problem in continuous–time. Williams [129] and Cvitanić, Wan and Zhang [30] used the stochastic maximum principle to characterize the optimal contract via coupled systems of Forward-Backward Stochastic Differential Equations (FBSDEs). Sannikov [109] used the martingale representation theorem to write the continuation utility of the Agent as a controlled process and then found heuristically a Hamilton-Jacobi-Bellman equation associated to the problem of the Principal. The former method is rigorous and very general but it results in difficult systems of FBSDEs where little can be said about the properties of the optimal contract. The latter method is tractable, but not justified rigorously. The methodology is an extension of the one used by Spear and Srivastava [118] in the discrete–time model. To solve the problem of the Principal, her continuation utility is assumed to be a deterministic function of the continuation utility of the Agent, which leads to an HJB equation.

Recently, Cvitanić, Possamaï and Touzi [28] provided a general, tractable and rigorous method for solving continuous-time Principal-Agent problems in finite horizon when the Agent is in charge of controlling both the drift and the volatility of the outcome. Their so–called dynamic programming approach, consists in restricting the set of contracts offered to the Agent to a convenient class, in which the problem of the Principal is reduced to a standard stochastic control problem and can be associated to a HJB equation. The authors proved that their restriction of the class of contracts is without loss of generality by making use of the theory of second-order BSDEs (2BSDEs), introduced by Soner, Touzi and Zhang [116] and extended by Possamaï, Tan and Zhou [93].

In the next sections, a description of the Principal-Agent models in continuous–time is given. First, the drift control model is presented together with the two most common approaches in the literature to solve it. Second, the recent extension of the model to the volatility control case and the dynamic programming approach of Cvitanic, Possamaï and Touzi [28] is presented. Finally, one of the works of this thesis, a Principal-Agent model under volatility uncertainty, is introduced. The model is studied in detail in Part II, Chapter 4 of this document.

1.2.1 Drift control: the model

The moral hazard in continuous–time was introduced in Hölmstrom and Milgrom [53] by considering that the Agent controls the drift of a diffusion, the outcome process, which is the only process observable by the Principal. This situation is modelled using the weak formulation of a controlled SDE, which is presented next.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W a one–dimensional Brownian motion on it. Let $\mathbb{F} := (\mathcal{F}_t^W)_{t\in[0,T]}$ be the filtration induced by W. Consider the volatility process σ, which is an $\mathbb{F}-$adapted process, square integrable and non-null. Define the outcome process by

$$X_t = \int_0^t \sigma_s \mathrm{d}W_s.$$

An $\mathbb{F}-$adapted process α is said to be admissible, denoted by $\alpha \in \mathcal{A}$, if it takes values in some set $A \subset \mathbb{R}$ and the process M defined below is an $(\mathbb{F}, \mathbb{P})-$uniformly integrable martingale.

$$M_t^\alpha = \mathcal{E}\left(\int_0^t \alpha_s \mathrm{d}W_s\right).$$

From Girsanov's theorem we have that

$$X_t = \int_0^t \alpha_s \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s^\alpha,\ \mathbb{P}^\alpha - \text{a.s.}, \tag{1.4}$$

where $\frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} = M_T^\alpha$ on $\mathcal{F}_T$ and $W^\alpha := W - \int_0^\cdot \alpha_s \mathrm{d}s$ is a $\mathbb{P}^\alpha-$Brownian motion.

The Principal wants to hire the Agent to be in charge of controlling the distribution of the outcome process, by choosing properly the process α in (1.4). Due to the presence of moral hazard, the Principal can observe merely X and not the actions of the Agent. To compensate his work, the Principal offers to the Agent continuous payments χ and a terminal remuneration ξ which are respectively an $\mathbb{F}-$adapted process and an $\mathcal{F}_T-$measurable random variable. Such a pair of (χ, ξ) are referred to as contracts.

Given a contract (χ, ξ) offered by the Principal, if the Agent accepts to work and chooses the control $\alpha \in \mathcal{A}$ his expected utility is given by

$$u_0^A(\chi, \xi, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\left[U_A(\xi) + \int_0^T (u_A(s, X, \chi_s) - c(s, X, \alpha_s))\mathrm{d}s\right],$$

where the real maps u_A and U_A are respectively his instantaneous and terminal utility functions and c is the cost function of his actions. Naturally, the Agent chooses the control $\alpha \in \mathcal{A}$ which maximizes his expected utility, which makes him face the following optimization problem

$$V^A(\chi, \xi) := \sup_{\alpha \in \mathcal{A}} u_0^A(\chi, \xi, \alpha). \tag{1.5}$$

Denote by $\mathcal{A}^\star(\chi, \xi)$ the set of optimal controls of the Agent when he is offered the contract (χ, ξ), *i.e.* the solutions to the previous problem. It is assumed that if this set has more than one element, the Agent chooses the control which benefits most the Principal. The Principal can anticipate the responses of the Agent to the contracts and faces the problem

$$V^P := \sup_{(\chi,\xi)\in\Xi} \sup_{\alpha\in\mathcal{A}^\star(\chi,\xi)} \mathbb{E}^{\mathbb{P}^\alpha}\left[U_P(\xi) + \int_0^T u_P(s, X, \chi_s)\mathrm{d}s\right], \tag{1.6}$$

where the real maps u_p and U_P are respectively the instantaneous and terminal utility function of the Principal and Ξ is the set of pairs (χ, ξ) sufficiently integrable, such that $\mathcal{A}^\star(\chi, \xi) \neq \emptyset$ and $V^A(\chi, \xi) \geq R_0$. The latter conditions are imposed because if the Principal wants to effectively anticipate the response to the contracts of the Agent, he must be able to optimally choose his most convenient actions. Moreover, as already explained in the discrete–time model, the Agent accepts only contracts which provide him more utility than his reservation value R_0.

Different approaches can be followed to solve both problems (1.5) and (1.6) above. In the following sections the main two of them are described, namely, the martingales and dynamic programming approach introduced by Sannikov [109] and the stochastic maximum principle approach introduced by Williams [129] and Cvitanić, Wan and Zhang [30].

1.2.1.1 Martingale and dynamic programming approach

In this section the approach initiated by Sannikov [109] is presented, with a small variant when solving the problem of the Agent. The difference and the reason to do so are detailed below.

Fix a contract $(\chi, \xi) \in \Xi$. For $t \in [0,T]$, define the continuation value of the Agent by

$$V_t^A(\chi, \xi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha} \left[U_A(\xi) + \int_t^T (u_A(s, X, \chi_s) - c(s, X, \alpha_s)) \mathrm{d}s \;\middle|\; \mathcal{F}_t \right].$$

Consider next the continuation utility of the Agent if he chooses the action $\alpha \in \mathcal{A}$

$$u_t^A(\chi, \xi, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[U_A(\xi) + \int_t^T (u_A(s, X, \chi_s) - c(s, X, \alpha_s)) \mathrm{d}s \;\middle|\; \mathcal{F}_t \right].$$

Observe that the process $u_t^A(\chi, \xi, \alpha) + \int_0^t (u_A(s, X, \chi_s) - c(s, X, \alpha_s)) \mathrm{d}s$ is an $(\mathbb{F}, \mathbb{P})-$martingale. Then, by the martingale representation theorem there exists a process $Z^{A,\alpha}$ such that

$$u_t^A(\chi, \xi, \alpha) = U_A(\xi) + \int_t^T \left(u_A(s, X, \chi_s) - c(s, X, \alpha_s) + \alpha_s Z_s^{A,\alpha} \right) \mathrm{d}s - \int_t^T Z_s^{A,\alpha} \mathrm{d}W_s, \; \mathbb{P} - \text{a.s.}$$

Therefore, the continuation utility of the Agent under the action $\alpha \in \mathcal{A}$ is the solution to a BSDE[3] with terminal condition $U_A(\xi)$ and generator

$$f(s, X, \chi, \alpha, z) = u_A(s, X, \chi) - c(s, X, \alpha) + \alpha z.$$

The previous representation of the continuation utility of the Agent is very important because it allows to solve the problem of the Agent under mild conditions, even in the non-markovian case, by means of the comparison theorems for BSDEs. To do so, start defining the maximal generator

$$f^\star(s, X, \chi, z) = \sup_{\alpha \in A} f(s, X, \chi, \alpha, z),$$

and consider the following BSDE

$$Y_t = U_A(\xi) + \int_t^T f^\star(s, X, \chi_s, Z_s) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \; t \in [0,T], \; \mathbb{P} - \text{a.s.} \tag{1.7}$$

Under appropriate integrability conditions on the components of the model, (1.7) is well-posed and has a unique solution (Y, Z). Moreover, if there exists a maximizer function $a^\star$ such that

$$f^\star(t, X, \chi, z) = f(t, X, \chi, a^\star(t, X, \chi, z), z) \text{ and } \alpha_t^\star = a^\star(t, X, \chi_t, Z_t) \in \mathcal{A},$$

then $V_t^A(\chi, \xi) = Y_t$ for every $t \in [0,T]$ and the optimal effort of the Agent is given by $\alpha^\star$. It is worthwhile to mention that in Sannikov [109] the problem of the Agent was not solved by using the comparison results for BSDEs. Instead, a specific proof for his particular model

[3] See the seminal works of Pardoux and Peng [86, 87] and El Karoui, Peng and Quenez [39] for the definitions and main results on this subject.

was made, leading of course to the same result as the one exposed here. The BSDEs variant is presented here for two reasons: it provides a better understanding of the problem of the Agent and it serves as a first-step to the more general 2BSDEs approach, introduced in Cvitanić, Possamaï and Touzi [28], used in the case of volatility control.

Moving to the problem of the Principal, it is necessary to restrict to the markovian case. The procedure to solve the problem is heuristic and it relies in making the *ansatz* that the continuation value of the Principal is a deterministic function of the continuation value of the Agent. Then, a Hamilton-Jacobi-Bellman equation associated to the problem of the Principal can be written and studied. The continuation value of the Agent plays the role of a state variable in the optimal contract, since its components are obtained through the maximizers appearing in the HJB equation.

To illustrate this, consider Sannikov's [109] model with infinite horizon, $T = \infty$, and

$$u_A(s, X, \chi_s) = \mathrm{e}^{-rs} u_A(\chi_s), \;\; c(s, X, \alpha_s) = \mathrm{e}^{-rs} c(\alpha_s).$$

This case is very convenient since it renders the contracting problem stationary and the time variable is not present in the associated HJB equation. Recall the representation (1.7) for the continuation value of the Agent. Assuming that $V_t^P = F(V_t^A)$ for some deterministic and smooth function F leads to the following HJB equation

$$\sup_{(a,k,Z)} \left(r(a-k) - rF(V) + F'(V) r(V - u_A(k) + c(a)) + \frac{1}{2} r^2 \sigma^2 Z^2 F''(V) \right) = 0.$$

It can be proved that the optimal contract for the Principal, in terms of the continuation utility of the Agent, is given by $\alpha_t^\star = \widehat{a}^\star(V_t^A)$ and $\chi_t^\star = k^\star(V_t^A)$, where $\widehat{a}^\star(V)$ and $k^\star(V)$ are the maximizers in the previous equation.

Although the characterization of the value of the Principal through an HJB equation gives great tractability (observe that in the previous example the obtained equation is just an ODE), the described method is not very rigorous. For instance, the crucial *ansatz* is not justified properly and it appears just as an attempt to extend the method of Spear and Srivastava [118] for discrete–time problems.

1.2.1.2 Stochastic maximum principle approach

In this section the stochastic maximum principle approach, introduced by Williams [129] and Cvitanić, Wan and Zhang [30], is presented. Roughly speaking, the approach consists in solving a problem of calculus of variation, where the (IR) constrain of the problem of the Principal is treated by a KKT multiplier.

Define then, for $\lambda > 0$, the relaxed problem

$$V^P(\lambda) := \sup_{(\chi,\xi)\in\Xi} \mathbb{E}^{\mathbb{P}^\alpha} \left[U_P(\xi) + \int_0^T u_P(s, X, \chi_s) \mathrm{d}s + \lambda (V_0^A - R_0) \right].$$

Observe that defining $\lambda_t = \lambda\delta_0(t)$, where δ_0 is the Dirac function, the previous problem can be rewritten as

$$V^P(\lambda) = \sup_{(\chi,\xi)\in\Xi} \mathbb{E}^{\mathbb{P}^\alpha}\left[U_P(\xi) + \int_0^T \big(u_P(s,X,\chi_s) + \lambda_s(V_s^A - R_0)\big)\,\mathrm{d}s\right]. \tag{1.8}$$

Define now the continuation utility of the Principal under the contract (χ,ξ) by

$$V_t^{P,\xi,\chi} := \mathbb{E}^{\mathbb{P}^\alpha}\left[U_P(\xi) + \int_t^T \big(u_P(s,X,\chi_s) + \lambda_s(V_s^A - R_0)\big)\,\mathrm{d}s\,\Big|\mathcal{F}_t\right].$$

From the martingale representation theorem, we have the existence of a process $Z^{P,\xi,\chi}$ such that

$$V_t^{P,\xi,\chi} = U_P(\xi) + \int_t^T \big(u_P(s,X,\chi_s) + \lambda_s(V_s^A - R_s)\alpha_s Z_s^{P,\xi,\chi}\big)\,\mathrm{d}s - \int_t^T Z_t^{P,\xi,\chi}\mathrm{d}W_t,\ \mathbb{P}-\text{a.s.} \tag{1.9}$$

The relaxed problem (1.8) corresponds then to a stochastic control problem of a 2-dimensional BSDE, given by (1.9) and (1.7), and with controls (χ,ξ). Optimality conditions to this problem can be obtained by applying the Stochastic Maximum Principle, which take the form of a coupled system of Forward-Backward stochastic differential equations (FBSDEs). For notational convenience, suppress all the dependences on the outcome X. Define then the maps

$$I_A = (c')^{-1},\ I_P^1 = \left(-\frac{U_P'}{U_A'}\right)^{-1},\ I_P^2 = \left(-\frac{u_P'}{u_A'}\right)^{-1}.$$

Under mild conditions, the continuation values of the Principal and the Agent can be obtained as solutions to the following coupled system of FBSDEs, where $D^\star$ is an adjoint process

$$\begin{aligned}
D_t^\star &= \lambda + \int_0^t Z_s^{P,\star} I_A'(s, Z_s^{A,\star})\big(\mathrm{d}W_s - I_A(s,Z_s^{A,\star})\mathrm{d}s\big).\\
V_t^{A,\star} &= U_A(I_P^1(D_T^\star)) + \int_t^T \big(u_A(s, I_P^2(s,D_s^\star)) - c(s, I_A(s,Z_s^{A,\star}))\big)\,\mathrm{d}s\\
&\quad - \int_t^T Z_s^{A,\star}\big(\mathrm{d}W_s - I_A(s,Z_s^{A,\star})\mathrm{d}s\big).\\
\tilde{V}_t^{P,\star} &= U_P(I_P^1(D_T^\star)) + \int_t^T u_P(s, I_P^2(s,D_s^\star))\mathrm{d}s - \int_t^T Z_s^{P,\star}\big(\mathrm{d}W_s - I_A(s,Z_s^{A,\star})\mathrm{d}s\big).
\end{aligned}$$

Moreover, the solution to the relaxed problem of the Principal is given by

$$\xi^\star = I_P^1(D_T^\star),\ \chi_t^\star = I_P^2(t, D_t^\star).$$

Although this approach is very rigorous and the methodology to follow is clear, it has the drawback that the optimality conditions obtained can be quite hard to solve. Even for the simple model presented in this section, one should solve a non-trivial coupled system of FBSDEs and the situation gets much more complicated as soon as one looks for more generality. Unfortunately, the stochastic maximum principle approach seems to be as hard to apply as the solutions to discrete–time models, considering that one of the motivations to move to the model in continuous–time was to obtain more tractability.

1.2.2 Volatility control: the model

Let $T > 0$ be the horizon of the project and d, a positive integer, be the dimension of the outcome process. Let $\Omega := \{\omega \in C([0,T], \mathbb{R}^{\mathrm{d}}) : \omega_0 = 0\}$ be the canonical space of continuous maps from $[0,T]$ into $\mathbb{R}^{\mathrm{d}}$, endowed with the uniform norm $\|\omega\|_\infty = \sup_{t\in[0,T]} \|\omega_t\|$. Denote by X the canonical process on Ω, representing the outcome of the project, *i.e.* $X_t(x) = x_t$, for all $x \in \Omega$ and $t \in [0,T]$. Set $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by X.

A control process is a pair (α, ν) of $\mathbb{F}$-adapted processes taking values in $A \times N$, where A and N are subsets of a finite dimensional space. The controlled drift coefficient $b : [0,T] \times \Omega \times A \times N \longrightarrow \mathbb{R}^n$ is bounded and satisfies that $b(\cdot, a, n)$ is an $\mathbb{F}$-progressively measurable process for every $(a,n) \in A \times N$. The controlled volatility coefficient $\sigma : [0,T] \times \Omega \times N \longrightarrow \mathcal{M}_{\mathrm{d},n}(\mathbb{R})$ is uniformly bounded and such that $\sigma\sigma^\top(\cdot, n)$ is an invertible $\mathbb{F}$-progressively measurable process for any $n \in N$.

The controlled state equation is defined from the following SDE

$$X_s^\nu = \int_0^s \sigma(r, X^\nu, \nu_r)\mathrm{d}W_r, \; s \in [0,T]. \tag{1.10}$$

A weak solution to (1.10) is a pair $(\mathbb{P}, \nu)$ such that the processes

$$X_\cdot \text{ and } X_\cdot X_\cdot^\top - \int_0^\cdot \sigma(r, X, \nu_r)\sigma^\top(r, X, \nu_r)\mathrm{d}r,$$

are $(\mathbb{P}, \mathbb{F})-$martingales on $[0,T]$. For such a weak solution $(\mathbb{P}, \nu)$ and every control α it follows from Girsanov's theorem that

$$X_s = \int_0^s b(r, X, \alpha_r, \nu_r)\mathrm{d}r + \int_0^s \sigma(r, X, \nu_r)\mathrm{d}W_r^\alpha, \; s \in [0,T], \; \mathbb{P}^\alpha - a.s., \tag{1.11}$$

where W^α is a $\mathbb{P}^\alpha-$Brownian motion and the measure $\mathbb{P}^\alpha$ is defined by

$$\frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_0^T \sigma^\top(\sigma\sigma^\top)^{-1}(s, X, \nu_s)b(s, X, \alpha_s, \nu_s) \cdot \mathrm{d}W_s^{\mathbb{P}}\right),$$

for a $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$.

Due to the presence of moral hazard, the Principal can observe merely the outcome process X and not the actions of the Agent. Therefore, as explained in the case of drift control, a contract consists in a pair (χ, ξ) of continuous payments χ which is an $\mathbb{F}-$adapted process and a terminal payment ξ which is an $\mathcal{F}_T-$measurable random variable.

Let $k : [0,T] \times \Omega \times A \times N \longrightarrow \mathbb{R}$ be a discount factor and $c : [0,T] \times \Omega \times A \longrightarrow \mathbb{R}$ be the cost function of the Agent. Denote by $u_A : [0,T] \times \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ the instantaneous utility function of the Agent and by $U_A : \mathbb{R} \longrightarrow \mathbb{R}$ his utility function at the horizon T. The objective function of the Agent is given by

$$u_0^A(\chi, \xi, \mathbb{P}, \alpha, \nu) := \mathbb{E}^{\mathbb{P}^\alpha}\left[\mathcal{K}_{0,T}^{\alpha,\nu} U_A(\xi) + \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu}(u_A(s, X, \chi_s) - c(s, X, \alpha_s))\mathrm{d}s\right],$$

where

$$\mathcal{K}_{s,t}^{\alpha,\nu} := \exp\left(-\int_s^t k(u, X, \alpha_u, \nu_u)\mathrm{d}u\right),\ 0 \leq s \leq t \leq T.$$

At this point, different criteria may be assigned to the Agent and the Principal and there is freedom for studying distinct models. For instance, Cvitanić, Possamaï and Touzi [27] studied the general case where the Agent controls effectively both the drift and the volatility of the outcome process and both players are utility-maximizers. This is an extension of the model presented in section 1.2.1 in which no volatility control is considered, that is $N = \{n\}$, and the Agent controls only the drift of the outcome process. In this model, the value function of the Agent when he is offered a certain contract and the value function of the Principal are respectively given by

$$V^A(\chi, \xi) := \sup_{(\mathbb{P},\alpha,\nu)\in\mathcal{M}} u_0^A(\chi, \xi, \mathbb{P}, \alpha, \nu), \tag{1.12}$$

$$V^P := \sup_{(\chi,\xi)\in\Xi}\ \sup_{(\mathbb{P},\alpha,\nu)\in\mathcal{M}^\star(\chi,\xi)} \mathbb{E}^{\mathbb{P}^\alpha}\left[U_P\left(L(X_T) - \xi - \int_0^T \chi_s \mathrm{d}s\right)\right], \tag{1.13}$$

where the real functions L and U_P are the liquidate and utility functions of the Principal. The set $\mathcal{M}$ of admissible triplets $(\mathbb{P}, \alpha, \nu)$ is obtained by imposing technical integrability conditions on the weak solutions to (1.10) in order to guarantee that all the stochastic integrals and change of measures considered so far are well defined. The set $\mathcal{M}^\star(\chi, \xi)$ is the set of optimal controls of the Agent when the contract (χ, ξ) is offered by the Principal. As usual in the literature, it is assumed that when the Agent is indifferent between many optimal controls he chooses one of the most convenient for the Principal. Finally, Ξ is the set of admissible contracts. The Principal is restricted to offer contracts sufficiently integrable, such that the Agent can optimally choose his actions and his value function is greater than his reservation utility R_0. Mathematically we have

$$\Xi := \left\{(\chi, \xi) \in \mathcal{C} : \mathcal{M}^\star(\chi, \xi) \neq \emptyset,\ V^A(\chi, \xi) \geq R_0\right\}.$$

In Chapter 4 of this thesis the problem of volatility uncertainty is studied. A brief description of the model and the differences with the system (1.12)-(1.13) is given in section 1.2.2.2. In the next section, the dynamic programming approach of Cvitanić, Possamaï and Touzi [28] for solving (1.12)-(1.13) is presented.

1.2.2.1 Dynamic programming approach

In the Principal-Agent relationship, the sequential interaction between the Principal and the Agent makes them play a Stackelberg game. Therefore, in the designing of the optimal contract, the Principal first anticipate what will be the optimal response of the Agent to every contract she may offer to him and next, taking this information into account, she decides what contract provides her the highest utility. However, mathematically, this strategy may be quite hard to apply since the problems which have to be solved on every step are non-standard stochastic control problems. In fact, the optimal control of the Agent may have a highly non-linear dependence on the control proposed by the Principal and moreover, the controls offered by the Principal may be as complicated as desired, since there is no reason *a priori* to impose any additional restriction on the set of admissible contracts.

The dynamic programming approach, proposed by Cvitanić, Possamaï and Touzi [28] avoids the difficulties just mentioned by showing that it is possible to restrict without loss of generality the class of contracts that the Principal offers to the Agent. The class of restricted contracts is chosen conveniently. Its elements correspond to the terminal values of a controlled SDE and this allows to reduce the problem of the Principal to a standard stochastic control problem where a Hamilton-Jacobi-Bellman equation can be written.

The authors also present a 2BSDE representation of the value function of the Agent. The intuition of this result is that, as shown in section 1.2.1, if the volatility of the outcome is fixed the value function of the Agent is the solution to a BSDE so when the Agent takes supremum on the possible volatilities his value function becomes the solution to a 2BSDE. The dynamic programming approach is presented next.

Define the function $F : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times A \times N \longrightarrow \mathbb{R}$ by

$$F(t,x,y,z,\chi,\alpha,\nu) := -k(t,x)y + u_A(t,x,\chi) - c(t,x,\alpha) + b(t,x,\alpha,\nu) \cdot z.$$

The Hamiltonian $H : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_{\mathrm{d}}^+ \longrightarrow \mathbb{R}$ associated with the problem of the Agent (1.12) is

$$H(t,x,y,z,\gamma,\chi) := \sup_{\Sigma \in \mathcal{S}_{\mathrm{d}}^+} \left\{ \frac{1}{2}\mathrm{Tr}(\Sigma\gamma) + \sup_{(\alpha,\nu)\in A\times V_t(x,\Sigma)} F(t,x,y,z,\chi,\alpha,\nu) \right\},$$

where $V_t(x,\Sigma) := \left\{\nu \in N, \sigma(t,x,\nu)\sigma^\top(t,x,\nu) = \Sigma\right\}$. Define the optimal map $F^\star$ by

$$F^\star(t,x,y,z,\chi,\Sigma) := \sup_{(\alpha,\nu)\in A\times V_t(x,\Sigma)} F(t,x,y,z,\chi,\alpha,\nu).$$

It holds that the value function of the Agent can be obtained from the solution to the following 2BSDE.

$$Y_t = U_A(\xi) + \int_t^T F^\star(s,X,Y_s,Z_s,\chi_s,\hat{\sigma}_s^2)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}X_s + \int_t^T \mathrm{d}K_s,\ \mathcal{P}_A - q.s. \tag{1.14}$$

More precisely, denote by (Y,Z,K) the solution to (1.14). Then, the value function of the Agent is equal to

$$V^A(\chi,\xi) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[Y_0\right],$$

where $\mathcal{P}$ is the set of probability measures on Ω such that $(\mathbb{P},\nu)$ is a weak solution to (1.10) for some ν. Moreover, the optimal controls of the Agent are given by the corresponding maximizers in $F^\star(t,X,Y_t,Z_t,\chi_t,\widehat{\sigma}_t^2)$.

Consider now a restricted class of terminal payments of the form $\xi = Y_T^{Z,\Gamma}$, where $Y^{Z,\Gamma}$ is defined as follows for sufficiently integrable processes (Z,Γ)

$$Y_t^{Z,\Gamma} := Y_0 + \int_0^t Z_r \cdot \mathrm{d}X_r + \frac{1}{2}\int_0^t \Gamma_r : \mathrm{d}\langle X\rangle_r - \int_0^t H_r\left(Y_r^{Z,\Gamma}, Z_r, \Gamma_r, \chi_r\right)\mathrm{d}r. \tag{1.15}$$

Under this nice class of contracts, it follows from simple arguments that the value function of the Agent is given by $V^A(\chi, Y_T^{Z,\Gamma}) = Y_0$, which leads to a representation of the problem of the Principal as a standard stochastic problem with controls (χ, Z, Γ) and state variables $(X, Y^{Z,\Gamma})$. It is shown in [28] that the value function of the Principal is the same under the restricted class of contracts, that is

$$V^P = \sup_{Y_0 \geq R_0} \sup_{(\chi,Z,\Gamma)\in\mathcal{V}} \sup_{(\mathbb{P},\alpha,\nu)\in\mathcal{M}^\star(\chi,Y_T^{Z,\Gamma})} \mathbb{E}^{\mathbb{P}^\alpha}\left[U_P\left(L(X_T) - Y_T^{Z,\Gamma} - \int_0^T \chi_s \mathrm{d}s\right)\right].$$

Where $\mathcal{V}$ denotes the class of sufficiently integrable processes such that (1.15) is well-posed. As explained earlier, this representation allows to solve the problem of the Principal by means of the associated HJB equation.

1.2.2.2 Contribution: the case of volatility uncertainty

In this section, a brief description of the Principal-Agent problem under volatility uncertainty is given. This problem is studied in detail in Part II, Chapter 4 of this Thesis

In the volatility uncertainty case, the Agent controls only the drift of the outcome process. However, the setting of volatility control is needed to model the problem, given the form in which both sides face the uncertainty. The Principal and the Agent do not know exactly what is the volatility of the outcome but they have some beliefs about it, which are represented by some sets of probability measures $\mathcal{P}_P$ and $\mathcal{P}_A$. Moreover, both players have a worst-case approach to the contract and they act as if a third player, the Nature, was choosing the worst possible volatility of the outcome process. Therefore, the value functions of the Agent and the Principal are

$$V^A(\chi,\xi) := \sup_{\alpha\in\mathcal{A}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A} u_0^A(\chi,\xi,\mathbb{P},\alpha,\nu), \tag{1.16}$$

$$V^P := \sup_{(\chi,\xi)\in\Xi} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_P} \mathbb{E}^{\mathbb{P}^{\alpha^\star(\chi,\xi)}}\left[U_P\left(L(X_T) - \xi - \int_0^T \chi_s \mathrm{d}s\right)\right], \tag{1.17}$$

where the sets $\mathcal{N}_A$ and $\mathcal{N}_P$ correspond to the set of weak solutions of (1.10) such that $\mathbb{P} \in \mathcal{P}_A$ and $\mathbb{P} \in \mathcal{P}_P$ respectively. The control $\alpha^\star(\chi,\xi)$ represents the optimal control of the Agent to the contract (χ,ξ), with the assumption that in case of multiple solutions he chooses one which is best for the Principal.

In this case, the Hamiltonian $H : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^{\mathrm{d}}\times\mathcal{S}_{\mathrm{d}}^+ \longrightarrow \mathbb{R}$ associated with the problem of the Agent (1.16) is defined by

$$H(t,x,y,z,\chi,\gamma) := \inf_{\Sigma\in\mathcal{S}_{\mathrm{d}}^+}\left\{\frac{1}{2}\mathrm{Tr}(\Sigma\gamma) + \sup_{\alpha\in A}\inf_{\nu\in V_t(x,\Sigma)} F(t,x,y,z,\chi,\alpha,\nu)\right\}.$$

Defining in this case, the map $F^\star$ by

$$F^\star(t,x,y,z,\chi,\Sigma) := \sup_{\alpha\in A} \inf_{\nu\in V_t(x,\Sigma)} F(t,x,y,z,\chi,\alpha,\nu),$$

the value function of the Agent can be obtained from the solution to the following 2BSDE

$$Y_t = U_A(\xi) + \int_t^T F^\star(s,X,Y_s,Z_s,\chi_s,\hat{\sigma}_s^2)\mathrm{d}s - \int_t^T Z_s\cdot \mathrm{d}X_s - \int_t^T \mathrm{d}K_s,\ \mathcal{P}_A - q.s. \tag{1.18}$$

More precisely, denoting by (Y,Z,K) the solution to (1.18), the value function of the Agent is given by

$$V^A(\chi,\xi) = \sup_{\alpha\in\mathcal{A}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A} \mathbb{E}^{\mathbb{P}^\alpha}\left[Y_0\right],$$

and the optimal controls $(\alpha^\star,\mathbb{P}^\star,\nu^\star)$ are given by the ones which attain the sup-inf in $F^\star(t,X,Y_t,Z_t,\chi_t,\widehat{\sigma}_t^2)$.

The problem of the Principal cannot be solved as a straightforward application of the result of Cvitanić, Possamaï and Touzi [28] because in this case it corresponds to a stochastic differential game. In Chapter 4 it is proved that the value function of the Principal is a viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation by following the Stochastic Perron's method of Bayraktar and Sîrbu [11, 12, 13, 115]. This method consists in proving that the value function of the Principal lies between two well–constructed functions, which are a viscosity super-solution and a viscosity sub-solution of the HJBI equation. Then, assuming that the PDE admits a comparison theorem, it follows that its unique solution is the value function of the Principal.

1.3 Perspectives

A natural extension of the model presented in section 1.2.2.2 is to consider the case in which the agent can control also the volatility of the outcome. By considering a volatility coefficient which depends on the control of the agent, $\sigma(t,x,\alpha,\nu)$, the value function of the agent cannot longer be represented as the solution to a 2BSDE. As explained in Remark 4.4.2 from Chapter 4, this case leads to a non-standard control of 2BSDEs for which a general theory is still lacking. It is necessary then to develop such theory in order to study more realistic situations when the agent can indeed have an impact on the volatility but in an uncertain way which makes him maintain his worst-case approach to the problem.

An interesting extension of the volatility control model is to add jumps to the output process. If the jumps of the output represented possible accidents, then the Agent would have an impact on the frequency of the accidents, by controlling the intensity of the jumps, and also on their severity, by controlling the size of the jumps. To the best of my knowledge, there is no literature in principal-agent models with an agent controlling both the size and the jumps of the outcome process. To develop this ideas, the theory of 2BSDEs with jumps (2BSDEJ) is required. Since such theory is very recent and is not completely developed yet, this project would involve important theoretic work. However, a model of this kind has natural applications in insurance contracts, what means a return to the original motivation of the principal-agent model, but in a much more general setting. Currently, I have started to work on the theoretical aspects of this project.

2 Adverse selection in the Principal-Agent problem

Two main sources of inefficiencies were identified in the Principal-Agent relationship, namely, hidden actions of the Agent (moral hazard) and hidden information on the Agent (adverse selection). In the adverse selection case, there is a population of Agents and each one of them has private characteristics which are resumed in a type $\rho \in \mathfrak{R}$. The Principal can initiate a contractual relationship with a single Agent or with the whole population. The key feature of the adverse selection is that, when the Principal offers a contract to a particular Agent, she does not know what is his type.

During the 1970s, the first studies in the pure adverse selection case were made in parallel with the theory of pure moral hazard. These works include Mirrlees [68], Mussa and Rosen [79], Roberts [101] and Spence [119]. Later works in the pure adverse selection case can be found in Baron and Myerson [10], Maskin and Riley [63], Guesnerie and Laffont [48], Salanié [108], Wilson [132], and Rochet and Choné [102] in the discrete–time setting, and in Zhang [135] and Williams [130] in the continuous–time setting. However, although they evidently represent more realistic situations, problems including both moral hazard and adverse selection have not been frequently studied, and the literature on the subject is a very small fraction of the rich literature on the Principal-Agent problem. The reason behind this, is probably the mathematical difficulty that adverse selection conveys in the optimal design of a contract. If the Principal-Agent problem including only moral hazard is already difficult to solve, the presence of adverse selection complicate things much more.

In the presence of adverse selection, the interaction between the Principal and Agent is slightly different from the one in the pure moral hazard case. Since the Principal is not informed about the type of the Agent when she makes him an offer, she designs a menu of contracts $(\Psi_\rho)_{\rho\in\mathfrak{R}}$, one for each possible type. The Principal offers the menu of contracts to the Agent and the Agent declares a type (not necessarily his true type). Then, he can accept or reject the contract corresponding to the declared type. The revelation principle states that if the Principal restricts herself to offer contracts in which the Agent reveals his true type, she incurs in no loss of utility. Mathematically, this condition means that

$$U_A(\rho, \Psi_\rho) = \sup_{\rho' \in \mathfrak{R}} U_A(\rho, \Psi_{\rho'}), \ \forall \rho \in \mathfrak{R}, \tag{2.1}$$

where $U_A(\rho, \Psi_{\rho'})$ denotes the maximal utility that the Agent can obtain from the contract of type ρ' if his real type is ρ. A menu of contracts satisfying (2.1) is referred to as truth–revealing. The presence of adverse selection thus, adds an additional constraint to the problem of the Principal. Depending on the setting, this constrain can take different forms and different approaches can be used to deal with it.

Discrete–time problems including both moral hazard and adverse selection are studied in Mirrlees [68], Weitzman [128], Baron and Holmström [9], Baron [6], Antle [2], Myerson [80], Dionne and Lasserre [36], Laffont and Tirole [59], McAfee and McMillan [65], Picard [91], Baron and Besanko [7, 8], Melumad and Reichelstein [66, 67], Guesnerie, Picard and Rey [49], Page [83], Zou[136], Caillaud, Guesnerie and Rey [22], Lewis and Sappington [61], or Bhattacharyya [15].

The first study of the continuous time problem with moral hazard and adverse selection was made by Sung [124], in an extension of the pure moral hazard model of Hölmstrom and Milgrom [53]. In that paper, the author studied the case of a risk–neutral Agent and an Agent with exponential utility who controls the drift and the volatility of the outcome. In a different framework, Cvitanić and Zhang [32] studied the case of an Agent who controls either the drift or the volatility of the outcome, with a continuum of types in the population. Cvitanic, Wan and Yang [29] extended the pure moral hazard model of Sannikov [109], where the Agent controls the drift of the outcome, to the case where there are two types of Agents in the market.

In the next sections, the problem of pure adverse selection and the third-best problem are discussed. Since there are many different approaches to these situations in the literature, one approach to the problem is presented in each case. Following each exposition, a related application in economics of the Principal-Agent model is briefly described. Such applications are part of this Thesis and the complete works can be found in Part I of this document, in Chapters 2 and 3.

2.1 Pure adverse selection

In the pure adverse selection setting, the case of a finite number of types becomes irrelevant, since equation (2.1) reduces to a finite number of inequalities which must be simply added to the participation constrain of the Agent (IR). Consider thus that the set of types $\mathfrak{R}$ is a bounded open convex subset of a finite dimensional space. Consider also, for simplicity, that the problem is static. The introduction of continuous time is discussed in the next section.

Different models have been proposed for studying particular applications of the pure moral hazard problem. However, all these models share the same mathematical structure, due to the design of truth–revealing contracts. By making some redefinitions, it is possible to move from one model to the other in most of the cases. Nonetheless, different approaches for solving the problem can be found in the literature, since different assumptions on the components of the model are made by different authors. In this section, the model introduced by Carlier [23] is presented, due to its great generality which allows to encompass most of the other models.

The Agent is hired by the Principal to perform an observable and contractible action $a \in A \subset \mathbb{R}^N_+$. The type of the Agent is unknown for the Principal, her only information being the density function f of the types in the population. The Principal offers to the Agent a menu of contracts $(a(\rho), w(\rho))_{\rho\in\mathfrak{R}}$, where for every Agent of type ρ, $a(\rho)$ is the action he must perform and $w(\rho)$ is the payment he receives for his work. Observe that different from the moral hazard setting, the payment is a direct function of the revealed type. In the same line, the Principal benefits directly from the action of the Agent through a function P which is usually assumed to be linear. The preferences of the Agent have the form $U_A(\rho, a, w) = u_A(\rho, a) + w$. In this case the truth–revealing condition (2.1) becomes

$$u_A(\rho, a(\rho)) + w(\rho) \geq u_A(\rho, a(\rho')) + w(\rho'),\ \forall \rho, \rho' \in \mathfrak{R}.$$

The Agent possesses an outside option which provides him reservation utility R_0. Then, recalling the revelation principle, the problem of the Principal becomes

$$(P_{AS}) \begin{cases} \underset{\rho, a(\cdot), w(\cdot)}{\text{maximize}} & \displaystyle\int_{\mathfrak{R}} \left(P(a(\rho)) - w(\rho)\right) f(\rho) \mathrm{d}\rho \\ \text{s.t.} & u_A(\rho, a(\rho)) + w(\rho) \geq R_0, \qquad \text{(IR)} \\ & u_A(\rho, a(\rho)) + w(\rho) \geq u_A(\rho, a(\rho')) + w(\rho'),\ \forall \rho, \rho' \in \mathfrak{R}. \quad \text{(TR)} \end{cases}$$

The main difficulty of this problem is the truth–revealing constrain (TR). As in the pure moral hazard theory, effort has been put in studying such constrain and finding equivalent formulations. Wilson [133] computed the first–order conditions for the equation (TR), which in some particular cases lead to a change of variables in the objective function. In the one–dimensional case, Mussa and Rosen [79] constructed an equivalent problem of calculus of variations where the (TR) becomes a non-decreasing constrain. Rochet and Choné [102] consider the multidimensional case when the preferences of the Agent are linear in type. By following the dual approach of Mirrlees [68], they constructed an equivalent problem of calculus of variations where (TR) becomes a convexity constrain.

The approaches just mentioned are particular cases of the one presented next. Before doing so, some basic concepts of the $u-$convex analysis are needed. A function $V : \mathfrak{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be u_A-convex if there exists a set $S \subset A \times \mathbb{R}$ such that

$$V(\rho) = \sup_{(a,w) \in S} u_A(\rho, a) + w,\ \forall \rho \in \mathfrak{R}.$$

For a function $V : \mathfrak{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\rho \in \mathfrak{R}$, $a \in A$ is called a u_A-subgradient of V at ρ if

$$V(\rho') \geq V(\rho) + u_A(\rho', a) - u_A(\rho, a),\ \forall \rho' \in \mathfrak{R}.$$

Define the set $\partial^\star V(\rho)$ as the one formed by all the u_A-subgradients of V at ρ. The function V is called u_A-subdifferentiable at ρ if $\partial^\star V(\rho) \neq \emptyset$.

It follows from the previous definitions that a menu of contracts $(a(\rho), w(\rho))_{\rho \in \mathfrak{R}}$ is truth–revealing if and only if

- $V(\rho) := u_A(\rho, a(\rho)) + w(\rho)$ is u_A-convex,
- u_A-subdifferentiable,
- $a(\rho) \in \partial^\star V(\rho)$, for every $\rho \in \mathfrak{R}$.

Consequently, problem (P_{AS}) is equivalent to

$$(\overline{P_{AS}}) \begin{cases} \underset{V(\cdot),x(\cdot)}{\text{minimize}} & \int_{\mathfrak{R}} \left(P(x(\rho)) + u_A(\rho, x(\rho)) - V(\rho)\right) f(\rho) \mathrm{d}\rho \\ \text{s.t.} & V \text{ is } u_A - \text{convex}, \\ & V(\rho) \geq R_0, \forall \rho \in \mathfrak{R}, \\ & x(\rho) \in \partial^\star V(\rho), \forall \rho \in \mathfrak{R}. \end{cases}$$

This formulation of the problem of the Principal opens the door to techniques from functional analysis and calculus of variation in the study of the existence of solutions and their characterization. Imposing integrability conditions on the parameters of the model, existence of solutions can be obtained in appropriate Sobolev spaces. Concerning the characterization of solutions, different tools are available depending on the particularities of the problem.

2.2 Application: power tarification under adverse selection

This section introduces a problem of power tarification, which is modelled as a Principal-Agent situation under adverse selection. It is important to remark that the model is not exactly of pure adverse selection, even if moral hazard is not present. However, the ideas presented in section 2.1 are the main tools used to approach it. The problem presented here is studied in detail in Part I, Chapter 2 of this Thesis.

An electricity company, the Principal, wants to determine the optimal instantaneous tariff $p(t, c)$ for the electrical consumption of its clients, the Agents. Such tariff is a function of the amount $c \in \mathbb{R}_+$ of electricity consumed and the time $t \in [0, T]$ at which the consumption is made. The Principal aggregated cost of production is $K : [0, T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$. The clients are heterogeneous, represented by a type $x \in [0, 1]$. The density of the distribution of types in the population is denoted by f and is known by the Principal. Finally, the instantaneous utility function of the Agents is given by $u : [0, T] \times X \times \mathcal{C} \longrightarrow \mathbb{R}$.

The action of every Agent is a consumption plan $c : [0, T] \longrightarrow \mathbb{R}_+$. The consumptions are observable by the Principal, so there is no moral hazard in this setting. Despite of this, each Agent can choose his own consumption, as the one which maximizes his benefits. This is the main difference with the case of pure adverse selection of section 2.1. A contract is a tariff p proposed by the Principal. In this model, the presence of types in the population generates adverse selection. However, due to legal reasons or to large costs of implementation, the Principal does not design a menu of tariffs, but instead a general p for all the clients. Let $\mathcal{P}$ be set of sufficiently integrable tariffs. Given $p \in \mathcal{P}$, the problem of the Agent of type x is

$$V^A(p, x) := \sup_{c(\cdot)} \int_0^T u(t, x, c(t)) - p(t, c(t)) \, \mathrm{d}t. \tag{2.2}$$

Complementing the basic concepts of $u-$convex analysis introduced in the previous section, for any function φ from $[0, T] \times \mathbb{R}_+$ to $\mathbb{R}$, its $u-$transform is defined as the map $\varphi^\star : [0, T] \times$

$X \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi^\star(t,x) := \sup_{c\in\mathbb{R}_+} \{u(t,x,c) - \varphi(t,c)\}\,, \text{ for any } (t,x) \in [0,T] \times X.$$

It follows then that for every $p \in \mathcal{P}$ and for almost every $x \in X$

$$V^A(p,x) = \int_0^T p^\star(t,x)\mathrm{d}t.$$

Since every type of Agent possesses private characteristics, their reservation utilities may differ. Consider then a non–decreasing function $H : [0,1] \longrightarrow \mathbb{R}$, representing the total benefits that the Agents could obtain by signing a contract with a competitor in the market, or even by producing energy locally. The set of Agents who accept the contract p offered by the Principal is given by

$$X^\star(p) := \left\{x \in [0,1],\ \int_0^T p^\star(t,x)\mathrm{d}t \geq H(x)\right\}.$$

Thus, optimal contracting problem faced by Principal is

$$V^P := \sup_{p\in\mathcal{P}} \sup_{c\in\partial^\star p^\star} \int_0^T \left[\int_{X^\star(p^\star)} p(t,c(t,x))f(x)\mathrm{d}x - K\left(t, \int_{X^\star(p^\star)} c(t,x)f(x)\mathrm{d}x\right)\right] \mathrm{d}t,$$

with the usual convention that when indifferent between two consumptions, the Agent chooses the one which benefits most the Principal. Recall from section 2.1 that $c(t,x) \in \partial^\star p^\star(t,x)$ if and only if $c(t,x)$ is the optimal consumption in problem (2.2).

The characterization of the value function of the Agent $V^A(p,x)$ and the set $X^\star(p)$ in terms of $p^\star$ suggest a reformulation of the problem V^P as a variational problem in terms of $p^\star$. With that purpose in mind, the tractable case of an Agent with CRRA utility is studied. Assume that

$$u(t,x,c) = g(x)\phi(t)\frac{c^\gamma}{\gamma},$$

where $g : [0,1] \longrightarrow \mathbb{R}_+$ represents the willingness of the Agents to pay for consumption, $\phi : [0,T] \longrightarrow \mathbb{R}_+$ represents that clients prefer to consume more at certain hours of the day, and $\gamma \in (0,1)$. In this case the optimal consumption of the Agent is uniquely determined by the equivalent condition

$$c(t,x) \in \partial^\star p^\star(t,x) \iff c(t,x) = \left(\frac{\gamma}{\phi\,(t)\,g'(x)}\frac{\partial p^\star}{\partial x}\,(t,x)\right)^{\frac{1}{\gamma}}.$$

Without entering into details, the problem of the Principal can be rewritten as

$$\begin{aligned} V^P = &\sup_{p\in\mathcal{A}} \int_0^T \left[\int_{X^\star(p)} \left(\frac{g(x)}{g'(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)\mathrm{d}x\right. \\ &\left. - K\left(t, \int_{X^\star(p)} \left(\frac{\gamma}{\phi(t)g'(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\right] \mathrm{d}t, \end{aligned} \tag{2.3}$$

where the set $\mathcal{A}$ is the set of tariffs p sufficiently integrable such that

- p is a $u-$convex function.
- $\partial^\star p^\star(t,x)$ is non-empty for every (t,x).
- $p^\star$ is increasing.

Compared to problem $(\overline{P_{AS}})$ from section 2.1, in problem V^P not all the Agents are necessary contracted. The presence of the set $X^\star(p)$ in (2.3) makes problem V^P very difficult. Since this set is determined directly by H, the problem is studied for different classes of reservation utilities.

If H is assumed to be constant, the set $X^\star(p)$ is an interval. Therefore, sufficient and necessary first-order conditions can be found when V^P is a concave problem (this is indeed the case by considering that K is a convex cost function). If H is an increasing and concave function, by applying multiple tools from variational analysis, the problem V^P can be reduced to an equivalent one in which the tariffs p offered by the Agent are restricted. The restriction consists in the set $X^\star(p)$ taking the form $X^\star(p) = [0,b_0] \cup [a_0,1]$, for some $a_0, b_0 \in [0,1]$. Again, this simple form of $X^\star$ leads to standard optimality conditions.

2.3 Third-best problem: the temptation-value approach

In this section, the third–best continuous–time model of Cvitanić, Wan and Yang [29] is presented. This work is an extension of Sannikov [109], by considering two types of Agents in the population.

Recall that in the pure moral hazard setting, the approach of Sannikov [109] consisted in assuming that the continuation value of the Principal was a deterministic function of the continuation value of the Agent, what allowed to write an HJB equation associated to the optimal contract. In the presence of a finite number of types, the previous approach can be generalized by increasing the number of state variables of the value of the Principal, and therefore the optimal contract. In the optimal contract designed for a particular type of Agent, the Principal also looks at the utilities that the other types would obtain if they lied and accepted the contract, the so–called temptation utilities. Therefore, under a fixed contract, the utility of the Principal is assumed to be a deterministic function of the potential values of all the types of Agents. This allows to associate each optimal contract to a high-dimensional HJB equation, with non-trivial domain and boundary conditions. A methodology to determine the latter components of the PDE is introduced in Cvitanić, Wan and Yang [29]. The details of their approach are given next.

Recall from section 1.2.1, the weak formulation of the drift-control performed by the Agent

$$X_t = \int_0^t \alpha_s \sigma_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W^\alpha_s,\ \mathbb{P}^\alpha - \text{a.s.}, \tag{2.4}$$

where W^α is a $\mathbb{P}^\alpha-$Brownian motion.

The two types of Agents in the population, are referred to as the good Agent and the bad Agent. The difference between the two types lies in how they profit from the payments, or equivalently, in how hard it is to provide them a certain level of utility. More precisely, the set of types is $\mathfrak{R} = \{\rho_g, \rho_b\}$, with $\rho_g > \rho_b$. The expected utility of the Agent of type $\rho_{\rm i}$ under the contract χ, when performing the action $\alpha \in \mathcal{A}$, is given by

$$u_0^A(\rho_{\rm i}, \chi, \alpha) := \mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^\infty (\rho_{\rm i} u_A(s, X, \chi_s) - c(s, X, \alpha_s))\mathrm{d}s\right], \ {\rm i} \in \{g, b\}.$$

Since the Agents are utility-maximizers, the problem of the Agent of type $\rho_{\rm i}$, when offered the contract χ, is

$$V^A(\rho_{\rm i}, \chi) := \sup_{\alpha \in \mathcal{A}} u_0^A(\rho_{\rm i}, \chi, \alpha). \tag{2.5}$$

The solution to this problem can be obtained exactly as in section 1.2.1.1, what provides a characterization of $V^A(\rho_{\rm i}, \chi)$ via BSDEs. Moving to the Principal, recall that she designs a menu of contracts (χ^g, χ^b). The Principal knows that when she meets an Agent, he is of type $\rho_{\rm i}$ with probability $p_{\rm i}$, for ${\rm i} \in \{g, b\}$. Therefore, her expected utility when offering the menu (χ^g, χ^b) is equal to

$$V^P(\chi^g, \chi^b) := \sup_{\alpha^g \in \mathcal{A}^{g,\star}(\chi^g)} p_g \mathbb{E}^{\mathbb{P}^{\alpha^g}}\left[\int_0^\infty u_P(s, X, \chi_s^g)\mathrm{d}s\right] + \sup_{\alpha^b \in \mathcal{A}^{b,\star}(\chi^b)} p_b \mathbb{E}^{\mathbb{P}^{\alpha^b}}\left[\int_0^\infty u_P(s, X, \chi_s^b)\mathrm{d}s\right]. \tag{2.6}$$

The so–called screening contract, is the solution to the following problem

$$V^P := \begin{cases} \underset{(\chi^g, \chi^b) \in \Xi^2}{\text{maximize}} & V^P(\chi^g, \chi^b) \\ \text{s.t.} & V^A(\rho^{\rm i}, \chi^{\rm i}) \geq R_0, \quad {\rm i} \in \{g, b\}, \\ & V^A(\rho^{\rm i}, \chi^{\rm i}) \geq V^A(\rho^{\rm i}, \chi^j), \quad {\rm i} \neq j, \ {\rm i}, j \in \{g, b\}. \end{cases}$$

Another interesting contract to study is the so–called shutdown contract, in which the Principal is interested in hiring only the good Agent. For instance, Cvitanić, Wan and Yang [29] showed that in cases where the reservation utility of the Agents is high, she obtains more benefits from the shutdown contract than from the screening contract. Under a contract which is accepted only for the good Agent, the expected utility of the Principal is

$$\widehat{V}^P(\chi^g) := \sup_{\alpha^g \in \mathcal{A}^{g,\star}(\chi^g)} p_g \mathbb{E}^{\mathbb{P}^{\alpha^g}}\left[\int_0^\infty u_P(s, X, \chi_s^g)\mathrm{d}s\right]. \tag{2.7}$$

The shutdown contract is the solution to the following problem

$$\widehat{V}^P := \begin{cases} \underset{\chi^g \in \Xi}{\text{maximize}} & \widehat{V}^P(\chi^g) \\ \text{s.t.} & V^A(\rho^g, \chi^g) \geq R_0 \geq V^A(\rho^b, \chi^g). \end{cases}$$

As explained earlier, the idea is to rewrite the previous problems as stochastic control problems with state variables given by the continuation utilities of the Agents when they are offered the same contract. By doing so, an associated HJB can be written for each problem, whose domain and boundary conditions are non-trivial and must be determined rigorously.

The approach to determine the domain and the boundary conditions of the HJB equations is the following. Suppose that for every admissible contract χ, the value functions of the good and the bad Agent take values in some sets $\mathcal{V}^g$ and $\mathcal{V}^b$ respectively (this sets can be explicitly determined depending on the components of the model). The so–called credible set $\mathcal{C}$ is defined as the subset of $\mathcal{V}^b \times \mathcal{V}^g$ containing the pairs of values of the bad and good Agents under every admissible contract offered by the Principal. Formally,

$$\mathcal{C} = \big\{(u^b, u^g) \in \mathcal{V}^b \times \mathcal{V}^g : \exists \chi \in \Xi, V^{A,b}(\chi) = u^b, V^{A,g}(\chi) = u^g, \\ (V_s^{A,b}(\chi), V_s^{A,g}(\chi)) \in \mathcal{V}^b \times \mathcal{V}^g \; \forall s \geq 0\big\},$$

with the notation $V^{A,\mathrm{i}}(\chi) = V^A(\rho_\mathrm{i}, \chi)$ for $\mathrm{i} \in \{g, b\}$. The credible set corresponds to the domain of the HJB equation associated to $\widehat{V}^P$. In problem V^P, since two contracts are designed, the domain of the HJB equation associated to it is $\mathcal{C} \times \mathcal{C}$. To determine the credible set it is convenient to denote by $\mathcal{U}(u^b)$ the largest value u^g that the good Agent can obtain from all the contracts χ such that $V^{A,b}(\chi) = u^b$, and denote by $\mathcal{L}(u^b)$ to the lowest value.

The functions $\mathcal{U}$ and $\mathcal{L}$ can be obtained as solutions to one–dimensional stochastic control problems, leading to associated HJB equations which are ODEs. $\mathcal{U}$ and $\mathcal{L}$ correspond respectively to the upper boundary and the lower boundary of the credible set. By proving attractive properties of $\mathcal{U}$ and $\mathcal{L}$, the value function of the Principal on the boundaries can be computed as the solution to one–dimensional stochastic control problems. By doing so, boundaries conditions for the HJB equation are obtained.

2.4 Application: third-best solution to bank monitoring

In this section, a brief description of a bank monitoring problem including both moral hazard and adverse selection is given. The problem studied is an extension of the model of Pagès and Possamaï [84], which studies the contracting problem between competitive investors and an impatient bank who monitors a pool of long-term loans subject to Markovian contagion. The model is in continuous time with a finite number of types, so the study of the problem is based on the techniques of Cvitanić, Wan and Yang [29] introduced in section 2.3. The problem is studied in detail in Part I, Chapter 3 of this Thesis

A bank monitors a pool of I identical loans, indexed by $j = 1, \dots, I$, each one of them yields cash flow μ per unit of time until it defaults. The bank is the Agent of the model and he raises funds from an investor, who is the Principal. There are two types of banks in the market: the good bank and the bad bank. As in the previous section, the set of types has the form $\mathfrak{R} = \{\rho_g, \rho_b\}$ with $\rho_g > \rho_b$. The proportions of banks of type ρ_g and ρ_b in the population are respectively p_g and p_b.

Let τ^j be the default time of loan j. The size of the pool at time t is equal to $I - N_t$, where

$$N_t := \sum_{j=1}^{I} \mathbf{1}_{\{\tau^j \leq t\}}, \; t \in [0, T].$$

The action of the Agent of type ρ_i is to decide how many loans he does not monitor. The action he chooses at time t is denoted by $k_t^\mathrm{i} \in \{0, \dots, I - N_t\}$. The Agent has incentives to shirk, since each non-monitored loan renders him a private benefit B per unit of time. However, by not monitoring, the default intensity of the loans increases. The latter is explicitly given by

$$\lambda_t^{k^{\mathrm{i}}} := \sum_{j=1}^{I-N_t} \alpha_t^{j,\mathrm{i}} = \alpha_{I-N_t}\left(I - N_t + \varepsilon k_t^{\mathrm{i}}\right),$$

where $\varepsilon > 0$ and the coefficients $\{\alpha_{\mathrm{i}}\}_{\mathrm{i}=1}^{I}$ represent the contagion effect, by assuming that $\alpha_{\mathrm{i}} \leq \alpha_{\mathrm{i}-1}$, $\mathrm{i} \leq I$.

As usual in the Principal–Agent literature, the Agent affects the distribution of the state process, in this case N. Since so far only the case of diffusion control had been considered, the details of the weak formulation in this case are presented. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which N is a Poisson process with intensity λ_t^0. Call τ the liquidation time of the whole pool and let $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ be the minimal filtration containing $(\mathcal{F}_t^N)_{t\geq 0}$ which makes τ a $\mathbb{G}$-stopping time. Define $\mathbb{P}^k$ on $\mathcal{G}_t$ by

$$\frac{\mathrm{d}\mathbb{P}^k}{\mathrm{d}\mathbb{P}} = Z_t^k,$$

where Z^k is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s-}^k \left(\frac{\lambda_s^k}{\lambda_s^0} - 1\right)\left(\mathrm{d}N_s - \lambda_s^0 \mathrm{d}s\right),\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s.$$

It follows from Girsanov's Theorem that $N_t - \int_0^t \lambda_t^k \mathrm{d}s$, is a $\mathbb{P}^k-$martingale.

The Principal designs a menu of contracts $(\Psi_{\mathrm{i}})_{\mathrm{i}\in\{g,b\}} := (k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}})_{\mathrm{i}\in\{g,b\}}$ consisting in:

- Predictable, non-decreasing payments D^{i}.
- Probabilities $(1-\theta^{\mathrm{i}})$ under which the project is liquidated given a default.
- Recommended level of effort k^{i}.

Denoting $H_t := \mathbf{1}_{t\geq\tau}$, then

$$\mathrm{d}H_t = \begin{cases} 0 \text{ with probability } \theta_t^{\mathrm{i}}, \\ \mathrm{d}N_t \text{ with probability } 1 - \theta_t^{\mathrm{i}}. \end{cases}$$

The utility functions of the Agent of type ρ^{i} and the Principal are respectively

$$u_0^{\mathrm{i}}(k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}}) := \mathbb{E}^{\mathbb{P}^{k^{\mathrm{i}}}}\left[\int_0^\tau \mathrm{e}^{-rs}(\rho_{\mathrm{i}} \mathrm{d}D_s^{\mathrm{i}} + Bk_s^{\mathrm{i}}\,\mathrm{d}s)\right],$$

$$v_0(\Psi_{\mathrm{i}}) := p_{\mathrm{i}}\mathbb{E}^{\mathbb{P}^{k^{\mathrm{i}}}}\left[\int_0^\tau (I - N_s)\,\mu \mathrm{d}s - \mathrm{d}D_s^{\mathrm{i}}\right].$$

Therefore, when offered a contract $(\theta^{\mathrm{i}}, D^{\mathrm{i}})$, the Agent faces the problem

$$V^{A,\mathrm{i}}(\theta^{\mathrm{i}}, D^{\mathrm{i}}) := \underset{k\in\mathfrak{K}}{\text{maximize}}\ u_0^{\mathrm{i}}(k, \theta^{\mathrm{i}}, D^{\mathrm{i}}).$$

On the other side, in the design of the screening contract, the Principal faces the problem

$$V^P := \begin{cases} \underset{(\Psi_g,\Psi_b)\in\Xi^2}{\text{maximize}} & v_0(\Psi_g)+v_0(\Psi_b) \\ \text{s.t.} & u_0^{\mathrm{i}}(k^{\mathrm{i}},\theta^{\mathrm{i}},D^{\mathrm{i}}) \geq R_0,\ \mathrm{i}\in\{g,b\}, \\ & u_0^{\mathrm{i}}(k^{\mathrm{i}},\theta^{\mathrm{i}},D^{\mathrm{i}}) = \sup_{k\in\mathfrak{K}} u_0^{\mathrm{i}}(k,\theta^{\mathrm{i}},D^{\mathrm{i}}),\ \mathrm{i}\in\{g,b\}, \\ & u_0^{\mathrm{i}}(k^{\mathrm{i}},\theta^{\mathrm{i}},D^{\mathrm{i}}) \geq \sup_{k\in\mathfrak{K}} u_0^{\mathrm{i}}(k,\theta^{j},D^{j}),\ \mathrm{i}\neq j,\ (\mathrm{i},j)\in\{g,b\}^2. \end{cases}$$

In the design of the shutdown contract, the Principal faces the problem

$$\widehat{V}^P := \begin{cases} \underset{\Psi_g\in\Xi}{\text{maximize}} & v_0(\Psi_g) \\ \text{s.t.} & u_0^{g}(k^{g},\theta^{g},D^{g}) \geq R_0 \geq \sup_{k\in\mathfrak{K}} u_0^{b}(k,\theta^{g},D^{g}), \\ & u_0^{g}(k^{g},\theta^{g},D^{g}) = \sup_{k\in\mathfrak{K}} u_0^{g}(k,\theta^{g},D^{g}). \end{cases}$$

Similar to the drift control case presented in 1.2.1, the presence of jumps in the model does not change the methodology to solve the problem of the Agent. By defining the continuation utility of the Agent, and applying the martingale representation Theorem conveniently, it follows the existence of processes $h^{1,k}$ and $h^{2,k}$ such that

$$\begin{aligned} \mathrm{d}u_t^{\mathrm{i}}(k,\theta^{\mathrm{i}},D^{\mathrm{i}}) = &\left(ru_t^{\mathrm{i}}(k,D^{\mathrm{i}},\theta^{\mathrm{i}}) - Bk_t\right)\mathrm{d}t - \rho_{\mathrm{i}}\mathrm{d}D_t^{\mathrm{i}} - h_t^{1,\mathrm{i},k}\left(\mathrm{d}N_t - \lambda_t^k\mathrm{d}t\right) \\ &- h_t^{2,\mathrm{i},k}\left(\mathrm{d}H_t - (1-\theta_t^{\mathrm{i}})\lambda_t^k\mathrm{d}t\right),\ 0\leq t<\tau,\ \mathbb{P}-a.s. \end{aligned}$$

Therefore, the optimal control of the Agent is obtained from the comparison theorems for BSDEs with jumps. In this case it is given by

$$k_t^{\star,\mathrm{i}} = (I-N_t)\mathbf{1}_{\{Z_t^{\mathrm{i}}\cdot(1,1-\theta_t^{\mathrm{i}})^\top<b_t\}},$$

where $(Y^{\mathrm{i}},Z^{\mathrm{i}})$ is the unique (super–)solution to the following BSDE

$$Y_t^{\mathrm{i}} = 0 - \int_t^\tau g^{\mathrm{i}}(s,Y_s^{\mathrm{i}},Z_s^{\mathrm{i}})\mathrm{d}s + \int_t^\tau Z_s^{\mathrm{i}}\cdot\mathrm{d}\widetilde{M}_s^{\mathrm{i}} + \int_t^\tau \mathrm{d}K_s^{\mathrm{i}},\ 0\leq t\leq\tau,\ \mathbb{P}-a.s.,$$

with

$$\begin{aligned} g^{\mathrm{i}}(t,y,z) &:= \inf_{k\in\{0,\dots,I-N_t\}} ry - Bk + k\alpha_{I-N_t}\varepsilon z\cdot(1,1-\theta_t^{\mathrm{i}})^\top, \\ \widetilde{M}_t^{\mathrm{i}} &:= M_t - \int_0^t \lambda_s^0(1,1-\theta_s^{\mathrm{i}})^\top\mathrm{d}s,\ K_t^{\mathrm{i}} := \rho_{\mathrm{i}}D_t^{\mathrm{i}}. \end{aligned}$$

Moreover, $V_t^{A,\mathrm{i}}(\theta^{\mathrm{i}},D^{\mathrm{i}}) = Y_t^{\mathrm{i}}$ for every $t\in[0,T]$.

Moving to the problem of the Principal, different from Cvitanić, Wan and Yang [29] and Sannikov [109], in this case it is not stationary. The size of the project depends on time, but only through the number of loans left. An important difference with Cvitanić, Wan and

Yang [29] is that in this setting the credible set is dynamic. The precise definition of the credible set $\mathcal{C}_{I-N_t}$, when there are $I - N_t$ loans left, is the following

$$\mathcal{C}_{I-N_t} = \big\{(u^b, u^g) \in \mathcal{V}_t^b \times \mathcal{V}_t^g : \exists \Psi \in \Xi, V^{A,b}(\Psi) = u^b, V^{A,g}(\Psi) = u^g, \\ (V_s^{A,b}(\Psi), V_s^{A,g}(\Psi)) \in \mathcal{V}_s^b \times \mathcal{V}_s^g \ \forall s \geq 0\big\},$$

where $\mathcal{V}_s^{\mathrm{i}}$ is the set where the continuation utility of the Agent of type ρ_{i} takes values at time s, the dynamic version of the set $\mathcal{V}^{\mathrm{i}}$ introduced in section 2.3.

Since the impact of time in the model is reduced only to a finite set (the values of $I - N_t$), it does not count exactly as a state variable of the value function of the Principal. Instead, the presence of the temporal variable leads to a system of value functions, one for every size of the project, associated to a recursive system of HJB equations. The domains of these equations are the analogous to the stationary problem. The credible set $\mathcal{C}_{I-N_t}$ is the domain of the HJB equation associated to the shutdown contract with $I - N_t$ loans left. Similarly, the domain of the HJB equation associated to the screening contract with $I - N_t$ loans left is $\mathcal{C}_{I-N_t} \times \mathcal{C}_{I-N_t}$.

2.5 Perspectives

The tarification model introduced in section 2.2 is the first-step to more general models in electricity contracts. To leave the intermediate position between a pure adverse selection problem and a third–best problem, it may be interesting to study the case in which the Principal is allowed to design a menu of contracts. Also, more general contracts can be considered. For instance, taking into account the availability of power, or with interactions between the consumptions of the different agents. Since the Principal negotiates with a great number of Agents, aggregate behaviour can have a crucial role and the problem can be approached from the theory of mean field games.

Staying in the electricity markets, there are many situations in which the principal-agent model can be applied. Recently I have started to work in the problem of a regulator of electricity providers, who gives incentives to the providers to reduce their pollutant emissions in their productive processes (for instance by preferring cleaner technologies). An interesting feature of the model, is that the regulator cannot punish the high-polluting technologies arbitrarily since she faces a social cost when the total energy generated by the providers is not enough to satisfy the demand of the population.

In the third–best problem, as happens in the bank monitoring problem introduced in section 2.4, adding a type of agent in the population increases exponentially the number of states variables of the optimal contract. Indeed, in the design of a contract for a particular type of Agent, the Principal takes care of all the temptation values of the other types. This makes the case with more than two agents intractable, due to the associated high-dimensional HJB equations. The study of the case with a continuum of types becomes relevant then, and the approach initiated by Cvitanić and Zhang [32] can be followed. In this setting, we can think about a Principal who is not perfectly informed about the distribution of types in the population. If she has a worst–case approach to the uncertainty, then the problem can be studied with the same techniques presented in section 1.2.2.2.

Part I

Economic applications of the principal-agent model.

Chapter 2

An adverse selection approach to power tarification

2.1 Introduction

Electricity is non-storable, except marginally: any quantity which is consumed now must be produced now, and conversely. Since there are no stocks to dampen shocks and smooth discrepancy, adjusting supply to demand is a difficult task. One way to do so is to use prices. Very early on, power companies have hit upon the idea of making electricity more expensive in peak hours, so that consumers who are able to do so would switch their demand to off-peak periods. Present-day electricity tariffs are based on (a) total consumption, and (b) maximum power available. But at least two important changes in the electricity sector make previous tariff structure questionable. First change is the development of smart meters which enables a precise metering of electricity consumption (USmartConsumer report [95] states that at the end of 2016 30% of overall European electricity meters were equipped by smart technology). Second change is the development of competition on electricity retail (see [75]). Previous analysis on electricity pricing had focused the point of view in a monopoly which needs to recover its costs, such as in the famous Ramsey-Boiteux tarification described in [19] or in [97]. Our setting considers electricity retailers which acquire electricity partly on the market and partly from their own production, if any, and whose purpose is to maximize their outputs.

This falls naturally within the framework of Principal-Agent problems: the Principal (here the power company) offers a variety of contracts, and each Agent picks the one which suits him best.

It does not seem, however, that such an analysis is available at the present time, and the aim of the present work is to fill this gap. Electricity pricing has a special feature, which distinguishes it from other Principal-Agent problems. Usually, the profit of the Principal is the sum of the profits she gets from all participating Agents. Here, the cost to the power plant is the cost of producing the aggregate demand, which is not the sum of the costs of producing the individual demands, because of decreasing returns to scale in production. This introduces a mathematical difficulty which, fortunately, can be superseded, as we will provide explicit solutions.

We now proceed to describe the mean features of our model. The Principal's cost, as mentioned above, is a convex function of aggregate production. The Agents' utilities are separable: the utility which an Agent of type x derives from consuming a quantity c of electricity at time t and being charged a (nonlinear) price p is:

$$u(t, x, c) - p$$

where $u(t, x, \cdot)$ is a concave function of c. This separability assumption is traditional in Principal-Agent problems. In this case, there are additional justifications, as a large part of the Agents are industrial users, who consume electricity in order to produce other goods, so that their utility simply identifies to the profit they derive from this activity. Note also the time-dependence, which reflects the seasonality of consumption.

In the sequel, we will consider CRRA utilities, of the type $\gamma^{-1}c^{\gamma}$, with $\gamma < 1$, and we will provide explicit solutions (except in the case $\gamma = 0$, or $u(c) = \ln c$). The case $\gamma < 0$ reflects the "household "behavior, where electricity fulfills some basic needs, such as lighting or appliances, and 0 consumption is not acceptable while high consumption is not needed. The case $0 < \gamma < 1$ reflects the "industrial" behavior, where high consumption is the norm, subject to decreasing returns to scale. Note, however, that in both cases there is a "fallback" option, a substitute to electricity when it becomes too expensive, for instance an alternative energy source, or simply another provider. This fallback option is expressed by a reservation utility, which may be constant or vary across Agents.

Despite the particular structure of the cost function, we are able to solve explicitly the problem at hand. We observe that the optimal contract rewrites as the combination of a fixed cost together with two variable costs, proportional to either the electricity consumption or a power function of it. This tariff structure happens to be quite simple and quite close to the classical tariff structures offered by most electricity providers.

Whenever the fallback option is the same for every Agent, we observe as usual in Principal-Agent problems, that the lower end of the market is not covered: the low types (meaning those households who are less dependent on electricity, or those industry users who are less efficient) will not be offered contracts which they are willing to accept, and will have to fall back on the outside option. More interestingly, we are also able to solve explicitly the case where the fallback option of the Agents depends on their type in a concave manner. In this case, the population of Agents accepting the optimal contract offered by the electricity provider may include the lower end as well as the higher end of the Agent types. In this case, getting more efficient Agents can be too costly, and the electricity provider has interest in including the less efficient but less expensive consumers.

2.2 General settings and optimal tariff

The model we are proposing is set up on Principal-Agent relationship where the Principal is an electricity provider and the Agents are electricity consumers. Since the electricity consumption is observed by the Principal, we suppose that there is no moral hazard. On the other hand, adverse selection is in force as the Agent's willingness to pay for electricity is not known by the Principal. This taste for electricity represents how much Agents price a given volume of electricity in term of usefulness. For an industrial Agent, this would represent the benefit he gets by running his industrial process with this given volume of electricity. For a residential Agent, this would represent the comfort he gets by using this given volume of electricity to perform domestic tasks (heating, lightning...). Of course, this depends on the Agent's efficiency of his equipment referred as his type X. As classically assumed in adverse selection setting, even if the Principal does not know the exact type of a particular Agent, he knows the repartition of Agents' type among the population. This hypothesis is realistic as the electricity provider can always make surveys in order to acquire this information.

2.2.1 Players' objectives and electricity particularity

Both players have their particular objectives:

- Agent's objective is to choose the level of electricity consumption c at any time t, which maximize his utility for electricity $u(t,x,c)$ with respect to his type x minus the tariff $p(t,c)$ that he needs to pay for the electricity consumed.

$$\max_{c}\left\{\int_{0}^{T} u\left(t,x,c\right)-p\left(t,c\right)\mathrm{d}t\right\}$$

- Principal's objective is to offer the tariffs which maximize his own profits: all payments she receives from consumers accepting the contract minus the costs for providing the total volume of electricity consumed by her clients. The provider can offer power either by buying on the electricity market or by producing it herself.

The tariffs designed by the Principal need to respect two conditions. The first one is the individual rationality of the Agents. Indeed, Agents are not forced to accept the contract offered by the Principal as they can pick alternative electricity providers, offering better conditions. This is taken into account in the model via a reservation utility H which represents the minimum level of satisfaction that an Agent needs to achieve in order to accept the contract. This reservation utility could interprets as an aggregation of competitors' offers or as a minimum level of utility imposed by public services, who offer a regulated tariff. The second condition that tariffs are required to respect is the so-called incentive compatibility condition:, which will be automatically satisfied, as there is no moral hazard in this model.

One particular feature of electricity product is the fact that it suffers from decreasing returns to scale: its marginal price increases with the total aggregate consumption. This comes from the fact that several technologies can be used in order to produce electricity. Some power plants have no or very low fuel costs such as renewable (hydro power plants, wind or solar production...) or nuclear production. These types of productions are chosen for satisfying base-load consumption. But when, the electricity consumption increases such as in peak hours, other power plants (coal, gas or fuel thermal plants for example) need to be turned on and their cost of production is much more expensive. The electricity spot price shows this strong base/peak patterns and can exhibit high spikes when capacity productions reach their limit with respect to consumption. For this reason, we impose in our model that the electricity production cost of the Principal depends on the consumption of all her clients. In fact, this cost should depend on the consumption of all consumers and not only on all clients of this provider but we assume that aggregate consumption of clients who select the provider is correlated to the consumption of the other consumers. This is justified since there are strong common preferences and behaviours among consumers, for example consumptions are higher during daytime than during the night.

2.2.2 Notations and model assumptions

We consider CRRA (Constant Relative Risk Aversion) utility function for the Agents.

$$u(t,x,c) = g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma} \tag{2.2.1}$$

where $\phi(t)$ represents the Agents' time preference for electricity. This factor is common to every Agent and typically represents for example the preference to have electricity during the daytime than during the middle of the night. $x \in [0,1]$ is the type of the Agent. We suppose that $\gamma < 1$ and we consider two different cases: $\gamma \in (0,1)$ and $\gamma < 0$. The function g_γ represents the willingness of the Agents to pay for their consumption depending on their type x and we take typically $g_\gamma(x) = x$ if $\gamma \in (0,1)$ and $g_\gamma(x) = 1-x$ if $\gamma < 0$. Graphic illustrations of the utility function are shown in Figure 2.1. The case $\gamma \in (0,1)$ corresponds to the modeling of industrial Agents, whose utility grows to infinity if they can have infinite volume of electricity: they can always make their industrial capacities grow and generate more benefits whenever they have extra electricity. On the contrary, they can stop producing if they could not get any electricity or substitute it by another energy, which corresponds to a zero utility whenever $c = 0$. The case $\gamma < 0$ illustrates the residential Agents utility for whom electricity is a staple product: they can not avoid consuming electricity (they would get $-\infty$ utility). They also have a saturation for electricity : above a high volume of electricity, they do not gain much satisfaction with an extra volume because all their electrical needs are already fullfilled.

The indirect utility $p^\star(x)$ is the best level of utility that the Agent can obtain by signing the contract, which is entitled on the period $[0,T]$.

$$P^\star(x) := \int_0^T p^\star(t,x)\mathrm{d}t = \sup_c \left\{ \int_0^T u(t,x,c) - p(t,c)\mathrm{d}t \right\}, \tag{2.2.2}$$

The Agent's decision to sign the contract depends on his reservation utility, denoted $H(x)$ for an Agent of type x: he signs the contract with the Principal if and only if $P^\star(x) \geq H(x)$.

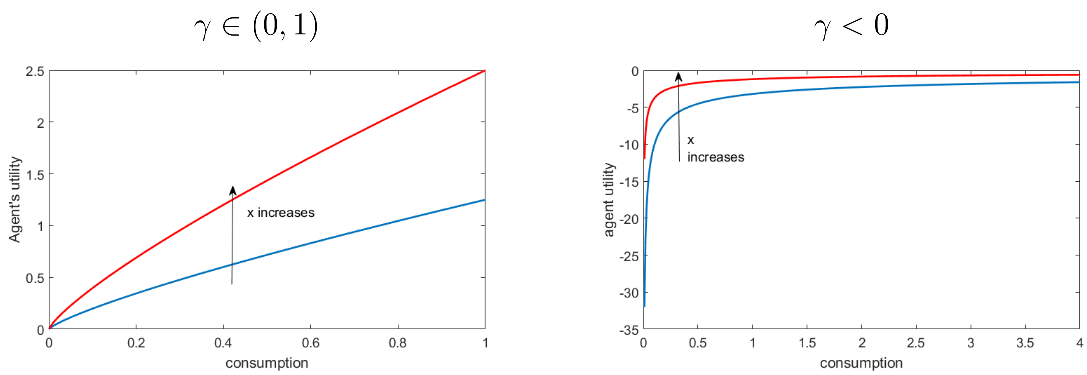


Figure 2.1: Agent's utility with respect to consumption.

We consider two cases for $H(x)$, either a constant function or a concave one verifying that the elasticity of reservation utility is smaller than the the elasticity of willingness to pay for consumption, i.e. $\frac{g_\gamma}{g'_\gamma} \le \frac{H}{H'}$. The concavity of H indicates that competitors target principally the more efficient Agents. We denote by X^* the set of Agents who end up signing the contract:

$$X^\star(p^\star) := \{x \in [0,1],\ P^\star(x) \ge H(x)\}\ .$$

We suppose that Agents are uniformly distributed among the population and, as already mentioned, this feature is known by the Principal. We assume a convex cost of power production K:

$$K(t,c) = k(t)\frac{c^n}{n} \tag{2.2.3}$$

The term $k(t)$ is positive and indicates the time dependence of electricity production costs: for example photovoltaic production occurs only at day and wind is blowing more during winter. The power $n > 1$ reflects the production fleet composition: for example when the fleet has expensive peak power plants n is high.

2.2.3 Optimal tariffs

In the setting we previously described, the Principal-Agent problem can be explicitly solved. We present in this part only a brief sketch of the argumentation and formal mathematical proofs are postponed to the remaining sections of this chapter. Let's stress out that the problem is solved without imposing a priori structure of the tariff function, except that it only depends on time and consumption. In order to be admissible, a tariff p should verified the individual rationality and incentive compatibility conditions, that is denoted $p \in \mathcal{P}$.

Let's write formally the objective function of the Agents U_A and Principal U_P:

$$U_A(p,x) := \sup_c \int_0^T (u(t,x,c(t)) - p(t,c(t)))\,\mathrm{d}t = \int_0^T p^\star(t,x)\mathrm{d}t.$$
$$U_P := \sup_{p\in\mathcal{P}} \int_0^T \left[\int_{X^\star(p^\star)} p(t,c^\star(t,x))f(x)\mathrm{d}x - K\left(t, \int_{X^\star(p^\star)} c^\star(t,x)f(x)\mathrm{d}x\right)\right]\mathrm{d}t.$$

The demonstration is performed into five main steps:

- For a given tariff p, the optimal responding consumption of the Agents can be determined as a function of $\frac{\partial p^\star}{\partial x}$. This is injected in the Principal problem which is now a problem expressed in term of $p^\star$ only.

- In order to solve the Principal problem, we first consider an alternative problem $\tilde{U}_P$, where we impose to the tariff p to be only continuous and non-decreasing, instead of being admissible.

- This alternative problem is simpler to solve. First, the structure of $X^\star$ is determined. Whenever H is constant, it is proven that $X^\star$ is of the form $[a_0, 1]$ meaning only the most efficient Agents select the contract. Whenever H is concave, $X^\star$ is of the form $[0, b_0] \cup [a_0, 1]$, meaning that the Principal not only selects the most efficient, but also the less efficient ones.

- Knowing the structure of $X^\star$, we rewrite $\frac{\partial p^\star}{\partial x}(t,x)$ as $f(t, a_0, b_0)$ using calculus of variations, we plug this expression into $\tilde{U}_P$ in order to determine a_0 and b_0.

- We finally verify whether the derives solution $p^\star$ for the alternative problem satisfies indeed the conditions of the initial problem, i.e. that it is admissible. By doing so, we are able to conclude that the solution $p^\star$ of the simpler problem, also solves the initial one of interest.

We compute that, whatever γ or H (constant or concave), the optimal tariff is a function of three components at most: a constant part p_3, a proportional part p_2 of the consumed power c and a proportional part p_1 of c^γ:

$$p(t,c) = p_1(t)c^\gamma + p_2(t)c + p_3(t)\ . \tag{2.2.4}$$

First, an important observation is that this tariff is quite simple and close to current tariff structures proposed by electricity providers. Indeed, these tariffs are commonly split into a fixed charge in Euro, a volumetric charge in Euro/MWh and possibly a demand charge in Euro/MW. The fixed and the volumetric charges can depend on the maximum subscribed power which is another way to price the demand charge. A detailed interpretation of the optimal tariff is given the next paragraph.

The optimal tariffs the Principal offers in our settings are summarized in the following table. The explicit expressions for the functions $(p_\mathrm{i}, \gamma)_\mathrm{i}$, $(p_\mathrm{i}^j)_{\mathrm{i},j}$ and $\hat{c}_\mathrm{i}^\gamma(t)$ are respectively provided in Theorem 2.5.2 and Theorem 2.6.3 hereafter.

	H constant	H concave
$X^\star$	$[a_0, 1]$	$[0, b_0] \cup [a_0, 1]$
$p(t,c)$	$p_{1,\gamma}(t)c^\gamma + p_{2,\gamma}(t)c + p_{3,\gamma}(t)$, $\gamma \in [0,1]$ $p_{2,\gamma}(t)c + p_{3,\gamma}(t)$, $\gamma < 0$	$p_2^1(t)c + p_3^1(t)$, $c < \hat{c}_1^\gamma(t)$ $p_1^2(t)c^\gamma + p_3^2(t)$, $\hat{c}_1^\gamma(t) < c < \hat{c}_2^\gamma(t)$ $p_1^3(t)c^\gamma + p_2^3(t)c + p_3^3(t)$, $\hat{c}_2^\gamma(t) < c$

Let's point out that the optimal tariffs are understandable and the three components can be connected to electricity pricing standard issues. Clearly, in the obtained optimal tariff p_3 represents the fixed charge. The volumetric charge is the combination of a standard term $p_2(t)c$ plus $p_1(t)c^\gamma$. This last term is a way to charge more high demand consumers (indeed it only features when c is high enough or when γ is positive). Finally, no explicit demand charge appears but the coefficients p_i depend on the maximum subscribed power $\hat{c}_i^\gamma$ which ensures to limit instantaneous power and to charge more high power consumers. In addition, the peak/off-peak issues are handled by the temporal structure of the tariff: each component depends on time t. Therefore, high power consumption within peak period will be overcharged compared to off-peak period. Let's notice that the proportional part $p_1(t)$ to c^γ only depends on the Agent's utility parameters. Therefore this part should be common to any Principal whatever her cost of production, or the utility of reservation of the consumers.

The selected Agents are the most efficient when H is constant, which a classical result. But whenever H is concave, the Principal should also select Agents among the less efficient one. Indeed, when H is concave, reaching most efficient Agents is costly and it happens that getting less efficient Agents can in this case be profitable as they are easily satisfied. This type of feature seems to be uncommon in the Principal-Agent literature.

When H is constant, the tariff is a unique simple function of consumption, which even happens to be linear in c when $\gamma < 0$. When H is concave, the tariff is the combination of three functions depending on the level of consumption c. Nevertheless, let's observe that Agents who sign the contract never consume in the range $[\hat{c}_1^\gamma, \hat{c}_2^\gamma]$. Hence, the tariff portion $p_1(t)^2 c^\gamma + p_3^2(t)$ only acts as a repellent part.

2.3 Economical interpretations and numerical results

2.3.1 Examples when the reservation utility is constant

For a constant utility reservation, more efficient Agents are selected and we draw numerical illustration in figure 2.2 for negative and positive γ. As previously explained, the tariff structure is linear with the consumption when $\gamma < 0$ and is concave otherwise which is represented in the upper graphics of figure 2.2. Middle graphics represent the utility Agents can obtain by signing the contract, depending on their type. If this utility level is smaller than their reservation utility (represented by the dashed line) they do not enter the contract and their consumption is null, as represented in the lower graphics. These utility representations also illustrate a classical result of informational rent: the most efficient Agents obtain a tariff inferior to what they are willing to pay, whereas the less efficient ones need to pay as much as they are able to, or are excluded.

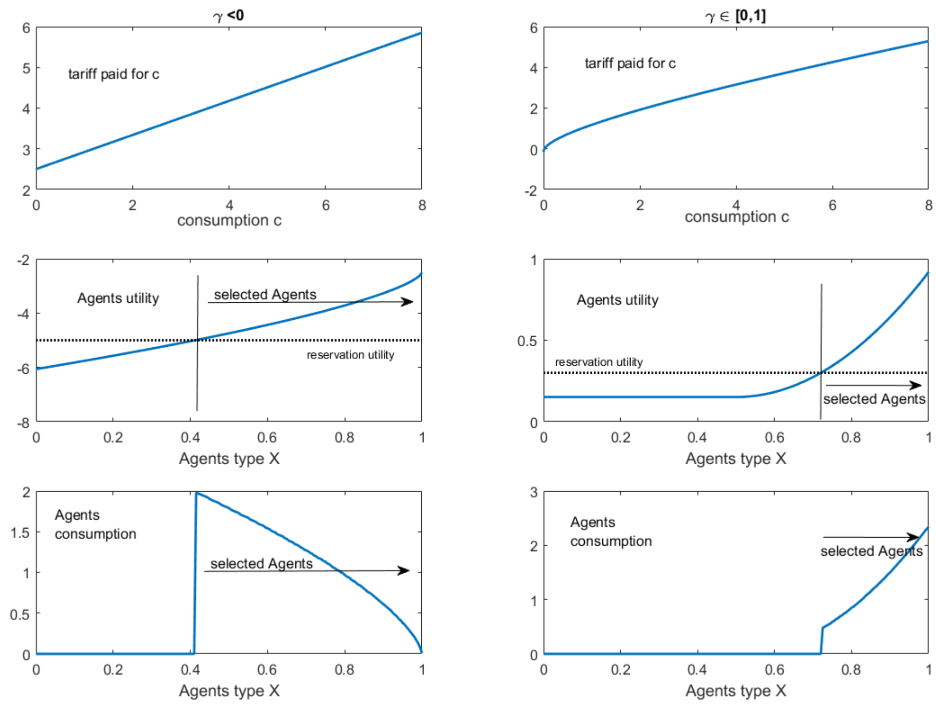


Figure 2.2: H is constant, examples of bill paid with respect to consumption (upper graphs), Agents's utility with respect to their type x and reservation utility in dashed lines (middle graphs) and selected consumption against Agents' type x (lower graphs) for $\gamma < 0$ (left figures) and $\gamma \in (0, 1)$ (right figures)

2.3.2 Examples when the reservation utility is concave

For a concave utility reservation, not only most efficient Agents are selected and we draw numerical illustrations of examples where either most efficient or less efficient Agents are selected on figure 2.3. First, let's analyze the example when $H(x) = \sqrt{x}$ and $\gamma \in (0, 1)$ which corresponds to the left column. In this example, only the most efficient Agents sign the contract as they are the only ones obtaining a higher utility than their reservation one. As presented in the previous section, the tariff structure is the combination of three functions of consumption (upper graphics) but Agents who sign the contract only choose consumption such that $\hat{c}_2^\gamma < c$ which corresponds to the concave tariff part $p_1(t)^3 c^\gamma + p_2^3(t)c + p_3^3(t)$.

When $H(x) = \log(x)$ and $\gamma < 0$ (the right column of figure 2.3), only less efficient Agents take the contract. Indeed, the concavity of the reservation utility makes it profitable for the Principal to select these Agents, rather than the most efficient ones. The tariff structure is again the combination of three functions of consumption (upper graphics) but Agents who sign the contract only take consumption such that $\hat{c}_2^\gamma < c$ (of course $\hat{c}_2$ is different from the

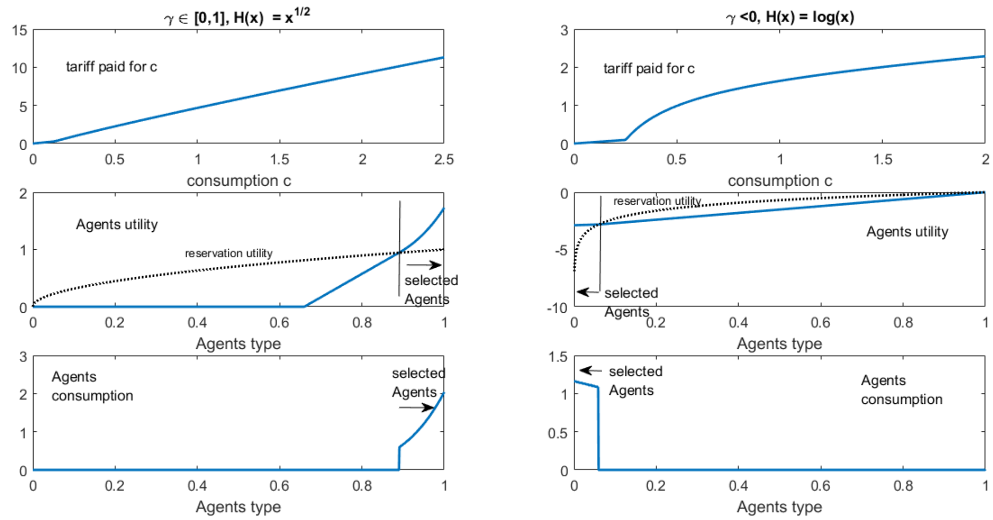


Figure 2.3: H is concave, examples of bill paid with respect to consumption (upper graphs), Agents's utility with respect to their type and reservation utility (dash lines) (middle graphs) and selected consumption against Agents' types (lower graphs)for $\gamma < 0$ and $H(x) = \sqrt{x}$ (left figures) and $\gamma \in (0, 1)$ $H(x) = \log x$ (right figure)

one in the previous example because we consider a different H). This again corresponds to the concave tariff part $p_1(t)^3 c^\gamma + p_2^3(t)c + p_3^3(t)$.

2.3.3 Impact of competition when the reservation utility is constant

When H increases because competition is for example more intense, the Principal adapts his tariff in order to remain competitive. In that case, the Principal mainly decreases the constant part $p_{3,\gamma}$ of its tariff in order to attract consumers, see the left graphic of figure 2.5 when $\gamma < 0$). The consumers selecting this new tariff obtain better conditions and as such consume more power because it is cheaper, see the same example on the left graphic of figure 2.4 when $\gamma < 0$. Therefore, when the Principal decreases his tariff, he does not decrease it enough in order to keep the same quantity of consumers: he accepts to retain less consumers but who consume more, as represented on the right graphic of figure 2.4. Nevertheless, the utility of the Principal decreases with competition, see the right part of figure 2.5. At the extreme, the Principal even offers no tariff whenever H is too high. Let's observe that in this example, the fixed part of the tariff represents more than a half of the total cost of electricity for the consumers.

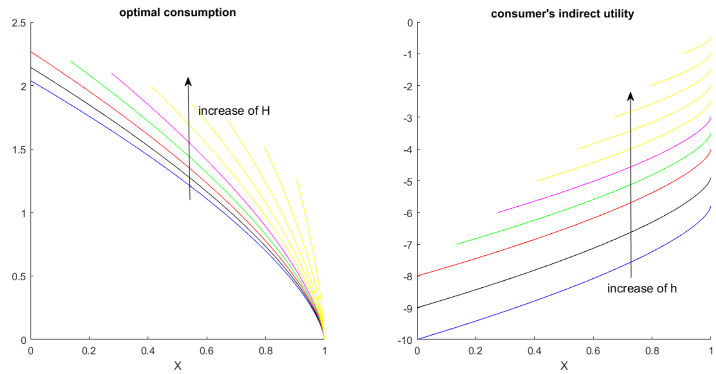


Figure 2.4: Evolution of Agents' utility (left) and consumption (right) against Agent's type X, when H increases and $\gamma < 0$

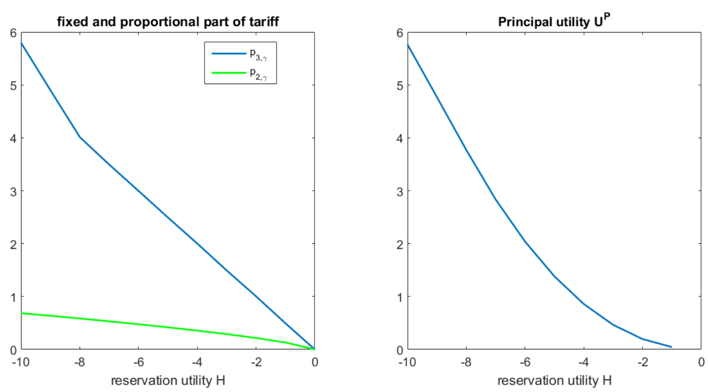


Figure 2.5: Evolution of tariff's components (left) and Principal utility U^P (right), when H increases and $\gamma < 0$

2.3.4 Impact of cost of production when the reservation utility is constant

For an increase of cost of production K, the Principal also adapts his tariff in order to reflect this cost increase. In that case, the Principal mainly increases the proportional part $p_{2,\gamma}$ of its tariff so that he continues to attract consumers, see an example for constant H and $\gamma < 0$ on the left part of figure 2.7. Consumers who select this new tariff are offered worse conditions and as such consumes less power because it is more expensive, see the same example on the left graphic of figure 2.6. In addition, less consumers select the contract. Therefore, the utility of the Principal decreases with the cost of production, as illustrated in the right part of figure of 2.6.

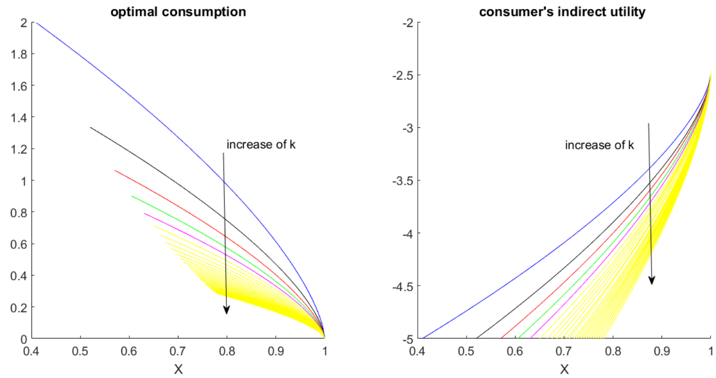


Figure 2.6: Agents' utility (left) and consumption (right figure) evolution againts Agent's type X when K increases and $\gamma < 0$

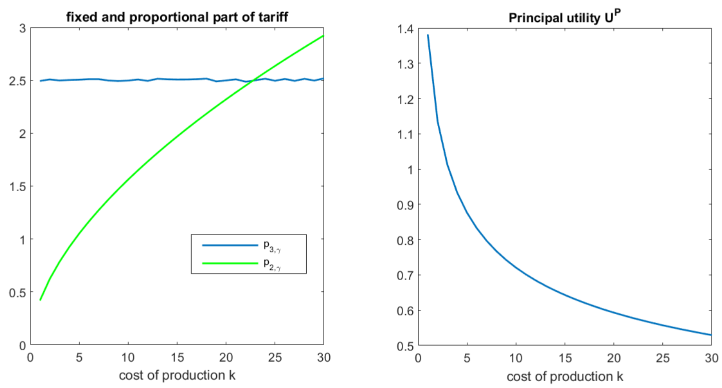


Figure 2.7: Evolution of tariff's components (left) and Principal utility U^P (right) against increase of K and $\gamma < 0$

2.4 Model specification

We now turn to a more precise presentation of the model and try to present it in a rather general setting. In particular, the specific assumptions on the shape on the utility, type distribution or cost functions will only be introduced later, in order present our results in a more explicit fashion.

In this model, the Principal is a power company, whose purpose is to offer to its clients a collection of tariffs in order to maximise its profits. The time horizon $T > 0$ is fixed. The following notations will be used throughout the article

- $\mathcal{C}$ represents the admissible levels of consumption for the Agents, and will either be $\mathbb{R}_+$ or $\mathbb{R}_+^*$ depending on the utility function of the Agents.
- $p : [0, T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$ is the tariff proposed by the Principal, such that $p(t, c)$ represents the price of electricity at time t corresponding to a level of consumption c.

- $K : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$ is the cost of production of electricity for the Principal, such that $K(t,c)$ represents the cost at time t for an aggregate level of production c. We assume that K is continuous in t, increasing, continuously differentiable and strictly convex in c.

- x is the Agent's type, assumed to take values in some subset X of $\mathbb{R}$.

- $c : [0,T] \times X \longrightarrow \mathcal{C}$ is the consumption function, such that $c(t,x)$ represents the consumption of electricity by an Agent of type x at time t.

- $u : [0,T] \times X \times \mathcal{C} \longrightarrow \mathbb{R}$ is the utility function of the Agents, such that $u(t,x,c)$ represents the utility obtained by an Agent of type x at time t when he consumes c. We will always assume that the map $c \longmapsto u(t,x,c)$ is non-decreasing and concave for every $(t,x) \in [0,T] \times X$. Moreover, the map u is assumed to be jointly continuous, such that $x \longmapsto u(t,x,c)$ is non-decreasing and differentiable Lebesgue almost everywhere for every $(t,c) \in [0,T] \times \mathcal{C}$, and such that $c \longmapsto \frac{\partial u}{\partial x}(t,x,c)$ is invertible. Finally, we assume that if $\mathcal{C} = \mathbb{R}_+$, the value $u(t,x,0) \in \mathbb{R}_+$ is independent of x, and if $\mathcal{C} = \mathbb{R}_+^*$ that $\lim_{c \to 0} u(t,x,c) = -\infty$, for every $(t,x) \in [0,T] \times X$. In other words, all the Agents have the same utility when they do not consume electricity.

- $f : X \longrightarrow \mathbb{R}_+$ is the distribution of the Agent's type over the population. As is customary in adverse selection problems, f is supposed to be known by the Principal.

2.4.1 Agent's problem

Let us start by defining the consumption strategies that the Agents are allowed to use. A consumption strategy c will be said to be admissible, which we denote by $c \in \mathfrak{C}$, if it is a Borel measurable map from $[0,T]$ to $\mathcal{C}$. Given a tariff p, that is a map from $[0,T] \times \mathbb{R}_+$ to $\mathbb{R}$, proposed by the Principal, an Agent of type $x \in X$ determines his consumption by solving the following problem

$$U_A(p,x) := \sup_{c \in \mathfrak{C}} \int_0^T \big(u(t,x,c(t)) - p(t,c(t))\big)\,\mathrm{d}t. \tag{2.4.1}$$

The tariff that the Principal can offer to the Agents has to satisfy the incentive compatibility (IC) and the individual rationality (IR) conditions. In our setting, there is no moral hazard, so that the incentive compatibility condition is automatically satisfied. Furthermore, the (IR) condition can be expressed through the set $X(p)$ of the types of Agents which accept the contract p, which can be defined as

$$X(p) := \{x \in X,\ U_A(p,x) \geq H(x)\},$$

with a continuous and non-decreasing function H which represents the reservation utility of the Agents of different types, that is to say the utility that the Agents can hope to obtain by subscribing their power contract with a competitor. Agents for which the map $U_A(p,\cdot)$ is smaller than H will not accept the contract offered by the Principal.

We are now ready to give our definition of admissible tariffs, which uses vocabulary from $u-$convex analysis. We have regrouped all the pertinent results and definitions in Appendix 2.7.1, for readers not necessary familiar with this theory.

Definition 2.4.1 A tariff $p:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}$ will be said to be admissible, denoted by $p\in\mathcal{P}$, if it satisfies

(i) For any $(t,x)\in[0,T]\times X(p)$ the set $\partial^\star p^\star(t,x)$ is non–empty.

(ii) The map $x\longmapsto p^\star(t,x)$ is continuous on X, differentiable Lebesgue almost everywhere, for every $t\in[0,T]$, and satisfies

$$\int_0^T\int_X\left|\frac{\partial p^\star}{\partial x}\right|(t,x)\mathrm{d}x\mathrm{d}t<+\infty.$$

(iii) If one defines the map $c^\star:[0,T]\times[0,1]\longrightarrow\mathbb{R}_+$ by

$$c^\star(t,x)=\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right), \tag{2.4.2}$$

then the restriction of p to $\{(t,c)\in[0,T]\times\mathcal{C},\ \exists x\in X(p),\ c=c^\star(t,x)\}$ is $u-$convex.

Let us comment on the above definition. First of all, the regularity assumptions are mainly technical. The main point here is that since only the clients with type in $X(p)$ are going to accept the contract, the Principal will only have to face consumptions chosen by these clients. Besides, as we are going to prove in the the next proposition, this optimal consumption is exactly $c^\star(t,x)$. Therefore, any consumption $c\in\mathcal{C}$ which does not belong to the pre–image of $X(p)$ will never have to be considered by the Principal. In particular, there is a degree of freedom when defining the value of p there. Indeed, if clients of some type x reject the contract p, they will reject any contract with a higher price. This is the reason why we do not impose the admissible tariffs to be $u-$convex on $\mathcal{C}$ but only on the corresponding pre–image of the set $X(p)$.

Our main result in this section is

Proposition 2.4.1 For every $p\in\mathcal{P}$ and for almost every $x\in X(p)$, we have

$$U_A(p,x)=\int_0^T p^\star(t,x)\mathrm{d}t,$$

and the optimal consumption of Agents of type x at any time $t\in[0,T]$ is given by $c^\star(t,x)$ defined in (2.4.2). In particular, $X(p)$ can be defined through $p^\star$ only as follows

$$X(p)=X^\star(p^\star):=\left\{x\in X,\ P^\star(x):=\int_0^T p^\star(t,x)\mathrm{d}t\geq H(x)\right\}.$$

Proof. Since the space of admissible strategies for the Agent is decomposable and the integrand is normal when p is admissible (see Definitions 14.59 and 14.27 in Rockafellar and

Wets [103] and also the particular case 14.29 of a Carethéodory integrand), we have from Theorem 14.60 in [103] that the solution of problem (2.4.1) is given by pointwise optimization. Moreover, $\partial^\star p^\star(t,x)$ is non–empty for every $(t,x) \in [0,T] \times X(p)$, so we have that every optimal consumption strategy $c^\star : [0,T] \longrightarrow \mathbb{R}_+$ satisfies $c^\star(t) \in \partial^\star p^\star(t,x)$ for almost every $t \in [0,T]$ and

$$p^\star(t,x) = u(t,x,c^\star(t)) - p(t,c^\star(t)).$$

Since $u(t,x,0)$ does not depend on x, the envelop Theorem ensures that the map $x \longmapsto p^\star(t,x)$ is differentiable Lebesgue almost everywhere and that we have for almost every $(t,x) \in [0,T] \times X(p)$

$$\frac{\partial u}{\partial x}(t,x,c^\star(t)) = \frac{\partial p^\star}{\partial x}(t,x). \tag{2.4.3}$$

Indeed, if $c^\star(t) > 0$, that is the classical envelop Theorem. Otherwise, when $\mathcal{C} = \mathbb{R}_+$, it is immediate to check, using the fact that $u(t,x,0)$ does not depend on x, that for any $(t,x) \in [0,T] \times X(p)$, we have

$$0 \in \partial^\star p^\star(t,x) \Longrightarrow 0 \in \partial^\star p^\star(t,x'), \text{ for all } x' \leq x,$$

so that both terms in (2.4.3) are then actually equal to 0.

Then, since the map $c \longmapsto \frac{\partial u}{\partial x}(t,x,c)$ is invertible, we have for almost every $(t,x) \in [0,T] \times X(p)$ that $\partial^\star p^\star(t,x)$ is a singleton, and the optimal consumption is $c^\star : [0,T] \times X(p) \longrightarrow \mathbb{R}_+$ defined in (2.4.2). □

2.4.2 The Principal's problem

The Principal sets a tariff $p \in \mathcal{P}$ as a solution to his maximization problem

$$U_P := \sup_{p \in \mathcal{P}} \int_0^T \left[\int_{X(p)} p(t, c^\star(t,x)) f(x) \mathrm{d}x - K\left(t, \int_{X(p)} c^\star(t,x) f(x) \mathrm{d}x\right)\right] \mathrm{d}t. \tag{2.4.4}$$

Using the results of Section 2.4.1, we can rewrite this problem in terms of $p^\star$ only as

$$\begin{aligned} U_P = \sup_{p \in \mathcal{P}} \int_0^T & \left[\int_{X^\star(p^\star)} \left(u\left(t, x, \left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)} \left(\frac{\partial p^\star}{\partial x}(t,x)\right)\right) - p^\star(t,x)\right) f(x)\mathrm{d}x \right. \\ & \left. -K\left(t, \int_{X^\star(p^\star)} \left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)} \left(\frac{\partial p^\star}{\partial x}(t,x)\right) f(x)\mathrm{d}x\right)\right] \mathrm{d}t. \end{aligned} \tag{2.4.5}$$

Now notice that by (2.4.3), $p^\star$ is actually non-decreasing (since $x \longmapsto u(t,x,c)$ is non-decreasing for every $(t,c) \in [0,T] \times \mathcal{C}$). Let us then consider the space C^+ of maps g, such that for every $t \in [0,T]$, $x \longmapsto g(t,x)$ is continuous and non-decreasing with

$$\int_0^T \int_X \left| \frac{\partial g}{\partial x}(t,y) \right| \mathrm{d}y \mathrm{d}t < +\infty.$$

We shall actually consider the problem $\widetilde{U}_P \geq U_P$, defined by

$$\widetilde{U}_P := \sup_{p^\star \in C^+} \int_0^T \left[\int_{X^\star(p^\star)} \left(u\left(t,x,\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)\right) - p^\star(t,x)\right) f(x)\mathrm{d}x \right.$$
$$\left. -K\left(t,\int_{X^\star(p^\star)}\left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)f(x)\mathrm{d}x\right)\right]\mathrm{d}t, \tag{2.4.6}$$

where we have forgotten the implicit link existing between p and $p^\star$, which explains why we have in general $\widetilde{U}_P \geq U_P$. We will see in the frameworks described below that we can give conditions under which the two problems are indeed equal. The main advantage of $\widetilde{U}_P$ is that it no longer contains the condition that $p^\star$ has to be $u-$convex, a constraint that is not easy to consider in full generality.

Besides, we also emphasize that since the elements of C^+ are non-decreasing with respect to x, they are also differentiable Lebesgue almost everywhere, so that $\widetilde{U}_P$ is indeed well defined. Our aim now will be to compute $\widetilde{U}_P$. However, the present framework is far too general to hope obtaining explicit solutions, which are of the utmost interest in our electricity pricing model, so that we will concentrate our attention on the case of Agents with power-type utilities. In the next Section, we will focus on the particular case where the reservation utility H is constant, and shall consider the more general case where it may depend on the Agents' type in Section 2.6.

2.5 Agents with CRRA utilities and constant reservation utility

In this section, we shall use the following standing assumptions

Assumption 2.5.1 (i) $X = [0,1]$.

(ii) We have for every $(t,x,c) \in [0,T] \times X \times \mathcal{C}$

$$u(t,x,c) = g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma},$$

for some $\gamma \in (-\infty,0) \cup (0,1)$, some map $g_\gamma : X \longrightarrow \mathbb{R}_+$ which is continuous, increasing if $\gamma \in (0,1)$, decreasing if $\gamma \in (-\infty,0)$, and for some continuous map $\phi : [0,T] \longrightarrow \mathbb{R}_+^\star$.

Let us comment on this modeling choice for the utility function. The term $g_\gamma(x)$ represents the willingness of the Agents to pay for their consumption, *i.e.* their need for energy depends on their type. The term ϕ is common to every type of Agents and represents the fact that (almost) everyone is eager to consume at the same time (for example during the day rather than at night). Furthermore, we consider both the cases $\gamma \in (0,1)$, which would be the classical power utility function, as well as the case $\gamma < 0$, which corresponds to a situation where Agents actually cannot avoid consuming electricity, as it would provide them a utility equal to $-\infty$, which may be seen as more realistic. As discussed previously, taking $\gamma \in (0,1)$ identifies to considering industrial Agents, as $\gamma < 0$ more typically refers to residential Agents.

Equation (2.4.2) now can be written as

$$c^\star(t,x) = \left(\frac{\gamma}{\phi(t)\, g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}}. \tag{2.5.1}$$

By inserting the previous expression in equation (2.4.4) and using that

$$p(t, c^\star(t,x)) = g_\gamma(x)\phi(t)\frac{c^\star(t,x)^\gamma}{\gamma} - p^\star(t,x),$$

the problem to solve can now be expressed as

$$\begin{aligned}\widetilde{U}_P = \sup_{p^\star \in C^+} \int_0^T \Bigg[& \int_{X^\star(p^\star)} \left(\frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)\mathrm{d}x \\ & - K\left(t, \int_{X^\star(p^\star)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\Bigg] \mathrm{d}t. \end{aligned} \tag{2.5.2}$$

We consider in this section a further simplification, related to the reservation utility of the Agents, which we suppose to be independent of their type. This assumption will be relieved in Section 2.6 hereafter.

Assumption 2.5.2 The reservation utility H is actually independent of x, that is

$$H(x) =: H, \text{ for every } x \in [0,1].$$

Under Assumptions 2.5.1 and 2.5.2, the (IR) condition reduces to

$$X^\star(p^\star) = \left\{x \in [0,1],\ \int_0^T p^\star(t,x)\mathrm{d}t \geq H\right\}.$$

Since $p^\star$ is non-decreasing in x we have for any $x_0 \in [0,1]$ that

$$\int_0^T p^\star(t,x_0)\mathrm{d}x \geq H \implies \int_0^T p^\star(t,x)\mathrm{d}x \geq H,\ \forall x \geq x_0.$$

Therefore, the set $X^\star(p^\star)$ has necessarily the form

$$X^\star(p^\star) = [x_0, 1],$$

where $x_0 \in [0,1]$ needs to be determined and verifies, by continuity, that $P^\star(x_0) = H$. This means that the Principal will only offer contracts to Agents of type greater than x_0. The problem (2.5.2) can therefore be written equivalently

$$\begin{aligned}\widetilde{U}_P = \sup_{x_0 \in [0,1]} \sup_{p^\star \in C^+(x_0)} \int_0^T \Bigg[& \int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)\mathrm{d}x \\ & - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t) g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\Bigg] \mathrm{d}t, \end{aligned} \tag{2.5.3}$$

with

$$C^+(x_0) := \left\{ p^\star \in C^+,\ \int_0^T p^\star(t,x_0)\mathrm{d}t = H \right\} = \left\{ p^\star \in C^+,\ X^\star(p^\star) = [x_0,1] \right\}.$$

Let us end this section with the following sufficient condition of $u-$convexity when Assumption 2.5.1 holds. Its proof is deferred to Appendix 2.7.2.

Lemma 2.5.1 Let Assumption 2.5.1 hold and suppose in addition that g_γ is concave if $\gamma \in (0,1)$ and convex if $\gamma \in (-\infty, 0)$. Let $\psi : [0,T] \times X \longrightarrow \mathbb{R}$ be a map such that $x \longmapsto \psi(t,x)$ is non-decreasing and convex. Then ψ is $u-$convex. Furthermore, if we take

$$g_\gamma(x) := \begin{cases} x, \text{ if } \gamma \in (0,1), \\ 1-x, \text{ if } \gamma < 0, \end{cases}$$

then any $u-$convex function is convex.

2.5.1 General distribution of costs and Agent types

Denote by F the cumulative distribution function of the types of Agents. By integration by parts we have for every $x_0 \in [0,1]$ and every $p^\star \in C^+(x_0)$

$$\begin{aligned}
&\int_0^T \left[\int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t \\
&= \int_0^T \left[\int_{x_0}^1 \left(\frac{g_\gamma(x)}{g'_\gamma(x)} f(x) + F(x) - 1 \right) \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t \\
&\quad + (F(x_0) - 1) \int_0^T p^\star(t,x_0)\mathrm{d}t.
\end{aligned}$$

We therefore end up with the maximization problem

$$\begin{aligned}
\widetilde{U}_P = \sup_{x_0 \in [0,1]} \sup_{p^\star \in C^+(x_0)} \int_0^T &\left[\int_{x_0}^1 \frac{\left(g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) \right)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x \right. \\
&\left. - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t + (F(x_0) - 1)H. \quad (2.5.4)
\end{aligned}$$

We can now state our main result of this section, wholes proof is postponed to Appendix 2.7.2. It requires the introduction of the following function

$$\ell(x_0) := \int_{x_0}^1 \left(\frac{\left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) \right]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x,\ x_0 \in [0,1].$$

Theorem 2.5.1 Let Assumptions 2.5.1 and 2.5.2 hold. We have

(i) The maximum in (2.5.3) is attained for the maps

$$p^\star(t,x) = p^\star(t,x_0^\star) + \int_{x_0^\star}^x \frac{g_\gamma'(y)}{\gamma}\left(\frac{\phi(t)^{\frac{1}{\gamma}}\left[g_\gamma(y)f(y)+g_\gamma'(y)F(y)-g_\gamma'(y)\right]^+}{f(y)\frac{\partial K}{\partial c}(t,A(t,x_0^\star))}\right)^{\frac{\gamma}{1-\gamma}}\mathrm{d}y,\ x\in[0,1],$$

where $A(t,x_0)$ is defined

$$A(t,x_0) := \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g_\gamma'(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x.$$

and $x_0^\star$ is any maximizer of the map and

$$[0,1]\ni x_0 \longmapsto \int_0^T \left(\frac{\partial K}{\partial c}(t,A(t,x_0))m(t,x_0) - K(t,\gamma m(t,x_0))\right)\mathrm{d}t + (F(x_0)-1)H,$$

with

$$m(t,x_0) := \frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{1}{1-\gamma}}},\ (t,x_0)\in[0,T]\times[0,1],$$

and $t\longmapsto p^\star(t,x_0^\star)$ is any map such that

$$\int_0^T p^\star(t,x_0^\star)\mathrm{d}t = H.$$

For instance, one can choose $p^\star(t,x_0^\star) := H/T$, $t\in[0,T]$.

(ii) Define p for any $(t,c)\in[0,T]\times\mathbb{R}_+$ by

$$p(t,c) = \sup_{x\in[0,1]}\left\{g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma} - p^\star(t,x)\right\}.$$

If the map defined on $[0,1]$ by

$$x\longmapsto g_\gamma'(x)\left(\frac{\left[g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x)\right]^+}{f(x)}\right)^{\frac{\gamma}{1-\gamma}},$$

is non-decreasing, then $p^\star$ is $u-$convex, and p is the optimal tariff for the problem (2.4.4). Furthermore, the Principal only signs contracts with the Agents of type $x\in[x_0^\star,1]$.

(iii) Finally, in the case $\gamma\in(0,1)$, if f is non-increasing and the map

$$\beta : x\longmapsto \frac{(g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x))}{f^\gamma(x)},$$

is increasing over the set $L:=\{x\in[0,1],\ \beta(x)>0\}$, then $x_0^\star$ is unique and is characterized by the equation

$$\left(\frac{1-\gamma}{\gamma}\right)\frac{\phi(t)^{\frac{1}{1-\gamma}}\beta(x_0^\star)}{\left(\frac{\partial K}{\partial c}(t,A(t,x_0^\star))\right)^{\frac{\gamma}{1-\gamma}}} = f(x_0^\star)H.$$

The same result holds in the case $\gamma\in(-\infty,0)$ if f is non-decreasing and β is decreasing over L.

2.5.2 An explicit example

We insist on the fact that the tariff p defined in Theorem 2.5.1 is $u-$convex by definition, and it is finite since it is written as a supremum of a continuous function over a compact set. In order to verify that $p \in \mathcal{P}$, one therefore only needs to make sure that $p^\star$ is indeed the $u-$transform of p (which is the case if $p^\star$ is $u-$convex) and satisfies the other required properties. We will consider here a simplified framework where all the computations can be done almost explicitly.

Assumption 2.5.3 The cost function K is given, for some $n > 1$, by

$$K(t,c) := k(t)\frac{c^n}{n},\ (t,c) \in [0,T] \times \mathbb{R}_+,$$

for some map $k : [0,T] \longrightarrow \mathbb{R}_+^\star$. Moreover, the distribution of the type of Agents is uniform, that is $f(x) = 1$, and we impose $g_\gamma(x) := x\mathbf{1}_{\gamma\in(0,1)} + (1-x)\mathbf{1}_{\gamma<0}$, for every $x \in [0,1]$.

Under Assumption 2.5.3, we then have

$$A(t,x_0) = \left(\frac{\phi(t)}{k(t)}\right)^{\frac{1}{n-\gamma}} \ell^{\frac{1-\gamma}{n-\gamma}}(x_0),$$

and the maximization problem becomes

$$\widetilde{U}_P = \sup_{x_0\in[0,1]} \left\{ \left(\frac{1}{\gamma} - \frac{1}{n}\right) \int_0^T \left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}} \mathrm{d}t\ \ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H \right\}.$$

Define

$$B_\gamma(T) := \left(\frac{1}{\gamma} - \frac{1}{n}\right) \int_0^T \left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}} \mathrm{d}t,\ \Phi(x_0) := B_\gamma(T)\ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0-1)H,$$

where we emphasize that since $n > 1$, when $\gamma \in (0,1)$, we easily have that $B_\gamma(T) > 0$, while $B_\gamma(T) < 0$ when $\gamma < 0$. Furthermore, we remind the reader that when $\gamma > 0$, the reservation utility of the Agents is necessarily non-negative, while it has to be negative when $\gamma < 0$, since the utility function itself is negative.

Our result rewrites in this case

Theorem 2.5.2 Let Assumptions 2.5.1, 2.5.2 and 2.5.3 hold.

(i) If $\gamma \in (0,1)$, then, the optimal tarif $p \in \mathcal{P}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = \phi(t)\frac{c^\gamma}{2\gamma} + \left(\left(\frac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c - \frac{H}{T} + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}},$$

where

$$M(t) = \frac{1-\gamma}{2\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}} \left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} \left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}},$$

and where $x_0^\star$ is the unique solution in $(1/2,1)$ of the equation

$$H = 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma}(2x_0^\star - 1)^{\frac{1}{1-\gamma}}\left(1-(2x_0^\star-1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

Furthermore, only the Agents of type $x \geq x_0^\star$ will accept the contract.

(ii) If $\gamma < 0$, then, the optimal tarif $p \in \mathcal{P}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = -\gamma c\left(-\frac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}}\left(\frac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \frac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}},$$

where

$$\widehat{M}(t) = -\frac{1-\gamma}{\gamma}\left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\frac{2^\gamma\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}}(1-\widehat{x}_0^\star)^{-\frac{\gamma(2-\gamma)(n-1)}{(n-\gamma)(1-\gamma)}},$$

and where

$$\widehat{x}_0^\star := \left(1-\left(\frac{n-\gamma}{n(1-\gamma)B_\gamma(T)}H\right)^{\frac{n-\gamma}{n(1-\gamma)+\gamma}}\left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{-\gamma(n-1)}{n(1-\gamma)+\gamma}}2^{\frac{-n}{n(1-\gamma)+\gamma}}\right)^+.$$

Furthermore, only the Agents of type $x \geq \widehat{x}_0^\star$ will accept the contract.

2.6 Agents with CRRA utilities and general reservation utility

In this part we study the case where the reservation utility H is a general continuous and non-decreasing function of the Agent type $x \in [0,1]$. This case strongly differs from the previous section as the existence of a solution to the infinite-dimensional problem faced by the Principal is not guaranteed and we need to impose some additional structure to the set of admissible tariffs. Specifically, we will consider a new set of admissible tariffs which is contained in a reflexive Banach space and we will use the classical results from functional analysis to prove the existence of solutions to the Principal's problem. With that purpose in mind, we introduce the following Sobolev-like spaces.

Definition 2.6.1 For any $\ell \geq 1$ and any open subset $\mathcal{O}$ of X, we denote by $W_x^{1,\ell}(\mathcal{O})$ the space of maps $q : [0,T] \times \mathcal{O} \longrightarrow \mathbb{R}$ for which there exists a null set $\mathcal{N}(q) \subset [0,T]$ (for the Lebesgue measure) satisfying that for every $t \in [0,T] \setminus \mathcal{N}(q)$ the map $x \longmapsto q(t,x)$ belongs to $W^{1,\ell}(\mathcal{O})$[1] and such that

$$||q||_{\ell,\mathcal{O}} := \left(\int_0^T\int_{\mathcal{O}}|q(t,x)|^\ell \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{\ell}} + \left(\int_0^T\int_{\mathcal{O}}\left|\frac{\partial q}{\partial x}(t,x)\right|^\ell \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{\ell}} < \infty.$$

[1] That is to say the usual Sobolev space of maps admitting a weak first order derivative.

Remark 2.6.1 For the rest of the chapter, for every map q belonging to some space $W_x^{1,\ell}(\mathcal{O})$, the set $\mathcal{N}(q)$ will make reference to the one mentioned in Definition 2.6.1.

For all the analysis of this section, we fix a number $m > 1$ such that $m\gamma < 1$. We are now ready to give our new definition of admissible tariffs.

Definition 2.6.2 A tariff $p : [0, T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is said to be admissible (in the case when H is not constant), denoted by $p \in \widehat{\mathcal{P}}$, if in addition to Definition 2.4.1, it satisfies that $p^\star \in W_x^{1,m}(\overset{o}{X})$.

In this new setting, the Principal offers a tariff $p \in \widehat{\mathcal{P}}$ which solves his maximization problem

$$\widehat{U}_P := \sup_{p\in\widehat{\mathcal{P}}} \int_0^T \left[\int_{X^\star(p^\star)} p(t, c^\star(t, x)) f(x) \mathrm{d}x - K\left(t, \int_{X^\star(p^\star)} c^\star(t, x) f(x) \mathrm{d}x\right)\right] \mathrm{d}t. \tag{2.6.1}$$

Following the previous sections, we will consider the problem $\overline{U}_P \geq \widehat{U}_P$, defined by

$$\begin{aligned}\overline{U}_P = \sup_{p^\star\in\widehat{C}^+} \int_0^T & \left[\int_{X^\star(p^\star)} \left(\frac{g_\gamma(x)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t, x) - p^\star(t, x) \right) f(x) \mathrm{d}x \right. \\ & \left. - K\left(t, \int_{X^\star(p^\star)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t, x) \right)^{\frac{1}{\gamma}} f(x) \mathrm{d}x \right) \right] \mathrm{d}t, \end{aligned} \tag{2.6.2}$$

where $\widehat{C}^+ = C^+ \cap W_x^{1,m}(\overset{o}{X})$. We aim at solving $\overline{U}_P$ and give conditions under which it coincides with $\widehat{U}_P$.

Now, moving to the reservation utility function H, recall that it determines the set $X^\star(p^\star)$. In order to avoid complex forms of this set we make the following assumption on g, H and f.

Assumption 2.6.1 The functions g and H are such that for every $x \in [0, 1]$

$$\frac{g_\gamma(x)}{g'_\gamma(x)} \leq \frac{H(x)}{H'(x)}. \tag{2.6.3}$$

Moreover, the following maps

$$\begin{aligned} v_1(x) &:= g'_\gamma(x) \left(\frac{\left[g_\gamma(x) f(x) + g'_\gamma(x) F(x)\right]^+}{f(x)} \right)^{\frac{\gamma}{1-\gamma}}, \\ v_2(x) &:= g'_\gamma(x) \left(\frac{\left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x)\right]^+}{f(x)} \right)^{\frac{\gamma}{1-\gamma}}, \end{aligned}$$

are non-decreasing on $[0, 1]$.

Remark 2.6.2 Condition (2.6.3) is equivalent to the elasticity of reservation utility being less than the elasticity of willingness to pay for consumption. For instance, in the case $\gamma \in (0,1)$, it is automatically satisfied when H is constant and if $g_\gamma(x) = x$ then (2.6.3) reduces to H being concave. Similarly, in the case $\gamma < 0$, (2.6.3) holds if $g_\gamma(x) = 1 - x$ and $H(x) = x^\alpha$ with $\alpha > 1$. On the other hand, if for instance, $f(x) = 1$ and $g(x) = x^\alpha$ with $\alpha \in (0,1]$, then v_1, v_2 are increasing when $\alpha \geq 1 - \gamma$.

The following proposition shows that when Condition (2.6.3) holds, it is actually never optimal for the Principal to propose a tariff for which the (IR) condition is binding on any measurable subset of $[0,1]$ with positive Lebesgue measure. Its proof is postponed to Appendix 2.7.3.

Proposition 2.6.1 Let Assumptions 2.5.1 and 2.6.1 hold, and let $p^\star \in \widehat{C}^+$ be any function such that the set

$$Y^\star(p^\star) := \{x \in [0,1],\ P^\star(x) = H(x)\},$$

has positive Lebesgue measure. Then $p^\star$ is not optimal for problem (2.6.2).

In this section, we show that the problem splits into subintervals. Thanks to the previous proposition, we can consider without loss of generality functions $p^\star \in \widehat{C}^+$ such that the Lebesgue measure of $Y^\star(p^\star)$ is zero. For these functions, we define the set

$$\widehat{X}^\star(p^\star) := X^\star(p^\star) \setminus Y^\star(p^\star) = \{x \in [0,1],\ P^\star(x) > H(x)\},$$

which by continuity is an open subset of $[0,1]$. As $X^\star$ and $p^\star$ are continuous, we can replace all the integrals over $X^\star(p^\star)$ by integrals over $\widehat{X}^\star(p^\star)$ and we can write the latter set as a countable union of open disjoint intervals, that is

$$\widehat{X}^\star(p^\star) := [0, b_0) \cup \bigcup_{n\geq 1}(a_n, b_n) \cup (a_0, 1],$$

for some $a_0 \in (0,1]$, $b_0 \in [0,1)$, and $0 < a_n < b_n < 1$, $\forall n \geq 0$. We denote $a := (a_n)_{n\geq 0}$, $b := (b_n)_{n\geq 0}$ and define $\mathcal{A}$ as the set of such that pairs (a,b). For any $(a,b) \in \mathcal{A}$, we also define the set

$$X^\star(a,b) = [0, b_0) \cup \bigcup_{n\geq 1}(a_n, b_n) \cup (a_0, 1].$$

Remark 2.6.3 The case $b_0 = 0$ stands for $P^\star(b_0) < H(b_0)$ and the case $a_0 = 1$ stands for $P^\star(a_0) < H(a_0)$. By continuity we have $P^\star(a_n) = H(a_n)$ and $P^\star(b_n) = H(b_n)$ for every $n \geq 1$.

We can therefore write

$$\overline{U}_P = \sup_{(a,b)\in\mathcal{A}} \sup_{p^\star \in C^+(a,b)} \int_0^T \left[\int_{X^\star(a,b)} \left(\frac{g(x)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x \right.$$
$$\left. - K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t) g'(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t, \qquad (2.6.4)$$

where $C^+(a,b)$ is given by all the maps $p^\star \in \widehat{C}^+$ such that $\widehat{X}^\star(p^\star) = X^\star(a,b)$. Define for every $(a,b) \in \mathcal{A}$ the operator $\Psi_{(a,b)} : C^+(a,b) \longrightarrow \mathbb{R}$ by

$$\Psi_{(a,b)}(p^\star) := \int_0^T \left[\int_{X^\star(a,b)} \left(\frac{g_\gamma(x)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x \right.$$
$$\left. - K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t.$$

We optimize first with respect to $p^\star$, and then study the problem

$$(P_{a,b}) \qquad \sup_{p^\star \in C^+(a,b)} \Psi_{(a,b)}(p^\star).$$

Our first result gives the existence of a solution to the above infinite-dimensional optimization problem, and requires the following assumption, which involves mainly the cost function. It is required in order to obtain nice coercivity properties

Assumption 2.6.2 The cost function K satisfies the following growth condition

$$K(t,c) \geq k(t)c^n, \ \forall c \in \mathcal{C},$$

where the map $k : [0,T] \longrightarrow \mathbb{R}_+$ is bounded from below by some constant $\underline{k} > 0$ and $n \geq 1$. Moreover, we have that

$$I := \inf \left\{ k(t) \left(\frac{\gamma f(x)}{\phi(t) g'_\gamma(x)} \right)^n, \ (t,x) \in [0,T] \times [0,1] \right\} > 0.$$

We now have

Proposition 2.6.2 Let Assumptions 2.5.1 and 2.6.2 hold. For every $(a,b) \in \mathcal{A}$, the optimization problem $(P_{a,b})$ has at least one solution.

Next, we obtain necessary optimality conditions for $(P_{a,b})$. Recalling from Remark 2.6.3 that the (IR) condition is binding at each a_n, b_n, by integration by parts we can rewrite $\Psi_{(a,b)}$ as

$$\begin{aligned}
\Psi_{(a,b)}(p^\star) = & \int_0^T \int_0^{b_0} \frac{\big(g_\gamma(x)f(x) + g'_\gamma(x)F(x)\big)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
& + \int_0^T \int_{a_0}^1 \frac{\big(g_\gamma(x)f(x) + g'_\gamma(x)F(x) - g'_\gamma(x)\big)}{g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
& + \sum_{n=1}^\infty \int_0^T \int_{a_n}^{b_n} \frac{\big(g_\gamma(x)f(x) + g'_\gamma(x)F(x)\big)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
& - K\left(t, \int_{(0,b_0)\cup\bigcup_{n\geq 1}(a_n,b_n)\cup(a_0,1)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right)\mathrm{d}t \\
& + \sum_{n=1}^\infty F(a_n)H(a_n) - \sum_{n=1}^\infty F(b_n)H(b_n) - F(b_0)H(b_0) + (F(a_0)-1)H(a_0).
\end{aligned} \tag{2.6.5}$$

To simplify notations, denote

$$A(t,a,b) := \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t) g'_\gamma(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x) \mathrm{d}x.$$

Theorem 2.6.1 Let Assumptions 2.5.1 and 2.6.2 hold and let $p^\star$ be a solution of $(P_{a,b})$. Consider an interval $I = (x_\ell, x_r) \subseteq X^\star(a,b)$ such that $P^\star(x) > H(x)$ for every $x \in I$, $P^\star(x_\ell) = H(x_l)$ and $P^\star(x_r) = H(x_r)$. Then there exists a null set $\mathcal{N} \subset [0,T]$ and a constant μ_t for every $t \in [0,T] \setminus \mathcal{N}$ such that the following optimality condition is satisfied

(i) In the case $I \subseteq (a_0, 1)$, for every $x \in I$ we have

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) + g'_\gamma(x) \mu_t \right]^+}{f(x) \frac{\partial K}{\partial c}\left(t, A(t,a,b)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \quad (2.6.6)$$

(ii) In the case $I \subseteq (0, b_0) \cup_{n \geq 1} (a_n, b_n)$, for every $x \in I$ we have

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) + g'_\gamma(x) \mu_t \right]^+}{f(x) \frac{\partial K}{\partial c}\left(t, A(t,a,b)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \quad (2.6.7)$$

The proof of the Theorem 2.6.1 consists in several technical propositions which are given and proved in Section 2.7.3.2 below.

The next and last proposition of this section proves the convexity of the solutions to problems $(P_{a,b})$ over intervals where the (IR) condition is not binding. This result will allow us to completely solve the Principal's problem in the next subsection, when the function H is strictly concave.

Proposition 2.6.3 Let Assumptions 2.5.1, 2.6.1 and 2.6.2 hold. Let $p^\star$ be a solution to problem $(P_{a,b})$. Then $P^\star$ is convex on every interval over which $P^\star$ is strictly greater than H.

Proof. From Assumption 2.6.1 and Theorem 2.6.1 we have that on every interval I over which $P^\star > H$, there exists a null set $\mathcal{N} \subset [0,T]$ such that for every $t \in [0,T] \setminus \mathcal{N}$, $x \longmapsto \frac{\partial p^\star}{\partial x}(t,x)$ is non-decreasing on I. Therefore $P^\star$ is convex on I since

$$\frac{\partial P^\star}{\partial x}(x) = \int_0^T \frac{\partial p^\star}{\partial x}(t,x) \mathrm{d}t.$$

□

2.6.1 Strictly concave reservation utility

In this section we show that there are at most that

Assumption 2.6.3 The map $x \longmapsto H(x)$ is strictly concave and non-decreasing.

The main interest of Assumption 2.6.3 is the following simple result, which shows that we can always restrict our attention to sets $X^\star$ with a very simple form.

Proposition 2.6.4 Let Assumptions 2.5.1, 2.6.1, 2.6.2 and 2.6.3 hold. Let $(a, b) \in \mathcal{A}$ be such that $0 < a_{n_0} < b_{n_0} < 1$ for some $n_0 \geq 1$. Then the solution to problem $(P_{a,b})$ is not optimal for problem (2.6.2).

Proof. Let $p^\star$ be the solution of problem $(P_{a,b})$. We will prove that $P^\star \equiv H$ in the interval (a_{n_0}, b_{n_0}) and the result will follow from Proposition 2.6.1. Suppose not, then there exists $x_0 \in (a_{n_0}, b_{n_0})$ such that $P^\star(x_0) > H(x_0)$ and $p^\star$ is given by (2.6.6) in a neighbourhood around x_0, so $P^\star$ is increasing in that neighbourhood. By Proposition 2.6.3 we have that $P^\star(b_{n_0}) > H(b_{n_0})$, because on every interval which is contained in the set $\{x,\ P^\star(x) \geq H(x)\}$ the convex map $P^\star$ and the strictly concave map H can intersect at most at one point. This contradicts the fact that $p^\star \in C^+(a, b)$. $\square$

In the rest of this section, we start by deriving a general solution under some implicit assumptions, and then show that the latter can be verified in the context of Assumption 2.5.3.

2.6.1.1 The general tariff

Proposition 2.6.4 implies that the solution of (2.6.4) is attained for some $p^\star$ satisfying $\widehat{X}^\star(p^\star) = [0, b_0) \cup (a_0, 1]$. We expect then the optimal tariff to look like the curve in Figure 2.8.

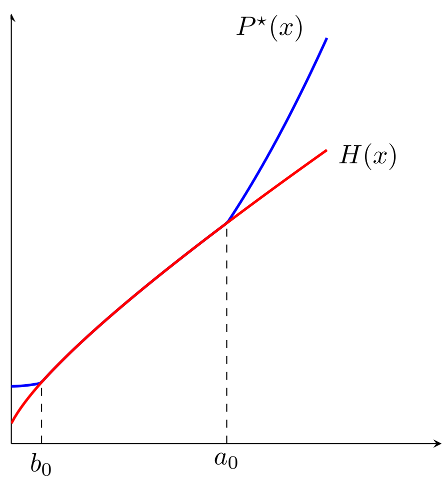


Figure 2.8: $\widehat{X}^\star(p^\star)$ for strictly concave H.

Let us then define the set

$$\mathcal{A}_2 := \left\{(a, b) \in [0, 1]^2,\ b \leq a\right\}.$$

Theorem 2.6.1 gives us only partial information about the solution of the problem $(P_{a,b})$, for

$(a,b) \in \mathcal{A}_2$. Now that we assume in addition that H is strictly concave, we can actually complete that information with the following proposition, which tells us that the value of the constants μ_t is zero in the intervals of the form $[0,b)$ and $(a,1]$. Its proof is postponed to the Appendix.

Proposition 2.6.5 Let Assumptions 2.5.1, 2.6.1, 2.6.2 and 2.6.3 hold. Let $p^\star$ be a solution of $(P_{a,b})$, for $(a,b) \in \mathcal{A}_2$, and I be as in Theorem 2.6.1. Then there exists a null set $\mathcal{N}' \subset [0,T]$ such that for every $t \in [0,T] \setminus \mathcal{N}'$ the optimality conditions from Theorem 2.6.1 hold with $\mu_t = 0$.

Following the computations of Section 2.5, we define

$$A(t,a_0,b_0) := g_K^{(-1)} \left(\phi(t)^{\frac{1}{1-\gamma}} \int_0^{b_0} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right.$$
$$\left. + \phi(t)^{\frac{1}{1-\gamma}} \int_{a_0}^1 \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right).$$

The aim of the next proposition is similar in spirit to that of Proposition 2.6.4, in the sense that it allows to exclude many specifications of $(a,b) \in \mathcal{A}_2$ when one is interested in solving problem (2.6.2).

Proposition 2.6.6 Let Assumptions 2.5.1, 2.6.1, 2.6.2 and 2.6.3 hold. Let $p^\star$ be a solution of $(P_{a,b})$. If either

$$\Xi_\gamma(a_0,b_0) := \int_0^T \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(a_0)f(a_0) + g'_\gamma(a_0)F(a_0) - g'_\gamma(a_0) \right]^+}{f(a_0) \frac{\partial K}{\partial c}(t, A(t,a_0,b_0))} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(a_0)}{\gamma} \mathrm{d}t < H'(a_0),$$

or

$$\Psi_\gamma(a_0,b_0) := \int_0^T \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(b_0)f(b_0) + g'_\gamma(b_0)F(b_0) \right]^+}{f(b_0) \frac{\partial K}{\partial c}(t, A(t,a_0,b_0))} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(b_0)}{\gamma} \mathrm{d}t > H'(b_0),$$

then the solution to problem $(P_{a,b})$ is not optimal for problem (2.6.2).

Judging by the results of Proposition 2.6.6, it is natural to define $\mathcal{A}'_2$ as the set of all the pairs $(a,b) \in \mathcal{A}_2$ for which

$$\Xi_\gamma(a_0,b_0) \geq H'(a_0), \;\; \Psi_\gamma(a_0,b_0) \leq H'(b_0).$$

Thanks to Proposition 2.6.6, we have thus reduced the problem $(P_{a,b})$ to

$$\sup_{(a_0,b_0)\in\mathcal{A}'_2} \int_0^T \left[\frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_0,b_0)}{\gamma \left(\frac{\partial K}{\partial c}(t, A(t,a_0,b_0)) \right)^{\frac{\gamma}{1-\gamma}}} - K \left(t, \frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_0,b_0)}{\left(\frac{\partial K}{\partial c}(t, A(t,a_0,b_0)) \right)^{\frac{1}{1-\gamma}}} \right) \right] \mathrm{d}t + \theta(a_0,b_0).$$

where we abused notations and defined the corresponding functions

$$\begin{aligned}\ell(a_0,b_0) :=& \int_0^{b_0}\left(\frac{[g(x)f(x)+g'(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x\\ &+\int_{a_0}^1\left(\frac{[g(x)f(x)+g'(x)F(x)-g'(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x,\\ \theta(a_0,b_0) :=& -F(b_0)H(b_0)+(F(a_0)-1)H(a_0),\end{aligned}$$

Since all these maps are continuous on $[0,1]^2$, the supremum over the compact set above is attained at some $(a_0^\star,b_0^\star)\in\mathcal{A}_2'$. We have therefore proved

Theorem 2.6.2 Let Assumptions 2.5.1, 2.6.1, 2.6.2 and 2.6.3 hold. We have

(i) The maximum in (2.6.2) is attained for the map

$$p^\star(t,x)=\begin{cases}\dfrac{H(b_0^\star)}{T}-\dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_0^\star,b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}}\displaystyle\int_x^{b_0^\star}v_1(y)\mathrm{d}y, \text{ if } x\in[0,b_0^\star),\\ \tilde{p}^\star(t,x), \text{ if } x\in[b_0^\star,a_0^\star],\\ \dfrac{H(a_0^\star)}{T}+\dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_0^\star,b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}}\displaystyle\int_{a_0^\star}^x v_2(y)\mathrm{d}y, \text{ if } x\in(a_0^\star,1],\end{cases}$$

where $\tilde{p}^\star(t,x)$ is any continuous and non-decreasing map (with respect to x) such that

$$\int_0^T\tilde{p}^\star(t,b_0^\star)\mathrm{d}t=H(b_0^\star),\ \int_0^T\tilde{p}^\star(t,a_0^\star)\mathrm{d}t=H(a_0^\star),\ \int_0^T\tilde{p}^\star(t,x)\mathrm{d}t<H(x), \text{ for all } x\in(b_0^\star,a_0^\star).$$

(ii) Define p, for any $(t,c)\in[0,T]\times\mathbb{R}_+$, by

$$p(t,c):=\sup_{x\in[0,1]}\left\{g_\gamma(x)\phi(t)\frac{c^\gamma}{\gamma}-p^\star(t,x)\right\}.$$

If $p^\star$ is $u-$convex on $X^\star(p^\star)$, then p is the optimal tariff for the problem (2.6.1). Furthermore, the Principal only signs contracts with the Agents of type $x\in[0,b_0^\star]\cup[a_0^\star,1]$.

2.6.1.2 Power type cost function

Exactly as in the case where H was independent of x, the computations become much simpler as soon as Assumption 2.5.3 holds. Let's note $R_\gamma(a_0,b_0)=1+(2b_0)^{\frac{2-\gamma}{1-\gamma}}-((2a_0-1)^+)^{\frac{2-\gamma}{1-\gamma}}$ if $\gamma\in(0,1)$ and $R_\gamma(a_0,b_0)=1-((1-2b_0)^+)^{\frac{2-\gamma}{1-\gamma}}+(2-2a_0)^{\frac{2-\gamma}{1-\gamma}}$ if $\gamma<0$. Then, the functions ℓ and A are given, for any $(t,a_0,b_0)\in[0,T]\times\mathcal{A}_2'$, by

$$\ell_\gamma(a_0,b_0)=\frac{1-\gamma}{2(2-\gamma)}R_\gamma(a_0,b_0),\ A(t,a_0,b_0)=\left(\frac{\phi(t)}{k(t)}\right)^{\frac{1}{n-\gamma}}\ell(a_0,b_0)^{\frac{1-\gamma}{n-\gamma}}.$$

So, in order to obtain $(a_0^\star, b_0^\star)$ we have to solve

$$\sup_{(a_0,b_0)\in\mathcal{A}_2'} \left(\frac{1}{\gamma}-\frac{1}{n}\right)\int_0^T \left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}} \mathrm{d}t\ \ell(a_0,b_0)^{\frac{n(1-\gamma)}{n-\gamma}} - b_0H(b_0) + (a_0-1)H(a_0). \tag{2.6.8}$$

Let us now compute the associated tariff p and check that p indeed belongs to $\mathcal{P}$ and that its $u-$transform is $p^\star$. Fix some $t\in[0,T]$ and define

$$N_\gamma := \frac{2^{\frac{\gamma}{1-\gamma}}(1-\gamma)}{\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} R_\gamma(a_0,b_0)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

Recall that by Proposition 2.6.6, the following inequalities must be satisfied

(i) If $\gamma\in(0,1)$

$$\frac{((2a_0^\star-1)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star,b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}\mathrm{d}t \geq H'(a_0^\star), \tag{2.6.9}$$

$$\frac{(2b_0^\star)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star,b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}\mathrm{d}t \leq H'(b_0^\star). \tag{2.6.10}$$

(ii) If $\gamma<0$

$$-\frac{(2(1-a_0^\star)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star,b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}\mathrm{d}t \geq H'(a_0^\star), \tag{2.6.11}$$

$$-\frac{((1-2b_0^\star)^+)^{\frac{\gamma}{1-\gamma}}}{\gamma\ell(a_0^\star,b_0^\star)^{\frac{\gamma(n-1)}{n-\gamma}}}\int_0^T\left(\frac{\phi(t)^n}{k(t)^\gamma}\right)^{\frac{1}{n-\gamma}}\mathrm{d}t \leq H'(b_0^\star). \tag{2.6.12}$$

Notice in particular that when $\gamma\in(0,1)$, (2.6.9) implies that $a_0^\star>1/2$, since H is increasing. With similar computations as in Section 2.5.2, we compute that

(i) If $\gamma\in(0,1)$

$$p^\star(t,x) = \begin{cases} \dfrac{H(b_0^\star)}{T} - N_\gamma\left((b_0^\star)^{\frac{1}{1-\gamma}} - x^{\frac{1}{1-\gamma}}\right), \text{ if } x\in[0,b_0^\star),\\ \tilde{p}^\star(t,x), \text{ if } x\in[b_0^\star,a_0^\star],\\ \dfrac{H(a_0^\star)}{T} + N_\gamma\left(\left(x-\dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \left(a_0^\star-\dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}\right), \text{ if } x\in(a_0^\star,1], \end{cases}$$

(ii) If $\gamma<0$

$$p^\star(t,x) = \begin{cases} \dfrac{H(b_0^\star)}{T} - N_\gamma\left(\left(\dfrac{1}{2}-b_0^\star\wedge\dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} - \left(\dfrac{1}{2}-x\wedge\dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}\right), \text{ if } x\in[0,b_0^\star),\\ \tilde{p}^\star(t,x), \text{ if } x\in[b_0^\star,a_0^\star],\\ \dfrac{H(a_0^\star)}{T} + N_\gamma\left((1-x)^{\frac{1}{1-\gamma}} - (1-a_0^\star)^{\frac{1}{1-\gamma}}\right), \text{ if } x\in(a_0^\star,1], \end{cases}$$

Actually, in this case, the map $p^\star$ will be $u-$convex if and only if the following implicit assumption holds.

Assumption 2.6.4 The solutions $(a_0^\star, b_0^\star)$ of (2.6.8) are such that

$$b_0^\star \leq a_0^\star - \frac{1}{2}.$$

Our main result in this case reads.

Theorem 2.6.3 Let Assumptions 2.5.1, 2.5.3, 2.6.1, 2.6.2, 2.6.3 and 2.6.4 hold, then the optimal tariff $p \in \widehat{\mathcal{P}}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$, when $\gamma \in (0,1)$ by

$$p(t,c) = \begin{cases} \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \frac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}, \end{cases}$$

and when $\gamma < 0$ by

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} - \frac{H(b_0^\star)}{T}, \text{ if } L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \end{cases}$$

where

$$L_\gamma(t) := \left(\frac{\gamma N_\gamma}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}$$

and where $(a_0^\star, b_0^\star)$ are maximizers of

$$\sup_{(a_0,b_0)\in\mathcal{A}_2'} C(T)R_\gamma(a_0,b_0)^{\frac{n(1-\gamma)}{2-\gamma}} - b_0H(b_0) + (a_0-1)H(a_0).$$

Furthermore, the Principal will only choose clients with type $x \in [0, b_0^\star] \cup [a_0^\star, 1]$.

2.6.2 Extension to other cases of reservation utility

In this section, we want to point out that the assumption of the reservation utility function H being strictly concave is not mandatory in order to solve problem (2.6.2), and we intend to explain in which other cases we can hope to solve it.

In order to reduce problem (2.6.2) to a finite dimensional problem, we just need H to have at most a finite number of intersecting points with a strictly convex function. If H were to satisfy this property, then we would be able to prove a result similar to Proposition 2.6.4, and we would know that the optimal set $\hat{X}^\star(p^\star)$ is a finite union of intervals contained in $[0,1]$.

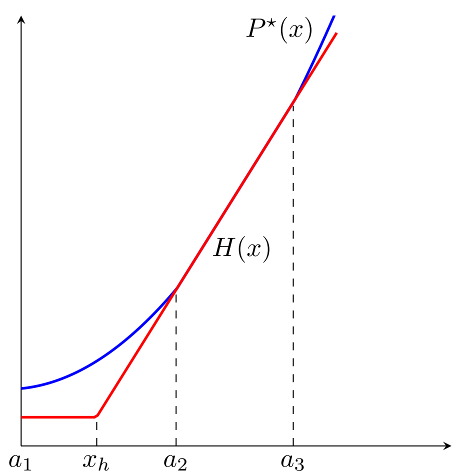
The next step then is to prove that the Lagrange multipliers μ_t in Theorem 2.6.1 are equal to zero, using for instance local perturbations as we did to prove Proposition 2.6.5. This would allow to solve the optimality conditions (2.6.6) and (2.6.7) by using the corresponding auxiliary map $A(a,b)$. We illustrate this in the following example.

Example 2.6.1 Suppose that H has the form

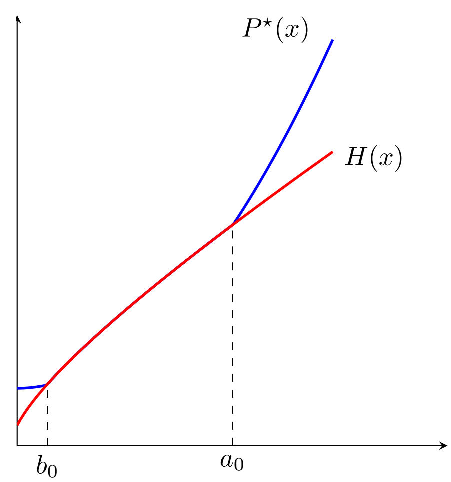
$$H(x) = \begin{cases} \beta, \text{ if } x \in [0, x_h], \\ \alpha(x - x_h) + \beta, \text{ if } x \in [x_h, 1], \end{cases}$$

where $\alpha, \beta \geq 0$ and where $x_h \in [0,1]$.

Such a reservation utility accounts for the fact that all the Agents, whatever their appetence for power consumption is, should at least receive a minimal level of utility, in this case β. Though in general two convex functions can intersect at countably many points, given the specific form of H, it can intersect an increasing and convex function at at most three points, as shown in Figure 2.9.



(a) Case $a_1 = 0$.



(b) Case $a_1 > 0$.

Figure 2.9: $X^\star(p^\star)$ for a "constant-linear" H.

Therefore, we deduce that $X^\star(p^\star)$ has the following form

$$X^\star(p^\star) = [a_1, a_2] \cup [a_3, 1], \text{ for some } 0 \leq a_1 \leq a_2 \leq a_3 \leq 1.$$

We define then the set

$$\mathcal{A}_3 := \left\{(a,b,c) \in [0,1]^2, \ a \leq b \leq c\right\}.$$

After proving that the Lagrange multipliers μ_t in Theorem 2.6.1 are equal to zero, Problem $(P_{a,b})$ becomes, abusing notations slightly

$$\sup_{(a_1,a_2,a_3)\in\mathcal{A}_3}\int_0^T\left[\frac{\phi(t)^{\frac{1}{1-\gamma}}\ell(a_1,a_2,a_3)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_1,a_2,a_3))\right)^{\frac{\gamma}{1-\gamma}}}-K\left(t,\frac{\phi(t)^{\frac{1}{1-\gamma}}\ell(a_1,a_2,a_3)}{\left(\frac{\partial K}{\partial c}(t,A(t,a_1,a_2,a_3)\right)^{\frac{1}{1-\gamma}}}\right)\right]\mathrm{d}t+\theta(a_1,a_2,a_3),$$

where for any $(t,a_1,a_2,a_3)\in[0,T]\times\mathcal{A}_3$

$$\ell(a_1,a_2,a_3):=\int_{a_1}^{a_2}\left(\frac{[g(x)f(x)+g'(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x+\int_{a_3}^{1}\left(\frac{[g(x)f(x)+g'(x)F(x)-g'(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x,$$

$$\theta(a_1,a_2,a_3):=F(a_1)H(a_1)-F(a_2)H(a_2)+(F(a_3)-1)H(a_3),$$

$$A(t,a_1,a_2,a_3):=g_K^{(-1)}\left(\phi(t)^{\frac{1}{1-\gamma}}\int_{a_1}^{a_2}\left(\frac{[g(x)f(x)+g'(x)F(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x\right.$$
$$\left.+\phi(t)^{\frac{1}{1-\gamma}}\int_{a_3}^{1}\left(\frac{[g(x)f(x)+g'(x)F(x)-g'(x)]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x\right).$$

2.7 Appendix

2.7.1 u-convex analysis

We first recall the definition of $u-$convexity (adapted to our context, we refer the reader to the monograph by Villani [127] on optimal transport theory for more details).

Definition 2.7.1 Let ψ be a map from $[0,T]\times X$ to $\mathbb{R}$. The $u-$transform of ψ, denoted by $\psi^\star:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is defined by

$$\psi^\star(t,c):=\sup_{x\in X}\{u(t,x,c)-\psi(t,x)\},\ \text{for any } (t,c)\in[0,T]\times\mathcal{C}.$$

Similarly, if φ is a map from $[0,T]\times\mathcal{C}$ to $\mathbb{R}$, its $u-$transform, still denoted by $\varphi^\star:[0,T]\times X\longrightarrow\mathbb{R}\cup\{+\infty\}$, is defined by

$$\varphi^\star(t,x):=\sup_{c\in\mathcal{C}}\{u(t,x,c)-\varphi(t,c)\},\ \text{for any } (t,x)\in[0,T]\times X.$$

A map $\phi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}\cup\{+\infty\}$ is then said to be $u-$convex if it is proper[2] and if there exists some $\psi:[0,T]\times X\longrightarrow\mathbb{R}$ such that

$$\phi(t,c)=\psi^\star(t,c),\ \text{for any } (t,c)\in[0,T]\times\mathcal{C}.$$

Similarly, a map $\Phi:[0,T]\times X\longrightarrow\mathbb{R}\cup\{+\infty\}$ is said to be $u-$convex if it is proper and there exists some $\Psi:[0,T]\times\mathcal{C}\longrightarrow\mathbb{R}$ such that

$$\Phi(t,x)=\Psi^\star(t,x),\ \text{for any } (t,x)\in[0,T]\times X.$$

[2] That is to say not identically equal to $+\infty$.

We recall the following easy characterization of $u-$convexity.

Lemma 2.7.1 A map $\phi : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is $u-$convex if and only if

$$\phi(t,c) = (\phi^\star)^\star(t,c), \text{ for any } (t,c) \in [0,T] \times \mathcal{C}.$$

A similar statement holds for maps $\Phi : [0,T] \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$.

Proof. We only prove the first statement, the other one being exactly similar. The result is an easy consequence of the fact that for any map $\phi : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$, we have the identity

$$\phi^\star(t,x) = ((\phi^\star)^\star)^\star(t,x), \text{ for any } (t,x) \in [0,T] \times X.$$

Indeed, we have by definition that for any $(t,x) \in [0,T] \times X$

$$\begin{aligned} ((\phi^\star)^\star)^\star(t,x) &= \sup_{c \in \mathcal{C}} \left\{ u(t,x,c) - \sup_{x' \in X} \left\{ u(t,x',c) - \sup_{c' \in \mathcal{C}} \{u(t,x',c') - \phi(t,c'))\} \right\} \right\} \\ &= \sup_{c \in \mathcal{C}} \inf_{x' \in X} \sup_{c' \in \mathcal{C}} \{u(t,x,c) - u(t,x',c) + u(t,x',c') - \phi(t,c')\}. \end{aligned}$$

Choosing $x' = x$, we immediately get that $((\phi^\star)^\star)^\star(t,x) \leq \phi^\star(t,x)$, while the converse inequality is obtained by choosing $c = c'$. □

Next, we can define the notion of the $u-$subdifferential of a $u-$convex function.

Definition 2.7.2 Let $\phi : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a $u-$convex function. For any $(t,c) \in [0,T] \times \mathcal{C}$, the $u-$subdifferential of ϕ at the point (t,c) is the set $\partial^\star\phi(t,c) \subset X$ defined by

$$\partial^\star\phi(t,c) := \{x \in X, \ \phi^\star(t,x) = u(t,x,c) - \phi(t,c)\}.$$

Similarly, let $\psi : [0,T] \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a $u-$convex function. For any $(t,x) \in [0,T] \times X$, the $u-$subdifferential of ψ at the point (t,x) is the set $\partial^\star\psi(t,x) \subset \mathbb{R}_+$ defined by

$$\partial^\star\psi(t,x) := \{c \in \mathcal{C}, \ \psi^\star(t,c) = u(t,x,c) - \psi(t,x)\}.$$

Notice that since the map u is continuous, a $u-$convex function is automatically lower-semicontinuous and its $u-$subdifferential is a closed set.

2.7.2 Proofs of Section 2.5

Proof of Lemma 2.5.1. By Lemma 2.7.1, we know that the $u-$convexity of ψ is equivalent to

$$\psi(t,x) = \sup_{c>0} \min_{y \in [0,1]} \left\{ (g_\gamma(x) - g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma} + \psi(t,y) \right\}. \tag{2.7.1}$$

First notice that since ψ is convex in y and $\frac{g_\gamma}{\gamma}$ is concave, then for any $(t,x,c) \in [0,T] \times [0,1] \times (0,+\infty)$, the map

$$f_{(t,x)}(y,c) := (g_\gamma(x) - g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma} + \psi(t,y),$$

is convex in y. Furthermore, for any $(t,x,y)\in[0,T]\times[0,1]^2$, the map $c\longmapsto f_{(t,x)}(y,c)$ is monotone on $(0,+\infty)$ and therefore quasiconcave. Since $[0,1]$ is convex and compact, we can apply Sion's minimax theorem [114] to obtain that

$$\begin{aligned}\sup_{c>0}\min_{y\in[0,1]}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\}&=\min_{y\in[0,1]}\sup_{c>0}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\}\\&=\min_{y\in[0,1]}\left\{+\infty\mathbf{1}_{x>y}+\psi(t,y)\right\}\\&=\psi(t,x),\end{aligned}$$

since ψ is non-decreasing.

Finally, it is easy to see that when g_γ is defined as in the statement of the lemma, we have

$$\sup_{c>0}\min_{y\in[0,1]}\left\{(g_\gamma(x)-g_\gamma(y))\phi(t)\frac{c^\gamma}{\gamma}+\psi(t,y)\right\}=\sup_{c>0}\min_{y\in[0,1]}\left\{(x-y)c+\psi(t,y)\right\},$$

which corresponds to the classical convex conjugate, hence the desired result by Fenchel-Moreau's theorem. □

Proof of Theorem 2.5.1. We optimize first with respect to $p^\star$. For fixed x_0, start by defining $\Psi_{x_0}:L^1([0,T]\times[0,1])\longrightarrow\mathbb{R}$ by

$$\begin{aligned}\Psi_{x_0}(p^\star):=\int_0^T\Bigg[&\int_{x_0}^1\frac{\left(g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right)}{g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\\&-K\left(t,\int_{x_0}^1\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}}f(x)\mathrm{d}x\right)\Bigg]\mathrm{d}t+(F(x_0)-1)H.\end{aligned}\tag{2.7.2}$$

Ψ_{x_0} is clearly continuous and Fréchet differentiable and it is also concave because K is convex in c. Furthermore, for any $q\in L^1([0,T]\times[0,1])$, we have

$$\begin{aligned}\Psi'_{x_0}(p^\star;q)=&\int_0^T\int_{x_0}^1\frac{\left(g_\gamma(x)f(x)+g'_\gamma(x)F(x)-g'_\gamma(x)\right)}{g'_\gamma(x)}\frac{\partial q}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t\\&-\int_0^T\int_{x_0}^1\frac{\partial q}{\partial x}(t,x)\frac{1}{\gamma}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}}f(x)\frac{\partial K}{\partial c}\left(t,A(t,x_0)\right)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where we defined

$$A(t,x_0):=\int_{x_0}^1\left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}}f(x)\mathrm{d}x.$$

Since Ψ_{x_0} is concave, the necessary and sufficient optimality condition for the problem with fixed x_0 is

$$\Psi'_{x_0}(p^\star;q)\leq 0,\ \forall q\in T_{C^+(x_0)}(p^\star),\tag{2.7.3}$$

where $T_{C^+(x_0)}(p^\star)$ denotes the tangent cone to the closed set $C^+(x_0)$ at the point $p^\star$ defined by

$$T_{C^+(x_0)}(p^\star):=\left\{z,\ \exists\varepsilon>0,\forall h\in[0,\varepsilon]\ \exists w(h)\in C^+(x_0),\ ||p^\star+hz-w(h)||=\circ(h)\right\}.$$

Using local functions we see that the inequality (2.7.3) must be satisfied almost everywhere on $[0,T]\times[x_0,1]$, that is, for every $q\in T_{C^+(x_0)}(p^\star)$ and almost every $(t,x)\in[0,T]\times[x_0,1]$ we have

$$\frac{\partial q}{\partial x}(t,x)\left[\frac{\left(g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x)\right)}{g_\gamma'(x)}-\frac{1}{\gamma}\left(\frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{\gamma}{\phi(t)g_\gamma'(x)}\right)^{\frac{1}{\gamma}}f(x)\frac{\partial K}{\partial c}\left(t,A(t,x_0)\right)\right]\leq 0.$$

Therefore, the optimal $p^\star\in C^+(x_0)$ should verify

$$\frac{\partial p^\star}{\partial x}(t,x)=\left(\frac{\phi(t)^{\frac{1}{\gamma}}\left[g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x)\right]^+}{f(x)\dfrac{\partial K}{\partial c}\left(t,A(t,x_0)\right)}\right)^{\frac{\gamma}{1-\gamma}}\frac{g_\gamma'(x)}{\gamma},\tag{2.7.4}$$

with $a^+=\max\{a,0\}$, the positive part operator. By the above equation, we must have

$$A(t,x_0)\left(\frac{\partial K}{\partial c}\left(t,A(t,x_0)\right)\right)^{\frac{1}{1-\gamma}}=\phi^{\frac{1}{1-\gamma}}(t)\int_{x_0}^1\left(\frac{\left[g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x)\right]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x.$$

Now, let

$$g_K(c):=c\left(\frac{\partial K}{\partial c}(t,c)\right)^{\frac{1}{1-\gamma}},\ c\geq 0.$$

Since K is strictly convex and increasing with respect to c, it can be checked directly that g_K is increasing as well (on $\mathbb{R}_+$), so that we deduce

$$A(t,x_0)=g_K^{(-1)}\left(\phi^{\frac{1}{1-\gamma}}(t)\int_{x_0}^1\left(\frac{\left[g_\gamma(x)f(x)+g_\gamma'(x)F(x)-g_\gamma'(x)\right]^+}{f^\gamma(x)}\right)^{\frac{1}{1-\gamma}}\mathrm{d}x\right).$$

Therefore the solution to the problem with fixed x_0 is given by (2.7.4). We have thus reduced the problem (2.5.4) to

$$\widetilde{U}_P=\sup_{x_0\in[0,1]}\int_0^T\left(\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\dfrac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{\gamma}{1-\gamma}}}-K\left(t,\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\dfrac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{1}{1-\gamma}}}\right)\right)\mathrm{d}t+(F(x_0)-1)H.$$

Seen as a function of x_0, the right-hand side above is clearly a continuous function. It therefore attains its maximum over the compact set $[0,1]$ at some (possibly non unique) $x_0^\star$. We will abuse notations and denote by $x_0^\star$ a generic maximizer.

If $p^\star$ is $u-$convex, since p is also $u-$convex (by definition), then $p^\star$ is necessarily the $u-$transform of p and therefore $p\in\mathcal{P}$, which means that we actually have $\widetilde{U}_P=U_P$. For the uniqueness result, define

$$\alpha(x_0):=\int_0^T\left(\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{\gamma}{1-\gamma}}}-K\left(t,\frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{1}{1-\gamma}}}\right)\right)\mathrm{d}t+(F(x_0)-1)H.$$

Note that α does not attain its maximum over any interval outside L, because there ℓ is constant (and therefore $A(t,\cdot)$ too) and F is increasing. Then, since over L we have

$$\begin{aligned}\alpha'(x_0) &= \int_0^T \left(\frac{1-\gamma}{\gamma}\right)\phi(t)^{\frac{1}{1-\gamma}}\ell'(x_0)\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{-\gamma}{1-\gamma}}\mathrm{d}t + f(x_0)H\\ &= -\int_0^T \left(\frac{1-\gamma}{\gamma}\right)\phi(t)^{\frac{1}{1-\gamma}}\beta(x_0)\left(\frac{\partial K}{\partial c}(t,A(t,x_0))\right)^{\frac{-\gamma}{1-\gamma}}\mathrm{d}t + f(x_0)H.\end{aligned}$$

Under the hypotheses of the theorem, α' is decreasing over L in each one of the two cases so α is strictly concave. □

Proof of Theorem 2.5.2. We divide the proof in two cases.

- **Case 1:** $\gamma \in (0,1)$

In this case we have

$$\ell(x_0) = \int_{x_0 \vee \frac{1}{2}}^1 (2x-1)^{\frac{1}{1-\gamma}}\,\mathrm{d}x = \frac{1-\gamma}{2(2-\gamma)}\left(1-\left((2x_0-1)^+\right)^{\frac{2-\gamma}{1-\gamma}}\right).$$

Hence, it is clear that $x_0 \longmapsto \Phi(x_0)$ is increasing in $[0,\frac{1}{2}]$, so that it suffices to solve

$$\sup_{x_0\in[1/2,1]}\left\{B_\gamma(T)\ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0-1)H\right\}.$$

Let

$$y_0 := (2x_0-1)^{\frac{1}{1-\gamma}},\ A_\gamma(T) := B_\gamma(T)\left(\frac{1-\gamma}{2(2-\gamma)}\right)^{\frac{n(1-\gamma)}{n-\gamma}}.$$

Defining the map $\overline{\Phi} : [0,1] \longrightarrow \mathbb{R}$ by

$$\overline{\Phi}(y_0) := \Phi\left(\frac{y_0^{1-\gamma}+1}{2}\right),$$

we deduce

$$\overline{\Phi}(y_0) = A_\gamma(T)\left(1-y_0^{2-\gamma}\right)^{\frac{n(1-\gamma)}{n-\gamma}} + \frac{1}{2}\left(y_0^{1-\gamma}-1\right)H.$$

Next, we can check directly that $\overline{\Phi}$ is concave on $[0,1]$, and we have for any $y_0 \in [0,1]$

$$\overline{\Phi}'(y_0) = \frac{1-\gamma}{2}y_0^{-\gamma}\left(H - 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma}y_0\left(1-y_0^{2-\gamma}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}\right).$$

Denote finally for any $y_0 \in [0,1]$

$$\chi(y_0) := H - 2nA_\gamma(T)\frac{2-\gamma}{n-\gamma}(2x_0-1)^{\frac{1}{1-\gamma}}\left(1-y_0^{2-\gamma}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

We have for any $y_0 \in [0,1]$

$$\chi'(y_0) = -\frac{2n(2-\gamma)}{(n-\gamma)^2}A_\gamma(T)\left(1-y_0^{2-\gamma}\right)^{-\frac{n+\gamma(n-2)}{n-\gamma}}\left(n-\gamma+\gamma(n-1)(2-\gamma)y_0^{2-\gamma}\right) < 0,$$

since $\gamma \in (0,1)$. Thus, since in addition we have

$$\chi(0) = H > 0, \text{ and } \lim_{y_0 \uparrow 1} \chi(y_0) = -\infty,$$

there is a unique $y_0^\star \in (0,1)$ (and thus a unique $x_0^\star \in (1/2,1)$) such that $\overline{\Phi}'(y_0^\star) = 0$, at which the maximum of $\overline{\Phi}$ is attained. Finally, we can compute explicitly $p^\star(t,x)$ for any $(t,x) \in [0,T] \times [0,1]$ as

$$p^\star(t,x) = \frac{H}{T} + M(t)\left(((2x-1)^+)^{\frac{1}{1-\gamma}} - (2x_0^\star - 1)^{\frac{1}{1-\gamma}}\right),$$

where we defined for simplicity

$$M(t) := \frac{1-\gamma}{2\gamma}\left(\frac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\frac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}}\left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

It can then be checked directly that for any $c \geq 0$, the map $x \longmapsto x\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is concave on $[0,1]$ and attains its maximum at the point

$$x^\star(c) := \mathbf{1}_{c > \left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}} + \frac{1}{2}\left(1 + \left(\frac{(1-\gamma)\phi(t)}{2\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c^{1-\gamma}\right)\mathbf{1}_{c \leq \left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}}.$$

Therefore, we deduce immediately that for any $(t,c) \in [0,T] \times \mathbb{R}_+$

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} + M(t)\left((2x_0^\star - 1)^{\frac{1}{1-\gamma}} - 1\right) - \dfrac{H}{T}, \text{ if } c > \left(\dfrac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + \left(\left(\dfrac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \dfrac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c - \dfrac{H}{T} + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}}, \text{ otherwise.} \end{cases}$$

Next, we notice that for any $(t,x) \in [0,T] \times [0,1]$, the map $c \longmapsto x\phi(t)c^\gamma/\gamma - p(t,c)$ is decreasing on $\mathbb{R}_+$ if $x < 1/2$, and that it is concave on $\mathbb{R}_+$ if $x \geq 1/2$, so that it attains its maximum at the point

$$c^\star(t,x) := \left(\frac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}} (2x-1)^{\frac{1}{1-\gamma}} \mathbf{1}_{x \in (1/2,1]}.$$

It is also immediate that $p^\star$ is always non-decreasing and is convex, and therefore $u-$convex by Lemma 2.5.1, so much so that we conclude that $p \in \mathcal{P}$.

It can easily be shown that the following suboptimal but simpler tariff will give the same results in terms of selected Agents, optimal consumption and Principal's utility: for any $(t,c) \in [0,T] \times \mathbb{R}_+$

$$p(t,c) = \phi(t)\frac{c^\gamma}{2\gamma} + \left(\left(\frac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \frac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} c - \frac{H}{T} + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}}$$

- **Case 2:** $\gamma \in (-\infty, 0)$

Now, we actually have

$$\ell(x_0) = 2^{\frac{1}{1-\gamma}} \int_{x_0}^1 (1-x)^{\frac{1}{1-\gamma}} \mathrm{d}x = 2^{\frac{1}{1-\gamma}} \left(\frac{1-\gamma}{2-\gamma}\right) (1-x_0)^{\frac{2-\gamma}{1-\gamma}}.$$

The problem to solve is now

$$\sup_{x_0 \in [0,1]} \left\{ B_\gamma(T) \ell(x_0)^{\frac{n(1-\gamma)}{n-\gamma}} + (x_0 - 1)H \right\}.$$

It can be checked directly that the above map is actually strictly concave for $x_0 \in [0,1]$, and therefore that it attains its maximum at

$$\widehat{x}_0^\star := \left(1 - \left(\frac{H}{B_\gamma(T)} \frac{n-\gamma}{n(1-\gamma)}\right)^{\frac{n-\gamma}{n(1-\gamma)+\gamma}} \left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{-\gamma(n-1)}{n(1-\gamma)+\gamma}} 2^{\frac{-n}{n(1-\gamma)+\gamma}}\right)^+.$$

Finally, we can compute explicitly $p^\star(t,x)$ for any $(t,x) \in [0,T] \times [0,1]$ as

$$p^\star(t,x) = \frac{H}{T} + \widehat{M}(t) \left((1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}} - (1-x)^{\frac{1}{1-\gamma}}\right),$$

where we defined for simplicity

$$\widehat{M}(t) := -\frac{1-\gamma}{\gamma} \left(\frac{2-\gamma}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}} \left(\frac{2^\gamma \phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}} (1-\widehat{x}_0^\star)^{-\frac{\gamma(2-\gamma)(n-1)}{(n-\gamma)(1-\gamma)}}.$$

We deduce directly that in this case the map $x \longmapsto (1-x)\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is concave, so that it attains its maximum on $[0,1]$ at

$$\widehat{x}^\star(c) := \left(1 - \left(-\frac{\phi(t)(1-\gamma)}{\gamma \widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} c^{1-\gamma}\right)^+,$$

so that

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} - \dfrac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}} + \widehat{M}(t), \text{ if } c > \left(-\dfrac{\gamma \widehat{M}(t)}{\phi(t)(1-\gamma)}\right)^{\frac{1}{\gamma}}, \\ -\gamma c \left(-\dfrac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}} \left(\dfrac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \dfrac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}}, \text{ otherwise.} \end{cases}$$

It is also immediate in this case that $p^\star$ is always non-decreasing and is convex, and therefore $u-$convex by Lemma 2.5.1, so much so that we conclude that $p \in \mathcal{P}$.

It can easily be shown that the following suboptimal but simpler tariff will give the same results in terms of selected Agents, optimal consumption and Principal's utility: for any $(t,c) \in [0,T] \times \mathbb{R}_+$

$$p(t,c) = -\gamma c \left(-\frac{\phi(t)}{\gamma}\right)^{\frac{1}{\gamma}} \left(\frac{1-\gamma}{\widehat{M}(t)}\right)^{\frac{1-\gamma}{\gamma}} - \frac{H}{T} - \widehat{M}(t)(1-\widehat{x}_0^\star)^{\frac{1}{1-\gamma}}$$

□

2.7.3 Proofs of Section 2.6

2.7.3.1 Technical results

Proof of Proposition 2.6.1. Define the following functionals on $\widehat{C}^+$

$$\mathfrak{K}(p) := \int_0^T K\left(t, \int_{X^\star(p)} \left(\frac{\gamma}{\phi(t)g'(x)}\frac{\partial p}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\mathrm{d}t,$$
$$\mathfrak{J}(p) := \int_0^T \left[\int_{X^\star(p)} \left(\frac{g(x)}{g'(x)}\frac{\partial p}{\partial x}(t,x) - p(t,x)\right) f(x)\mathrm{d}x\right]\mathrm{d}t.$$

Proposition 2.6.1 states that if $Y^\star(p^\star)$ has positive measure, then $p^\star$ is not a maximizer of $p \longmapsto \mathfrak{J}(p) - \mathfrak{K}(p)$ over the set $\widehat{C}^+$. Indeed, in this case we can find an interval $[c, \mathrm{d}] \subset Y^\star(p^\star)$ (remember that this is an open set with positive Lebesgue measure and that the latter is regular) and thus

$$\int_0^T p^\star(t,x)\mathrm{d}t = H(x), \text{ for every } x \in [c,\mathrm{d}].$$

Next, define

$$T^+ = \{t \in [0,T] : p^\star(t,c) < p^\star(t,\mathrm{d})\}.$$

Since H is strictly increasing we have that T^+ has positive Lebesgue measure. For every $t \in T^+$, define over $[c,\mathrm{d}]$ a continuous and increasing function q satisfying $q(t,c) := p^\star(t,c)$, $q(t,\mathrm{d}) := p^\star(t,\mathrm{d})$ and $q(t,x) < p^\star(t,x)$ over (c,d). Consider the following modification of $p^\star$.

$$\hat{p}(t,x) := \begin{cases} q(t,x), \text{ if } (t,x) \in T^+ \times [c,\mathrm{d}], \\ p^\star(t,x), \text{ if } (t,x) \notin T^+ \times [c,\mathrm{d}]. \end{cases}$$

We have that $X^\star(\hat{p}) = X^\star(p^\star) \setminus (c,\mathrm{d})$ and therefore $\mathfrak{K}(\hat{p}) < \mathfrak{K}(p^\star)$. Moreover,

$$\begin{aligned} \mathfrak{J}(\hat{p}) &= \mathfrak{J}(p^\star) - \int_0^T \left[\int_c^{\mathrm{d}} \left(\frac{g(x)}{g'(x)}\frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x)\right) f(x)\mathrm{d}x\right]\mathrm{d}t \\ &= \mathfrak{J}(p^\star) - \int_c^{\mathrm{d}} \left(\frac{g(x)}{g'(x)}H'(x) - H(x)\right) f(x)\mathrm{d}x \\ &> \mathfrak{J}(p^\star), \end{aligned}$$

where we used Assumption 2.6.1. Since $\hat{p}$ is also non-decreasing in x, $\hat{p} \in \widehat{C}^+$, and we conclude that $p^\star$ is not optimal. □

Proof of Proposition 2.6.2. Note that in the optimization problem $(P_{a,b})$ we can without loss of generality restrict our attention to the feasible maps on $[0,T] \times X^\star(a,b)$. In other words, for fixed $(a,b) \in \mathcal{A}$, we define the closed and convex set $F_{a,b}$ as the set of maps $q \in W_x^{1,m}(X^\star(a,b))$ such that for every $t \in [0,T]$, $x \longmapsto q(t,x)$ is continuous and non-decreasing, $Q(x) := \int_0^T q(t,x)\mathrm{d}t \geq H(x)$ for every $x \in X^\star(a,b)$ and $Q(a_n) = H(a_n)$, $Q(b_n) = H(b_n)$ for every $n \geq 1$.

We show that $\Psi_{(a,b)}$, seen on the Banach space $\left(W_x^{1,m}(X^\star(a,b)), ||\cdot||_{m,X^\star(a,b)}\right)$, is coercive on $F_{a,b}$.

- **Case 1:** $\gamma \in (0,1)$

Observe first that if $(q_n)_{n\in\mathbb{N}} \subset F_{a,b}$ is such that $||q_n||_{L^m([0,T]\times X^\star(a,b))} \underset{n\to\infty}{\longrightarrow} \infty$, then since for every $n \in \mathbb{N}$ the map $x \longmapsto \int_0^T q_n(t,x)\mathrm{d}t$ is bounded from below by H on $X^\star(a,b)$, we have that

$$-\int_0^T \int_{X^\star(a,b)} q_n(t,x) f(x)\mathrm{d}x\mathrm{d}t \underset{n\to\infty}{\longrightarrow} -\infty. \tag{2.7.5}$$

Next, define

$$A := \int_{X^\star(a,b)} f(x) \left(\frac{\gamma}{\phi(t)g'(x)}\right)^{\frac{1}{\gamma}} \mathrm{d}x,\ B := \int_0^T k(t)\mathrm{d}t.$$

From Jensen's inequality for the maps $\psi_A(x) = x^{\frac{1}{\gamma m}}$, $\psi_B(x) = x^{\frac{n}{\gamma m}}$, we have that (recall that m is such that $\gamma m < 1$ and that $n > 1$)

$$\begin{aligned}
&\int_0^T K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)} \frac{\partial q_n}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\mathrm{d}t \\
\geq &\int_0^T k(t) \left(\int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)} \frac{\partial q_n}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)^n \mathrm{d}t \\
\geq &A^{n(1-\frac{1}{\gamma m})} \int_0^T k(t) \left(\int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}} \left|\frac{\partial q_n}{\partial x}(t,x)\right|^m f(x)\mathrm{d}x\right)^{\frac{n}{\gamma m}} \mathrm{d}t \\
\geq &A^{n(1-\frac{1}{\gamma m})} B^{(1-\frac{n}{\gamma m})} \left(\int_0^T \int_{X^\star(a,b)} k(t) \left(\frac{\gamma}{\phi(t)g'_\gamma(x)}\right)^{\frac{1}{\gamma}} \left|\frac{\partial q_n}{\partial x}(t,x)\right|^m f(x)\mathrm{d}x\mathrm{d}t\right)^{\frac{n}{\gamma m}} \\
\geq &A^{n(1-\frac{1}{\gamma m})} B^{(1-\frac{n}{\gamma m})} I \left|\left|\frac{\partial q_n}{\partial x}\right|\right|^{\frac{n}{\gamma}}_{L^m([0,T]\times X^\star(a,b))}.
\end{aligned}$$

We have therefore proved that, denoting by m' the conjugate of m

$$\Psi_{(a,b)}(q_n) \leq \left|\left|\frac{g_\gamma}{g'_\gamma}\right|\right|_{L^{m'}([0,T]\times X^\star(a,b))} \left|\left|\frac{\partial q_n}{\partial x}\right|\right|_{L^m([0,T]\times X^\star(a,b))} - A^{n(1-\frac{1}{\gamma m})} B^{1-\frac{n}{\gamma m}} I \left|\left|\frac{\partial q_n}{\partial x}\right|\right|^{\frac{n}{\gamma}}_{L^m([0,T]\times X^\star(a,b))}.$$

This with (2.7.5) implies clearly that $\Psi_{(a,b)}$ is indeed coercive in $F_{a,b}$.

- **Case 2:** $\gamma < 0$

In this case $\frac{g_\gamma}{g'_\gamma} < 0$. Since $\frac{\partial q_n}{\partial x}$ is non-negative, if $||\frac{\partial q_n}{\partial x}||_{L^m([0,T]\times X^\star(a,b))} \longrightarrow \infty$ we have that

$$\int_0^T \int_{X^\star(a,b)} \frac{g_\gamma(x)}{g'_\gamma(x)} \frac{\partial q_n}{\partial x}(t,x) f(x)\mathrm{d}x\ \mathrm{d}t \longrightarrow -\infty.$$

Then, from (2.7.5) and the positiveness of K we conclude that $\Psi_{(a,b)}$ is coercive in $F_{a,b}$.

To conclude the proof, note that $W_x^{1,m}(X^\star(a,b))$ is a reflexive Banach space, so the coercivity of $\Psi_{(a,b)}$ implies that it possesses at least a maximizer $p^\star$ in $F_{a,b}$. Therefore any $q \in W_x^{1,m}(0,1)$, continuous and non-decreasing in x, which coincides with $p^\star$ in $X^\star(a,b)$ and such that $Q(x) = \int_0^T q(t,x)\mathrm{d}t$ satisfies $Q(x) < H(x)$ for $x \in [0,1] \setminus X^\star(a,b)$ is a solution to $(P_{a,b})$. □

We state the following Lemma before proving Proposition 2.6.5.

Lemma 2.7.2 Let $p^\star$ be a solution of $(P_{a,b})$. Define

$$T^{b_0} = \left\{ t \in [0,T] : \frac{\partial p^\star}{\partial x}(t,x) = 0, \forall x \in (0,b_0) \right\}, \; T^{a_0} = \left\{ t \in [0,T] : \frac{\partial p^\star}{\partial x}(t,x) = 0, \forall x \in (a_0,1) \right\}.$$

If T^{a_0} has positive Lebesgue measure, then for every $x \in (0,b_0)$, $g(x)f(x) + g'(x)F(x) \leq 0$. If T^{b_0} has positive Lebesgue measure, then for every $x \in (a_0,1)$, $g(x)f(x) + g'(x)F(x) - g'(x) \leq 0$.

Proof. We consider the case in which T^{a_0} has positive Lebesgue measure. Suppose there exist $[x_1,x_2] \subset [a_0,1]$ such that for every $x \in [x_1,x_2]$

$$g(x)f(x) + g'(x)F(x) - g'(x) > 0.$$

Then, for any $q \in W_x^{1,m}(0,1)$ satisfying $q(t,x) = 0, \forall (t,x) \notin T^{a_0} \times [x_1,1]$, $x \longmapsto q(t,x)$ is increasing in $[x_1,x_2]$, $\forall t \in T^{a_0}$, and $q(t,x) = q(t,x_2)$, $\forall (t,x) \in T^{a_0} \times [x_2,1]$, the map $p^\star + \varepsilon q$ belongs to $C^+(a,b)$ for $\varepsilon \geq 0$. Therefore $\Psi'_{(a,b)}(p^\star;q) \leq 0$,, which means

$$\int_{T^{a_0}} \int_{x_1}^{x_2} \frac{\partial q}{\partial x}(t,x) \frac{f(x)g(x) + g'(x)F(x)}{g'(x)} \leq 0,$$

hence a contradiction. □

Proof of Proposition 2.6.5. Let us prove the case $I \subset (a_0,1)$, the case $I \subset (0,b_0)$ being similar. From the convexity of $P^\star$ on every interval over which $P^\star$ is strictly greater than H we deduce the existence of $c_0 \in [a_0,1)$ such that $P^\star(x) > H(x)$ for every $x \in (c_0,1]$ and $P^\star(x) = H(x)$ for every $x \in [a_0,c_0]$. It follows from Lemma 2.7.2 that either T^{a_0} is a null set or for every $t \in T^{a_0}$ the optimality conditions from Theorem 2.6.1 hold with $\mu_t = 0$. Call $T_1 = [0,T] \setminus (\mathcal{N} \cup T^{a_0})$ and define for every $t \in T_1$

$$x_1(t) = \inf \left\{ x \in (c_0,1) : \frac{\partial p^\star}{\partial x}(t,x) > 0 \right\}.$$

We have that $p^\star(t,\cdot)$ is strictly increasing in $[x_1(t),1]$ and it is given by (2.6.6). Define next

$$T_1^+ := \{t \in T_1, \; \mu_t > 0\}, \; T_1^- := \{t \in T_1, \; \mu_t < 0\}.$$

We will prove that T_1^- and T_1^+ have Lebesgue measure equal to zero. Consider any map $q \in W_x^{1,m}(0,1)$ satisfying

$$\begin{cases} q(t,x) = 0, \forall (t,x) \notin T_1^- \times (x_1(t),1], \\ x \longmapsto q(t,x) \text{ is increasing in } [x_1(t),1], \forall t \in T_1^-. \end{cases}$$

Then $p^\star + \varepsilon q \in C^+(a,b)$ for every $\varepsilon \geq 0$, so $\Psi'(p^\star; q) \leq 0$. Since

$$\Psi'(p^\star; q) = \int_{T_1^-} \int_{x_1(t)}^1 -\frac{\partial q}{\partial x}(t,x)\mu_t \mathrm{d}x\mathrm{d}t,$$

we conclude that T_1^- is a null set. Next, take any $\bar{x} \in (c_0, 1)$ such that $P^\star(\bar{x}) > H(\bar{x})$ and for every $t \in T_1^+$ redefine if necessary the point $x_1(t)$ in order to satisfy $x_1(t) \geq \bar{x}$. Define then $q : [0,T] \times [0,1] \longrightarrow \mathbb{R}$ by

$$q(t,x) := 1_{\{t \in T_1^+,\ x \geq x_1(t)\}} \frac{\frac{\partial p^\star}{\partial x}(t, x_1(t))(x - x_1(t)) + p^\star(t, x_1(t)) - p^\star(t,x)}{p^\star(t,1) - p^\star(t,x_1(t)) - \frac{\partial p^\star}{\partial x}(t, x_1(t))(1 - x_1(t))} \Delta.$$

Since $p^\star(t,\cdot)$ is convex, we have that q is non-increasing, $p^\star(t,\cdot) + \varepsilon q(t,\cdot)$ is non-decreasing for $\varepsilon \sim 0$ and $p(t,1) + q(t,1) = p(t,1) - \Delta$. Therefore $p^\star + \varepsilon q \in C^+(a,b)$ for $\varepsilon \sim 0$ so $\Psi'(p^\star; q) \leq 0$. Since

$$\Psi'(p^\star; q) = \int_{T_1^+} \int_{x_1(t)}^1 -\frac{\partial q}{\partial x}(t,x)\mu_t \ \mathrm{d}x \ \mathrm{d}t,$$

we conclude that the set T_1^+ has Lebesgue measure equal to zero. □

Proof of Proposition 2.6.6. We show that under the conditions of the proposition, $P^\star \equiv H$ over some subset of $(0, a_0] \cup [b_0, 1)$ with positive Lebesgue measure and the result follows from Proposition 2.6.1. Suppose not, then for almost every $t \in [0,T]$ we have

$$\frac{\partial p^\star}{\partial x}(t,x) = \begin{cases} \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t, a_0, b_0))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'_\gamma(x)}{\gamma}, \ x \in (0, b_0), \\ \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x) \right]^+}{f(x) \frac{\partial K}{\partial c}(t, A(t, a_0, b_0))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'_\gamma(x)}{\gamma}, \ x \in (a_0, 1). \end{cases}$$

Thus either $\frac{\partial P^\star}{\partial x}(a_0) > H'(a_0)$ or $\frac{\partial P^\star}{\partial x}(b_0) < H'(b_0)$, which contradicts that $\hat{X}^\star(p^\star) = [0, b_0) \cup (a_0, 1]$. □

Proof of Theorem 2.6.3. First of all, we recall that we have a degree of freedom in choosing the map $\tilde{p}$ to which p is equal on $[b_0^\star, a_0^\star]$, since it does not play any role in criterion that $p^\star$ maximises. Of course, if we want to be able to conclude, this map has to be $u-$convex in the end. Therefore, if we can choose it so that $p^\star$ is C^1 and convex in x, we can apply Lemma 2.5.1 and conclude that $p^\star$ is indeed $u-$convex. This can be made if and only if the derivative of $p^\star$ at $a_0^\star$ is greater or equal to the derivative of $p^\star$ at $b_0^\star$, which can be shown immediately to be equivalent to, regardless of the value of γ,

$$a_0^\star - \frac{1}{2} \geq b_0^\star.$$

Furthermore, if this is not satisfied, then $p^\star$ is not convex, and we can apply the second part of Lemma 2.5.1 to conclude that $p^\star$ is not $u-$convex.

We now divide the proof in two steps.

- **Case** (i): $\gamma \in (0,1)$.

Given the discussion above, in this case the only thing we have to do is to compute p. Denote for simplicity

$$L_\gamma(t) := \left(\frac{\gamma N_\gamma}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}.$$

We know that the map $x \longmapsto x\phi(t)c^\gamma/\gamma - p^\star(t,x)$ is concave on $[0,1]$. Notice as well that since $a_0^\star > 1/2$, we have $1/2 \geq a_0^\star - 1/2 \geq b_0^\star$. We can then compute its maximum and obtain directly that it is attained at

$$x^\star(c) := \begin{cases} 1, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \dfrac{1}{2} + L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \dfrac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}, \end{cases}$$

where $\tilde{x}^\star(c)$ is any point in $[b_0^\star, a_0^\star]$ such that

$$\frac{\partial \tilde{p}^\star}{\partial x}(t, \tilde{x}^\star(c)) = \phi(t)\frac{c^\gamma}{\gamma}.$$

We deduce that

$$p(t,c) = \begin{cases} \phi(t)\frac{c^\gamma}{\gamma} - N_\gamma\left(2^{-\frac{1}{1-\gamma}} - \left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}}\right) - \frac{H(a_0^\star)}{T}, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \frac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}. \end{cases}$$

As in the case H constant, it can easily be shown that the following simpler tariff will produced the same results:

$$p(t,c) = \begin{cases} \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t, \tilde{x}(c)), \text{ if } L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(a_0^\star - \frac{1}{2}\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)L_\gamma(t)^{\gamma-1}c - \frac{H(b_0^\star)}{T} + N_\gamma(b_0^\star)^{\frac{1}{1-\gamma}}, \text{ if } 0 \leq c \leq L_\gamma(t)(b_0^\star)^{\frac{1}{1-\gamma}}. \end{cases}$$

- **Case** (ii): $\gamma < 0$. As in the previous case, our assumptions imply that $a_0^\star \geq 1/2$ and $1/2 \geq a_0^\star - 1/2 \geq b_0^\star$. We can then prove that the maximum of the map $x \longmapsto x\phi(t)c^\gamma/\gamma -$

$p^\star(t,x)$ is attained at

$$x^\star(c) := \begin{cases} 1 - L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \frac{1}{2} - L_\gamma(t)^{\gamma-1}c^{1-\gamma}, \text{ if } L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ 0, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \end{cases}$$

where $\tilde{x}^\star(c)$ is any point in $[b_0^\star, a_0^\star]$ such that

$$\frac{\partial \tilde{p}^\star}{\partial x}(t, \tilde{x}^\star(c)) = \phi(t)\frac{c^\gamma}{\gamma}.$$

We deduce that

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} - \frac{H(b_0^\star)}{T}, \text{ if } L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}, \\ \phi(t)\frac{c^\gamma}{\gamma} + N_\gamma\left(\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} - 2^{-\frac{1}{1-\gamma}}\right) - \frac{H(b_0^\star)}{T}, \text{ if } c > L_\gamma(t)2^{-\frac{1}{1-\gamma}}. \end{cases}$$

As previously, it can easily be shown that the following simpler tariff will produced the same results:

$$p(t,c) = \begin{cases} \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma(1-a_0^\star)^{\frac{1}{1-\gamma}} - \frac{H(a_0^\star)}{T}, \text{ if } 0 < c \leq L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}}, \\ \tilde{x}^\star(c)\phi(t)\frac{c^\gamma}{\gamma} - \tilde{p}^\star(t,\tilde{x}(c)), \text{ if } L_\gamma(t)(1-a_0^\star)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}}, \\ \phi(t)\frac{c^\gamma}{2\gamma} + \phi(t)L_\gamma(t)^{\gamma-1}c + N_\gamma\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} - \frac{H(b_0^\star)}{T}, \text{ if } L_\gamma(t)\left(\frac{1}{2} - b_0^\star\right)^{\frac{1}{1-\gamma}} < c \leq L_\gamma(t)2^{-\frac{1}{1-\gamma}}. \end{cases}$$

□

2.7.3.2 Proof of Theorem 2.6.1

We give here a series of result which once combined prove Theorem 2.6.1. To simplify the statements, we give them in a generic set (a_n, b_n), the generalization being straightforward. The first proposition shows that the existence of the interval I in the theorem allows us to localize Problem $(P_{a,b})$, and replace it by a simpler one, in which the constraint $P^\star(x) \geq H(x)$ for every $x \in X^\star(a,b)$ can be ignored.

Proposition 2.7.1 Let $p^\star$ be a solution of $(P_{a,b})$ and suppose there exists $x_1 \in (a_n, b_n)$ such that $P^\star(x_1) > H(x_1)$. Then, there exists $x_0 \in (a_n, b_n)$, $x_0 < x_1$, such that $p^\star$ is solution to the following problem

$$(P_{x_0,x_1}) \qquad \sup_{q \in C(x_0,x_1)} \Psi_{(a,b)}^{x_0,x_1,p^\star}(q), \tag{2.7.6}$$

where

$$\Psi_{(a,b)}^{x_0,x_1,p^\star}(q) := \int_0^T \int_{x_0}^{x_1} \frac{g_\gamma(x)f(x) + g_\gamma'(x)F(x)}{g_\gamma'(x)} \frac{\partial q}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T K\left(t, \int_{x_0}^{x_1} \left(\frac{\gamma}{\phi(t)g_\gamma'(x)} \frac{\partial q}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x + I_{(a,b)}^{x_0,x_1}(p^\star)\right)\mathrm{d}t,$$
$$I_{(a,b)}^{x_0,x_1}(p^\star) := \int_{X^\star(a,b)\setminus(x_0,x_1)} \left(\frac{\gamma}{\phi(t)g_\gamma'(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x,$$

and $C(x_0, x_1)$ denotes the set of maps $q \in W_x^{1,m}(x_0, x_1)$ such that

- $x \longmapsto q(t,x)$ is continuous and increasing for every $t \in [0,T] \setminus \mathcal{N}(q)$.
- $p^\star(t,x_0) + \displaystyle\int_{x_0}^{x_1} \frac{\partial q}{\partial x}(t,x)\,\mathrm{d}x = p^\star(t,x_1)$ for every $t \in [0,T] \setminus \mathcal{N}(q)$.

Proof. Define

$$x_0 := \inf\left\{z \in X^\star(a,b),\ P^\star(x) \geq H(x_1) \text{ for every } x \in [z, x_1]\right\}.$$

By continuity we have that $x_0 < x_1$ and $P^\star(x_0) = H(x_1)$. Notice that the restriction of $p^\star$ to the set $[x_0, x_1]$ belongs to $C(x_0, x_1)$. Suppose the restriction is not a solution of (P_{x_0,x_1}), then there exists $q^\star \in C(x_0,x_1)$ such that $\Psi_{(a,b)}^{x_0,x_1,p^\star}(q^\star) > \Psi_{(a,b)}^{x_0,x_1,p^\star}(p^\star)$. Define then $\bar{p} : [0,T] \times [0,1] \longrightarrow \mathbb{R}$ by

$$\bar{p}(t,x) := \begin{cases} p^\star(t,x), \ x \notin [x_0,x_1], \\ p^\star(t,x_0) + \displaystyle\int_{x_0}^{x} \frac{\partial q^\star}{\partial x}(t,x)\mathrm{d}x, \ x \in (x_0,x_1). \end{cases}$$

Then, for every $x \in [x_0, x_1]$

$$\int_0^T \bar{p}(t,x)\mathrm{d}t \geq \int_0^T \bar{p}(t,x_0)\mathrm{d}t \geq H(x_1) \geq H(x),$$

and it is straightforward that $\bar{p} \in C^+(a,b)$. This is a contradiction with the optimality of $p^\star$ in problem $(P_{a,b})$ because

$$\Psi_{(a,b)}(\bar{p}) = \Psi_{(a,b)}(p^\star) - \Psi_{(a,b)}^{x_0,x_1,p^\star}(p^\star) + \Psi_{(a,b)}^{x_0,x_1,p^\star}(q^\star).$$

□

Now we state the optimality conditions for the problem (P_{x_0,x_1}).

Proposition 2.7.2 Let $p^\star$ be a solution of (P_{x_0,x_1}) with x_0, x_1 as in Proposition 2.7.1. Then there exists a null set $\mathcal{N} \subset [0,T]$ and a constant μ_t for every $t \in [0,T] \setminus \mathcal{N}$ such that for every $x \in (x_0, x_1)$

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) + g'_\gamma(x) \mu_t \right]^+}{f(x) \frac{\partial K}{\partial c}\left(t, A(t,a,b)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}. \tag{2.7.7}$$

Proof. Notice that the set $C(x_0,x_1)$ can be written as

$$C(x_0,x_1) = \left\{ q \in W_x^{1,m}(x_0,x_1),\ g(q) \in C,\ h(q) = 0 \right\},$$

where $g : W_x^{1,m}(x_0,x_1) \longrightarrow L^m([0,T] \times [x_0,x_1])$ is defined by $g(q) = \frac{\partial q}{\partial x}$, where C is the following convex cone $C := \{ q \in L^m([0,T] \times [x_0,x_1]),\ q(t,x) \geq 0,\ \text{a.e.} \}$ and $h : W_x^{1,m}(x_0,x_1) \longrightarrow L^m([0,T])$ is defined by

$$h(q) := \int_{x_0}^{x_1} \frac{\partial q}{\partial x}(\cdot, x) \mathrm{d}x + p^\star(\cdot, x_0) - p^\star(\cdot, x_1).$$

It can be checked in the same way as in Remark 5 from [33], that their Assumption S is satisfied in this context. Furthermore, it is a classical result that the dual of $W_x^{1,m}(x_0,x_1)$ is $W_x^{1,m/(m-1)}(x_0,x_1)$.

Define now the Lagrangian $L : W_x^{1,m}(x_0,x_1) \times W_x^{1,m/(m-1)}(x_0,x_1) \times L^{\frac{m}{m-1}}(0,T) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} L(q,\lambda,\mu) := {} & \Psi_{(a,b)}^{x_0,x_1,p^\star}(q) + \int_0^T \int_{x_0}^{x_1} \lambda(t,x) \frac{\partial q}{\partial x}(t,x) \mathrm{d}x \mathrm{d}t \\ & + \int_0^T \mu(t) \left(\int_{x_0}^{x_1} \frac{\partial q}{\partial x}(t,x) \mathrm{d}x + p^\star(t,x_0) - p^\star(t,x_1) \right) \mathrm{d}t. \end{aligned}$$

Then, from Corollary 2 in [37] it follows that there exists $\lambda \in W_x^{1,m/(m-1)}(x_0,x_1)$, $\mu \in L^m(0,T)$ such that

$$\begin{cases} 0 = \dfrac{g_\gamma(x) f(x) + g'_\gamma(x) F(x)}{g'_\gamma(x)} - \dfrac{1}{\gamma} \left(\dfrac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1-\gamma}{\gamma}} \left(\dfrac{\gamma}{\phi(t) g'_\gamma(x)} \right)^{\frac{1}{\gamma}} f(x) \dfrac{\partial K}{\partial c}\left(t, A(t,a,b)\right) \\ \quad +\mu(t) + \lambda(t,x),\ \text{a.e. in } [0,T] \times [x_0,x_1], \\ \lambda(t,x) \dfrac{\partial p^\star}{\partial x}(t,x) = 0,\ \lambda(t,x) \geq 0,\ \text{a.e. in } [0,T] \times [x_0,x_1]. \end{cases}$$

Then, when $\frac{\partial p^\star}{\partial x}(t,x) > 0$ we have that $\lambda(t,x) = 0$ and

$$\frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) + g'_\gamma(x) \mu(t) \right]}{f(x) \frac{\partial K}{\partial c}\left(t, A(t,a,b)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'_\gamma(x)}{\gamma}.$$

In case $\frac{\partial p^\star}{\partial x}(t,x) = 0$ we have that

$$\frac{g_\gamma(x) f(x) + g'_\gamma(x) F(x)}{g'_\gamma(x)} + \mu(t) = -\lambda(t,x) \leq 0,$$

which ends the proof. □

We prove finally that the map μ does not depend on x_0, x_1 and is the same in the interval $I = (x_\ell, x_r)$.

Proposition 2.7.3 Let $I = (x_\ell, x_r) \subset (a_n, b_n)$ be as in Theorem 2.6.1. Then for any x_0, $x_1 \in I$, there exist a null set $\mathcal{N} \subset [0, T]$ and a constant μ_t for every $t \in [0, T] \setminus \mathcal{N}$ such that for every $x \in (x_0, x_1)$ (2.6.7) is satisfied.

Proof. Let $y_0 := x_1$ and define by induction for $k \geq 0$

$$z_k := \inf\{z \in (a_n, b_n),\ P^\star(x) \geq H(y_k),\ \forall x \in [z, y_k]\},\ y_{k+1} := \frac{z_k + y_k}{2}.$$

By continuity we have that $P^\star(z_k) = H(y_k)$, so $y_{k+1} < y_k$ and the sequence $(y_k)_k$ converges necessarily to a_n. We conclude by applying Proposition 2.7.2 to every interval (z_k, y_k) and noting that these intervals overlap themselves. □

Chapter 3

Bank monitoring incentives under moral hazard and adverse selection

3.1 Introduction

Principal-Agent problems with moral hazard have an extremely rich history, dating back to the early static models of the 70s, see among many others Zeckhauser [134], Spence and Zeckhauser [120], or Mirrlees [69, 70, 72, 74], as well as the seminal papers by Grossman and Hart [47], Jewitt, [55], Holmström [52] or Rogerson [105]. If moral hazard results from the inability of the Principal to monitor, or to contract upon, the actions of the Agent, there is a second fundamental feature of the Principal-Agent relationship which has been very frequently studied in the literature, namely that of adverse selection, corresponding to the inability to observe private information of the Agent, which is often referred to as his type. In this case, the Principal offers to the Agent a menu of contracts, each having been designed for a specific type. The so-called *revelation principle*, states then that it is always optimal for the Principal to propose menus for which it is optimal for the Agent to truthfully reveal his type. Pioneering research in the latter direction were due to Mirrlees [68], Mussa and Rosen [79], Roberts [101], Spence [119], Baron and Myerson [10], Maskin and Riley [63], Guesnerie and Laffont [48], and later by Salanié [108], Wilson [132], or Rochet and Choné [102]. However, despite the early realisation of the importance of considering models involving both these features at the same time, the literature on Principal-Agent problems involving both moral hazard and adverse selection has remained, in comparison, rather scarce. As far as we know, they were considered for the first time by Antle [2], in the context of auditor contracts, and then, under the name of generalised Principal-Agent problems, by Myerson [80][1]. These generalised agency problems were then studied in a wide variety of economic settings, notably by Dionne and Lasserre [36], Laffont and Tirole [59], McAfee and McMillan [65], Picard [91], Baron and Besanko [7, 8], Melumad and Reichelstein [66, 67], Guesnerie, Picard and Rey [49], Page [83], Zou[136], Caillaud, Guesnerie and Rey [22], Lewis and Sappington [61], or

[1]There were earlier attempts in this direction, but providing a less systematic treatment of the problem; see the income tax model of Mirrlees [68], the Soviet incentive scheme study of Weitzman [128], or the papers by Baron and Holmström [9] and Baron [6].

Bhattacharyya [15][2].

All the previous models are either in static or discrete-time settings. The first study of the continuous time problem with moral hazard and adverse selection was made by Sung [124], in which the author extends the seminal finite horizon and continuous-time model of Holmström and Milgrom [53]. A more recent work, to which this chapter is mostly related has been treated by Cvitanić, Wan and Yang [29], where the authors extend the famous infinite horizon model of Sannikov [109] to the adverse selection setting. If one of the main contributions of Sannikov [109] was to have identified that the *continuation value* of the Agent was a fundamental state variable for the problem of the Principal, [29] shows that in a context with both moral hazard and adverse selection, the Principal has also to keep track of the so-called *temptation value*, that is to say the continuation utility of the Agent who would not reveal his true type. Although close to the latter paper, our work is foremost an extension of the bank incentives model of Pagès and Possamaï [85], which studies the contracting problem between competitive investors and an impatient bank who monitors a pool of long-term loans subject to Markovian contagion (we also refer the reader to the companion paper by Pagès [84] for the economic intuitions and interpretations of the model). In the model of [85], moral hazard emerges because the bank has more "skin a game" than the investors, and has the opportunity, *ex ante* and *ex post*, to exercise a (costly) monitoring of the non-defaulted loans. This is a stylised way to sum up all the actions than the bank can enter into to ensure itself of the solvability of the borrowers. Since the investors cannot observe the monitoring effort of the bank, they offer CDS type contracts offering remuneration to the bank, and giving it incentives through postponement of payments and threat of stochastic liquidation of the contract (similarly to the seminal paper of Biais, Mariotti, Rochet and Villeneuve [16]). In the present work, we assume furthermore that there are two types of banks, which we coin good and bad, co-existing in the market, differing by their efficiency in using their remuneration (or equivalently differing by their monitoring costs). Even if the investor is supposed to know the distribution of the type of banks, he cannot know whether the one is entering into a contract with is good or bad.

Mathematically speaking, we follow both the general dynamic programming approach of Cvitanić, Possamaï and Touzi [27], as well as the take on adverse selection problems initiated by [29]. Intuitively, these approaches require first, using martingale (or more precisely backward SDEs) arguments, to solve the (non-Markovian) optimal control problem faced by the two type of banks when choosing each contracts. This requires obviously, using the terminology introduced above, to keep track of both the *continuation value* and the *temptation value* of the banks, when they choose the contract designed for them or not. The problem of the Principal rewrites then as two standard stochastic control problems, one in which he hires the good bank, and one in which he hires the bad one. Each of these problems uses in turn the aforementioned two state variables (and these two only, because the horizon is infinite and the Principal is risk-neutral), with truth-telling constraint, asserting that the continuation value should always be greater than the temptation value. This leads to optimal control problems with state constraints, and thus to Hamilton-Jacobi-Bellman (HJB for short) equations (or more precisely variational inequalities with gradient constraints, since our problem

[2]We refer the interested reader to the more recent works of Faynzilberg and Kumar [43], Theilen [126], Jullien, Salanié and Salanié [57], Gottlieb and Moreira [46].

is actually a singular stochastic control problem) in a domain, which, following [29], we call the credible set. This set is defined as the set containing the pair of value functions of the good and bad bank under every admissible contract offered by the investor. The determination of this set is the first fundamental step in our approach. Following the the orignal ideas of [29], we prove that the determination of the boundaries of this set can be achieved by solving two so-called double-sided moral hazard problems, in which one of the type of banks is actually hiring the other one. Fortunately for us, it turned out to be possible to obtain rigorously[3] explicit expressions for these boundaries by solving the associated system of HJB equations and using verification type arguments. We also would like to emphasise that unlike in [29], there is certain dynamic component in our model, since we have to keep track of the number of non-defaulted loans, through a time inhomogeneous Poisson process. This leads to a dynamic credible set, as well as, in the end, to a *recursive system* of HJB equations characterising the value function of the Principal.

After having determined the credible set itself, we pursue our study by concentrating on two specific forms of contracts: the shutdown contract in which the investor designs a contract which will be accepted only by the good bank, and the more classical screening contract, corresponding to a menu of contracts, one for each type of bank, which provides incentives to reveal her true type and choose the contract designed for her. These two contracts correspond simply to the offering, over the correct domain of expected utilities of the banks (so as to satisfy the proper truth-telling and participation constraints), of the best contracts that the investor can design independently for hiring the good and the bad bank.

Since we characterise, under classical verification type arguments, the value function of the investor through a system of HJB equations, we also have classically access to the optimal contracts through this value function and its derivatives. This allows us to provide an associated qualitative and quantitative analysis. It turns out that he optimal contracts designed for the good and the bad bank share the same attributes, and are close in spirit to the ones derived in the pure moral hazard case in [85]. On the boundaries of the credible set, the value function of the bad bank plays the role of a state process. The payments of the optimal contracts are postponed until the moment the state process reaches a sufficiently high level, depending on the current size of the project. Similarly, when one of the loans of the pool defaults, the project is liquidated with a probability that decreases with the value of the state process. If the value function of the bad bank at the default time is below some critical level, the project will be liquidated for sure under the optimal contracts. On the other side, if the value function of the bad bank is high enough at the default time, the project will be maintained. In the interior of the credible set, the continuation value and the temptation value of the banks are the state processes for the optimal contracts. It is possible to identify zones of *good performance* inside of the credible set, where the Agents are remunerated and the project is maintained in case a default occurs. It is also possible to identify zones of *bad performance*, where the Agents are not paid and the project is liquidated in case of default. In the rest of the credible set the optimal contracts provide intermediary situations.

Notations: Let $\mathbb{N}$ denote the set of non–negative integers. For any $n \in \mathbb{N}\backslash\{0\}$, we

[3] Notice that in this respect the study in [29] was more formal, and our work provides, as far as we know, the first rigorous derivation of this credible set.

identify $\mathbb{R}^n$ with the set of $n-$dimensional column vectors. The associated inner product between two elements $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$ will be denoted by $x\cdot y$. For simplicity of notations, we will sometimes write column vectors in a row form, with the usual transposition operator $\top$, that is to say $(x_1,\dots,x_n)^\top\in\mathbb{R}^n$ for some $x_{\rm i}\in\mathbb{R}$, $1\leq {\rm i}\leq n$. Let $\mathbb{R}_+$ denote the set of non–negative real numbers, and $\mathcal{B}(\mathbb{R}_+)$ the associated Borel $\sigma-$algebra. For any fixed non–negative measure ν on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$, the Lebesgue–Stieljes integral of a measurable map $f:\mathbb{R}_+\longrightarrow\mathbb{R}$ will be denoted indifferently

$$\int_{[u,t]}f(s)\mathrm{d}\nu_s \text{ or } \int_u^t f(s)\mathrm{d}\nu_s,\ 0\leq u\leq t.$$

3.2 The model

This section is dedicated to the description of the model we are going to study, presenting the contracts as well as the criterion of both the Principal and the Agent. As recalled in the Introduction, it is actually an adverse selection extension of the model introduced first by Pagès in [84] and studied in depth by Pagès and Possamaï [85].

3.2.1 Preliminaries

We consider a model in continuous time, indexed by $t\in[0,\infty)$. Without loss of generality and for simplicity, the risk–free interest rate is taken to be 0[4]. Our first player will be a bank (the Agent, referred to as "she"), who has access to a pool of I unit loans indexed by $j=1$, $\dots$, I which are *ex ante* identical. Each loan is a perpetuity yielding cash flow μ per unit time until it defaults. Once a loan defaults, it gives no further payments. As is commonplace in the Principal-Agent literature, especially since the paper of Sannikov [109], the infinite maturity assumption is here for simplicity and tractability, since it makes the problem stationary, in the sense that the value function of the Principal will not be time–dependent. We assume that the banks in the market are different, and that two types of banks coexist, each one being characterised by a parameter taking values in the set $\mathfrak{R}:=\{\rho_g,\rho_b\}$ with $\rho_g>\rho_b$. We call the bank good (respectively bad) if its type is ρ_g (respectively ρ_b). Furthermore, it is considered to be common knowledge that the proportion of the banks of type $\rho_{\rm i}$, ${\rm i}\in\{g,b\}$, is $p_{\rm i}$.

Denote by

$$N_t:=\sum_{j=1}^{I}\mathbf{1}_{\{\tau^j\leq t\}},$$

the sum of individual loan default indicators, where τ^j is the default time of loan j. The current size of the pool is, at some time $t\geq 0$, $I-N_t$. Since all loans are *a priori* identical,

[4]As already pointed out in the seminal paper of Biais, Mariotti, Rochet and Villeneuve [16], see also [85], the only quantity of interest here is the difference between the discounting factors of the Principal and the Agent.

they can be reindexed in any order after defaults. The action of the banks consists in deciding at each time $t \geq 0$ whether they monitor any of the loans which have not defaulted yet. These actions are summarised by the functions $\mathrm{e}_t^{j,\mathrm{i}}$, where for $1 \leq j \leq I - N_t$, $\mathrm{i} \in \{g, b\}$, $\mathrm{e}_t^{j,\mathrm{i}} = 1$ if loan j is monitored at time t by the bank of type ρ_i, and $\mathrm{e}_t^{j,\mathrm{i}} = 0$ otherwise. Non-monitoring renders a private benefit $B > 0$ per loan and per unit time to the bank, regardless of its type. The opportunity cost of monitoring is thus proportional to the number of monitored loans. Once more, more general cost structures could be considered, but this choice has been made for the sake of simplicity.

The rate at which loan j defaults is controlled by the hazard rate α_t^j specifying its instantaneous probability of default conditional on history up to time t. Individual hazard rates are assumed to depend on the monitoring choice of the bank and on the size of the pool. In particular, this allows to incorporate a type of contagion effect in the model. Specifically, we choose to model the hazard rate of a non–defaulted loan j at time t, when it is monitored (or not) by a bank of type ρ_i as

$$\alpha_t^{j,\mathrm{i}} := \alpha_{I-N_t} \left(1 + \left(1 - \mathrm{e}_t^{j,\mathrm{i}}\right) \varepsilon\right), \ t \geq 0, \ j = 1, \dots, I - N_t, \ \mathrm{i} \in \{b, g\}, \tag{3.2.1}$$

where the parameters $\{\alpha_j\}_{1 \leq j \leq I}$ represent individual "baseline" risk under monitoring when the number of loans is j and $\varepsilon > 0$ is the proportional impact of shirking on default risk. We assume that the impact of shirking is independent of the type of the bank. Actually, we found out that differentiating between the banks in this regard created degeneracy in the model. We refer the reader to Section 3.6.8 in the Appendix for a more detailed explanation.

For $\mathrm{i} \in \{b, g\}$, we define the shirking process k^i as the number of loans that the bank of type ρ_i fails to monitor at time $t \geq 0$. Then, according to (3.2.1), the corresponding aggregate default intensity is given by

$$\lambda_t^{k^\mathrm{i}} := \sum_{j=1}^{I-N_t} \alpha_t^{j,\mathrm{i}} = \alpha_{I-N_t} \left(I - N_t + \varepsilon k_t^\mathrm{i}\right). \tag{3.2.2}$$

The banks can fund the pool internally at a cost $r \geq 0$. They can also raise funds from a competitive investor (the Principal, referred to as "he") who values income streams at the prevailing risk–less interest rate of zero. We assume that both the banks and the investor observe the history of defaults and liquidations, as well as the parameters p_b and p_g, but the monitoring choices and the type of the bank are unobservable for the investor.

3.2.2 Description of the contracts

Before going on, let us now describe the stochastic basis on which we will be working. We will always place ourselves on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which N is a Poisson process with intensity λ_t^0 (which is defined by (3.2.2)). We denote by $\mathbb{F} := (\mathcal{F}_t^N)_{t \geq 0}$ the $\mathbb{P}-$completion of the natural filtration of N. We call τ the liquidation time of the whole pool and let $H_t := \mathbf{1}_{\{t \geq \tau\}}$ be the liquidation indicator of the pool. We denote by $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ the minimal filtration containing $\mathbb{F}$ and that makes τ a $\mathbb{G}-$stopping time. We note that this filtration satisfies the usual hypotheses of completeness and right–continuity.

Contracts are offered by the investor to the bank and agreed upon at time 0. As usual in contracting theory, the bank can accept or refuse the contract, but once accepted, both the bank and the investor are fully committed to the contract. More precisely, the investor offers a menu of contracts $\Psi_{\rm i} := (k^{\rm i}, \theta^{\rm i}, D^{\rm i})$, ${\rm i} \in \{g, b\}$ specifying on the one hand a desired level of monitoring $k^{\rm i}$ for the bank of type $\rho_{\rm i}$, which is a $\mathbb{G}-$predictable process such that for any $t \geq 0$, $k^{\rm i}_t$ takes values in $\{0, \dots, I - N_t\}$ (this set is denoted by $\mathfrak{K}$), as well as a flow of payment $D^{\rm i}$. These payments belong to set $\mathcal{D}$ of processes which are càdlàg, non–decreasing, non–negative, $\mathbb{G}-$predictable and such that

$$\mathbb{E}^{\mathbb{P}}[D^{\rm i}_\tau] < +\infty.$$

We do not rule out the possibility of immediate lump–sum payments at the initialisation of the contract, and therefore the processes in $\mathcal{D}$ are assumed to satisfy $D_{0^-} = 0$. Hence, if $D_0 \neq 0$, it means that a lump–sum payment has indeed been made.

The contract also specifies when liquidation occurs. We assume that liquidations can only take the form of the stochastic liquidation of all loans following immediately default[5] Hence, the contract specifies the probability $\theta^{\rm i}_t$, which belongs to the set Θ of $[0,1]-$valued, $\mathbb{G}-$predictable processes, with which the pool is maintained given default ($\mathrm{d}N_t = 1$), so that at each point in time, if the bank has indeed chosen the contract $\Psi_{\rm i}$

$$\mathrm{d}H_t = \begin{cases} 0 \text{ with probability } \theta^{\rm i}_t, \\ \mathrm{d}N_t \text{ with probability } 1 - \theta^{\rm i}_t. \end{cases}$$

With our notations, given a contract $\Psi_{\rm i}$, the hazard rates associated with the default and liquidation processes N_t and H_t are, if the bank does choose the contract $\Psi_{\rm i}$, $\lambda^{k^{\rm i}}_t$ and $\left(1 - \theta^{\rm i}_t\right) \lambda^{k^{\rm i}}_t$, respectively.

The above properties translate into

$$\mathbb{P}\left(\tau \in \left\{\tau^1, ..., \tau^I\right\}\right) = 1, \text{ and } \mathbb{P}(\tau = \tau^j | \mathcal{F}_{\tau^j}, \tau > \tau^{j-1}) = 1 - \theta^{\rm i}_{\tau^j},\ j \in \{1, \dots, I\}.$$

For ease of notations, a contract $\Psi := (k, \theta, D)$ will be said to be admissible if $(k, \theta, D) \in \mathfrak{K} \times \Theta \times \mathcal{D}$. As is commonplace in the Principal–Agent literature, we assume that the monitoring choices of the banks affect only the distribution of the size of the pool. To formalise this, recall that, by definition, any shirking process $k \in \mathfrak{K}$ is $\mathbb{G}-$predictable and bounded. Then, by Girsanov Theorem, we can define a probability measure $\mathbb{P}^k$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, such that $N_t - \int_0^t \lambda^k_t \mathrm{d}s$, is a $\mathbb{P}^k-$martingale. More precisely, we have on $\mathcal{G}_t$

$$\frac{\mathrm{d}\mathbb{P}^k}{\mathrm{d}\mathbb{P}} = Z^k_t,$$

[5] Obviously, several other liquidations procedures could be considered. In the pure moral hazard case treated in [85] (see also the thesis [92, Chapter 8, Section 4]), which will be reviewed below in Section 3.3, some heuristic justifications are given, which lead to thinking that this should in general be, at least, not too far from optimality.

where Z^k is the unique solution of the following SDE

$$Z_t^k = 1 + \int_0^t Z_{s^-}^k \left(\frac{\lambda_s^k}{\lambda_s^0} - 1 \right) \left(\mathrm{d}N_s - \lambda_s^0 \mathrm{d}s \right), \ 0 \leq t \leq \tau, \ \mathbb{P} - a.s.$$

Then, if the bank of type ρ_{i} chooses the contract Ψ_{i}, her utility at $t = 0$, if she follows the recommendation k^{i}, is given by

$$u_0^{\mathrm{i}}(k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}}) := \mathbb{E}^{\mathbb{P}^{k^{\mathrm{i}}}} \left[\int_0^\tau \mathrm{e}^{-rs} (\rho_{\mathrm{i}} \mathrm{d}D_s^{\mathrm{i}} + B k_s^{\mathrm{i}} \, \mathrm{d}s) \right], \tag{3.2.3}$$

while that of the investor is

$$v_0\left((\Psi_{\mathrm{i}})_{\mathrm{i} \in \{g,b\}} \right) := \sum_{\mathrm{i} \in \{g,b\}} p_{\mathrm{i}} \mathbb{E}^{\mathbb{P}^{k^{\mathrm{i}}}} \left[\int_0^\tau \left(I - N_s \right) \mu \mathrm{d}s - \mathrm{d}D_s^{\mathrm{i}} \right]. \tag{3.2.4}$$

The parameter ρ_{i} actually discriminates between the two types of banks through the way they derive utility from the cash–flows delivered by the investor. Hence, for a same level of salary, the good bank will get more utility than a bad bank. Such a form of adverse selection is also considered in the paper of Cvitanić, Wan and Yang [29].

3.2.3 Formulation of the investor's problem

We assume for simplicity that the reservation utility for banks of both type is R_0. The investor's problem is to offer a menu of admissible contracts $(\Psi_{\mathrm{i}})_{\mathrm{i} \in \{g,b\}} := (k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}})_{\mathrm{i} \in \{g,b\}}$ which maximises his utility (3.2.4), subject to the three following constraints

$$u_0^{\mathrm{i}}(k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}}) \geq R_0, \ \mathrm{i} \in \{g, b\}, \tag{3.2.5}$$

$$u_0^{\mathrm{i}}(k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}}) = \sup_{k \in \mathfrak{K}} u_0^{\mathrm{i}}(k, \theta^{\mathrm{i}}, D^{\mathrm{i}}), \ \mathrm{i} \in \{g, b\}, \tag{3.2.6}$$

$$u_0^{\mathrm{i}}(k^{\mathrm{i}}, \theta^{\mathrm{i}}, D^{\mathrm{i}}) \geq \sup_{k \in \mathfrak{K}} u_0^{\mathrm{i}}(k, \theta^j, D^j), \ \mathrm{i} \neq j, \ (\mathrm{i}, j) \in \{g, b\}^2. \tag{3.2.7}$$

Condition (3.2.5) is the usual participation constraint for the banks. Condition (3.2.6) is the so–called incentive compatibility condition, stating that given $(\theta^{\mathrm{i}}, D^{\mathrm{i}})$ the optimal monitoring choice of the bank of type ρ_{i} is the recommended effort k^{i}. Finally, Condition (3.2.7) means that if a bank adversely selects a contract, she cannot get more utility than if she had truthfully revealed her type at time 0. Following the literature, we call such a contract a *screening* contract.

In the sequel, we will start by deriving the optimal contract in the pure moral-hazard case, then we will look into the so–called optimal shutdown contract, for which the investor deliberately excludes the bad bank, before finally investigating the optimal screening contract.

3.3 The pure moral hazard case

In this section, we assume that the type of the bank is publicly known and is fixed to be some ρ_{i}, $\mathrm{i} \in \{g, b\}$, which makes the problem exactly similar to the one considered in [85] (up to the modification of some constants). In particular, the investor only offers one contract. We will briefly explain how to solve the general maximisation problem for the bank and then recall the results obtained in [85]. Furthermore, the results we obtain here, in particular the dynamics of the continuation utilities of the banks, will be crucial to the study of the shutdown and screening contracts later on. Therefore, they will be used throughout without further references.

In this setting, the utility of the investor, when he offers a contract $(k^{\rm i}, \theta^{\rm i}, D^{\rm i}) \in \mathfrak{K} \times \Theta \times \mathcal{D}$ is given by

$$v_0^{\rm pm}(k^{\rm i}, \theta^{\rm i}, D^{\rm i}) := \mathbb{E}^{\mathbb{P}^{k^{\rm i}}}\left[\int_0^\tau (I - N_s)\,\mu {\rm d}s - {\rm d}D_s^{\rm i}\right], \tag{3.3.1}$$

for which we define the following dynamic version for any $t \geq 0$

$$v_t^{\rm pm}(k^{\rm i}, \theta^{\rm i}, D^{\rm i}) := \mathbb{E}^{\mathbb{P}^{k^{\rm i}}}\left[\int_{t\wedge\tau}^\tau (I - N_s)\,\mu {\rm d}s - {\rm d}D_s^{\rm i}\bigg|\,\mathcal{G}_t\right].$$

3.3.1 The bank's problem

3.3.1.1 Dynamics of the bank's value function

As usual, the so–called continuation value of the bank (that is to say her future expected payoff) when offered $(\theta^{\rm i}, D^{\rm i}) \in \Theta \times \mathcal{D}$ plays a central role in the analysis. It is defined, for any $(t, k) \in \mathbb{R}_+ \times \mathfrak{K}$ by

$$u_t^{\rm i}(k, \theta^{\rm i}, D^{\rm i}) := \mathbb{E}^{\mathbb{P}^k}\left[\int_{t\wedge\tau}^\tau {\rm e}^{-r(s-t)}\left(\rho_{\rm i}{\rm d}D_s^{\rm i} + k_s B{\rm d}s\right)\bigg|\,\mathcal{G}_t\right].$$

We also define the value function of the bank for any $t \geq 0$

$$U_t^{\rm i}(\theta^{\rm i}, D^{\rm i}) := \underset{k\in\mathfrak{K}}{\text{ess sup}}\ u_t^{\rm i}(k, \theta^{\rm i}, D^{\rm i}).$$

Departing slightly from the usual approach in the literature, initiated notably by Sannikov [109, 110], we reinterpret the problem of the bank in terms of BSDEs, which, we believe, offers an alternative approach which may be easier to apprehend for the mathematical finance community. Of course, such an interpretation of optimal stochastic control problem with control on the drift is far from being original, and we refer the interested reader to the seminal papers of Hamadène and Lepeltier [50] and El Karoui and Quenez [40] for more information, as well as to the recent articles by Cvitanić, Possamaï and Touzi [26, 27] for more references and a systematic treatment of Principal–Agent type problems with this backward SDE approach. Before stating the related result, let us denote by $(Y^{\rm i}, Z^{\rm i})$ the unique (super–)solution (existence and uniqueness will be justified below) to the following BSDE

$$Y_t^{\rm i} = 0 - \int_t^\tau g^{\rm i}(s, Y_s^{\rm i}, Z_s^{\rm i}){\rm d}s + \int_t^\tau Z_s^{\rm i}\cdot {\rm d}\widetilde{M}_s^{\rm i} + \int_t^\tau {\rm d}K_s^{\rm i},\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s., \tag{3.3.2}$$

where

$$M_t := (N_t, H_t)^\top, \; \widetilde{M}^{\mathrm{i}}_t := M_t - \int_0^t \lambda^0_s (1, 1-\theta^{\mathrm{i}}_s)^\top \mathrm{d}s, \; K^{\mathrm{i}}_t := \rho_{\mathrm{i}} D^{\mathrm{i}}_t,$$

$$g^{\mathrm{i}}(t,y,z) := \inf_{k\in\{0,\dots,I-N_t\}} f^{\mathrm{i}}(t,k,y,z) = ry - (I-N_t)\left(\alpha_{I-N_t}\varepsilon z \cdot (1,1-\theta^{\mathrm{i}}_t)^\top - B\right)^- .$$

We have the following proposition, which is basically a reformulation of [85, Proposition 3.2]. The proof is postponed to Appendix 3.6.1

Proposition 3.3.1 For any $(\theta^{\mathrm{i}}, D^{\mathrm{i}}) \in \Theta\times\mathcal{D}$, the value function of the bank has the dynamics, for $t \in [0,\tau]$, $\mathbb{P}-a.s.$

$$\mathrm{d}U^{\mathrm{i}}_t(\theta^{\mathrm{i}}, D^{\mathrm{i}}) = \left(rU^{\mathrm{i}}_t(\theta^{\mathrm{i}}, D^{\mathrm{i}}) - Bk^{\star,\mathrm{i}}_t + \lambda^{k^{\star,\mathrm{i}}}_t Z^{\mathrm{i}}_t \cdot (1,1-\theta^{\mathrm{i}}_t)^\top\right)\mathrm{d}t - \rho_{\mathrm{i}}\mathrm{d}D^{\mathrm{i}}_t - Z^{\mathrm{i}}_t \cdot \mathrm{d}\widetilde{M}^{\mathrm{i}}_t,$$

where Z^{i} is the second component of the solution to the BSDE (3.3.2). In particular, the optimal monitoring choice of the bank is given by

$$k^{\star,\mathrm{i}}_t = (I-N_t)\mathbf{1}_{\{Z^{\mathrm{i}}_t\cdot(1,1-\theta^{\mathrm{i}}_t)^\top < b_t\}}.$$

Notice that the above result implies that the monitoring choices of the bank are necessarily of bang–bang type, in the sense that she either monitors all the remaining loans, or none at all, which in turn implies that the investor can never give the bank incentives to monitor only a fraction of the loans at a given time[6].

3.3.1.2 Introducing feasible sets

Following the terminology of Cvitanić, Wan and Yang [29], let us discuss the so–called feasible set for the banks.

Definition 3.3.1 We call $\mathcal{V}^{\mathrm{i}}_t$ the feasible set for the expected payoff of banks of type ρ_{i}, starting from some time $t \geq 0$, that is to say all the possible utilities that a bank of type ρ_{i} can get from all the admissible contracts offered by the investor from time t on.

Our first result gives an explicit form of the the feasible set $\mathcal{V}^{\mathrm{i}}_t$, which turns out to be independent of the type of the bank. The proof is relegated to Appendix 3.6.1

Lemma 3.3.1 For $\mathrm{i} \in \{g, b\}$ and for any $t \geq 0$, we have that $\mathcal{V}^{\mathrm{i}}_t = \mathcal{V}_t$, with

$$\mathcal{V}_t := \left[\frac{B(I-N_t)}{r+\lambda^{I-N_t}_t}, +\infty\right).$$

To describe the results of [85], we need to limit our subsequent analysis (for this section only), to contracts enforcing a constant monitoring from the banks, that is to say contracts

[6] We assume here, as is commonplace in the Principal–Agent literature, that in the case where the bank is indifferent with respect to her monitoring decision, that is when $Z^{\mathrm{i}}_t \cdot (1,1-\theta^{\mathrm{i}}_t)^\top = b_t$, she acts in the best interest of the investors, and thus monitors all the $I-N_t$ remaining loans.

incentive–compatible with $k=0$. Obviously, for such contracts, the feasible set of the banks are not equal to $\mathcal{V}_t$, although we will see next that in this case again, it does not depend on the type of the bank.

Definition 3.3.2 The set $\mathcal{V}_t^{0,\mathrm{i}} \subset \mathcal{V}_t$ is called the feasible set for the expected payoff of the banks of type ρ_{i}, starting from some time $t \geq 0$, when the investors can only offer contracts enforcing $k=0$.

This sets can also be obtained explicitly, see Appendix 3.6.1 for the proof.

Lemma 3.3.2 We have for $\mathrm{i} \in \{g, b\}$ and for any $t \geq 0$ that $\mathcal{V}_t^{0,\mathrm{i}} = \mathcal{V}_t^0$, with $\mathcal{V}_t^0 := [b_t, +\infty)$.

3.3.2 The investor's problem and the optimal full–monitoring contract

As mentioned above, in this section only, we follow [85] and consider that the only acceptable behaviour for the bank, from the social point of view, is that she never shirks away from her monitoring responsibilities[7]. In other words, we only allow contracts with a recommendation of $k=0$. Therefore, the value function of the investor becomes

$$V_t^{\mathrm{pm},0}(R_0) := \underset{(D^{\mathrm{i}},\theta^{\mathrm{i}})\in\mathcal{A}^{0,\mathrm{i}}(t,R_0)}{\operatorname{ess\,sup}} \mathbb{E}^{\mathbb{P}^0}\left[\int_{t\wedge\tau}^{\tau}(I-N_s)\mu \mathrm{d}s - \mathrm{d}D_s^{\mathrm{i}}\bigg|\mathcal{G}_t\right],$$

where the set of admissible contracts $\mathcal{A}^{0,\mathrm{i}}(t,R_0)$ is defined for $R_0 \geq b_t$, by

$$\mathcal{A}^{0,\mathrm{i}}(t,R_0) := \left\{(\theta^{\mathrm{i}}, D^{\mathrm{i}}) \in \Theta \times \mathcal{D},\ \text{s.t.}\ (\theta^{\mathrm{i}}, D^{\mathrm{i}}) \text{ enforces } k=0 \text{ and } U_t^{\mathrm{i}}(\theta^{\mathrm{i}}, D^{\mathrm{i}}) \geq R_0\right\}.$$

The main findings of [85] require the following assumptions. Define for any $t \geq 0$ and $j \in \{1, \dots, I\}$,

$$\frac{j}{\overline{\alpha}_j} := \sum_{\mathrm{i}=1}^{j} \frac{1}{\alpha_{\mathrm{i}}},\ \widehat{\lambda}_j^0 := \alpha_j j,\ \widehat{b}_j := \frac{B}{\alpha_j \varepsilon}.$$

Assumption 3.3.1 (i) $\mu \geq \overline{\alpha}_I$.

(ii) We have for all $j \leq I$, $rB(1+\varepsilon) \leq (\mu\varepsilon - B)\varepsilon\overline{\alpha}_j$.

(iii) Individual default risk is non–decreasing with past default, $\alpha_j \leq \alpha_{j-1}$, for all $j \leq I$.

Define next for $x > 0$

$$\phi(x) := \left(\frac{1+x}{1+2x}\right)^{\frac{1}{x}-1},\ \psi(x) := \frac{\phi(x) - x}{(1-x)\phi(x)}.$$

[7]We refer however to Example 3.3.1 below, where we show that this may not always be optimal for the investor, which is reason why we will forego this assumption later in this work.

Let us then define some family of concave functions, unique solutions to the following system of ODEs

$$\begin{cases}\left(ru+\widehat{\lambda}_j^0\widehat{b}_j\right)(v_j^{\rm i})'(u)+j\mu-\widehat{\lambda}_j^0\left(v_j^{\rm i}(u)-\dfrac{u-\widehat{b}_j}{\widehat{b}_{j-1}}v_{j-1}^{\rm i}(\widehat{b}_{j-1})\right)=0,\ u\in\left(\widehat{b}_j,\widehat{b}_j+\widehat{b}_{j-1}\right],\\ \left(ru+\widehat{\lambda}_j^0\widehat{b}_j\right)(v_j^{\rm i})'(u)+j\mu-\widehat{\lambda}_j^0\left(v_j^{\rm i}(u)-v_{j-1}^{\rm i}(u-\widehat{b}_j)\right)=0,\ u\in\left(\widehat{b}_j+\widehat{b}_{j-1},\gamma_j^{\rm i}\right],\\ \rho_{\rm i}(v_j^{\rm i})'(u)+1=0,\ u>\gamma_j^{\rm i},\end{cases} \tag{3.3.3}$$

with initial values $\gamma_1^{\rm i}:=\widehat{b}_1$ and

$$v_1^{\rm i}(u):=\overline{v}_1^{\rm i}-\frac{1}{\rho_{\rm i}}(u-\widehat{b}_1),u\geq\widehat{b}_1,\ \overline{v}_1^{\rm i}:=\frac{\mu}{\widehat{\lambda}_1^0}-\frac{\widehat{b}_1(r+\widehat{\lambda}_1^0)}{\rho_{\rm i}\widehat{\lambda}_1^0},$$

and where for $j\geq 2$, $\gamma_j^{\rm i}$ is defined recursively by $r/\widehat{\lambda}_j^0-1\in\partial v_{j-1}^{\rm i}(\gamma_j^{\rm i}-\widehat{b}_j)$, where $\partial v_{j-1}^{\rm i}$ is the super–differential of the concave function $v_{j-1}^{\rm i}$. The main result of [85] is

Theorem 3.3.1 Assume that the $\left(\widehat{\lambda}_j^0\right)_{1\leq j\leq I}$ satisfy the following recursive conditions for $j\geq 2$

$$\frac{r}{\widehat{\lambda}_j^0}-1\leq\frac{v_{j-1}^{\rm i}\left(\widehat{b}_{j-1}\right)}{\widehat{b}_{j-1}}\ \text{and}\ \left((v_{j-1}^{\rm i})'\left(\widehat{b}_{j-1}\right)\right)^+\frac{\widehat{b}_{j-1}}{v_{j-1}^{\rm i}\left(\widehat{b}_{j-1}\right)}\leq\psi\left(\frac{r}{\widehat{\lambda}_j^0}\right).$$

Then, under Assumption 3.3.1, the system (3.3.3) is well–posed and we have

$$V_t^{\rm pm,0}(R_0)=\sup_{u_t\geq R_0}v_{I-N_t}^{\rm i}(u_t),$$

where $(u_s)_{s\geq t}$ is defined as the unique solution to the SDE on $[t,\tau)$

$$\begin{aligned}\mathrm{d}u_s=&\left(ru_s+\lambda_{I-N_s}^0\widehat{b}_{I-N_s}\right)\mathrm{d}s-\rho_{\rm i}\mathrm{d}D_s^{\star,\rm i}\\&-\left(\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s},\widehat{b}_{I-N_s-1}+\widehat{b}_{I-N_s})\}}(u_s-\widehat{b}_{I-N_s-1})+\widehat{b}_{I-N_s}\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s}+\widehat{b}_{I-N_s-1},\gamma_{I-N_s}^{\rm i})\}}\right)\mathrm{d}N_s\\&-\left(\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s},\widehat{b}_{I-N_s-1}+\widehat{b}_{I-N_s})\}}\widehat{b}_{I-N_s-1}+(u_s-\widehat{b}_{I-N_s})\mathbf{1}_{\{u_s\in[\widehat{b}_{I-N_s}+\widehat{b}_{I-N_s-1},\gamma_{I-N_s}^{\rm i})\}}\right)\mathrm{d}H_s,\end{aligned}$$

with initial value u_t at t, and where we defined for $s\in[t,\tau)$ and $j=1,\dots,I$

$$\begin{aligned}&D_s^{\star,\rm i}:=\mathbf{1}_{\{s=t\}}\frac{(u_t-\gamma_{I-N_t}^{\rm i})^+}{\rho_{\rm i}}+\int_t^s\delta_{\rm i}^{I-N_r}(u_r)\mathrm{d}r,\ \theta_s^\star:=\theta_{\rm i}^{I-N_s}(u_s),\\&\delta_{\rm i}^j(u):=\mathbf{1}_{\{u=\gamma_j^{\rm i}\}}\frac{\widehat{\lambda}_j^0\widehat{b}_j+r\gamma_j^{\rm i}}{\rho_{\rm i}},\ \theta_{\rm i}^j(u):=\mathbf{1}_{\{u\in[\widehat{b}_j,\widehat{b}_{j-1}+\widehat{b}_j)\}}\frac{u-\widehat{b}_j}{\widehat{b}_{j-1}}+\mathbf{1}_{\{u\in[\widehat{b}_j+\widehat{b}_{j-1},\gamma_j^{\rm i})\}}.\end{aligned}$$

We finish this section with an example showing that forcing the bank to always monitor all the loans may not always be optimal for the Principal, which we explain why we forego this assumption in the rest of the chapter.

Example 3.3.1 Consider the case when there is one loan in the project, $I = 1$. The value function of the investor is given by

$$V_t^{\mathrm{pm},0}(R_0) = \sup_{u_t \geq R_0} v_1^{\mathrm{i}}(u_t) = \begin{cases} \overline{v}_1^{\mathrm{i}} - \frac{1}{\rho_{\mathrm{i}}}(R_0 - \widehat{b}_1), & R_0 \geq \widehat{b}_1, \\ \overline{v}_1^{\mathrm{i}}, & R_0 < \widehat{b}_1. \end{cases}$$

It follows from Lemma 3.3.1 that the contract given by $\theta \equiv 0$, $D \equiv 0$ is the only one such that the banks get utility equal to $\frac{B}{r+\widehat{\lambda}_1^1}$ under it. Therefore, the value function of the investor at the point of minimum utility is equal to

$$V_t^{\mathrm{pm}}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right) = \frac{\mu}{\widehat{\lambda}_1^1}.$$

If $R_0 \leq \frac{B}{r+\widehat{\lambda}_1^1}$ and $\mu < \frac{\widehat{\lambda}_1^0 \widehat{b}_1 (r+\widehat{\lambda}_1^0)}{\rho_{\mathrm{i}}(\widehat{\lambda}_1^1 - \widehat{\lambda}_0^1)}$, then $V_t^{\mathrm{pm}}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right) > V_t^{\mathrm{pm},0}\left(\frac{B}{r+\widehat{\lambda}_1^1}\right)$ and it is not optimal for the investor to offer contracts under which the banks never shirks.

3.4 Credible set

In this section we come back to the case in which there are two types of banks in the market, and study the so–called credible set, which is formed by the pairs of value functions of the banks under the admissible contracts. We follow the ideas developed in section 2.4 of the Introduction, to define a dynamic version of the credible set.

As in [29], we do not expect all the points in the feasible set to correspond to a pair of reachable values of the banks under some admissible contract. We will therefore follow the approach initiated by [29] and we will characterize the credible set. We emphasise an important difference with [29] though, in the sense that in our context, the credible set becomes dynamic as it depends on the current size of the pool.

In this section we work with generic contracts $(\theta, D) \in \Theta \times \mathcal{D}$, not necessarily designed for a particular type of bank.

3.4.1 Definition of the credible set and its boundaries

We first define $\widehat{\mathcal{V}}_j := [Bj/(r + \widehat{\lambda}_j^{SH}), \infty)$. Observe that the feasible set

$$\mathcal{V}_t = \left[\frac{B(I - N_t)}{r + \lambda_t^{I-N_t}}, +\infty\right),$$

satisfies $\mathcal{V}_t = \widehat{\mathcal{V}}_{I-N_t}$ for every t, so the only dependence of the feasible set in time is due to the number of loans left. The formal definition of the credible set is the following.

Definition 3.4.1 For any time $t \geq 0$, we define the credible set $\mathcal{C}_{I-N_t}$ as the set of $(u^b, u^g) \in \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that there exists some admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$ satisfying $U_t^b(\theta, D) = u^b$, $U_t^g(\theta, D) = u^g$ and $(U_s^b(\theta, D), U_s^g(\theta, D)) \in \widehat{\mathcal{V}}_{I-N_s} \times \widehat{\mathcal{V}}_{I-N_s}$ for every $s \in [t, \tau)$, $\mathbb{P} - a.s.$

Given a starting time $t \geq 0$ and $u^b \in \widehat{\mathcal{V}}_{I-N_t}$, define the set of contracts under which the value function of the bad bank at time t is equal to u^b,

$$\mathcal{A}^b(t, u^b) = \left\{(\theta, D) \in \Theta \times \mathcal{D},\ U_t^b(\theta, D) = u^b\right\}.$$

We denote by $\mathfrak{U}_t(u^b)$ the largest value $U_t^g(\theta, D)$ that the good bank can obtain from all the contracts $(\theta, D) \in \mathcal{A}^b(t, u^b)$. Once again, this set only depends on t through the value of $I - N_t$, so that we will also use the notation $\widehat{\mathfrak{U}}_{I-N_t}(u^b) := \mathfrak{U}_t(u^b)$. We also denote the lowest one by $\mathfrak{L}_t(u^b)$ and $\widehat{\mathfrak{L}}_{I-N_t}(u^b)$ indifferently. Next, define

$$\widehat{\mathcal{C}}_j := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{L}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{U}}_j(u^b)\right\}.$$

We will prove in Proposition 3.4.3 below that $\widehat{\mathcal{C}}_j = \mathcal{C}_j$ for every $j = 1, \dots I$. Therefore, we will call respectively the functions $\widehat{\mathfrak{L}}_j$ and $\widehat{\mathfrak{U}}_j$ the lower and upper boundary of the credible set when there are j loans left. The aim of the next sections is to obtain explicit formulas for these boundaries.

3.4.2 Utility of not monitoring

We introduce some notations, and denote by k^{SH} the strategy of a bank which does not monitor any loan at any time, *i.e.* $k_s^{SH} = I - N_s$ for every $s \geq 0$. We also denote by $\widehat{\lambda}_j^{SH}$ the default intensity under k^{SH} when there are j loans left, *i.e.* $\widehat{\lambda}_j^{SH} := \alpha_j j(1+\varepsilon)$. We observe that $\widehat{\lambda}_j^{SH} = \lambda_t^{k^{SH}} = \alpha_{I-N_t}(I - N_t)(1+\varepsilon)$, for every $t \geq 0$ such that $I - N_t = j$. Now consider any starting time t such that $I - N_t = j$ and any $\theta \in \Theta$. The continuation utility that the banks get from always shirking (without considering the payments) is

$$u_t^g(k^{SH}, \theta, 0) = u_t^b(k^{SH}, \theta, 0) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_{t\wedge\tau}^{\tau} \mathrm{e}^{-r(s-t)} B k_s^{SH} \mathrm{d}s \middle| \mathcal{G}_t\right]. \tag{3.4.1}$$

This quantity is obviously increasing in θ, so that (3.4.1) attains its minimum value under any contract with $\theta \equiv 0$, which is equal to $c(j,1) := Bj/(r + \widehat{\lambda}_j^{SH})$. Moreover, if the pool is liquidated exactly after the next m defaults, with $m \in \{2, \dots, j\}$, (3.4.1) is equal to (see Appendix 3.6.2)

$$c(j,m) := \frac{Bj}{r + \widehat{\lambda}_j^{SH}} + \sum_{\mathrm{i}=j-m+1}^{j-1} \frac{B\mathrm{i}}{r + \widehat{\lambda}_{\mathrm{i}}^{SH}} \prod_{\ell=\mathrm{i}+1}^{j} \frac{\widehat{\lambda}_\ell^{SH}}{r + \widehat{\lambda}_\ell^{SH}}.$$

In particular, under any contract such that $\theta \equiv 1$, (3.4.1) attains its maximum value, which is equal to

$$C(j) := c(j,j) = \frac{Bj}{r + \widehat{\lambda}_j^{SH}} + \sum_{\mathrm{i}=1}^{j-1} \frac{B\mathrm{i}}{r + \widehat{\lambda}_{\mathrm{i}}^{SH}} \prod_{\ell=\mathrm{i}+1}^{j} \frac{\widehat{\lambda}_\ell^{SH}}{r + \widehat{\lambda}_\ell^{SH}}. \tag{3.4.2}$$

3.4.3 Lower boundary of the credible set

The lower boundary of the credible set is the simplest of the two boundaries and it can be computed directly. We will see that it is a piecewise linear function corresponding to two lines with different slopes. The next proposition states the main inequalities that determine the lower boundary.

Lemma 3.4.1 For any $t \in [0, \tau]$ and any admissible contract $(\theta, D) \in \Theta \times \mathcal{D}$, the value functions of the good and the bad banks satisfy, $\mathbb{P} - a.s.$

$$U_t^g(\theta, D) \geq U_t^b(\theta, D), \tag{3.4.3}$$

$$U_t^g(\theta, D) \geq \frac{\rho_g}{\rho_b} U_t^b(\theta, D) - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t), \tag{3.4.4}$$

where the function $C(j)$ is defined in (3.4.2).

Using Lemma 3.4.1, we prove the following characterisation of the lower boundary of the credible set.

Proposition 3.4.1 For any $j \in \{1, \ldots, I\}$, the lower boundary when there are j loans left is given by

$$\widehat{\mathfrak{L}}_j(u^b) = \begin{cases} u^b, \ c(j,1) \leq u^b \leq C(j), \\ \dfrac{\rho_g}{\rho_b} u^b - \dfrac{(\rho_g - \rho_b)}{\rho_b} C(j), \ C(j) \leq u^b < +\infty. \end{cases}$$

3.4.4 Upper boundary of the credible set

The upper boundary of the credible set is not as simple to obtain as the lower boundary and we have to solve a specific stochastic control problem to identify it. Notice that this approach is similar to the one used in [29].

Let us fix any contract $(\theta, D) \in \Theta \times \mathcal{D}$. We remind the reader that thanks to Proposition 3.3.1, we know that there exist $\mathbb{G}-$predictable and integrable processes $(h^{1,g}(\theta, D), h^{2,g}(\theta, D))$ such that

$$\begin{aligned} \mathrm{d}U_s^g(\theta, D) = & \ (rU_s^g(\theta, D) - Bk_s^{\star,g}(\theta, D))\,\mathrm{d}s - \rho_g \mathrm{d}D_s - h_s^{1,g}(\theta, D)\left(\mathrm{d}N_s - \lambda_s^{k^{\star,g}(\theta,D)}\mathrm{d}s\right) \\ & - h_s^{2,g}(\theta, D)\left(\mathrm{d}H_s - (1-\theta_s)\lambda_s^{k^{\star,g}(\theta,D)}\mathrm{d}s\right), \ s \in [0, \tau], \end{aligned} \tag{3.4.5}$$

where we recall that the optimal monitoring choice $k^{\star,g}(\theta, D)$ is given by

$$k_s^{\star,g}(\theta, D) = (I - N_s)\mathbf{1}_{\{h_s^{1,g}(\theta,D) + (1-\theta_s)h_s^{2,g}(\theta,D) < b_s\}}.$$

Similarly, there exist $\mathbb{G}-$predictable and integrable processes $(h^{1,b}(\theta, D), h^{2,b}(\theta, D))$ such that

$$\begin{aligned} \mathrm{d}U_s^b(\theta, D) = & \ \left(rU_s^b(\theta, D) - Bk_s^{\star,b}(\theta, D)\right)\mathrm{d}s - \rho_b \mathrm{d}D_s - h_s^{1,b}(\theta, D)\left(\mathrm{d}N_s - \lambda_s^{k^{\star,b}(\theta,D)}\mathrm{d}s\right) \\ & - h_s^{2,b}(\theta, D)\left(\mathrm{d}H_s - (1-\theta_s)\lambda_s^{k^{\star,b}(\theta,D)}\mathrm{d}s\right), \ s \in [0, \tau], \end{aligned} \tag{3.4.6}$$

with $k_s^{\star,b}(\theta,D) = (I-N_s)\mathbf{1}_{\{h_s^{1,b}(\theta,D)+(1-\theta_s)h_s^{2,b}(\theta,D)<b_s\}}$. We will use the dynamics (3.4.5)–(3.4.6) to define a simple set of admissible contracts in which we will reinterpret both the value functions of the Agents as controlled diffusion processes, where the controls are $(D,\theta,h^{1,g},h^{2,g},h^{1,b},h^{2,b})$ and satisfying the instantaneous conditions (3.6.2). Obviously, doing so makes us look at a larger class of "contracts", in the sense that in the above representation of the value functions of the bank, the choice of the processes $(h^{1,g},h^{2,g},h^{1,b},h^{2,b})$ is not free, since they are completely determined by the choice of (θ,D). Nonetheless, we will prove later a verification result that will ensure us that the solution of the stochastic control problem we consider provides us the upper boundary of the credible set.

Let us therefore denote by $\mathcal{H}$ the set of non–negative, $\mathbb{G}-$predictable and integrable processes. We abuse notations and define, for every $\Psi := (D,\theta,h^{1,g},h^{2,g},h^{1,b},h^{2,b}) \in \mathcal{D}\times\Theta\times\mathcal{H}^4$, the processes $U^g(\Psi)$ and $U^b(\Psi)$ which satisfy the following (linear) SDEs (well–posedness is trivial)

$$\mathrm{d}U_s^g(\Psi) = rU_s^g(\Psi) - Bk_s^{\star,g}(\Psi) - \rho_g\mathrm{d}D_s - h_s^{1,g}\left(\mathrm{d}N_s - \lambda_s^{k^{\star,g}(\Psi)}\mathrm{d}s\right) - h_s^{2,g}\left(\mathrm{d}H_s - (1-\theta_s)\lambda_s^{k^{\star,g}(\Psi)}\mathrm{d}s\right), \tag{3.4.7}$$

$$\mathrm{d}U_s^b(\Psi) = rU_s^b(\Psi) - Bk_s^{\star,b}(\Psi) - \rho_b\mathrm{d}D_s - h_s^{1,b}\left(\mathrm{d}N_s - \lambda_s^{k^{\star,b}(\Psi)}\mathrm{d}s\right) - h_s^{2,b}\left(\mathrm{d}H_s - (1-\theta_s)\lambda_s^{k^{\star,b}(\Psi)}\mathrm{d}s\right), \tag{3.4.8}$$

where we defined

$$k_s^{\star,g}(\Psi) := (I-N_s)\mathbf{1}_{\{h_s^{1,g}+(1-\theta_s)h_s^{2,g}<b_s\}},\ k_s^{\star,b}(\Psi) := (I-N_s)\mathbf{1}_{\{h_s^{1,b}+(1-\theta_s)h_s^{2,b}<b_s\}}.$$

Remark 3.4.1 In the model, there is no need to consider $h^{1,g}$ and $h^{1,b}$ as positive processes and we do this just for technical reasons. Intuitively, the optimal contracts should satisfy this additional constraint because the investor does not benefit from earlier defaults and if a contract increases the banks' continuation utilities after one of the defaults, the banks should increase the default intensity as much as possible.

For any starting time $t\in[0,\tau]$ and for every $u^b \geq B(I-N_t)/(r+\widehat{\lambda}_{I-N_t}^{SH})$ define

$$\overline{\mathcal{A}}^b(t,u^b) := \Bigg\{ \Psi = (D,\theta,h^{1,b},h^{2,b}) \in \mathcal{D}\times\Theta\times\mathcal{H}^2,\ \text{such that } \forall s\in[t,\tau],$$
$$U_{s^-}^b(\Psi) = h_s^{1,b}+h_s^{2,b},\ U_{s^-}^b(\Psi) - h_s^{1,b} \geq \frac{B(I-N_s)}{r+\lambda_s^{I-N_s}},\ U_t^b(\Psi) = u^b \Bigg\}.$$

We will abuse notations and also call elements of $\overline{\mathcal{A}}^b(t,u^b)$ contracts. The upper boundary $\mathfrak{U}_t$ solves the following control problem

$$\mathfrak{U}_t(u^b) = \underset{\Psi\in\overline{\mathcal{A}}^b(t,u^b)}{\text{ess sup}}\ \mathbb{E}^{\mathbb{P}^{k^g(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_g\mathrm{d}D_s + Bk_s^g(\Psi)\mathrm{d}s)\bigg|\mathcal{G}_t\right],$$

subject to the dynamics

$$U_r^b(\Psi_g) = u^b + \int_t^r \Big(ru_s^b - Bk_s^{\star,b}(\Psi) + h_s^{1,b}\lambda_s^{k^{\star,b}} + h_s^{2,b}(1-\theta_s)\lambda_s^{k^{\star,b}} \Big) \mathrm{d}s - \rho_b \int_t^r \mathrm{d}D_s$$
$$- \int_t^r h_s^{1,b} \mathrm{d}N_s - \int_t^r h_s^{2,b} \mathrm{d}H_s,\ t \leq r \leq \tau,$$

with

$$k_s^{\star,b}(\Psi) = (I - N_s)\mathbf{1}_{\{h_s^{1,b}+(1-\theta_s)h_s^{2,b}<\widehat{b}_{I-N_s}\}},\ k^g(\Psi) \in \underset{k\in\mathfrak{K}}{\arg\max}\,\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_g \mathrm{d}D_s + Bk_s \mathrm{d}s) \middle| \mathcal{G}_t \right].$$

Indeed, the above stochastic control problem corresponds to the highest value that the good bank can obtain from any admissible contract, while ensuring that when the bad bank takes it, she receives exactly u^b, which is exactly the definition of the upper boundary of the credible set. Also, notice that the dependence of $\mathfrak{U}$ on the time is only through the number of loans left at time t.

The next subsections are devoted to first obtaining the HJB equation associated with the above problem, its resolution and then finally to the proof of a verification theorem adapted to our framework. Notice that the above is actually a singular stochastic control problem, since the control D is a non–decreasing process, which is not necessarily absolutely continuous with respect to the Lebesgue measure. We refer the reader to the monograph by Fleming and Soner [44] for more details. In particular, this implies that the HJB equation associated to the problem will be a variational inequality with gradient constraints.

3.4.4.1 HJB equation for the upper boundary

Fix some $1 \leq j \leq I$, and define for every $k = 0, 1, \cdots, j$, $\widehat{\lambda}_j^k := \alpha_j(j + k\varepsilon)$. The system of HJB equations associated to the previous control problem is given by $\widehat{\mathcal{U}}_0 \equiv 0$, and for any $1 \leq j \leq I$

$$\min\left\{ -\sup_{(\theta,h^1,h^2)\in C^j} \left\{ \begin{array}{c} \widehat{\mathcal{U}}_j'(u^b)\left(ru^b - Bk^b + [h^1 + (1-\theta)h^2]\widehat{\lambda}_j^{k^b}\right) \\ +\widehat{\lambda}_j^{k^g}\theta\widehat{\mathcal{U}}_{j-1}(u^b - h^1) - (\widehat{\lambda}_j^{k^g} + r)\widehat{\mathcal{U}}_j(u^b) + Bk^g \end{array} \right\},\ \widehat{\mathcal{U}}_j'(u^b) - \frac{\rho_g}{\rho_b} \right\} = 0, \tag{3.4.9}$$

for every $u^b \geq \frac{Bj}{r+\widehat{\lambda}_j^{SH}}$, with the boundary condition $\widehat{\mathcal{U}}_j(Bj/(r+\widehat{\lambda}_j^{SH})) = Bj/(r+\widehat{\lambda}_j^{SH})$, where $k^b := j\mathbf{1}_{\{h^1+(1-\theta)h^2<\widehat{b}_j\}}$, $k^g := j\mathbf{1}_{\{\widehat{\mathcal{U}}_j(u^b)-\theta\widehat{\mathcal{U}}_{j-1}(u^b-h^1)<\widehat{b}_j\}}$, and the set of constraints is defined by

$$C^j := \left\{ (\theta, h^1, h^2) \in [0,1] \times \mathbb{R}_+^2,\ h^1 + h^2 = u^{b,c},\ h^2 \geq \frac{B(j-1)}{r + \widehat{\lambda}_{j-1}^{SH}} \right\}.$$

Remark 3.4.2 Note that the incentive compatibility condition for the good bank is implicit in the HJB equation. Indeed, at every s we have

$$\widehat{\mathcal{U}}_{I-N_s}(U_s^b(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^b(\Psi)) = \left(\widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^b(\Psi) - h_s^{1,b}(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^b(\Psi))\right)\Delta N_s$$
$$- \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^b(\Psi) - h_s^{1,b}(\Psi))\Delta H_s,$$

which implies that on the upper boundary $h_s^{1,g}(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^b(\Psi)) - \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^b(\Psi) - h_s^{1,b}(\Psi))$ and $h_s^{2,g}(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^b(\Psi) - h_s^{1,b}(\Psi))$. Therefore

$$h_s^{1,g}(\Psi) + (1-\theta_s^g)h_s^{2,g}(\Psi) = \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^b(\Psi)) - \theta_s^g\widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^b(\Psi) - h_s^{1,b}(\Psi)).$$

At the points where $\widehat{\mathcal{U}}_j'(u^b) > \rho_g/\rho_b$, the first term of the variational inequality (3.4.9) must be equal to zero, so the upper boundary must satisfy the following equation

$$r\widehat{\mathcal{U}}_j(u^b) = \sup_{(\theta,h^1,h^2)\in C^j}\left\{\begin{array}{c}\widehat{\mathcal{U}}_j'(u^b)\left(ru^b - Bk^b + [h^1 + (1-\theta)h^2]\widehat{\lambda}_j^{k^b}\right)\\ +[\widehat{\mathcal{U}}_{j-1}(u^b - h^1)\theta - \widehat{\mathcal{U}}_j(u^b)]\widehat{\lambda}_j^{k^g} + Bk^g\end{array}\right\}. \tag{3.4.10}$$

We will refer to this equation as the diffusion equation.

- **Step 1: case of 1 loan, solving the diffusion equation**

Before dealing with the variational inequality (3.4.9), we will solve the diffusion equation (3.4.10). When $j = 1$, it reduces to

$$r\widehat{\mathcal{U}}_1(u^b) = \widehat{\mathcal{U}}_1'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) - \widehat{\mathcal{U}}_1(u^b)\widehat{\lambda}_1^{k^g} + Bk^g, \tag{3.4.11}$$

with $k^b = \mathbf{1}_{\{u^b<\widehat{b}_1\}}$, $k^g = \mathbf{1}_{\{\widehat{\mathcal{U}}(u^b)<\widehat{b}_1\}}$.

Remark 3.4.3 Notice that the boundary condition $\widehat{\mathcal{U}}_1\left(\frac{B}{r+\widehat{\lambda}_1^1}\right) = \frac{B}{r+\widehat{\lambda}_1^1}$ is implicit in the equation.

Our first result is the following, whose proof is deferred to Appendix 3.6.6

Lemma 3.4.2 There is a family of continuously differentiable solutions to the diffusion equation, indexed by some constant $C > 0$, which are given by

$$\widehat{\mathcal{U}}_1^C(u^b) := \begin{cases} C^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}_1^1}\right) + \dfrac{B}{r+\widehat{\lambda}_1^1},\ u^b \in \left[\dfrac{B}{r+\widehat{\lambda}_1^1}, x_C^\star\right), \\ C\widehat{b}_1^{\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(\dfrac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}},\ u^b \in [x_C^\star, \widehat{b}_1), \\ Cu^b,\ u^b \in [\widehat{b}_1, +\infty), \end{cases}$$

where $x_1^{C,\star} := \left(\dfrac{1}{C}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}}\widehat{b}_1\dfrac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} + \dfrac{B}{r+\widehat{\lambda}_1^1}.$

- **Step 2: case of 1 loan, solving the HJB equation**

In this case the variational inequality (3.4.9) reduces to

$$\min\left\{r\widehat{\mathcal{U}}_1(u^b) - \widehat{\mathcal{U}}'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1(u^b)\widehat{\lambda}_1^{k^g} - Bk^g,\ \widehat{\mathcal{U}}_1'(u^b) - \frac{\rho_g}{\rho_b}\right\} = 0. \tag{3.4.12}$$

We already found the solutions of the diffusion equation inside of this variational inequality and now we will take care of the whole HJB equation. We expect the upper boundary to saturate the second term in the variational inequality for big values of u^b, so we will search for a solution of (3.4.12) satisfying the following condition: there exists $x^\star \in [B/(r+\widehat{\lambda}_1^1), \infty)$ such that

$$\widehat{\mathcal{U}}_1'(x^\star) = \frac{\rho_g}{\rho_b} \text{ and } \widehat{\mathcal{U}}_1'(u^b) > \frac{\rho_g}{\rho_b},\ \text{for } u^b < x^\star. \tag{3.4.13}$$

At first sight it could seem that by doing this we face the risk of not finding the correct solution of the dynamic programming equation. Nevertheless, this is not the case and we will prove later a verification result which assures us that the solution that we find under this condition corresponds indeed to the upper boundary of the credible set. The proof of the following Lemma will be given in Appendix 3.6.6.

Lemma 3.4.3 The unique solution of the HJB equation which satisfies condition (3.4.13) is given by, defining $x_1^\star := x_1^{\rho_g/\rho_b,\star}$

$$\widehat{\mathcal{U}}_1^\star(u^b) := \widehat{\mathcal{U}}_1^{\rho_g/\rho_b,\star}(u^b) = \begin{cases} \left(\dfrac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}_1^1}\right) + \dfrac{B}{r+\widehat{\lambda}_1^1},\ u^b \in \left[\dfrac{B}{r+\widehat{\lambda}_1^1}, x_1^\star\right), \\ \dfrac{\rho_g}{\rho_b} b_1^{\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(\dfrac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(u^b - \dfrac{B}{r+\widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}},\ u^b \in [x_1^\star, \widehat{b}_1), \\ \dfrac{\rho_g}{\rho_b}u^b,\ u^b \in [\widehat{b}_1, +\infty). \end{cases} \tag{3.4.14}$$

As an illustration, in Figure 3.1 we show the credible set which corresponds to the region delimited by its upper and lower boundaries. In this example, we considered $r = 0.02$, $B = 0.002$, $\varepsilon = 0.25$, $\alpha_1 = 0.055$, $\frac{\rho_g}{\rho_b} = 2$.

- **Step 3: solving the HJB equation in the general case**

In the general case, when $j > 1$, we can reduce the number of variables and rewrite the diffusion equation (3.4.10) in an equivalent form

$$r\widehat{\mathcal{U}}_j(u^b) = \sup_{(\theta,h^1)\in\widehat{C}^j}\left\{\begin{array}{c}\widehat{\mathcal{U}}_j'(u^b)\left(ru^b - Bk^b + [u^b - \theta(u^b-h^1)]\widehat{\lambda}_j^{k^b}\right) \\ +\left(\widehat{\mathcal{U}}_{j-1}(u^b-h^1)\theta - \widehat{\mathcal{U}}_j(u^b)\right)\widehat{\lambda}_j^{k^g} + Bk^g\end{array}\right\}, \tag{3.4.15}$$

where we recall that $k^b = \mathbf{1}_{\{u^b-\theta(u^b-h^1)<\widehat{b}_j\}}$, $k^g = \mathbf{1}_{\{\widehat{\mathcal{U}}_j(u^b)-\theta\widehat{\mathcal{U}}_{j-1}(u^b-h^1)<\widehat{b}_j\}}$ and the set of con-

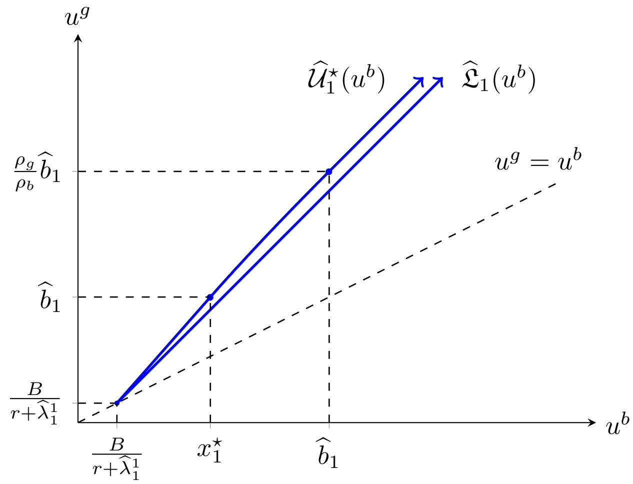


Figure 3.1: Credible set with one loan left.

straints is now given by

$$\widehat{C}^j := \left\{ (\theta, h^1) \in [0,1] \times \mathbb{R}_+,\ u^b \geq h^1 + \frac{B(j-1)}{r + \widehat{\lambda}_{j-1}^{SH}} \right\}. \tag{3.4.16}$$

When we proved that the lower boundary of the credible set is reachable we used contracts of maximum duration, which maintain the pool until the last default. This gives us the intuition that the longer the contract lasts, the smaller the difference between the utilities of the banks will be. Therefore the upper boundary of the credible set, where the difference between both utilities is maximal, should be reachable with contracts of minimum duration, which terminate the contractual relationship immediately after the first default. In the model this means that θ is equal to zero and the resulting HJB equation for the upper boundary has the same form that the one in the case with one loan left. We expect then that the solution of the diffusion equation will be the of the same form as (3.4.14). The object of the next proposition is to prove our guess rigorously. We postpone the proof to Appendix 3.6.6.

Proposition 3.4.2 For any $j \geq 1$, the function $\widehat{\mathcal{U}}_j^\star$ defined by

$$\widehat{\mathcal{U}}_j^\star(u^b) := \begin{cases} \left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \left(u^b - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) + \frac{Bj}{r+\widehat{\lambda}_j^{SH}},\ u^b \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right), \\ \frac{\rho_g}{\rho_b} b_j^{\frac{\widehat{\lambda}_j^{SH}-\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(u^b - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}},\ u^b \in [x_j^\star, \widehat{b}_j), \\ \frac{\rho_g}{\rho_b} u^b,\ u^b \in [\widehat{b}_j, +\infty), \end{cases} \tag{3.4.17}$$

where $x_j^\star := \left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \widehat{b}_j \frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}} + \frac{Bj}{r+\lambda_j^{SH}}$, is a solution of the HJB equation (3.4.9).

3.4.4.2 Verification Theorem

According to the maximisers in equation (3.4.15) we define the following controls

$$\begin{cases} \delta^j(u^b) := \mathbf{1}_{\{u^b \geq \widehat{b}_j\}} \frac{u^b(r+\widehat{\lambda}_j^0)}{\rho_b}, \\ \theta^j(u^b) := 0, \\ h^{1,b,j}(u^b) := u^b, \ h^{2,b,j}(u^b) := 0, \\ k^{b,j}(u^b) := j\mathbf{1}_{\{u^b < \widehat{b}_j\}}, \ k^{g,j}(u^b) := j\mathbf{1}_{\{\widehat{\mathcal{U}}_j^\star(u^b) < \widehat{b}_j\}}. \end{cases} \tag{3.4.18}$$

Before stating the verification result for the upper boundary, we make a comment about the domain of the functions $\widehat{\mathcal{U}}_j^\star$. Rigorously speaking, it is possible for the utilities of the banks to be zero but this happens only at time τ when all the pools are liquidated. The domain of $\widehat{\mathcal{U}}_j^\star$ is the set $\widehat{\mathcal{V}}_j$ but in the proof of the verification theorem it will be implicitly understood that $\widehat{\mathcal{U}}_j^\star(0) = 0$. In any case, we do not need the functions $\widehat{\mathcal{U}}_j^\star$ to be defined at zero because Itô's formula will be used on intervals which do not contain τ.

Theorem 3.4.1 Consider any starting time $t \geq 0$. For any $u^b \geq \frac{B(I-N_t)}{r+\widehat{\lambda}_{I-N_t}^{SH}}$, let the process $(u_s^b)_{s\in[t,\tau]}$ be the unique solution of the following SDE

$$u_v^b = u^b + \int_t^v \left[(r + \lambda_s^{k^{b,I-N_t}}) u_s^b - Bk^{b,I-N_t}(u_s^b) - \rho_b \delta^{I-N_t}(u_s^b) \right] \mathrm{d}s - \int_t^v u_{s^-}^b \mathrm{d}N_s, \ v \in [t, \tau]. \tag{3.4.19}$$

Then, under the contract $\Psi^\star := (D^{g,\star}, \theta^{g,\star}, h^{1,b,\star}, h^{2,b,\star}) \in \mathcal{D} \times \Theta \times \mathcal{H}^2$ defined for $s \in [t, \tau]$ by

$$\mathrm{d}D_s^\star := \delta^{I-N_t}(u_s^b)\mathrm{d}s, \ \theta_s^\star \equiv 0, \ h_s^{1,b,\star} := h^{1,b,I-N_t}(u_s^b), \ h_s^{2,b,\star} \equiv 0,$$

the value function of the bad bank is $U_t^b(\Psi^\star) = u^b$ and the one of good bank is $U_t^g(\Psi^\star) = \widehat{\mathcal{U}}_{I-N_t}^\star(u^b)$. Moreover, $\Psi^\star \in \overline{\mathcal{A}}^b(t, u^b)$ and for any other contract which belongs to $\overline{\mathcal{A}}^b(t, u^b)$, the value function of the good bank under such a contract is less or equal to $\widehat{\mathcal{U}}_{I-N_t}^\star(u^b)$. In particular, this implies that

$$\widehat{\mathcal{U}}_{I-N_t}^\star(u^b) = \widehat{\mathfrak{U}}_{I-N_t}(u^b).$$

To conclude the section, we state that $\overline{\mathcal{C}_j}$ is indeed equal to the credible set with j loans left and therefore the functions $\widehat{\mathfrak{U}}_j$ and $\widehat{\mathfrak{L}}_j$ correspond to its upper and lower boundaries.

Proposition 3.4.3 For every $1 \leq j \leq I$, $\widehat{\mathcal{C}}_j = \mathcal{C}_j$.

3.5 Optimal contracts

In this section we study two kind of contracts that the investor can offer to the bank, the shutdown contract, which corresponds to a single contract designed to be accepted only for the good bank and the screening contract, corresponding to a menu of contracts, one for each type of Agent, providing incentives to the bank to accept the contract designed for her true type.

3.5.1 Shutdown contract

In the so–called shutdown contract, the investor designs a contract $\Psi_g = (k^g, D^g, \theta^g)$ only for the good bank and makes sure that the bad bank will not accept it. Under these conditions the utility of the investor at time $t = 0$ is

$$v_0^{\mathrm{g,Shut}}(\Psi_g) = p_g \mathbb{E}^{\mathbb{P}^{k^g}} \left[\int_0^\tau \mu(I - N_s)\mathrm{d}s - \mathrm{d}D_s^g \right]. \tag{3.5.1}$$

So the investor will offer a contract which maximises (3.5.1) subject to the constraints

$$u_0^g(k^g, \theta^g, D^g) \geq R_0 \geq \sup_{k \in \mathfrak{K}} u_0^b(k, \theta^g, D^g), \tag{3.5.2}$$

$$u_0^g(k^g, \theta^g, D^g) = \sup_{k \in \mathfrak{K}} u_0^g(k, \theta^g, D^g). \tag{3.5.3}$$

Recalling the dynamics (3.4.5)–(3.4.6), we can rewrite the investor's maximisation problem as follows

$$v_0^{\mathrm{Shut}} := \sup_{(\theta^g, D^g) \in \mathcal{A}^g_{\mathrm{Shut}}} p_g \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\theta^g, D^g)}} \left[\int_0^\tau \mu(I - N_s)\mathrm{d}s - \mathrm{d}D_s^g \right],$$

where

$$\mathcal{A}^g_{\mathrm{Shut}} := \left\{ (\theta^g, D^g) \in \Theta \times \mathcal{D},\ U_0^{b,c}(\theta^g, D^g) \leq R_0 \leq U_0^g(\theta^g, D^g) \right\}.$$

Remark 3.5.1 We will use the notation $U^{b,c}(\theta^g, D^g)$ for the value function that the bad bank gets if she does not reveal her true type and accepts the contract designed for the good bank. We make a distinction between this process and $U^b(\theta^b, D^b)$, which corresponds to the value function that the bad bank obtain if she accepts the contract designed for her by the investor. We make the same distinction between the associated processes $h^{1,b,c}(\theta, D)$, $h^{2,b,c}(\theta, D)$ and $h^{1,b}(\theta, D)$, $h^{2,b}(\theta, D)$.

As in the previous section, we will define a simple set of contracts and consider the value functions of the Agents as diffusion processes controlled by $(D, \theta, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c})$. As explained before, by doing so we will look at a larger class of "contracts". Nonetheless, we will prove later that under reasonable assumption the solution of the problem we consider do coincide with the optimal shutdown contract.

Define for any $(t, u^g, u^{b,c}) \in [0, +\infty) \times \mathcal{C}_{I-N_t}$

$$\widehat{\mathcal{A}}^g(t, u^g, u^{b,c}) := \Bigg\{ \Psi_g = (D^g, \theta^g, h^{1,g}, h^{2,g}, h^{1,b,c}, h^{2,b,c}) \in \mathcal{D} \times \Theta \times \mathcal{H}^4, \text{ such that } \forall s \in [t, \tau],$$

$$U_{s^-}^g(\Psi_g) = h_s^{1,g} + h_s^{2,g},\ U_{s^-}^g(\Psi_g) - h_s^{1,g} \geq \frac{B(I - N_s)}{r + \lambda_s^{I-N_s}},\ U_t^g(\Psi_g) = u^g$$

$$U_{s^-}^{b,c}(\Psi_g) = h_s^{1,b,c} + h_s^{2,b,c},\ U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c} \geq \frac{B(I - N_s)}{r + \lambda_s^{I-N_s}},\ U_t^{b,c}(\Psi_g) = u^{b,c} \Bigg\}.$$

We will also consider in the sequel the following standard control problem, for any $(u^{b,c},u^g)\in\mathcal{C}_I$

$$\widehat{v}_0^g(u^{b,c},u^g):=\sup_{\Psi_g\in\widehat{\mathcal{A}}^g(0,u^g,u^{b,c})} p_g\mathbb{E}^{\mathbb{P}^{k^\star,g(\Psi_g)}}\left[\int_0^\tau \mu(I-N_s)\mathrm{d}s-\mathrm{d}D_s^g\right].$$

We abuse notations and also call elements of $\widehat{\mathcal{A}}^g(t,u^g,u^{b,c})$ contracts.

3.5.1.1 Value function of the investor

In this section, we characterise the value function of the investor when he offers only shutdown contracts. We will start by computing the value function on the boundaries of the credible set, before explaining how it can be characterised by a specific HJB equation in the interior of the credible set, under reasonable assumptions.

3.5.1.1.1 Value function of the investor on the lower boundary Recall the lower boundary with j loans left

$$\widehat{\mathfrak{L}}_j(u^{b,c})=\begin{cases}u^{b,c},\ c(j,1)\le u^{b,c}\le C(j),\\ \dfrac{\rho_g}{\rho_b}u^{b,c}-\dfrac{(\rho_g-\rho_b)}{\rho_b}C(j),\ C(j)\le u^{b,c}<\infty.\end{cases}$$

Consider any starting time $t\ge 0$. For $u^{b,c}\in\mathcal{C}_{I-N_t}$, we denote by $V^{\mathfrak{L},g}(u^{b,c})$ the value function of the investor in the lower boundary, that is

$$V_t^{\mathfrak{L},g}(u^{b,c}):=\operatorname*{ess\,sup}_{\Psi_g\in\widehat{\mathcal{A}}^g(t,\widehat{\mathfrak{L}}_{I-N_t}(u^{b,c}),u^{b,c})}\mathbb{E}^{\mathbb{P}^{k^\star,g(\Psi_g)}}\left[\int_t^\tau (\mu(I-N_s)\mathrm{d}s-\mathrm{d}D_s^g)\middle|\mathcal{G}_t\right].\qquad(3.5.4)$$

The following two propositions are proved in Appendix 3.6.7 and give explicitly the value of $V_t^{\mathfrak{L},g}(u^{b,c})$.

Proposition 3.5.1 For every $u^{b,c}\in\mathcal{C}_{I-N_t}$, if $u^{b,c}\ge C(I-N_t)$ then the value function of the investor in the lower boundary is given by

$$V_t^{\mathfrak{L},g}(u^{b,c})=\sum_{\mathrm{i}=N_t}^{I-1}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}_{I-\mathrm{i}}^{SH}}-\left(\frac{u^{b,c}-C(I-N_t)}{\rho_b}\right).$$

Proposition 3.5.2 Fix some $t\ge 0$. For every $u^{b,c}\in\mathcal{C}_{I-N_t}$, with $c(I-N_t,1)\le u^{b,c}<C(I-N_t)$, let $\nu(u^{b,c})$ be the unique solution of the following equation in ν

$$\left(\frac{B(I-N_t)}{r+\widehat{\lambda}_{I-N_t}^{SH}}-u^{b,c}\right)+\sum_{\mathrm{i}=N_t+1}^{I-1}\int_{s_\mathrm{i}(\nu)}^{\infty}\left(\frac{B(I-\mathrm{i})}{r+\widehat{\lambda}_{I-\mathrm{i}}^{SH}}\mathrm{e}^{-rx}\right)f_{\tau_\mathrm{i}}(x)\mathrm{d}x=0,$$

where $f_{\tau_{\mathrm{i}}}$ is the density of the law of τ_{i} under $\mathbb{P}^{k^{SH}}$ and where

$$s_{\mathrm{i}}(\nu) := \begin{cases} 0, \ \nu \leq \dfrac{\mu(r+\widehat{\lambda}^{SH}_{I-\mathrm{i}})}{B\widehat{\lambda}^{SH}_{I-\mathrm{i}}}, \\ \dfrac{1}{r}\ln\left(\dfrac{\nu B\widehat{\lambda}^{SH}_{I-\mathrm{i}}}{\mu(r+\widehat{\lambda}^{SH}_{I-\mathrm{i}})}\right), \ \nu \geq \dfrac{\mu(r+\widehat{\lambda}^{SH}_{I-\mathrm{i}})}{B\widehat{\lambda}^{SH}_{I-\mathrm{i}}}. \end{cases}$$

Then the value function of the investor in the lower boundary is given by

$$V_t^{\mathfrak{L},g}(u^{b,c}) = \frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{\mathrm{i}=N_t+1}^{I-1}\int_{s_{\mathrm{i}}(\nu(u^{b,c}))}^{\infty}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}^{SH}_{I-\mathrm{i}}}f_{\tau_{\mathrm{i}}}(x)\,\mathrm{d}x.$$

Remark 3.5.2 Observe that the function $V_t^{\mathfrak{L},g}$ computed in Propositions 3.5.1 and 3.5.2 depends on t only through the quantity $I-N_t$. Define, for any $j=1,\dots,J$ the map

$$\widehat{V}_j^{\mathfrak{L},g}(u^{b,c}) := \begin{cases} \displaystyle\sum_{\mathrm{i}=1}^{j}\frac{\mu\mathrm{i}}{\widehat{\lambda}^{SH}_{\mathrm{i}}} - \left(\frac{u^{b,c}-C(j)}{\rho_b}\right), \ u^{b,c} \geq C(j), \\ \displaystyle\frac{\mu j}{\widehat{\lambda}^{SH}_j} + \sum_{\mathrm{i}=1}^{j-1}\int_{s_{I-j}(\nu(u^{b,c}))}^{\infty}\frac{\mu\mathrm{i}}{\widehat{\lambda}^{SH}_{\mathrm{i}}}f_{\tau_{I-\mathrm{i}}}(x)\,\mathrm{d}x, \ u^{b,c} \in (c(j,1),C(j)). \end{cases}$$

We have therefore, that $V_t^{\mathfrak{L},g}(u^{b,c}) = \widehat{V}^{\mathfrak{L},g}_{I-N_t}(u^{b,c})$.

3.5.1.1.2 Value function of the investor on the upper boundary The next proposition states that the upper boundary of the credible set is absorbing in the following sense: if under any contract the pair of value functions of the banks reaches the upper boundary at some time, the pair will stay on the upper boundary until the pool is liquidated.

Proposition 3.5.3 Consider $(t,u^g,u^{b,c}) \in [0,+\infty)\times\mathcal{C}_{I-N_t}$ such that $u^g = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$ and any contract $\Psi_g = (D^g,\theta^g,h^{1,g},h^{2,g},h^{1,b,c},h^{2,b,c}) \in \widehat{\mathcal{A}}^g(t,u^g,u^{b,c})$. Then $U_s^g(\Psi_g) = \widehat{\mathfrak{U}}_{I-N_s}(U_s^{b,c}(\Psi_g))$ for every $s\in[t,\tau)$.

The next proposition states an important property satisfied by the contracts which make the continuation utilities of the banks lie in the upper boundary of the credible set.

Proposition 3.5.4 Consider $(t,u^g,u^{b,c}) \in [0,+\infty)\times\mathcal{C}_{I-N_t}$ such that $u^g = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$ and any contract $\Psi_g = (D^g,\theta^g,h^{1,g},h^{2,g},h^{1,b,c},h^{2,b,c}) \in \widehat{\mathcal{A}}^g(t,u^g,u^{b,c})$. Then

(i) $\theta_s^g = 0$ for every $s\in[t,\tau)$ such that $U_s^{b,c}(\Psi_g) < b_s$.

(ii) If $U_{s_0}^{b,c}(\Psi_g) \geq b_{s_0}$ for some $s_0\in[t,\tau)$ then $k_s^{\star,b,c}(\Psi_g) = 0$ and $U_s^{b,c}(\Psi_g)\geq b_s$ for every $s\in[s_0,\tau)$.

We are now ready to give the value function of the investor on the upper boundary of the credible set, under the assumptions of Theorem 3.3.1.

Proposition 3.5.5 Under Assumption 3.3.1, we have that for any $t \geq 0$ and any $u^{b,c} \in \widehat{\mathcal{V}}_{I-N_t}$, the value function of the investor on the upper boundary, defined by

$$V_t^{\mathfrak{U},g}(u^{b,c}) := \underset{\Psi_g \in \widehat{\mathcal{A}}^g(t, \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c}), u^{b,c})}{\text{ess sup}} \mathbb{E}^{\mathbb{P}^{k^\star, g}(\Psi_g)} \left[\int_t^\tau (\mu(I - N_s)\mathrm{d}s - \mathrm{d}D_s^g) \bigg| \mathcal{G}_t \right], \tag{3.5.5}$$

verifies $V_t^{\mathfrak{U},g}(u^{b,c}) = \widehat{V}_{I-N_t}^{\mathfrak{U},g}(u^{b,c})$, where for any $j = 1, \cdots, I$

$$\widehat{V}_j^{\mathfrak{U},g}(u^{b,c}) := \begin{cases} \dfrac{\mu j}{\widehat{\lambda}_j^{SH}} + \widehat{C}^j \left(u^{b,c} - \dfrac{Bj}{r + \widehat{\lambda}_j^{SH}} \right)^{\frac{\widehat{\lambda}_j^{SH}}{r + \widehat{\lambda}_j^{SH}}}, \ u^{b,c} < x_j^\star, \\ \dfrac{\mu j}{\widehat{\lambda}_j^0} + \left(v_j^b(\widehat{b}_j) - \dfrac{\mu j}{\widehat{\lambda}_j^0} \right) \left(\widehat{b}_j \dfrac{r + \widehat{\lambda}_j^0}{r + \widehat{\lambda}_j^{SH}} \right)^{-\frac{\widehat{\lambda}_j^0}{r + \widehat{\lambda}_j^{SH}}} \left(u^{b,c} - \dfrac{Bj}{r + \widehat{\lambda}_j^{SH}} \right)^{\frac{\widehat{\lambda}_j^0}{r + \widehat{\lambda}_j^{SH}}}, \ u^{b,c} \in \left[x_j^\star, \widehat{b}_j \right), \\ v_j^b(u^{b,c}), \ u^{b,c} \geq \widehat{b}_j, \end{cases}$$

with

$$\widehat{C}^j := \left(\frac{\mu j}{\widehat{\lambda}_j^0} - \frac{\mu j}{\widehat{\lambda}_j^{SH}} + \left(\frac{\rho_b}{\rho_g} \right)^{\frac{\widehat{\lambda}_j^0}{r + \widehat{\lambda}_j^0}} \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0} \right) \right) \left(\frac{\rho_b}{\rho_g} \right)^{-\frac{\widehat{\lambda}_j^{SH}}{r + \widehat{\lambda}_j^0}} \left(\frac{\widehat{b}_j (r + \widehat{\lambda}_j^0)}{r + \widehat{\lambda}_j^{SH}} \right)^{-\frac{\widehat{\lambda}_j^{SH}}{r + \widehat{\lambda}_j^{SH}}}.$$

3.5.1.1.3 Value function of the investor in the credible set We define, for any $t \geq 0$ and any $(u^{b,c}, u^g) \in \widehat{\mathcal{C}}_{I-N_t}$, the value function of the investor in the credible set by

$$V_t^g(u^{b,c}, u^g) := \underset{\Psi_g \in \widehat{\mathcal{A}}^g(t, u^g, u^{b,c})}{\text{ess sup}} \mathbb{E}^{\mathbb{P}^{k^\star, g}(\Psi_g)} \left[\int_t^\tau (\mu(I - N_s)\mathrm{d}s - \mathrm{d}D_s^g) \bigg| \mathcal{G}_t \right]. \tag{3.5.6}$$

The system of HJB equations associated to this control problem is given by $\widehat{V}_0^g \equiv 0$, and for any $1 \leq j \leq I$

$$\min \left\{ -\sup_{\overline{C}^j} \left\{ \begin{array}{l} \partial_{u^{b,c}} \widehat{V}_j^g(u^{b,c}, u^g) \left(r u^{b,c} - B k^{b,c} + [h^{1,b,c} + (1-\theta) h^{2,b,c}] \widehat{\lambda}_j^{k^{b,c}} \right) \\ + \partial_{u^g} \widehat{V}_j^g(u^{b,c}, u^g) \left(r u^g - B k^g + [h^{1,g} + (1-\theta) h^{2,g}] \widehat{\lambda}_j^{k^g} \right) \\ + [\widehat{V}_{j-1}^g(u^{b,c} - h^{1,b,c}, u^g - h^{1,g}) - \widehat{V}_j^g(u^{b,c}, u^g)] \widehat{\lambda}_j^{k^g} \\ - \widehat{V}_{j-1}^g(u^{b,c} - h^{1,b,c}, u^g - h^{1,g})(1-\theta) \widehat{\lambda}_j^{k^g} + \mu j \end{array} \right\} \right.$$
$$\left. , \rho_b \partial_{u^{b,c}} \widehat{V}_j^g(u^{b,c}, u^g) + \rho_g \partial_{u^g} \widehat{V}_j^g(u^{b,c}, u^g) + 1 \right\} = 0. \tag{3.5.7}$$

With $k^{b,c} = j \cdot 1_{\{h^{1,b,c} + (1-\theta)h^{2,b,c} < \widehat{b}_j\}}$, $k^g = j \cdot 1_{\{h^{1,g} + (1-\theta)h^{2,g} < \widehat{b}_j\}}$ and the set of constraints

$$\overline{C}^j = \left\{ (\theta, h^{1,b,c}, h^{2,b,c}, h^{1,g}, h^{2,g}), \theta \in [0,1], u^g = h^{1,g} + h^{2,g}, u^{b,c} = h^{1,b,c} + h^{2,b,c}, h^{2,g}; h^{2,b,c} \geq \frac{B(j-1)}{r + \widehat{\lambda}_{j-1}^{SH}} \right\}.$$

The boundary conditions of (3.5.7) are given by

$$\widehat{V}_j^g(u^{b,c}, \widehat{\mathfrak{U}}_j(u^{b,c})) = \widehat{V}_j^{\mathcal{U},g}(u^{b,c}), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j,$$
$$\widehat{V}_j^g(u^{b,c}, \widehat{\mathfrak{L}}_j(u^{b,c})) = \widehat{V}_j^{\mathcal{L},g}(u^{b,c}), \text{ for every } u^{b,c} \in \widehat{\mathcal{V}}_j.$$

The last step would now be to make a rigorous link between a solution in an appropriate sense to the above system and the value function V^g. We have then two possibilities at hand.

(i) First, we can use classical arguments to prove that $\widehat{V}_j^g$ is a viscosity solution of the above PDE for every $j = 1, \ldots, I$, a result we should then complement with a comparison theorem ensuring uniqueness of the viscosity solution. This would provide a complete characterisation of the value function of the investor, and more importantly would make the problem amenable to numerical computations, using for instance classical finite difference methods. As for the optimal contract, it will correspond to the maximisers in the Hamiltonian above, and therefore would require that we prove that $\widehat{V}_j^g$ is at least weakly differentiable (for instance if $\widehat{V}_j^g$ is concave or Lipschitz continuous, which we expect from the form of the problem) to be well defined. This program can in principle be carried out using standard arguments in viscosity theory of Hamilton–Jacobi equations. However, given the length of the present work, we believe that it would not serve a specific purpose and decided to leave these arguments out.

(ii) Another possibility would be to show existence of a smooth solution to the PDE, and prove a comparison theorem similar to Theorem 3.4.1. However, since the above recursive system involves elliptic PDEs in dimension 2 in a non–trivial domain, we do not expect to be able to obtain explicit solutions in general, which means that existence would have to be proved through abstract arguments. Once again, we believe that such considerations are outside the scope of this work. We will therefore simply state without proof (since it would be extremely similar to that of Theorem 3.4.1) a verification theorem adapted to our framework.

Theorem 3.5.1 Assume that the system of HJB equations (3.5.7) has a C^1-solution and that $(\theta_{\mathrm{i}}^\star(u^{b,c}, u^g), h_{\mathrm{i}}^{\star,1,b,c}(u^{b,c}, u^g), h_{\mathrm{i}}^{\star,2,b,c}(u^{b,c}, u^g), h_{\mathrm{i}}^{\star,1,g}(u^{b,c}, u^g), h_{\mathrm{i}}^{\star,2,g}(u^{b,c}, u^g))$ attains the supremum in the Hamiltonian. Define then

$$\delta_s^{\star,g}(u^{b,c}, u^g) := \frac{1}{\rho_g}\Big(ru^g - Bk_s^{\star,g} + \widehat{\lambda}_{I-N_s}^{k_s^{\star,g}} h_{I-N_s}^{\star,1,g}(u^{b,c}, u^g) + (1-\theta_{I-N_s}^\star(u^{b,c}, u^g))\widehat{\lambda}_{I-N_s}^{k_s^{\star,g}} h_{I-N_s}^{\star,2,g}(u^{b,c}, u^g)\Big)\Big)$$
$$\times \mathbf{1}_{\{\rho_b \partial_{u^{b,c}} \widehat{V}_{I-N_s}^g(u^{b,c},u^g) + \rho_g \partial_{u^g} \widehat{V}_{I-N_s}^g(u^{b,c},u^g) + 1\}},$$

where

$$k_s^{\star,g}(u^{b,c}, u^g) : = (I - N_s)\mathbf{1}_{\{h_{I-N_s}^{\star,1,g}(u^{b,c},u^g) + (1-\theta_{I-N_s}^{\star,g}(u^{b,c},u^g))h_{I-N_s}^{\star,2,g}(u^{b,c},u^g) < \widehat{b}_{I-N_s}\}},$$
$$k_s^{\star,b,c}(u^{b,c}, u^g) : = (I - N_s)\mathbf{1}_{\{h_{I-N_s}^{\star,1,b,c}(u^{b,c},u^g) + (1-\theta_{I-N_s}^{\star,g}(u^{b,c},u^g))h_{I-N_s}^{\star,2,b,c}(u^{b,c},u^g) < \widehat{b}_{I-N_s}\}}.$$

If the corresponding contract is admissible

$$\Psi^{\star,g} := \Big(\Big(\delta_{I-N_\cdot}^{\star,g}, \theta_{I-N_\cdot}^\star, h_{I-N_\cdot}^{\star,1,b,c}, h_{I-N_\cdot}^{\star,2,b,c}, h_{I-N_\cdot}^{\star,1,g}, h_{I-N_\cdot}^{\star,2,g}\Big)(U^{\star,b,c}, U^{\star,g})\Big),$$

where $(U^{\star,b,c}, U^{\star,g})$ are weak solutions to the corresponding SDEs

$$\begin{aligned}
\mathrm{d}U_s^{\star,g} = &\ \left(rU_s^{\star,g} - Bk_s^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}) - \rho_g \delta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g})\right) \mathrm{d}s \\
&- h_{I-N_s}^{\star,1,g}(U_s^{\star,b,c}, U_s^{\star,g}) \left(\mathrm{d}N_s - \lambda_s^{k^{\star,g}((U^{\star,b,c},U^{\star,g}))}\mathrm{d}s\right) \\
&- h_{I-N_s}^{\star,2,g}(U_s^{\star,b,c}, U_s^{\star,g}) \left(\mathrm{d}H_s - (1-\theta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}))\lambda_s^{k^{\star,g}((U^{\star,b,c},U^{\star,g}))}\mathrm{d}s\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}U_s^{\star,b,c} = &\ \left(rU_s^{\star,b,c} - Bk_s^{\star,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) - \rho_b \delta_{I-N_s}^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g})\right) \mathrm{d}s \\
&- h_{I-N_s}^{\star,1,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) \left(\mathrm{d}N_s - \lambda_s^{k^{\star,b,c}((U^{\star,b,c},U^{\star,g}))}\mathrm{d}s\right) \\
&- h_{I-N_s}^{\star,2,b,c}(U_s^{\star,b,c}, U_s^{\star,g}) \left(\mathrm{d}H_s - (1-\theta_s^{\star,g}(U_s^{\star,b,c}, U_s^{\star,g}))\lambda_s^{k^{\star,b,c}((U^{\star,b,c},U^{\star,g}))}\mathrm{d}s\right),
\end{aligned}$$

then we have

$$v_0^{\mathrm{Shut}} = \sup_{u^{b,c} \leq R_0 \leq u^g} \widehat{v}_0^g(u^{b,c}, u^g) = \sup_{u^{b,c} \leq R_0 \leq u^g} p_g \widehat{V}_I^g(u^{b,c}, u^g),$$

and $\Psi^{\star,g}$ is an optimal contract for the investor.

3.5.2 Screening contract

Recall that in the screening contract the investor designs a menu of contracts, one for each Agent, and his expected utility is given by

$$v_0\left((\Psi_{\mathrm{i}})_{\mathrm{i}\in\{g,b\}}\right) = \sum_{\mathrm{i}\in\{g,b\}} p_{\mathrm{i}} \mathbb{E}^{\mathbb{P}^{k^{\mathrm{i}}}}\left[\int_0^\tau (I - N_s)\,\mu \mathrm{d}s - \mathrm{d}D_s^{\mathrm{i}}\right]. \tag{3.5.8}$$

In this case, we will have to keep track of the continuation utilities of both banks, when they choose the contract designed for them, as well as when they do not truthfully reveal their type. We will denote by v_0 the maximal utility that the investor can get out of the screening contract.

$$v_0 := \sup_{(\theta^g,\theta^b,D^g,D^b)\in\mathcal{A}_{\mathrm{Scr}}} p_g \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\theta^g,D^g)}}\left[\int_0^\tau \mu(I-N_s)\mathrm{d}s - \mathrm{d}D_s^g\right] + p_b \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\theta^b,D^b)}}\left[\int_0^\tau \mu(I-N_s)\mathrm{d}s - \mathrm{d}D_s^b\right],$$

where

$$\mathcal{A}_{\mathrm{Scr}} := \left\{(\theta^g, \theta^b, D^g, D^b) \in \Theta^2 \times \mathcal{D}^2, U_0^b(\theta^b, D^b) \geq R_0, U_0^j(\theta^j, D^j) \geq U_0^{j,c}(\theta^{\mathrm{i}}, D^{\mathrm{i}}), (\mathrm{i}, j) \in \{g, b\}^2,\ \mathrm{i} \neq j\right\}.$$

Remark 3.5.3 Observe that we can omit the condition $U_0^g(\theta^g, D^g) \geq R_0$ in the definition of $\mathcal{A}_{\mathrm{Scr}}$. Indeed, it is implied by the inequality $U_0^{g,c}(\theta^b, D^b) \geq U_0^b(\theta^b, D^b)$, which follows from Lemma 3.4.1.

Different from the study of the shutdown contract, where the investor contracts only the good bank, in order to obtain the optimal screening contract we need to characterise also the value function of the investor when he contracts the bad bank. We will therefore follow

Section 3.5.1.1, but by replacing the good bank by the bad bank. Hence, we define for any $(t, u^b, u^{g,c}) \in [0, +\infty) \times \mathcal{C}_{I-N_t}$ the set

$$\widehat{\mathcal{A}}^b(t, u^{g,c}, u^b) := \Big\{ \Psi_b = (D^b, \theta^b, h^{1,g,c}, h^{2,g,c}, h^{1,b}, h^{2,b}) \in \mathcal{D} \times \Theta \times \mathcal{H}^4, \text{ such that } \forall s \in [t, \tau],$$

$$U^b_{s^-}(\Psi_b) = h^{1,b}_s + h^{2,b}_s,\ U^b_{s^-}(\Psi_b) - h^{1,b}_s \geq \frac{B(I - N_s)}{r + \lambda_s^{I-N_s}},\ U^b_t(\Psi_b) = u^b,$$

$$U^{g,c}_{s^-}(\Psi_b) = h^{1,g,c}_s + h^{2,g,c}_s,\ U^{g,c}_{s^-}(\Psi_b) - h^{1,g,c}_s \geq \frac{B(I - N_s)}{r + \lambda_s^{I-N_s}},\ U^{g,c}_t(\Psi_b) = u^{g,c} \Big\}.$$

We also introduce the following stochastic control problem for any $(u^b, u^{g,c}) \in \mathcal{C}_I$

$$\widehat{v}^{\mathrm{b}}_0(u^b, u^{g,c}) := \sup_{\Psi_b \in \widehat{\mathcal{A}}^b(0, u^{g,c}, u^b)} p_b \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)} \left[\int_0^\tau \mu(I - N_s) \mathrm{d}s - \mathrm{d}D^b_s \right].$$

The aim of the next sections is to compute the function $\widehat{v}^{\mathrm{b}}_0(u^{g,c}, u^b)$, representing the utility of the investor when hiring the bad bank. We start by studying it on the boundary of the credible set.

3.5.2.1 Boundary study

We denote by $V^{\mathfrak{L},b}(u^{g,c})$ the value function of the investor in the lower boundary, when hiring the bad bank, defined by

$$V^{\mathfrak{L},b}_t(u^b) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t, \widehat{\mathfrak{L}}_{I-N_t}(u^b), u^b)} \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)} \left[\int_t^\tau \left(\mu(I - N_s) \mathrm{d}s - \mathrm{d}D^b_s \right) \Big| \mathcal{G}_t \right]. \tag{3.5.9}$$

The first result is that the value function of the investor on the lower boundary of the credible set is the same when hiring either the bad or the good bank. This is mainly due to the fact that both banks shirk on the lower boundary.

Proposition 3.5.6 For every $u^b \in \mathcal{C}_{I-N_t}$, we have $V^{\mathfrak{L},b}_t(u^b) = V^{\mathfrak{L},g}_t(u^b)$.

Proof. By definition we have the set equality $\widehat{\mathcal{A}}^g(t, \widehat{\mathfrak{L}}_{I-N_t}(u^b), u^b) = \widehat{\mathcal{A}}^b(t, \widehat{\mathfrak{L}}_{I-N_t}(u^b), u^b)$. From Lemmas 3.6.1 and 3.6.2 we know that for every $\Psi_b \in \widehat{\mathcal{A}}^b(t, \widehat{\mathfrak{L}}_{I-N_t}(u^b), u^b)$, both Agents always shirk under Ψ_b, therefore the objective functions in the definitions of $V^{\mathfrak{L},g}_t(u^b)$ and $V^{\mathfrak{L},b}_t(u^b)$ are also the same and equality holds. □

Let us now consider the upper boundary. We denote by $V^{\mathfrak{U},b}(u^b)$ the value function of the investor on the upper boundary when hiring the bad Agent.

$$V^{\mathfrak{U},b}_t(u^b) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t, \widehat{\mathfrak{U}}_{I-N_t}(u^b), u^b)} \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)} \left[\int_t^\tau \left(\mu(I - N_s) \mathrm{d}s - \mathrm{d}D^b_s \right) \Big| \mathcal{G}_t \right]. \tag{3.5.10}$$

We have the following result.

Proposition 3.5.7 Under the assumptions of Theorem 3.3.1, for any $t \geq 0$ and any $u^b \in \widehat{\mathcal{V}}_{I-N_t}$, we have that $V_t^{\mathfrak{U},b}(u^b) = \widehat{V}_{I-N_t}^{\mathfrak{U},b}(u^b)$, where for any $j = 1, \cdots, I$

$$\widehat{V}_j^{\mathfrak{U},b}(u^b) := \begin{cases} \dfrac{\mu j}{\widehat{\lambda}_j^{SH}} + \tilde{C}^j \left(u^b - \dfrac{Bj}{r + \widehat{\lambda}_j^{SH}} \right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}, \ u^b < \widehat{b}_j, \\ v_j^b(u^b), \ u^b \geq \widehat{b}_j, \end{cases}$$

with

$$\tilde{C}^j = \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^{SH}} \right) \left(\frac{\widehat{b}_j(r + \widehat{\lambda}_j^0)}{r + \widehat{\lambda}_j^{SH}} \right)^{\frac{-\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}.$$

Proof. The proof is identical to the proof of Proposition 3.5.5, with the only difference that since the Principal is hiring the bad Agent, for $u^b < \widehat{b}_j$ the ODE associated to the value function is

$$0 = \widehat{\mathcal{V}}_j'(u^b) \left(\left(r + \widehat{\lambda}_j^{SH} \right) u^b - Bj \right) - \widehat{\mathcal{V}}_j(u^b) \widehat{\lambda}_j^{SH} + \mu j,$$

with the boundary condition $\widehat{\mathcal{V}}_j \left(\frac{Bj}{r+\widehat{\lambda}_j^{SH}} \right) = \frac{\mu j}{\widehat{\lambda}_j^{SH}}$. $\square$

3.5.2.2 Study of the credible set

We define, for any $t \geq 0$ and any $(u^b, u^{g,c}) \in \widehat{\mathcal{C}}_{I-N_t}$, the value function of the investor in the credible set when hiring the bad bank by

$$V_t^b(u^b, u^{g,c}) := \operatorname*{ess\,sup}_{\Psi_b \in \widehat{\mathcal{A}}^b(t, u^{g,c}, u^b)} \mathbb{E}^{\mathbb{P}^{k^\star,b}(\Psi_b)} \left[\int_t^\tau \left(\mu(I - N_s) \mathrm{d}s - \mathrm{d}D_s^b \right) \middle| \mathcal{G}_t \right]. \tag{3.5.11}$$

The system of HJB equations associated to this control problem is given by $\widehat{V}_0^b \equiv 0$, and for any $1 \leq j \leq I$

$$\min \left\{ -\sup_{\overline{C}^j} \left\{ \begin{array}{l} \partial_{u^b} \widehat{V}_j^b(u^b, u^{g,c}) \left(ru^b - Bk^b + [h^{1,b} + (1-\theta)h^{2,b}] \widehat{\lambda}_j^{k^b} \right) \\ + \partial_{u^{g,c}} \widehat{V}_j^b(u^b, u^{g,c}) \left(ru^{g,c} - Bk^{g,c} + [h^{1,g,c} + (1-\theta)h^{2,g,c}] \widehat{\lambda}_j^{k^{g,c}} \right) \\ + [\widehat{V}_{j-1}^b(u^b - h^{1,b}, u^{g,c} - h^{1,g,c}) - \widehat{V}_j^b(u^b, u^{g,c})] \widehat{\lambda}_j^{k^b} \\ - \widehat{V}_{j-1}^b(u^b - h^{1,b}, u^{g,c} - h^{1,g,c})(1-\theta) \widehat{\lambda}_j^{k^b} + \mu j \end{array} \right\} \right.$$
$$\left. , \rho_b \partial_{u^b} \widehat{V}_j^b(u^b, u^{g,c}) + \rho_g \partial_{u^{g,c}} \widehat{V}_j^b(u^b, u^{g,c}) + 1 \right\} = 0. \tag{3.5.12}$$

With $k^b = j \cdot 1_{\{h^{1,b}+(1-\theta)h^{2,b} < \widehat{b}_j\}}$, $k^{g,c} = j \cdot 1_{\{h^{1,g,c}+(1-\theta)h^{2,g,c} < \widehat{b}_j\}}$ and the same set of constraints $\overline{C}^j$ as in the system of HJB equations associated to the functions $\widehat{V}_j^g(u^{b,c}, u^g)$. The boundary conditions of (3.5.12) are given by

$$\widehat{V}_j^b(u^b, \widehat{\mathfrak{U}}_j(u^b)) = \widehat{V}_j^{\mathcal{U},b}(u^b), \ \text{for every } u^{b,c} \in \widehat{\mathcal{V}}_j,$$
$$\widehat{V}_j^b(u^b, \widehat{\mathfrak{L}}_j(u^b)) = \widehat{V}_j^{\mathcal{L},g}(u^b), \ \text{for every } u^{b,c} \in \widehat{\mathcal{V}}_j.$$

Theorem 3.5.2 Assume that the conditions of Theorem 3.5.1 hold, that (3.5.12) admits a C^1-solution and that $(\theta^\star_{\rm i}, h^{\star,1,g,c}_{\rm i}, h^{\star,2,g,c}_{\rm i}, h^{\star,1,b}_{\rm i}, h^{\star,2,b}_{\rm i})(u^{g,c}, u^b)$ attains the supremum in the Hamiltonian. Define then

$$\delta^{\star,b}_s(u^{g,c}, u^b) := \frac{1}{\rho_b}\Big(ru^b - Bk^{\star,b}_s + \widehat{\lambda}^{k^{\star,b}_s}_{I-N_s} h^{\star,1,b}_{I-N_s}(u^{g,c}, u^b) + (1 - \theta^\star_{I-N_s}(u^{g,c}, u^b))\widehat{\lambda}^{k^{\star,b}_s}_{I-N_s} h^{\star,2,b}_{I-N_s}(u^{g,c}, u^b))\Big)$$
$$\times \mathbf{1}_{\{\rho_g \partial_{u^{g,c}} \widehat{V}^b_{I-N_s}(u^{g,c},u^b) + \rho_b \partial_{u^b} \widehat{V}^b_{I-N_s}(u^{g,c},u^b) + 1\}},$$

where

$$k^{\star,b}_s(u^{g,c}, u^b) := (I - N_s) \cdot 1_{\{h^{\star,1,b}_{I-N_s}(u^{g,c},u^b) + (1-\theta^{\star,b}_{I-N_s}(u^{g,c},u^b))h^{\star,2,b}_{I-N_s}(u^{g,c},u^b) < \widehat{b}_{I-N_s}\}},$$
$$k^{\star,g,c}_s(u^{g,c}, u^b) := (I - N_s) \cdot 1_{\{h^{\star,1,g,c}_{I-N_s}(u^{g,c},u^b) + (1-\theta^{\star,b}_{I-N_s}(u^{g,c},u^b))h^{\star,2,g,c}_{I-N_s}(u^{g,c},u^b) < \widehat{b}_{I-N_s}\}}.$$

If the corresponding contract is admissible

$$\Psi^{\star,b} := \Big(\Big(\delta^{\star,b}_{I-N_\cdot}, \theta^\star_{I-N_\cdot}, h^{\star,1,g,c}_{I-N_\cdot}, h^{\star,2,g,c}_{I-N_\cdot}, h^{\star,1,b}_{I-N_\cdot}, h^{\star,2,b}_{I-N_\cdot}\Big)(U^{\star,g,c}, U^{\star,b})\Big),$$

where $(U^{\star,g,c}, U^{\star,b})$ are weak solutions to the corresponding SDEs

$$\begin{aligned}
\mathrm{d}U^{\star,b}_s = &\Big(rU^{\star,b}_s - Bk^{\star,b}_s(U^{\star,g,c}_s, U^{\star,b}_s) - \rho_b \delta^{\star,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big)\,\mathrm{d}s \\
&- h^{\star,1,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big(\mathrm{d}N_s - \lambda_s^{k^{\star,b}((U^{\star,g,c},U^{\star,b}))}\mathrm{d}s\Big) \\
&- h^{\star,2,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big(\mathrm{d}H_s - (1 - \theta^{\star,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s))\lambda_s^{k^{\star,b}((U^{\star,g,c},U^{\star,b}))}\mathrm{d}s\Big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}U^{\star,g,c}_s = &\Big(rU^{\star,g,c}_s - Bk^{\star,g,c}_s(U^{\star,g,c}_s, U^{\star,b}_s) - \rho_g \delta^{\star,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big)\,\mathrm{d}s \\
&- h^{\star,1,g,c}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big(\mathrm{d}N_s - \lambda_s^{k^{\star,g,c}((U^{\star,g,c},U^{\star,b}))}\mathrm{d}s\Big) \\
&- h^{\star,2,g,c}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s)\Big(\mathrm{d}H_s - (1 - \theta^{\star,b}_{I-N_s}(U^{\star,g,c}_s, U^{\star,b}_s))\lambda_s^{k^{\star,g,c}((U^{\star,g,c},U^{\star,b}))}\mathrm{d}s\Big),
\end{aligned}$$

then $(\Psi^{\star,g}, \Psi^{\star,b})$ is an optimal menu of contracts for the investor, and we have

$$\begin{aligned}
v_0 &= \sup_{\{R_0 \le u^b, u^{b,c} \le u^b, u^{g,c} \le u^g\}} \widehat{v}^g_0(u^{b,c}, u^g) + \widehat{v}^{\rm b}_0(u^b, u^{g,c}) \\
&= \sup_{\{R_0 \le u^b, u^{b,c} \le u^b, u^{g,c} \le u^g\}} p_g \widehat{V}^g_I(u^{b,c}, u^g) + p_b \widehat{V}^b_I(u^b, u^{g,c}).
\end{aligned}$$

3.5.3 Description of the optimal contracts

In this section we describe the optimal contracts for the investor when he designs a contract for the good or the bad bank. We explain in detail the optimal contracts on the boundaries of the credible set, which can be obtained explicitly from the value function of the investor. In the interior of the credible set, we discuss the properties we expect the optimal contracts to have when the verification theorems 3.5.1 and 3.5.2 hold.

3.5.3.1 Optimal contracts on the boundaries of the credible set

We start with the upper boundary of the credible set. The following result is a direct consequence of the proofs of Proposition 3.5.5 and 3.5.7, and the optimal contract for the pure moral hazard case described in Theorem 3.3.1.

Proposition 3.5.8 Under Assumption 3.3.1, consider for any $t \geq 0$ and $(u^b, \mathfrak{U}_t(u^b)) \in \mathcal{C}_{I-N_t}$ the process $(u^b_s)_{s\geq t}$ as the solution of the following SDE on $[t,\tau)$

$$\mathrm{d}u^b_s = \Big((ru^b_s - Bk^{b,\star}_s + \lambda^{k^{b,\star}}_s(h^{1,b,\star}_s + (1-\theta^\star_s)h^{2,b,\star}_s)\Big)\,\mathrm{d}s - \rho_b \mathrm{d}D^\star_s - h^{1,b,\star}_s \mathrm{d}N_s - h^{2,b,\star}_s \mathrm{d}H_s, \tag{3.5.13}$$

with initial value u^b at t, and with

$$D^\star_s := \mathbf{1}_{\{s=t\}} \frac{(u^b - \gamma^b_{I-N_t})^+}{\rho_b} + \int_t^s \delta^{I-N_r}(u^b_r)\mathrm{d}r,\ \theta^\star_s := \theta^{I-N_s}(u^b_s),$$
$$h^{1,b,\star}_s := h^{1,b,I-N_s}(u^b_s),\ h^{2,b,\star}_s := h^{2,b,I-N_s}(u^b_s), k^{b,\star}_s := k^{b,j}(u^b_s),$$

for $s \in [t,\tau)$ and $j = 1,\dots,I$, where

$$\delta^j(u) := \mathbf{1}_{\{u=\gamma^b_j\}} \frac{\widehat{\lambda}^0_j \widehat{b}_j + r\gamma^b_j}{\rho_{\mathrm{i}}},\ \theta^j(u) := \mathbf{1}_{\{u\in[\widehat{b}_j, \widehat{b}_{j-1}+\widehat{b}_j)\}} \frac{u-\widehat{b}_j}{\widehat{b}_{j-1}} + \mathbf{1}_{\{u\in[\widehat{b}_j+\widehat{b}_{j-1},\gamma^b_j)\}},$$
$$h^{1,b,j}(u) := \mathbf{1}_{\{u\in[c(j,1),\widehat{b}_j)\}}u + \mathbf{1}_{\{u\in[\widehat{b}_j,\widehat{b}_{j-1}+\widehat{b}_j)\}}(u-\widehat{b}_{j-1}) + \mathbf{1}_{\{u\in[\widehat{b}_j+\widehat{b}_{j-1},\gamma^b_j)\}}\widehat{b}_j,$$
$$h^{2,b,j}(u) := u - h^{1,b,j}(u),\ k^{b,j}(u) = j\mathbf{1}_{\{h^{1,b,j}(u)+(1-\theta^j(u))h^{2,b,j}(u)<\widehat{b}_j\}}.$$

Then, the contract $\Psi^\star = (D^\star, \theta^\star, h^{1,b,\star}, h^{2,b,\star})$ is the unique solution of problems (3.5.5) and (3.5.10).

Let us comment the optimal contract for the investor on the upper boundary of the credible set. It is the same if he designs a contract for the good or the bad bank. The state process $(u^b_s)_{s\geq t}$ defined by (3.5.13) corresponds to the value function of the bad bank under the optimal contract. The optimal contract offers no payments to the banks when u^b_s is smaller than $\gamma^b_{I-N_s}$. In this case the continuation utility of the bad bank is an increasing process and eventually reaches the value $\gamma^b_{I-N_s}$, if no default happens in the meantime. Payments are postponed until this moment. If the initial value for the bad Agent u^b is greater than $\gamma^b_{I-N_t}$, a lump-sum payment is made at t^- in order to have $u_t = \gamma^b_{I-N_t}$. When $u^b_s = \gamma^b_{I-N_s}$, the banks receive constant payments which keep the value function of the bad bank constant at this level. Concerning the liquidation of the project, if at the default time τ_j, it holds that $u^b_{\tau_j} < \widehat{b}_j$ the project is liquidated. In case $u^b_{\tau_j} \in [\widehat{b}_j + \widehat{b}_{j-1}, \gamma^b_j)$, the project will continue with probability $\theta_j \in (0,1)$ which will be closer to one as $u^b_{\tau_j}$ gets closer to γ^b_j. If $u^b_{\tau_j} \geq \gamma^b_j$, the project will be maintained. Finally, the bad bank will monitor all the loans only when her value function is greater than $\widehat{b}_{I-N_s}$, whereas the good bank will monitor when the value of the bad bank is greater than $x^\star_{I-N_s}$. Figure 3.2 depicts the optimal contract of the investor on the upper boundary of the credible set, denoting $\widehat{B}_j := \widehat{b}_j + \widehat{b}_{j-1}$.

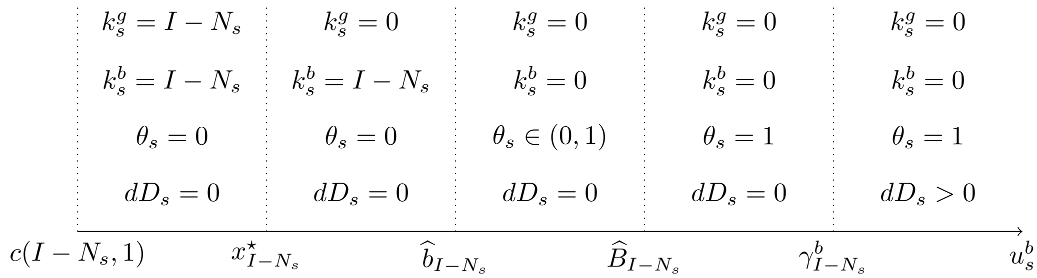


Figure 3.2: Optimal contract on the upper boundary.

For the lower boundary of the credible set, we have the following result.

Proposition 3.5.9 Under Assumption 3.3.1, consider for any $t \geq 0$ and $(u^b, \mathfrak{L}_t(u^b)) \in \mathcal{C}_{I-N_t}$ the process $(u_s^b)_{s\geq t}$ as the solution of the following SDE on $[t, \tau)$

$$\mathrm{d}u_s^b = \Big((ru_s^b - Bk_s^{b,\star} + \lambda_s^{k_s^{b,\star}}(h_s^{1,b,\star} + (1-\theta_s^\star)h_s^{2,b,\star}) \Big) \mathrm{d}s - \rho_b \mathrm{d}D_s^\star - h_s^{1,b,\star}\mathrm{d}N_s - h_s^{2,b,\star}\mathrm{d}H_s, \tag{3.5.14}$$

with initial value u^b at t, and with

$$D_s^\star := \mathbf{1}_{\{s=t\}} \frac{(u^b - C(I-N_s))^+}{\rho_b}, \; \theta_s^\star := \mathbf{1}_{\{u_s^b \geq C(I-N_s)\}},$$

$$h_s^{1,b,\star} := u_s^b - C(I - N_s - 1)\mathbf{1}_{\{u_s^b \geq C(I-N_s)\}}, \; h_s^{2,b,\star} := C(I - N_s - 1)\mathbf{1}_{\{u_s^b \geq C(I-N_s)\}},$$
$$k_s^{b,\star} = (I - N_s)\mathbf{1}_{\{h_s^{1,b,\star} + (1-\theta_s^\star)h_s^{2,b,\star} < b_s\}},$$

for $s \in [t, \tau)$. Then, the contract $\Psi^\star = (D^\star, \theta^\star, h^{1,b,\star}, h^{2,b,\star})$ is the unique solution of (3.5.4) and (3.5.9).

Proof. The payments and the value of $\theta^\star$ in the case $u^b \geq C(I - N_t)$ are a direct consequence of the proof of Proposition 3.5.1. From the proof of Proposition 3.5.2 we have that if $u^b < C(I - N_t)$ then

$$\theta_s^\star = \mathbf{1}_{\left\{ s-t > \frac{1}{r}\ln\left(\frac{\nu(u^b)B\widehat{\lambda}^{SH}_{I-N_t}}{\mu(r+\widehat{\lambda}^{SH}_{I-N_t})} \right) \right\}},$$

where $\nu(u^b)$ the solution of the associated dual problem. Since the quantity inside of the logarithm decreases with time, we have that $\theta^\star$ is a process which starts at zero, jumps to one at some instant and keeps constant afterwards. This means that if $\theta^\star$ jumps to one at some time s and the project is still running, necessarily the continuation utility of the bad Agent is equal to $C(I - N_s)$ because the project will continue until the last default. □

On the lower boundary of the credible set, the optimal contract for the investor also does not depend on the type of the bank. If the initial value of the bad bank u^b is greater than $C(I - N_t)$, the banks receive a lump-sum payment such that $u^b_{t+} = C(I - N_t)$. This is the only payment offered by the contract. If there is a default at some time s such that $u_s^b < C(I - N_s)$, the project is liquidated. When $u_s^b = C(I - N_s)$ the contract maintains the project until the last default. Since the optimal contract does not provides incentives to the banks to monitor the loans, the good and the bad bank shirk until the liquidation of the project. Figure 3.3 depicts the optimal contract of the investor on the lower boundary of the credible set.

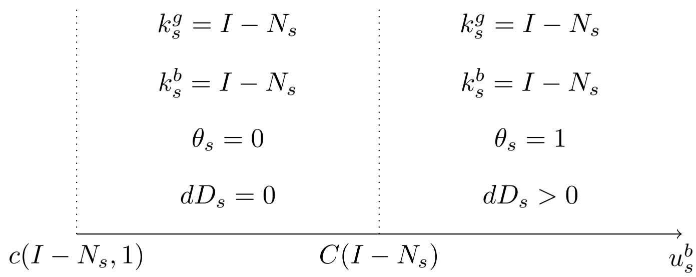


Figure 3.3: Optimal contract on the lower boundary.

3.5.3.2 Optimal contracts in the interior of the credible set

Figure 3.4 represents the optimal contracts on the boundaries of the credible set as well as the movements of the values of the banks along these curves. The green zone corresponds to the region where the contract offers payments to the Agents and the project is maintained if there is a default. The red zone corresponds to the region where there are no payments and the project is liquidated immediately after a default. Intermediate situations correspond to the yellow zone. We remark that the banks are paid only on the green zone.

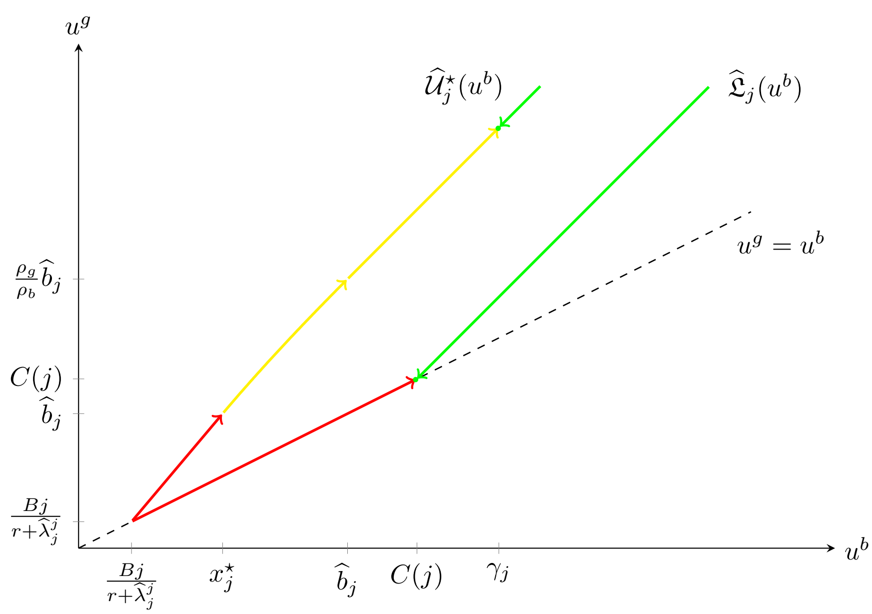


Figure 3.4: Optimal contract on the boundaries of the credible set.

Let us now consider the whole credible set and explain how we expect the green and red zones on the boundaries to propagate towards the interior region. If the verification theorems 3.5.1 and 3.5.2 hold, then the optimal contracts for problems (3.5.6) and (3.5.11) correspond to the maximisers in the Hamiltonian of the systems (3.5.7) and (3.5.12). Moreover, payments only take place when the value function of the investor saturates the gradient constraint. Therefore, it is natural to expect that if at some point of the credible set the banks are paid, this will also be the case under movements in the direction (ρ_b, ρ_g). The interpretation of this property is that the green region, where the banks are paid and the project is maintained after a default, is formed by the points where the banks have a good performance and they are rewarded. A movement in the direction (ρ_b, ρ_g) correspond to a better performance of both banks, so it seems unnatural to deprive them of the reward. We can do the opposite interpretation for the red region, consisting of the points where the banks receive no payments and the project is liquidated after a default. In consequence, we expect that under the optimal contracts, it will be possible to identify red and green areas in the credible set, where the characteristics described in the boundaries will remain, and that will be delimited by some curves similar to those shown in figure 3.5 below.

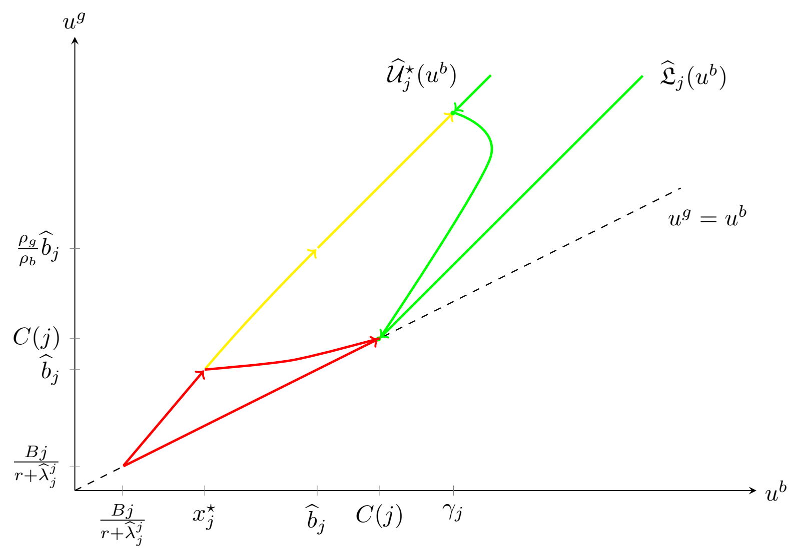


Figure 3.5: Optimal contract on the credible set.

3.6 Appendix

3.6.1 Proofs for the pure moral hazard case

We provide in this section all the proofs of the results of Section 3.3. We start with the

Proof. [Proof of Proposition 3.3.1] Using the martingale representation theorem[8] (recall that D is supposed to be integrable and that k is bounded by definition), we deduce that for

[8]We emphasise that since the filtration $\mathbb{G}$ is augmented and generated by inhomogeneous Poisson processes, the predictable martingale representation holds for any of the probability measures $(\mathbb{P}^k)_{k\in\mathfrak{K}}$.

any $k \in \mathfrak{K}$ there exist $\mathbb{G}-$predictable processes $h^{1,\mathrm{i},k}$ and $h^{2,\mathrm{i},k}$ such that

$$\mathrm{d}u_t^{\mathrm{i}}(k,\theta^{\mathrm{i}},D^{\mathrm{i}}) = \left(ru_t^{\mathrm{i}}(k,D^{\mathrm{i}},\theta^{\mathrm{i}}) - Bk_t\right)\mathrm{d}t - \rho_{\mathrm{i}}\mathrm{d}D_t^{\mathrm{i}} - h_t^{1,\mathrm{i},k}\left(\mathrm{d}N_t - \lambda_t^k\mathrm{d}t\right) \\ - h_t^{2,\mathrm{i},k}\left(\mathrm{d}H_t - (1-\theta_t^{\mathrm{i}})\lambda_t^k\mathrm{d}t\right),\ 0 \leq t < \tau,\ \mathbb{P} - a.s. \tag{3.6.1}$$

Let us then define

$$Y_t^{\mathrm{i},k} := u_t^{\mathrm{i}}(k,\theta^{\mathrm{i}},D^{\mathrm{i}}),\ Z_t^{\mathrm{i},k} := (h_t^{1,\mathrm{i},k},h_t^{2,\mathrm{i},k})^\top,\ M_t := (N_t,H_t)^\top, \\ \widetilde{M}_t^{\mathrm{i}} := M_t - \int_0^t \lambda_s^0(1,1-\theta_s^{\mathrm{i}})^\top\mathrm{d}s,\ K_t^{\mathrm{i}} := \rho_{\mathrm{i}}D_t^{\mathrm{i}},$$

so that we can rewrite (3.6.1) as follows

$$Y_t^{\mathrm{i},k} = 0 - \int_t^\tau f^{\mathrm{i}}(s,k_s,Y_s^{\mathrm{i},k},Z_s^{\mathrm{i},k})\mathrm{d}s + \int_t^\tau Z_s^{\mathrm{i},k}\cdot\mathrm{d}\widetilde{M}_s^{\mathrm{i}} + \int_t^\tau \mathrm{d}K_s^{\mathrm{i}},\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s.,$$

where

$$f^{\mathrm{i}}(t,k,y,z) := ry - Bk + k\alpha_{I-N_t}\varepsilon z\cdot(1,1-\theta_t^{\mathrm{i}})^\top.$$

In other words, $(Y^{\mathrm{i},k},Z^{\mathrm{i},k})$ appears as a (super–)solution to a BSDE with (finite) random terminal time, as studied for instance by Peng [89] or Darling and Pardoux [34]. Following then Hamadène and Lepeltier [50] and El Karoui and Quenez [40]. By direct computations, it is easy to see that g^{i} satisfies, for any $(t,y,y',z,z') \in \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$

$$\left|g^{\mathrm{i}}(t,y,z) - g^{\mathrm{i}}(t,y',z)\right| = r\left|y-y'\right|,$$

$$g^{\mathrm{i}}(t,y,z) - g^{\mathrm{i}}(t,y,z') \leq \sup_{0\leq k\leq I-N_t}\left\{k\alpha_{I-N_t}\varepsilon(z-z')\cdot(1,1-\theta_t^{\mathrm{i}})^\top\right\} = \gamma_t(z,z')\lambda_t^0(z-z')\cdot(1,1-\theta_t^{\mathrm{i}})^\top,$$

where $\gamma_t(z,z') := \varepsilon\mathbf{1}_{\{(z-z')\cdot(1,1-\theta_t^{\mathrm{i}})^\top>0\}}$, verifies $0 \leq \gamma_t(z,z') \leq \varepsilon$. In particular, this means that the generator g^{i} satisfies the classical sufficient condition, introduced by Royer [107, Condition (A_γ)], ensuring that a comparison theorem holds for the corresponding BSDE with jumps (see [107, Theorem 2.5]). Moreover, since the intensity of the Poisson process M under $\mathbb{P}$ is bounded, it is clear that τ has exponential moments of any order. Since in addition we have $g^{\mathrm{i}}(t,0,0) = -B(I-N_t)$, it is clear that the generator and the terminal condition of the BSDE (3.3.2) admit moments of any order and thus satisfy all the requirements ensuring wellposedness. Therefore, we deduce immediately that for any $k \in \mathfrak{K}$

$$Y_t^{\mathrm{i},k} \leq Y_t^{\mathrm{i}} = Y_t^{\mathrm{i},k^{\star,\mathrm{i}}},\ \mathbb{P} - a.s.,$$

where we defined

$$k_t^{\star,\mathrm{i}} := (I-N_t)\mathbf{1}_{\{Z_t^{\mathrm{i}}\cdot(1,1-\theta_t^{\mathrm{i}})^\top<b_t\}},\ \text{and } b_t := \frac{B}{\alpha_{I-N_t}\varepsilon},\ t \geq 0.$$

This means that Y^{i} is the value function of the bank, and that her optimal response given $(\theta^{\mathrm{i}},D^{\mathrm{i}}) \in \Theta\times\mathcal{D}$ is $k^{\star,\mathrm{i}}$. □

We continue with the

Proof. [Proof of Lemma 3.3.1] First of all, it is clear that the bank of type ρ_{i} can get arbitrarily large levels of utility (it suffices for the investor to set $\mathrm{d}D^{\mathrm{i}}_s := n\mathrm{d}s$ for n large enough, starting from time t). The bank's maximal level of utility is therefore $+\infty$, which corresponds to a utility equal to $-\infty$ for the investor. Then, coming back to the definition of the bank's problem, or to the BSDE (3.3.2), it is clear, for instance by using the comparison theorem for super solutions to (3.3.2) (see [107, Theorem 2.5]), that in order to minimise the utility that the bank obtains, the investor has to set $D^{\mathrm{i}} = 0$. Moreover, since by definition we must always have $Y^{\mathrm{i}}_t \geq 0$ and $Y^{\mathrm{i}}_\tau = 0$, and since the totally inaccessible jumps of Y (recall that D is assumed to be predictable) are given by $\Delta Y^{\mathrm{i}}_t = -Z^{\mathrm{i}}_t \cdot \Delta M_t$, we must have that

$$Y^{\mathrm{i}}_{t^-} = Z^{\mathrm{i}}_t \cdot (1,1)^\top, \text{ and } Y^{\mathrm{i}}_{t^-} \geq Z^{\mathrm{i}}_t \cdot (1,0)^\top, \ t > 0, \ \mathbb{P} - a.s., \tag{3.6.2}$$

Indeed, the support of the laws of τ and the τ^j under $\mathbb{P}$ is $[0, +\infty)$. This implies in particular that we must have $Z^{\mathrm{i}}_t \cdot (0,1)^\top \geq 0$, which in turn implies that the generator g^{i} is then non–increasing with respect to θ^{i}, and thus that the minimal utility for the bank is attained, as expected, when $\theta^{\mathrm{i}} = 0$. Then, if $(\theta^{\mathrm{i}}, D^{\mathrm{i}}) = (0,0)$ (which is obviously in $\Theta \times \mathcal{D}$) starting from time t, it is clear that the bank will never monitor and will obtain

$$\begin{aligned}
U^{\mathrm{i}}_t(0,0) = B(I-N_t)\mathbb{E}^{\mathbb{P}^{I-N_\cdot}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}\mathrm{d}s \middle| \mathcal{G}_t\right] &= \frac{B(I-N_t)}{r}\left(1 - \mathbb{E}^{\mathbb{P}^{I-N_\cdot}}\left[\mathrm{e}^{-r(\tau-t)}\middle| \mathcal{G}_t\right]\right) \\
&= \frac{B(I-N_t)}{r}\left(1 - \int_0^{+\infty} \lambda_t^{I-N_t}\mathrm{e}^{-x(r+\lambda_t^{I-N_t})}\mathrm{d}x\right) \\
&= \frac{B(I-N_t)}{r+\lambda_t^{I-N_t}}.
\end{aligned}$$

Notice that this corresponds to the investor getting

$$\mu(I-N_t)\mathbb{E}^{\mathbb{P}^{I-N_\cdot}}[\tau - t | \mathcal{G}_t] = \frac{\mu(I-N_t)}{\lambda_t^{I-N_t}}.$$

□

We finish with the

Proof. [Proof of Lemma 3.3.2] Let us show that for any $(\theta^{\mathrm{i}}, D^{\mathrm{i}}) \in \Theta \times \mathcal{D}$ enforcing $k = 0$ from time t, we have $U^{\mathrm{i}}_t(\theta^{\mathrm{i}}, D^{\mathrm{i}}) \geq b_t$. With such a contract, we must have

$$Z^{\mathrm{i}}_s \cdot (1, 1-\theta^{\mathrm{i}}_s)^\top \geq b_s, \ s \geq t.$$

By (3.6.2), this implies that for $s \geq t$, $Y^{\mathrm{i}}_{s^-} \geq b_s$, which, by right-continuity at time t leads to the desired result. Notice also that this result implies the so–called *limited liability* property of the bank, which reads

$$Y^{\mathrm{i}}_{t^-} - Z^{\mathrm{i}}_t \cdot (1,0)^\top \geq b_t.$$

Now, in order for the investor to ensure that $U^{\mathrm{i}}_t(\theta^{\mathrm{i}}, D^{\mathrm{i}}) = b_t$, it suffices for him, after time t, to offer the optimal contract derived in [85] (with initial condition b_t at time t), which we recall below (see Theorem 3.3.1). By [85, Proposition 3.16], the utility of the bank will then be b_t. □

3.6.2 Utility of not monitoring

In this section we compute the utilities that the banks get from always shirking (without considering the payments) under contracts which liquidates the pool after some fixed number of defaults. Observe first that we have

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathrm{e}^{-r(\tau_{N_t+1}-t)}\middle|\mathcal{G}_t\right] = \int_0^\infty \mathrm{e}^{-rx}\lambda^{SH}_{I-N_t}\mathrm{e}^{-\lambda^{SH}_{I-N_t}x}\,\mathrm{d}x = \frac{\lambda^{SH}_{I-N_t}}{r+\lambda^{SH}_{I-N_t}},$$

and for any $l \in \{N_t+1,\dots,I-1\}$

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathrm{e}^{-r(\tau_{l+1}-\tau_l)}\middle|\mathcal{G}_t\right] = \int_0^\infty \mathrm{e}^{-rx}\lambda^{SH}_{I-l}\mathrm{e}^{-\lambda^{SH}_{I-l}x}\,\mathrm{d}x = \frac{\lambda^{SH}_{I-l}}{r+\lambda^{SH}_{I-l}}.$$

For $m \in \{2,\dots,I-N_t\}$, consider $\theta \in \Theta$ given by

$$\theta_s = \begin{cases} 1, & t \le s \le \tau_{N_t+m}, \\ 0, & s > \tau_{N_t+m}. \end{cases}$$

It means that the pool will be liquidated exactly after the following m defaults, so that the utility that the bank gets from shirking is

$$\begin{aligned}
u_t(k^{SH},\theta,0) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}B(I-N_s)\,\mathrm{d}s\middle|\mathcal{G}_t\right] \\
&= \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau_{N_t+1}} \mathrm{e}^{-r(s-t)}B(I-N_t)\,\mathrm{d}s + \sum_{\mathrm{i}=N_t+1}^{N_t+m-1}\int_{\tau_\mathrm{i}}^{\tau_{\mathrm{i}+1}} \mathrm{e}^{-r(s-t)}B(I-\mathrm{i})\,\mathrm{d}s\middle|\mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[1-\mathrm{e}^{-r(\tau_{N_t+1}-t)}\middle|\mathcal{G}_t\right] \\
&\quad + \sum_{\mathrm{i}=N_t+1}^{N_t+m-1}\frac{B(I-\mathrm{i})}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathrm{e}^{-r(\tau_\mathrm{i}-t)} - \mathrm{e}^{-r(\tau_{\mathrm{i}+1}-t)}\middle|\mathcal{G}_t\right] \\
&= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{\mathrm{i}=N_t+1}^{N_t+m-1}\frac{B(I-\mathrm{i})}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\left(1-\mathrm{e}^{-r(\tau_{\mathrm{i}+1}-\tau_\mathrm{i})}\right)\prod_{l=N_t}^{\mathrm{i}-1}\mathrm{e}^{-r(\tau_{l+1}-\tau_l)}\middle|\mathcal{G}_t\right].
\end{aligned}$$

Therefore, by independence we have

$$\begin{aligned}
u_t(k^{SH},\theta,0) &= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{\mathrm{i}=N_t+1}^{N_t+m-1}\frac{B(I-\mathrm{i})}{r+\lambda^{SH}_{I-\mathrm{i}}}\prod_{l=N_t}^{\mathrm{i}-1}\frac{\lambda^{SH}_{I-l}}{r+\lambda^{SH}_{I-l}} \\
&= \frac{B(I-N_t)}{r+\lambda^{SH}_{I-N_t}} + \sum_{\mathrm{i}=I-N_t-m+1}^{I-N_t-1}\frac{B\mathrm{i}}{r+\lambda^{SH}_\mathrm{i}}\prod_{l=\mathrm{i}+1}^{I-N_t}\frac{\lambda^{SH}_l}{r+\lambda^{SH}_l}.
\end{aligned}$$

3.6.3 Short–term contracts with constant payment

In this section we analyse the optimal responses and the value functions of the banks at a starting time $t \ge 0$, under contracts with constant payments of the form $\mathrm{d}D_s = c\mathrm{d}s$, where c is any $\mathcal{G}_t$-measurable random variable, and with $\theta \equiv 0$, so that the pool is liquidated immediately after the first default.

3.6.3.1 Optimal responses and feasible set

In this section we compute the optimal responses of the Agents to the described contracts, depending on the value of c. We also show that for this class of contracts the set of expected payoff of the Agents, starting of time t, is exactly $\mathcal{V}_t = \Big[B(I-N_t)/(r+\lambda_t^{k^{SH}}), \infty\Big)$.

(i) Let $k^0 := 0$. If the bank of type $\rho_{\rm i}$ always monitors, we have

$$u_t^{\rm i}(k^0,\theta,D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}\rho_{\rm i}c\mathrm{d}s\middle|\mathcal{G}_t\right] = \frac{\rho_{\rm i}c}{r+\lambda_t^{k^0}}.$$

Hence, the continuation utility is constant in time and if the payment c is exactly equal to $u^{\rm i}(r+\lambda_t^{k^0})/\rho_{\rm i}$, then the bank receives exactly $u^{\rm i}$. In this case, k^0 is incentive compatible if and only if $u^{\rm i} \geq b_{I-N_t}$. The minimum payment such that the bank of type $\rho_{\rm i}$ will always work is therefore

$$\underline{c_{\rm i}} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_{\rm i}}.$$

(ii) If the bank of type $\rho_{\rm i}$ always shirks, her continuation utility is constant and equal to

$$u_t^{\rm i}(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_{\rm i}c+B)\mathrm{d}s\middle|\mathcal{G}_t\right] = \frac{\rho_{\rm i}c+B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

Then, if one takes c equal

$$\frac{u^{\rm i}(r+\lambda_t^{k^{SH}})-B(I-N_t)}{\rho_{\rm i}},$$

the bank receives $u^{\rm i}$. Therefore k^{SH} is incentive compatible if and only if $u^{\rm i} < b_{I-N_t}$. Nevertheless, since the payment c must be positive, $u^{\rm i}$ must be greater than $B(I-N_t)/(r+\lambda_t^{k^{SH}})$. The supremum of the payments such that the bank of type $\rho_{\rm i}$ will always shirk is therefore equal to

$$\overline{c_{\rm i}} = \frac{b_{I-N_t}(r+\lambda_t^{k^{SH}})-B(I-N_t)}{\rho_{\rm i}} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_{\rm i}} = \underline{c_{\rm i}}.$$

Therefore the set of expected payoff under this class of contracts is $\mathcal{V}_t$. Let us summarise our findings.

Response of the bank of type $\rho_{\rm i}$ to the contract $\theta\equiv 0,\ \mathrm{d}D_s = c\mathrm{d}s$, after time t:

With $\overline{c_{\rm i}} = \dfrac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_{\rm i}}$,

- if $c \leq \overline{c_{\rm i}} \implies k^{\star,{\rm i}}(\theta,D) = k^{SH},\ U_t^{\rm i}(\theta,D) = \dfrac{\rho_{\rm i}c+B(I-N_t)}{r+\lambda_t^{k^{SH}}}$.
- if $c \geq \overline{c_{\rm i}} \implies k^{\star,{\rm i}}(\theta,D) = k^0,\ U_t^{\rm i}(\theta,D) = \dfrac{\rho_{\rm i}c}{r+\lambda_t^{k^0}}$.

3.6.3.2 Credible region under short–term contracts with constant payments

Once we know the optimal responses of the good and the bad bank for every payment c, we can study the relationship between their value functions for any short–term contract with constant payments.

(i) Suppose $c \in [0, \overline{c_g})$. Since $\overline{c_g} < \overline{c_b}$, we have that $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^{SH}$ and

$$U_t^g(\theta, D) = \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}} + \frac{\rho_g c}{r + \lambda_t^{k^{SH}}},\ U_t^b(\theta, D) = \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}} + \frac{\rho_b c}{r + \lambda_t^{k^{SH}}}.$$

Thus, the value functions verify the following equation

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D) + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\left(1 - \frac{\rho_g}{\rho_b}\right),$$

as well as

$$U_t^g(\theta, D) \in \left[\frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}, b_{I-N_t}\right),\ U_t^b(\theta, D) \in \left[\frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}, \frac{\rho_b}{\rho_g} b_{I-N_t} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\left(1 - \frac{\rho_b}{\rho_g}\right)\right).$$

(ii) If $c \in [\overline{c_g}, \overline{c_b})$, then $k^{\star,g}(\theta, D) = k^0$, $k^{\star,b}(\theta, D) = k^{SH}$ and the value functions of the banks are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}},\ U_t^b(\theta, D) = \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}} + \frac{\rho_b c}{r + \lambda_t^{k^{SH}}}.$$

Hence, they verify

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b}\left(\frac{r + \lambda_t^{k^{SH}}}{r + \lambda_t^{k^0}}\right) U_t^b(\theta, D) - \frac{\rho_g}{\rho_b}\frac{B(I - N_t)}{r + \lambda_t^{k^0}},$$

with

$$U_t^g(\theta, D) \in \left[b_{I-N_t}, \frac{\rho_g}{\rho_b} b_{I-N_t}\right),\ U_t^b(\theta, D) \in \left[\frac{\rho_b}{\rho_g} b_{I-N_t} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\left(1 - \frac{\rho_b}{\rho_g}\right), b_{I-N_t}\right).$$

(iii) Finally, if $c \in [\overline{c_b}, \infty)$ then $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^0$ and

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}},\ U_t^b(\theta, D) = \frac{\rho_b c}{r + \lambda_t^{k^{SH}}}.$$

Hence

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D),$$

with

$$U_t^g(\theta, D) \in \left[\frac{\rho_g}{\rho_b} b_{I-N_t}, \infty\right),\ U_t^b(\theta, D) \in [b_{I-N_t}, \infty).$$

Figure 3.6 shows the pair of values of the banks that can be obtained using contracts with constant payments. For simplicity, u^g denotes the value function of the good bank and u^b that of the bad bank, and $j := I - N_t$. Depending on the payments, the values of the banks belong to one of the three lines represented, the last one being unbounded.

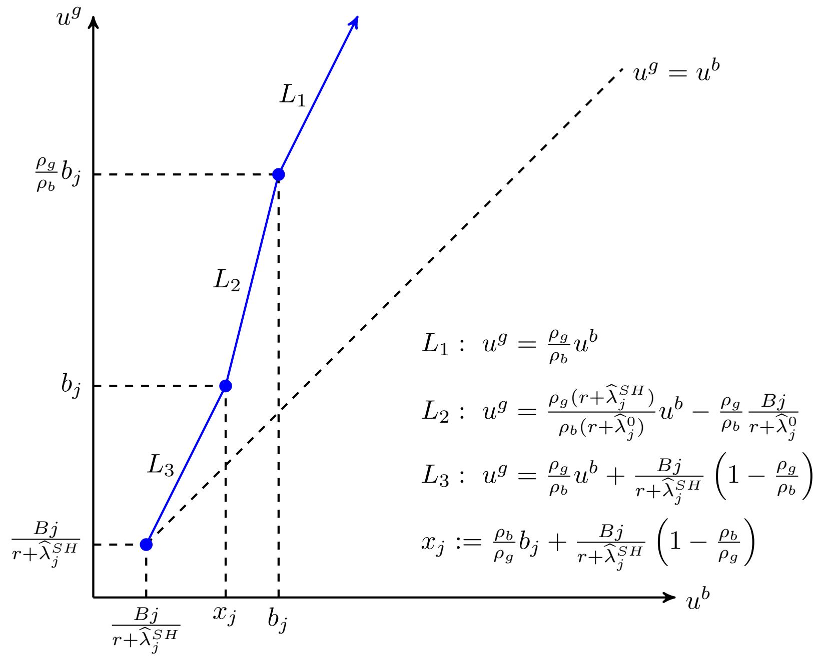


Figure 3.6: Credible region under short-term contracts with constant payments.

3.6.3.3 Initial lump–sum payment

Take any point $(u^b, u^g) \in L_1 \cup L_2 \cup L_3$. We know that there exists a contract $\theta \equiv 0$, $\mathrm{d}D_s = c\mathrm{d}s$, starting from time t, such that $U_t^b(\theta, D) = u^b$ and $U_t^g(\theta, D) = u^g$. Consider the payments D^ℓ which differ from D only at time t, where a lump-sum payment of size $\ell > 0$ is made. This added lump-sum payment will not change the banks' incentives and the new value functions at time t will be

$$U_t^g(\theta, D^\ell) = u^g + \rho_g \ell, \ U_t^b(\theta, D^\ell) = u^b + \rho_b \ell.$$

Hence, the new pair of values of the banks belong to the line with slope $\frac{\rho_g}{\rho_b}$ which passes through the point (u^b, u^g). Since in our setting there is no upper bound on the payment, by increasing the value of ℓ it is possible to reach every point of the ray which starts at (u^b, u^g) and goes in the positive direction. The subregion of the credible set that can be obtained by short-term contracts with constant payments and initial lump–sum payments is shown in Figure 3.7, with the same conventions as in Figure 3.6.

3.6.4 Short-term contracts with delay

In this section we study the optimal responses of the banks and their value functions at a starting time $t \geq 0$, under contracts with constant payment after a certain time $t^\star > t$, and $\theta \equiv 0$. The case $t^\star = t$ corresponds to the situation of Appendix 3.6.3.

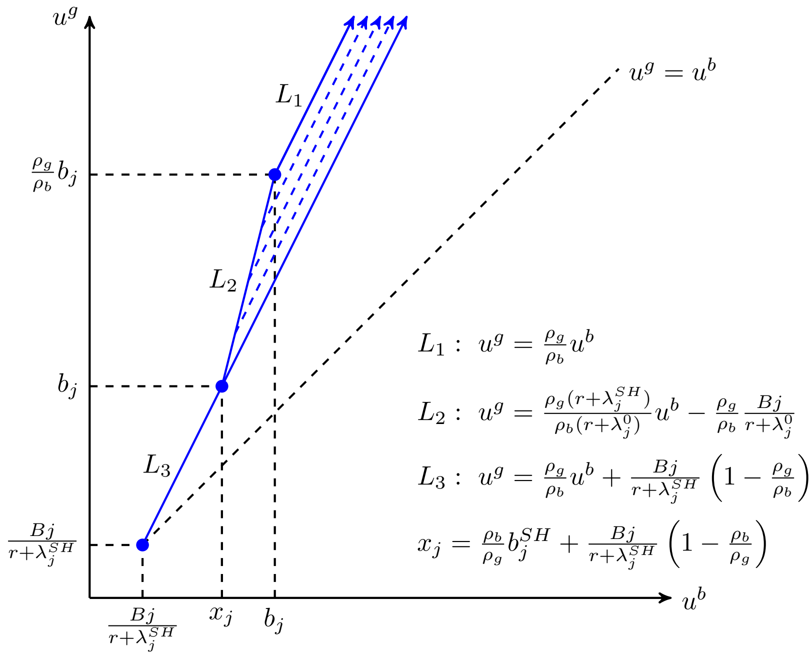


Figure 3.7: Credible region under short-term contracts with constant payment and lump-sum payments.

3.6.4.1 Optimal responses and feasible set

In this section we compute the optimal responses of the Agents to the described contracts, depending on the values of c and $t^\star$. We also show that under this class of contracts the set of expected payoff of the Agents, starting at time t, is exactly $\mathcal{V}_t = \Big[B(I - N_t)/(r + \lambda_t^{k^{SH}}), \infty\Big)$.

(i) If the bank of type $\rho_{\rm i}$ always works, at any time $t \leq s < t^\star$, her continuation utility is, noticing that since $\theta = 0$, we have that $(\lambda_u^{k^0})_{u \geq t}$ is constant,

$$u_s^{\rm i}(k^0, \theta, D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_{t^\star \wedge \tau}^{\tau} {\rm e}^{-r(u-s)} \rho_{\rm i} c {\rm d}u \middle| \mathcal{G}_s\right] = \frac{{\rm e}^{-(r+\lambda_t^{k^0})(t^\star - s)} \rho_{\rm i} c}{r + \lambda_t^{k^0}} = u_t^{\rm i}(k^0, \theta, D) {\rm e}^{(r+\lambda_t^{k^0})(s-t)}.$$

Therefore, at $s = t^\star$ the continuation utility of the bank is $u_{t^\star}^{\rm i}(k^0, \theta, D) = u_t^{\rm i}(k^0, \theta, D) {\rm e}^{(r+\lambda_t^{k^0})(t^\star - t)}$. Next, for any $s > t^\star$, the continuation utility of the bank will be

$$u_s^{\rm i}(k^0, \theta, D) = \mathbb{E}^{\mathbb{P}^0}\left[\int_s^{\tau} {\rm e}^{-r(u-s)} \rho_{\rm i} c {\rm d}s \middle| \mathcal{G}_s\right] = \frac{\rho_{\rm i} c}{r + \lambda_t^{k^0}}.$$

Then, we see that once the bank starts being paid, her continuation utility becomes constant and it must be equal to $u_{t^\star}^{\rm i}(k^0, \theta, D)$. Then, if for some $u^{\rm i} \geq 0$, one chooses c equal to

$$\frac{u^{\rm i} {\rm e}^{(r+\lambda_t^{k^0}) t^\star} (r + \lambda_t^{k^0})}{\rho_{\rm i}}, \tag{3.6.3}$$

the continuation utility of the bank will be an increasing process with initial value $u^{\rm i}$. Therefore, k^0 is incentive compatible if and only if $u^{\rm i} \geq b_{I-N_t}$. The minimum payment and delay such that the bank always works are $t^\star = 0$ and

$$\underline{c_\mathrm{i}} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_\mathrm{i}}.$$

(ii) If the bank of type ρ_i always shirks, at any time $t \le s < t^\star$, her continuation utility is

$$u_s^\mathrm{i}(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{SH}}\left[\int_{t^\star\wedge\tau}^{\tau} \mathrm{e}^{-r(u-s)}\rho_\mathrm{i} c \mathrm{d}u + \int_s^\tau B\mathrm{d}u \middle| \mathcal{G}_s\right] = \frac{\mathrm{e}^{-(r+\lambda_t^{k^{SH}})(t^\star - s)}\rho_\mathrm{i} c}{r+\lambda_t^{k^{SH}}} + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

Therefore

$$u_s^\mathrm{i}(k^{SH},\theta,D) = \mathrm{e}^{(r+\lambda_t^{k^{SH}})(s-t)}\left(u_t^\mathrm{i}(k^{SH},\theta,D) - \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\right) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}},$$

and the continuation utility is an increasing process. Recall that k^{SH} is incentive compatible if and only if $u_s^\mathrm{i}(k^{SH},\theta,D) < b_{I-N_t}$ for every $s \ge t$. However, if $t^\star$ is large, there will exist t_w such that $u_{t_w}^\mathrm{i}(k^{SH},\theta,D) = b_{I-N_t}$ and the bank will start to work. More precisely, t_w depends on the initial value $u_t^\mathrm{i}(k^{SH},\theta,D)$ and is given by

$$t_w = t + \frac{1}{r+\lambda_t^{k^{SH}}}\log\left(\frac{b_{I-N_t}(r+\lambda_t^{k^{SH}}) - B(I-N_t)}{u_t^\mathrm{i}(k^{SH},\theta,D)(r+\lambda_t^{k^{SH}}) - B(I-N_t)}\right).$$

Notice that $t_w \ge t$ if and only if $b_{I-N_t} \ge u_t^\mathrm{i}(k^{SH},\theta,D)$. Therefore, k^{SH} is incentive compatible if and only if $t^\star < t_w$. Under this condition, at $t = t^\star$ the continuation utility of the bank is

$$u_{t^\star}^\mathrm{i}(k^{SH},\theta,D) = \mathrm{e}^{(r+\lambda_t^{k^{SH}})(t^\star-t)}\left(u_t^\mathrm{i}(k^{SH},\theta,D) - \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\right) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}} < b_{I-N_t}.$$

Once the bank starts being paid her continuation utility is constant and equal to

$$u_s^\mathrm{i}(k^{SH},\theta,D) = \mathbb{E}^{\mathbb{P}^{SH}}\left[\int_s^\tau \mathrm{e}^{-r(u-s)}(\rho_\mathrm{i} c + B(I-N_t))\mathrm{d}s\right] = \frac{\rho_\mathrm{i} c + B(I-N_t)}{r+\lambda_t^{k^{SH}}}.$$

So if the payment c is equal to

$$\frac{\mathrm{e}^{(r+\lambda_j^{SH})(t^\star-t)}\left(u^\mathrm{i}(r+\lambda_t^{k^{SH}}) - B(I-N_t)\right)}{\rho_\mathrm{i}}, \tag{3.6.4}$$

the expected payoff of the bank at time t is u^i. The supremum of the delays and payments such that the bank always shirks are t_w and

$$\overline{c_\mathrm{i}} = \frac{\mathrm{e}^{(r+\lambda_t^{k^{SH}})(t_w-t)}\left[b_{I-N_t}(r+\lambda_t^{k^{SH}}) - B(I-N_t)\right]}{\rho_\mathrm{i}} = \frac{b_{I-N_t}(r+\lambda_t^{k^0})}{\rho_\mathrm{i}} = \underline{c_\mathrm{i}}.$$

(iii) Finally, consider the case when $t^\star$ is greater than t_w. Under this contract, the bank will shirk until time t_w and will work afterwards. Indeed, from the previous analysis we know that this strategy is incentive compatible. At time t_w we have that $u^{\mathrm{i}}_{t_w}(k^{SH},\theta,D)=b_{I-N_t}$ and for $s\in[t_w,t^\star)$ the continuation utility is given by

$$u^{\mathrm{i}}_s(k^0,\theta,D)=\mathbb{E}^{\mathbb{P}^0}\left[\int_{t^\star\wedge\tau}^{\tau}\mathrm{e}^{-r(u-s)}\rho_{\mathrm{i}}c\mathrm{d}u\middle|\mathcal{G}_s\right]=\frac{\mathrm{e}^{-(r+\lambda^{k^0}_t)(t^\star-s)}\rho_{\mathrm{i}}c}{r+\lambda^{k^0}_t}$$
$$=\mathrm{e}^{(r+\lambda^{k^0}_t)(s-t_w)}u^{\mathrm{i}}_{t_w}(k^{SH},\theta,D)=b_{I-N_t}\mathrm{e}^{(r+\lambda^{k^0}_t)(s-t_w)}.$$

Therefore, at $t=t^\star$ the continuation utility of the bank is

$$u^{\mathrm{i}}_{t^\star}(k^0,\theta,D)=b_{I-N_t}\mathrm{e}^{(r+\lambda^{k^0}_t)(t^\star-t_w)},$$

and for any $s>t^\star$, the continuation utility of the bank is constant and equal to

$$u^{\mathrm{i}}_s(k^0,\theta,D)=\mathbb{E}^{\mathbb{P}^0}\left[\int_{s}^{\tau}\mathrm{e}^{-r(u-s)}\rho_{\mathrm{i}}c\mathrm{d}u\middle|\mathcal{G}_s\right]=\frac{\rho_{\mathrm{i}}c}{r+\lambda^{k^0}_t}.$$

So if the payment c is equal to

$$\frac{b_{I-N_t}(r+\lambda^{k^0}_t)\mathrm{e}^{(r+\lambda^{k^0}_t)(t^\star-t)}}{\rho_{\mathrm{i}}}\left(\frac{u^{\mathrm{i}}(r+\lambda^{k^{SH}}_t)-B(I-N_t)}{b_{I-N_t}(r+\lambda^{k^0}_t)}\right)^{\frac{r+\lambda^{k^0}_t}{r+\lambda^{k^{SH}}_t}}, \tag{3.6.5}$$

the expected payoff of the bank at time t is u^{i}. The minimum payment and delay such that the bank shirks first and works afterwards are $t^\star=t_w$ and

$$\underline{c_{\mathrm{i}}}=\frac{b_{I-N_t}(r+\lambda^{k^0}_t)}{\rho_{\mathrm{i}}}=\overline{c_{\mathrm{i}}}.$$

The following box summarizes our findings in this case. Here, $\overline{t_{\mathrm{i}}}(c)$ is the corresponding expression for t_w as a function of the payments c.

Response of the bank of type ρ_{i} to the contract $\theta\equiv 0$, $\mathrm{d}D_s=1_{\{s\geq t^\star\}}c\mathrm{d}s$ after t:

Let $\overline{c_{\mathrm{i}}}=\dfrac{b_{I-N_t}(r+\lambda^{k^0}_t)}{\rho_{\mathrm{i}}}$, $\overline{t_{\mathrm{i}}}(c):=t+\dfrac{1}{r+\lambda^{k^0}_t}\log\left(\dfrac{\rho_{\mathrm{i}}c}{b_{I-N_t}(r+\lambda^{k^0}_t)}\right)$.

- If $c\leq\overline{c_{\mathrm{i}}}\implies k^{\star,\mathrm{i}}(\theta,D)=k^{SH}$, $U^{\mathrm{i}}_t(\theta,D)=\mathrm{e}^{-(r+\lambda^{k^{SH}}_t)(t^\star-t)}\dfrac{\rho_{\mathrm{i}}c}{r+\lambda^{k^{SH}}_t}+\dfrac{B(I-N_t)}{r+\lambda^{k^{SH}}_t}$.
- If $c>\overline{c_{\mathrm{i}}}$, $t^\star\leq\overline{t_{\mathrm{i}}}(c)\implies k^{\star,\mathrm{i}}(\theta,D)=k^0$, $U^{\mathrm{i}}_t(\theta,D)=\mathrm{e}^{-(r+\lambda^{k^0}_t)(t^\star-t)}\dfrac{\rho_{\mathrm{i}}c}{r+\lambda^{k^0}_t}$.
- If $c>\overline{c_{\mathrm{i}}}$, $t^\star>\overline{t_{\mathrm{i}}}(c)\implies k^{\star,\mathrm{i}}_s(\theta,D)=k^{SH}_s\mathbf{1}_{\{s<\overline{t_j}(c)\}}+k^0_s\mathbf{1}_{\{s\geq\overline{t_j}(c)\}}$ and

$$U^{\mathrm{i}}_t(\theta,D)=\mathrm{e}^{-(r+\lambda^{k^{SH}}_t)(t^\star-t)}\left[\frac{\rho_{\mathrm{i}}c}{b_{I-N_t}(r+\lambda^{k^0}_t)}\right]^{\frac{r+\lambda^{k^{SH}}_t}{r+\lambda^{k^0}_t}}\frac{b_{I-N_t}(r+\lambda^{k^0}_t)}{r+\lambda^{k^{SH}}_t}+\frac{B(I-N_t)}{r+\lambda^{k^{SH}}_t}.$$

3.6.4.2 The upper boundary can be reached with contracts with delay

In this section we show that in some cases the short-term contracts with delay provide to the Agents a pair of value functions lying in the upper boundary of the credible set.

(i) Let $c > \overline{c_b} > \overline{c_g}$ and $t^\star \leq \overline{t_b}(c) < \overline{t_g}(c)$. Then $k^{\star,b}(\theta, D) = k^{\star,g}(\theta, D) = k^0$ and the values of the banks are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}} \mathrm{e}^{-(r+\lambda_t^{k^0})(t^\star - t)}, \; U_t^b(\theta, D) = \frac{\rho_b c}{r + \lambda_t^{k^0}} \mathrm{e}^{-(r+\lambda_t^{k^0})(t^\star - t)}.$$

Therefore the utilities satisfy

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} U_t^b(\theta, D), \text{ with } U_t^g(\theta, D) \in \left[\frac{\rho_g}{\rho_b} b_{I-N_t}, \infty\right), \; U_t^b(\theta, D) \in [b_{I-N_t}, \infty).$$

(ii) If $c > \overline{c_b}$ and $\overline{t_b}(c) < t^\star \leq \overline{t_g}(c)$, we have that the good bank will always work and the bad bank will start to work at time $\overline{t_b(c)}$. Their value functions are

$$U_t^g(\theta, D) = \frac{\rho_g c}{r + \lambda_t^{k^0}} \mathrm{e}^{-(r+\lambda_t^{k^0})(t^\star - t)},$$

$$U_t^b(\theta, D) = \mathrm{e}^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_b c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

so they belong to the curve

$$U_t^g(\theta, D) = \frac{\rho_g}{\rho_b} b_{I-N_t}^{\frac{\lambda_t^{k^{SH}} - \lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}} \left(U_t^b(\theta, D) - \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}}\right)^{\frac{r+\lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}} \left(\frac{r + \lambda_t^{k^{SH}}}{r + \lambda_t^{k^0}}\right)^{\frac{r+\lambda_t^{k^0}}{r+\lambda_t^{k^{SH}}}},$$

and take values in the sets (recall the definition of $x_j^\star$ in proposition 3.4.2)

$$U_t^g(\theta, D) \in \left[b_{I-N_t}, \frac{\rho_g}{\rho_b} b_{I-N_t}\right), \; U_t^b(\theta, D) \in [x_{I-N_t}^\star, b_{I-N_t}).$$

(iii) If $c > \overline{c_b}$ and $\overline{t_g}(c) < t^\star$, the good bank will start to work at time $\overline{t_g(c)}$ and the bad bank will start to work at time $\overline{t_b(c)}$. Their value functions are

$$U_t^g(\theta, D) = \mathrm{e}^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_g c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

$$U_t^b(\theta, D) = \mathrm{e}^{-(r+\lambda_t^{k^{SH}})(t^\star - t)} \left[\frac{\rho_b c}{b_{I-N_t}(r + \lambda_t^{k^0})}\right]^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \frac{b_{I-N_t}(r + \lambda_t^{k^0})}{r + \lambda_t^{k^{SH}}} + \frac{B(I - N_t)}{r + \lambda_t^{k^{SH}}},$$

so they belong to the line

$$U_t^g(\theta,D) = \left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\lambda_t^{k^{SH}}}{r+\lambda_t^{k^0}}} \left(U_t^b(\theta,D) - \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}\right) + \frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}},$$

with

$$U_t^g(\theta,D) \in \left[\frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}, b_{I-N_t}\right),\ U_t^b(\theta,D) \in \left[\frac{B(I-N_t)}{r+\lambda_t^{k^{SH}}}, x^\star_{I-N_t}\right).$$

3.6.4.3 Credible region under contracts with delay

From the previous subsection we know that for every point (u^b, u^g) on the upper boundary there exists a pair $(c, t^\star)$, with $c > \overline{c_b}$, such that under the contract $(\theta \equiv 0, \mathrm{d}D_s = c\mathbf{1}_{\{s \geq t^\star\}}\mathrm{d}s)$ we have $U_t^b(\theta,D) = u^b$ and $U_t^g(\theta,D) = u^g$. As explained in 3.6.3.3, if we consider the contract (θ, D^ℓ) with an additional initial lump-sum payment, the incentives of the banks will not change and the new value functions of the Agents will be $U_t^b(\theta,D^\ell) = u^b + \rho_b\ell$, $U_t^g(\theta,D) = u^g + \rho_g\ell$. Therefore under short-term contracts with delay which reach the upper boundary and lump-sum payments, all the subregion of the credible set delimited by the lines shown in Figure 3.8 can be reached. We will not enter into details but it can be proved that under all the short-term contracts with delay (not only the ones who reach the upper boundary) and lump-sum payments, the subregion of the credible set which can be reached is exactly the same. When there is only one loan left, this region is equal to the whole credible set but when $j > 1$ the credible set is strictly bigger due to the pair of utilities that can be achieved in situations when $\theta \not\equiv 0$.

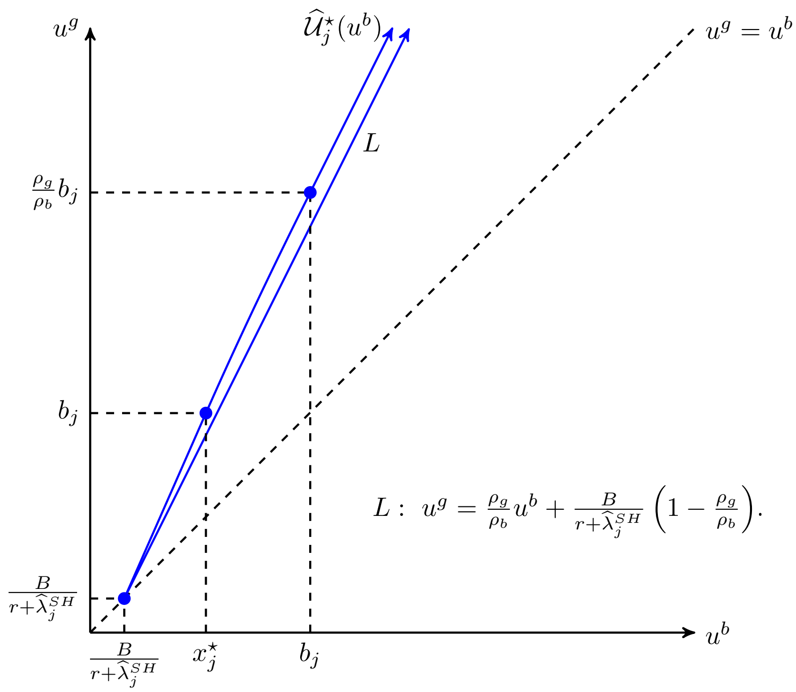


Figure 3.8: Credible region under short-term contracts with delay and lump-sum payment.

3.6.5 Technical results for the lower boundary

We begin this section with the

Proof. [Proof of Lemma 3.4.1] The value functions of the banks under $\Psi := (\theta, D)$ are given by

$$U_t^g(\Psi) = \mathbb{E}^{\mathbb{P}^{k^{\star,g}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_g \mathrm{d}D_s + Bk_s^{\star,g}(\Psi)\mathrm{d}s)\bigg|\,\mathcal{G}_t\right],$$
$$U_t^b(\Psi) = \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_b \mathrm{d}D_s + Bk_s^{\star,b}(\Psi)\mathrm{d}s)\bigg|\,\mathcal{G}_t\right].$$

Thus, we first have, $\mathbb{P}-a.s.$

$$\begin{aligned}U_t^g(\Psi) &\geq \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_g \mathrm{d}D_s + Bk_s^{\star,b}(\Psi)\mathrm{d}s)\bigg|\,\mathcal{G}_t\right]\\ &\geq \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_b \mathrm{d}D_s^g + Bk_s^{\star,b}(\Psi)\mathrm{d}s)\bigg|\,\mathcal{G}_t\right] = U_t^b(\Psi).\end{aligned}$$

But we also have

$$\begin{aligned}U_t^g(\Psi) &\geq \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}(\rho_g \mathrm{d}D_s + Bk_s^{\star,b}(\Psi)\mathrm{d}s)\bigg|\,\mathcal{G}_t\right]\\ &= U_t^b(\Psi) + (\rho_g-\rho_b)\mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}\mathrm{d}D_s\bigg|\,\mathcal{G}_t\right]\\ &= U_t^b(\Psi) + \frac{(\rho_g-\rho_b)}{\rho_b}\left(U_t^b(\Psi) - \mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}Bk_s^{\star,b}(\Psi)\mathrm{d}s\bigg|\,\mathcal{G}_t\right]\right)\\ &= \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}\mathbb{E}^{\mathbb{P}^{k^{\star,b}(\Psi)}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}Bk_s^{\star,b}(\Psi)\mathrm{d}s\bigg|\,\mathcal{G}_t\right].\end{aligned}$$

Observe next that

$$\sup_{k\in\mathfrak{K}}\mathbb{E}^{\mathbb{P}^k}\left[\int_t^\tau \mathrm{e}^{-r(t-s)}Bk_s\mathrm{d}s\bigg|\,\mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mathrm{e}^{-r(t-s)}Bk_s^{SH}\mathrm{d}s\bigg|\,\mathcal{G}_t\right],$$

because the left–hand side is the value function of a bank who is offered a contract with no payments. Therefore, we have that

$$U_t^g(\Psi) \geq \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}Bk_s^{SH}\mathrm{d}s\bigg|\,\mathcal{G}_t\right] \geq \frac{\rho_g}{\rho_b}U_t^b(\Psi) - \frac{(\rho_g-\rho_b)}{\rho_b}C(I-N_t),$$

because the utility that the banks get from shirking is non–decreasing with respect to the process θ and its maximum value is equal to $C(I-N_t)$, attained when $\theta\equiv 1$ (see (3.4.2)).
□

We continue this section with the

Proof. [Proof of Proposition 3.4.1] Due to Lemma 3.4.1, it suffices to prove the existence of contracts under which the value functions of the banks satisfy the equalities.

- **Step 1:** First, fix some $t \geq 0$, take any $u^b \in [c(I - N_t, 1), C(I - N_t)]$ and fix $m \in \{1, \ldots, I - N_t - 1\}$ such that $c(I - N_t, m) \leq u^b \leq c(I - N_t, m + 1)$. Next, take $\theta_t^0(u^b) \in [0, 1]$ such that

$$u^b = c(I - N_t, m) + \theta_t^0(u^b) \left(c(I - N_t, m + 1) - c(I - N_t, m) \right).$$

Then, there is a contract $(\theta, D) \in \Theta \times \mathcal{D}$ such that $U_t^g(\theta, D) = U_t^b(\theta, D) = u^b$. Such a contract can be defined as follows

$$\mathrm{d}D_s := 0, \ \theta_s := \mathbf{1}_{\left\{t \leq s \leq \tau_{N_t+m}\right\}} + (1 - \theta_t^0(u^b))\mathbf{1}_{\left\{\tau_{N_t+m} < s \leq \tau_{N_t+m+1}\right\}}, \quad \text{for every } s \geq t.$$

The contract has no payments, it always maintains the pool after the first m defaults, maintains the pool with probability θ_0 after default $m + 1$, and liquidates the pool at default $m + 2$. It is clear that under this contract both banks always shirk in $[t, \tau]$, since they are not paid, and their value functions satisfy

$$\begin{aligned} U_t^g(\theta, D) = U_t^b(\theta, D) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}} \left[\int_t^\tau \mathrm{e}^{-r(s-t)} B k_s^{SH} \mathrm{d}s \bigg| \mathcal{G}_t \right] \\ &= c(I - N_t, m) + \theta_t^0(u^b) \left(c(I - N_t, m + 1) - c(I - N_t, m) \right) = u^b. \end{aligned}$$

- **Step 2:** Fix again some $t \geq 0$, and choose now any $u^b \geq C(I - N_t)$ and define

$$u^g := \frac{\rho_g}{\rho_b} u^b - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t).$$

Let $\ell_t := (u^b - C(I - N_t))/\rho_b$ and consider the admissible contract satisfying, $\theta_s = 1, \ \mathrm{d}D_s = \ell_t \mathbf{1}_{\{s=t\}}$, for every $s \geq t$. The optimal strategy for both banks under this contract is to always shirk and then

$$\begin{aligned} U_t^b(\theta, D) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}} \left[\int_t^\tau \mathrm{e}^{-r(s-t)} (\rho_b \mathrm{d}D_s + B k_s^{SH} \mathrm{d}s) \bigg| \mathcal{G}_t \right] = \rho_b \ell_t + C(I - N_t) = u^b, \\ U_t^g(\theta, D) &= \mathbb{E}^{\mathbb{P}^{k^{SH}}} \left[\int_t^\tau \mathrm{e}^{-r(s-t)} (\rho_g \mathrm{d}D_s + B k_s^{SH} \mathrm{d}s) \bigg| \mathcal{G}_t \right] = \rho_g \ell_t + C(I - N_t) = u^g. \end{aligned}$$

□

We conclude this section by proving some useful results that will be used in Section 3.5.1.1 in the study of the value function of the investor on the lower boundary. We show that there are several ways of reaching the lower boundary and that all the contracts which can achieve it have the same structure as the ones used in the proof of Proposition 3.4.1.

Lemma 3.6.1 Consider any $(t, u^b, u^g) \in [0, \tau] \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that in addition $u^b = u^g$. Any contract $\Psi = (\theta, D) \in \Theta \times D$ such that $U_t^b(\Psi) = u^b$ and $U_t^g(\Psi) = u^g$, has no payments on $[t, \tau]$ and consequently both banks always shirk under Ψ.

Proof. Looking at the proof of (3.4.3) we deduce that necessarily

$$k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi), \ \mathrm{d}D_s = 0, \ \forall s \geq t.$$

Since there are no payments, we have that $k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi) = k_s^{SH}$ for $s \in [t,\tau]$ and indeed have

$$U_t^g(\Psi) = U_t^b(\Psi) = \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} B(I-N_s)\mathrm{d}s \middle| \mathcal{G}_t\right].$$

□

Lemma 3.6.2 Consider any $(t, u^g, u^b) \in \mathbb{R}_+ \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ such that in addition

$$u^g = \frac{\rho_g}{\rho_b} u^b - \frac{(\rho_g - \rho_b)}{\rho_b} C(I - N_t).$$

Under any contract $\Psi = (\theta, D) \in \Theta \times D$ such that $U_t^b(\Psi) = u^b$ and $U_t^g(\Psi) = u^g$, the pool is not liquidated until the last default ($\tau = \tau^I$) and both banks always shirk on $[t,\tau]$.

Proof. Looking at the proof of (3.4.4), we deduce that necessarily $k_s^{\star,g}(\Psi) = k_s^{\star,b}(\Psi) = k_s^{SH}$, $\theta_s = 1$, for every $s \geq t$. Thus, the value functions of the banks are given by

$$U_t^g(\Psi) = \rho_g \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} \mathrm{e}^{-r(s-t)} \mathrm{d}D_s \middle| \mathcal{G}_t\right] + C(I - N_t),$$

$$U_t^b(\Psi) = \rho_b \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} \mathrm{e}^{-r(s-t)} \mathrm{d}D_s \middle| \mathcal{G}_t\right] + C(I - N_t).$$

□

3.6.6 Technical results for the upper boundary

Lemma 3.6.3 For every $j \geq 1$, $x_j^\star > \dfrac{\rho_b}{\rho_g} b_j$.

Proof. For any $j \geq 1$, define the functions $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) := x^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} b_j \frac{r + \widehat{\lambda}_j^0}{r + \widehat{\lambda}_j^{SH}} + \frac{Bj}{r + \widehat{\lambda}_j^{SH}}, \ h(x) := b_j x.$$

Then g is strictly convex in $\mathbb{R}_+$ and we have that $g(1) = h(1) = b_j$ and $g'(1) = h'(1) = b_j$. Thus, h is the tangent line to g at $x = 1$ so $g(x) > h(x)$ for every $x \neq 1$ and therefore

$$x_j^\star = g\left(\frac{\rho_b}{\rho_g}\right) > h\left(\frac{\rho_b}{\rho_g}\right) = \frac{\rho_b}{\rho_g} b_j.$$

□

Proposition 3.6.1 For every $j \geq 1$, the function $\widehat{\mathcal{U}}_j^\star$ defined by (3.4.17) satisfies

$$\frac{\widehat{\mathcal{U}}_j^\star(x)}{x} \leq \frac{\rho_g}{\rho_b}, \ \forall x \geq \frac{Bj}{r + \widehat{\lambda}_j^{SH}}.$$

Moreover, equality holds if and only if $x \geq \widehat{b}_j$.

Proof. Define $A(x) := \dfrac{\widehat{\mathcal{U}}_j^\star(x)}{x}$. If $x \geq \widehat{b}_{j-1}$ then $A(x) = \rho_g/\rho_b$. If now $x \in [x_j^\star, \widehat{b}_j)$, we have

$$A(x) = \frac{\rho_g}{\rho_b}(\widehat{b}_j)^{\frac{\widehat{\lambda}_j^{SH}-\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \left(\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}} \frac{1}{x}\left(x - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)^{\frac{r+\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}.$$

This function is decreasing so that A reaches its maximum value over $[x_j^\star, \widehat{b}_j)$ at $x_j^\star$. Next, we have

$$A(x_j^\star) = \frac{\widehat{b}_j}{x_j^\star} < \frac{\rho_g}{\rho_b} \iff x_j^\star > \frac{\rho_b}{\rho_g} b_j,$$

and the last inequality holds as a consequence of Lemma 3.6.3.

Finally, if $x \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right)$ then

$$A(x) = \frac{1}{x}\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}}\left(x - \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right) + \frac{1}{x}\frac{Bj}{r+\widehat{\lambda}_j^{SH}}.$$

This function is increasing, hence $A(x) \leq A(x_j^\star) < \frac{\rho_g}{\rho_b}$, $\forall x \in \left[\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, x_j^\star\right]$. □

Corollary 3.6.1 Let $j \geq 2$ and $\widehat{\mathcal{U}}_j^\star$, $\widehat{\mathcal{U}}_{j-1}^\star$ defined by (3.4.17), and assume that $\widehat{\lambda}_j^{k^g} \leq \widehat{\lambda}_j^{k^b}$. Then, for any $u^b \geq h^{1,b} + \frac{B(j-1)}{r+\widehat{\lambda}_{j-1}^{SH}}$ we have

$$\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b})\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{k^b}(u^b - h^{1,b}) \leq 0.$$

Furthermore, equality holds if and only if $u^b - h^{1,b} \geq \widehat{b}_j$, $u^b \geq \widehat{b}_j$ and $\widehat{\lambda}_j^{k^b} = \widehat{\lambda}_j^{k^g}$.

Proof. Under the conditions of the corollary, the following allows us to conclude immediately

$$\frac{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b})}{u^b - h^{1,b}} \leq \frac{\rho_g}{\rho_b} \leq \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b).$$

□

Corollary 3.6.2 For $j \geq 1$, let $\widehat{C}_j$ and $\widehat{\mathcal{U}}_j^\star$ be defined by (3.4.16) and (3.4.17) respectively. If $(\theta, h^{1,b}) \in \widehat{C}_j$ is such that $u^b - \theta(u^b - h^{1,b}) \geq \widehat{b}_j$ then $\widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b}) \geq \widehat{b}_j$. As a consequence, in the context of equation (3.4.15), for every $(\theta, h^{1,b}) \in \widehat{C}_j$ we have $k^g \leq k^b$ and $\widehat{\lambda}_j^{k^g} \leq \widehat{\lambda}_j^{k^b}$.

Proof. First observe that $u^b - \theta(u^b - h^{1,b}) \geq \widehat{b}_j$ implies $u^b \geq \widehat{b}_j$. Then we have

$$\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j \geq \frac{\rho_g}{\rho_b}(u^b - \widehat{b}_j) \geq \frac{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b})}{u^b - h^{1,b}}(u^b - \widehat{b}_j).$$

Also, $\theta \le \dfrac{u^b - \widehat{b}_j}{u^b - h^{1,b}}$ and thus

$$\widehat{\mathcal{U}}_j^\star(u^b) - \theta \widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b}) \ge \widehat{\mathcal{U}}_j^\star(u^b) - \left(\frac{u^b - \widehat{b}_j}{u^b - h^{1,b}}\right) \widehat{\mathcal{U}}_{j-1}^\star(u^b - h^{1,b}) \ge \widehat{b}_j.$$

□

We now proceed with the

Proof. [Proof of Lemma 3.4.2] We start with the region $u^b < \widehat{b}_1$, $\widehat{\mathcal{U}}_1(u^b) < \widehat{b}_1$. For these points, we have that $k^b = k^g = 1$, so (3.4.11) can be solved easily and leads to, for some $C_1 \in \mathbb{R}$,

$$\widehat{\mathcal{U}}_1(u^b) = C_1 \left(u^b - \frac{B}{r + \widehat{\lambda}_1^1}\right) + \frac{B}{r + \widehat{\lambda}_1^1}.$$

If $u^b < \widehat{b}_1$ and $\widehat{\mathcal{U}}_1(u^b) \ge \widehat{b}_1$, then $k^b = 1$, $k^g = 0$ and we can solve (3.4.11) to obtain for some $C_2 \in \mathbb{R}$

$$\widehat{\mathcal{U}}_1(u^b) = C_2 \left(u^b - \frac{B}{r + \widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}.$$

Finally, when $u^b \ge \widehat{b}_1$ and $\widehat{\mathcal{U}}(u^b) \ge \widehat{b}_1$ the optimal strategies are $k^b = k^g = 0$ and we have for some $C_3 \in \mathbb{R}$, $\widehat{\mathcal{U}}_1(u^b) = C_3 u^b$. We are interested in smooth solutions of (3.4.11). Denote by $\widehat{\mathcal{U}}_1^{(1)}$, $\widehat{\mathcal{U}}_1^{(2)}$ and $\widehat{\mathcal{U}}_1^{(3)}$ the following functions

$$\widehat{\mathcal{U}}_1^{(1)}(u^b) := C_1 \left(u^b - \frac{B}{r + \widehat{\lambda}_1^1}\right) + \frac{B}{r + \widehat{\lambda}_1^1},\ \widehat{\mathcal{U}}_1^{(2)}(u^b) := C_2 \left(u^b - \frac{B}{r + \widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}},\ \widehat{\mathcal{U}}_1^{(3)}(u^b) := C_3 u^b.$$

We will determine the relations between the constants which allow the smooth fitting of $\widehat{\mathcal{U}}_1$. First we impose $\widehat{\mathcal{U}}_1^{(2)}(\widehat{b}_1) = \widehat{\mathcal{U}}_1^{(3)}(\widehat{b}_1)$ and we get

$$C_2 \left(\widehat{b}_1 \frac{r + \widehat{\lambda}_1^0}{r + \widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}} = C_3 \widehat{b}_1.$$

It can be checked that this relation between C_1 and C_2 ensures also that $(\widehat{\mathcal{U}}_1^{(2)})'(\widehat{b}_1) = (\widehat{\mathcal{U}}_1^{(3)})'(\widehat{b}_1)$. Next, define x_1 as the point such that $\widehat{\mathcal{U}}_1^{(1)}(x_1) = \widehat{b}_1$, *i.e.*

$$x_1 = \frac{\widehat{b}_1}{C_1} \left(\frac{r + \widehat{\lambda}_1^0}{r + \widehat{\lambda}_1^1}\right) + \frac{B}{r + \widehat{\lambda}_1^1}.$$

Also, define x_2 as the point such that $\widehat{\mathcal{U}}_1^{(2)}(x_2) = \widehat{b}_1$, *i.e.*

$$x_2 = \left(\frac{\widehat{b}_1}{C_2}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}} + \frac{B}{r + \widehat{\lambda}_1^1}.$$

We impose $x_1 = x_2$ and we get

$$\frac{\widehat{b}_1}{C_1}\left(\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}\right) = \left(\frac{\widehat{b}_1}{C_2}\right)^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}},$$

and this relation ensures also that $(\widehat{\mathcal{U}}_1^{(1)})'(x_1) = (\widehat{\mathcal{U}}_1^{(2)})'(x_2)$. Expressing both C_1 and C_2 in terms of C_3 we get $\widehat{\mathcal{U}}_1^{(3)}(u^b) = C_3 u^b$, and

$$\widehat{\mathcal{U}}_1^{(1)}(u^b) = C_3^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1},$$

$$\widehat{\mathcal{U}}_1^{(2)}(u^b) = C_3 \widehat{b}_1^{\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}}.$$

□

We pursue with the

Proof. [Proof of Lemma 3.4.3] For $C > 0$, define the following modification $\widehat{\mathcal{U}}_1^{C,\star}$ of $\widehat{\mathcal{U}}_1^C$

$$\widehat{\mathcal{U}}_1^{C,\star}(u^b) := \begin{cases} \widehat{\mathcal{U}}_1^C(u^b),\ u^b \leq x_1^{C,\star}, \\ \frac{\rho_g}{\rho_b}(u^b - x_1^{C,\star}) + \widehat{\mathcal{U}}_1^C(x_1^{C,\star}),\ u^b \geq x_1^{C,\star}, \end{cases}$$

where

$$x_1^{C,\star} := \inf\left\{u^b \in \left[\frac{B}{r+\widehat{\lambda}_1^1}, +\infty\right),\ \left(\widehat{\mathcal{U}}_1^C\right)'(u^b) \leq \frac{\rho_g}{\rho_b}\right\}.$$

The function $\widehat{\mathcal{U}}_1^{C,\star}$ is continuously differentiable, solves the diffusion equation in $[B/(r+\widehat{\lambda}_1^1), x_1^{C,\star})$ and satisfies $\left(\widehat{\mathcal{U}}_1^{C,\star}\right)' = \rho_g/\rho_b$ in $(x_1^{C,\star}, \infty)$. In the following we will study for which values of C this function indeed solves the HJB equation.

– First of all, if $C^{\frac{r+\widehat{\lambda}_1^1}{r+\widehat{\lambda}_1^0}} \leq \frac{\rho_g}{\rho_b}$, we have that

$$x_1^{C,\star} = \frac{B}{r+\widehat{\lambda}_1^1},\ \widehat{\mathcal{U}}_1^{C,\star}(u^b) = \frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1},\ \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\rho_b - \rho_g = 0,$$

so that we need to check that for every u^b in $[B/(r+\widehat{\lambda}_1^1), \infty)$

$$r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \geq 0.$$

Take $u^b > \widehat{b}_1$. Then $k^g = k^b = 0$, and we have

$$
\begin{aligned}
& r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \\
&= r\left[\frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}\right] - \frac{\rho_g}{\rho_b}\left[(r+\widehat{\lambda}_1^0)u^b\right] + \widehat{\lambda}_1^0\left[\frac{\rho_g}{\rho_b}\left(u^b - \frac{B}{r+\widehat{\lambda}_1^1}\right) + \frac{B}{r+\widehat{\lambda}_1^1}\right] \\
&= (r+\widehat{\lambda}_1^0)\frac{B}{r+\widehat{\lambda}_1^1}\left(1 - \frac{\rho_g}{\rho_b}\right) < 0.
\end{aligned}
$$

Hence $\widehat{\mathcal{U}}_1^{C,\star}$ is not a solution of (3.4.12).

– If $\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}} < C \leq \frac{\rho_g}{\rho_b}$, then $x_1^{C,\star} = \widehat{b}_1 \frac{r+\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1}\left(C\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}} + \frac{B}{r+\widehat{\lambda}_1^1}$. Take $u^b > \widehat{b}_1$, then $k^g = k^b = 0$ and

$$
\begin{aligned}
& r\widehat{\mathcal{U}}_1^{C,\star}(u^b) - \left(\widehat{\mathcal{U}}_1^{C,\star}\right)'(u^b)\left(ru^b - Bk^b + u^b\widehat{\lambda}_1^{k^b}\right) + \widehat{\mathcal{U}}_1^{C,\star}(u^b)\widehat{\lambda}_1^{k^g} - Bk^g \\
&= (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1 C^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^0}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{\rho_g}{\rho_b}\frac{B}{r+\widehat{\lambda}_1^1}\right) \\
&\leq (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1\left(\frac{\rho_g}{\rho_b}\right)^{\frac{r+\widehat{\lambda}_1^1}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\left(\frac{\rho_b}{\rho_g}\right)^{\frac{r+\widehat{\lambda}_1^0}{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{\rho_g}{\rho_b}\frac{B}{r+\widehat{\lambda}_1^1}\right) \\
&= (r+\widehat{\lambda}_1^0)\left(\widehat{b}_1\frac{\rho_g}{\rho_b}\frac{\widehat{\lambda}_1^1-\widehat{\lambda}_1^0}{r+\widehat{\lambda}_1^1} - \frac{B}{r+\widehat{\lambda}_1^1}\frac{\rho_g}{\rho_b}\right) = 0.
\end{aligned}
$$

The inequality is strict if $C < \frac{\rho_g}{\rho_b}$ so the only value of C such that $\widehat{\mathcal{U}}_1^{C,\star}$ solves the HJB equation is $C = \frac{\rho_g}{\rho_b}$.

– For large values of C, *i.e.* $C > \frac{\rho_g}{\rho_b}$, we have that $x_1^{C,\star} = +\infty$ and then $\widehat{\mathcal{U}}_1^{C,\star} = \widehat{\mathcal{U}}_1^C$. We exclude this case because these functions do not satisfy condition (3.4.13). □

We end this section with the

Proof. [Proof of Proposition 3.4.2] The proof is by induction. For $j = 1$ the result is proved in Step 2, so we take any $j > 1$ and assume that $\widehat{\mathcal{U}}_{j-1}^\star$ solves its corresponding diffusion equation. We will need to consider three different cases to prove that $\widehat{\mathcal{U}}_j^\star$ solves the equation (3.4.15). In each one of them we prove that the supremum in the right–hand side of (3.4.15) is attained with $\theta = 0$, so therefore the diffusion equation takes the same form as the one in the case with one loan left. Then, it follows from the analysis in Step 2 that its solution satisfies also the variational inequality (3.4.9).

– **Case 1:** $u^b < \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) < \widehat{b}_j$.

In this case for any $(\theta, h^1) \in \widehat{C}^j$, we have that $k^g = k^b = j$. To simplify the notations, let us define $c_j(u^b) := \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b - Bj + u^b\widehat{\lambda}_j^{SH}\right)$, then the term inside the sup in (3.4.15) becomes

$$c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\widehat{\lambda}_j^{SH}\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right],$$

and the optimal choice of θ in this case is 0 (uniquely) because from Corollary 3.6.1 we have

$$\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1) < 0.$$

– **Case 2:** $u^b < \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$.

In this case $k^b = j$ for every $(\theta, h^1) \in \widehat{C}^j$. The term inside the sup in (3.4.15) becomes

$$c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right].$$

Define the following sets

$\widehat{C}_j^0 := \{(\theta, h^1) \in \widehat{C}^j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\}$, $\widehat{C}_j^j := \{(\theta, h^1) \in \widehat{C}^j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j\}$,

and note that $k^g = 0$ for every $(\theta, h^1) \in \widehat{C}_j^0$ and $k^g = j$ for every $(\theta, h^1) \in \widehat{C}_j^j$. Also, the pair $(0, h^1)$ belongs to $\widehat{C}_j^0$ for every feasible h^1.

- If $(\theta, h^1) \in \widehat{C}_j^0$ we have

$$\begin{aligned}&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right]\\ &= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,\end{aligned}$$

where the inequality is due to Corollary 3.6.1.

- If $(\theta, h^1) \in \widehat{C}_j^j$ we have

$$\begin{aligned}&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{k^g} + Bk^g + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{k^g} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right]\\ &= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^{SH} - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right]\\ &< c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj\\ &= c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + b_j(\widehat{\lambda}_j^{SH} - \widehat{\lambda}_j^0)\\ &\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + \widehat{\mathcal{U}}_j^\star(u^b)(\widehat{\lambda}_j^{SH} - \widehat{\lambda}_j^0) = c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,\end{aligned}$$

where the first inequality is a consequence of Corollary 3.6.1 and the second one holds because $\widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$. So we conclude that the optimal value for θ in this case is also 0 (uniquely).

– **Case 3:** $u^b \geq \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) \geq \widehat{b}_j$.

Thanks to Proposition 3.6.2 , we know that there are only three possibilities for the value of (k^b, k^g). Define the sets

$$\begin{aligned}
\widehat{C}_j^{0,0} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) \geq \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\right\}, \\
\widehat{C}_j^{j,0} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) < \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) \geq \widehat{b}_j\right\}, \\
\widehat{C}_j^{j,j} &:= \left\{(\theta, h^1) \in \widehat{C}^j,\ u^b - \theta(u^b - h^1) < \widehat{b}_j,\ \widehat{\mathcal{U}}_j^\star(u^b) - \theta\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j\right\}.
\end{aligned}$$

Then, $(k^b, k^g) = (0, 0)$ for every $(\theta, h^1) \in \widehat{C}_j^{0,0}$, $(k^b, k^g) = (j, 0)$ for every $(\theta, h^1) \in \widehat{C}_j^{j,0}$ and $(k^b, k^g) = (j, j)$ for every $(\theta, h^1) \in \widehat{C}_j^{j,j}$. Also, $(0, h^1)$ belongs to $\widehat{C}_j^{0,0}$ for any feasible h^1.

- If $(\theta, h^1) \in \widehat{C}_j^{0,0}$ then the term inside the sup in (3.4.15) is, because of Corollary 3.6.1, equal to

$$\begin{aligned}
&\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)u^b\left(r + \widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\widehat{\lambda}_j^0\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b - h^1)\right] \\
&\leq \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)u^b\left(r + \widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0,
\end{aligned}$$

- If $(\theta, h^1) \in \widehat{C}_j^{j,0}$, then $h^1 < \widehat{b}_j$ and $\frac{u^b - b_j}{u^b - h^1} < \theta \leq \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)}$. The term in the sup in (3.4.15) is

$$\begin{aligned}
&c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \theta\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&< c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0 + \left(\frac{u^b - \widehat{b}_j}{u^b - h^1}\right)\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}(u^b - h^1)\right] \\
&\leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\lambda_j^0 + (u^b - \widehat{b}_j)\left[\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^0 - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\lambda}_j^{SH}\right] \\
&= \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + u^b\widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0.
\end{aligned}$$

Both inequalities are direct consequences of Corollary 3.6.1.

- Finally, if $(\theta, h^1) \in \widehat{C}_j^{j,j}$, note that $h^1 < \widehat{b}_j$, $\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1) < \widehat{b}_j$ and

$$\frac{u^b - \widehat{b}_j}{u^b - h^1} \leq \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b - h^1)} < \theta.$$

Then, the term inside the sup in (3.4.15) becomes

$$
\begin{aligned}
& c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \theta\widehat{\lambda}_j^{SH}\left[\widehat{\mathcal{U}}_{j-1}^\star(u^b-h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b-h^1)\right] \\
& \leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\widehat{\mathcal{U}}_{j-1}^\star(u^b-h^1)}\widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_{j-1}^\star(u^b-h^1) - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)(u^b-h^1)\right) \\
& \leq c_j(u^b) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^{SH} + Bj + \widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j - \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\frac{\widehat{\mathcal{U}}_j^\star(u^b) - \widehat{b}_j}{\frac{\rho_g}{\rho_b}}\right) \\
& = c_j(u^b) - \widehat{b}_j\widehat{\lambda}_j^0 + \widehat{\lambda}_j^{SH}\left(-\frac{\rho_b}{\rho_g}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{\mathcal{U}}_j^\star(u^b) + \frac{\rho_b}{\rho_g}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\widehat{b}_j\right) \\
& = \widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(u^b - \frac{\rho_b}{\rho_g}\widehat{\mathcal{U}}_j^\star(u^b)\right) + \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + \frac{\rho_b}{\rho_g}\widehat{\lambda}_j^{SH}\widehat{b}_j - Bj\right) - \widehat{\lambda}_j^0\widehat{b}_j.
\end{aligned}
$$

The first inequality is a consequence of Corollary 3.6.1 and the second one of the fact that the function $h^1 \longmapsto \widehat{\mathcal{U}}_{j-1}^\star(u^b-h^1)/(u^b-h^1)$ is non–decreasing and constant for large values of h^1, which implies that $\widehat{\mathcal{U}}_{j-1}^\star(u^b-h^1)/(u^b-h^1) \leq \rho_g/\rho_b$. Now we use the explicit form of $\widehat{\mathcal{U}}_j^\star$ and compute

$$
\begin{aligned}
& \widehat{\lambda}_j^{SH}\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(u^b - \frac{\rho_b}{\rho_g}\widehat{\mathcal{U}}_j^\star(u^b)\right) + \left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + \frac{\rho_b}{\rho_g}\widehat{\lambda}_j^{SH}\widehat{b}_j - Bj\right) - \widehat{\lambda}_j^0\widehat{b}_j \\
& = \frac{\rho_g}{\rho_b}ru^b + \widehat{\lambda}_j^{SH}\widehat{b}_j - \frac{\rho_g}{\rho_b}Bj - \widehat{\lambda}_j^0\widehat{b}_j = \frac{\rho_g}{\rho_b}ru^b + Bj\left(1 - \frac{\rho_g}{\rho_b}\right) < \frac{\rho_g}{\rho_b}ru^b.
\end{aligned}
$$

The term in the last line corresponds to $\left(\widehat{\mathcal{U}}_j^\star\right)'(u^b)\left(ru^b + u^b\widehat{\lambda}_j^0\right) - \widehat{\mathcal{U}}_j^\star(u^b)\widehat{\lambda}_j^0$ and therefore the optimal θ in this case is also 0. Observe that in this case every $(\theta, h^1) \in \widehat{C}_j^{0,0}$ such that $u^b - h^1 \geq \widehat{b}_j$ is optimal. □

We next continue with the

Proof. [Proof of Theorem 3.4.1] We divide the proof in 3 steps.

• **Step 1:** Let us prove first that the SDE (3.4.19) has a unique solution, keeping in mind that $\Psi^\star$ liquidates the pool immediately after the first default. We consider two cases: if $u^b < \widehat{b}_{I-N_t}$, by right–continuity we can find for every solution of (3.4.19) some $\varepsilon \in (0, \tau - t)$ such that $u_s^b < \widehat{b}_{I-N_t}$ for $s \in [t, t+\varepsilon]$. Consequently u^b solves the ODE

$$
\mathrm{d}u_s^b = \left[(r + \widehat{\lambda}_{I-N_t}^{SH})u_s^b - B(I-N_t)\right]\mathrm{d}s,\ s \in [t, t+\varepsilon],
$$

whose unique solution is given by

$$
u_s^b = \mathrm{e}^{(r+\widehat{\lambda}_{I-N_t}^{SH})(s-t)}\left(u^b - \frac{B(I-N_t)}{r + \widehat{\lambda}_{I-N_t}^{SH}}\right) + \frac{B(I-N_t)}{r + \widehat{\lambda}_{I-N_t}^{SH}},\ s \in [t, t+\varepsilon].
$$

So, as long as there is no default and the project keeps running u^b_s will be deterministic until it reaches the value $\widehat{b}_{I-N_t}$. That will eventually happen at time

$$t^\star(u^b) := t + \frac{1}{r+\widehat{\lambda}^{SH}_{I-N_t}} \log\left(\frac{\widehat{b}_{I-N_t}(r+\widehat{\lambda}^0_{I-N_t})}{u^b(r+\widehat{\lambda}^{SH}_{I-N_t}) - B(I-N_t)}\right),$$

and we see from (3.4.19) that at time $t^\star(u^b)$ we will have $\mathrm{d}u^b_s = 0$, so $u^b_s = \widehat{b}_{I-N_t}$ for every $s \in [t^\star(u^b), \tau)$. In the second case, if $u^b \geq \widehat{b}_{I-N_t}$ then (3.4.19) becomes $\mathrm{d}u^b_s = -u^b_{s-}\mathrm{d}N_s,\ s \in [t,\tau]$, and necessarily $u^b_s = u^b$ for every $s \in [t,\tau)$. This proves the existence and uniqueness of the solution of (3.4.19) in both cases.

• **Step** 2**:** Now we turn to the values of the banks under $\Psi^\star$. If $u^b \geq \widehat{b}_{I-N_t}$, we know from the previous analysis that $u^b_s = u^b \geq \widehat{b}_{I-N_t}$ for every $s \in [t,\tau)$, so in this case $\Psi^\star$ is a short–term contract with constant payment, see Section 3.6.3.1. Using the notations of this section, since $c \geq \overline{c_b} \geq \overline{c_g}$ both banks will always work, the value function of the bad bank is $U^b_t(\Psi^\star) = \rho_b c/(r+\widehat{\lambda}^0_{I-N_t}) = u^b$ and the one of the good bank is $U^g_t(\Psi^\star) = \rho_g c/(r+\widehat{\lambda}^0_{I-N_t}) = \rho_g/\rho_b u^b = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b)$.

In the case where $u^b < \widehat{b}_{I-N_t}$, $\Psi^\star$ is a short–term contract with delay $t^\star(u^b)$ and constant payment, see Section 3.6.4.1. Using the notations of this section, since $c = \overline{c_b}$ the bad bank will always shirk and her value function is

$$U^b_t(\Psi^\star) = \rho_b c \frac{\mathrm{e}^{-(r+\widehat{\lambda}^{SH}_{I-N_t})t^\star(u^b)}}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \frac{B}{r+\widehat{\lambda}^{SH}_{I-N_t}} = u^b.$$

For the good bank we have two sub-cases. First, if $u^b \in [x^\star_{I-N_t}, \widehat{b}_{I-N_t})$ then $\overline{t_g}(c) \geq t^\star(u^b)$, so the good bank will always work and her value function is

$$U^g_t(\Psi^\star) = \frac{\rho_g}{\rho_b}\widehat{b}_{I-N_t}^{\frac{\widehat{\lambda}^{SH}_{I-N_t}-\widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} \left(\frac{r+\widehat{\lambda}^{SH}_{I-N_t}}{r+\widehat{\lambda}^0_{I-N_t}}\right)^{\frac{r+\widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} \left(u^b - \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}\right)^{\frac{r+\widehat{\lambda}^0_{I-N_t}}{r+\widehat{\lambda}^{SH}_{I-N_t}}} = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b).$$

If $u^b \in \left[\frac{B}{r+\widehat{\lambda}^{SH}_{I-N_t}}, x^\star_{I-N_t}\right)$ then $\overline{t_g}(c) < t^\star(u^b)$, so the good bank will start working at time $t^\star(u^b)$ and her value function is

$$U^g_t(\Psi^\star) = \frac{\rho_g}{\rho_b}^{\frac{r+\widehat{\lambda}^{SH}_{I-N_t}}{r+\widehat{\lambda}^0_{I-N_t}}} \left(u^b - \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}\right) + \frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}} = \widehat{\mathcal{U}}^\star_{I-N_t}(u^b).$$

• **Step** 3**:** Since $U^b_t(\Psi^\star) = u^b$, it is trivial that $\Psi^\star \in \overline{\mathcal{A}}^b(t,u^b)$. Consider now any contract $\Psi = (D,\theta,h^{1,b},h^{2,b}) \in \overline{\mathcal{A}}^b(t,u^b)$. We recall that the value function of the bad bank under Ψ satisfies

$$\mathrm{d}U^b_s(\Psi) = \left(rU^b_s(\Psi) - Bk^{\star,b}_s(\Psi) + [h^{1,b}_s + h^{2,b}_s(1-\theta_s)]\lambda^{k^{\star,b}(\Psi)}_s\right)\mathrm{d}s - \rho_b\mathrm{d}D_s - h^{1,b}_s\mathrm{d}N_s - h^{2,b}_s\mathrm{d}H_s,$$

with $k_s^{\star,b}(\Psi) = \mathbf{1}_{\{h_s^{1,b}+(1-\theta_s)h_s^{2,b}<b_s\}}$. Define the process

$$G_w := \int_t^w \mathrm{e}^{-r(s-t)}\left[\rho_g \mathrm{d}D_s + k_s^{\star,g}(\Psi)B\mathrm{d}s\right] + \mathrm{e}^{-r(w-t)}\widehat{\mathcal{U}}^\star_{I-N_w}(U_w^b(\Psi)),\ w \in [t,\tau].$$

Observe we can rewrite the second term in the following form (with the convention $\tau_{N_t} = t$, $\tau_{N_w+1} = w$)

$$\begin{aligned}\mathrm{e}^{-r(w-t)}\widehat{\mathcal{U}}^\star_{I-N_w}(U_w^b(\Psi)) &= \sum_{\mathrm{i}=N_t}^{N_w} \mathrm{e}^{-r(\tau_{\mathrm{i}+1}-t)}\widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi)\right) - \mathrm{e}^{-r(\tau_\mathrm{i}-t)}\widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{\tau_\mathrm{i}}(\Psi)\right)\\ &+ \sum_{\mathrm{i}=N_t}^{N_w-1} \mathrm{e}^{-r(\tau_{\mathrm{i}+1}-t)}\left(\widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{\tau_{\mathrm{i}+1}}(\Psi)\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi)\right)\right) + \widehat{\mathcal{U}}^\star_{I-N_t}\left(U^b_t(\Psi)\right).\end{aligned}$$

Since the functions $\widehat{\mathcal{U}}^\star_j$ are C^1, we can apply Itô's formula on the intervals $[\tau_\mathrm{i}\wedge\tau, \tau_{\mathrm{i}+1}\wedge\tau)$ with $\mathrm{i} \in \{N_t,\dots,N_w\}$ to obtain an integral expression for the first sum. Regarding the second sum, observe that

$$\begin{aligned}&\widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{\tau_{\mathrm{i}+1}}(\Psi)\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi)\right)\\ &= \left(\widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi) - h^{1,b}_{\tau_{\mathrm{i}+1}}\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi)\right)\right)\Delta N_{\tau_{\mathrm{i}+1}} - \widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{\tau^-_{\mathrm{i}+1}}(\Psi) - h^{1,b}_{\tau_{\mathrm{i}+1}}\right)\Delta H_{\tau_{\mathrm{i}+1}}\\ &= \int_{\tau_\mathrm{i}}^{\tau_{\mathrm{i}+1}} \left(\widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{s^-}(\Psi)\right)\right)\mathrm{d}N_s - \int_{\tau_\mathrm{i}}^{\tau_{\mathrm{i}+1}} \widehat{\mathcal{U}}^\star_{I-(\mathrm{i}+1)}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right)\mathrm{d}H_s.\end{aligned}$$

Hence

$$\begin{aligned}G_{\tau\wedge v} &= \widehat{\mathcal{U}}^\star_{I-N_t}(u^b) + \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\left(\rho_g - \rho_b\left(\widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\right)'\left(U^b_s(\Psi)\right)\right)\mathrm{d}D_s\\ &+ \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\left(k_s^{\star,g}(\Psi)B - r\widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_s(\Psi)\right)\right)\mathrm{d}s\\ &+ \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\lambda_s^{k^{\star,g}(\Psi)}\left(\theta_s\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_s(\Psi)\right)\right)\mathrm{d}s\\ &+ \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\widehat{\mathcal{U}}^{\star\prime}_{I-\mathrm{i}}\left(U^b_s(\Psi)\right)\left(rU^b_s(\Psi) - Bk_s^{\star,b}(\Psi) + \lambda_s^{k^{\star,b}(\Psi)}(h^{1,b}_s + (1-\theta_s)h^{2,b}_s)\right)\mathrm{d}s\\ &+ \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\left(\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{s^-}(\Psi)\right)\right)(\mathrm{d}N_s - \lambda_s^{k^{\star,g}(\Psi)}\mathrm{d}s)\\ &- \sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)}\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right)(\mathrm{d}H_s - \lambda_s^{k^{\star,g}(\Psi)}(1-\theta_s)\mathrm{d}s).\end{aligned}$$

We know that the derivative of every $\widehat{\mathcal{U}}^\star_j$ is greater than ρ_g/ρ_b by definition, and since D is non–decreasing, the first sum of integrals is non–positive. Also, the functions $\widehat{\mathcal{U}}^\star_j$ are solutions of the system of HJB equations, which implies that for any admissible contract the second and the third sum of integrals are also non–positive. We deduce

$$G_{\tau\wedge v} \leq \widehat{\mathcal{U}}^\star_{I-N_t}(u^b) + \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{r(t-s)} \Big(\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\Big(U^b_{s^-}(\Psi) - h^{1,b}_s\Big) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\Big(U^b_{s^-}(\Psi)\Big)\Big)(\mathrm{d}N_s - \lambda^{k^\star,g}_s \mathrm{d}s)$$
$$- \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_\mathrm{i}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \mathrm{e}^{-r(s-t)} \widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right)\left(\mathrm{d}H_s - \lambda^{k^\star,g(\Psi)}_s(1-\theta_s)\mathrm{d}s\right). \tag{3.6.6}$$

Next, for every i we have that, recalling that the functions $\widehat{\mathcal{U}}^\star_j$ are non–decreasing and null at 0

$$\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \left|\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_s(\Psi) - h^{1,b}_s\right)\right| \mathrm{d}s \middle| \mathcal{G}_t\right] \leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \frac{\rho_g}{\rho_b} U^b_s(\Psi)\mathrm{d}s \middle| \mathcal{G}_t\right]$$
$$\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \frac{\rho_g}{\rho_b} u^b \mathrm{e}^{(r+\lambda)s}\mathrm{d}s \middle| \mathcal{G}_t\right] < \infty,$$

with $\lambda := \max_{1\leq j\leq I} \widehat{\lambda}^{SH}_j$. Indeed, we have between two consecutive jump times of N

$$\begin{aligned}
\mathrm{d}U^b_s(\Psi) &= \left(rU^b_s(\Psi) - Bk^{\star,b}_s(\Psi) + (h^{1,b}_s + (U^b_s(\Psi) - h^{1,b}_s)(1-\theta_s))\lambda^{k^\star,b(\Psi)}_s\right)\mathrm{d}s - \rho_b \mathrm{d}D_s \\
&\leq \left(rU^b_s(\Psi) + h^{1,b}_s \lambda^{k^\star,b(\Psi)}_s + (U^b_s(\Psi) - h^{1,b}_s)(1-\theta_s)\lambda^{k^\star,b(\Psi)}_s\right)\mathrm{d}s \\
&= U^b_s(\Psi)\left(r + (1-\theta_s)\lambda^{k^\star,b(\Psi)}_s\right)\mathrm{d}s + h^{1,b}_s \theta_s \lambda^{k^\star,b(\Psi)}_s \mathrm{d}s \\
&\leq U^b_s(\Psi)\left(r + \lambda^{k^\star,b(\Psi)}_s\right)\mathrm{d}s,
\end{aligned}$$

where we used the facts that $h^{1,b}_s \in [0, U^b_s(\Psi)]$, the functions $\widehat{\mathcal{U}}^\star_j$ are non–decreasing and $U^b_s(\Psi)$ is bounded from below and has positive jumps. Similarly

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \left|\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right) - \widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{s^-}(\Psi)\right)\right| \mathrm{d}s \middle| \mathcal{G}_t\right] \\
&\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \left|\widehat{\mathcal{U}}^\star_{I-\mathrm{i}-1}\left(U^b_{s^-}(\Psi) - h^{1,b}_s\right)\right| \mathrm{d}s \middle| \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \left|\widehat{\mathcal{U}}^\star_{I-\mathrm{i}}\left(U^b_{s^-}(\Psi)\right)\right| \mathrm{d}s \middle| \mathcal{G}_t\right] \\
&\leq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \frac{\rho_g}{\rho_b} U^b_s(\Psi)\mathrm{d}s \middle| \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \frac{\rho_g}{\rho_b} U^b_s(\Psi)\mathrm{d}s \middle| \mathcal{G}_t\right] \\
&\leq 2\mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\int_t^\tau \mathrm{e}^{-r(s-t)} \frac{\rho_g}{\rho_b} u^b \mathrm{e}^{(r+\lambda)s}\mathrm{d}s \middle| \mathcal{G}_t\right] < \infty.
\end{aligned}$$

Then, the stochastic integrals appearing above are martingales, and taking conditional expectation in (3.6.6) we get $\mathbb{E}^{\mathbb{P}^{k^\star,g}}[G_{\tau\wedge v}|\,\mathcal{G}_t] \leq \widehat{\mathcal{U}}^\star_{I-N_t}(u^b)$ and from Fatou's Lemma we obtain

$$\widehat{\mathcal{U}}^\star_{I-N_t}(u^b) \geq \varliminf_{v\to\infty} \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[G_{\tau\wedge v}|\,\mathcal{G}_t\right] \geq \mathbb{E}^{\mathbb{P}^{k^\star,g}}\left[\varliminf_{v\to\infty} G_{\tau\wedge v}\middle| \mathcal{G}_t\right] = U^g_t(\Psi),$$

where we used that, $\mathbb{P}^{k^\star,g} - a.s.$

$$\begin{aligned}
\varliminf_{v\to\infty} G_{\tau\wedge v} &= \lim_{v\to\infty} \int_t^{\tau\wedge v} \mathrm{e}^{-r(s-t)}\left[\rho_g \mathrm{d}D_s + k^{\star,g}_s(\Psi)B\mathrm{d}s\right] + 1_{\{v<\tau\}}\mathrm{e}^{-r(v-t)}\widehat{\mathcal{U}}^\star_{I-N_v}(U^b_v(\Psi)) \\
&= \int_t^\tau \mathrm{e}^{-r(s-t)}\left[\rho_g \mathrm{d}D_s + k^{\star,g}_s(\Psi)B\mathrm{d}s\right].
\end{aligned}$$

□

We end this section with the

Proof. [Proof of Proposition 3.4.3] The definition of $\widehat{\mathcal{C}}_j$ does not necessarily match with the credible set $\mathcal{C}_j$, however we can notice that the inclusion $\mathcal{C}_j \subseteq \widehat{\mathcal{C}}_j$ holds and therefore we only need to prove that $\widehat{\mathcal{C}}_j \subseteq \mathcal{C}_j$. We will make use of contracts with lump–sum payments to prove that every point from $\widehat{\mathcal{C}}_j$ belongs to the credible set $\mathcal{C}_j$. We start by defining the line with slope ρ_g/ρ_b which passes through the point $(u^b, u^g) = \left(\frac{Bj}{r+\widehat{\lambda}_j^{SH}}, \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\right)$,

$$\widehat{\mathfrak{M}}_j(u^b) := \frac{\rho_g}{\rho_b}u^b + \frac{Bj}{r+\widehat{\lambda}_j^{SH}}\left(1 - \frac{\rho_g}{\rho_b}\right),$$

and the sets

$$\widehat{\mathcal{C}}_j^1 := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{M}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{U}}_j(u^b)\right\},$$
$$\widehat{\mathcal{C}}_j^2 := \left\{(u^b, u^g) \in \widehat{\mathcal{V}}_j \times \widehat{\mathcal{V}}_j,\ \widehat{\mathfrak{L}}_j(u^b) \leq u^g \leq \widehat{\mathfrak{M}}_j(u^b)\right\}.$$

From Section 3.6.4.3 in the Appendix, we know that $\widehat{\mathcal{C}}_j^1 \subseteq \mathcal{C}_j$. The reason of this is that every point from the upper boundary $\widehat{\mathfrak{U}}_j$ belongs to the credible set and that if we perturb a contract $\Psi = (\theta, D)$ only by adding a lump–sum payment ε at time t, that is $\mathrm{d}D_s^{\Psi'} = \mathbf{1}_{\{s=t\}}\varepsilon + \mathrm{d}D_s^{\Psi}$, then the values of the banks under Ψ' are $U_t^g(\Psi') = u^g + \varepsilon\rho_g$ and $U_t^b(\Psi') = u^b + \varepsilon\rho_b$, so $(U_t^b(\Psi'), U_t^g(\Psi')) = (u^b, u^g) + \varepsilon(\rho_b, \rho_g)$. We use this idea to prove also that $\widehat{\mathcal{C}}_j^2 \subseteq \mathcal{C}_j$. From Proposition 3.4.1, we know that the graph of $\widehat{\mathfrak{L}}_j$ is contained in $\mathcal{C}_j$. Therefore any point of the following form belongs to $\mathcal{C}_j$

$$(\widehat{u}^b, \widehat{u}^g) = (u^b, u^g) + \ell(\rho_b, \rho_g),\ \ell \geq 0,\ u^g = \widehat{\mathfrak{L}}_j(u^b). \tag{3.6.7}$$

By the geometry of the lower boundary $\widehat{\mathfrak{L}}_j$, the set of points of the form (3.6.7) is exactly $\widehat{\mathcal{C}}_j^2$.
□

3.6.7 Principal's value function on the boundary of the credible set

We start this section with the

Proof. [Proof of Proposition 3.5.1] Consider any time $t \geq 0$ and take any $u^{b,c} \geq C(I - N_t)$, as well as some $\Psi_g \in \widehat{\mathcal{A}}^g(t, \widehat{\mathfrak{L}}_{I-N_t}(u^{b,c}), u^{b,c})$. From Lemma 3.6.2, we know that the components of Ψ_g must satisfy $\theta^g \equiv 1$ and that both banks shirk under Ψ_g. The payments determine the utility of the banks and the following holds by definition

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I} \mathrm{e}^{-r(s-t)}\mathrm{d}D_s^g \middle| \mathcal{G}_t\right] = \frac{u^{b,c} - C(I - N_t)}{\rho_b}.$$

Besides, the utility of the investor under the contract Ψ_g is

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I}(\mu(I-N_s)\mathrm{d}s-\mathrm{d}D_s^g)\bigg|\,\mathcal{G}_t\right]=\sum_{\mathrm{i}=N_t}^{I-1}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}_{I-\mathrm{i}}^{SH}}-\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I}\mathrm{d}D_s^g\bigg|\,\mathcal{G}_t\right].$$

Now, observe that

$$\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I}\mathrm{d}D_s^g\bigg|\,\mathcal{G}_t\right]\geq\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau^I}\mathrm{e}^{-r(s-t)}\mathrm{d}D_s^g\bigg|\,\mathcal{G}_t\right]=\frac{u^{b,c}-C(I-N_t)}{\rho_b},$$

and the equality holds if and only if D^g has a jump at time t of size $\frac{u^{b,c}-C(I-N_t)}{\rho_b}$ and $\mathrm{d}D_s^g=0$ for every $s>t$. That means that it is optimal for the investor to use a contract with an initial lump–sum payment and to pay nothing afterwards. Consequently, the value function of the investor on the lower boundary is given by

$$V_t^{\mathfrak{L},g}(u^{b,c})=\sum_{\mathrm{i}=N_t}^{I-1}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}_{I-\mathrm{i}}^{SH}}-\left(\frac{u^{b,c}-C(I-N_t)}{\rho_b}\right).$$

□

We continue this section with the

Proof. [Proof of Proposition 3.5.2] Consider any time $t\geq 0$. Take any $u^{b,c}\in[c(I-N_t,1),C(I-N_t))$, and $\Psi_g\in\widehat{\mathcal{A}}^g(t,u^{b,c},u^{b,c})$. From Lemma 3.6.1, we know that the components of Ψ_g must satisfy $\mathrm{d}D_s^g=0$ for all $s\geq t$ and that both banks will shirk under this contract. Then, θ^g determines the continuation utilities of the banks in the following way

$$u^{b,c}=\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau}\mathrm{e}^{-r(s-t)}B(I-N_s)\mathrm{d}s\bigg|\,\mathcal{G}_t\right],$$

so in this case, the problem (3.5.4) reduces to

$$(P)\quad\sup_{\theta\in\Theta}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau}\mu(I-N_s)\mathrm{d}s\bigg|\,\mathcal{G}_t\right],\ \text{s.t}\ \mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau}\mathrm{e}^{-r(s-t)}B(I-N_s)\mathrm{d}s\bigg|\,\mathcal{G}_t\right]=u^{b,c}.$$

Next, we rewrite the objective function in a more convenient way

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau}\mu(I-N_s)\mathrm{d}s\bigg|\,\mathcal{G}_t\right]\\
&=\mu(I-N_t)\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\tau_{N_t+1}-t|\mathcal{G}_t\right]+\sum_{\mathrm{i}=N_t+1}^{I-1}\mu(I-\mathrm{i})\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}(\tau_{\mathrm{i}+1}-\tau_\mathrm{i})\big|\,\mathcal{G}_t\right]\\
&=\frac{\mu(I-N_t)}{\widehat{\lambda}_{I-N_t}^{SH}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\mu(I-\mathrm{i})\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}(\tau_{\mathrm{i}+1}-\tau_\mathrm{i})\big|\,\mathcal{G}_{\tau_\mathrm{i}}\right]\Big|\,\mathcal{G}_t\right]\\
&=\frac{\mu(I-N_t)}{\widehat{\lambda}_{I-N_t}^{SH}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\mu(I-\mathrm{i})\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}\big|\,\mathcal{G}_{\tau_\mathrm{i}}\right]\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\tau_{\mathrm{i}+1}-\tau_\mathrm{i}\big|\,\mathcal{G}_{\tau_\mathrm{i}}\right]\Big|\,\mathcal{G}_t\right]\\
&=\frac{\mu(I-N_t)}{\widehat{\lambda}_{I-N_t}^{SH}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}_{I-\mathrm{i}}^{SH}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\big|\,\mathcal{G}_t\right].
\end{aligned}$$

We do the same with the constraint

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau} \mathrm{e}^{-r(s-t)} B(I-N_s)\mathrm{d}s \middle| \mathcal{G}_t\right]\\
&=\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\int_t^{\tau_{N_t+1}} B(I-N_t)\mathrm{e}^{-r(s-t)}\mathrm{d}s+\sum_{\mathrm{i}=N_t+1}^{I-1}\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}\int_{\tau_\mathrm{i}}^{\tau_{\mathrm{i}+1}} \mathrm{e}^{-r(s-t)}B(I-\mathrm{i})\mathrm{d}s\middle| \mathcal{G}_t\right]\\
&=\frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\frac{B(I-\mathrm{i})}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}\left(\mathrm{e}^{-r(\tau_\mathrm{i}-t)}-\mathrm{e}^{-r(\tau_{\mathrm{i}+1}-t)}\right)\middle|\mathcal{G}_{\tau_\mathrm{i}}\right]\middle|\mathcal{G}_t\right]\\
&=\frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\frac{B(I-\mathrm{i})}{r}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathrm{e}^{-r(\tau_\mathrm{i}-t)}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\mathbf{1}_{\{\tau>\tau_\mathrm{i}\}}\middle|\mathcal{G}_{\tau_\mathrm{i}}\right]\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[1-\mathrm{e}^{-r(\tau_{\mathrm{i}+1}-\tau_\mathrm{i})}\middle|\mathcal{G}_{\tau_\mathrm{i}}\right]\middle|\mathcal{G}_t\right]\\
&=\frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}+\sum_{\mathrm{i}=N_t+1}^{I-1}\frac{B(I-\mathrm{i})}{r+\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\mathrm{e}^{-r(\tau_\mathrm{i}-t)}\middle|\mathcal{G}_t\right].
\end{aligned}
$$

So we obtain the following expression for our problem

$$
(P)\begin{cases}\sup\limits_{\theta\in\Theta} & \dfrac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}}+\sum\limits_{\mathrm{i}=N_t+1}^{I-1}\dfrac{\mu(I-\mathrm{i})}{\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\middle|\mathcal{G}_t\right]\\ \text{s.t} & \dfrac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}+\sum\limits_{\mathrm{i}=N_t+1}^{I-1}\dfrac{B(I-\mathrm{i})}{r+\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\mathrm{e}^{-r(\tau_\mathrm{i}-t)}\middle|\mathcal{G}_t\right]=u^{b,c}.\end{cases}
$$

We do not know how to solve (P) directly, so we will define its dual problem, characterise its solution and show that the duality gap is zero. In order to do that, we define the Lagrangian function $L:\Theta\times\mathbb{R}\times\Omega\longrightarrow\mathbb{R}$ as follows

$$
\begin{aligned}
L(\theta,\nu,\omega):=&-\frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}}-\sum_{\mathrm{i}=N_t(\omega)+1}^{I-1}\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\middle|\mathcal{G}_t\right](\omega)\\
&+\nu\left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}}+\sum_{\mathrm{i}=N_t(\omega)+1}^{I-1}\frac{B(I-\mathrm{i})}{r+\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta_{\tau_\mathrm{i}}\mathrm{e}^{-r(\tau_\mathrm{i}-t)}\middle|\mathcal{G}_t\right](\omega)-u^{b,c}\right),
\end{aligned}
$$

and also define the dual function and the dual problem respectively as

$$
g(\nu,\omega):=\inf_{\theta\in\Theta}L(\theta,\nu,\omega),\ (D)\ \sup_{\nu\in\mathbb{R}}g(\nu,\omega)\ .
$$

Then, we have the weak duality inequality (where val denotes the value of the optimisation problem)

$$
-\mathrm{val}(P)=\inf_{\theta\in\Theta}\sup_{\nu\in\mathbb{R}}L(\theta,\nu,\omega)\geq\sup_{\nu\in\mathbb{R}}\inf_{\theta\in\Theta}L(\theta,\nu,\omega)=\mathrm{val}(D).
$$

We rewrite the dual function as follows

$$
\begin{aligned}
g(\nu,\omega)=&-\frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}}+\nu\left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}}-u^{b,c}\right)\\
&+\inf_{\theta\in\Theta}\sum_{\mathrm{i}=N_t(\omega)+1}^{I-1}\int_\Omega\theta_{\tau_\mathrm{i}}(\widetilde{\omega})\left(\nu\frac{B(I-\mathrm{i})}{r+\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathrm{e}^{-r(\tau_\mathrm{i}(\widetilde{\omega})-t)}-\frac{\mu(I-\mathrm{i})}{\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\right)\mathrm{d}\mathbb{P}^{SH}_{t,\omega}(\widetilde{\omega}),
\end{aligned}
$$

where $\mathbb{P}^{SH}_{t,\omega}$ is a regular conditional probability distribution for the conditional expectation with respect to the raw (that is to say not completed) version of $\mathcal{G}_t$. We have easily that it is optimal to set the optimal control θ^ν to be $\theta^\nu_{\tau_{\rm i}}(\widetilde{\omega}) := \mathbf{1}_{\widetilde{\omega}\in A^{\rm i}_\nu}(\widetilde{\omega})$, where the set $A^{\rm i}_\nu$ is defined by

$$A^{\rm i}_\nu := \begin{cases} \Omega, \text{ if } \nu < \dfrac{\mu}{B}\dfrac{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}{\widehat{\lambda}^{SH}_{I-{\rm i}}}, \\ \left\{\widetilde{\omega},\ \tau_{\rm i}(\widetilde{\omega}) - t > \dfrac{1}{r}\ln\left(\dfrac{\nu B\widehat{\lambda}^{SH}_{I-{\rm i}}}{\mu(r+\widehat{\lambda}^{SH}_{I-{\rm i}})}\right)\right\},\ \text{if } \nu \geq \dfrac{\mu}{B}\dfrac{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}{\widehat{\lambda}^{SH}_{I-{\rm i}}}. \end{cases}$$

Therefore, for any $\nu \in \mathbb{R}$ the dual function has the following form, using that the conditional law of $\tau_{\rm i} - t$ given $\mathcal{G}_t$ is the same as the law of $\tau_{\rm i}$

$$\begin{aligned} g(\nu,\omega) = &-\frac{\mu(I-N_t(\omega))}{\widehat{\lambda}^{SH}_{I-N_t(\omega)}} + \nu\left(\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t(\omega)}} - u^{b,c}\right) \\ &+ \sum_{{\rm i}=N_t(\omega)+1}^{I-1}\int_{s_{\rm i}(\nu)}^{\infty}\left(\frac{\nu B(I-{\rm i})\mathrm{e}^{-rx}}{r+\widehat{\lambda}^{SH}_{I-{\rm i}}} - \frac{\mu(I-{\rm i})}{\widehat{\lambda}^{SH}_{I-{\rm i}}}\right)f_{\tau_{\rm i}}(x)\mathrm{d}x. \end{aligned} \tag{3.6.8}$$

It is not difficult to see that g is a continuous and differentiable function. As we want to maximise g in the dual problem, we compute its derivative with respect to ν and we get

$$g'(\nu,\omega) = \frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t}} - u^{b,c} + \sum_{{\rm i}=N_t+1}^{I-1}\int_{s_{\rm i}(\nu)}^{\infty}\frac{B(I-{\rm i})}{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}\mathrm{e}^{-rx}f_{\tau_{\rm i}}(x)\mathrm{d}x.$$

Since $\nu \longmapsto s_{\rm i}(\nu)$ is non–decreasing for any ${\rm i} = 1,\dots,I$, g' is non–increasing in ν. Furthermore, since $u^{b,c} \geq c(I-N_t,1)$, we have the limit at $+\infty$ of g' is non–positive, and that its value for small ν is positive because $u^{b,c} < C(I-N_t)$ and

$$\frac{B(I-N_t(\omega))}{r+\widehat{\lambda}^{SH}_{I-N_t}} + \sum_{{\rm i}=N_t+1}^{I-1}\int_0^{\infty}\frac{B(I-{\rm i})}{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}\mathrm{e}^{-rx}f_{\tau_{\rm i}}(x)\mathrm{d}x = C(I-N_t).$$

Therefore, there is a unique value of ν that makes g' equal to 0.

Now, we compute for any ν the value of the constraint from the primal problem for the control θ^ν

$$\sum_{{\rm i}=N_t+1}^{I-1}\frac{B(I-{\rm i})}{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}\left[\theta^\nu_{\tau_{\rm i}}\mathrm{e}^{-r(\tau_{\rm i}-t)}\middle|\mathcal{G}_t\right] = \sum_{{\rm i}=N_t+1}^{I-1}\int_{s_{\rm i}(\nu)}^{\infty}\frac{B(I-{\rm i})}{r+\widehat{\lambda}^{SH}_{I-{\rm i}}}\mathrm{e}^{-rx}f_{\tau_{\rm i}}(x)\mathrm{d}x,$$

so θ^ν is feasible in problem (P) if and only if $g'(\nu,\omega)=0$. Next, we compute for θ^ν the value of the objective function in the primal (minimisation) problem

$$-\frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} - \sum_{{\rm i}=N_t+1}^{I-1}\frac{\mu(I-{\rm i})}{\widehat{\lambda}^{SH}_{I-{\rm i}}}\mathbb{E}^{\mathbb{P}^{k^{SH}}}_t\left[\theta^\nu_{\tau_{\rm i}}\right] = -\frac{\mu(I-N_t)}{\widehat{\lambda}^{SH}_{I-N_t}} - \sum_{{\rm i}=N_t+1}^{I-1}\int_{s_{\rm i}(\nu)}^{\infty}\frac{\mu(I-{\rm i})}{\widehat{\lambda}^{SH}_{I-{\rm i}}}f_{\tau_{\rm i}}(x)\mathrm{d}x.$$

If this quantity is equal to $g(\nu,\cdot)$, the duality gap is zero. From (3.6.8) we see that this happens if and only if

$$\nu\left(\frac{B(I-N_t)}{r+\widehat{\lambda}^{SH}_{I-N_t}}-u^{b,c}+\sum_{\mathrm{i}=N_t+1}^{I-1}\int_{s_{\mathrm{i}}(\nu)}^{\infty}\frac{B(I-\mathrm{i})}{r+\widehat{\lambda}^{SH}_{I-\mathrm{i}}}\mathrm{e}^{-rx}f_{\tau_{\mathrm{i}}}(x)\mathrm{d}x\right)=0 \iff \nu g'(\nu,\cdot)=0.$$

We conclude that if $\nu\in\mathbb{R}$ is such that $g'(\nu)=0$ then the control θ^ν is optimal in the primal problem. □

We continue with the

Proof. [Proof of Proposition 3.5.3] Define the process $\ell_s=\widehat{\mathfrak{U}}_{I-N_s}(U^{b,c}_s(\Psi_g))-U^g_s(\Psi_g)$ and note that $\ell_s\geq 0$ for every $s\geq 0$. We will prove that $\ell_t=0$ implies $\ell_v=0$ for every $v\geq t$. Assume thus that $\ell_t=0$. Following the same idea as in the proof of Theorem 3.4.1, we have for $v\geq t$

$$\begin{aligned}
\ell_v=&\sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v}-\left(rU^g_s(\Psi_g)-Bk^{\star,g}_s(\Psi_g)+[h^{1,g}_s+(1-\theta^g_s)h^{2,g}_s]\lambda_s^{k^{\star,g}(\Psi_g)}\right)\mathrm{d}s\\
&+\sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v}\widehat{\mathfrak{U}}'_{I-\mathrm{i}}(U^{b,c}_s(\Psi_g))\left(rU^{b,c}_s(\Psi_g)-Bk^{\star,b,c}_s(\Psi_g)+\lambda_{I-\mathrm{i}}^{k^{\star,b,c}(\Psi_g)}(h^{1,b,c}_s+(1-\theta^g_s)h^{2,b,c}_s)\right)\mathrm{d}s\\
&+\sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v}\left(h^{1,g}_s+\widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U^{b,c}_{s^-}(\Psi_g)-h^{1,b,c}_s)-\widehat{\mathfrak{U}}_{I-\mathrm{i}}(U^{b,c}_{s^-}(\Psi_g))\right)\mathrm{d}N_s\\
&+\sum_{\mathrm{i}=N_t}^{I-1}\int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v}\left(h^{2,g}_s-\widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U^{b,c}_{s^-}(\Psi_g)-h^{1,b,c}_s)\right)\mathrm{d}H_s+\left(\rho_g-\rho_b\widehat{\mathfrak{U}}'_{I-\mathrm{i}}(U^{b,c}_s(\Psi_g))\right)\mathrm{d}D^g_s.
\end{aligned}$$

Since the functions $\widehat{\mathfrak{U}}_{\mathrm{i}}$ solve the system of HJB equations (3.4.9), and $\left(\rho_g-\rho_b\widehat{\mathfrak{U}}'_{\mathrm{i}}(U^{b,c}_s(\Psi_g))\right)\mathrm{d}D^g_s\leq$

0 for every s, we have

$$
\begin{aligned}
\ell_v \leq& \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \Big(r\widehat{\mathfrak{U}}_{I-\mathrm{i}}(U_s^{b,c}(\Psi_g)) - rU_s^g(\Psi_g) - [h_s^{1,g} + (1-\theta_s^g)h_s^{2,g}]\lambda_s^{k^\star,g(\Psi_g)} \Big) \mathrm{d}s \\
&- \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \lambda_s^{k^\star,g(\Psi_g)} \Big(\theta_s \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-\mathrm{i}}(U_s^{b,c}(\Psi_g)) \Big) \mathrm{d}s \\
&+ \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \Big(h_s^{1,g} + \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-\mathrm{i}}(U_{s^-}^{b,c}(\Psi_g)) \Big) \mathrm{d}N_s \\
&+ \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \Big(h_s^{2,g} - \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) \Big) \mathrm{d}H_s \\
=& \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} (r+\lambda_s^{k^\star,g})(\widehat{\mathfrak{U}}_{I-\mathrm{i}}(U_s^{b,c}(\Psi_g)) - U_s^g(\Psi_g)) + (h_s^{2,g} - \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_s^{b,c}(\Psi_g) - h_s^{1,b,c}))\theta_s^g \lambda_s^{k^\star,g} \mathrm{d}s \\
&+ \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \Big(h_s^{1,g} + \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) - \widehat{\mathfrak{U}}_{I-\mathrm{i}}(U_{s^-}^{b,c}(\Psi_g)) \Big) \mathrm{d}N_s \\
&+ \sum_{\mathrm{i}=N_t}^{I-1} \int_{\tau_{\mathrm{i}}\wedge v}^{\tau_{\mathrm{i}+1}\wedge v} \Big(h_s^{2,g} - \widehat{\mathfrak{U}}_{I-\mathrm{i}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}) \Big) \mathrm{d}H_s.
\end{aligned}
$$

Recall from Remark 3.4.2 that on the upper boundary, we have

$$
h_s^{1,g} = \widehat{\mathcal{U}}_{I-N_{s^-}}(U_{s^-}^{b,c}(\Psi_g)) - \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}(\Psi_g)),\ h_s^{2,g} = \widehat{\mathcal{U}}_{I-N_{s^-}-1}(U_{s^-}^{b,c}(\Psi_g) - h_s^{1,b,c}(\Psi_g)),
$$

so that for $\mathrm{i} = N_t$ the drift of the right–hand side is 0 in $[\tau_{\mathrm{i}}, \tau_{\mathrm{i}+1})$ and the jump at time $\tau_{\mathrm{i}+1}$ is also 0. It is easy to see that the same happens for every $\mathrm{i} \in \{N_t, \dots, I\}$ and therefore $\ell_v \leq 0$ for every $v \geq 0$ which means $\ell_v = 0$ for every $v \geq t$. □

We go on with the

Proof. [Proof of Proposition 3.5.4]

(i) We have from the proof of Proposition 3.5.3 that the processes $(\theta^g, h^{1,b,c}, h^{2,b,c})$ are necessarily maximisers of the system of HJB equations (3.4.9). We can go back to the proof of Proposition 3.4.2, which is based on Corollary 3.6.1, to observe that for $u^{b,c} < \widehat{b}_j$ the optimal $\theta \in C^j$ is uniquely given by $\theta = 0$.

(ii) Observe that for every $(t, u^{b,c}, u^g) \in [0,\tau] \times \widehat{\mathcal{V}}_{I-N_t} \times \widehat{\mathcal{V}}_{I-N_t}$ and $\Psi_g \in \widehat{\mathcal{A}}^g(t, u^g, u^{b,c})$ we have

$$
\begin{aligned}
U_t^{b,c}(\Psi_g) &\geq \mathbb{E}^{\mathbb{P}^{k^\star,g(\Psi_g)}} \left[\int_t^\tau \mathrm{e}^{-r(s-t)} (\rho_b \mathrm{d}D_s^g + Bk_s^{\star,g}(\Psi_g)\mathrm{d}s) \middle| \mathcal{G}_t \right] \\
&= \frac{\rho_b}{\rho_g} U_t^g(\Psi_g) + \mathbb{E}^{\mathbb{P}^{k^\star,g(\Psi_g)}} \left[\int_t^\tau \mathrm{e}^{-r(s-t)} Bk_s^{\star,g}(\Psi_g)\mathrm{d}s \middle| \mathcal{G}_t \right] \left(1 - \frac{\rho_b}{\rho_g}\right) \geq \frac{\rho_b}{\rho_g} U_t^g(\Psi_g).
\end{aligned}
$$

Then $U^{b,c}_{s_0}(\Psi_g) = \frac{\rho_g}{\rho_b} U^g_{s_0}(\Psi_g)$ implies that $k^{\star,g}_s(\Psi_g) = k^{\star,b,c}_s(\Psi_g) = 0$, for every $s \in [s_0, \tau)$, and in consequence

$$U^{b,c}_s(\Psi_g) = \frac{\rho_g}{\rho_b} U^g_s(\Psi_g) \geq b_s, \text{ for every } s \in [s_0, \tau).$$

□

We end this section with the

Proof. [Proof of Proposition 3.5.5] We divide the proof in 2 steps.

- **Step 1:** We start with the region $u^{b,c} > \widehat{b}_{I-N_t}$. Let $\Psi_g = (D^g, \theta^g, h^{1,b,c}, h^{2,b,c}) \in \overline{\mathcal{A}}^g(t, u^{b,c})$ be such that $U^{b,c}_t(\Psi_g) = u^{b,c} \geq \widehat{b}_{I-N_t}$, $U^g_t(\Psi_g) = \widehat{\mathfrak{U}}_{I-N_t}(u^{b,c})$. From Proposition 3.5.4 we know that

$$U^{b,c}_s(\Psi_g) \geq \widehat{b}_{I-N_s},\ k^{\star,b,c}(\Psi_g) = 0,\ s \in [t, \tau).$$

Therefore, Problem (3.5.5) is equivalent to

$$V^{\mathfrak{U},g}_t(u^{b,c}) = \sup_{\Psi_g \in \overline{\mathcal{A}}^g(t,u^{b,c})} \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau \mu(I - N_s)\mathrm{d}s - \int_t^\tau \mathrm{d}D^g_s\right], \text{ s.t} \begin{cases} U^{b,c}_s(\Psi_g) \geq \widehat{b}_{I-N_s}, s \in [t, \tau), \\ \mathbb{E}^{\mathbb{P}^0}\left[\int_t^\tau \mathrm{e}^{-r(s-t)}\mathrm{d}D^g_s\right] = \dfrac{u^{b,c}}{\rho_b}. \end{cases}$$

This is exactly the problem of [85], recalled in Section 3.3.2, so we conclude that $V^{\mathfrak{U},g}_t(u^{b,c}) = v^b_{I-N_t}(u^{b,c})$.

- **Step 2:** For the rest of the upper boundary, observe that the system of HJB equations associated to (3.5.5) is is given by $\widehat{\mathcal{V}}_0 \equiv 0$, and for any $1 \leq j \leq I$

$$\min\left\{ - \sup_{(\theta,h^1,h^2)\in C^{\mathfrak{U},j}} \left\{ \begin{array}{c} \widehat{\mathcal{V}}'_j(u^{b,c}) \left(ru^{b,c} - Bk^{b,c} + [h^1 + (1-\theta)h^2]\widehat{\lambda}^{k^{b,c}}_j\right) \\ +\mu j + \widehat{\lambda}^{k^g}_j \theta \widehat{\mathcal{V}}_{j-1}(u^{b,c} - h^1) - \widehat{\lambda}^{k^g}_j \widehat{\mathcal{V}}_j(u^{b,c}) \end{array} \right\},\ \widehat{\mathcal{V}}'_j(u^{b,c}) + \frac{1}{\rho_b} \right\} = 0, \tag{3.6.9}$$

for every $u^{b,c} \geq \frac{Bj}{r+\widehat{\lambda}^{SH}_j}$, with the boundary condition $\widehat{\mathcal{V}}_j(Bj/(r+\widehat{\lambda}^{SH}_j)) = \mu j/\widehat{\lambda}^{SH}_j$, and where

$$k^{b,c} := j\mathbf{1}_{\{h^1+(1-\theta)h^2<\widehat{b}_j\}},\ k^g := j\mathbf{1}_{\{\widehat{\mathcal{U}}^\star_j(u^{b,c})-\theta\widehat{\mathcal{U}}^\star_{j-1}(u^{b,c}-h^1)<\widehat{b}_j\}},$$

and the set of constraints $C^{\mathfrak{U},j}$ determined by Proposition 3.5.4 is defined by

$$C^{\mathfrak{U},j} := \left\{(\theta, h^1, h^2) \in [0,1]\times\mathbb{R}^2_+, h^1+h^2 = u^{b,c}, h^2 \geq \frac{B(j-1)}{r+\widehat{\lambda}^{SH}_{j-1}}, \theta\mathbf{1}_{\{u^{b,c}<\widehat{b}_j\}} = (k^{b,c}+k^g)\mathbf{1}_{\{u^{b,c}\geq\widehat{b}_j\}} = 0\right\}.$$

Then, for any $u^{b,c} < \widehat{b}_j$, the diffusion equation in (3.6.9) reduces to the ODE

$$0 = \widehat{\mathcal{V}}'_j(u^{b,c}) \left(\left(r + \widehat{\lambda}^{SH}_j\right) u^{b,c} - Bj\right) - \widehat{\mathcal{V}}_j(u^{b,c})\widehat{\lambda}^{k^g}_j + \mu j, \tag{3.6.10}$$

with the boundary condition $\widehat{\mathcal{V}}_j\left(\frac{Bj}{r+\widehat{\lambda}^{SH}_j}\right) = \frac{\mu j}{\widehat{\lambda}^{SH}_j}$. If $u^{b,c} < x^\star_j$, we get that

$$\widehat{\mathcal{V}}_j(u^{b,c}) = \frac{\mu j}{\widehat{\lambda}^{SH}_j} + C_1\left(\left(\frac{r+\widehat{\lambda}^{SH}_j}{\widehat{\lambda}^{SH}_j}\right)u^{b,c} - \frac{Bj}{\widehat{\lambda}^{SH}_j}\right)^{\frac{\widehat{\lambda}^{SH}_j}{r+\widehat{\lambda}^{SH}_j}},$$

for some $C_1 \in \mathbb{R}$. If $u^{b,c} \in \left[x_j^\star, \widehat{b}_j\right)$, equation (3.6.10) is solved by

$$\widehat{\mathcal{V}}_j(u^{b,c}) = \frac{\mu j}{\widehat{\lambda}_j^0} + C_2 \left(\left(\frac{r + \widehat{\lambda}_j^{SH}}{\widehat{\lambda}_j^0} \right) u^{b,c} - \frac{Bj}{\widehat{\lambda}_j^0} \right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}},$$

for some $C_2 \in \mathbb{R}$. The values of C_1 and C_2 for which the solution of equation (3.6.10) is continuous are

$$C_1 = \frac{\frac{\mu j}{\widehat{\lambda}_j^0} - \frac{\mu j}{\widehat{\lambda}_j^{SH}} + \left(\frac{\rho_b}{\rho_g}\right)^{\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^0}} \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0} \right)}{\left(\frac{\rho_b}{\rho_g}\right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^0}} \left(\frac{\widehat{b}_j(r+\widehat{\lambda}_j^0)}{\widehat{\lambda}_j^{SH}} \right)^{\frac{\widehat{\lambda}_j^{SH}}{r+\widehat{\lambda}_j^{SH}}}}, \; C_2 = \left(v_j^b(\widehat{b}_j) - \frac{\mu j}{\widehat{\lambda}_j^0} \right) \left(\widehat{b}_j \frac{r + \widehat{\lambda}_j^0}{\widehat{\lambda}_j^0} \right)^{-\frac{\widehat{\lambda}_j^0}{r+\widehat{\lambda}_j^{SH}}}.$$

It follows from the properties of the map v_j^b, that the resulting function $\widehat{\mathcal{V}}_j$ is a concave map with slope greater than $-\frac{1}{\rho_b}$ and therefore the family $\{\widehat{\mathcal{V}}_j\}_{1 \le j \le I}$ is a solution of the system of HJB equations (3.6.9). It can be proved similarly as in the proof of Theorem 3.4.1 (see also Theorem 3.15 in [85]), that the verification result holds for this family of functions. We therefore omit the proof of this result. □

3.6.8 Extension: unbounded relationship between utilities of the banks

One possible extension of our model could rely on a further differentiation between the work of the two banks, *i.e.* when both banks work, the good one would be more efficient in the sense that the associated default intensity is strictly smaller than that of the bad bank. We can do this by introducing an extra type variable with values m_g and m_b, with $m_g < m_b$ and modelling the hazard rate of a non-defaulted loan i at time t, when it is monitored by a bank of type j as $\alpha_t^{\mathrm{i},j} = \alpha_{I-N_t}(1 + \mathrm{e}_t^{\mathrm{i},j} m_j + (1 - \mathrm{e}_t^{\mathrm{i},j})\varepsilon)$. Then, if the banks fails to monitor k loans, the default intensity will be

$$\lambda_t^{k,j} = \alpha_{I-N_t}((I - N_t)(1 + m_j) + (\varepsilon - m_j)k_t).$$

We did not consider such a situation because it creates a degeneracy, in the sense that the credible set no longer has an upper boundary. Indeed, consider for simplicity the case $j = 1$ and take any $u_0^b \ge b_1^j$, $t^\star \ge 0$ and choose the corresponding payment

$$c(t^\star) := u_0^b \frac{\mathrm{e}^{(r+\widehat{\lambda}_1^{0,b})t^\star}(r + \widehat{\lambda}_1^{0,b})}{\rho_b} \ge \frac{b_1^b(r + \widehat{\lambda}_1^{0,b})}{\rho_b} \ge \frac{b_1^g(r + \widehat{\lambda}_1^{0,g})}{\rho_g}.$$

Then, under the contract with delay and constant payments given by $\mathrm{d}D_s = c(t^\star)1_{\{s>t^\star\}}\mathrm{d}s$ the bad bank will always work and her value function will be equal to u_0^b (see section 3.6.4.1). Notice that the optimal strategy for the good bank will be also to work at every time. Then, her value function is equal to

$$u_0^g := u_0^b \frac{\rho_g(r + \widehat{\lambda}_1^{0,b})}{\rho_b(r + \widehat{\lambda}_1^{0,g})} \mathrm{e}^{(\widehat{\lambda}_1^{0,b} - \widehat{\lambda}_1^{0,g})t^\star}.$$

We see that by increasing $t^\star$, it is possible to make u_0^g as big as we want and keep fixed the value of the bad bank. This means that the credible set will have no upper boundary in the interval $[b_1^b, \infty)$. Moving to any $j > 1$ and considering short-term contracts with delay, with $\theta = 0$ and the analogous payments, we observe the same degeneracy and the credible set will have no upper boundary in the interval $[b_j^b, \infty)$.

One way out of this problem would be to consider different discount rates for the banks, r_b and r_g, and assume that the default intensities are such that $\lambda_t^{0,b} + r_b \leq \lambda_t^{0,g} + r_g$. However, this complicates things a lot because simple statements that we expect to be true are very difficult to prove or need assumptions on the parameters of the problem. For example the inequality $U_t^g(D, \theta) \geq U_t^b(D, \theta)$ is no longer clear at all. We therefore refrained from going into that direction, and leave it for potential future research.

Part II

Contributions to the continuous-time model.

Chapter 4

Moral hazard under volatility uncertainty

4.1 Introduction

Invented by the US army and used afterwards in the business glossary, the acronym VUCA reflects through four concepts to deal with a challenging problem: **V**olatility, **U**ncertainty, **C**omplexity and **A**mbiguity. A nice article[1] of Bennett and Lemoine aims at defining all the situations and issues characterized by these concepts. The notion of **V**olatility is at the heart of mathematical finance, by reflecting unstable properties of financial products, such as prices, which are often modelled through the presence of noise in their dynamics. **U**ncertainty can be viewed as a lack of information for an active Agent with information asymmetry between what this Agent observes and the intrinsic effects of her observations. **C**omplexity holds in general when several interconnected entities interact, leading to issues whose solutions are not obvious at first sight. These three first concepts typically provide the difficulty of general problems appearing in contract theory with moral hazard.

The study developed in this chapter is at the heart of a VUCA model by focusing on the last concept, the **A**mbiguity, which can be seen as a pattern of uncertainty by making unknown events fully unclear and not controlled. This last concept express itself in our model with the introduction of a third player in the Principal-Agent relationship, named the *Nature*, which randomly modifies the volatility of the output without any control on it. The present work is an extension of the models proposed by Mastrolia and Possamaï [64] and Sung [125] to a more general framework. The Agent is hired by the Principal to manage a project, by controlling the drift of an output process, which is a diffusion representing generally the value of the firm. The Principal can not observe the actions of the Agent, so moral hazard is present. Moreover, the Principal and the Agent are not informed about the volatility of the project and they just have some beliefs about it. Since we work under weak formulation, the uncertainty on the volatility is represented by assigning to the Principal and the Agent

[1] *What VUCA Really Means for You,* N. Bennett and GJ. Lemoine, *Harvard Business Review*, January-February 2014.

different sets of probability measures under which they make their decisions. Moreover, both individuals present an extreme ambiguity aversion and they have a worst case approach to the problem. They act as if a third individual, the Nature, was playing against them and choosing the worst possible volatility. As a consequence, both the Principal and the Agent play a stochastic differential game against the Nature.

Cvitanić, Possamaï and Touzi [28] provided a general method for solving continuous-time Principal-Agent problems in finite horizon when the Agent is in charge of controlling both the drift and the volatility of the outcome. Their so–called dynamic programming approach, consists in restricting the set of contracts offered to the Agent to a convenient class, in which the problem of the Principal is reduced to a standard stochastic control problem and can be associated to a Hamilton-Jacobi-Bellman equation. The authors proved that their restriction of the class of contracts is without loss of generality by making use of the theory of second-order BSDEs. It is well known that when the Agent controls only the drift of a diffusion process, the value function of his utility maximization problem corresponds to the solution of a BSDE (we refer to the works of Rouge and El Karoui [106], Hu, Imkeller and Müller [54], Hamadéne, Lepeltier and Peng [51] and references therein for more details). Therefore, when the Agent takes also supremum over the possible volatilities his value function is the solution to a 2BSDE. The restricted class of contracts in [28] is chosen so conveniently, that the value function of the Agent under these contracts is the solution of a BSDE instead. Thus, the main difference between the unrestricted and the restricted class of contracts lies in the absolutely continuity of the increasing process appearing in the 2BSDE. The proof consists then in constructing absolutely continuous approximations of the increasing process and showing that the restricted and the unrestricted problems have the same value.

In our problem, we follow partially the dynamic programming approach introduced in [28]. The Agent does not choose the volatility of the outcome process but his worst case approach makes his value function be the solution to a 2BSDE, since it can be identified with an infimum of BSDEs. For the problem of the Principal, the same kind of proof as in [28] does not work because we deal with a non-standard stochastic differential game instead. However, we write the Hamilton-Jacobi-Bellman-Isaacs equation associated to the class of restricted contracts and we prove that the value function of the Principal is a viscosity solution to it by following the stochastic Perron's method of Bayraktar and Sîrbu [11, 12, 13, 115]. The Perron's method amounts to a verification result and it consists in proving that the value function of the Principal lies between a viscosity super-solution (the supremum of the stochastic sub-solutions) and a viscosity sub-solution (the infimum of the stochastic super-solutions) of the HJBI equation. Thus, as soon as a comparison theorem holds for such PDE, and the set of stochastic semi-solutions are non-empty, it follows that the value of the Principal coincides with the unique viscosity solution. Moreover, the dynamic programming principle is obtained as a consequence of the definition of the stochastic semi-solutions. This is another contribution with respect to [64], where the DPP is assumed.

An interesting point to mention is that the problem degenerates if the sets of beliefs of the Principal and the Agent are disjoint. In this case the Principal can obtain the highest possible expected utility by offering contracts which penalize the Agent extremely hard on the support of the beliefs of the Principal and on the support of the beliefs of the Agent his reservation utility is met.

4.2 Preliminares

4.2.1 Canonical process, semi-martingale measure and quadratic variation

For $T > 0$ and a positive integer d, let $\Omega := \{\omega \in C\left([0,T], \mathbb{R}^{\mathrm{d}}\right) : \omega_0 = 0\}$ be the canonical space of continuous maps from $[0,T]$ into $\mathbb{R}^{\mathrm{d}}$, endowed with the uniform norm

$$\|\omega\|_\infty = \sup_{t\in[0,T]} \|\omega_t\|.$$

We denote by X the canonical process on Ω, *i.e.* $X_t(x) = x_t$, for all $x \in \Omega$ and $t \in [0,T]$. We set $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by X and $\mathbb{F}^+ := \mathcal{F}_t^+, 0 \le t \le T$, the right limit of $\mathbb{F}$, where $\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s$ for $s \in [0,T)$ and $\mathcal{F}_T^+ := \mathcal{F}_T$. We denote by $\mathbb{P}_0$ the Wiener measure on $(\Omega, \mathcal{F}_T)$. Let $\mathbf{M}(\Omega)$ be the set of all probability measures on $(\Omega, \mathcal{F}_T)$. Recall the so–called universal filtration $\mathbb{F}^\star := \{\mathcal{F}_t^\star\}_{0\le t\le T}$ defined as follows

$$\mathcal{F}_t^\star := \bigcap_{\mathbb{P}\in\mathbf{M}(\Omega)} \mathcal{F}_t^{\mathbb{P}},$$

where $\mathcal{F}_t^{\mathbb{P}}$ is the usual completion under $\mathbb{P}$.

For any subset $\mathcal{P} \subset \mathbf{M}(\Omega)$, a $\mathcal{P}-$polar set is a $\mathbb{P}-$negligible set for all $\mathbb{P} \in \mathcal{P}$, and we say that a property holds $\mathcal{P}-$quasi–surely if it holds outside some $\mathcal{P}-$polar set. We introduce the following filtration $\mathbb{G}^{\mathcal{P}} := \{\mathcal{G}_t^{\mathcal{P}}\}_{0\le t\le T}$ which will be useful in the sequel

$$\mathcal{G}_t^{\mathcal{P}} := \mathcal{F}_t^\star \vee \mathcal{T}^{\mathcal{P}},\ t \le T,$$

where $\mathcal{T}^{\mathcal{P}}$ is the collection of $\mathcal{P}-$polar sets, and its right-continuous limit, denoted $\mathbb{G}^{\mathcal{P},+}$.

For any subset $\mathcal{P} \subset \mathbf{M}(\Omega)$ and any $(t,\mathbb{P}) \in [0,T] \times \mathcal{P}$ we denote

$$\mathcal{P}[\mathbb{P}, \mathbb{F}^+, t] := \left\{\mathbb{P}' \in \mathcal{P},\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\right\}.$$

We also recall that for every probability measure $\mathbb{P}$ on Ω and $\mathbb{F}-$stopping time τ taking values in $[0,T]$, there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}_x^\tau)_{x\in\Omega}$ (see e.g. Stroock and Varadhan [121]), satisfing Properties (i) $-$ (iv) of [93] and we refer to [93, Section 2.1.3] for more details on it.

We set $\mathcal{M}_{\mathrm{d},n}(\mathbb{R})$ the space of matrices with d rows and n columns with real entries. We define a semi-martingale measure on $(\Omega, \mathcal{F}_T)$ as any probability measure $\mathbb{P}$ such that X is a semi-martingale under $\mathbb{P}$. We denote by $\mathcal{P}_S$ the set of all semi-martingale measures. It is well-known, see for instance the result of [58], that there exists an $\mathbb{F}$-progressively measurable process denoted by $\langle X\rangle := (\langle X\rangle_t)_{t\in[0,T]}$ coinciding with the quadratic variation of X, $\mathbb{P}-a.s.$ for any $\mathbb{P}\in\mathcal{P}_S$. We introduce the non-negative symmetric matrix $\widehat{\sigma}_t\in\mathcal{M}_{\mathrm{d,d}}(\mathbb{R})$ defined by

$$\widehat{\sigma}_t := \limsup_{\substack{\varepsilon\longrightarrow 0\\ \varepsilon>0}} \frac{\langle X\rangle_t - \langle X\rangle_{t-\varepsilon}}{\varepsilon}.$$

The formal definition of all the functional spaces mentioned in this chapter can be found in Appendix 4.7.1.

4.2.2 Weak formulation of the output process

We start defining $\mathfrak{A}$ and $\mathfrak{N}$ as the sets of $\mathbb{F}$-adapted processes taking values in A and N respectively, where A, N are compact subsets of some finite dimensional space. We will call control process to every pair $(\alpha,\nu)\in\mathfrak{A}\times\mathfrak{N}$. To clarify the notations for the rest of the chapter, α has to be understood as the control of the Agent and ν as the control of the Nature. Consider next the volatility coefficient for the controlled process

$$\sigma : [0,T]\times\Omega\times N \longrightarrow \mathcal{M}_{\mathrm{d},n}(\mathbb{R}),$$

which is assumed to be uniformly bounded and such that $\sigma\sigma^\top(\cdot,n)$ is an invertible $\mathbb{F}$-progressively measurable process for any $n\in N$. For every $(t,x)\in[0,T]\times\Omega$ and $\nu\in\mathfrak{N}$, we set the following SDE driven by an n-dimensional Brownian motion W

$$\begin{aligned} X_s^{t,x,\nu} &= x(t) + \int_t^s \sigma(r, X^{t,x,\nu}, \nu_r)\mathrm{d}W_r,\ s\in[t,T], \\ X_r^{t,x,\nu} &= x(r),\ r\in[0,t]. \end{aligned} \tag{4.2.1}$$

We present now the definition of a weak solution to SDE (4.2.1).

Definition 4.2.1 Given $(t,x)\in[0,T]\times\Omega$, a weak solution to SDE (4.2.1) is a pair $(\mathbb{P},\nu)\in\mathbf{M}(\Omega)\times\mathfrak{N}$ such that

- $X_{\cdot\wedge t} = x_{\cdot\wedge t}$, $\mathbb{P}-a.s.$
- The processes

  $$X_\cdot \text{ and } X_\cdot X_\cdot^\top - \int_t^\cdot \sigma(r,X,\nu_r)\sigma^\top(r,X,\nu_r)\mathrm{d}r,$$

  are $(\mathbb{P},\mathbb{F})$-martingales on $[t,T]$.

Recall, for instance from [121, Theorem 4.5.2], that for any weak solution $(\mathbb{P},\nu)$, there exists a $\mathbb{P}-$Brownian motion, denoted by $W^{\mathbb{P}}$, such that

$$X_s = x(t) + \int_t^s \sigma(r,X,\nu_r)\mathrm{d}W_r^{\mathbb{P}},\ s\in[t,T],\ \mathbb{P}-a.s. \tag{4.2.2}$$

We will denote by $\mathcal{N}(t,x)$ the set of weak solutions to SDE (4.2.1). We also define the set $\mathcal{P}(t,x)$ of probability measures which are components of weak solutions by

$$\mathcal{P}(t,x) := \bigcup_{\nu \in \mathfrak{N}} \mathcal{P}^\nu(t,x), \text{ where } \mathcal{P}^\nu(t,x) := \{\mathbb{P} \in \mathbf{M}(\Omega),\ (\mathbb{P},\nu) \in \mathcal{N}(t,x)\}.$$

We conclude this section by showing that the sets $\mathcal{P}(t,x)$ satisfy an important property which is essential to deal with the wellposedness of 2BSDEs, the main tool we will use later to solve the problem of the Agent. We recall first the definition of a saturated set of probability measures (see [93, Definition 5.1]).

Definition 4.2.2 A set $\mathcal{P} \subset \mathbf{M}(\Omega)$ is said to be saturated if for every $\mathbb{P} \in \mathcal{P}$, any probability $\mathbb{Q} \in \mathbf{M}(\Omega)$ which is equivalent to $\mathbb{P}$ and under which X is a local martingale, belongs to $\mathcal{P}$.

We thus have the following Lemma, whose proof follows the same lines that [28, Proof of Proposition 5.3, step (i)]

Lemma 4.2.1 The family $\{\mathcal{P}(t,x),\ (t,x) \in [0,T] \times \Omega\}$ is saturated.

4.2.3 Estimate sets of volatility

The beliefs of the Agent and the Principal about the volatility of the project will be summed up in the families of measures $(\mathcal{P}_A(t,x))_{(t,x)\in[0,T]\times\Omega}$ and $(\mathcal{P}_P(t,x))_{(t,x)\in[0,T]\times\Omega}$ respectively, which satisfy that $\mathcal{P}_A(t,x) \cup \mathcal{P}_P(t,x) \subset \mathcal{P}(t,x)$ for every $(t,x) \in [0,T] \times \Omega$. We emphasize that the families $\mathcal{P}_A$ and $\mathcal{P}_P$ cannot be chosen completely arbitrarily, and have to satisfy a certain number of stability and measurability properties, classical in stochastic control theory, for allowing us to use the theory of 2BSDEs developed in [93]. The following assumption guarantees the well-posedness of 2BSDEs defined in the set of beliefs of the Principal and the Agent.

Assumption 4.2.1 For $\Psi = A, P$, we have

(i) For every $(t,x) \in [0,T] \times \Omega$, one has $\mathcal{P}_\Psi(t,x) = \mathcal{P}_\Psi(t, x_{\cdot\wedge t})$ and $\mathbb{P}(\Omega_t^x) = 1$ whenever $\mathbb{P} \in \mathcal{P}_\Psi(t,x)$. The graph $[[\mathcal{P}_\Psi]]$ of $\mathcal{P}_\Psi$, defined by $[[\mathcal{P}_\Psi]] := \{(t,x,\mathbb{P}) : \mathbb{P} \in \mathcal{P}_\Psi(t,x)\}$, is upper semi–analytic in $[0,T] \times \Omega \times \mathbf{M}(\Omega)$.

(ii) $\mathcal{P}_\Psi$ is stable under conditioning, *i.e.* for every $(t,x) \in [0,T] \times \Omega$ and every $\mathbb{P} \in \mathcal{P}_\Psi(t,x)$ together with an $\mathbb{F}-$stopping time τ taking values in $[t,T]$, there is a family of r.c.p.d. $(\mathbb{P}_x)_{x\in\Omega}$ such that $\mathbb{P}_x \in \mathcal{P}_\Psi(\tau(x),x)$, for $\mathbb{P}-a$.e. $x \in \Omega$.

(iii) $\mathcal{P}_\Psi$ is stable under concatenation, *i.e.* for every $(t,x) \in [0,T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}_\Psi(t,x)$ together with a $\mathbb{F}-$stopping time τ taking values in $[t,T]$, let $(\mathbb{Q}_x)_{x\in\Omega}$ be a family of probability measures such that $\mathbb{Q}_x \in \mathcal{P}_\Psi(\tau(x),x)$ for all $x \in \Omega$ and $x \longmapsto \mathbb{Q}_x$ is $\mathcal{F}_\tau-$measurable, then the concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{P}_\Psi(t,x)$.

Notice that under property (i) the sets $\mathcal{P}_A(t,x)$ and $\mathcal{P}_P(t,x)$ at time $t=0$ are independent of x. Define $\mathcal{P}_A := \mathcal{P}_A(0,x)$, $\mathcal{P}_P := \mathcal{P}_P(0,x)$ for every $x \in \Omega$, and consider also

(iv) The set $\mathcal{P}_\Psi$ is saturated.

An example of estimate sets of volatility which satisfy Assumption 4.2.1 is the learning model presented in Mastrolia and Possamaï [64].

Example 4.2.1 Consider, for $\Psi = A, P$, set–valued processes $\mathbf{D}_\Psi : [0,T] \times \Omega \longmapsto 2^{\mathbb{R}_+^\star}$ such that for every $t \in [0,T]$

$$\left\{(s,\omega,A) \in [0,t] \times \Omega \times \mathbb{R}_+^\star,\ A \in \mathbf{D}_\Psi(s,\omega)\right\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^\star),$$

where $\mathcal{B}([0,t])$ and $\mathcal{B}(\mathbb{R}_+^\star)$ denote the Borel $\sigma-$algebra of $[0,t]$ and $\mathbb{R}_+^\star$ respectively. Define next, for every $(t,w) \in [0,T]\times\Omega$, the set $\mathcal{P}_\Psi(t,\omega)$ as the set of probability measures $\mathbb{P} \in \mathbf{M}(\Omega)$ such that

$$\widehat{\sigma}_s(w') \in \mathbf{D}_\Psi(s+t, \omega \otimes_t w'),\ \text{for } \mathrm{d}s \otimes \mathrm{d}\mathbb{P} - a.\text{e. } (s,w') \in [0, T-t] \times \Omega.$$

It is shown by Nutz and van Handel [82] that the sets $\mathbf{P}_\Psi(t,\omega)$ satisfy Assumption 4.2.1.

In the context of the previous example, Mastrolia and Possamaï [64] study the case where

$$\mathbf{D}_A(t,\omega) = [\underline{\sigma}_t^A(\omega), \overline{\sigma}_t^A(\omega)],\ \mathbf{D}_P(t,\omega) = [\underline{\sigma}_t^P(\omega), \overline{\sigma}_t^P(\omega)],$$

for certain processes $(\underline{\sigma}^P, \underline{\sigma}^A, \overline{\sigma}^P, \overline{\sigma}^A) \in \left(\mathbb{H}^0(\mathbb{R}_+^*, \mathbb{F})\right)^4$. We refer to their paper for an interpretation of such model. To conclude this section, we define the set of weak solutions to the SDE (4.2.1) associated to the beliefs of the Principal of the Agent

$$\mathcal{N}_A(t,x) = \{(\mathbb{P},\nu) \in \mathcal{N}(t,x) : \mathbb{P} \in \mathcal{P}_A(t,x)\},\ \mathcal{N}_P(t,x) = \{(\mathbb{P},\nu) \in \mathcal{N}(t,x) : \mathbb{P} \in \mathcal{P}_P(t,x)\}.$$

We define the sets $\mathcal{N}_A$ and $\mathcal{N}_P$ equivalently. The importance of these sets is that, as explained in the next section, both the Principal and the Agent consider that the volatility of the outcome process is chosen from one of them, according to their beliefs.

4.3 The contracting problem

We study a generalization of both the classical problem of Holmström and Milgrom [53] and the problem of moral hazard under volatility uncertainty studied in [64, 125]. In our model, the Agent is hired by the Principal to control the drift of the outcome process X, but neither side have certainty about what is the volatility of the project. Both individuals have a "worst-case" approach to the contract, in the sense that they act as if a third player, the "Nature", was playing against them by choosing the worst possible volatility.

4.3.1 Admissible efforts

As usual in the literature, we work under the weak formulation of the Principal-Agent problem. Therefore, the set of controls of the Agent is restricted to the ones for which an appropriate change of measure can be applied to the weak solutions to SDE (4.2.1). In this section we precise the condition required on a control to be an admissible effort and the impact of the actions of the Agent in the outcome process.

The Agent exerts an effort $\alpha \in \mathfrak{A}$ to manage the project, unobservable by the Principal, impacting the outcome process through the drift coefficient $b : [0,T] \times \Omega \times A \times N \longrightarrow \mathbb{R}^n$, which satisfies that $b(\cdot, a, n)$ is an $\mathbb{F}$-progressively measurable process for every $(a,n) \in A \times N$. The actions of the Agent are costly for him, so his benefits are penalized by a cost function $c : [0,T] \times \Omega \times A \longrightarrow \mathbb{R}$ such that for every $a \in A$, $c(\cdot, a)$ is an $\mathbb{F}-$progressively measurable process. We assume that for some $p > 1$ there exists $\kappa \in (1,p]$ such that

$$\sup_{\mathbb{P}\in\mathcal{P}_A} \mathbb{E}^{\mathbb{P}}\left[\operatorname{ess\,sup}^{\mathbb{P}}_{0\leq t\leq T} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^T \sup_{a\in A}|c(s,X,a)|^{\kappa}\mathrm{d}s\right)^{\frac{p}{\kappa}}\Big|\mathcal{F}_t^+\right]\right] < +\infty. \tag{4.3.1}$$

The Agent discounts the future through a map $k : [0,T] \times \Omega \times A \times N \longrightarrow \mathbb{R}$, such that $k(\cdot, a, n)$ is an $\mathbb{F}-$progressively measurable process for every $(a,n) \in A \times N$. For some $(\ell, m, \underline{m}) \in [1,+\infty) \times [\ell, +\infty) \times (0, \ell + m - 1]$, we impose the following conditions on the maps b, c and k.

Assumption 4.3.1 ($\mathbf{H}^{\ell,m,\underline{m}}$) There exists $0 < \underline{\kappa} < \kappa$ such that for any $(t,x,a,\eta) \in [0,T] \times \Omega \times A \times N$

(i) The drift b satisfies

$$\|b(t,x,a,\eta)\| \leq \kappa\left(1 + \|x\|_{t,+\infty} + \|a\|^{\ell}\right), \ \|\partial_a b(t,x,a,\eta)\| \leq \kappa\left(1 + \|a\|^{\ell-1}\right).$$

(ii) The map $a \longmapsto c(t,x,a)$ is increasing, strictly convex and continuously differentiable for any $(t,x) \in [0,T] \times \Omega$ and satisfies

$$0 \leq c(t,x,a) \leq \kappa\left(1 + \|x\|_{t,\infty} + \|a\|^{\ell+m}\right),$$

$$\underline{\kappa}\|a\|^{\underline{m}} \leq \|\partial_a c(t,x,a)\| \leq \kappa\left(1 + \|a\|^{\ell+m-1}\right) \text{ and } \overline{\lim}_{|a|\to\infty}\frac{c(t,x,a)}{\|a\|^{\ell}} = +\infty.$$

(iii) The discount factor k is uniformly bounded by κ.

Remark 4.3.1 Observe that for $(\ell, m, \underline{m}) = (1,1,1)$ the model studied in [64] is recovered.

We present finally the definition of admissible efforts of the Agent.

Definition 4.3.1 (Admissible efforts) A control process $\alpha \in \mathfrak{A}$ is said to be admissible, if for every $(\mathbb{P}, \nu) \in \mathcal{N}_A$ the following process is an $(\mathbb{F}, \mathbb{P})$-martingale

$$\left(\mathcal{E}\left(\int_0^t \sigma^\top(\sigma\sigma^\top)^{-1}(s, X, \nu_s) b(s, X, \alpha_s, \nu_s) \cdot \mathrm{d}W_s^{\mathbb{P}}\right)\right)_{t\in[0,T]}.$$

We denote by $\mathcal{A}$ the set of admissible efforts.

To conclude, we present the impact of the actions of the Agent in the outcome process. Consider an admissible effort $\alpha \in \mathcal{A}$ and $(t, x) \in [0, T] \times \Omega$. For every subset $\mathcal{N} \subset \mathcal{N}(t, x)$ define

$$\mathcal{N}^\alpha := \left\{(\mathbb{P}^\alpha, \nu), \frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_t^T \sigma^\top(\sigma\sigma^\top)^{-1}(s, X, \nu_s) b(s, X, \alpha_s, \nu_s) \cdot \mathrm{d}W_s^{\mathbb{P}}\right), (\mathbb{P}, \nu) \in \mathcal{N}\right\}.$$

Thus, under Assumption $(\mathbf{H}^{\ell,m,\underline{m}})$, by Girsanov's Theorem we have for any $\alpha \in \mathcal{A}$, and for any $(\mathbb{P}^\alpha, \nu) \in \mathcal{N}^\alpha$

$$X_s = x_t + \int_t^s b(r, X, \alpha_r, \nu_r)\mathrm{d}r + \int_t^s \sigma(r, X, \nu_r)\mathrm{d}W_r^\alpha,\ s \in [t, T],\ \mathbb{P}^\alpha - a.s., \tag{4.3.2}$$

where W^α is a $\mathbb{P}^\alpha-$Brownian motion. More precisely,

$$W^\alpha := W^{\mathbb{P}} - \int_t^\cdot \sigma^\top(\sigma\sigma^\top)^{-1}(r, X, \nu_r) b(r, X, \alpha_r, \nu_r)\mathrm{d}r,$$

for some $\mathbb{P} \in \mathcal{P}$.

4.3.2 Admissible contracts

The Principal offers to the Agent a final salary taking place on the horizon T. Since the Principal can observe merely the outcome process X, a contract corresponds to an $\mathcal{F}_T$-measurable random variable ξ. The Agent benefits from the payments of the Principal through his utility function $U_A : \mathbb{R} \longrightarrow \mathbb{R}$, which depends on his terminal remuneration and is a continuous, increasing and concave map. The Principal benefits from her wealth, penalized by the salary given to the Agent, through her utility function $U_P : \mathbb{R} \longrightarrow \mathbb{R}$ which is a continuous, increasing and concave map. The outcome process is not necessary monetary so the Principal possesses a liquidation function $L : \mathbb{R} \longrightarrow \mathbb{R}$ which is assumed to be continuous with linear growth. The following (classical) notion of admissibility for the set of contracts proposed by the Principal is due to the fact that we will reduce later the problem of the Agent to solve a 2BSDE.

Definition 4.3.2 (Admissible contracts) A contract ξ is called admissible, if

- For some $p > 1$ there exists $\kappa \in [1, p)$ such that $U_A(\xi) \in \mathbb{L}_0^{p,\kappa}(\mathcal{P}_A)$.
- For any $(\mathbb{P}, \nu) \in \mathcal{N}_P$ we have $\mathbb{E}^{\mathbb{P}}\left[U_P\left(L(X_T) - \xi\right)\right] < +\infty$.

We denote by $\mathfrak{C}$ the class of admissible contracts.

4.3.3 The problem of the Agent

For a given contract $\xi \in \mathfrak{C}$ offered by the Principal, the utility of the Agent at time $t = 0$, if he performs the action $\alpha \in \mathcal{A}$, is given by his worst-case approach over the set $\mathcal{N}_A^\alpha$ of weak solutions to (4.2.1) associated to his beliefs. That is

$$u_0^A(\xi, \alpha) := \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}^{\alpha,\nu} U_A(\xi) - \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu} c(s, X, \alpha_s)\mathrm{d}s\right],$$

where

$$\mathcal{K}_{s,t}^{\alpha,\nu} := \exp\left(-\int_s^t k(u, X, \alpha_u, \nu_u)\mathrm{d}u\right),\ 0 \leq s \leq t \leq T.$$

The problem of the Agent, consisting in finding the action which maximizes his utility, is therefore

$$U_0^A(\xi) := \sup_{\alpha\in\mathcal{A}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}}\left[\mathcal{K}_{0,T}^{\alpha,\nu} U_A(\xi) - \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu} c(s, X, \alpha_s)\mathrm{d}s\right]. \tag{4.3.3}$$

We will denote by $\mathcal{A}^\star(\xi)$ the set of optimal $\alpha \in \mathcal{A}$ when ξ is offered, and define the set of optimal weak solutions

$$\mathcal{N}_A^\star(\xi) := \bigcup_{\alpha^\star\in\mathcal{A}^\star(\xi)} \mathcal{N}_A^{\alpha^\star}.$$

4.3.4 The problem of the Principal

Since the strategy of the Principal is to anticipate the response of the Agent to the offered contracts, she is restricted to offer contracts such that the Agent can optimally choose his actions. Moreover, the Agent accepts only contracts under which he obtains more benefits than his reservation utility R_0. All of this implies that the set of admissible contracts is restricted to the following set

$$\Xi := \{\xi \in \mathfrak{C},\ \mathcal{A}^\star(\xi) \neq \emptyset,\ U_0^A(\xi) \geq R_0\}.$$

Notice that for any $\xi \in \Xi$, the set $\mathcal{A}^\star(\xi)$ is not necessarily reduced to a singleton. As is common in the literature, we will assume that when there is more than one optimal strategy for the Agent, he chooses one which is best for the Principal. We denote such a strategy by $\alpha^\star(x, \xi)$. Thus, the problem of the Principal is to solve

$$U_0^p := \sup_{\xi\in\Xi} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_P^{\alpha^\star(x,\xi)}} \mathbb{E}^{\mathbb{P}}\left[U_P\left(L(X_T) - \xi\right)\right]. \tag{4.3.4}$$

Remark 4.3.2 For the sake of simplicity, we do not add any discount factor for the Principal's problem (4.3.4). A model dealing with a discount factor $k^P : [0, T] \times \Omega \longrightarrow \mathbb{R}$ could be easily studied and does not add any difficulties, as soon as k^P is sufficiently integrable, by modifying the HJBI equation (4.5.11) below.

4.4 The Agent's problem: a 2BSDE's story

In this section we study the Agent's problem (4.3.3). We follow both the study made in Section 4.1 of [64] by extending it to a more general framework, and [28] by adding uncertainty on the volatility. We mention also that another approach which does not use the theory of 2BSDEs has been proposed in [125].

4.4.1 Definition of the Hamiltonian

Define the function $F : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times A \times N \longrightarrow \mathbb{R}$ by

$$F(t,x,y,z,\alpha,\nu) := -k(t,x)y - c(t,x,\alpha) + b(t,x,\alpha,\nu) \cdot z.$$

Define also for every $(t,x,\Sigma) \in [0,T] \times \Omega \times \mathcal{S}_{\mathrm{d}}^{+}$ the set

$$V_t(x,\Sigma) := \left\{\nu \in N, \sigma(t,x,\nu)\sigma^{\top}(t,x,\nu) = \Sigma\right\},$$

and denote by $\mathcal{V}(\hat{\sigma}^2)$ the set of controls ν with values in $V_t(x,\hat{\sigma}_t^2)$, $\mathrm{d}t \otimes \mathbb{P}$-a.e. for every $\mathbb{P} \in \mathcal{P}_A$.

The Hamiltonian $H : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times \mathcal{S}_{\mathrm{d}}^{+} \longrightarrow \mathbb{R}$ associated with the problem of the Agent (4.3.3) is defined by (see [20])

$$H(t,x,y,z,\gamma) := \inf_{\Sigma \in \mathcal{S}_{\mathrm{d}}^{+}} \left\{ \frac{1}{2}\mathrm{Tr}(\Sigma\gamma) + \sup_{\alpha \in A} \inf_{\nu \in V_t(x,\Sigma)} F(t,x,y,z,\alpha,\nu) \right\}.$$

Let us define the map $F^{\star} : [0,T] \times \Omega \times \mathbb{R}^{\mathrm{d}+1} \times \mathcal{S}_{\mathrm{d}}^{+} \longrightarrow \mathbb{R}$ by

$$F^{\star}(t,x,y,z,\Sigma) := \sup_{\alpha \in A} \inf_{\nu \in V_t(x,\Sigma)} F(t,x,y,z,\alpha,\nu).$$

We thus state a fundamental lemma on the growth of any control $\alpha^{\star}$ which maximizes $F^{\star}(\cdot)$. We refer to the proof of [42, Lemma 4.1] which fits our setting.

Lemma 4.4.1 Let Assumption $(\mathbf{H}^{\ell,m,\underline{m}})$ hold. Then, for any $(t,x,y,z,\Sigma) \in [0,T] \times \Omega \times \mathbb{R}^{\mathrm{d}+1} \times \mathcal{S}_{\mathrm{d}}^{+}$ and for any maximiser $\alpha^{\star}$ of $F^{\star}(t,x,y,z,\Sigma)$ with $\nu \in V_t(x,\Sigma)$, there exists some positive constant C such that

$$\|\alpha^{\star}(t,x,y,z,\Sigma)\| \leq C\left(1 + \|z\|^{\frac{1}{\underline{m}+1-\ell}}\right),$$

$$|F^{\star}(t,x,y,z,\Sigma)| \leq C\left(1 + \|x\|_{t,\infty} + |y| + \|z\|^{\frac{\ell+m}{\underline{m}+1-\ell}}\right).$$

4.4.2 2BSDEs representation of the Agent's problem

Consider the following 2BSDE

$$Y_t = U_A(\xi) + \int_t^T F^\star(s, X, Y_s, Z_s, \hat{\sigma}_s^2)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}X_s - \int_t^T \mathrm{d}K_s,\ \mathcal{P}_A - q.s. \tag{4.4.1}$$

Recall now the notion of solution to this 2BSDE introduced in [116] and extended in [93].

Definition 4.4.1 We say that a triplet (Y, Z, K) is a solution to the 2BSDE (4.4.1) if there exists $p > 1$ such that $(Y, Z, K) \in \mathbb{S}_0^p(\mathbb{F}_+^{\mathcal{P}_A}, \mathcal{P}_A) \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{P}_A}, \mathcal{P}_A) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{P}_A}, \mathcal{P}_A)$ satisfies (4.4.1) and K satisfies the minimality condition

$$K_t = \underset{\mathbb{P}' \in \mathcal{P}_A[\mathbb{P}, \mathbb{F}^+, t]}{\operatorname{ess\,inf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T \middle| \mathcal{F}_t^{\mathbb{P},+}\right],\ t \in [0, T],\ \mathbb{P} - a.s.,\ \forall \mathbb{P} \in \mathcal{P}_A. \tag{4.4.2}$$

Remark 4.4.1 Similarly to [28], we use here the result of [81] for stochastic integral by considering the aggregative version of the non-decreasing process K.

From now, we set the standing assumption to be used in all the following results

Assumption 4.4.1 (**S**) For some $(\ell, m, \underline{m}) \in [1, +\infty) \times [\ell, +\infty) \times (0, \ell + m - 1]$ with $\frac{\ell+m}{\underline{m}+1-\ell} \leq 2$, Assumption $(\mathbf{H})^{\ell,m,\underline{m}}$ holds together with Assumptions 4.2.1.

We have the following result which ensures that the 2BSDE (4.4.1) is well-posed. Its proof is postponed to the Appendix.

Lemma 4.4.2 Under Assumption (**S**), the 2BSDE (4.4.1) has a unique solution (Y, Z, K) for any ξ in $\mathfrak{C}$.

The next Theorem is the main result of this section and it provides an equivalence between solving the Agent's problem (4.3.3) and the 2BSDE (4.4.1). Its proof is postponed to the Appendix and is similar to the proof of [28, Proposition 5.4], being its extension to the worst-case volatility case.

Theorem 4.4.1 Let Assumption (**S**) hold and denote by (Y, Z, K) the solution to the 2BSDE (4.4.1). Then, the value function of the Agent is given by

$$U_0^A(\xi) = \sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P}, \nu) \in \mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}}[Y_0]. \tag{4.4.3}$$

Moreover, $(\alpha^\star, \mathbb{P}^\star, \nu^\star) \in \mathcal{A}^\star(\xi) \times \mathcal{N}_A^\star(\xi)$ if and only if $(\alpha^\star, \mathbb{P}^\star, \nu^\star) \in \mathcal{A} \times \mathcal{N}_A$ and satisfies

(i) $(\alpha^\star, \nu^\star)$ attains the sup-inf in the definition of $F^\star(\cdot, X, Y, Z, \widehat{\sigma}^2)$, $\mathrm{d}t \otimes \mathbb{P}^\star$−a.e.,

(ii) $K_T = 0$, $\mathbb{P}^\star$−a.s.

To conclude this section, let us comment the intuition behind this result and the limitations of our model.

Remark 4.4.2 If the volatility of the outcome process is fixed and the Agent controls only the drift, it is well-known that his value function is the solution to a BSDE (see section 1.2.1 in the Introduction of the Thesis). The worst-case approach of the Agent makes his value function be the infimum of BSDEs and therefore the solution to a 2BSDE. This reasoning works because the Agent controls only the drift and not the volatility of the outcome. Indeed, by considering a controlled volatility coefficient $\sigma(t, x, \alpha, \nu)$, the worst-case approach of the Agent induces a first 2BSDE and the control α induces a second 2BSDE on top of that. Currently, such kind of 2BSDEs control has not been studied in the literature.

4.5 The Principal's problem

In this section, we aim at solving the contracting problem (4.3.4). This corresponds to an extension of both [28] to the uncontrolled volatility case and [64] in a more general model, without assuming that a dynamic programming principle holds for the value function of the Principal. We follow the ideas of [11, 12, 115].

4.5.1 A pathological stochastic control problem

To facilitate the understanding of this section, we provide a general overview of the method we use, dividing it in the following steps.

Step 1. In Section 4.5.2, we rewrite the set of admissible contracts and the Principal's problem (4.3.4) making use of the results obtained in Section 4.4. We also make a distinction between the case in which the estimation sets of the Principal and the Agent are disjoint and the case in which they are not.

Step 2. In Section 4.5.3, we show that if the beliefs of the Principal and the Agent are disjoints, there is a degeneracy in the sense that the Principal can propose to the Agent a sequence of admissible contracts such that asymptotically she gets her maximal utility.

Step 3. We solve next the problem of the Principal in Section 4.5.4 when the beliefs about the volatility of the Principal and the Agent are not disjoint by restricting the study to piece-wise constant controls and by using Perron's method.

In the following, we suppose that (**S**) and the next assumption are enforced.

Assumption 4.5.1 (Markovian case) All the objects considered are Markovian, *i.e.* they depend on $(t, X_\cdot)$ only through (t, X_t).

Remark 4.5.1 Assumption 4.5.1 may be removed if we deal with the theory of path dependent PDEs (see among others [38, 98]). Here, we assume that it holds for the sake of simplicity and to focus on the procedure to solve the Principal's problem.

4.5.2 Reformulation of the problem and the set of admissible contracts

The solution to the problem of the Agent provides a very particular form for $U_A(\xi)$. Indeed, let (Y, Z, K) be the solution of 2BSDE (4.4.1), then

$$U_A(\xi) = Y_0 - \int_0^T F^\star(s, X_s, Y_s, Z_s, \hat{\sigma}_s^2)\mathrm{d}s + \int_0^T Z_s \cdot \mathrm{d}X_s + \int_0^T \mathrm{d}K_s, \ \mathcal{P}_A - q.s., \tag{4.5.1}$$

the process K satisfies the minimality condition (4.4.2), and

$$\sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}}[Y_0] \geq R_0.$$

Let us define the set of $\mathcal{F}_0-$measurable random variables

$$\mathbb{Y}_0 := \left\{ Y_0, \ \sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}}[Y_0] \geq R_0 \right\}.$$

Then, for any contract $\xi \in \Xi$ there exists a triplet $(Y_0, Z, K) \in \mathbb{Y}_0 \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{N}_A}) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{N}_A})$ such that (4.4.2) and (4.5.1) hold. Since such a triplet is unique, we can establish a one-to-one correspondence between the set of admissible contracts Ξ and an appropriate subset of $\mathbb{Y}_0 \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{N}_A}) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{N}_A})$. However, as explained in [64], decomposition (4.5.1) only holds $\mathcal{P}_A-$quasi surely and we have to take this fact into account in order to provide a suitable characterization of the set of admissible contracts by means of this formula.

For any $(Y_0, Z, K) \in \mathbb{Y}_0 \times \mathbb{H}_0^p(\mathbb{F}^{\mathcal{N}_A}) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{N}_A})$ such that K satisfies (4.4.2) and every $(\mathbb{P}, t) \in \mathcal{P}_A \times [0, T]$, we define the process $Y^{Y_0,Z,K}$ by

$$Y_t^{Y_0,Z,K} := Y_0 - \int_0^t F^\star(s, X_s, Y_s^{Y_0,Z,K}, Z_s, \hat{\sigma}_s^2)\mathrm{d}s + \int_0^t Z_s \cdot \mathrm{d}X_s + \int_0^t \mathrm{d}K_s, \ \mathbb{P} - a.s. \tag{4.5.2}$$

Recall that since k is bounded, $F^\star$ is Lipschitz with respect to y, thus $Y^{Y_0,Z,K}$ is well defined. The definition is independent of the probability $\mathbb{P}$ because the stochastic integrals can be defined pathwise (see [28, Definition 3.2] and the following paragraph).

Fix now $Y_0 \in \mathbb{Y}_0$ and let $\mathcal{K}_{Y_0}$ be the set of pairs $(Z, K) \in \mathbb{H}_0^p(\mathbb{F}^{\mathcal{N}_A}) \times \mathbb{K}_0^p(\mathbb{F}^{\mathcal{N}_A})$ sufficiently integrable such that $U_A^{-1}(Y_T^{Y_0,Z,K}) \in \mathfrak{C}_{\mathcal{P}_A}$ and with K satisfying (4.4.2). The Principal has thus to propose a contract with the form $U_A^{-1}(Y_T^{Y_0,Z,K})$ under every probability measure in the space $\mathcal{P}_A$. Outside of the support of this space, the Principal is completely free on the salary given to the Agent.

We denote by $\mathcal{D}$ the set of $\mathcal{F}_T-$measurable random variables ξ such that

$$\xi = \begin{cases} U_A^{-1}(Y_T^{Y_0,Z,K}), \ \mathcal{P}_A - q.s., \\ \widehat{\xi}, \ \mathcal{P}_P \backslash \mathcal{P}_A - q.s., \end{cases} \tag{4.5.3}$$

for some triplet $(Y_0, Z, K) \in \mathbb{Y}_0 \times \mathcal{K}_{Y_0}$ and some $\widehat{\xi} \in \mathfrak{C}_{\mathcal{P}_P \backslash \mathcal{P}_A}$. The integrability conditions imposed on Z, K and $\widehat{\xi}$ ensure us that $\mathcal{D} \subset \Xi$. In fact, from the reasoning given in the paragraphs above we have that $\mathcal{D}$ coincides with Ξ and (4.5.3) corresponds to a characterization of the set of admissible contracts. Therefore, the problem of the Principal (4.3.4) becomes

$$U_0^P = \sup_{(Y_0,Z,K,\widehat{\xi}) \in \mathbb{Y}_0 \times \mathcal{K}_{Y_0} \times \mathfrak{C}_{\mathcal{P}_P \backslash \mathcal{P}_A}} U_0^P(U_A^{-1}(Y_T^{Y_0,Z,K}), \widehat{\xi}), \tag{4.5.4}$$

with the following slight abuse of notations

$$U_0^P(\mathcal{X}, \widehat{\xi}) := \min \left\{ \inf_{(\mathbb{P},\nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})} \cap \mathcal{N}_A^{\alpha^\star(\mathcal{X})}} \mathbb{E}^{\mathbb{P}} \left[U_P(L(X_T) - \mathcal{X}) \right], \inf_{(\mathbb{P},\nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})} \backslash \mathcal{N}_A^{\alpha^\star(\mathcal{X})}} \mathbb{E}^{\mathbb{P}} \left[U_P(L(X_T) - \widehat{\xi}) \right] \right\}.$$

4.5.3 Degeneracy for disjoint beliefs

Similarly to the study made in [64, Section 4.3.1.], if the believes of the Agent and the Principal are disjoint, we face a pathological case caused by the fact that the Agent and the Principal do not somehow live in the same world. Indeed, if $\mathcal{P}_A \cap \mathcal{P}_P = \emptyset$ we have

$$U_0^P = \sup_{(Y_0,Z,K,\widehat{\xi}) \in \mathbb{Y}_0 \times \mathcal{K}_{Y_0} \times \mathfrak{C}_{\mathcal{P}_P}} \inf_{(\mathbb{P},\nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})}} \mathbb{E}^{\mathbb{P}} \left[U_P(L(X_T) - \widehat{\xi}) \right], \tag{4.5.5}$$

with $\mathcal{X} = U_A^{-1}(Y_T^{Y_0,Z,K})$. We then have the following proposition.

Proposition 4.5.1 If $\mathcal{P}_P \cap \mathcal{P}_A = \emptyset$ then $U_0^P = \lim_{x \to \infty} U_P(x)$.

Proof. Let n be any positive integer and define $\widehat{\xi}^n := L(X_T) - n$. Take any $(Y_0, Z, K) \in \mathbb{Y}_0 \times \mathcal{K}_{Y_0}$ and set the admissible contract

$$\xi^n := \begin{cases} U_A^{-1}(Y_T^{Y_0,Z,K}), \ \mathcal{P}_A - q.s., \\ \widehat{\xi}^n, \ \mathcal{P}_P - q.s. \end{cases}$$

Then, we have

$$U_0^P \geq \inf_{(\mathbb{P},\nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})}} \mathbb{E}^{\mathbb{P}} \left[U_P(L(X_T) - L(X_T) + n) \right] = U_P(n).$$

By making $n \to \infty$ we conclude, since the other inequality is trivial.

This result is the same as in [64, Proposition 4.2]. Since the Agent does not see the random variables defined outside of his set of beliefs $\mathcal{P}_A$, the Principal is completely free on the design of the contract on $\mathcal{P}_P$. Thus, the Principal can offer a contract which satisfies the reservation utility constraint on $\mathcal{P}_A$ and which attains asymptotically her maximal utility on $\mathcal{P}_P$. By doing this the Principal cancel all her risk. This situation is not realistic, since a Principal should not hire an Agent with a completely different point of view on the market behaviour.

4.5.4 Solution for common beliefs.

We now turn to a more realistic situation and we study the problem when $\mathcal{P}_A \cap \mathcal{P}_P \neq \emptyset$. In this case, as showed in [64, Proposition 4.3], (4.5.4) becomes

$$U_0^P = \sup_{Y_0 \in \mathbb{Y}_0} U_0^P(Y_0), \tag{4.5.6}$$

with the abuse of notation

$$U_0^P(Y_0) := \sup_{(Z,K)\in\mathcal{K}_{Y_0}} \inf_{(\mathbb{P},\nu)\in\mathcal{N}_P^{\alpha^\star(\mathcal{X})}\cap\mathcal{N}_A^{\alpha^\star(\mathcal{X})}} \mathbb{E}^{\mathbb{P}}\left[U_P\left(L(X_T) - U_A^{-1}(Y_T^{Y_0,Z,K})\right)\right], \tag{4.5.7}$$

with $\mathcal{X} = U_A^{-1}(Y_T^{Y_0,Z,K})$.

4.5.4.1 A natural restriction to piece-wise constant controls

As explained in [109], then in [26, 28], the problem (4.5.7) coincides with the weak formulation of a (non standard) zero-sum stochastic differential game with the following characteristics

- control variables: $(Z, K) \in \mathcal{K}_{Y_0}$ for the Principal and $(\mathbb{P}, \nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})} \cap \mathcal{N}_A^{\alpha^\star(\mathcal{X})}$ for the Nature,
- state variables: the output process $X^{x,\Theta}$ and the continuation utility of the Agent $Y^{y,\Theta}$, with dynamic given for any $t \leq s \leq T$, $\mathbb{P}-a.s.$, by

$$\begin{cases} X_s^{t,x,\Theta} = x + \int_t^s b\left(r, X_r^{t,x,\Theta}, \alpha^\star(\mathcal{X})\right), \nu_r\right) \mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x,\Theta}, \nu_r)\mathrm{d}W_r^{\alpha^\star(\mathcal{X})}, \\ Y_s^{t,y,\Theta} = y + \int_t^s Z_r \cdot b\left(r, X_r^{t,x,\Theta}, \alpha^\star(\mathcal{X})\right), \nu_r\right) - F^\star(r, X_r^{t,x,\Theta}, Y_r^{t,y,\Theta}, Z_r, \hat{\sigma}_r^2)\mathrm{d}r \\ \qquad\qquad + \int_t^s Z_r \cdot \sigma(r, X_r^{t,x,\Theta}, \nu_r)\mathrm{d}W_r^{\alpha^\star(\mathcal{X})} + \int_t^s \mathrm{d}K_r. \end{cases},$$

  with $\Theta \equiv (Z, K, \nu)$ and $\mathcal{X} = U_A^{-1}(Y_T^{Y_0,Z,K})$.

We now fix an arbitrary $Y_0 \in \mathbb{Y}_0$ and turn to the procedure to solve (4.5.7). The main issue is that the class of controls $\mathcal{K}_{Y_0}$ is too general since, as explained in [64, Section 4.3.2] and

[26, 28], the non-decreasing process K impacts the dynamic of $Y^{Y_0,Z,K}$ only throught the minimality condition (4.4.2) and more information on this process is required to solve the problem. As emphasized in [115, Remark 3.4], we need to consider *piece-wise controls* and restrict our investigation on elementary strategies. This issue is intrinsically linked to the fact that we are looking for a zero-sum game between the Principal and the Nature. We now consider a restricted set of controls piece-wise constant included in $\mathcal{K}_{Y_0}$.

Definition 4.5.1 (Elementary controls starting at a stopping time) Let $t \in [0,T]$ and τ a stopping time $\mathcal{G}^t_s$-adapted for any $s \in [t,T]$. We say that an $\mathbb{R}^d \times \mathbb{R}^+$-valued process (Z,K) (resp. $\nu \in \mathfrak{N}$) is an elementary control starting at τ for the Principal (resp. the Nature) if there exist

- a finite sequence $(\tau_i)_{0\leq i\leq n}$ of $\mathcal{F}_t$-adapted stopping times such that
$$\tau = \tau_0 \leq \cdots \leq \tau_n = T,$$
- a sequence $(z_i,k_i)_{1\leq i\leq n}$ of $\mathbb{R}^d \times \mathbb{R}^+$-valued random variables such that z_i, k_i are $\mathcal{F}^t_{\tau_{i-1}}$-measurable and
$$Z_t = \sum_{i=1}^n z_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i},\ K_t = \sum_{i=1}^n k_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i},$$
resp. a sequence $(n_i)_{1\leq i\leq n}$ of N-valued random variables such that n_i is $\mathcal{F}^t_{\tau_{i-1}}-$measurable and
$$\nu_t = \sum_{i=1}^n n_i \mathbf{1}_{\tau_{i-1}<t\leq\tau_i}.$$

We denote by $\mathcal{U}(t,\tau)$ (resp. $\mathcal{V}(t,\tau)$) the set of elementary controls of the Principal (resp. the Nature). If $\tau = t = 0$, we just write $\mathcal{U}$ (resp. $\mathcal{V}$).

We now set
$$\mathcal{U}_{Y_0} := \mathcal{K}_{Y_0} \cap \mathcal{U},$$
and for any $(Z,K) \in \mathcal{U}_{Y_0}$
$$\mathcal{V}_{Y_0,Z,K} := \left\{ (\mathbb{P},\nu) \in \mathcal{N}_P^{\alpha^\star(\mathcal{X})} \cap \mathcal{N}_A^{\alpha^\star(\mathcal{X})} \middle| \nu \in \mathcal{V} \right\}.$$
We thus consider the following restricted problem

$$V_0^P = \sup_{Y_0 \in \mathbb{Y}_0} V_0^P(Y_0), \tag{4.5.8}$$
with the abuse of notation
$$V_0^P(Y_0) := \sup_{(Z,K)\in\mathcal{U}_{Y_0}} \inf_{(\mathbb{P},\nu)\in\mathcal{V}_{Y_0,Z,K}} \mathbb{E}^{\mathbb{P}}\left[U_P\left(L(X_T) - U_A^{-1}(Y_T^{Y_0,Z,K}) \right) \right]. \tag{4.5.9}$$

The literature in stochastic control problems, in particular [115, 90, 117], leads us to expect that in our problem the equality $U_0^P = V_0^P$ holds. That is, the value of the general problem (4.5.6) coincides with its restriction (4.5.8) to piece-wise defined controls. In the following, we will focus on the restricted problem (4.5.8), that we aim to solve completely.

4.5.4.2 The intuitive HJBI equation

Assumption **(PPD)** in [64] seems to be too complicated to prove[2] for a general class of processes K, since it requires a deep study of the measurability of the dynamic version of the value function associated with the problem (4.5.8). To avoid this difficulty linked directly to the ambiguity on the volatility of the model, we will deal with the so-called Perron's method by following the same ideas as in [11, 13, 12, 115]. Recall that if one aims at associating (4.5.7) with an HJBI equation, as usual in the stochastic control theory, the problem seems to be ill-posed and we need more information on the process K. We thus expect to have an optimal contract $\xi := U_A^{-1}(Y_T^{Y_0,Z,K})$ for which the process K is absolutely continuous. More precisely, by following [28, Remark 5.1] we expect the optimal contract to belong to the subclass of contracts for which there exists a $\mathbb{G}^{\mathcal{N}_A}$-predictable process Γ with values in $\mathcal{M}_{\mathrm{d,d}}(\mathbb{R})$ such that

$$K_t = \int_0^t \left(F^\star(s, Y_s, Z_s, \widehat{\sigma}_s^2) + \frac{1}{2}\mathrm{Tr}\left(\widehat{\sigma}_s^2 \Gamma_s\right) - H(s, X_s, Y_s, Z_s, \Gamma_s) \right) \mathrm{d}s. \tag{4.5.10}$$

This intuition leads us to set the following Hamiltonian function $G : [0,T] \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \times \mathcal{S}^{\mathrm{d,d}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}$ defined by

$$G(t,x,y,u,p,\tilde{p},q,\tilde{q},r) := \sup_{(z,\gamma)\in\mathbb{R}^{\mathrm{d}}\times\mathcal{M}_{\mathrm{d,d}}(\mathbb{R})} \inf_{n\in N} g(t,x,y,u,p,\tilde{p},q,\tilde{q},r,z,\gamma,n),$$

where

$$\begin{aligned} g(t,x,y,u,p,\tilde{p},q,\tilde{q},r,z,\gamma,n) := {} & p\cdot b(t,x,\alpha^\star(t,x,y,z,\widehat{\sigma}_t),n) + \frac{1}{2}\mathrm{Tr}\left(\sigma(t,x,n)\sigma(t,x,n)^\top q\right) \\ & + \tilde{p}\left(\frac{1}{2}\mathrm{Tr}\left(\sigma(t,x,n)\sigma(t,x,n)^\top\gamma\right) - H(t,x,y,z,\gamma)\right) \\ & + \tilde{p}\, b(t,x,\alpha^\star(t,x,y,z,\widehat{\sigma}_t),n)\cdot z + \mathrm{Tr}\left(z^\top\sigma(t,x,n)\sigma(t,x,n)^\top r\right) \\ & + \frac{1}{2}\tilde{q}\ \mathrm{Tr}\left(z^\top\sigma(t,x,n)\sigma(t,x,n)^\top z\right). \end{aligned}$$

We can now set the HJBI equation which is hopefully strongly connected to the problem of the Principal (4.5.9)

$$\begin{cases} -\partial_t u(t,x,y) - G(t,x,y,u,\nabla_x u,\partial_y u,\Delta_{xx}u,\partial_{yy}u,\nabla_{xy}u) = 0,\ (t,x,y)\in[0,T)\times\mathbb{R}^{\mathrm{d}}\times\mathbb{R} \\ u(T,x,y) = U_P(L(x) - U_A^{-1}(y)). \end{cases} \tag{4.5.11}$$

4.5.4.3 The one dimensional case

In this section we lose generality by considering the one-dimensional case and imposing additional assumptions on the drift coefficient b and the volatility coefficient σ. We enforce

[2]Another approach not considered in this work, may consist in proving a *weak* dynamic programming principle by following [18, 14].

these conditions to prove that the supremum over (z,γ) in the definition of G can be reduced to a supremum over a compact set. This result is fundamental in the proof of Theorem 4.5.1 below, but not the one-dimensionality. We point out that if we can extend Lemma 4.5.1 below to the general finite dimensional case, then Theorem 4.5.1 follows directly. We indeed believe we can extend this result under appropriate conditions for b and σ.

Assumption 4.5.2 *b and σ are continuous functions which satisfy the following properties.*

1. For every $(t,x,\alpha) \in [0,T] \times \mathbb{R} \times A$ and for every $\underline{x}$ global minimum of $\sigma(t,x,\cdot)$, the following limit is finite
$$\lim_{\nu \to \underline{\nu}} \frac{b(t,x,\alpha,\nu) - b(t,x,\alpha,\underline{\nu})}{\sigma(t,x,\nu) - \sigma(t,x,\underline{\nu})}.$$

2. For every $(t,x,\alpha) \in [0,T] \times \mathbb{R} \times A$ and for every $\overline{x}$ global maximum of $\sigma(t,x,\cdot)$, the following limit is finite
$$\lim_{\nu \to \overline{\nu}} \frac{b(t,x,\alpha,\nu) - b(t,x,\alpha,\overline{\nu})}{\sigma(t,x,\nu) - \sigma(t,x,\overline{\nu})}.$$

Under Assumption 4.5.2, we have the following result, whose proof is postponed to the Appendix 4.8.

Lemma 4.5.1 *Let $\tilde{q} < 0$. Then, for every $(t,x,y,u,p,\tilde{p},q,r) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ there exists $R := R(t,x,y,u,p,\tilde{p},q,\tilde{q},r)$ such that*

$$G(t,x,y,u,p,\tilde{p},q,\tilde{q},r) = \sup_{|z| \le R} \sup_{|\gamma| \le R} \inf_{n \in N} g(t,x,y,u,p,\tilde{p},q,\tilde{q},r,z,\gamma,n).$$

Remark 4.5.2 As can be seen in the proof of the previous Lemma, when $\tilde{p} < 0$, there are values of $\gamma \in \mathbb{R}$ such that the optimal $\nu_G^\star$ in the definition of the Hamiltonian $G(t,x,y,u,p,\tilde{p},q,\tilde{q},r)$ is different from the optimal $\nu_H^\star$ in the definition of $H(t,x,y,z,\gamma)$. This means that if the volatility of the outcome process takes one of these values, then the worst-case response of the Nature is different from the point of view of the Principal and the Agent. This is not the case in [125], where the author proves that in his particular model the two worst-case responses of the Nature coincide.

4.5.4.4 Perron's method

We now focus on a deep study of PDE (4.5.11). We assume that b and σ are continuous functions and that Lemma 4.5.1 holds. We recall that the proof of Theorem 4.5.1 does not rely on the one–dimensional setting and all its arguments work as soon as Lemma 4.5.1 holds.

In this section we drop the assumptions made in [64] and we prove a verification result for a non-smooth value function, by following the Stochastic Perron's method introduced by Bayraktar and Sîrbu [11, 13, 12, 115]. More precisely, we show that the value function of the Principal associated with the problem (4.5.9) is a viscosity solution to the HJBI equation

(4.5.11). The approach we follow avoids to prove (or assume) a dynamic programming principle and only deals with comparison results. Moreover, the dynamic programming principle is a consequence of the used method. We adapt now the definition of stochastic semi-solutions to stochastic differential games [115] to our framework under the weak formulation.

Definition 4.5.2 (Stopping rule) For $t \in [0,T]$, let $\hat{X}$ be the canonical process on $C\left([t,T], \mathbb{R}^{\mathrm{d}+1}\right)$. Define the filtration $\mathbb{B}^t = (\mathcal{B}^t_s)_{t\le s\le T}$ by

$$\mathcal{B}^t_s := \sigma(\hat{X}(u), t \le u \le s), t \le s \le T.$$

$\tau \in C\left([t,T], \mathbb{R}^{\mathrm{d}+1}\right)$ is a stopping rule starting at t if it is a stopping time with respect to $\mathbb{B}^t$.

Definition 4.5.3 (Stochastic semisolutions of the HJBI equation) Let $Y_0 \in \mathbb{Y}_0$.

- A function $v : [0,T] \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a stochastic sub-solution of HJBI equation (4.5.11) if

  (i-) v is continuous and $v(T,x,y) \le U_P(L(x) - U_A^{-1}(y))$ for any $(x,y) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$,

  (ii-) for any $t \in [0,T]$ and for any stopping rule $\tau \in \mathbb{B}^t$, there exists an elementary control $(\tilde{Z},\tilde{K}) \in \mathcal{U}_{Y_0}(t,\tau)$ such that for any $(Z,K) \in \mathcal{U}_{Y_0}(t,t)$, any $(\mathbb{P},\nu) \in \mathcal{V}_{Y_0}(t,t)$ and each stopping rule $\rho \in \mathbb{B}^t$ with $\tau \le \rho \le T$ we have

  $$v(\tau', X_{\tau'}, Y_{\tau'}) \le \mathbb{E}^{\mathbb{P}}\left[v(\rho', X_{\rho'}, Y_{\rho'})\middle|\mathcal{F}^t_{\tau'}\right], \ \mathbb{P}-a.s., \tag{4.5.12}$$

  where for any $(x,y,\omega) \in \mathbb{R}^2 \times \Omega$,

  $$X := X^{t,x,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu}, \ Y := Y^{t,y,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu},$$
  $$\tau'(\omega) := \tau(X^{t,x,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu}_\cdot(\omega), Y^{t,y,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu}_\cdot(\omega)),$$
  $$\rho'(\omega) := \rho(X^{t,x,(Z,K)\otimes_\rho(\tilde{Z},\tilde{K}),\nu}_\cdot(\omega), Y^{t,y,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu}_\cdot(\omega)).$$

  We denote by $\mathbb{V}^-$ the set of stochastic sub-solution of (4.5.11).

- A function $v : [0,T] \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a stochastic super-solution of HJBI equation (4.5.11) if

  (i+) v is continuous and $v(T,x,y) \ge U_P(L(x) - U_A^{-1}(y))$ for any $(x,y) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$

  (ii+) for any $t \in [0,T]$, for any stopping rule $\tau \in \mathbb{B}^t$ and for any $(Z,K) \in \mathcal{U}_{Y_0}(t,t)$, there exists an elementary control $(\hat{\mathbb{P}},\tilde{\nu}) \in \mathcal{V}_{Y_0}(t,\tau)$ such that for every $\nu \in \mathcal{V}(t,t)$ satisfying $(\hat{\mathbb{P}},\nu) \in \mathcal{V}_{Y_0}(t,t)$ and every stopping rule $\rho \in \mathbb{B}^t$ with $\tau \le \rho \le T$ we have

  $$v(\tau', X_{\tau'}, Y_{\tau'}) \ge \mathbb{E}^{\hat{\mathbb{P}}}\left[v(\rho', X_{\rho'}, Y_{\rho'})\middle|\mathcal{F}^t_{\tau'}\right], \ \hat{\mathbb{P}}-a.s., \tag{4.5.13}$$

  where for any $(x,y,\omega) \in \mathbb{R}^2 \times \Omega$,

  $$X := X^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}}, \ Y := Y^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}},$$
  $$\tau'(\omega) := \tau(X^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}}_\cdot(\omega), Y^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}}_\cdot(\omega)),$$
  $$\rho'(\omega) := \rho(X^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}}_\cdot(\omega), Y^{t,x,Z,K,\nu\otimes_\tau\tilde{\nu}}_\cdot(\omega)).$$

We denote by $\mathbb{V}^+$ the set of stochastic super-solution of (4.5.11).

To apply Perron's method we need the following assumption, assuring the existence of stochastic semi-solutions to the HJBI equation (4.5.11) (see Assumptions 3.4 and 4.3 in [12]).

Assumption 4.5.3 The sets $\mathbb{V}^+$ and $\mathbb{V}^-$ are non-empty.

As explained in [11, 13] the set $\mathbb{V}^+$ is trivially non empty if the function U_P is bounded by above, whereas $\mathbb{V}^-$ is non empty if U_P is bounded by below.

Now we follow the stochastic Perron's method proposed in [115]. Let us define

$$v^- := \sup_{v \in \mathbb{V}^-} v, \ v^+ := \inf_{v \in \mathbb{V}^+} v,$$

and notice that we have from Definition 4.5.3 that for any $Y_0 \in \mathbb{Y}_0$

$$v^-(0, x, Y_0) \leq V_0^P(Y_0) \leq v^+(0, x, Y_0). \tag{4.5.14}$$

We thus get the main theorem of this work and we refer to the Appendix 4.9 for the proof.

Theorem 4.5.1 v^- is a lower semi-continuous viscosity super-solution of HJBI equation (4.5.11) and v^+ is an upper semi-continuous viscosity sub-solution of HJBI equation (4.5.11). Moreover, if there exists a comparison result for HJBI equation (4.5.11), *i.e.* for any lower semi-continuous viscosity sub-solution $\underline{v}$ and for any upper semi-continuous viscosity super-solution $\overline{v}$, we have $\underline{v} \leq \overline{v}$, then

$$v^-(0, x, Y_0) = V_0^P(Y_0) = v^+(0, x, Y_0).$$

4.6 Conclusion

In this work we provide the first comprehensive methodology for Principal-Agent problems with volatility uncertainty and worst-case approach from both sides. We consider a general framework in which we characterize the value function of the Agent as the solution to a second–order BSDE. Concerning the problem of the Principal, we rewrite it as a non–standard stochastic differential game and we restrict our attention to the sub–problem where only piece–wise constant controls are allowed. This restricted class of controls is common in the literature, as in Pham and Zhang [90] and Sîrbu [115]. We expect this restriction does not suppose any loss of utility for the Principal, based on the results of El Karoui and Tan [41], in which the approximation of different controlled diffusion problems by piecewise constant controls is studied. In the restricted problem, we prove that the value function of the Principal is the unique viscosity solution of the associated HJBI equation by assuming that this equation satisfies a comparison result. To do so, we follow the Stochastic Perron's method of Bayraktar and Sîrbu [11, 12, 13, 115]. This is an improvement with respect to the work of Mastrolia and Possamaï [64], in which a dynamic programming principle is assumed. In fact, in our case the dynamic programming principle is a consequence of the previous result. We also extend the work of Sung [125] by considering a more general model without any restrictions on the form of the contracts.

4.7 Appendix

4.7.1 Functional spaces

We introduce the spaces used in this chapter, by following [93]. Let $t \in [0,T]$ and $x \in \Omega$ and a family $(\mathcal{P}(t,x))_{t\in[0,T]\times x\in\Omega}$ of sets of probability measures on $(\Omega, \mathcal{F}_T)$. In this section, we denote by $\mathbb{X} := (\mathcal{X}_s)_{s\in[0,T]}$ a general filtration on $(\Omega, \mathcal{F}_T)$. For any $\mathcal{X}_T-$measurable real valued random variable ξ such that $\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}[|\xi|] < +\infty$, we set for any $s \in [t,T]$

$$\mathbb{E}_s^{\mathbb{P},t,x,\mathbb{X}^+}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}(t,x)[\mathbb{P},\mathbb{X}^+,t]} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{X}_s].$$

Let $p \geq 1$ and $\mathbb{P} \in \mathcal{P}(t,x)$ and $\mathbb{X}_{\mathbb{P}}$ the usual $\mathbb{P}$-augmented filtration associated with $\mathbb{X}$.

- Let $\kappa \in [1,p]$, $\mathbb{L}^{p,\kappa}_{t,x}(\mathbb{X},\mathcal{P})$ denotes the space of $\mathcal{X}_T-$measurable $\mathbb{R}-$valued random variables ξ such that

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,x}(\mathbb{X},\mathcal{P})} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{t\leq s\leq T}\left(\mathbb{E}_s^{\mathbb{P},t,x,\mathbb{F}^+}[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right] < +\infty.$$

- $\mathbb{H}^p_{t,x}(\mathbb{X},\mathbb{P})$ denotes the spaces of $\mathbb{X}$-predictable $\mathbb{R}^{\mathrm{d}}$-valued processes Z such that

$$\|Z\|^p_{\mathbb{H}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \|\widehat{\sigma}_s^{\frac{1}{2}} Z_s\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] < +\infty.$$

  We denote by $\mathbb{H}^p_{t,x}(\mathbb{X},\mathcal{P})$ the spaces of $\mathbb{X}$-predictable $\mathbb{R}^{\mathrm{d}}$-valued processes Z such that

$$\|Z\|^p_{\mathbb{H}^p_{t,x}(\mathbb{X},\mathcal{P})} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|Z\|^p_{\mathbb{H}^p_{t,x}(\mathbb{P})} < +\infty.$$

- $\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})$ denotes the spaces of $\mathbb{X}$-progressively measurable $\mathbb{R}$-valued processes Y such that

$$\|Y\|^p_{\mathbb{S}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\sup_{s\in[t,T]} |Y_s|^p\right] < +\infty.$$

  We denote by $\mathbb{S}^p_{t,x}(\mathbb{X},\mathcal{P})$ of $\mathbb{X}$-progressively measurable $\mathbb{R}$-valued processes Y such that

$$\|Y\|^p_{\mathbb{S}^p_{t,x}(\mathbb{X},\mathcal{P})} := \sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|Y\|^p_{\mathbb{S}^p_{t,x}(\mathbb{P})} < +\infty.$$

- $\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})$ denotes the spaces of $\mathbb{X}$-optional $\mathbb{R}$-valued processes K with $\mathbb{P}-$a.s. càdlàg and non-decreasing paths on $[t,T]$ with $K_t = 0$, $\mathbb{P}-a.s.$ and

$$\|K\|^p_{\mathbb{K}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K^p_T\right] < +\infty.$$

We denote by $\mathbb{K}^p_{t,x}((\mathbb{X}_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)})$ the set of families of processes $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)}$ such that for any $\mathbb{P}\in\mathcal{P}(t,x)$, $K^{\mathbb{P}}\in\mathbb{K}^p_{t,x}(\mathbb{X}_{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|K^{\mathbb{P}}\|_{\mathbb{K}^p_{t,x}(\mathbb{P})} < +\infty.$$

- $\mathbb{M}^p_{t,x}(\mathbb{X},\mathbb{P})$ denotes the spaces of $\mathbb{X}$-optional $\mathbb{R}$-valued martingales M with $\mathbb{P}-$a.s. càdlàg paths on $[t,T]$ with $M_t = 0$, $\mathbb{P}-a.s.$ and

$$\|M\|^p_{\mathbb{M}^p_{t,x}(\mathbb{X},\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[[M]^{\frac{p}{2}}_T\right] < +\infty.$$

We denote by $\mathbb{M}^p_{t,x}((\mathbb{X}_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)})$ the set of families of processes $(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,x)}$ such that for any $\mathbb{P}\in\mathcal{P}(t,x)$, $M^{\mathbb{P}}\in\mathbb{M}^p_{t,x}(\mathbb{X}_{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,x)} \|M^{\mathbb{P}}\|_{\mathbb{M}^p_{t,x}(\mathbb{P})} < +\infty.$$

When $t = 0$ we simplify the previous notations by omitting the dependence on x.

4.7.2 Proofs for the Agent's problem

Proof. [Proof of Lemma 4.4.2.] Since $\frac{\ell+m}{\underline{m}+1-\ell} \leq 2$, we have from Lemma 4.4.1 that the 2BSDE (4.4.1) has quadratic growth with respect to z and coincide with the framework of [94]. In view of Remark 4.2 in [93], we aim at applying Theorem 4.1 in [93] by slightly changing its assumptions. More precisely, we replace (i) of Assumption 2.1 in [93] by Assumption 2.1 in [94], excepting part (iii). Condition (iv) in [94] is a consequence of Lemma 4.4.1. Conditions (v)-(vi) in [94] holds in our setting because k is bounded. Therefore, Assumption 2.1 in [94] is satisfied.

Finally, we turn to parts (ii)–(v) of Assumption 2.1 in [93]. The terminal condition $U_A(\xi)$ belongs to $\mathbb{L}^{p,\kappa}_{0,x}(\mathcal{P}_A)$ by definition of the admissible contracts and the conditions imposed on c in $(\mathbf{H}^{\ell,m,\underline{m}})$ ensure that (ii) holds. The parts (iii), (iv) and (v) correspond exactly to our Assumption 4.2.1. □

Proof. [Proof of Theorem 4.4.1.] We first prove that (4.4.3) holds with a characterization of the optimal effort of the Agent as a maximizer of the 2BSDE (4.4.1). The proof is divided in 4 steps.

• **Step 1:** For every $(\alpha,\nu)\in\mathcal{A}\times\mathcal{V}(\hat{\sigma}^2)$ denote by $(Y^{\alpha,\nu}, Z^{\alpha,\nu}, K^{\alpha,\nu})$ the solution of the following controlled 2BSDE, defined $\mathcal{P}_A - q.s.$ (well-posedness holds by the same arguments employed in the proof of Lemma 4.4.2)

$$Y_t^{\alpha,\nu} = U_A(\xi) + \int_t^T F(s, X, Y_s^{\alpha,\nu}, Z_s^{\alpha,\nu}, \alpha_s, \nu_s)\mathrm{d}s - \int_t^T Z_s^{\alpha,\nu} \cdot \mathrm{d}X_s - \int_t^T \mathrm{d}K_s^{\alpha,\nu} - \int_0^T \mathrm{d}M_s^{\alpha,\nu}. \tag{4.7.1}$$

Consider also, for every $\alpha \in \mathcal{A}$, the solution $(Y^\alpha, Z^\alpha, K^\alpha)$ of the following 2BSDE, defined $\mathcal{P}_A - q.s.$

$$Y_t^{\alpha} = U_A(\xi) + \int_t^T \inf_{\nu \in V_s(x,\hat{\sigma}_s^2)} F(s, X, Y_s^{\alpha}, Z_s^{\alpha}, \alpha_s, \nu)\mathrm{d}s - \int_t^T Z_s^{\alpha} \cdot \mathrm{d}X_s - \int_t^T \mathrm{d}K_s^{\alpha} - \int_0^T \mathrm{d}M_s^{\alpha}. \tag{4.7.2}$$

We have from comparison theorems for 2BSDEs (which are inherited by the classical comparison results for BSDEs)

$$\begin{aligned} Y_0 &= \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}}{}^{\mathbb{P}}\ Y_0^{\alpha},\ \mathbb{P} - \text{a.s. for every } \mathbb{P} \in \mathcal{P}_A \\ &= \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}}{}^{\mathbb{P}}\ \operatorname*{ess\,inf}_{\nu \in \mathcal{V}(\hat{\sigma}^2)}{}^{\mathbb{P}}\ Y_0^{\alpha,\nu},\ \mathbb{P} - \text{a.s. for every } \mathbb{P} \in \mathcal{P}_A. \end{aligned} \tag{4.7.3}$$

• **Step 2:** Next, consider for any $\mathbb{P} \in \mathcal{P}_A$ the triple $(\mathcal{Y}_t^{\mathbb{P},\alpha,\nu}, \mathcal{Z}_t^{\mathbb{P},\alpha,\nu}, \mathcal{M}_t^{\mathbb{P},\alpha,\nu})_{t\in[0,T]}$ which is the solution of the (well-posed) linear BSDE

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} = U_A(\xi) + \int_0^T F(s, X, \mathcal{Y}_s^{\mathbb{P},\alpha,\nu}, \mathcal{Z}_s^{\mathbb{P},\alpha,\nu}, \alpha_s, \nu_s)\mathrm{d}s - \int_0^T \mathcal{Z}_s^{\mathbb{P},\alpha,\nu} \cdot \mathrm{d}X_s - \int_0^T \mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu},\ \mathbb{P} - a.s. \tag{4.7.4}$$

We will follow the idea of Theorem 4.2 in [93], to prove that for every $(\alpha,\nu) \in \mathcal{A} \times \mathcal{V}(\hat{\sigma}^2)$, the solution of the 2BSDE (4.7.1) satisfies the following representation

$$Y_0^{\alpha,\nu} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_A[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}}\ \mathcal{Y}_0^{\mathbb{P}',\alpha,\nu},\ \mathbb{P} - a.s. \text{ for every } \mathbb{P} \in \mathcal{P}_A. \tag{4.7.5}$$

First, notice that since $K^{\alpha,\nu}$ is non-decreasing, we have for every $\mathbb{P} \in \mathcal{P}_A$ and $\mathbb{P}' \in \mathcal{P}_A[\mathbb{P},\mathbb{F}^+,0]$

$$Y_0^{\alpha,\nu} \leq\ \mathcal{Y}_0^{\mathbb{P}',\alpha,\nu},\ \mathbb{P} - a.s.$$

thus

$$Y_0^{\alpha,\nu} \leq \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_A[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}}\ \mathcal{Y}_0^{\mathbb{P}',\alpha,\nu},\ \mathbb{P} - a.s. \text{ for every } \mathbb{P} \in \mathcal{P}_A.$$

To the reverse inequality, compute for every $\mathbb{P} \in \mathcal{P}_A$

$$\begin{aligned} \mathcal{Y}_t^{\mathbb{P},\alpha,\nu} - Y_t^{\alpha,\nu} = &\int_t^T \left(F(s, X, \mathcal{Y}_s^{\mathbb{P},\alpha,\nu}, \mathcal{Z}_s^{\mathbb{P},\alpha,\nu}, \alpha_s, \nu_s) - F(s, X, Y_s^{\alpha,\nu}, Z_s^{\alpha,\nu}, \alpha_s, \nu_s)\right)\mathrm{d}s \\ &- \int_t^T \left(\mathcal{Z}_s^{\mathbb{P},\alpha,\nu} - Z_s^{\alpha,\nu}\right) \cdot \mathrm{d}X_s + \int_t^T \mathrm{d}K_s^{\alpha,\nu} - \int_t^T \left(\mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu} - \mathrm{d}M_s^{\alpha,\nu}\right), \mathbb{P} - a.s. \end{aligned}$$

Which is equivalent to

$$\mathcal{Y}_t^{\mathbb{P},\alpha,\nu} - Y_t^{\alpha,\nu} = \int_t^T \left(-k(s,X,\alpha_s,\nu_s)(\mathcal{Y}_s^{\mathbb{P},\alpha,\nu} - Y_s^{\alpha,\nu}) + b(s,X,\alpha_s)\left(\mathcal{Z}_s^{\mathbb{P},\alpha,\nu} - Z_s^{\alpha,\nu}\right)\right)\mathrm{d}s$$
$$- \int_t^T \left(\mathcal{Z}_s^{\mathbb{P},\alpha,\nu} - Z_s^{\alpha,\nu}\right)\cdot \mathrm{d}X_s + \int_t^T \mathrm{d}K_s^{\alpha,\nu} - \int_t^T \left(\mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu} - \mathrm{d}M_s^{\alpha,\nu}\right), \mathbb{P}-a.s.$$

Using a linearization (see for instance [39]), we get

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} - Y_0^{\alpha,\nu} = \mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^T \mathcal{K}_{0,s}^{\alpha,\nu}\mathrm{d}K_s^{\alpha,\nu}\,\middle|\,\mathcal{F}_0\right],\ \mathbb{P}-a.s.$$

Then, from Assumption ($\mathbf{H}^{\ell,m,\underline{m}}$) (iii) we deduce that

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} - Y_0^{\alpha,\nu} \geq \mathrm{e}^{-\kappa T}\mathbb{E}^{\mathbb{P}^\alpha}\left[\int_0^T \mathrm{d}K_s^{\alpha,\nu}\,\middle|\,\mathcal{F}_0\right],\ \mathbb{P}-a.s.$$

Since $K^{\alpha,\nu}$ satisfies the minimality condition (4.4.2), we deduce that $\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} - Y_0^{\alpha,\nu} \geq 0,\ \mathbb{P}-a.s.$ for every $\mathbb{P}\in\mathcal{P}_A$ and (4.7.5) holds.

- **Step 3:** Finally, by denoting $c_s^\alpha := c(s,X,\alpha_s)$, $k_s^{\alpha,\nu} = k(s,X,\alpha_s,\nu_s)$, $b_s^{\alpha,\nu} = b(s,X,\alpha_s,\nu_s)$, we can rewrite the BSDE (4.7.4) $\mathbb{P}-a.s.$ as

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} = U_A(\xi) + \int_0^T \left(-k_s^{\alpha,\nu}\mathcal{Y}_s^{\mathbb{P},\alpha,\nu} - c_s^\alpha + \mathcal{Z}_s^{\mathbb{P},\alpha,\nu}\cdot b_s^{\alpha,\nu}\right)\mathrm{d}s - \int_0^T \mathcal{Z}_s^{\mathbb{P},\alpha,\nu}\cdot\mathrm{d}X_s - \int_0^T \mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu}.$$

Which is equivalent to

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} = U_A(\xi) + \int_0^T \left(-k_s^{\alpha,\nu}\mathcal{Y}_s^{\mathbb{P},\alpha,\nu} - c_s^\alpha + (\sigma_s^\nu)^\top\mathcal{Z}_s^{\mathbb{P},\alpha,\nu}\cdot(\sigma_s^\nu)^\top\left(\sigma_s^\nu{\sigma_s^\nu}^\top\right)^{-1}b_s^{\alpha,\nu}\right)\mathrm{d}s$$
$$- \int_0^T (\sigma_s^\nu)^\top\mathcal{Z}_s^{\mathbb{P},\alpha,\nu}\cdot\mathrm{d}W_s^{\mathbb{P}} - \int_0^T \mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu}.$$

Defining $\hat{\mathcal{Z}}_s^{\mathbb{P},\alpha,\nu} = (\sigma_s^\nu)^\top\mathcal{Z}_s^{\mathbb{P},\alpha,\nu}$, we obtain

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} = U_A(\xi) + \int_0^T \left(-k_s^{\alpha,\nu}\mathcal{Y}_s^{\mathbb{P},\alpha,\nu} - c_s^\alpha + \hat{\mathcal{Z}}_s^{\mathbb{P},\alpha,\nu}\cdot(\sigma_s^\nu)^\top\left(\sigma_s^\nu{\sigma_s^\nu}^\top\right)^{-1}b_s^{\alpha,\nu}\right)\mathrm{d}s$$
$$- \int_0^T \hat{\mathcal{Z}}_s^{\mathbb{P},\alpha,\nu}\cdot\mathrm{d}W_s^{\mathbb{P}} - \int_0^T \mathrm{d}\mathcal{M}_s^{\mathbb{P},\alpha,\nu},$$

whose solution is

$$\mathcal{Y}_0^{\mathbb{P},\alpha,\nu} = \mathbb{E}^{\mathbb{P}^{\alpha,\nu}}\left[\mathcal{K}_{0,T}^{\alpha,\nu}U_A(\xi) - \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu}c_s^\alpha\mathrm{d}s\,\middle|\,\mathcal{F}_0\right],\ \mathbb{P}-a.s.,$$

where the measure $\mathbb{P}^{\alpha,\nu}$ is equivalent to $\mathbb{P}$ and is defined by

$$\frac{\mathrm{d}\mathbb{P}^{\alpha,\nu}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(\int_0^T \sigma^\top(\sigma\sigma^\top)^{-1}(s,X,\nu_s)b(s,X,\alpha_s,\nu_s)\cdot\mathrm{d}W_s^{\mathbb{P}}\right).$$

• **Step 4:** We have from the previous steps that the measure $\mathbb{P}^{\alpha,\nu} \in \mathcal{P}_A^\alpha$ and for every measure $\mathbb{P} \in \mathcal{P}_A$ we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
Y_0 &= \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{\nu \in \mathcal{V}(\hat{\sigma}^2)}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_A[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'^{\alpha,\nu}} \left[\mathcal{K}_{0,T}^{\alpha,\nu} U_A(\xi) - \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu} c_s^\alpha \mathrm{d}s \,\middle|\, \mathcal{F}_0 \right] \\
&= \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}}{}^{\mathbb{P}} \operatorname*{ess\,inf}_{(\mathbb{P}',\nu) \in \mathcal{N}_A^\alpha[\mathbb{P},\mathbb{F}^+,0]}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} \left[\mathcal{K}_{0,T}^{\alpha,\nu} U_A(\xi) - \int_0^T \mathcal{K}_{0,s}^{\alpha,\nu} c_s^\alpha \mathrm{d}s \,\middle|\, \mathcal{F}_0 \right].
\end{aligned}
$$

By similar arguments to the ones used in the proofs of Lemma 3.5 and Theorem 5.2 of [93], it follows that

$$
\begin{aligned}
\sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P},\nu) \in \mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}} [Y_0] &= \sup_{\alpha \in \mathcal{A}} \inf_{(\mathbb{P},\nu) \in \mathcal{N}_A^\alpha} \mathbb{E}^{\mathbb{P}} \left[\mathcal{K}_{0,T} U_A(\xi) - \int_0^T \mathcal{K}_{0,s} c_s^\alpha \mathrm{d}s \right] \\
&= U_0^A(\xi).
\end{aligned}
$$

We now turn to the second part of the Theorem with the characterization of an optimal triplet $(\alpha, \mathbb{P}, \nu)$ for the optimization problem (4.4.3). From the proof of the first part, it is clear that a control $(\alpha^\star, \mathbb{P}^\star, \nu^\star)$ is optimal if and only if it attains all the essential suprema and infima above. The infimum in (4.7.5) is attained if (ii) holds and equality (4.7.3) holds if $\alpha^\star$ and $\nu^\star$ satisfy (i). □

4.8 Proof of Lemma 4.5.1

The proof of Lemma 4.5.1 is based on the following Lemma.

Lemma 4.8.1 Let $\sigma : [c, \mathrm{d}] \to \mathbb{R}$ be continuous, strictly positive and let $q : [c, \mathrm{d}] \to \mathbb{R}$ be continuous. Define for every $\gamma \in \mathbb{R}$ the map $f_\gamma(x) := \gamma \sigma(x)^2 - q(x)$.

1. Suppose that for every $\underline{x}$ global minimum of σ, the following limit is finite

$$\ell := \lim_{x \to \underline{x}} \frac{q(x) - q(\underline{x})}{\sigma(x) - \sigma(\underline{x})}.$$

   Then there exists $M > 0$ such that f_γ attains its minimum over $[c, \mathrm{d}]$ at $\underline{x}$ for every $\gamma > M$.

2. Suppose that for every $\overline{x}$ global maximum of σ, the following limit is finite

$$L := \lim_{x \to \overline{x}} \frac{q(x) - q(\overline{x})}{\sigma(x) - \sigma(\overline{x})}.$$

   Then there exists $m < 0$ such that f_γ attains its minimum over $[c, \mathrm{d}]$ at $\overline{x}$ for every $\gamma < m$.

Proof.

1. We suppose without loss of generality that σ attains its minimum over $[c, \mathrm{d}]$ at a unique point $\underline{x}$. Define $g : [c, \mathrm{d}] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{q(x)-q(\underline{x})}{\sigma(x)-\sigma(\underline{x})}, & x \neq \underline{x}, \\ \ell, & x = \underline{x}. \end{cases}$$

We have that g is continuous on $[c, \mathrm{d}]$ and therefore there exists M_g such that

$$|g(x)| \leq M_g, \ \forall x \in [c, \mathrm{d}].$$

Then, for every $\gamma > M := \frac{M_g}{2\sigma(\underline{x})}$ we have

$$\begin{aligned} &\gamma > \frac{q(x) - q(\underline{x})}{2\sigma(\underline{x})(\sigma(x) - \sigma(\underline{x}))}, && \forall x \in [c, \mathrm{d}], x \neq \underline{x}, \\ \iff &\gamma \cdot 2\sigma(\underline{x})(\sigma(x) - \sigma(\underline{x})) > q(x) - q(\underline{x}), && \forall x \in [c, \mathrm{d}], x \neq \underline{x}, \\ \implies &\gamma(\sigma(x) + \sigma(\underline{x}))(\sigma(x) - \sigma(\underline{x})) > q(x) - q(\underline{x}), && \forall x \in [c, \mathrm{d}], x \neq \underline{x}, \\ \iff &\gamma\sigma(x)^2 - q(x) > \gamma\sigma(\underline{x})^2 - q(\underline{x}), && \forall x \in [c, \mathrm{d}], x \neq \underline{x}. \end{aligned}$$

2. We suppose without loss of generality that σ attains its maximum over $[c, \mathrm{d}]$ at a unique point $\overline{x}$. Define $G : [c, \mathrm{d}] \to \mathbb{R}$ by

$$G(x) = \begin{cases} \frac{q(x)-q(\overline{x})}{\sigma(x)-\sigma(\overline{x})}, & x \neq \overline{x}, \\ L, & x = \overline{x}. \end{cases}$$

We have that G is continuous on $[c, \mathrm{d}]$ and therefore there exists M_G such that

$$|G(x)| \leq M_G, \ \forall x \in [c, \mathrm{d}].$$

Then, for every $\gamma < m := -\frac{M_G}{2\sigma(\underline{x})}$ we have

$$\begin{aligned} &\gamma < \frac{q(\overline{x}) - q(x)}{2\sigma(\underline{x})(\sigma(\overline{x}) - \sigma(x))}, && \forall x \in [c, \mathrm{d}], x \neq \underline{x}, \\ \iff &\gamma \cdot 2\sigma(\underline{x})(\sigma(\overline{x}) - \sigma(x)) < q(\overline{x}) - q(x), && \forall x \in [c, \mathrm{d}], x \neq \underline{x}, \\ \implies &\gamma(\sigma(\overline{x}) + \sigma(x))(\sigma(\overline{x}) - \sigma(x)) < q(\overline{x}) - q(x), && \forall x \in [c, \mathrm{d}], x \neq \overline{x}, \\ \iff &\gamma\sigma(\overline{x})^2 - q(\overline{x}) < \gamma\sigma(x)^2 - q(x), && \forall x \in [c, \mathrm{d}], x \neq \overline{x}. \end{aligned}$$

□

Proof. [Proof of Lemma 4.5.1.] If $\tilde{q} < 0$, the boundedness of b and σ makes g coercive in z and the supremum in this variable can be restricted to a compact. The property on γ is independent of the sign of $\tilde{q}$ and is presented next. Recall the Hamiltonian

$$H(t, x, y, z, \gamma) = \sup_{\alpha \in A} \inf_{\nu \in N} \left\{ \frac{1}{2}\gamma\sigma(t, x, \nu)^2 - k(t, x)y - c(t, x, \alpha) + b(t, x, \alpha, \nu)z \right\}.$$

It follows from Lemma 4.8.1 the existence of $m, M \in \mathbb{R}$ such that if $\gamma > M$ then the infimum in H is attained at the minimizer $\underline{\nu}$ of $\sigma(t,x,\cdot)$ and if $\gamma < m$ then the infimum in H is attained at the maximizer $\overline{\nu}$ of $\sigma(t,x,\cdot)$.

Suppose now that $\tilde{p} > 0$. It follows again from Lemma 4.8.1, that for γ big enough, the infimum in G is attained at the minimizer $\underline{\nu}$. For γ negative enough, the infimum in G is attained at the maximizer $\overline{\nu}$. This means that there exists some $R := R(t,x,y,u,p,\tilde{p},q,\tilde{q},r)$ such that G and H attain its minima at the same value $n \in N$ for $|\gamma| > R$. Therefore G does not depend on γ and the supremum on γ can be restricted to the set $|\gamma| \leq R$.

Suppose finally that $\tilde{p} < 0$. Then for γ big enough, the infimum in G is attained at the maximizer $\overline{\nu}$. For γ small enough, the infimum in G is attained at the minimizer $\underline{\nu}$. In both cases, the dependence of G on γ is given by the term $\tilde{p}|\gamma|(\sigma(t,x,\overline{\nu})^2 - \sigma(t,x,\underline{\nu})^2)$ so it follows that g is coercive in γ. □

4.9 Proof of Theorem 4.5.1 for the problem of the Principal

The following Lemma is used in the proof of Theorem 4.5.1. Its proof is omitted, being a path-wise approximation of deterministic Lebesgue integrals.

Lemma 4.9.1 Define the process $K(Z,\Gamma)$ by

$$K_t(Z,\Gamma) = \int_0^t \left(F^\star(s, Y_s, Z_s, \widehat{\sigma}_s^2) + \frac{1}{2}\mathrm{Tr}\left(\widehat{\sigma}_s^2 \Gamma_s\right) - H(s, X_s, Y_s, Z_s, \Gamma_s) \right) \mathrm{d}s. \tag{4.9.1}$$

Then, for any bounded map ψ, there exists a sequence k^n of elementary controls such that for any $\varepsilon > 0$ and any n big enough

$$\left| \int_0^t \psi_s \mathrm{d}K_s(Z,\Gamma) - \int_0^t \psi_s \mathrm{d}k_s^n \right| \leq \varepsilon, \ \mathcal{P}_P - q.s. \tag{4.9.2}$$

Proof. [Proof of Theorem 4.5.1] We follow the ideas of [115]. Intuitively, V_0 has to be greater than v^- since v^- is roughly speaking the HJBI equation associated with the problem of the Principal when K has the particular decomposition (4.5.10). In other words, the value of the unrestricted problem for the Principal has to be a super-solution of such HJBI equation.

Step 1. v^- is a viscosity super-solution of (4.5.11)**.**

We prove that v^- is a viscosity super-solution of (4.5.11) by contradiction.

1. **The viscosity supersolution property on** $[0,T)$

   a. **Setting the contradiction.** Let φ be some map from $[0,T] \times \mathbb{R}^\mathrm{d} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuously differentiable in time and twice continuously differentiable in space.

Let $(t_0, x_0, y_0) \in [0,T) \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$ be such that $v^- - \varphi$ attains a strict local minimum equal to 0 at this point. We assume (by contradiction) that

$$\partial_t \varphi(t_0, x_0, y_0) + G(t_0, x_0, y_0, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx}\varphi, \partial_{yy}\varphi, \nabla_{xy}\varphi) > 0 \tag{4.9.3}$$

In particular, there exists some $(\hat{z}, \hat{\gamma}) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{M}_{\mathrm{d,d}}(\mathbb{R})$ and a small $\varepsilon > 0$ such that

$$\partial_t \varphi(t_0, x_0, y_0) + \inf_{n \in N} g(t_0, x_0, y_0, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx}\varphi, \partial_{yy}\varphi, \nabla_{xy}\varphi, \hat{z}, \hat{\gamma}, n) > \varepsilon.$$

Recall that g is continuous and N is a compact subset of some finite dimensional space. From Heine's Theorem, we deduce that there exists some $\varepsilon' > 0$ such that for any $(t,x,y) \in \mathcal{B}((t_0, x_0, y_0); \varepsilon')$ we have

$$\partial_t \varphi(t,x,y) + \inf_{n \in N} g(t, x, y, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx}\varphi, \partial_{yy}\varphi, \nabla_{xy}\varphi, \hat{z}, \hat{\gamma}, n) > \varepsilon'. \tag{4.9.4}$$

We denote $\mathcal{T}_{\varepsilon'} := \overline{\mathcal{B}((t_0, x_0, y_0); \varepsilon')} \setminus \mathcal{B}((t_0, x_0, y_0); \frac{\varepsilon'}{2})$. On $\mathcal{T}_{\varepsilon'}$, we have $v^- > \varphi$ so that the maximum of $\varphi - v^-$ is attained and is negative. Thus, there exists some $\eta > 0$ such that $\varphi < v^- - \eta$ on $\mathcal{T}_{\varepsilon'}$. From [115, Lemma 3.8] there exists a non decreasing sequence w_n in $\mathbb{V}^-$ converging to v^-. Then, there exists $n_0 \geq 1$ such that for any $n \geq n_0$ large enough, $\varphi + \frac{\eta}{2} < w_n$ on $\mathcal{T}_{\varepsilon'}$. We denote by w_{n_0+} such w_n. Thus, for $0 < \delta < \frac{\eta}{2}$ we define

$$w^\delta := \begin{cases} (\varphi + \delta) \vee w_{n_0+}, & \text{on } \mathcal{B}((t_0, x_0, y_0); \varepsilon'), \\ w_{n_0+}, & \text{outside } \mathcal{B}((t_0, x_0, y_0); \varepsilon'). \end{cases}$$

Notice that

$$\begin{aligned} w^\delta(t_0, x_0, y_0) &= (\varphi(t_0, x_0, y_0) + \delta) \vee w_{n_0+}(t_0, x_0, y_0) \\ &\geq \varphi(t_0, x_0, y_0) + \delta \\ &> v^-(t_0, x_0, y_0). \end{aligned} \tag{4.9.5}$$

Thus proving that $w^\delta \in \mathbb{V}^-$ provides the desired contradiction. From now, we fix some $t \in [0,T]$ and $\tau \in \mathbb{B}^t$. We need to build a strategy $(\tilde{Z}, \tilde{K}) \in \mathcal{U}_{Y_0}(t, \tau)$ such that Property (ii−) in Definition 4.5.3 holds. Recall that $w_{n_0+} \in \mathbb{V}^-$, thus there exists some elementary strategy $(\tilde{Z}^1(\tau), \tilde{K}^1(\tau)) \in \mathcal{U}_{Y_0}(t, \tau)$ such that Property (ii−) in Definition 4.5.3 holds.

b. **Building the elementary strategy and Property (ii-)** We consider the following strategy that we denote by $(\tilde{\mathcal{Z}}, \tilde{\mathcal{K}})$.

   * If $\varphi + \delta > w_{n_0+}$ at time τ, we choose the strategy $(\hat{z}, \hat{k}^p(\hat{z}, \hat{\gamma}))$, where $\hat{k}^p(\hat{z}, \hat{\gamma})$ is such that inequality (4.9.2) holds with $\frac{\varepsilon}{2}$.

   * Otherwise we follow the elementary strategy $(\tilde{Z}^1(\tau), \tilde{K}^1(\tau))$.

Let τ_1 be the first time when (t, X_t, Y_t) exits from $\mathcal{B}((t_0, x_0, y_0); \varepsilon)$ (which can be τ itself). On the boundary of this ball, we know that $w^\delta = w_{n_0+}$, thus we choose the strategy $(\tilde{Z}^1(\tau_1), \tilde{K}^1(\tau_1)) \in \mathcal{U}(t, \tau_1)$, coinciding with the strategy associated with w_{n_0+} starting at τ_1.

Rigorously speaking, define

$$\tilde{\mathcal{Z}}(s, x(\cdot), y(\cdot)) := \hat{z}\mathbf{1}_{\{\varphi(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))+\delta > w_{n_0+}(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))\}} + \tilde{Z}^1_s(\tau)\mathbf{1}_{\{\varphi(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))+\delta \le w_{n_0+}(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))\}},$$

$$\tilde{\mathcal{K}}(s, x(\cdot), y(\cdot)) := \hat{k}^p_s(\hat{z}, \hat{\gamma})\mathbf{1}_{\{\varphi(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))+\delta > w_{n_0+}(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))\}} + \tilde{K}^1_s(\tau)\mathbf{1}_{\{\varphi(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))+\delta \le w_{n_0+}(\tau(x,y),x(\tau(x,y)),y(\tau(x,y)))\}},$$

and the stopping rule $\tau_1 : C([t,T], \mathbb{R}^{\mathrm{d}+1}) \longrightarrow [t,T]$ by

$$\tau_1(x,y) = \inf_{\tau(x,y) \le s \le T} (s, x(s), y(s)) \in \partial\mathcal{B}((t_0, x_0, y_0); \varepsilon).$$

Then, we consider the following strategy

$$\tilde{Z} := \tilde{\mathcal{Z}} \otimes_{\tau_1} \tilde{Z}^1(\tau_1), \ \tilde{K} := \tilde{\mathcal{K}} \otimes_{\tau_1} \tilde{K}^1(\tau_1). \tag{4.9.6}$$

From Lemma 2.8 in [115], we have $(\tilde{Z}, \tilde{K}) \in \mathcal{U}(t, \tau)$. It remains to prove that $\tilde{K}$ satisfies the minimality condition (4.4.2) to conclude that the strategy defined by (4.9.6) is in $\mathcal{U}_{Y_0}(t, \tau)$. Using a measurability selection argument as in the proof of Theorem 5.3 in [116], for any $\varepsilon > 0$ there exists a weak solution $\mathbb{P}^\varepsilon$ such that $K(\hat{z}, \hat{\gamma}) \le \varepsilon$, $\hat{\mathbb{P}}^\varepsilon - a.s.$. By Lemma 4.9.1, we deduce that for any $\varepsilon > 0$, any p big enough and any $t \in [0,T]$, $|\hat{k}^p_t(\hat{z}, \hat{\gamma})| \le 2\varepsilon$, $\hat{\mathbb{P}}^\varepsilon - a.s.$ Hence, $(\tilde{Z}, \tilde{K}) \in \mathcal{U}_{Y_0}(t, \tau)$.

Fix $(Z, K) \in \mathcal{U}_{Y_0}(t,t)$, $(\mathbb{P}, \nu) \in \mathcal{V}_{Y_0}(t,t)$ and ρ a stopping rule in $\mathbb{B}^t$ such that $\tau \le \rho \le T$. With the notations in Definition 4.5.3 (ii-), we define the event

$$A := \{\varphi(\tau', X_{\tau'}, Y_{\tau'}) + \delta > w_{n_0+}(\tau', X_{\tau'}, Y_{\tau'})\}.$$

Applying Ito's formula to $\varphi + \delta$ on A, and setting $\sigma_r := \sigma(r, X^{\hat{z},\hat{k}^p}_r, \nu)$, we get for any $t \le \tau' \le s' \le s \le \tau'_1$

$$\begin{aligned}\varphi(s, X^{\hat{z},\hat{k}^p}_s, Y^{\hat{z},\hat{k}^p}_s) = {}& \varphi(s', X^{\hat{z},\hat{k}^p}_{s'}, Y^{\hat{z},\hat{k}^p}_{s'}) + \int_{s'}^s (\nabla_x\varphi + \partial_y\varphi\hat{z}) \cdot \sigma_r \mathrm{d}W^\star_r \\ &+ \int_{s'}^s \partial_t\varphi + g(r, X^{\hat{z},\hat{k}^p}_r, Y^{\hat{z},\hat{k}^p}_r, \varphi, \nabla_x\varphi, \partial_y\varphi, \Delta_{xx}\varphi, \partial_{yy}\varphi, \nabla_{xy}\varphi, \hat{z}, \hat{\gamma}, \nu)\mathrm{d}r \\ &+ \int_{s'}^s \partial_y\varphi(r, X^{\hat{z},\hat{k}^p}_r, Y^{\hat{z},\hat{k}^p}_r)\left(\mathrm{d}\hat{k}^p_s - \mathrm{d}K_r(\hat{z}, \hat{\gamma})\right).\end{aligned}$$

From Lemma 4.9.1 together with (4.9.4), we get (for p big enough)

$$\begin{aligned}\varphi(s, X^{\hat{z},\hat{k}^p}_s, Y^{\hat{z},\hat{k}^p}_s) > {}& \varphi(s', X^{\hat{z},\hat{k}^p}_{s'}, Y^{\hat{z},\hat{k}^p}_{s'}) + \int_{s'}^s (\nabla_x\varphi + \partial_y\varphi\hat{z}) \cdot \sigma_r \mathrm{d}W^\star_r \\ &+ (s - s')\frac{\varepsilon'}{2}.\end{aligned}$$

Thus, φ is a sub-martingale on $[\tau, \tau_1]$ under $\mathbb{P}$ and Property (ii$-$) is satisfied on $[\tau', \tau_1']$. On A^c, w_{n_0+} automatically satisfies Property (ii$-$). By noticing that for any $\tau' \leq s \leq \tau_1'$

$$X_s^{t,x,(Z,K)\otimes_\tau(\tilde{Z},\tilde{K}),\nu} = \mathbf{1}_A X_s^{t,x,(Z,K)\otimes_\tau(\hat{z},\hat{k}^p),\nu} + \mathbf{1}_{A^c} X_s^{t,x,(Z,K)\otimes_\tau(\tilde{Z}^1(\tau),\tilde{K}^1(\tau)),\nu},$$

using iterated conditioning and by following the same lines in proof 1.1 of Theorem 3.5 in [115], we deduce that $w^\delta \in \mathbb{V}^-$, which contradicts (4.9.5). Thus,

$$\partial_t\varphi(t_0, x_0, y_0) + G(t_0, x_0, y_0, \varphi, \nabla_x\varphi, \partial_y\varphi, \Delta_{xx}\varphi, \partial_{yy}\varphi, \nabla_{xy}\varphi) \leq 0.$$

2. **The viscosity supersolution property at time T.** We now aim at proving that $v^-(T, x, y) \geq U_P(L(x) - U_A^{-1}(y))$ for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}$. This proof follows the same lines that the step 3 of the proof of Theorem 3.1 in [12] or the proof of Theorem 3.5, 1.2 in [115]. We assume by contradiction that there exists $(x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}$ such that $v^-(T, x_0, y_0) < U_P(L(x_0) - U_A^{-1}(y_0))$. Since U_P is continuous, there exists $\varepsilon > 0$ such that

$$U_P(L(x) - U_A^{-1}(y)) \geq v^-(T, x, y) + \varepsilon,\ (x, y) \in \mathcal{B}((x_0, y_0); \varepsilon).$$

We define $\mathcal{T}_\varepsilon := \overline{\mathcal{B}((T, x_0, y_0); \varepsilon)} \setminus \mathcal{B}((T, x_0, y_0); \frac{\varepsilon}{2})$. Let $\eta > 0$ be small enough such that

$$v^-(T, x_0, y_0) + \varepsilon < \frac{\varepsilon^2}{4\eta} + \inf_{(t,x,y)\in\mathcal{T}_\varepsilon} v^-(t, x, y).$$

Thus, using exactly the same Dini type arguments that in [115, 13], there exists n_0 big enough such that for some $w_{n_0} \in \mathbb{V}^-$ we have

$$v^-(T, x_0, y_0) + \varepsilon < \frac{\varepsilon^2}{4\eta} + \inf_{(t,x,y)\in\mathcal{T}_\varepsilon} w_{n_0}(t, x, y).$$

We now define for any $\lambda > 0$

$$\varphi^{\varepsilon,\eta,\lambda}(t, x, y) := v^-(T, x_0, y_0) - \frac{\|(x, y) - (x_0, y_0)\|^2}{\eta} - \lambda(T - t).$$

By using the result of Lemma 4.5.1, for some $\lambda > 0$ large enough, we get for any $(t, x, y) \in \overline{\mathcal{B}((T, x_0, y_0); \varepsilon)}$

$$-\partial_t\varphi^{\varepsilon,\eta,\lambda}(t, x, y) - G(t, x, y, \varphi^{\varepsilon,\eta,\lambda}, \nabla_x\varphi^{\varepsilon,\eta,\lambda}, \partial_y\varphi^{\varepsilon,\eta,\lambda}, \Delta_{xx}\varphi^{\varepsilon,\eta,\lambda}, \partial_{yy}\varphi^{\varepsilon,\eta,\lambda}, \nabla_{xy}\varphi^{\varepsilon,\eta,\lambda}) < 0.$$

Moreover, such $\varphi^{\varepsilon,\eta,\lambda}$ satisfies on $\mathcal{T}_\varepsilon$

$$\begin{aligned}\varphi^{\varepsilon,\eta,\lambda}(t, x, y) &\leq v^-(T, x_0, y_0) - \frac{\varepsilon^2}{4\eta} \\ &\leq w_{n_0}(t, x, y) - \varepsilon,\end{aligned}$$

and on $\mathcal{B}((x_0, y_0); \varepsilon)$,

$$\varphi^{\varepsilon,\eta,\lambda}(T, x, y) \leq v^-(T, x, y) \leq U_P(L(x) - U_A^{-1}(y)) - \varepsilon.$$

Thus, for $0 < \delta < \frac{\eta}{2}$ we define

$$w^{\varepsilon,\eta,\lambda,\delta} := \begin{cases} (\varphi^{\varepsilon,\eta,\lambda} + \delta) \vee w_{n_0+}, & \text{on } \mathcal{B}((T, x_0, y_0); \varepsilon), \\ w_{n_0+}, & \text{outside } \mathcal{B}((T, x_0, y_0); \varepsilon). \end{cases}$$

The rest of the proof is analogous similar to the step 1, we show that $w^{\varepsilon,\eta,\lambda,\delta} \in \mathbb{V}^-$ and

$$\begin{aligned} w^{\varepsilon,\eta,\lambda,\delta}(T, x_0, y_0) &= v^-(T, x_0, y_0) + \delta \\ &> v^-(T, x_0, y_0), \end{aligned}$$

which leads to a contradiction. We conclude that

$$v^-(T, x, y) \geq U_P(L(x) - U_A^{-1}(y))$$

for any $(x, y) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$.

Step 2. v^+ is a viscosity sub-solution of (4.5.11).

We now prove that v^+ is a viscosity sub-solution of (4.5.11) by contradiction.

1. **The viscosity subsolution property on $[0, T)$**

   a. **Setting the contradiction.** Let φ be some map from $[0, T] \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuously differentiable in time and twice continuously differentiable in space. Let $(t_0, x_0, y_0) \in [0, T) \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$ be such that $v^+ - \varphi$ attains a strict local maximum equal to 0 at this point. We assume (by contradiction) that

$$\partial_t \varphi(t_0, x_0, y_0) + G(t_0, x_0, y_0, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi) < 0 \qquad (4.9.7)$$

   Then, for any $(z, \gamma) \in \mathbb{R}^{\mathrm{d}} \times \mathcal{M}_{\mathrm{d,d}}(\mathbb{R})$, we have

$$\partial_t \varphi(t_0, x_0, y_0) + \inf_{\nu \in N} g(t_0, x_0, y_0, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi, z, \gamma, \nu) < 0.$$

   Therefore, there exists $\varepsilon > 0$ and $\hat{\nu}(z, \gamma) \in N$ such that

$$\partial_t \varphi(t_0, x_0, y_0) + g(t_0, x_0, y_0, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi, z, \gamma, \hat{\nu}(z, \gamma)) < -\varepsilon.$$

   Using the continuity of our applications, we deduce that on $\mathcal{B}((t_0, x_0, y_0); \varepsilon)$ we have, for a small $\varepsilon' > 0$,

$$\partial_t \varphi(t, x, y) + g(t, x, y, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi, z, \gamma, \hat{\nu}(z, \gamma)) < -\varepsilon'.$$

   We denote $\mathcal{T}_{\varepsilon'} := \overline{\mathcal{B}((t_0, x_0, y_0); \varepsilon')} \setminus \mathcal{B}((t_0, x_0, y_0); \frac{\varepsilon'}{2})$. On $\mathcal{T}_{\varepsilon'}$, we have $v^+ < \varphi$ so that the minimum of $\varphi - v^+$ is attained and is positive. Thus, there exists some $\eta > 0$ such that $\varphi > v^+ + \eta$ on $\mathcal{T}_{\varepsilon'}$. From [115, Lemma 3.8] there exists a non increasing sequence w_n in $\mathbb{V}^+$ converging to v^+. Then, there exists $n_0 \geq 1$ such that for any $n \geq n_0$ large enough, $\varphi - \frac{\eta}{2} > w_n$ on $\mathcal{T}_{\varepsilon'}$. We denote by w_{n_0+} such w_n. Thus, for $0 < \delta < \frac{\eta}{2}$ we define

$$w^\delta := \begin{cases} (\varphi - \delta) \wedge w_{n_0+}, \text{ on } \mathcal{B}((t_0, x_0, y_0); \varepsilon'), \\ w_{n_0+}, \text{ outside } \mathcal{B}((t_0, x_0, y_0); \varepsilon'). \end{cases}$$

Notice that

$$\begin{aligned} w^\delta(t_0, x_0, y_0) &= (\varphi(t_0, x_0, y_0) - \delta) \wedge w_{n_0+}(t_0, x_0, y_0) \\ &\leq \varphi(t_0, x_0, y_0) - \delta \\ &< v^+(t_0, x_0, y_0). \end{aligned} \tag{4.9.8}$$

Thus proving that $w^\delta \in \mathbb{V}^+$ provides the desired contradiction. From now, we fix $t \in [0, T]$, a stopping rule $\tau \in \mathbb{B}^t$ and $(Z, K) \in \mathcal{U}_{Y_0}(t, \tau)$. We need to build a strategy $(\mathbb{P}, \tilde{\nu}) \in \mathcal{V}_{Y_0}(t, \tau)$ such that Property (ii+) in the definition 4.5.3 of a super-solution holds. Recall that $w_{n_0+} \in \mathbb{V}^+$, thus for the fixed $(Z, K) \in \mathcal{U}_{Y_0}(t, \tau)$, there exists some elementary strategy $(\tilde{\mathbb{P}}, \tilde{\nu}^1) \in \mathcal{V}_{Y_0}(t, \tau)$ such that Property (ii+) in Definition 4.5.3 holds.

b. **Building the elementary strategy and Property (ii+)** We consider the following strategy that we denote by $\tilde{\nu}$.

   * If $\varphi - \delta < w_{n_0+}$ at time τ, we choose the strategy $(\hat{\mathbb{P}}, \hat{\nu}(Z, 0))$, where $\hat{\mathbb{P}} \in \mathcal{P}_A$ is such that the minimality condition (ii) in Theorem 4.4.1 holds with control K,

   * Otherwise we follow the elementary strategy $(\tilde{\mathbb{P}}, \tilde{\nu}^1)$.

   The rest of this part is completely similar to Step 1., paragraph 1.b. with control

   $$\tilde{\nu}_t := \hat{\nu}(Z, 0)\mathbf{1}_{\{\varphi - \delta < w_{n_0+}\}} + \tilde{\nu}_t^1 \mathbf{1}_{\{\varphi - \delta \geq w_{n_0+}\}},$$

   and considering the event

   $$\tilde{A} := \left\{\varphi(\tau', X_{\tau'}, Y_{\tau'}) - \delta < w_{n_0+}(\tau', X_{\tau'}, Y_{\tau'})\right\}.$$

   Applying Ito's formula to $\varphi + \delta$ on $\tilde{A}$, and setting $\sigma_r := \sigma(r, X_r^{\tilde{\nu}}, \hat{\nu}(Z, 0))$, we get for any $t \leq \tau' \leq s' \leq s \leq \tau_1'$

   $$\begin{aligned} \varphi(s, X_s^{\tilde{\nu}}, Y_s^{\tilde{\nu}}) &= \varphi(s', X_{s'}^{\tilde{\nu}}, Y_{s'}^{\tilde{\nu}}) + \int_{s'}^s (\nabla_x \varphi + \partial_y \varphi Z) \cdot \sigma_r \mathrm{d}W_r^\star + \int_{s'}^s \partial_t \varphi \mathrm{d}r \\ &+ g(r, X_r^{\tilde{\nu}}, Y_r^{\tilde{\nu}}, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi, Z, 0, \hat{\nu}(Z, 0))\mathrm{d}r \\ &+ \int_{s'}^s \partial_y \varphi(r, X_r^{\tilde{\nu}}, Y_r^{\tilde{\nu}}) \mathrm{d}K_s. \end{aligned}$$

   Since $K = 0$ under $\hat{\mathbb{P}}$ we have that φ is a super–martingale under $\hat{\mathbb{P}}$ and (ii+) is satisfied on $[\tau, \tau_1]$. We thus deduce similarly that $w^\delta \in \mathbb{V}^+$ which contradicts (4.9.8) so that for any $(t, x, y) \in [0, T) \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$

   $$-\partial_t \varphi(t, x, y) - G(t, x, y, \varphi, \nabla_x \varphi, \partial_y \varphi, \Delta_{xx} \varphi, \partial_{yy} \varphi, \nabla_{xy} \varphi) \leq 0$$

2. **The viscosity supersolution property at time T.** We now aim at proving that $v^+(T,x,y) \leq U_P(L(x) - U_A^{-1}(y))$ for any $(x,y) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$. This proof follows the same lines that the previous step 1.2. We assume by contradiction that there exists $(x_0,y_0) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$ such that $v^+(T,x_0,y_0) > U_P(L(x_0) - U_A^{-1}(y_0))$. By continuity, there exists $\varepsilon > 0$ such that

$$U_P(L(x) - U_A^{-1}(y)) \leq v^+(T,x,y) - \varepsilon,\ (x,y) \in \mathcal{B}((x_0,y_0);\varepsilon).$$

We define $\mathcal{T}_\varepsilon := \overline{\mathcal{B}((T,x_0,y_0);\varepsilon)} \setminus \mathcal{B}((T,x_0,y_0);\frac{\varepsilon}{2})$. Let $\eta > 0$ be small enough such that

$$v^+(T,x_0,y_0) + \frac{\varepsilon^2 + 4\ln(1+\frac{\varepsilon}{2})}{4\eta} > \varepsilon + \sup_{(t,x,y)\in\mathcal{T}_\varepsilon} v^+(t,x,y).$$

Thus, using exactly the same Dini type arguments that in [115, 13], there exists n_0 big enough such that for some $w_{n_0} \in \mathbb{V}^+$ we have

$$v^+(T,x_0,y_0) + \frac{\varepsilon^2 + 4\ln(1+\frac{\varepsilon}{2})}{4\eta} > \varepsilon + \sup_{(t,x,y)\in\mathcal{T}_\varepsilon} w_{n_0}(t,x,y).$$

We now define for any $\lambda > 0$

$$\varphi^{\varepsilon,\eta,\lambda}(t,x,y) := v^+(T,x_0,y_0) + \frac{\|x-x_0\|^2 + \ln(1+|y-y_0|)}{\eta} + \lambda(T-t).$$

By using the result of Lemma 4.5.1, for some $\lambda > 0$ large enough, we get for any $(t,x,y) \in \overline{\mathcal{B}((T,x_0,y_0);\varepsilon)}$

$$-\partial_t\varphi^{\varepsilon,\eta,\lambda}(t,x,y) - G(t,x,y,\varphi^{\varepsilon,\eta,\lambda},\nabla_x\varphi^{\varepsilon,\eta,\lambda},\partial_y\varphi^{\varepsilon,\eta,\lambda},\Delta_{xx}\varphi^{\varepsilon,\eta,\lambda},\partial_{yy}\varphi^{\varepsilon,\eta,\lambda},\nabla_{xy}\varphi^{\varepsilon,\eta,\lambda}) > 0.$$

In particular, such $\varphi^{\varepsilon,\eta,\lambda}$ satisfies on $\mathcal{T}_\varepsilon$

$$\begin{aligned}\varphi^{\varepsilon,\eta,\lambda}(t,x,y) &\geq v^+(T,x_0,y_0) + \frac{\varepsilon^2 + 4\ln(1+\frac{\varepsilon}{2})}{4\eta}\\ &\geq w_{n_0}(t,x,y) + \varepsilon,\end{aligned}$$

and on $\mathcal{B}((x_0,y_0);\varepsilon)$,

$$\varphi^{\varepsilon,\eta,\lambda}(T,x,y) \geq v^-(T,x,y) \geq U_P(L(x) - U_A^{-1}(y)) + \varepsilon.$$

Thus, for $0 < \delta < \varepsilon$ small enough we define

$$w^{\varepsilon,\eta,\lambda,\delta} := \begin{cases}(\varphi^{\varepsilon,\eta,\lambda} - \delta) \wedge w_{n_0+},\ \text{on } \mathcal{B}((T,x_0,y_0);\varepsilon),\\ w_{n_0+},\ \text{outside } \mathcal{B}((T,x_0,y_0);\varepsilon).\end{cases}$$

The rest of the proof is completely similar to the step 1.b, we show that $w^{\varepsilon,\eta,\lambda,\delta} \in \mathbb{V}^+$ and

$$\begin{aligned}w^{\varepsilon,\eta,\lambda,\delta}(T,x_0,y_0) &= v^+(T,x_0,y_0) + \delta\\ &> v^+(T,x_0,y_0),\end{aligned}$$

which leads to a contradiction. We deduce that

$$v^+(T,x,y) \leq U_P(L(x) - U_A^{-1}(y))$$

for any $(x,y) \in \mathbb{R}^{\mathrm{d}} \times \mathbb{R}$.

General conclusion and verification. In step 1 (resp. in step 2) we have proved that v^- is a viscosity super-solution (resp. v^+ is a viscosity sub-solution) of the HJBI equation (4.5.11). If a comparison theorem in the viscosity sense holds, then we deduce from (4.5.14) that

$$v^-(0,x,Y_0) \leq V_0^P(Y_0) \leq v^+(0,x,Y_0) \leq v^-(0,x,Y_0),\ x \in \mathbb{R}^{\mathrm{d}},$$

which proves the theorem.

□

Bibliography

[1] R. Aïd, D. Possamaï, and N. Touzi. A principal–agent model for pricing electricity volatility demand. *preprint*, 2016.

[2] R. Antle. *Moral hazard and auditor contracts : an approach to auditors' legal liability and independence*. PhD thesis, Stanford university, 1980.

[3] K. Arrow. The role of securities in the optimal allocation of risk bearing. *Colloques Internationaux du Centre National de la Recherche Scientifique*, 50:41–49, 1953.

[4] K. Arrow. The economics of moral hazard: Further comment. *The American Economic Review*, 58(3):537–539, 1968.

[5] J. S. Banks and R. K. Sundaram. Moral hazard and adverse selection in a model of repeated elections, 1993.

[6] D.P. Baron. A model of the demand for investment banking advising and distribution services for new issues. *The Journal of Finance*, 37(4):955–976, 1982.

[7] D.P. Baron and D. Besanko. Monitoring, moral hazard, asymmetric information, and risk sharing in procurement contracting. *The RAND Journal of Economics*, 18(4):509–532, 1987.

[8] D.P. Baron and D. Besanko. Monitoring of performance in organizational contracting: the case of defense procurement. *The Scandinavian Journal of Economics*, 90(3):329–356, 1988.

[9] D.P. Baron and B. Holmström. The investment banking contract for new issues under asymmetric information: Delegation and the incentive problem. *The Journal of Finance*, 35(5):1115–1138, 1980.

[10] D.P. Baron and R.B. Myerson. Regulating a monopolist with unknown costs. *Econometrica: Journal of the Econometric Society*, 50(4):911–930, 1982.

[11] E. Bayraktar and M. Sîrbu. Stochastic Perron's method and verification without smoothness using viscosity comparison: the linear case. *Proceedings of the American Mathematical Society*, 140(10):3645–3654, 2012.

[12] E. Bayraktar and M. Sîrbu. Stochastic Perron's method for Hamilton–Jacobi–Bellman

equations. *SIAM Journal on Control and Optimization*, 51(6):4274–4294, 2013.

[13] E. Bayraktar and M. Sîrbu. Stochastic Perron's method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games. *Proceedings of the American Mathematical Society*, 142(4):1399–1412, 2014.

[14] E. Bayraktar and S. Yao. A Weak Dynamic Programming Principle for Zero-Sum Stochastic Differential Games with Unbounded Controls. *SIAM Journal on Control and Optimization*, 51(3):2036–2080, 2013.

[15] N. Bhattacharyya. Good managers work more and pay less dividends – a model of dividend policy. Technical report, University of British Columbia, 1997.

[16] B. Biais, T. Mariotti, J.-C. Rochet, and S. Villeneuve. Large risks, limited liability, and dynamic moral hazard. *Econometrica*, 78(1):73–118, 2010.

[17] K. Borch. Equilibrium in a reinsurance market. *Econometrica*, 30(3):424–444, 1962.

[18] B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM Journal on Control and Optimization*, 49(3):948–962, 2011.

[19] M. Boyer, M. Moreau, and M. Truchon. Partage des coûts et tarification des infrastructures. 2006.

[20] R. Buckdahn and J. Li. Stochastic differential games and viscosity solutions of Hamilton–Jacobi–Bellman–Isaacs equations. *SIAM Journal on Control and Optimization*, 47(1):444–475, 2008.

[21] A. Cadenillas, J. Cvitanić, and F. Zapatero. Optimal risk–sharing with effort and project choice. *Journal of Economic Theory*, 133(1):403–440, 2007.

[22] B. Caillaud, R. Guesnerie, and P. Rey. Noisy observation in adverse selection models. *The Review of Economic Studies*, 59(3):595–615, 1992.

[23] G. Carlier. A general existence result for the principal-agent problem with adverse selection. *Journal of Mathematical Economics*, 35(1):129–150, 2001.

[24] G. Carlier, I. Ekeland, and N. Touzi. Optimal derivatives design for mean–variance agents under adverse selection. *Mathematics and Financial Economics*, 1(1):57–80, 2007.

[25] P. Chiappori, I. Macho, P. Rey, and B. Salanié. Repeated moral hazard: The role of memory, commitment, and the access to credit markets. *European Economic Review*, 38(8):1527–1553, 1994.

[26] J. Cvitanić, D. Possamaï, and N. Touzi. Moral hazard in dynamic risk management. *Management Science*, to appear, 2014.

[27] J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–

agent problems. *arXiv preprint arXiv:1510.07111*, 2015.

[28] J. Cvitanić, D. Possamaï, and N. Touzi. Dynamic programming approach to principal–agent problems. *arXiv preprint arXiv:1510.07111v3*, 2017.

[29] J. Cvitanić, X. Wan, and H. Yang. Dynamics of contract design with screening. *Management Science*, 59(5):1229–1244, 2013.

[30] J. Cvitanić, X. Wan, and J. Zhang. Optimal contracts in continuous–time models. *International Journal of Stochastic Analysis*, 2006(095203), 2006.

[31] J. Cvitanić, X. Wan, and J. Zhang. Optimal compensation with hidden action and lump–sum payment in a continuous–time model. *Applied Mathematics and Optimization*, 59(1):99–146, 2009.

[32] J. Cvitanić and J. Zhang. Optimal compensation with adverse selection and dynamic actions. *Mathematics and Financial Economics*, 1(1):21–55, 2007.

[33] P. Daniele, S. Giuffrè, G. Idone, and A. Maugeri. Infinite dimensional duality and applications. *Mathematische Annalen*, 339(1):221–239, 2007.

[34] R.W.R. Darling and É. Pardoux. Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *The Annals of Probability*, 25(3):1135–1159, 1997.

[35] P.M. Demarzo and Y. Sannikov. Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724, 2006.

[36] G. Dionne and P. Lasserre. Dealing with moral hazard and adverse selection simultaneously. Cahier de recherche 8559, Département de sciences économiques, université de Montréal, 1985.

[37] M.B. Donato. The infinite dimensional Lagrange multiplier rule for convex optimization problems. *Journal of Functional Analysis*, 261(8):2083–2093, 2011.

[38] I. Ekren, N. Touzi, and J. Zhang. Viscosity solutions of fully nonlinear parabolic path dependent PDEs: part i. *The Annals of Probability*, 44(2):1212–1253, 2016.

[39] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[40] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM journal on Control and Optimization*, 33(1):29–66, 1995.

[41] N. El Karoui and X. Tan. Capacities, measurable selection and dynamic programming part I: abstract framework. *arXiv preprint arXiv:1310.3363*, 2013.

[42] R. Élie, T. Mastrolia, and D. Possamaï. A tale of a principal and many many agents.

arXiv preprint arXiv:1608.05226, 2016.

[43] P.S. Faynzilberg and P. Kumar. On the generalized principal–agent problem: decomposition and existence results. *Review of Economic Design*, 5(1):23–58, 2000.

[44] W.H. Fleming and H.M. Soner. *Controlled Markov processes and viscosity solutions*. Springer, 2006.

[45] D. Fudenberg, B. Holmstrom, and P. Milgrom. Short-term contracts and long-term agency relationships. *Journal of Economic Theory*, 51(1):1 – 31, 1990.

[46] D. Gottlieb and H. Moreira. Simultaneous adverse selection and moral hazard. Technical report, The university of Pennsylvania and escola Brasileira de economia e finanças, 2011.

[47] S.J. Grossman and O.D. Hart. An analysis of the principal–agent problem. *Econometrica*, 51(1):7–45, 1983.

[48] R. Guesnerie and J.-J. Laffont. A complete solution to a class of principal–agent problems with an application to the control of a self-managed firm. *Journal of Public Economics*, 25(3):329–369, 1984.

[49] R. Guesnerie, P. Picard, and P. Rey. Adverse selection and moral hazard with risk neutral agents. *European Economic Review*, 33(4):807–823, 1989.

[50] S. Hamadène and J.-P. Lepeltier. Backward equations, stochastic control and zero–sum stochastic differential games. *Stochastics: An International Journal of Probability and Stochastic Processes*, 54(3-4):221–231, 1995.

[51] S. Hamadène, J.-P. Lepeltier, and S. Peng. BSDEs with continuous coefficients and stochastic differential games. In N. El Karoui and L. Mazliak, editors, *Backward stochastic differential equations*, volume 364 of *Chapman & Hall/CRC Research Notes in Mathematics Series*, pages 115–128. Longman, 1997.

[52] B. Hölmstrom. Moral hazard and observability. *The Bell Journal of Economics*, 10(1):74–91, 1979.

[53] B. Holmström and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987.

[54] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.

[55] I. Jewitt. Justifying the first–order approach to principal–agent problems. *Econometrica*, 56(5):1177–1190, 1988.

[56] I. Jewitt, O. Kadan, and J. Swinkels. Moral hazard with bounded payments. *Journal of Economic Theory*, 143:59–82, 2008.

[57] B. Jullien, B. Salanié, and F. Salanié. Screening risk–averse agents under moral hazard: single–crossing and the CARA case. *Economic Theory*, 30(1):151–169, 2007.

[58] R.L. Karandikar. On pathwise stochastic integration. *Stochastic Processes and their Applications*, 57(1):11–18, 1995.

[59] J.-J. Laffont and J. Tirole. Using cost observation to regulate firms. *The Journal of Political Economy*, 94(3, part 1):614–641, 1986.

[60] R. Lambert. Long-term contracts and moral hazard. *The Bell Journal of Economics*, 14(2):441–452, 1983.

[61] T.R. Lewis and D.E.M. Sappington. Optimal capital structure in agency relationships. *The RAND Journal of Economics*, 26(3):343–361, 1995.

[62] M. Malcomson and F. Spinnewyn. The multiperiod principal-agent problem. *The Review of Economic Studies*, 55(3):391–407, 1988.

[63] E. Maskin and J. Riley. Monopoly with incomplete information. *The RAND Journal of Economics*, 15(2):171–196, 1984.

[64] T. Mastrolia and D. Possamaï. Moral hazard under ambiguity. *arXiv preprint arXiv:1511.03616*, 2015.

[65] R.P. McAfee and J. McMillan. Bidding for contracts: a principal–agent analysis. *The RAND Journal of Economics*, 17(3):326–338, 1986.

[66] N.D. Melumad and S. Reichelstein. Centralization versus delegation and the value of communication. *Journal of Accounting Research*, 25:1–18, 1987.

[67] N.D. Melumad and S. Reichelstein. Value of communication in agencies. *Journal of Economic Theory*, 47(2):334–368, 1989.

[68] J. A. Mirrlees. An exploration in the theory of optimum income taxation. *The review of economic studies*, 38(2):175–208, 1971.

[69] J.A. Mirrlees. Population policy and the taxation of family size. *Journal of Public Economics*, 1(2):169–198, 1972.

[70] J.A. Mirrlees. Notes on welfare economics, information and uncertainty. In M.S. Balch, D.L. McFadden, and S.Y. Wu, editors, *Essays on economic behavior under uncertainty*, pages 243–261. Amsterdam: North Holland, 1974.

[71] J.A. Mirrlees. The theory of moral hazard and unobservable behaviour: part I. *mimeo*, 1975.

[72] J.A. Mirrlees. The optimal structure of incentives and authority within an organization. *The Bell Journal of Economics*, 7(1):105–131, 1976.

[73] J.A. Mirrlees. The implications of moral hazard for optimal insurance. *mimeo*, 1979.

[74] J.A. Mirrlees. The theory of moral hazard and unobservable behaviour: part i (reprint of the unpublished 1975 version). *The Review of Economic Studies*, 66(1):3–21, 1999.

[75] M. J. Morey and L. D. Kirsch. Retail choice in electricity: what have we learned in 2à years? 2016.

[76] S. Moroni and J. Swinkels. Existence and non-existence in the moral hazard problem. *Journal of Economic Theory*, 150:668–682, 2014.

[77] H.M. Müller. The first–best sharing rule in the continuous–time principal–agent problem with exponential utility. *Journal of Economic Theory*, 79(2):276–280, 1998.

[78] H.M. Müller. Asymptotic efficiency in dynamic principal–agent problems. *Journal of Economic Theory*, 91(2):292–301, 2000.

[79] M. Mussa and S. Rosen. Monopoly and product quality. *Journal of Economic Theory*, 18(2):301–317, 1978.

[80] R.B. Myerson. Optimal coordination mechanisms in generalized principal–agent problems. *Journal of Mathematical Economics*, 10(1):67–81, 1982.

[81] M. Nutz. Pathwise construction of stochastic integrals. *Electronic Communications in Probability*, 17(24):1–7, 2012.

[82] M. Nutz and R. van Handel. Constructing sublinear expectations on path space. *Stochastic Processes and their Applications*, 123(8):3100–3121, 2013.

[83] F.H.Jr. Page. Optimal contract mechanisms for principal–agent problems with moral hazard and adverse selection. *Economic Theory*, 1(4):323–338, 1991.

[84] H. Pagès. Bank monitoring incentives and optimal ABS. *Journal of Financial Intermediation*, 22(1):30–54, 2013.

[85] H. Pagès and D. Possamaï. A mathematical treatment of bank monitoring incentives. *Finance and Stochastics*, 18(1):39–73, 2014.

[86] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[87] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In B.L. Rozovskii and R.B. Sowers, editors, *Stochastic partial differential equations and their applications. Proceedings of IFIP WG 7/1 international conference University of North Carolina at Charlotte, NC June 6–8, 1991*, volume 176 of *Lecture notes in control and information sciences*, pages 200–217. Springer, 1992.

[88] M. Pauly. The economics of moral hazard. *American Economic Review*, 58(3):531–537, 1985.

[89] S. Peng. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics and Stochastic Reports*, 37(1-2):61–74, 1991.

[90] T. Pham and J. Zhang. Two person zero–sum game in weak formulation and path dependent Bellman–Isaacs equation. *SIAM Journal on Control and Optimization*, 52(4):2090–2121, 2014.

[91] P. Picard. On the design of incentive schemes under moral hazard and adverse selection. *Journal of Public Economics*, 33(3):305–331, 1987.

[92] D. Possamaï. *A journey through second–order BSDEs and other contemporary problems of mathematical finance*. PhD thesis, École Polytechnique, 2011.

[93] D. Possamaï, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *arXiv preprint arXiv:1510.08439*, 2015.

[94] D. Possamaï and C. Zhou. Second order backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 123(10):3770–3799, 2013.

[95] USmartConsumer Project. European smart metering landscape report ütilities and consumers ̈. 2016.

[96] R. Radner. Repeated principal-agent games with discounting. *Econometrica*, 53(5):1173–98, 1985.

[97] M. Rasanen, J. Ruusunen, and R.P. Hamalainen. Optimal tariff design under consumer self-selection,. *Energy Economics*, 1997.

[98] Z. Ren, N. Touzi, and J. Zhang. Comparison of viscosity solutions of fully nonlinear degenerate parabolic path–dependent PDEs. *arXiv preprint arXiv:1511.05910*, 2015.

[99] P. Rey. A note on the result of rogerson (1985) on the desirability of constrained savings. *mimeo*, 1988.

[100] P. Rey and B. Salanié. Long-term, short-term and renegotiation: On the value of commitment in contracting. *Econometrica*, 58(3):597–619, 1990.

[101] K.W.S. Roberts. Welfare considerations on nonlinear pricing. *The Economic Journal*, 89(353):66–83, 1979.

[102] J.-C. Rochet and P. Choné. Ironing, sweeping, and multidimensional screening. *Econometrica*, 66(4):783–826, 1998.

[103] R.T. Rockafellar and R.J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschafte*. Springer, 3rd edition, 1998.

[104] W. Rogerson. Repeated moral hazard. *Econometrica*, 53(1):69–76, 1985.

[105] W. Rogerson. The first-order approach to principal-agent problems. *Econometrica*,

53(6):1357–1368, 1985.

[106] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[107] M. Royer. Backward stochastic differential equations with jumps and related non–linear expectations. *Stochastic Processes and their Applications*, 116(10):1358–1376, 2006.

[108] B. Salanié. Sélection adverse et aversion pour le risque. *Annales d'Économie et de Statistique*, 18:131–149, 1990.

[109] Y. Sannikov. A continuous–time version of the principal–agent problem. *The Review of Economic Studies*, 75(3):957–984, 2008.

[110] Y. Sannikov. Contracts: the theory of dynamic principal–agent relationships and the continuous–time approach. In D. Acemoglu, M. Arellano, and E. Dekel, editors, *Advances in economics and econometrics, 10th world congress of the Econometric Society, volume 1, economic theory*, number 49 in Econometric Society Monographs, pages 89–124. Cambridge University Press, 2013.

[111] H. Schättler and J. Sung. The first–order approach to the continuous–time principal–agent problem with exponential utility. *Journal of Economic Theory*, 61(2):331–371, 1993.

[112] H. Schättler and J. Sung. On optimal sharing rules in discrete–and continuous–time principal–agent problems with exponential utility. *Journal of Economic Dynamics and Control*, 21(2):551–574, 1997.

[113] S. Shavell. Risk sharing and incentives in the principal and agent relationship. *The Bell Journal of Economics*, 10(1):55–73, 1979.

[114] M. Sion. On general minimax theorems. *Pacific Journal of Mathematics*, 8(1):171–176, 1958.

[115] M. Sîrbu. Stochastic Perron's method and elementary strategies for zero–sum differential games. *SIAM Journal on Control and Optimization*, 52(3):1693–1711, 2014.

[116] H.M. Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward SDEs. *Probability Theory and Related Fields*, 153(1-2):149–190, 2012.

[117] H.M. Soner, N. Touzi, and J. Zhang. Dual formulation of second order target problems. *The Annals of Applied Probability*, 23(1):308–347, 2013.

[118] S.E. Spear and S. Srivastava. On repeated moral hazard with discounting. *The Review of Economic Studies*, 54(4):599–617, 1987.

[119] A.M. Spence. Multi–product quantity–dependent prices and profitability constraints. *The Review of Economic Studies*, 47(5):821–842, 1980.

[120] M. Spence and R. Zeckhauser. Insurance, information, and individual action. *The American Economic Review*, 61(2):380–387, 1971.

[121] D.W. Stroock and S.R.S. Varadhan. *Multidimensional diffusion processes*. Springer, 2007.

[122] J. Sung. Linearity with project selection and controllable diffusion rate in continuous–time principal–agent problems. *The RAND Journal of Economics*, 26(4):720–743, 1995.

[123] J. Sung. Corporate insurance and managerial incentives. *Journal of Economic Theory*, 74(2):297–332, 1997.

[124] J. Sung. Optimal contracts under adverse selection and moral hazard: a continuous–time approach. *Review of Financial Studies*, 18(3):1021–1073, 2005.

[125] J. Sung. Optimal contracting under mean–volatility ambiguity uncertainties: an alternative perspective on managerial compensation. *Available at SSRN 2601174*, 2015.

[126] B. Theilen. Simultaneous moral hazard and adverse selection with risk averse agents. *Economics Letters*, 79(2):283–289, 2003.

[127] C. Villani. *Optimal transport: old and new*, volume 338 of *Grundlehren der Mathematischen Wissenschafte*. Springer, 2008.

[128] M.L. Weitzman. The new Soviet incentive model. *The Bell Journal of Economics*, 7(1):251–257, 1976.

[129] N. Williams. On dynamic principal–agent problems in continuous time. University of Wisconsin, Madison, 2009.

[130] N. Williams. Persistent private information. *Econometrica*, 79(4):1233–1275, 2011.

[131] R. Wilson. The theory of syndicates. *Econometrica*, 36(1):119–132, 1968.

[132] R. Wilson. *Non linear pricing*. Oxford University Press, 1993.

[133] Robert B Wilson. *Nonlinear pricing*. Oxford University Press on Demand, 1993.

[134] R. Zeckhauser. Medical insurance: a case study of the tradeoff between risk spreading and appropriate incentives. *Journal of Economic Theory*, 2(1):10–26, 1970.

[135] Y. Zhang. Dynamic contracting with persistent shocks. *Journal of Economic Theory*, 144(2):635–675, 2009.

[136] L. Zou. Threat-based incentive mechanisms under moral hazard and adverse selection. *Journal of Comparative Economics*, 16(1):47–74, 1992.