Inverse Problems

PAPER

Size estimates of an obstacle in a stationary Stokes fluid

To cite this article: E Beretta *et al* 2017 *Inverse Problems* **33** 025008

IOP Publishing Inverse Problems

Inverse Problems **33** (2017) 025008 (29pp) doi:10.1088/1361-6420/33/2/025008

Size estimates of an obstacle in a stationary Stokes fluid

E Beretta[1], **C Cavaterra**[2], **J H Ortega**[3] **and S Zamorano**[3]

[1] Dipartimento di Matematica, Politecnico di Milano, Milano 20133, Italy
[2] Dipartimento di Matematica, Università degli Studi di Milano, Milano 20133, Italy
[3] Centro de Modelamiento Matemático (CMM) and Departamento de Ingeniería Matemática, Universidad de Chile (UMI CNRS 2807), Avenida Beauchef 851, Ed. Norte, Casilla 170-3, Correo 3, Santiago, Chile

E-mail: elena.beretta@polimi.it, cecilia.cavaterra@unimi.it, jortega@dim.uchile.cl and szamorano@dim.uchile.cl

Received 23 December 2015, revised 7 November 2016
Accepted for publication 30 November 2016
Published 9 January 2017

Abstract

In this work we are interested in estimating the size of a cavity D immersed in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, filled with a viscous fluid governed by the Stokes system, by means of velocity and Cauchy forces on the external boundary $\partial\Omega$. More precisely, we establish some lower and upper bounds in terms of the difference between the external measurements when the obstacle is present and without the object. The proof of the result is based on interior regularity results and quantitative estimates of unique continuation for the solution of the Stokes system.

Keywords: inverse problems, Stokes system, size estimate, interior regularity, boundary value problems, numerical analysis, Rellich's identity

(Some figures may appear in colour only in the online journal)

1. Introduction

We consider an obstacle D immersed in a region $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is filled with a viscous fluid. Then, the velocity vector u and the scalar pressure p of the fluid in the presence of the obstacle D fulfill the following boundary value problem for the Stokes system:

$$\begin{cases} -\mathrm{div}(\sigma(u,p)) &= 0, \quad \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div}\, u &= 0, \quad \text{in } \Omega \setminus \overline{D}, \\ u &= g, \quad \text{on } \partial\Omega, \\ u &= 0, \quad \text{on } \partial D, \end{cases} \tag{1.1}$$

where $\sigma(u,p) = 2\mu e(u) - pI$ is the stress tensor, $e(u) = \frac{(\nabla u + \nabla u^T)}{2}$ is the strain tensor, I is the identity matrix of order $d \times d$, n denotes the exterior unit normal to $\partial\Omega$ and $\mu > 0$ is the kinematic viscosity. The condition $u|_{\partial D} = 0$ is the so called *no-slip* condition.

Given the boundary velocity $g \in (H^{1/2}(\partial\Omega))^d$ satisfying the compatibility condition

$$\int_{\partial\Omega} g \cdot n = 0,$$

we consider the solution of problem (1.1), $(u,p) \in (H^1(\Omega \setminus \overline{D}))^d \times L^2(\Omega \setminus \overline{D})$, and measure the corresponding Cauchy force on $\partial\Omega$, $\psi = \sigma(u,p)n|_{\partial\Omega}$, in order to recover the obstacle D. Then, it is well known that this inverse problem has a unique solution. In fact, in [8], the authors prove uniqueness in the case of the steady-state and evolutionary Stokes system using unique continuation property of solutions. By uniqueness we mean the following fact: if u_1 and u_2 are two solutions of (1.1) corresponding to a given boundary data g, for obstacles D_1 and D_2 respectively, and we consider that the Cauchy forces satisfy $\sigma(u_1,p_1)n = \sigma(u_2,p_2)n$ on an open subset $\Gamma_0 \subset \partial\Omega$, then $D_1 = D_2$. Moreover, in [12], log–log type stability estimates for the Hausdorff distance between the boundaries of two cavities in terms of the Cauchy forces have been derived. Reconstruction algorithms for the detection of the obstacle have been proposed in [9, 16] and in [24]. The method used in [24] relies on the construction of special complex geometrical optics solutions for the stationary Stokes equation with a variable viscosity. In [9], the reconstruction algorithm released in a nonconvex optimization algorithm (simulating annealing) for the reconstruction of parametric objects. In [16], the detection algorithm is based on topological sensitivity and shape derivatives of a suitable functional. We would like to mention that there hold log type stability estimates for the Hausdorff distance between the boundaries of two cavities in terms of boundary data, also in the case of conducting cavities and elastic cavities (see [3, 17] and [30]). These very weak stability estimates reveal that the problem is severely ill posed limiting the possibility of efficient reconstruction of the unknown object. The above problem motives the study or the identification of partial information on the unknown obstacle D like, for example, the size.

In literature we can find several results concerning the determination of inclusions or cavities and the estimate of their sizes related to different kind of models. Without being exhaustive, we quote some of them. For example in [26] and [27] the problem of estimating the volume of inclusions is analyzed using a finite number of boundary measurements in electrical impedance tomography. In [20], the authors prove uniqueness, stability and reconstruction of an immersed obstacle in a system modeled by a linear wave equation. These results are obtained applying the unique continuation property for the wave equation and in the two dimensional case the inverse problem is transformed in a well-posed problem for a suitable cost functional. We can also mention [24], in which it is analyzed the problem of reconstructing obstacles inside a bounded domain filled with an incompressible fluid by means of special complex geometrical optics solutions for the stationary Stokes equation.

Here we follow the approach introduced by Alessandrini *et al* in [5] and in [29] and we establish a quantitative estimate of the size of the obstacle D, i.e. $|D|$, in terms of suitable boundary measurements. More precisely, let us denote by $(u_0, p_0) \in (H^1(\Omega))^d \times L^2(\Omega)$ the velocity vector of the fluid and the pressure in the absence of the obstacle D, namely the solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}(\sigma(u_0,p_0)) = 0, & \text{in } \Omega, \\ \mathrm{div}\, u_0 = 0, & \text{in } \Omega, \\ u_0 = g, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

and let $\psi_0 = \sigma(u_0, p_0)n|_{\partial\Omega}$. We consider now the following quantities

$$W_0 = \int_{\partial\Omega} g \cdot \psi_0 \qquad \text{and} \qquad W = \int_{\partial\Omega} g \cdot \psi,$$

representing the measurements at our disposal. Observe that the following identities hold true

$$W_0 = 2\int_{\Omega} |e(u_0)|^2 \qquad \text{and} \qquad W = 2\int_{\Omega\setminus\overline{D}} |e(u)|^2,$$

giving us the information on the total deformation of the fluid in the corresponding domains, Ω and $\Omega \setminus \overline{D}$. We will establish a quantitative estimate of the size of the obstacle D, $|D|$, in terms of the difference $W - W_0$. In order to accomplish this goal, we will follow the main track of [5] and [29] applying fine interior regularity results, Poincaré type inequalities and quantitative estimates of unique continuation for solutions of the stationary Stokes system. The plan of the paper is as follows. In section 2 we provide the rigorous formulations of the direct problem and state the main results, theorems 2.11 and 2.12. Section 3 is devoted to some auxiliar results and to give the proofs of theorems 2.11 and 2.12. In section 4 we prove proposition 3.5 which deals with some estimates for the trace of the Cauchy force on the boundary of the cavity D. Finally, in section 5 we show some computational examples of the behavior of the rate with respect to the shape and the size of the interior obstacle.

2. Main results

In this section we introduce some definitions and some preliminary results we will use through the paper and we will state our main theorems. Let $x \in \mathbb{R}^d$, we denote by $B_r(x)$ the ball in $\mathbb{R}^d$ centered in x of radius r and $B'_r(0)$ the ball in $\mathbb{R}^{d-1}$. In what follows we will consider the notation $\cdot$ for the scalar product between vectors in $\mathbb{R}^d$, $:$ for the inner product between matrices, and $\otimes$ for the tensorial product between vectors. We set $x = (x_1, \ldots, x_d)$ as $x = (x', x_d)$, where $x' = (x_1, \ldots, x_{d-1})$.

Definition 2.1 (Definition 2.1 [5]). Let $\Omega \subset \mathbb{R}^d$ be bounded domain. We say that $\partial\Omega$ is of class $C^{k,\alpha}$, with constants ρ_0, $M_0 > 0$, where k is a nonnegative integer and $\alpha \in [0, 1)$, if, for any $x_0 \in \partial\Omega$, there exists a rigid transformation of coordinates, in which $x_0 = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x \in B_{\rho_0}(0) : \ x_n > \varphi(x')\},$$

where φ is a function of class $C^{k,\alpha}(B'_\rho(0))$, such that

$$\begin{aligned} &\varphi(0) = 0, \\ &\nabla\varphi(0) = 0, \quad \text{if } k \geqslant 1 \\ &\|\varphi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} \leqslant M_0\rho_0. \end{aligned}$$

When $k = 0$ and $\alpha = 1$ we will say that $\partial\Omega$ is of Lipschitz class with constants ρ_0, M_0.

Remark 2.2. We normalize all norms in such a way that they are dimensionally equivalent to their argument, and coincide with the usual norms when $\rho_0 = 1$. In this setup, the norm taken in the previous definition is intended as follows:

$$\|\phi\|_{C^{k,\alpha}(B'_{\rho_0}(0))} = \sum_{i=0}^{k} \rho_0^i \|D^i\phi\|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+\alpha}|D^k\phi|_{\alpha, B'_{\rho_0}(0)},$$

where $|\cdot|$ represents the α-Hölder seminorm

$$|D^k\phi|_{\alpha,B'_{\rho_0}(0)} = \sup_{x',y'\in B'_{\rho_0}(0),x'\neq y'} \frac{|D^k\phi(x') - D^k\phi(y')|}{|x'-y'|^{\alpha}},$$

and $D^k\phi = \{D^\beta\phi\}_{|\beta|=k}$ is the set of derivatives of order k. Similarly we set the norms

$$\|u\|^2_{L^2(\Omega)} = \frac{1}{\rho_0^d}\int_\Omega |u|^2,$$

$$\|u\|^2_{H^1(\Omega)} = \frac{1}{\rho_0^d}\left(\int_\Omega |u|^2 + \rho_0^2\int_\Omega |\nabla u|^2\right).$$

2.1. Some classical results for Stokes problem

We now define the following quotient space since, if we consider incompressible models, the pressure is defined only up to a constant.

Definition 2.3. Let Ω be a bounded domain in $\mathbb{R}^d$. We define the quotient space

$$L^2_0(\Omega) = L^2(\Omega)/\mathbb{R},$$

represented by the class of functions of $L^2(\Omega)$ which differ by an additive constant. We equip this space with the quotient norm

$$\|v\|_{L^2_0(\Omega)} = \inf_{\alpha\in\mathbb{R}} \|v+\alpha\|_{L^2(\Omega)}.$$

The Stokes problem has been studied by several authors and, since it is impossible to quote all the related relevant contributions, we refer the reader to the extensive surveys [23] and [33], and the references therein. We limit ourselves to present some classical results, useful for the treatment of our problem, concerning existence, uniqueness, stability and regularity of solutions to the following boundary value problem for the Stokes system

$$\begin{cases} -\mathrm{div}(\sigma(u,p)) = f, & \text{in } \Omega, \\ \mathrm{div}\, u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where, for the sake of simplicity, from now on we assume $\mu(x) \equiv 1$, $\forall x \in \Omega$. Concerning the well-posedness of this problem we have

Theorem 2.4 (Existence and uniqueness, [33]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary, with $d \geqslant 2$. Let $f \in (H^{-1}(\Omega))^d$ and $g \in (H^{1/2}(\partial\Omega))^d$ satisfying the compatibility condition*

$$\int_{\partial\Omega} g\cdot n = 0. \tag{2.2}$$

Then, there exists a unique $(u,p) \in ((H^1(\Omega))^d \times L^2_0(\Omega))$ solution to problem (2.1). Besides, for any $f \in (L^2(\Omega))^d$ and $g \in (H^{3/2}(\partial\Omega))^d$ satisfying (2.2), the unique solution to (2.1) is such that, see [10],

$$(u,p) \in (H^2(\Omega))^d \times H^1(\Omega). \tag{2.3}$$

Moreover, we have

$$\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leqslant C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)}), \tag{2.4}$$

where C is a positive constant depending only on Ω*.*

2.2. Preliminaries

In order to prove our main results we need the following *a priori* assumptions on Ω, D and the boundary data g.

(H1) $\Omega \subset \mathbb{R}^d$ is a bounded domain with a connected boundary $\partial\Omega$ of Lipschitz class with constants ρ_0, M_0. Further, there exists $M_1 > 0$ such that

$$|\Omega| \leqslant M_1 \rho_0^d. \tag{2.5}$$

(H2) $D \subset \Omega$ is such that $\Omega \setminus \overline{D}$ is connected and it is strictly contained in Ω, that is there exists a positive constant d_0 such that

$$d(D, \partial\Omega) \geqslant d_0 > 0. \tag{2.6}$$

Moreover, D has a connected boundary ∂D of Lipschitz class with constants ρ, L.

(H3) D satisfies **(H2)** and the scale-invariant fatness condition with constant $Q > 0$, that is

$$\mathrm{diam}(D) \leqslant Q\rho. \tag{2.7}$$

(H4) The boundary condition g is such that

$$g \in (H^{1/2}(\partial\Omega))^d, \quad g \not\equiv 0, \quad \frac{\|g\|_{H^{1/2}(\partial\Omega)}}{\|g\|_{L^2(\partial\Omega)}} \leqslant c_0,$$

for a given constant $c_0 > 0$, and satisfies the compatibility condition

$$\int_{\partial\Omega} g \cdot n = 0.$$

Also suppose that there exists a point $P \in \partial\Omega$, such that,

$$g = 0 \text{ on } \partial\Omega \cap B_{\rho_0}(P).$$

(H5) Since one measurement g is enough in order to detect the size of D, we choose g in such a way that the corresponding solution u satisfies the following condition

$$\int_{\partial\Omega} \sigma(u, p) n = 0. \tag{2.8}$$

(H6) There exists a constant $h_1 > 0$, such that the *fatness condition* holds, namely

$$|D_{h_1}| \geqslant \frac{1}{2}|D|. \tag{2.9}$$

Concerning assumption **(H5)**, the following result holds.

Proposition 2.5. *There exists at least one function g satisfying* **(H4)** *and* **(H5)***.*

Proof. Consider $(d+1)$ linearly independent functions g_i satisfying **(H4)**, $i = 1, \ldots, d+1$.

Let

$$\int_{\partial\Omega} \sigma(u_i, p_i)n = v_i \in \mathbb{R}^d,$$

where (u_i, p_i) is the corresponding solution of (1.1) associated to g_i, $i = 1, \ldots, d + 1$.

If, for some i, we have that $v_i = 0$, then the result follows. So, assume that all the v_i are different from the null vector. Then, there exist some constants λ_i, with $i = 1, \ldots, d + 1$, not all zero, such that

$$\sum_{i=1}^{d+1} \lambda_i v_i = 0$$

and we can choose our Dirichlet boundary data as

$$g = \sum_{i=1}^{d+1} \lambda_i g_i.$$

Therefore, g satisfies (**H4**) and since the Cauchy force is linear with respect to the Dirichlet boundary condition we have

$$\int_{\partial\Omega} \sigma(u, p)n = 0,$$

where (u, p) is the corresponding solution to (1.1), associated to g. $\square$

With respect to these hypotheses, we make some remarks.

Remark 2.6. Integrating the first equation of (1.1) on $\Omega \setminus \overline{D}$, applying the divergence theorem and using (2.8), we obtain

$$\int_{\partial D} \sigma(u, p)n = 0. \tag{2.10}$$

Remark 2.7. Notice that the constant ρ in (**H2**) already incorporates information on the size of D. In fact, an easy computation shows that if D has a Lipschitz boundary class, with positive constants ρ and L, then we have

$$|D| \geqslant C(L)\rho^d.$$

Moreover, if also condition (**H3**) is satisfied, then it holds

$$|D| \leqslant C(Q)\rho^d.$$

Then, it will be necessary to consider ρ as an unknown parameter while the constants L and Q will be assumed as given pieces of *a priori* information on the unknown inclusion D.

Remark 2.8. The fatness condition assumption (**H6**) is classic in the context of the size estimates (see [6, 7, 31]), and is satisfied when mild *a priori* regularity assumptions are made on D. For instance, if D has a boundary of class $C^{1,\alpha}$, then there exists a constant $h_1 > 0$, such that (see [1])

$$|D_{h_1}| \geqslant \frac{1}{2}|D|. \tag{2.11}$$

where we set, for any $A \subset \mathbb{R}^d$ and $h > 0$,

$$A_h = \{x \in A : \ d(x, \partial A) > h\}$$

Remark 2.9. The non-slip condition for viscous fluids establishes that, on the boundary of the solid, the fluid has zero speed. The fluid velocity in any liquid–solid boundary is the same as that of the solid surface. Conceptually, we can think that the molecules of the fluid closest to the surface of the solid 'stick' to the molecules of the solid on which it flows. For that reason, the condition $g = 0$, on $\partial\Omega \cap B_{\rho_0}(P)$, in the assumption (**H4**) is a congruent hypothesis with the non-slip condition on the boundary data. On the other hand, in our case this condition is also a technical assumption. This can be seen in the proof of the main theorems (section 3), where we need to use the classical Poincaré inequality and one result of Ballerini [12] about the Lipschitz propagation of smallness.

Remark 2.10. Condition (**H5**) is merely technic and it is used in the proof of theorem 2.11. We can see that in the case where there is no obstacle in the interior, the condition holds directly. Moreover, we mention that replacing the Dirichlet boundary condition by $\sigma(u, p)n = g$, then assumption (**H5**) is straightforward, due to the compatibility condition.

2.3. Main results

Under the previous assumptions we consider the following boundary value problems. When the obstacle $D \subset \Omega$ is present, the pair given by the velocity and the pressure of the fluid in $\Omega \setminus \overline{D}$ is the weak solution $(u, p) \in (H^1(\Omega \setminus \overline{D}))^d \times L^2(\Omega \setminus \overline{D})$ to

$$\begin{cases} -\mathrm{div}(\sigma(u,p)) &= 0, \quad \text{in } \Omega \setminus \overline{D}, \\ \mathrm{div}\, u &= 0, \quad \text{in } \Omega \setminus \overline{D}, \\ u &= g, \quad \text{on } \partial\Omega, \\ u &= 0, \quad \text{on } \partial D. \end{cases} \tag{2.12}$$

Then we can define the function ψ by

$$\psi = \sigma(u, p)n|_{\partial\Omega} \in (H^{-1/2}(\partial\Omega))^d \tag{2.13}$$

and the quantity

$$W = \int_{\partial\Omega} (\sigma(u, p)n) \cdot u = \int_{\partial\Omega} \psi \cdot g.$$

When the obstacle D is absent, we shall denote by $(u_0, p_0) \in (H^1(\Omega))^d \times L^2(\Omega)$ the unique weak solution to the Dirichlet problem

$$\begin{cases} -\mathrm{div}(\sigma(u_0,p_0)) &= 0, \quad \text{in } \Omega, \\ \mathrm{div}\, u_0 &= 0, \quad \text{in } \Omega, \\ u_0 &= g, \quad \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Let us define

$$\psi_0 = \sigma(u_0, p_0)n|_{\partial\Omega} \in (H^{-1/2}(\partial\,\Omega))^d, \tag{2.15}$$

and

$$W_0 = \int_{\partial\Omega} (\sigma(u_0, p_0)n) \cdot u_0 = \int_{\partial\Omega} \psi_0 \cdot g.$$

Our goal is to derive estimates of the size of D, $|D|$, in terms of W and W_0.

Theorem 2.11. *Assume* **(H1)**, **(H4)**–**(H6)**, *and* (2.6). *Then, we obtain*

$$|D| \leqslant K\left(\frac{W - W_0}{W_0}\right), \tag{2.16}$$

where the constant $K > 0$ *depends on* $\Omega, d, d_0, h_1, M_0, M_1$, *and* $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$.

Theorem 2.12. *Assume* **(H1)**–**(H4)**. *Then, it holds*

$$C\frac{(W - W_0)^2}{WW_0} \leqslant |D|, \tag{2.17}$$

where $C > 0$ *depends on* $|\Omega|, d, d_0, L$, *and* Q.

Remark 2.13. We expect that a similar result to the one obtained in theorems 2.11 and 2.12 can be derived when we replace the Dirichlet boundary data with

$$\sigma(u, p)n = g, \quad \text{on } \partial\,\Omega\,,$$

g satisfying suitable regularity assumptions and the compatibility condition

$$\int_{\partial\Omega} g = 0.$$

Remark 2.14. In the work [2], the authors showed that the upper bound without assuming *a priori* information on D, has the form

$$|D| \leqslant K\left(\frac{W - W_0}{W_0}\right)^{1/p},$$

where $p > 1$. The proof of this inequality is strongly based on the fact that the gradient of the solution of the background conductivity problem, namely u_0, is a Muckenhoupt weight, [22]. Namely, for any $\tilde{r} > 0$ there exists $B > 0$ and $p > 1$ such that

$$\left(\frac{1}{|B_r|}\int_{B_r} |\nabla u_0|^2\right)\left(\frac{1}{|B_r|}\int_{B_r} |\nabla u_0|^{-\frac{2}{p-1}}\right)^{p-1} \leqslant B,$$

for any ball B_r such that $B_{4r} \subset \Omega_{\tilde{r}}$. This estimate is based on the Caccioppoli inequality, Poincaré–Sobolev inequality, and the called Doubling inequality. It is known that the Doubling inequality holds for some classes of elliptic systems [4]. Unfortunately, as far as we know, for the Stokes system the doubling inequality has not been proved. For instance, see the paper by Lin, Uhlmann and Wang [28] where the authors explain that they were not able to prove a doubling inequality for the Stokes systems, but only to derive a certain optimal three spheres inequality, which is also a strong unique continuation property.

3. Proofs of the main theorems

The main idea of the proof of theorem 2.11 is an application of a three spheres inequality. In particular, we apply a result contained in [28] concerning the solutions to the following Stokes systems

$$\begin{cases} -\Delta u + A(x)\cdot\nabla u + B(x)u + \nabla p &= 0, \quad \text{in } \Omega\,, \\ \operatorname{div} u &= 0, \quad \text{in } \Omega\,. \end{cases} \tag{3.1}$$

Indeed it holds:

Theorem 3.1 (Theorem 1.1 [28]). *Consider* $0 \leqslant R_0 \leqslant 1$ *satisfying* $B_{R_0}(0) \subset \Omega \subset \mathbb{R}^d$*. Then, there exists a positive number* $\tilde{R} < 1$*, depending only on d, such that, if* $0 < R_1 < R_2 < R_3 \leqslant R_0$ *and* $R_1/R_3 < R_2/R_3 < \tilde{R}$*, we have*

$$\int_{|x|<R_2} |u|^2 \mathrm{d}x \leqslant C\left(\int_{|x|<R_1} |u|^2 \mathrm{d}x\right)^{\tau}\left(\int_{|x|<R_3} |u|^2 \mathrm{d}x\right)^{1-\tau},$$

for $(u, p) \in (H^1(B_{R_0}(0)))^d \times H^1(B_{R_0}(0))$ *solution to* (3.1)*. Here* $C > 0$ *depends on* R_2/R_3*, d, and* $\tau \in (0, 1)$ *depends on* R_1/R_3*,* R_2/R_3*, d. Moreover, for fixed* R_2 *and* R_3*, the exponent* τ *behaves like* $1/(-\log R_1)$*, when* R_1 *is sufficiently small.*

Based on this result, the following proposition holds:

Proposition 3.2 (Lipschitz propagation of smallness, proposition 3.1 [12]). *Let* Ω *satisfy (* **H1***) and g satisfies (* **H4***). Let u be a solution to the problem*

$$\begin{cases} -\operatorname{div}(\sigma(u_0, p)) &= 0, \quad \text{in } \Omega\,, \\ \operatorname{div} u_0 &= 0, \quad \text{in } \Omega\,, \\ u_0 &= g, \quad \text{on } \partial\Omega\,. \end{cases} \tag{3.2}$$

Then, there exists a constant $s > 1$*, depending only on d and* M_0*, such that for every* $r > 0$ *there exists a constant* $C_r > 0$*, such that for every* $x \in \Omega_{sr}$*, we have*

$$\int_{B_r(x)} |\nabla u_0|^2 \mathrm{d}x \geqslant C_r \int_{\Omega} |\nabla u_0|^2 \mathrm{d}x, \tag{3.3}$$

where the constant $C_r > 0$ *depends only on* $d, M_0, M_1, \rho_0, r, \frac{\| g \|_{H^{1/2}(\partial\Omega)}}{\| g \|_{L^2(\partial\Omega)}}$.

Following the ideas developed in [5], we establish a key variational inequality relating the boundary data $W - W_0$ with the L^2 norm of the gradient of u_0 inside the cavity D.

Lemma 3.3. *Let* $u_0 \in (H^1(\Omega))^d$ *be the solution to problem* (2.14) *and* $u \in (H^1(\Omega \setminus \overline{D}))^d$ *be the solution to problem* (2.12)*. Then, there exists a positive constant* $C = C(\Omega)$ *such that*

$$\int_D |\nabla u_0|^2 \leqslant C(W - W_0) = C\int_{\partial D} u_0 \cdot \sigma(u, p)n, \tag{3.4}$$

where n denotes the exterior unit normal to ∂D*.*

Proof. Let (u, p) and (u_0, p_0) be the solutions to problems (2.12) and (2.14), respectively. We multiply the first equation of (2.12) by u_0 and after integrating by parts, we have

$$\int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} (\sigma(u,p)n) \cdot u_0 + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 = 0, \tag{3.5}$$

where n denotes either the exterior unit normal to $\partial\Omega$ or to ∂D.

In a similar way, multiplying the first equation of (2.14) by u_0, we obtain

$$\int_{\Omega} \sigma(u_0,p_0) : \nabla u_0 - \int_{\partial\Omega} (\sigma(u_0,p_0)n) \cdot u_0 = 0. \tag{3.6}$$

Now, replacing $\psi = \sigma(u,p)n|_{\partial\Omega}$ and $\psi_0 = \sigma(u_0,p_0)n|_{\partial\Omega}$ into the equations (3.5) and (3.6), we get

$$\begin{cases} \int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 = 0, \\ \int_{\Omega} \sigma(u_0,p_0) : \nabla u_0 - \int_{\partial\Omega} \psi_0 \cdot g = 0. \end{cases} \tag{3.7}$$

Let us define

$$\tilde{u}(x) = \begin{cases} u & \text{if } x \in \Omega\setminus\overline{D}, \\ 0 & \text{if } x \in \overline{D}. \end{cases}$$

Since $u = 0$ on ∂D, we have $\tilde{u} \in (H^1(\Omega))^d$. So, multiplying (2.12) and (2.14) by $\tilde{u}$, we obtain

$$\begin{cases} \int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla\tilde{u} - \int_{\partial\Omega} \psi \cdot g + \underbrace{\int_{\partial D} (\sigma(u,p)n) \cdot \tilde{u}}_{=0} = 0, \\ \int_{\Omega\setminus\overline{D}} \sigma(u_0,p_0) : \nabla\tilde{u} - \int_{\partial\Omega} \psi_0 \cdot g = 0. \end{cases} \tag{3.8}$$

Using that $\sigma(u,p) = 2e(u) - pI$, where $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, in the first equation of (3.7), we have

$$\begin{aligned} 0 &= \int_{\Omega\setminus\overline{D}} \sigma(u,p) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\ &= \int_{\Omega\setminus\overline{D}} (2e(u) - pI) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\ &= \int_{\Omega\setminus\overline{D}} 2e(u) : \nabla u_0 - \int_{\Omega\setminus\overline{D}} p(\operatorname{div} u_0) - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0 \\ &= \int_{\Omega\setminus\overline{D}} 2e(u) : \nabla u_0 - \int_{\partial\Omega} \psi \cdot g + \int_{\partial D} (\sigma(u,p)n) \cdot u_0, \end{aligned}$$

where we use the fact that $\operatorname{div} u_0 = 0$. For the next step, we need a different expression for the term $e(u) : \nabla u_0$. We claim that, for every $v \in (H^1(\Omega))^d$ such that $\operatorname{div} v = 0$, we have $e(u) : \nabla v = e(u) : e(v)$. Indeed,

$$
\begin{aligned}
2e(u):\nabla v &= \left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j}\\
&= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j}+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial v_j}{\partial x_i}\\
&= e(u):\nabla v+e(u):\nabla v^T=2e(u):e(v).
\end{aligned}
\tag{3.9}
$$

Therefore, equalities (3.7) and (3.8) can be rewritten as

$$2\int_{\Omega\setminus\overline{D}} e(u):e(u_0)-\int_{\partial\Omega}\psi\cdot g+\int_{\partial D}u_0\cdot(\sigma(u,p)n)=0, \tag{3.10}$$

$$2\int_{\Omega}|e(u_0)|^2-\int_{\partial\Omega}\psi_0\cdot g=0, \tag{3.11}$$

$$2\int_{\Omega\setminus\overline{D}}|e(u)|^2-\int_{\partial\Omega}\psi\cdot g=0, \tag{3.12}$$

$$2\int_{\Omega\setminus\overline{D}} e(u_0):e(u)-\int_{\partial\Omega}\psi_0\cdot g=0. \tag{3.13}$$

We note that if we subtract (3.13) from (3.10) we get

$$\int_{\partial\Omega}(\psi-\psi_0)\cdot g=\int_{\partial D}u_0\cdot(\sigma(u,p)n). \tag{3.14}$$

Now, let us consider the quadratic form

$$
\begin{aligned}
\int_{\Omega} e(\tilde{u}-u_0):e(\tilde{u}-u_0) &= \int_{\Omega}|e(u_0)|^2+\int_{\Omega\setminus\overline{D}}|e(u)|^2-2\int_{\Omega\setminus\overline{D}}e(u):e(u_0)\\
&= \frac{1}{2}\int_{\partial\Omega}\psi_0\cdot g+\frac{1}{2}\int_{\partial\Omega}\psi\cdot g-\int_{\partial\Omega}\psi_0\cdot g\\
&= \frac{1}{2}\int_{\partial\Omega}(\psi-\psi_0)\cdot g.
\end{aligned}
$$

By Korn's inequality there exists a constant $C=C(\Omega)>0$, such that

$$\int_{\Omega}|\nabla(\tilde{u}-u_0)|^2\leqslant C\int_{\Omega}|e(\tilde{u}-u_0)|^2.$$

Finally, by the chain of inequalities

$$
\begin{aligned}
\int_{D}|\nabla u_0|^2 &= \int_{D}|\nabla(\tilde{u}-u_0)|^2\leqslant\int_{\Omega}|\nabla(\tilde{u}-u_0)|^2\\
&\leqslant C\int_{\Omega}|e(\tilde{u}-u_0)|^2=C\int_{\partial\Omega}(\psi-\psi_0)\cdot g=C(W-W_0),
\end{aligned}
$$

and (3.14) the claim follows. □

Now, using the previous results, we are able to prove theorem 2.11.

Proof. The proof is based on arguments similar to those used in [5] and [6]. Let us consider the intermediate domain $\Omega_{d_0/2}$. Recalling that $d(D,\partial\Omega)\geqslant d_0$, we have $d(D,\partial\Omega_{d_0/2})\geqslant\frac{d_0}{2}$. Let $\varepsilon=\min\left(\frac{d_0}{2},\frac{h_1}{\sqrt{d}}\right)>0$. Let us cover the domain D_{h_1} with cubes Q_l of side ε, for $l=1,\ldots,N$. By the choice of ε, the cubes Q_l are contained in D. Then,

$$\int_D|\nabla u_0|^2\geqslant\int_{\bigcup_{l=1}^N Q_l}|\nabla u_0|^2\geqslant\frac{|D_{h_1}|}{\varepsilon^d}\int_{Q_{\bar{l}}}|\nabla u_0|^2, \tag{3.15}$$

where $\bar{l}$ is chosen in such way that

$$\int_{Q_{\bar{l}}}|\nabla u_0|^2=\min_l\int_{Q_l}|\nabla u_0|^2>0.$$

We observe that the previous minimum is strictly positive, in fact, if the minimum is zero, then u_0 would be constant in $Q_{\bar{l}}$. Thus, from the unique continuation property, u_0 would be constant in Ω and since there exists a point $P\in\partial\Omega$, such that,

$$g=0 \text{ on } \partial\Omega\cap B_{\rho_0}(P),$$

we would have that $u_0\equiv 0$ in Ω, contradicting the fact that g is different from zero. Then, the minimum is strictly positive.

Let $\bar{x}$ be the center of $Q_{\bar{l}}$. From the estimate (3.3) in proposition 3.2 with $x=\bar{x}$, $r=\frac{\varepsilon}{2}$, we deduce that

$$\int_{Q_{\bar{l}}}|\nabla u_0|^2\geqslant C\int_\Omega|\nabla u_0|^2. \tag{3.16}$$

On account of remark 2.8, we obtain

$$\int_D|\nabla u_0|^2\geqslant\frac{\frac{1}{2}|D|}{\varepsilon^d}C\int_\Omega|\nabla u_0|^2=|D|C'\int_\Omega|\nabla u_0|^2. \tag{3.17}$$

We estimate the right hand side of (3.17). First, using (3.11) we have

$$\int_{\partial\Omega}\psi_0\cdot g=2\int_\Omega|e(u_0)|^2=2\int_\Omega\frac{|\nabla u_0+\nabla u_0^T|^2}{4} \tag{3.18}$$

$$=2\left(\int_\Omega\frac{|\nabla u_0|^2+|\nabla u_0^T|^2+2\nabla u_0:\nabla u_0^T}{4}\right) \tag{3.19}$$

Now, Hölder's inequality implies

$$\int_{\partial\Omega}\psi_0\cdot g\leqslant 2\int_\Omega|\nabla u_0|^2. \tag{3.20}$$

Then, coming back to (3.17), we obtain that there exists a constant K, depending on $\Omega, d, d_0, h_1, \rho_0, M_0, M_1$, and $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$ such that

$$\int_D |\nabla u_0|^2 \geqslant |D| K \int_{\partial\Omega} \psi_0 \cdot g. \tag{3.21}$$

Combining (3.21) and lemma 3.3 we have

$$C \int_{\partial\Omega} (\psi - \psi_0) \cdot g \geqslant \int_D |\nabla u_0|^2 \geqslant \left(K \int_{\partial\Omega} \psi_0 \cdot g \right) |D|. \tag{3.22}$$

Therefore, we can conclude that

$$|D| \leqslant K \frac{W - W_0}{W_0},$$

where $\tilde{K}$ is a positive constant depending on $\Omega, d, d_0, h_1, \rho_0, M_0, M_1$, and $\|g\|_{H^{1/2}(\partial\Omega)}/\|g\|_{L^2(\partial\Omega)}$. □

In order to prove theorem 2.12, we make use of the following two propositions. The first proposition can be found in [5] and the second proposition will be shown in the next section.

Proposition 3.4 (Poincaré type inequality, proposition 3.2 [5]. *Let D be a bounded domain in* $\mathbb{R}^d$ *of Lipschitz class with constants* ρ, L *and such that* (2.7) *holds. Then, for every* $u \in (H^1(D))^d$ *we have*

$$\int_{\partial D} |u - u_{\partial D}|^2 \leqslant \overline{C_1} \rho \int_D |\nabla u|^2, \tag{3.23}$$

$$\int_D |u - u_D|^2 \leqslant \overline{C_2} \rho^2 \int_D |\nabla u|^2, \tag{3.24}$$

where

$$u_{\partial D} = \frac{1}{|\partial D|} \int_{\partial D} u \qquad \textit{and} \qquad u_D = \frac{1}{|D|} \int_D u,$$

and the constants $\overline{C_1}, \overline{C_2} > 0$ *depend only on L,Q.*

Proposition 3.5. *Assume* **(H1)**–**(H4)***. The Cauchy force* $\sigma(u,p)n$ *on* ∂D *belongs to* $L^2(\partial D)$ *and the following estimate holds:*

$$\int_{\partial D} |\sigma(u,p)n|^2 \leqslant \frac{C}{\min\{\rho, 1\}} \int_{\Omega \setminus \overline{D}} |\nabla u|^2, \tag{3.25}$$

where $C > 0$ *only depends on* $|\Omega|$*, L, Q and* d_0*.*

Using this results and lemma 3.3, we can prove now theorem 2.12.

Proof. Let $\bar{u}_0$ be the following number

$$\bar{u}_0 = \frac{1}{|\partial D|} \int_{\partial D} u_0. \tag{3.26}$$

Then, we deduce that

$$\int_{\partial D} (\sigma(u,p)n) \cdot u_0 = \int_{\partial D} (\sigma(u,p)n) \cdot u_0 - \int_{\partial D} (\sigma(u,p)n) \bar{u}_0, \tag{3.27}$$

because $\int_{\partial D}\sigma(u,p)n=0$. From equality (3.14) in lemma 3.3, we have

$$W-W_0=\int_{\partial D}(\sigma(u,p)n)\cdot u_0=\int_{\partial D}(\sigma(u,p)n)\cdot(u_0-\bar{u}_0). \tag{3.28}$$

Applying Hölder inequality in the right hand side of (3.28) we obtain

$$W-W_0\leqslant\left(\int_{\partial D}|u_0-\overline{u_0}|^2\right)^{1/2}\left(\int_{\partial D}|\sigma(u,p)n|^2\right)^{1/2}. \tag{3.29}$$

Now, using Poincaré inequality (3.23) and inequality (3.25) on the right hand side of (3.29), we get

$$W-W_0\leqslant C\left(\int_D|\nabla u_0|^2\right)^{1/2}\left(\int_{\Omega\backslash\overline{D}}|\nabla u|^2\right)^{1/2}, \tag{3.30}$$

where $C>0$ depends on $|\Omega|$, Q, L, and d_0. The first integral on the right hand side of (3.30) can be estimated as

$$\int_D|\nabla u_0|^2\leqslant|D|\sup_D|\nabla u_0|^2. \tag{3.31}$$

Now, we need to give an interior estimate for the gradient of u_0. We known that the pressure is an harmonic function. This implies that each component of u_0 is a biharmonic function. Then, using interior regularity estimates for fourth order equations, we deduce that

$$\sup_D|\nabla u_0|\leqslant C\|u_0\|_{L^2(\Omega)}, \tag{3.32}$$

where the constant C depends on Q, $|\Omega|$ and d_0. Estimate (3.32) can be obtained considering the following results. We know that the embedding from $H^4(\Omega)$ to $C^k(\Omega)$ is continuous for $0\leqslant k<4-\frac{d}{2}$, with $d=2,3$. Then, in particular,

$$\|u_0\|_{C^1(D)}\leqslant C\|u_0\|_{H^4(D)}.$$

Moreover, from the interior regularity of fourth order equations, see [31, theorem 8.3], we obtain

$$\|u_0\|_{H^4(D)}\leqslant C\|u_0\|_{H^2(\Omega_{d_0/2})}.$$

Finally, considering the estimates in [11] and [14], we have

$$\|u_0\|_{H^2(\Omega_{d_0/2})}\leqslant C\|u_0\|_{L^2(\Omega_{d_0/4})}\leqslant C\|u_0\|_{L^2(\Omega)},$$

and (3.32) holds. We refer to [11, 13, 18], and references therein, for more details on interior estimates for elliptic operators.

As the boundary data g satisfies (**H4**), we use the classical Poincaré inequality and obtain

$$\|u_0\|_{L^2(\Omega)}\leqslant C\|\nabla u_0\|_{L^2(\Omega)}. \tag{3.33}$$

Therefore, by means of the inequality $\int_\Omega|\nabla u_0|^2\leqslant C\int_{\partial\Omega}\psi_0\cdot g$, we deduce

$$\left(\int_D |\nabla u_0|^2\right)^{1/2} \leqslant C|D|^{1/2} W_0^{1/2}. \tag{3.34}$$

Now, concerning the second integral in (3.30), by (3.12), we get

$$\int_{\Omega\setminus\overline{D}} |\nabla u|^2 \leqslant C \int_{\Omega\setminus\overline{D}} |e(u)|^2 \leqslant CW. \tag{3.35}$$

Therefore, it holds

$$C\frac{(W-W_0)^2}{WW_0} \leqslant |D|,$$

where C depends on $|\Omega|, d, L, d_0$ and Q. This completes the proof. □

Remark 3.6. We note that the last inequality can be rewritten in the form

$$C\phi\left(\frac{W-W_0}{W_0}\right) \leqslant |D|,$$

where the function ϕ is given by

$$\phi(t) = \frac{t^2}{1+t}, \ \forall\, t \in [0, 1].$$

The previous expression is identical to the one obtained in [5].

4. Proof of proposition 3.5

The proof closely follows the arguments of [5]. For technical reason, we introduce the following notation. Given $\tau, L > 0$, and a Lipschitz function $\varphi : B_{2\tau}(0) \subset \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\varphi(0) = 0, \|\varphi\|_{C^{0,1}(B_{2\tau}(0))} \leqslant 2\tau L$. We define for every t, with $0 < t \leqslant 2\tau$, the following sets

$$C_t^+ := \{x = (x', x_d) \in \mathbb{R}^d \ : |x'| < t, \ \varphi(x') < x_d < Lt\},$$
$$\Delta_t := \{x = (x', x_d) \in \mathbb{R}^d \ : |x'| < t, \ x_d = \varphi(x')\}.$$

Before proving proposition 3.5, we need some auxiliary result.

We start by some algebraic formalisms associated with the Stokes system. Let us consider the following family of coefficients, with δ_{jk} denoting the Kronecker symbol,

$$a_{jk}^{\alpha\beta} := \delta_{jk}\delta_{\alpha\beta} + \delta_{j\beta}\delta_{k\alpha}, \ 1 \leqslant j, k, \alpha, \beta \leqslant d.$$

We denote by A the fourth order tensor associated to the family of coefficients $a_{jk}^{\alpha\beta}$, namely $A = (a_{jk}^{\alpha\beta})$. Let B be any matrix in $\mathbb{R}^d$. Adopting the summation convention over repeated indices, we obtain that this tensor A applied on matrix B, component-wise, is

$$(AB)_{\alpha j} = a_{jk}^{\alpha\beta} b_{k\beta} = \sum_{k,\beta} \delta_{jk}\delta_{\alpha\beta} b_{k\beta} + \sum_{k,\beta} \delta_{j\beta}\delta_{k\alpha} b_{k\beta} = b_{j\alpha} + b_{\alpha j}, \ 1 \leqslant \alpha, j \leqslant d. \tag{4.1}$$

From the previous considerations, we can write the strain tensor $e(u) = \frac{\nabla u + \nabla u^T}{2}$, for $u = (u_\beta)_{1 \leqslant \beta \leqslant d}$, component-wise, as

$$(e(u))_{\alpha j} = \frac{a_{jk}^{\alpha\beta}\partial_k u_\beta}{2} = \frac{\partial_\alpha u_k + \partial_k u_\alpha}{2}, \ 1 \leqslant \alpha, j \leqslant d, \tag{4.2}$$

and in matrix form as

$$2e(u) = A\nabla u, \tag{4.3}$$

where ∇u is the Jacobian matrix associated to u. Then, the Stokes system in a domain $\Omega \subset \mathbb{R}^d$ can be written as

$$\mathrm{div}(A\nabla u - qI) = 0, \quad \mathrm{div}\ \ u = 0, \ \text{in}\ \Omega\,. \tag{4.4}$$

From the previous computations, it follows that the αth component of the normal derivative is

$$[(\nabla u)n]_\alpha = \sum_l (\partial_l u_\alpha) n_l, \ 1 \leqslant \alpha \leqslant d. \tag{4.5}$$

Then, the tangential component of the gradient of u, $\nabla_T u$, can be expressed by

$$(\nabla_T u)_{\alpha j} = \partial_j u_\alpha - \sum_l (\partial_l u_\alpha) n_l n_j, \ 1 \leqslant \alpha, \ j \leqslant d. \tag{4.6}$$

Lemma 4.1. *Let* $(u, q) \in (H^{3/2}(C_{2\tau}^+))^d \times L^2(C_{2\tau}^+)$ *such that* $\mathrm{div}(\sigma(u, q)) \in L^2(C_{2\tau}^+)$ *and* $\mathrm{div}\ u = 0$ *in* $C_{2\tau}^+$. *Then, there exists* $C > 0$ *such that*

$$\int_{\partial C_{2\tau}^+} (|\nabla u|^2 + q^2) \leqslant C \int_{\partial C_{2\tau}^+ \setminus \Delta_{2\tau}} |\sigma(u, q)n|^2 + C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}^+} |\mathrm{div}(\sigma(u, q))| |\nabla u|, \tag{4.7}$$

where we indicate by $\nabla_T u$ *the tangential gradient of* u *(see (4.6)).*

Proof. The proof is based on the Rellich's identity for the Stokes system [15, 21] and elliptic system [32] from which it holds, for any vector valued field $f \in C^\infty(\mathbb{R}^n)$,

$$\int_{C_{2\tau}^+} \mathrm{div}(\sigma(u, q)) \cdot ((\nabla u)f) = \int_{\partial C_{2\tau}^+} \sigma(u, q)n \cdot ((\nabla u)f) - \frac{1}{2}\int_{\partial C_{2\tau}^+} (f \cdot n)|\nabla u|^2 + \frac{1}{2}\int_{C_{2\tau}^+} (\mathrm{div}\, f)|\nabla u|^2 + \int_{C_{2\tau}^+} q\partial_i u_k \partial_k f_i - \partial_i u_k(\partial_j u_k + \partial_k u_j)\partial_j f_i. \tag{4.8}$$

More precisely, in theorem 4.1 and corollary 4.2 of [15] the authors studied a mixed problem for the Stokes system and they established a technical estimate. This estimate (where we have taken $r = 1$) implies in our particular case

$$\int_{\partial C_{2\tau}^+} (|\nabla u|^2 + q^2) \leqslant C \int_{\partial C_{2\tau}^+ \setminus \Delta_{2\tau}} |\sigma(u, q)n|^2 + C \int_{\Delta_{2\tau}} |\nabla_T u|^2 + C \int_{C_{2\tau}^+} (|\nabla u|^2 + q^2) + C \int_{C_{2\tau}^+} |\mathrm{div}(\sigma(u, q))| |\nabla u|.$$

Then, following the proof of theorem 4.1 in [15] and choosing $f = e_d$ in the Rellich's identity (4.8), we obtain that any terms involving derivatives of f vanish. So that, as in corollary 4.2 in [15], we have

$$\int_{\partial C_{2\tau}^+}(|\nabla u|^2+q^2)\leqslant C\int_{\partial C_{2\tau}^+\setminus\Delta_{2\tau}}|\sigma(u,q)n|^2+C\int_{\Delta_{2\tau}}|\nabla_T u|^2+C\int_{C_{2\tau}^+}|\mathrm{div}(\sigma(u,q))||\nabla u|.$$

□

Proposition 4.2. *Let* $(u,q)\in(H^{3/2}(C_{2\tau}^+))^d\times L^2(C_{2\tau}^+)$ *such that* $\mathrm{div}(\sigma(u,q))\in L^2(C_{2\tau}^+)$, $q=u=|\nabla u|=0$ *on* $\partial C_{2\tau}^+\setminus\Delta_{2\tau}$, *and* $\mathrm{div}\, u=0$ *in* $C_{2\tau}^+$. *Then, we have*

$$\int_{\Delta_\tau}|\sigma(u,q)n|^2\leqslant C\left(\int_{\Delta_{2\tau}}|\nabla_T u|^2+\int_{C_{2\tau}^+}\big(|\nabla u|^2+|\nabla u||\mathrm{div}(\sigma(u,q))|\big)\right),\tag{4.9}$$

where the constant $C>0$ *only depends on L.*

Proof. Using again the Rellich identity (4.8) with $f=e_d$ and recalling that $q=u=|\nabla u|=0$ on $\partial C_{2\tau}^+\setminus\Delta_{2\tau}$, $u=(u_1,\dots,u_d)$ and $n=(n_1,\dots,n_d)$, then we obtain

$$\int_{\Delta_{2\tau}}\left(\sigma(u,q)n\cdot(\partial_d u)-\frac{1}{2}n_d|\nabla u|^2\right)=\int_{C_{2\tau}^+}\mathrm{div}(\sigma(u,q))\cdot(\partial_d u).\tag{4.10}$$

Now, we express the matrix ∇u in terms on its tangential component $\nabla_T u$ and the Cauchy forces $\sigma(u,q)n$. From (4.6), we have

$$(\nabla u)_{\alpha j}=\sum_l(\partial_l u_\alpha)n_l n_j+\sum_l(\partial_\alpha u_l)n_l n_j+(\nabla_T u)_{\alpha j}-\sum_l(\partial_\alpha u_l)n_l n_j.\tag{4.11}$$

Recalling that the tensorial product is denoted by $\otimes$, we obtain

$$\nabla u=(A\nabla u-qI)n\otimes n+\nabla_T u-(\nabla u-qI)^T n\otimes n.\tag{4.12}$$

Using the above expression we can write the scalar terms $\sigma(u,q)n\cdot(\partial_d u)$ and $\frac{1}{2}n_d|\nabla u|^2$ as

$$\begin{aligned}\sigma(u,q)n\cdot(\partial_d u)&=n_d|\sigma(u,q)n|^2+\sum_j(\sigma(u,q)n)_j(\nabla_T u)_{jd}\\&\quad-\sum_j(\sigma(u,q)n)_j((\nabla u-qI)^T n:n)_{jd}\\&=n_d|\sigma(u,q)n|^2+\sigma(u,q)n\cdot(\nabla_T u)_{\cdot d}-\sigma(u,q)n\cdot((\nabla u-qI)^T n)_{\cdot d},\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\frac{1}{2}n_d|\nabla u|^2&=\frac{1}{2}n_d|\sigma(u,q)n|^2+\frac{1}{2}n_d|\nabla_T u|^2+\frac{1}{2}n_d|(\nabla u-qI)^T n|^2\\&\quad+n_d\big(|[\sigma(u,q)n\otimes n]:\nabla_T u|+|[\sigma(u,q)n\otimes n]:[(\nabla u-qI)^T n\otimes n]|\\&\quad+|\nabla_T u:[(\nabla u-qI)^T n\otimes n]|\big).\end{aligned}\tag{4.14}$$

Replacing (4.13) and (4.14) in (4.10), we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{\Delta_{2\tau}} n_d|\sigma(u,q)n|^2 = \int_{\Delta_{2\tau}} \Bigg(&\frac{1}{2}n_d|\nabla_T u|^2 + \frac{1}{2}n_d|(\nabla u - qI)^T n|^2 \\
&+ \frac{1}{2}n_d|[\sigma(u,q)n\otimes n] : \nabla_T u| + \frac{1}{2}n_d|[\sigma(u,q)n\otimes n] : [(\nabla u - qI)^T n\otimes n]| \\
&+ \frac{1}{2}n_d|\nabla_T u : [(\nabla u - qI)^T n\otimes n]| + \sigma(u,q)n\cdot(\nabla_T u)_{\cdot d} - \sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d}\Bigg) \\
&+ \int_{C^+_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u). \qquad (4.15)
\end{aligned}
$$

Now, we apply the Young inequality with weight $\epsilon > 0$, namely $ab \leqslant \frac{\epsilon}{2}a^2 + \frac{1}{\epsilon}b^2$, to the last five terms on $\Delta_{2\tau}$ on the right hand side in (4.15). We obtain that

$$
|[\sigma(u,q)n\otimes n] : \nabla_T u| \leqslant \frac{\epsilon}{2}|\sigma(u,q)n|^2 + \frac{1}{2\epsilon}|\nabla_T u|^2 = \frac{\epsilon}{2}|\sigma(u,q)n|^2 + C_\epsilon|\nabla_T u|^2, \qquad (4.16)
$$

$$
|[\sigma(u,q)n\otimes n] : [(\nabla u - qI)^T n\otimes n]| \leqslant \frac{\epsilon}{2}|\sigma(u,q)n|^2 + C_\epsilon|(\nabla u - qI)^T n|^2, \qquad (4.17)
$$

$$
|\nabla_T u : [(\nabla u - qI)^T n\otimes n]| \leqslant \frac{\epsilon}{2}|(\nabla u - qI)^T n|^2 + C_\epsilon|\nabla_T u|^2, \qquad (4.18)
$$

$$
\sigma(u,q)n\cdot(\nabla_T u)_{\cdot d} \leqslant \frac{\epsilon}{2}|\sigma(u,q)n|^2 + C_\epsilon|\nabla_T u|^2, \qquad (4.19)
$$

$$
\begin{aligned}
-\sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d} &\leqslant |\sigma(u,q)n\cdot((\nabla u - qI)^T n)_{\cdot d}| \\
&\leqslant \frac{\epsilon}{2}|\sigma(u,q)n|^2 + C_\epsilon|(\nabla u - qI)^T n|^2. \qquad (4.20)
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{1}{2}\int_{\Delta_{2\tau}} n_d|\sigma(u,q)n|^2 &\leqslant C_\epsilon\int_{\Delta_{2\tau}} \left(|\nabla_T u|^2 + |\nabla u|^2 + q^2\right) \\
&+ \epsilon\left(C\int_{\Delta_{2\tau}}\left(|\sigma(u,q)n|^2 + |\nabla u|^2 + q^2\right)\right) + \int_{C^+_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u), \qquad (4.21)
\end{aligned}
$$

where $C > 0$. From lemma 4.1 and using the assumptions $q = u = |\nabla u| = 0$ on $\partial C^+_{2\tau}\setminus\Delta_{2\tau}$, we obtain

$$
\int_{\Delta_{2\tau}} |\nabla u|^2 + q^2 \leqslant C\int_{\Delta_{2\tau}} |\nabla_T u|^2 + C\int_{C_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u). \qquad (4.22)
$$

Combining (4.21) and (4.22), and using the inequality $|n_d| \geqslant \frac{1}{\sqrt{1+L^2}}$, then we derive

$$
\begin{aligned}
&\frac{1}{2\sqrt{1+L^2}}\int_{\Delta_{2\tau}} |\sigma(u,q)n|^2 - \epsilon\int_{\Delta_{2\tau}} |\sigma(u,q)n|^2 \\
&\leqslant C_\epsilon\left(\int_{\Delta_{2\tau}} |\nabla_T u|^2 + \int_{C_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u)\right) \\
&+ \epsilon\left(C\int_{\Delta_{2\tau}} |\nabla_T u|^2 + C\int_{C_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u)\right) + \int_{C_{2\tau}} \operatorname{div}(\sigma(u,q))\cdot(\partial_d u).
\end{aligned}
$$

Choosing $\epsilon > 0$ small enough, we have

$$\int_{\Delta_\tau} |\sigma(u,q)n|^2 \leqslant C\left(\int_{\Delta_{2\tau}} |\nabla_T(u)|^2 + \int_{C^+_{2\tau}} (|\nabla u|^2 + |\nabla u||\mathrm{div}(\sigma(u,q))|)\right),$$

where the constant $C > 0$ depends on L, and the proof is finished. $\square$

Proposition 4.3. *Let $(v,p) \in (H^1(C^+_{2\tau}))^d \times L^2(C^+_{2\tau})$ be the solution of the problem*

$$\begin{cases} -\mathrm{div}(\sigma(v,p)) = 0, & \text{in } C^+_{2\tau}, \\ \mathrm{div}\, v = 0, & \text{in } C^+_{2\tau}. \end{cases} \tag{4.23}$$

If $v|_{\Delta_{2\tau}} \in H^1(\Delta_{2\tau})$, then $\sigma(v,p)n \in L^2(\Delta_\tau)$ and

$$\int_{\Delta_\tau} |\sigma(v,p)n|^2 \leqslant C\left[\int_{\Delta_{2\tau}} |\nabla_T v|^2 + \left(1 + \frac{1}{\tau}\right) \int_{C^+_{2\tau}} |\nabla v|^2 \right], \tag{4.24}$$

where the constant $C > 0$ only depends on L.

Proof. First, we assume that the function v is more regular, namely $v \in H^{3/2}(C^+_{2\tau})$. We consider the following vector field cut-off function $\eta = (\eta_1, \dots, \eta_d)$ in $\mathbb{R}^d$

$$\eta_i(x', x_d) = \varphi_i(x')\psi_i(x_d),\ \forall\, i = 1, \dots, d,\ \mathrm{div}\,\eta = 0, \tag{4.25}$$

where

$$\varphi_i \in C^\infty_0(\mathbb{R}^{d-1}),\ \varphi_i(x') = 1 \text{ if } |x'| \leqslant \tau,\ \varphi_i(x') = 0 \text{ if } |x'| \geqslant \frac{3}{2}\tau, \tag{4.26}$$

$$\|\nabla\varphi_i\|_\infty \leqslant C_1\tau^{-1},\ \|\nabla^2\varphi_i\|_\infty \leqslant C_1\tau^{-2}, \tag{4.27}$$

$$\psi_i \in C^\infty_0(\mathbb{R}),\ \psi_i(x_d) = 1 \text{ if } |x_d| \leqslant \tau L,\ \psi_i(x_d) = 0 \text{ if } |x_d| \geqslant \frac{3}{2}\tau L, \tag{4.28}$$

$$\|\nabla\psi_i\|_\infty \leqslant C_2\tau^{-1},\ \|\nabla^2\psi_i\|_\infty \leqslant C_2\tau^{-2}. \tag{4.29}$$

Here C_1 is an absolute constant and C_2 is a constant only depending on L. For $u = (u_1, \dots, u_d)$ and $c \in \mathbb{R}$, we consider the function

$$u_i = \eta_i(v_i - c),\ q = \eta_j\, p,\ i = 1, \dots, d,\ \text{for some } j \in \{1, \dots, d\}. \tag{4.30}$$

We note that if we take $\tau = t$ in proposition 4.2, for every $\frac{3}{4}\tau < t < \tau$, we obtain that $|x'| \in (\frac{3}{2}\tau, 2\tau)$. This implies $\varphi_i = 0$, for every $i = 1, \dots, d$. Then, the pair $(u,\ q)$ satisfies the hypotheses of proposition 4.2, with $\tau = t$, for every $\frac{3}{4}\tau < t < \tau$. Namely,

$$\int_{\Delta_\tau} |\sigma(u,q)n|^2 \leqslant C\left(\int_{\Delta_{2\tau}} |\nabla_T u|^2 + \int_{C^+_{2\tau}} (|\nabla u|^2 + |\nabla u||\mathrm{div}(\sigma(u,q))|)\right).$$

Recalling that $(v,\, p)$ satisfies equation (4.23) and the definition of the cut-off function (4.26)–(4.29), we obtain

$$\int_{\Delta_t} |\sigma(v,p)n|^2 \leqslant C\left(\int_{\Delta_{2t}} |\nabla_T v|^2 + \left(1+\frac{1}{t}\right) \int_{C^+_{2t}} \left[\frac{(v-c)^2}{t^2} + |\nabla v|^2 \right] \right),$$

for every $\frac{3}{4}\tau < t < \tau$. Choosing the constant c such that

$$c = \frac{1}{|C^+_{2t}|} \int_{C^+_{2t}} v,$$

and applying the Poincaré inequality (3.24), we obtain

$$\int_{\Delta_t} |\sigma(v,p)n|^2 \leqslant C\left(\int_{\Delta_{2t}} |\nabla_T v|^2 + \left(1+\frac{1}{t}\right) \int_{C^+_{2t}} |\nabla v|^2 \right).$$

Then passing to the limit for $t \to \tau$, we deduce (4.24). We observe that the assumption of the regularity on v is satisfied when $\varphi \in C^\infty$, by the regularity of the Stokes problem.

Now, given a Lipschitz function φ, let $\{\varphi_m\}_m$ be a sequence of C^∞ equi-Lipschitz functions with constant L, such that

$$\begin{aligned} &\varphi_m(0) = 0, \quad \varphi_m \to \varphi \text{ uniformly}, \\ &\nabla \varphi_m \to \nabla \varphi \text{ in } L^p, \ \forall p < \infty, \ \text{as } m \to \infty. \end{aligned}$$

Therefore, we have that (4.24) is valid when φ is replaced by φ_m, for every m.

For every m and for every t, with $0 < t \leqslant 2\tau$, let us consider the following sets

$$\begin{aligned} C^+_{t,m} &:= \{x = (x', x_d) \in \mathbb{R}^d \ : |x'| < t, \ \varphi_m(x') < x_d < Lt\}, \\ \Delta_{t,m} &:= \{x = (x', x_d) \in \mathbb{R}^d \ : |x'| < t, \ x_d = \varphi_m(x')\}. \end{aligned}$$

Let $(u_m, p_m) \in (H^1(C^+_{2\tau}))^d \times L^2 C^+_{2\tau}$ be the solution to the following Stokes problem

$$\begin{cases} -\mathrm{div}(\sigma(v_m, p_m)) = 0, & \text{in } C^+_{2\tau}, \\ \mathrm{div}\, v_m = 0, & \text{in } C^+_{2\tau}, \\ v_m = v, & \text{on } \partial C^+_{2\tau}. \end{cases} \tag{4.31}$$

Then, multiplying the first equation in (4.31) by v_m and integrating by parts, we obtain

$$\begin{aligned} \int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla v_m &= \int_{\partial C^+_{2\tau}} [\sigma(v_m, p_m)n] v_m = \int_{\partial C^+_{2\tau}} [\sigma(v_m, p_m)n] v \\ &= \int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla v. \end{aligned}$$

Equivalently, from (3.9) and the fact that div $v_m = 0$, we deduce

$$\int_{C^+_{2\tau}} e(v_m) : e(v_m) = \int_{C^+_{2\tau}} e(v_m) : e(v).$$

Therefore, $e(v_m)$ is a bounded sequence in $L^2(C^+_{2\tau})$ and

$$\int_{C^+_{2\tau}} |e(v_m)|^2 \leqslant C \int_{C^+_{2\tau}} |e(v)|^2, \tag{4.32}$$

where $C > 0$. We note that $(v_m - v)|_{\partial C^+_{2\tau}} = 0$, then applying the Poincaré inequality to $v_m - v$ and the Korn's inequality, we deduce

$$\int_{C^+_{2\tau}} |v_m - v|^2 \leqslant C \int_{C^+_{2\tau}} |\nabla v_m|^2 \leqslant C \int_{C^+_{2\tau}} |e(v_m)|^2 \leqslant C \int_{C^+_{2\tau}} |e(v)|^2.$$

Namely, $\{v_m\}_m$ is a bounded sequence in $(H^1(C^+_{2\tau}))^d$. Then, there exists a subsequence, still denoted by v_m, such that $\{v_m\}_m$ converges weakly in $(H^1(C^+_{2\tau}))^d$ to some function $u \in (H^1(C^+_{2\tau}))^d$. From estimate (2.4), we obtain that the sequence $\{p_m\}_m$ also converges weakly in $L^2(C^+_{2\tau})$ to some $q \in L^2(C^+_{2\tau})$. Besides, div $v_m = 0$ in $C^+_{2\tau}$, for every m, and then we have div $u = 0$ in $C^+_{2\tau}$. Recalling (4.31), for every $\xi \in V = \{f \in (H^1_0(C^+_{2\tau}))^d \ : \ \mathrm{div} f = 0\}$, and using again (3.9), we obtain

$$\begin{aligned} -\int_{C^+_{2\tau}} \mathrm{div}(\sigma(u, q)) \cdot \xi &= \int_{C^+_{2\tau}} \sigma(u, q) : \nabla\xi - \int_{\partial C^+_{2\tau}} \sigma(u, q)n \cdot \xi \\ &= \int_{C^+_{2\tau}} e(u) : \nabla\xi, \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} 0 = -\int_{C^+_{2\tau}} \mathrm{div}(\sigma(v_m, p_m)) \cdot \xi &= \int_{C^+_{2\tau}} \sigma(v_m, p_m) : \nabla\xi - \int_{\partial C^+_{2\tau}} \sigma(v_m, p_m)n \cdot \xi \\ &= \int_{C^+_{2\tau}} e(v_m) : \nabla\xi. \end{aligned} \tag{4.34}$$

This implies

$$\begin{aligned} -\int_{C^+_{2\tau}} \mathrm{div}(\sigma(u, q)) \cdot \xi &= \int_{C^+_{2\tau}} e(u) : \nabla\xi - \int_{C^+_{2\tau}} e(v_m) : \nabla\xi \\ &= \int_{C^+_{2\tau}} e(u - v_m) : \nabla\xi. \end{aligned} \tag{4.35}$$

From the weak convergence in $(H^1(C^+_{2\tau}))^d$ of $\{v_m\}m$ to u, we obtain that the right hand side of (4.35) converges to zero, as m tends to infinity, for every $\xi \in V$. Then, (u, q) is a weak solution to

$$\begin{cases} -\mathrm{div}(\sigma(u, q)) = 0, & \text{in } C^+_{2\tau}, \\ \mathrm{div}\, u = 0, & \text{in } C^+_{2\tau}. \end{cases}$$

On the other hand, on account of the trace theorem we have $v_m \rightharpoonup u$ weakly in $(H^{1/2}(\partial C^+_{2\tau}))^d$. So that $v_m = v$ on $\partial C^+_{2\tau}$ implies $u = v$ on $\partial C^+_{2\tau}$. From the uniqueness of the solution to the Stokes problem we obtain that $u = v$ and $q = p$ in $C^+_{2\tau}$. Therefore, we get

$$v_m \rightharpoonup v \text{ weakly in } (H^1(C^+_{2\tau}))^d,$$

and, by compactness,

$$v_m \to v \text{ in } (L^2(C_{2\tau}^+))^d.$$

Now, as noticed before, the equation (4.24) holds for $v = v_m$ and $p = p_m$, then

$$\int_{\Delta_\tau} |\sigma(v_m, p_m)n|^2 \leqslant C\left(\int_{\Delta_{2\tau}} |\nabla_T v_m|^2 + \left(1 + \frac{1}{\tau}\right) \int_{C_{2\tau}^+} |\nabla v_m|^2 \right), \tag{4.36}$$

where $C > 0$ is independent of m. We observe that $v = v_m$ on $\Delta_{2\tau}$, $\nabla_T v \in (L^2(\Delta_{2\tau}))^d$ by hypotheses. So that, using the equation (4.32) we deduce

$$\int_{\Delta_\tau} |\sigma(v_m, p_m)n|^2 \leqslant C,$$

where $C > 0$ is independent of m. Hence, up to asubsequence, $\sigma(v_m, p_m)n$ converges weakly in $(L^2(\Delta_\tau))^d$ to some $h \in (L^2(\Delta_\tau))^d$. On the other hand, let us take any $\xi \in \tilde{V} = \{f \in (H^1(C_{2\tau}))^d \ : \ \operatorname{div} f = 0\}$. Using (3.9), it follows

$$0 = \int_{C_{2\tau}^+} \sigma(v_m, p_m) : \nabla\xi - \int_{\partial C_{2\tau}^+} \sigma(v_m, p_m)n \cdot \xi,$$

$$0 = \int_{C_{2\tau}^+} \sigma(v, p) : \nabla\xi - \int_{\partial C_{2\tau}^+} \sigma(v, p)n \cdot \xi.$$

Therefore,

$$\int_{\partial C_{2\tau}^+} \sigma(v_m, p_m)n \cdot \xi - \int_{\partial C_{2\tau}^+} \sigma(v, p)n \cdot \xi = \int_{C_{2\tau}^+} e(v_m - v) : \nabla\xi,$$

and the last integral converges to zero, as m tends to infinity. Namely,

$$\sigma(v_m, p_m)n \rightharpoonup \sigma(v, p)n \text{ weakly in } (L^2(\partial C_{2\tau}^+))^d. \tag{4.37}$$

Finally, we obtain $\sigma(v, p)n = h \in (L^2(\Delta_\tau))$. Then, by definition

$$\|\sigma(v, p)n\|_{L^2(\Delta_\tau)} \leqslant \liminf_{m\to\infty} \|\sigma(v_m, p_m)n\|_{L^2(\Delta_\tau)}.$$

On account of (4.32) and (4.36), we deduce (3.25). □

Using the previous result, we are able to prove the proposition 3.5.

Proof. First, assume that $\rho < d_0$. Let us cover ∂D with internally nonoverlapping closed cubes Q_j, $j = 1, \ldots, J$, with side $\tilde{\rho} = \gamma(L)\rho$, where $\gamma(L) = \frac{\min\{1, L\}}{2\sqrt{d}\sqrt{1 + L^2}}$. From the result of [5], we have

$$J \leqslant C\frac{|D|}{\rho^d} \leqslant CQ^d, \tag{4.38}$$

where $C > 0$ only depends on L. For every $j = 1, \ldots, J$ there exists $x_0 \in \partial D \cap Q_j$ such that $Q_j \cap (\Omega \setminus \overline{D}) \subset C^+_{\overline{\rho}}$, where $\overline{\rho} = \frac{\rho}{2\sqrt{1+L^2}}$ and $C^+_t = \{y = (y', y_d) \in \mathbb{R}^d : |y'| < t \ \ \varphi(y') < y_d < tL\}$, for every t, with $0 < t \leqslant 2\overline{\rho}$. In this case, φ is a Lipschitz function in $B_{2\overline{\rho}}(0) \subset \mathbb{R}^{d-1}$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{0,1}(B_{2\overline{\rho}}(0))} \leqslant 2\overline{\rho}L$, representing locally the boundary of D in a suitable coordinate system $y = (y_1, \ldots, y_d)$, $y = Rx$, with R an orthogonal transformation and $x = (x_1, \ldots, x_d)$ the reference coordinate system. We note that from (4.4), the functions $u \in (H^1(\Omega \setminus \overline{D}))^d$, $p \in L^2(\Omega \setminus \overline{D})$ satisfies

$$-\mathrm{div}(\tilde{\sigma}(u, p)) = 0, \quad \text{in } C^+_{2\overline{\rho}},$$

where $\tilde{\sigma}(u, p) = (RA(R^T \nabla u)R^T - RpIR^T)$. We have that $u = 0$ on ∂D, then applying equation (4.24) with $\tau = \overline{\rho}$, we obtain

$$\int_{\partial D \cap Q_j} |\sigma(u, p)|^2 \leqslant C\left(1 + \frac{1}{\rho}\right) \int_{C^+_{2\overline{\rho}}} |\nabla u|^2,$$

where $C > 0$ only depends on L. Following the same arguments as in the proof of proposition 3.3 in [5], we deduce (3.25). □

5. Computational examples

In this section we will perform some numerical experiments to compute $|\frac{W - W_0}{W_0}|$ for classes of cavities for which our result holds. In particular, we expect to collect numerical evidence that the ratio between $\frac{|D|}{|\Omega|}$ and $|\frac{W - W_0}{W_0}|$ is bounded from below and above by two constants, representing the ones appearing in our estimates.

Moreover, we are interested in studying the dependence of this ratio on d_0, which bounds from below the distance of D from $\partial\Omega$, and the size of the inclusions.

A more systematic analysis would require the knowledge of explicit solutions u and u_0. This would allow to compute analytically the constants in the upper and lower bounds, at least for some particular geometries. On the contrary to the case in [5], for the Stokes system it is difficult to find explicit solutions.

For the experiments we use the free software *FreeFem++* (see [25]). Moreover, in all numerical tests we consider a square domain Ω, discretized with a mesh of 100×100 elements, and with boundary condition $u|_{\partial\Omega} = g$ as in figure 1. The datum g satisfies the assumptions (**H4**) and (**H5**).

The first series of numerical tests has been performed by varying the position and the size of a circle inclusion D with volume up to 8% of the total size of the domain. In particular, we consider a circle inclusion with volume 0.2%, 3.1% and 7.1% with respect to $|\Omega|$. We have placed these circles in eight different positions, see figure 2. The results are collected in figures 3–5, for different values of the distance d_0 between the object D and the boundary of Ω. Also, the averages of all this simulations are collected in figure 6.

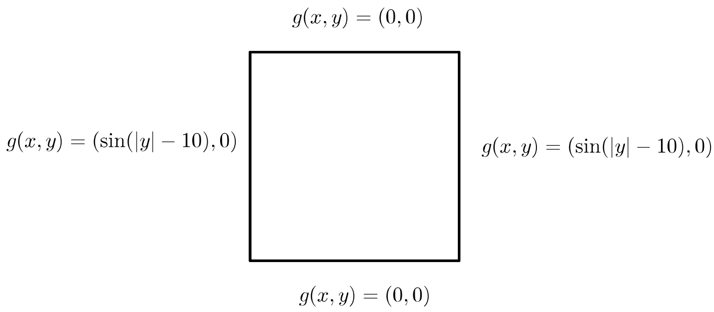

Figure 1. Square domain in 2D with boundary condition g.

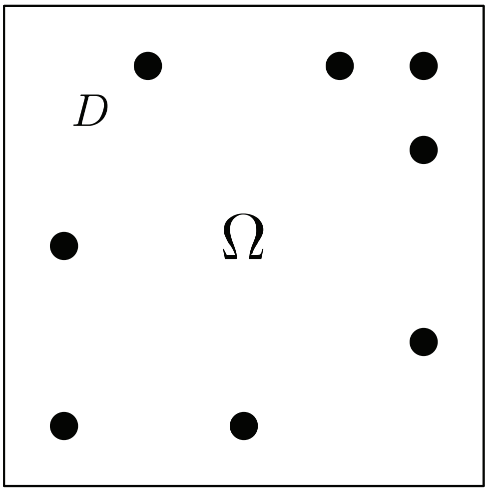

Figure 2. The eight positions of the circle inclusion D.

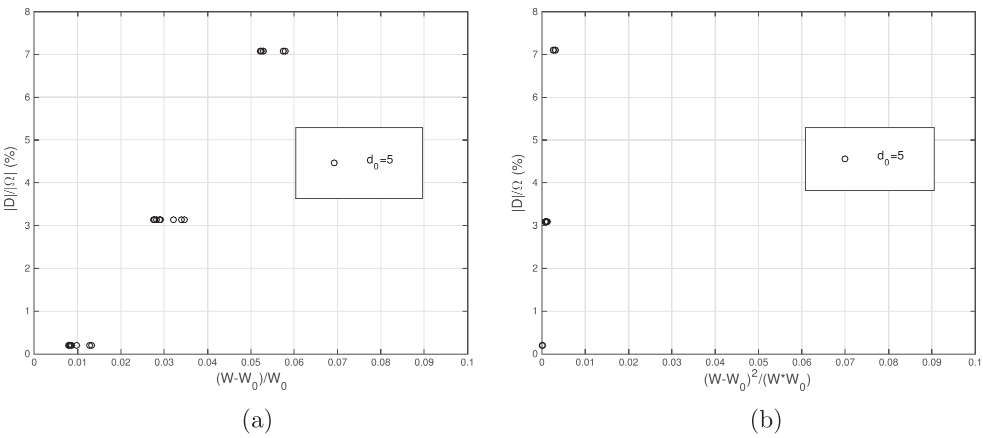

Figure 3. Case $d_0 = 5$ for circle inclusion. (a) Upper estimate. (b) Lower estimate.

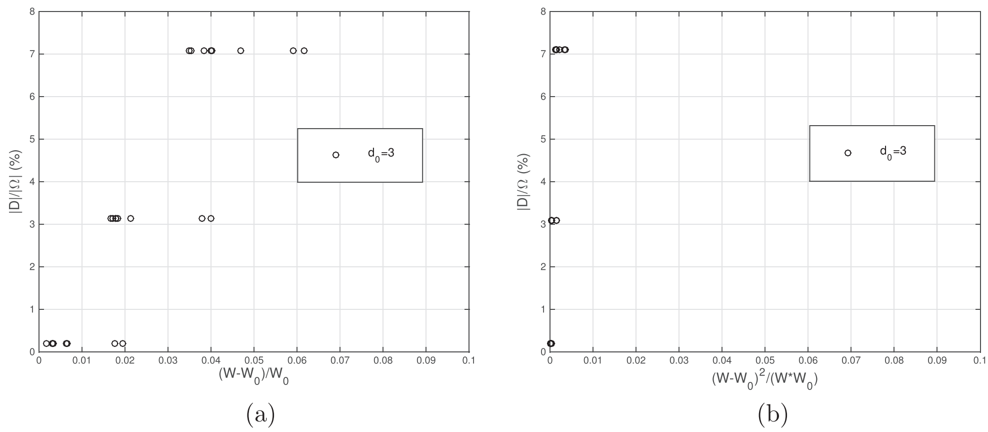

Figure 4. Case $d_0 = 3$ for circle inclusion. (a) Upper estimate. (b) Lower estimate.

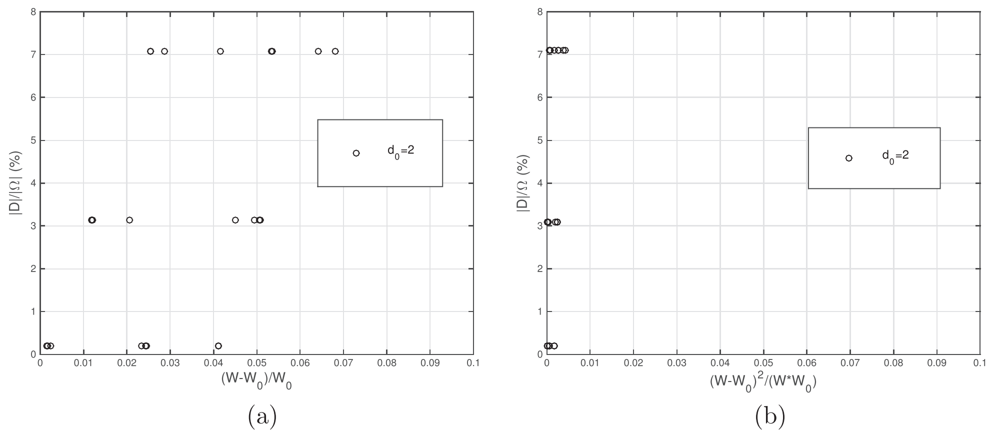

Figure 5. Case $d_0 = 2$ for circle inclusion. (a) Upper estimate. (b) Lower estimate.

In order to compare our numerical results with the theoretical upper and lower bounds (2.16) and (2.17), it is interesting to study the relationship between $\frac{|D|}{|\Omega|}$ and $|\frac{W-W_0}{W_0}|$. As we expected from the theory, the points $\left(\frac{W-W_0}{W_0}, \frac{|D|}{|\Omega|}\right)$ are confined inside an angular sector delimited by two straight lines.

However, it is quite clear that when d_0 decreases, then the lower bound becomes worse. To illustrate this situation, we simulate also the case when the distance is $d_0 = 1$, see figure 7.

As a second class of experiments, we consider what happens when the size of the circle increases. In this case we can observe that the number $|\frac{W-W_0}{W_0}|$ grows rapidly when the volume occupies almost the entire domain. The result is collected in figure 8.

Again it is observed the relationship between the volume of the object with the quotient $(W - W_0)/W_0$. This gives us an indication that the estimates found in theorems 2.11 and 2.12 involve constants that do not depend on the inclusion.

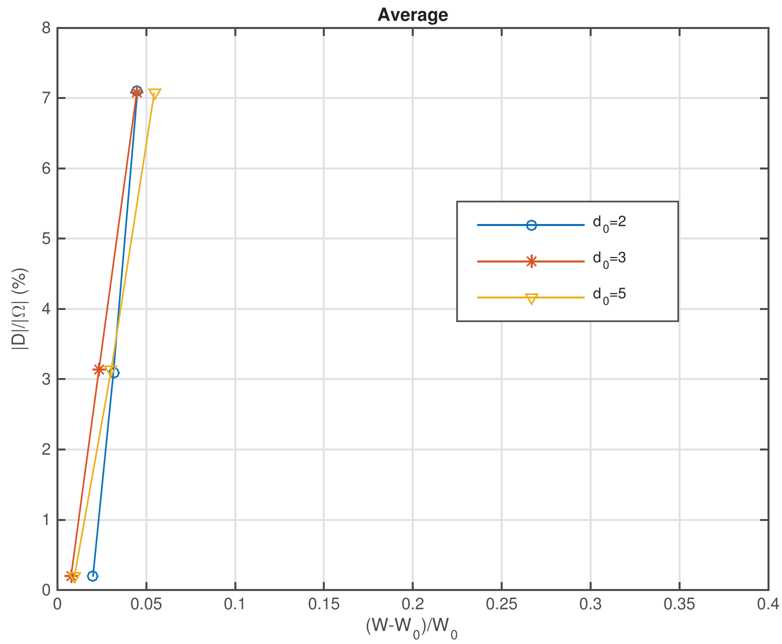

Figure 6. Averages of the ratio $\frac{W-W_0}{W_0}$ with different d_0 for circle inclusion.

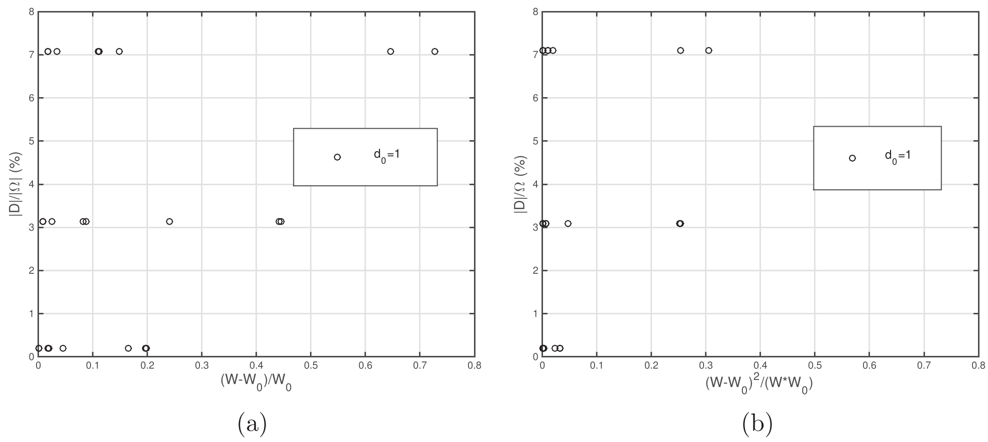

Figure 7. Case $d_0 = 1$ for circle inclusion. (a) Upper estimate. (b) Lower estimate.

Remark 5.1. From the previous analysis an interesting problem would be to find optimal lower and upper bounds for this model. Another interesting issue would be to weaken the *a priori* assumptions imposed on the obstacle, as for example the fatness condition (see, for instance, [2, 19], where this restriction is removed in the case of the conductivity and shallow shell equations, respectively).

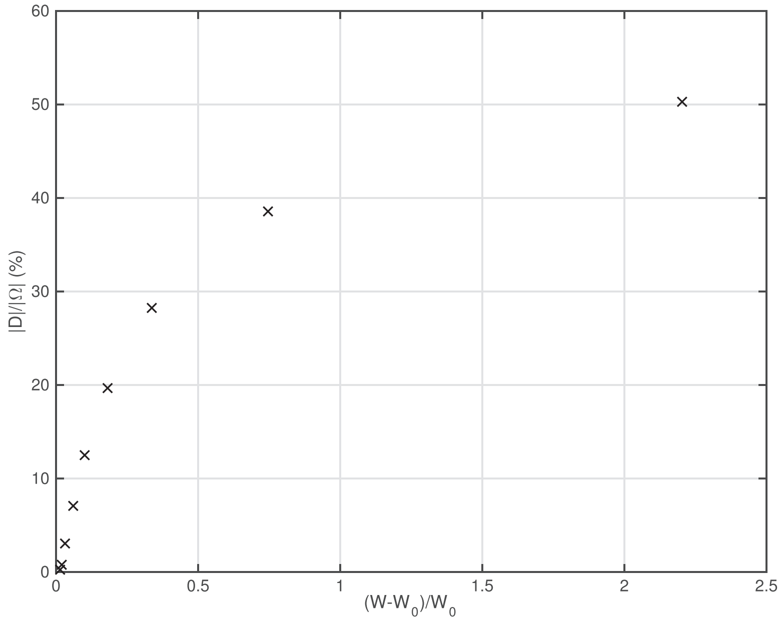

Figure 8. Influence of the size of the circle.

Acknowledgments

This work was partially supported by PFB03-CMM and Fondecyt 1111012. The work of E Beretta was supported by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica) and part of it was done while the author was visiting New York University Abu Dhabi. The work of C Cavaterra was supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase) and by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM. Part of this work was done while J Ortega was visiting the Departamento de Matemática, Universidad Autónoma de Madrid—UAM and the Instituto de Ciencias Matemáticas ICMAT-CSIC, Madrid, Spain. The work of S Zamorano was supported by CONICYT-Doctorado nacional 2012-21120662, and this work was also partially supported by the Advanced Grant NUMERIWAVES/FP7-246775 of the European Research Council Executive Agency, the FA9550-15-1-0027 of AFOSR, the MTM2011-29306 and MTM2014-52347 Grants of the MINECO.

References

[1] Alessandrini G and Rosset E 1999 The inverse conductivity problem with one measurement: bounds on the size of the unknown object *Siam J. Appl. Math.* **58** 1060–71

[2] Alessandrini G, Rosset E and Seo J K 1999 Optimal size estimates for the inverse conductivity problem with one measurement *Proc. Am. Math. Soc.* **128** 53–64

[3] Alessandrini G, Beretta E, Rosset E and Vessella S 2001 Optimal stability for inverse elliptic boundary value problems with unknown boundary *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* **4** 755–806

[4] Alessandrini G, Morassi A, Rosset E and Vessella S 2009 On doubling inequalities for elliptic systems *J. Math. Anal. Appl.* **357** 349–55

[5] Alessandrini G, Morassi A and Rosset E 2002 Detecting cavities by electrostatic boundary measurements *Inverse Problems* **18** 1333–53

[6] Alessandrini G, Morassi A and Rosset E 2004 Detecting an inclusion in an elastic body by boundary measurements *SIAM Rev.* **46** 477–98

[7] Alessandrini G, Bilotta A, Formica G, Morassi A, Rosset E and Turco E 2005 Numerical size estimates of inclusion in elastic bodies *Inverse Problems* **21** 133–51

[8] Álvarez C, Conca C, Friz L, Kavian O and Ortega J H 2005 Identification of immersed obstacles via boundary measurements *Inverse Problems* **21** 1531–52

[9] Álvarez C, Conca C, Lecaros R and Ortega J H 2008 On the identification of a rigid body immersed in a fluid: a numerical approach *Eng. Anal. Bound. Elem.* **32** 919–25

[10] Amrouche C and Rodriguez-Bellido M A 2012 On the regularity for the Laplace equation and the Stokes system *Monogr. Real Acad. Cienc. Zaragoza* **38** 1–20

[11] Auscher P and Qafsaoui M 2000 Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form *J. Funct. Anal.* **177** 310–64

[12] Ballerini A 2010 Stable determination of an immersed body in a stationary Stokes fluid *Inverse Problems* **26** 125015–39

[13] Barton A 2016 Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients *Manuscr. Mathematica* **151** 375–418

[14] Boyer F and Fabrie P 2013 *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models* (Applied Mathematical Sciences vol 183) (New York: Springer)

[15] Brown R, Mitrea I, Mitrea M and Wright M 2010 Mixed boundary value problems for the Stokes system *Trans. Am. Math. Soc.* **362** 1211–30

[16] Caubet F, Conca C and Godoy M 2016 On the detection of several obstacles in 2*D* Stokes flow: topological sensitivity and combination with shape derivatives *Inverse Problems Imaging* **10** 327–67

[17] Chenh J, Hou Y C and Yamamoto M 2001 Conditional stability estimates for an inverse boundary problem with non-smooth boundary in $\mathbb{R}^3$ *Trans. Am. Math. Soc.* **353** 4123–38

[18] Cordes H O 1956 Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen *Math. Ann.* **131** 278–312

[19] Di Cristo M, Lin C L, Vessella S and Wang J N 2013 Size estimates of the inverse inclusion problem for the shallow shell equation *SIAM J. Math. Anal.* **45** 88–100

[20] Doubova A and Fernández-Cara E 2015 Some geometric inverse problems for the linear wave equation *Inverse Problems Imaging* **9** 371–93

[21] Fabes E B, Kenig C E and Verchota G C 1988 The Dirichlet problem for the Stokes system on Lipschitz domains *Duke Math. J.* **57** 769–93

[22] Garofalo N and Lin F 1986 Monotonicity properties of variational integrals, A_p weights and unique continuation *Indiana Univ. Math. J.* **35** 245–68

[23] Girault V and Raviart P-A 2012 *Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms* vol 5 (New York: Springer)

[24] Heck H, Uhlmann G and Wang J N 2007 Reconstruction of obstacles immersed in an incompressible fluid *Inverse Problems Imaging* **1** 63–76

[25] Hecht F 2012 New development in FreeFem++ *J. Numer. Math.* **20** 251–65

[26] Kang H, Kim E and Milton G 2012 Sharp bounds on the volume fractions of two materials in a two-dimensional body from electrical boundary measurements: the translation method *Calculus Variations PDE* **45** 367–401

[27] Kang H and Milton G 2013 Bounds on the volume fractions of two materials in a three-dimensional body from boundary measurements by the translation method *SIAM J. Appl. Math.* **73** 475–92

[28] Lin C, Uhlmann G and Wang J N 2010 Optimal three-ball inequalities and quantitative uniqueness for the Stokes system *Discrete Continuous Dyn. Syst.* **28** 1273–90

[29] Morassi A and Rosset E 2003 Detecting rigid inclusions, or cavities, in an elastic body *J. Elast.* **73** 101–26

[30] Morassi A and Rosset E 2004 Stable determination of cavities in elastic bodies *Inverse Problems* **20** 453–80
[31] Morassi A, Rosset E and Vessella S 2007 Size estimates for inclusions in an elastic plate by boundary measurements *Indiana Univ. Math. J.* **56** 2325–84
[32] Payne L E and Weinberger H F 1958 New bounds for solutions of second order elliptic partial differential equations *Pacific J. Math.* **8** 551–73
[33] Temam R 2001 *Navier–Stokes Equations: Theory and Numerical Analysis* vol 343 (Providence, RI: American Mathematical Society)