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A non-perverse Soergel bimodule in type A

Un bimodule de Soergel non pervers de type A

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ABSTRACT

A basic question concerning indecomposable Soergel bimodules is to understand their endomorphism rings. In characteristic zero all degree-zero endomorphisms are isomorphisms (a fact proved by Elias and the second author) which implies the Kazhdan–Lusztig conjectures. More recently, many examples in positive characteristic have been discovered with larger degree zero endomorphisms. These give counter-examples to expected bounds in Lusztig's conjecture. Here we prove the existence of indecomposable Soergel bimodules in type A having non-zero endomorphisms of negative degree. This gives the existence of a non-perverse parity sheaf in type A.

RÉSUMÉ

L'étude de l'anneau des endomorphismes des bimodules de Soergel indécomposables est une question importante. En caractéristique zéro, tous les endomorphismes de degré zero sont des isomorphismes (comme démontré par Elias et le deuxième auteur). Ceci implique les conjectures de Kazhdan–Lusztig. Plus récemment, en caractéristique positive, de nombreux exemples ont été trouvés d'endomorphismes de degré zero qui ne sont pas des isomorphismes. Ceci donne des contre-exemples aux bornes dans la conjecture de Lusztig. Dans cette Note, nous prouvons l'existence de bimodules de Soergel indécomposables, de type A, ayant un endomorphisme de degré négatif. Ceci prouve l'existence d'un faisceau de parité non pervers de type A.

1. Introduction

Kazhdan–Lusztig polynomials play a central role in highest weight representation theory. It is gradually becoming clear that in modular (i.e. characteristic p) representation theory a similarly central role is played by p-Kazhdan–Lusztig polynomials [12,9,16,15,1]. Just as Kazhdan–Lusztig polynomials describe the stalks of intersection cohomology complexes on flag

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varieties, p-Kazhdan–Lusztig polynomials describe the stalks of parity sheaves with coefficients in a field of characteristic p [12, Part 3].

Kazhdan–Lusztig polynomials are characterised by a self-duality condition and a degree bound, which mimics the defining properties of the intersection cohomology sheaf. Currently there exists no similar combinatorial characterisation of p-Kazhdan–Lusztig polynomials, however they always satisfy the self-duality condition. An important problem concerning p-Kazhdan–Lusztig polynomials is whether the degree bound for p-Kazhdan–Lusztig polynomials can "fail by more than one". In the language of parity sheaves, this translates into the question as to whether an indecomposable parity sheaf is necessarily perverse. In the language of Soergel bimodules, it translates into the question as to whether an indecomposable Soergel bimodule can have non-zero (and necessarily nilpotent) endomorphisms of negative degree. In this paper an indecomposable Soergel bimodule possessing such an endomorphism is called *non-perverse*.

Since the beginning of the theory of parity sheaves, it was known that parity sheaves need not be perverse on nilpotent cones and on the affine flag variety (see [7, §4.3] and [8, Lemma 3.7]). In 2009, the second author found an example of a parity sheaf on a finite flag variety of type C_3 in characteristic 2 which is not perverse [9, §5.4]. Moreover, a recent conjecture of Lusztig and the second author implies that parity sheaves can be arbitrarily far from being perverse on the affine flag manifold of SL_3 [10]. However, in [8,11] it is proved that parity sheaves on the affine Grassmannian *are* perverse as long as p is a good prime. (Recall that a prime p is good for a fixed root system if it does not divide any coefficient of the highest root when expressed in the simple roots.) Extensive calculations on finite flag varieties have suggested that parity sheaves are perhaps perverse in good characteristic. Recently, Achar and Riche proved that this would have nice consequences (existence of "Koszul like gradings") on modular category $\mathcal{O}$ [2].

In this note we prove the existence of a parity sheaf in characteristic $p = 2$[1] on the flag variety GL_{15}/B which is not perverse. In other words, non-perverse Soergel bimodule for S_{15} exist. Our construction is a variation of the method of [16]. We expected to be able to produce many examples in this way (and thus obtain results similar to [16] for non-perverse Soergel bimodules), however extensive computer calculations only produced a few more examples, all in characteristic 2.

Another interesting consequence of our construction is that it gives a Schubert variety for the general linear group with no semi-small (generalised) Bott–Samelson resolution. More generally, the Schubert variety in question admits no semi-small even (in the sense of [7, §2.4]) resolution. Probably the Schubert variety admits no semi-small resolution at all.

2. Soergel and singular Soergel diagrammatics

2.1. Hecke algebra and spherical module

Fix $n \geq 0$. Let $W := S_n$ denote the symmetric group on n letters, viewed as a Coxeter group (W, S) where $S = \{s_i\}_{1 \leq i \leq n-1}$ is the set of simple transpositions (i.e. $s_i := (i, i+1)$), with length function ℓ and Bruhat order $\leq$. Let H denote the Hecke algebra of (W, S) with standard $\mathbb{Z}[v^{\pm 1}]$-basis $\{h_x\}_{x \in W}$ and Kazhdan–Lusztig basis $\{b_x\}_{x \in W}$ (e.g., $b_s = h_s + v h_{\mathrm{id}}$ for all $s \in S$). We write $b_x := \sum \beta_{y,x} h_y$ (so $\beta_{y,x}$ are Kazhdan–Lusztig polynomials). For any expression $\underline{x} := (s_1, s_2, \ldots, s_m)$ we set $b_{\underline{x}} := b_{s_1} b_{s_2} \ldots b_{s_m}$.

For any subset $A \subset S$ we denote by W_A the (standard parabolic) subgroup it generates, by $w_A \in W_A$ the longest element and by W^A the minimal coset representatives for W/W_A. Corresponding to A we have the spherical (left) module M with its standard basis $\{m_x\}_{x \in W^A}$ and Kazhdan–Lusztig basis $\{c_x\}_{x \in W^A}$ (see [13, §3], note however that we work with left modules throughout). We write $c_x := \sum_{y,x \in W^A} \gamma_{y,x} m_y$ (so $\gamma_{y,x}$ are spherical Kazhdan–Lusztig polynomials). The map $m_x \mapsto h_{x w_A}$ gives an embedding $\phi : M \hookrightarrow H$ of left H-modules mapping $c_x \mapsto b_{x w_A}$ [13, Proposition 3.4]. Given any expression $\underline{x} := (s_1, s_2, \ldots, s_m)$ we set $c_{\underline{x}} := b_{s_1} b_{s_2} \ldots b_{s_m} \cdot m_{\mathrm{id}} \in M$.

2.2. Diagrammatic Soergel bimodules

Fix a field $\Bbbk$ of characteristic $p \geq 0$ and let $R := \Bbbk[x_1, \ldots, x_n]$ be graded with $\deg x_i = 2$. The symmetric group W acts naturally on R via permutation of variables. With W and R one may associate an additive graded and Karoubian monoidal category $\mathcal{H}$ of "diagrammatic Soergel bimodules" as in [5]. We denote the shift functor on $\mathcal{H}$ by $B \mapsto B(i)$ for $i \in \mathbb{Z}$. For any expression $\underline{x}$ we denote by $B_{\underline{x}}$ the corresponding Bott–Samelson object in $\mathcal{H}$ and (if $\underline{x}$ is reduced) by B_x its maximal indecomposable summand. By [5, Theorem 6.26] the set $\{B_x \mid x \in W\}$ is a complete set of isomorphism classes of indecomposable objects in $\mathcal{H}$, up to shift and isomorphism. We denote by $[\mathcal{H}]$ the split Grothendieck ring of $\mathcal{H}$ (a $\mathbb{Z}[v^{\pm 1}]$-module via $v \cdot [M] := [M(1)]$) and by $\mathrm{ch} : \mathcal{H} \to H$ the character [5, §6.5] (ch is uniquely characterised by $\mathrm{ch}(B_{\underline{x}}) = b_{\underline{x}}$ and $\mathrm{ch}(M(1)) = v\,\mathrm{ch}(M)$). It induces an isomorphism $\mathrm{ch} : [\mathcal{H}] \xrightarrow{\sim} H$. We denote by ${}^p b_x := \mathrm{ch}(B_x) \in H$ the character of B_x. The set $\{{}^p b_x\}_{x \in W}$ only depends on the characteristic p of $\Bbbk$ and yields the *p-canonical basis* of H. We have ${}^0 b_x = b_x$ for all $x \in W$. The expression ${}^p b_x = \sum {}^p \beta_{y,x} h_y$ defines the *p-Kazhdan–Lusztig polynomials* ${}^p \beta_{y,x}$.

[1] Note that $p = 2$ is good for GL_n.

2.3. Diagrammatic Spherical category

For any subset $A \subset S$ we denote by $\mathcal{M}$ the spherical category associated with A (see [3, §5]). It is a graded additive Karoubian left $\mathcal{H}$-module category with shift functor $C \mapsto C(m)$. We denote by C_{id} the "identity" of $\mathcal{M}$ (in the notation of [3], C_{id} is given by the empty diagram consisting only of the A-colored membrane). For any expression $\underline{x}$ we denote by $C_{\underline{x}}$ the object $B_{\underline{x}} \cdot C_{\mathrm{id}}$ and (if $\underline{x}$ is a reduced expression for $x \in W^A$) by C_x its maximal indecomposable summand. The set $\{C_x \mid x \in W^A\}$ is a complete set of isomorphism classes of indecomposable objects in $\mathcal{M}$, up to shift and isomorphism. We denote by $[\mathcal{M}]$ the split Grothendieck group of $\mathcal{M}$ (a $\mathbb{Z}[v^{\pm 1}]$-module as above) and by $\mathrm{ch} : \mathcal{M} \to M$ the character (it is uniquely characterised by $\mathrm{ch}(C_{\underline{x}}) = c_{\underline{x}}$ and $\mathrm{ch}(C(1)) = v\,\mathrm{ch}(C)$). The character map satisfies $\mathrm{ch}(BC) = \mathrm{ch}(B)\,\mathrm{ch}(C)$ for all $B \in \mathcal{H}$ and $C \in \mathcal{M}$, and induces an isomorphism $\mathrm{ch} : [\mathcal{M}] \xrightarrow{\sim} M$ of left $H \cong [\mathcal{H}]$-modules. We denote by ${}^p c_x := \mathrm{ch}(C_x) \in M$ the character of C_x. The set $\{{}^p c_x\}_{x \in W^A}$ yields the *p-canonical basis* of M. We have ${}^0 c_x = c_x$ for all $x \in W^A$. The expression ${}^p c_x = \sum_{y \in W^A} {}^p \gamma_{y,x} m_y$ defines the *spherical p-Kazhdan–Lusztig polynomials* ${}^p \gamma_{y,x}$.

There is a functor $\Phi : \mathcal{M} \to \mathcal{H}$ of left $\mathcal{H}$-module categories which sends C_{id} to B_{w_A} (see [3, Definition 5.4]) and satisfies $\Phi(C_x) = B_{xw_A}$ for all $x \in W^A$. Passing to split Grothendieck groups as above it realises the embedding $\phi : M \hookrightarrow H$. In particular $\phi({}^p c_x) = {}^p b_{xw_A}$ and hence

$$ {}^p \gamma_{y,x} = {}^p \beta_{yw_A, xw_A} \tag{2.1} $$

for all $x, y \in W^A$.

2.4. Soergel's hom formulas

Consider the bilinear form $(-,-) : H \times H \to \mathbb{Z}[v^{\pm 1}]$ on H defined in [5, §2.4]. It satisfies $(ph, qh') = \overline{p}q(h,h')$, $(b_s h, h') = (h, b_s h')$ and $(hb_s, h') = (h, h'b_s)$ for all $p, q \in \mathbb{Z}[v^{\pm 1}]$, $h, h' \in H$ and $s \in S$ (see [5, §2.4]). Similarly, there is a unique bilinear form $(-,-) : M \times M \to \mathbb{Z}[v^{\pm 1}]$ defined by $(m, m') := (\phi(m), \phi(m'))/\widetilde{\pi}(A) \in \mathbb{Z}[v^{\pm 1}]$, where $\widetilde{\pi}(A) := \sum_{x \in W_A} v^{2\ell(x)}$. It satisfies $(pm, qm') = \overline{p}q(m, m')$ and $(b_s m, m') = (m, b_s m')$ for all $p, q \in \mathbb{Z}[v^{\pm 1}]$, $m, m' \in M$ and $s \in S$. (It is not immediately obvious that $(\phi(m), \phi(m'))/\widetilde{\pi}(A)$ always belongs to $\mathbb{Z}[v^{\pm 1}]$, but this is the case by [14, (2.9)].)

Given $B, B' \in \mathcal{H}$ we denote by $\mathrm{Hom}^\bullet(B, B') := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{H}}(B, B'(n))$, which is naturally a graded R-bimodule. Similarly, given $C, C' \in \mathcal{M}$ we denote by $\mathrm{Hom}^\bullet(C, C') := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(C, C'(n))$, which is naturally a graded (R, R^A)-bimodule (see [3, Definition 5.1]). *Soergel's hom formulas* (crucial below) are the statements:

$$ \begin{aligned} &\text{For } B, B' \in \mathcal{H}, \mathrm{Hom}^\bullet(B, B') \text{ is graded free as a left } R\text{-module,} \\ &\qquad\qquad \text{of graded rank } (\mathrm{ch}(B), \mathrm{ch}(B')). \end{aligned} \tag{2.2} $$

$$ \begin{aligned} &\text{For } C, C' \in \mathcal{M}, \mathrm{Hom}^\bullet(C, C') \text{ is graded free as a left } R\text{-module,} \\ &\qquad\qquad \text{of graded rank } (\mathrm{ch}(C), \mathrm{ch}(C')). \end{aligned} \tag{2.3} $$

As in the introduction, we say that an indecomposable object $X \in \mathcal{H}$ (resp. $X \in \mathcal{M}$) is *perverse* if it has no non-zero endomorphisms of negative degree. (This terminology comes from [4], where such bimodules play a key role in the proof of Soergel's conjecture.) The following lemmas are a direct consequence of the hom formulas above (see [4, (6.1)]):

Lemma 2.1. *A self-dual Soergel bimodule B is perverse* $\Leftrightarrow \mathrm{ch}(B) = \bigoplus_{z \in W} \mathbb{Z} b_z$.

Lemma 2.2. *A self-dual element $C \in \mathcal{M}$ is perverse* $\Leftrightarrow \mathrm{ch}(C) = \bigoplus_{z \in W^A} \mathbb{Z} c_z$.

Below we will prove that there exists a non-perverse object in $\mathcal{M}$. By (2.1) non-perverse objects in $\mathcal{M}$ produce non-perverse objects in $\mathcal{H}$ by application of the functor Φ.

2.5. Intersection forms in $\mathcal{M}$

In what follows we identify W^A and W/W_A via the canonical isomorphism given by the composition $W^A \hookrightarrow W \twoheadrightarrow W/W_A$. If $I \subset W/W_A$ is an ideal (i.e. $x \le y \in I \Rightarrow x \in I$) we denote by $\mathcal{M}_I$ the ideal of $\mathcal{M}$ generated by all morphisms which factor through an object $C_{\underline{y}}$, for any reduced expression $\underline{y}$ for $y \in I$.

Given $x \in W^A$ we denote by $\mathcal{M}^{\ge x}$ the quotient category $\mathcal{M}/\mathcal{M}_{\not\ge x}$ where $\not\ge x := \{y \in W^A \mid y \not\ge x\}$. We write $\mathrm{Hom}_{\ge x}$ for (degree zero) morphisms in $\mathcal{M}^{\ge x}$. All objects $C_{\underline{x}}$ corresponding to reduced expressions $\underline{x}$ for x become canonically isomorphic to C_x in $\mathcal{M}^{\ge x}$. For $C \in \mathcal{M}$ and any $x \in W^A$ the spaces

$$ \mathrm{Hom}^\bullet_{\ge x}(C_x, M) = \bigoplus \mathrm{Hom}_{\ge x}(C_x, M(i)) \quad \text{and} \quad \mathrm{Hom}^\bullet_{\ge x}(M, C_x) = \bigoplus \mathrm{Hom}_{\ge x}(M, C_x(i)) $$

are free as graded left R-modules of graded rank p_x where $\mathrm{ch}(M) = \sum_{x \in W^A} p_x m_x$. In particular, we have $\mathrm{End}_{\ge x}(C_x) = R$. Given an expression $\underline{w}$ and an element $x \in W^A$, the *intersection form* is the canonical pairing

$$I^{\Bbbk}_{x,\underline{w},d} : \mathrm{Hom}_{\geq x}(C_x(d), C_{\underline{w}}) \times \mathrm{Hom}_{\geq x}(C_{\underline{w}}, C_x(d)) \to \Bbbk = \mathrm{End}_{\geq x}(C_x(d))/(R^+)$$

where $R^+ \subset R$ denotes the ideal of elements of positive degree.

Lemma 2.3. *The multiplicity of $C_x(d)$ as a summand of $C_{\underline{w}}$ equals the rank of $I^{\Bbbk}_{x,\underline{w},d}$.*

Proof. This claim is standard for a $\Bbbk$-linear Krull–Schmidt category with finite dimensional Hom spaces. For one proof see [7, Lemma 3.1]. □

2.6. Parabolic defect

Fix a word $\underline{y} = s_{i_1} \dots s_{i_m}$ in S representing an element $y \in W$. A *subexpression* of $\underline{y}$ is a sequence $\underline{e} = e_1 \dots e_m$ with $e_i \in \{0,1\}$ for all i. We set $\underline{y}^{\underline{e}} := s_{i_1}^{e_1} \dots s_{i_m}^{e_m} \in W$. Any subexpression $\underline{e}$ determines a sequence $y_0, y_1, \dots, y_m \in W$ via $y_0 := \mathrm{id}$, $y_j := s_{i_{m+1-j}}^{e_{m+1-j}} y_{j-1}$ for $1 \leq j \leq m$ (so $y_m = \underline{y}^{\underline{e}}$). Given a subexpression $\underline{e}$ we associate a sequence $d_j \in \{U, D, S\}$ (for *U*p, *D*own, *S*tay) via

$$d_j := \begin{cases} U & \text{if } y_{m-j} < s_{i_j} y_{m-j} \in W/W_A, \\ D & \text{if } y_{m-j} > s_{i_j} y_{m-j} \in W/W_A, \\ S & \text{if } s_{i_j} y_{m-j} = y_{m-j} \text{ in } W/W_A. \end{cases}$$

We usually view $\underline{e}$ as the decorated sequence $(d_1 e_1, \dots, d_m e_m)$. The *parabolic defect* of $\underline{e}$ is

$$\mathrm{pdf}(\underline{e}) := |\{i \mid d_i e_i = U0 \text{ or } S1\}| - |\{i \mid d_i e_i = D0 \text{ or } S0\}|.$$

One can see readily from the formula for $b_s c_x$ (see [13, §3] for example) the following formula of Dehodar

$$m_{\underline{x}} = \sum_{\underline{e} \subset \underline{x}} v^{\mathrm{pdf}(\underline{e})} m_{\underline{x}^{\underline{e}}}, \tag{2.4}$$

where $\underline{x}^{\underline{e}}$ is viewed as an element of W/W_A.

3. Existence of a non-perverse indecomposable Soergel bimodule

3.1. Strategy of the proof

Consider a reduced expression $\underline{w}$ representing an element $w \in W^A$ and another element $x \in W^A$, such that

$$\mathrm{rk}(I^{\mathbb{Q}}_{x,\underline{w},-1}) = 1\ ,\ \mathrm{rk}(I^{\mathbb{F}_2}_{x,\underline{w},-1}) = 0 \ \text{ and} \tag{3.1}$$

$$m_{\underline{w}} \in \bigoplus_{x<z\leq w} \mathbb{Z}[v] m_z \oplus \bigoplus_{x \not< z} \mathbb{Z}[v, v^{-1}] m_z. \tag{3.2}$$

Lemma 3.1. *If the above hypotheses are satisfied, then there is a non-perverse indecomposable object in $\mathcal{M}$ over the field $\mathbb{F}_2$.*

Proof. Suppose, for contradiction, that there is no non-perverse indecomposable object in $\mathcal{M}$ over the field $\mathbb{F}_2$. By Lemma 2.2 we have

$${}^2\underline{m}_y \in \bigoplus_{z \in W^A} \mathbb{Z} \underline{m}_z \quad \text{for all } y < w. \tag{3.3}$$

By Lemma 2.3 we have the following formulae

$$c_w = m_{\underline{w}} - \sum_{\substack{x<w \\ x \in W^A}} \Big(\sum_{d \in \mathbb{Z}} \mathrm{rk}(I^{\mathbb{Q}}_{x,\underline{w},d}) v^d \Big) c_x \quad \text{and} \quad {}^2c_w = m_{\underline{w}} - \sum_{\substack{x<w \\ x \in W^A}} \Big(\sum_{d \in \mathbb{Z}} \mathrm{rk}(I^{\mathbb{F}_2}_{x,\underline{w},d}) v^d \Big) {}^2c_x.$$

By Equations (3.1), (3.2) and (3.3) we have that $v^{-1} m_x$ appears in the expansion of ${}^2c_w$ in the standard basis. Now apply Lemma 2.2 again. □

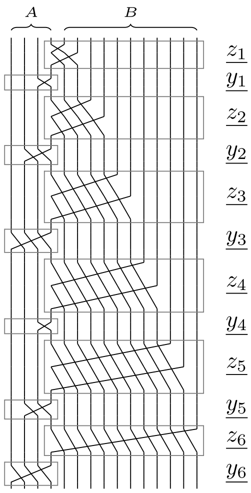

Fig. 1. The reduced expression $\underline{w}$.

3.2. *The elements $\underline{w}$ and x*

Consider the string diagram in Fig. 1. We denote the corresponding reduced expression (obtained by reading from bottom to top) by $\underline{w}$, so that

$$\underline{w} = (s_1, s_2, \dots, s_{14}, s_2, s_3, \dots, s_{13}, s_4, s_5, \dots, s_{12}, s_3, s_4, \dots, s_{11}, \dots) =: (t_1, \dots, t_{78}).$$

Let $w \in W^A$ be the element represented by $\underline{w}$. By [16, Lemma 5.5] (or by simple inspection), $\underline{w}$ is reduced. Let us define $x := w_B \in W^A$.

3.3. *Proof of equations (3.1) and (3.2)*

If $\underline{e} = (e_1, e_2, \dots, e_{78})$ is a reduced expression of $\underline{w}$ with $\underline{w}^{\underline{e}} \in w_B W_A$, then, by [16, Lemma 5.6] $t_i = s_4 \implies e_i = 0$ and for length reasons one has that if $t_i = s_j$ with $j \geq 5$ then $e_i = 1$. In both these cases d_i is U. So a subexpression $\underline{e}$ of $\underline{w}$ for x is completely determined by its "y-part" (see Fig. 1). In other words, it is determined by the sets $I^0 := \{i \,|\, t_i = s_j \text{ with } j \leq 3 \text{ and } e_i = 0\}$ and $I^1 := \{i \,|\, t_i = s_j \text{ with } j \leq 3 \text{ and } e_i = 1\}$ (in these cases $d_i = S$). By the definition of the parabolic defect, one has that

$$\mathrm{pdf}(\underline{e}) = 11 - |I^0| + |I^1|.$$

With this formula we see that if $\underline{e}$ is a subexpression of $\underline{w}$ for x we have $\mathrm{pdf}(\underline{e}) = -1 \Leftrightarrow |I^0| = 12$, and thus there is only one subexpression satisfying this. On the other hand, $\mathrm{pdf}(\underline{e}) = 1 \Leftrightarrow |I^0| = 11$, thus there are 12 subexpressions satisfying this. Thus, in this case, the intersection form $I^{\mathbb{Q}}_{x,\underline{w},-1}$ is a 1×12-matrix. One can calculate explicitly that this matrix is given by

$$(-2, -2, 0, -2, -2, 0, -2, -2, -2, 2, 0, 0).$$

To perform this calculation one uses the main result of [6] together with the same reductions used at the end of [16, §5]. For $1 \leq i < 15$ let $\alpha_i := x_{i+1} - x_i \in R$ denote the simple root and let $\partial_i : R \to R(-2) : f \mapsto (f - s_i(f))/\alpha_i$ denote the Demazure operator. Each of the 12 entries above is the result of erasing one ∂_i from the following expression (which is equal to 0 for degree reasons)

$$\partial_1\partial_2\partial_3(\alpha_4\partial_2\partial_3(\alpha_4^2\partial_3(\alpha_4^2\partial_1\partial_2\partial_3(\alpha_4^2\partial_2\partial_3(\alpha_4^2\partial_3(\alpha_4^2)))))).$$

For example, if we erase the fourth ∂ (i.e. ∂_2), we obtain the fourth entry of the intersection form

$$\partial_1\partial_2\partial_3(\alpha_4\partial_3(\alpha_4^2\partial_3(\alpha_4^2\partial_1\partial_2\partial_3(\alpha_4^2\partial_2\partial_3(\alpha_4^2\partial_3(\alpha_4^2)))))) = -2.$$

Note that this matrix has rank 1 over $\mathbb{Q}$, but rank 0 over a field of characteristic 2. This proves (3.1).

To check (3.2) one needs to do a big computer check. But there is one point that needs explanation about this calculation. If one considers all the subexpressions of $\underline{w}$ one has 2^{78} possibilities. This is too big even for our computer! Suppose $\underline{e} = (e_1, e_2, \dots, e_{78})$ is a subexpression such that $\underline{w}^{\underline{e}} W_A$ belongs to the interval $[xW_A, wW_A]$, then for any i such that $t_i \in W_B$ we must have $e_i = 1$. Thus we "only" need to check 2^{23} subexpressions, which is feasible by computer (and takes our machine 20 minutes).

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