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General equilibrium in CLO markets

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Abstract

We address a two-period equilibrium model with securitization of collateral-backed promises. Borrowers may suffer extra-economic default penalties and debts are pooled into collateralized loans obligations (CLO), allowing different seniority levels among tranches in a same CLO.

As securities with lower priority receive nothing unless those with higher priority are fully paid, we will have a non-convex set of feasible payment rates. Even in this context, equilibrium always exists. Moreover, although CLO have endogenous payments, the durability of the collateral will avoid pessimistic beliefs about the future rates of default.

JEL classification: D52; D91

Keywords: Asset-backed securitization; Extra-economic default penalties; Non-trivial equilibrium

1. Introduction

In financial markets, securitization of debt contracts has been a mean for financial institutions to reduce risk in their balance sheets. It allows better portfolio diversification, as investors have access to a broader pool of contracts, also called asset-backed securities (ABS). In this sense, this type of financial innovation has an important role to improve efficiency.

From the risk distribution perspective, one of the usual ways in which a given poll of assets is securitized into a family of ABS is allowing senior-subordinated structures among the different

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derivatives at the moment of payments. These structures are called collateralized loan obligations (CLO).

Our objective is to insert CLO in a general equilibrium model, generalizing the seminal works that extend the traditional general equilibrium model to allow for credit risk, collateral and extra-economic penalties (see Geanakoplos, 1996; Geanakoplos and Zame, 2002; Dubey et al., 2005). Moreover, we are interested in study the role of physical collateral requirements to avoid excess of investor pessimism about the futures rates of default.

Our economy has two time periods and there is uncertainty about the state of nature in the second period. Commodities may be durable, assets are real, and a finite number of agents can trade on spot markets. There are two types of contracts available in financial markets: (i) primitive assets, that are sold by the borrowers and are backed by physical collateral requirements; and (ii) derivatives, which are bought by the lenders and are backed by classes of primitives. In case of default, borrowers are burdened by the seizure of collateral requirements. Also, as in Dubey et al. (2005), individuals can suffer extra-economic penalties, which reflect the existence of legal or moral enforcements and may differ among agents.[1]

We suppose that claims are pooled into derivatives. These ABS are families of tranches, which receive payments following a senior-subordinated structure, guaranteeing that tranches with lower priority receive nothing unless those with higher priority are fully paid. As financial markets are anonymous, lenders take the rates of payment of the derivatives as given. At equilibrium, these rates are determined in such manner that the total value of deliveries matches the total value of payments.

Now, because financial returns are endogenous, it is possible to prove the existence of an equilibrium in a trivial way. If primitive assets have zero prices and lenders expect to suffer total default, any pure spot market equilibrium constitutes an equilibrium for our economy. Note that a pure spot markets equilibrium always exists, under usual neoclassical assumptions on preferences.

However, it is not rational that lenders expect to suffer total default in a derivative, whenever both the depreciated collateral and the original promises of their primitives have positive values. For these reasons, we propose and show the existence of an equilibrium refinement in which over-pessimistic beliefs are ruled out. The refined equilibrium will be one in which agents can expect to suffer total default, in a given state of nature, only on derivatives that are backed by primitives that have either zero real promises or zero depreciated bundle of collateral requirements.

1.1. Insertion in the literature and contributions

The study of securitization structures in a general equilibrium context, in which agents can default in their promises, has experienced an increasing importance over the last few years.

Geanakoplos (1996) and Geanakoplos and Zame (2002) studied the existence of an equilibrium in models in which borrowers are burdened by collateral requirements in order to protect lenders from credit risk. In these models, the only enforcement in case of default is the seizure of the collateral. Therefore, borrowers make strategic default and trade directly with lenders, that expect to receive the minimum between the depreciated value of the collateral and the value of the original promises. In this context, equilibrium existence follows from the scarceness of physical collateral requirements, which guarantees that short sales are bounded at equilibrium. Our financial structure

[1] In an early version of this article we internalize non-economic default penalties into an structure of non-ordered preferences for consumption. Also, we treat the case of pass-trough securities, i.e. asset-backed securities that distribute default *pro-rata*. For more details, see Steinert and Torres-Martínez (2004).

extend these models, as (abstract) intermediaries can issue more than one derivative, allowing for senior-subordinated structures.

Other types of models allow default without physical collateral requirements, but they burden borrowers by extra-economic penalties, proportional to the real value of default. In this context, Zame (1993) studies the advantages of default in order to promote efficiency, and Dubey et al. (2005) prove the existence of an equilibrium in a two-period model with incomplete markets.

As pointed out by Dubey et al. (2005), in models in which agents take payment rates of assets as given, the existence of a *pure spot market equilibrium* can be easily proved. This problem comes from very pessimistic beliefs of lenders about future rates of payment. In order to avoid this pathology and to allow assets to be traded at equilibrium, it is interesting to refine this equilibrium concept.

In our model, as primitives are backed by physical bundles, we can introduce a refinement concept using the fact that primitive assets deliver, in case of default, at least the depreciated value of the collateral. Thus, lenders expect to receive a positive payment when the physical collateral, associated to the underlying primitives, does not disappear from the economy. Although we cannot guarantee that assets are traded at equilibrium, our refinement concept assures that the absence of negotiation is not a consequence of over-pessimistic beliefs.

The rest of the paper is organized as follows: Section 2 describes the model; in Section 3 we discuss the role of collateral to avoid over-pessimistic beliefs, and state our refinement concept; and Section 4 is devoted to analyze the assumptions and to state our main result about existence. Finally, we make the proof of equilibrium existence in Appendix A.

2. Model

We consider a two-period economy in which there is no uncertainty at the first period, $t = 0$ (i.e., only one state of nature, denoted by $s = 0$, is reached). At the second period, $t = 1$, a state of nature is revealed among a finite number of possibilities, $s \in S$. For convenience of notation, we put $S^* = \{0\} \cup S$.

At each state $s \in S^*$ a finite number of perfect divisible commodities $l \in L$ are negotiated in spot markets. These goods can be durable and they may suffer depreciation contingent to the state of nature. This structure is given by matrixes $Y_s \in \mathbb{R}_+^{L\times L}$. Thus, when an agent chooses a bundle x at $t = 0$, he expects to receive a bundle $Y_s x$ if the state of nature $s \in S$ is reached.

Commodities in L are traded, at each $s \in S^*$, at prices $p_s \in \mathbb{R}_+^L$. Let $p = (p_s; s \in S^*)$ be the commodity price process and $p_{-0} = (p_s; s \in S)$.

A finite number of agents, $h \in H$, trades commodities at every state, choosing consumption allocations in $X_s = \mathbb{R}_+^L$. Moreover, at each s, agents receive an initial endowment $w_s^h \in \mathbb{R}_{++}^L$. Let $X = \Pi_{s\in S^*} X_s$ and $W_s = \sum_{h\in H} w_s^h$ for all $s \in S^*$.

We consider a financial structure in which assets are subject to credit risk. Borrowers can negotiate real securities, called primitive assets, which are subject to default and backed by *physical* collateral requirements. On the other side, financial intermediaries, *which are limited to pool individual claims*, make an asset-backed securitization of these debts contracts, selling derivatives to the lenders.

Formally, a finite number of collateral-backed primitive assets $k \in K$ can be sold at $t = 0$ for unitary prices $(q_k; k \in K) \in \mathbb{R}_+^K$. These assets make real promises $A_{s,k} \in \mathbb{R}_+^L$ at each state $s \in S$. When an agent h sells φ_k^h units of the primitive k, he pays an amount $q_k\varphi_k^h$ and he is burden to constitute a bundle $C_k\varphi_k^h$, where C_k is the unitary collateral requirement of k, which all agents take as given.

Furthermore, we suppose that in case of default agents can be burdened not only by the seizure of the depreciated bundle of collateral, but also by utility penalties, which are incorporated in their preferences, as in Dubey et al. (2005). With this penalties the market can induce agents to pay more than the collateral value at $t = 1$.

Hence, an agent h, who borrows φ_k^h units of k, delivers, at each state $s \in S$, a non-negative amount $\delta_{s,k}^h$, which is chosen jointly with the portfolio and consumption allocations and satisfies $\delta_{s,k}^h \geq \min\{p_s A_{s,k}; p_s Y_s C_k\}\varphi_k^h$.

In the other side, lenders can negotiate securities backed by the promises made by the borrowers. Also, families of securities are backed by classes of primitives. Thus, we suppose that the set K is partitioned, exogenously, into a *finite* number of disjoint classes, denoted by $\mathbb{A} \subset K$.

Let J be the collection of all derivatives that can be traded on the market. The promises within each class $\mathbb{A}$ are pooled by a financial intermediary, that issues a finite collection $J(\mathbb{A}) := \{j^1(\mathbb{A}), j^2(\mathbb{A}), \ldots, j^{n(\mathbb{A})}(\mathbb{A})\} \subset J$ of short-lived real assets.

Each $j \in J(\mathbb{A})$ makes individual real promises $A_{s,j} \in \mathbb{R}_+^L$ at each $s \in S$ and can be bought at price q_j at the first period. We assume that $j^m(\mathbb{A})$ has priority over the assets $(j^r(\mathbb{A}))_{r>m}$ in relation to promise payments. We denote by θ_j^h the number of units of asset j that a lender h buys.

Lenders know the securitization structure (i.e., they know what are the priorities among assets in the same family) and markets are anonymous. Thus, they expect to receive for each unit of the asset $j \in J(\mathbb{A})$ a percentage of the original promises, given by a payment rate $r_{s,j} \in [0, 1]$.

As tranches with lower priority suffer default before those with higher priority levels, if $j^m(\mathbb{A})$ pays in full at state $s \in S$ (i.e. $r_{s,j^m(\mathbb{A})} = 1$), then all the derivatives $j^{m'}(\mathbb{A})$, with $m' < m$, pay in full too (i.e. $r_{s,j^{m'}(\mathbb{A}_C)} = 1$). Moreover, if an asset $j^m(\mathbb{A})$ gives a partial default (i.e. $r_{s,j^m(\mathbb{A})} \in (0, 1)$), then all the tranches with higher priority over it pay in full (i.e. $r_{s,j^{m'}(\mathbb{A}_C)} = 1$, for $m' < m$) and all the derivatives that are subordinated to $j^m(\mathbb{A})$ give total default (i.e. $r_{s,j^{m'}(\mathbb{A})} = 0$, for $m' > m$).[2]

Therefore, we suppose that, at each state of nature, anonymous payment rates associated to derivatives in $J(\mathbb{A})$, belong to the non-empty, compact and *non-convex* set

$$\mathcal{R}(\mathbb{A}) := \{(r_1, \ldots, r_{n(\mathbb{A})}) \in [0, 1]^{n(\mathbb{A})} : \exists m, (r_{m'} = 1 \ \forall \ m' < m) \wedge (r_{m'} = 0 \ \forall \ m' > m)\}.$$

As we said above, each $h \in H$ is characterized by preferences that may depend on the real amount of default. Formally, the utility associated to an allocation $(x^h, \varphi^h, \delta^h, \theta^h)$ is given by

$$V^h(p_{-0}; (x^h, \varphi^h, \delta^h, \theta^h)) = U^h(x^h) - \sum_{s \in S} \sum_{k \in K} \frac{\lambda_{s,k}^h}{p_s v_s} [p_s A_{s,k} \varphi_k^h - \delta_{s,k}^h]^+,$$

where $U^h : X \to \mathbb{R}_+$ is the utility for consumption and $\lambda_{s,k}^h \geq 0$ the penalty that agent h suffer, at state s, for each unit of default on primitive $k \in K$.[3] Moreover, as in Dubey et al. (2005), vectors $(v_s; s \in S) \in \mathbb{R}_{++}^{L \times S}$ are exogenously fixed to transform nominal default into real terms.

Finally, as agents are price takers, given commodity prices p, given a price vector for both primitive and derivative assets $q = (q_k, q_j)_{k \in K, j \in J}$, and given anonymous payment rates for the

[2] Note that, when primitives are securitized into only one derivative, we could assume that agents take as given the nominal payment $N_{s,j} = r_{s,j} p_s A_{s,j}$. Thus, the financial intermediary would issue *endogenous asset-backed derivatives*. However, as a consequence of the senior-subordinated structure, in the general case we need to define separately both original promises and rates of payment.

[3] Given $z \in \mathbb{R}$, $[z]^+ = \max\{z, 0\}$.

derivatives $r = (r_{s,j})_{(s,j)\in S\times J}$, an agent $h \in H$ can choose non-negative consumption-financial allocations $[x^h, \varphi^h, \delta^h, \theta^h]$ subject to

- First period budget constraint

$$p_0 x_0^h + \sum_{\mathbb{A}\subset K}\left(\sum_{j\in J(\mathbb{A})} q_j\theta_j^h - \sum_{k\in\mathbb{A}} q_k\varphi_k^h\right) \le p_0 w_0^h, \tag{1}$$

- Collateral requirements constraint

$$x_0^h \ge \sum_{k\in K} C_k\varphi_k^h, \tag{2}$$

- Payments constraints

$$\delta_{s,k}^h \ge \min\{p_s A_{s,k}; p_s Y_s C_k\}\varphi_k^h \quad \forall\ (s,k)\in S\times K. \tag{3}$$

- Second period budget constraints

$$p_s x_s^h \le p_s w_s^h + p_s Y_s x_o^h + \sum_{\mathbb{A}\subset K}\left(\sum_{j\in J(\mathbb{A})} r_{s,j} p_s A_{s,j}\theta_j^h - \sum_{k\in\mathbb{A}}\delta_{s,k}^h\right) \quad \forall\ s\in S. \tag{4}$$

When prices are (p, q), and rates of payment are r, the budget set of the agent $h \in H$, denoted by $B^h(p, q, r)$, is given by the collection of non-negative consumption-financial allocations $(x^h, \varphi^h, \delta^h, \theta^h)$ that satisfy conditions (1)–(4) above.

It follows that our *economy with CLO markets* $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$ is characterized by the set of states of nature S^*, the set of agents characteristics $\mathcal{H} = (X, V^h, w^h)_{h\in H}$, the physical market structure $\mathcal{L} = (L, (Y_s)_{s\in S}, (W_s)_{s\in S^*})$ and the financial structure $\mathcal{F} = [\mathbb{A}, J(\mathbb{A}), (A_{s,k}, A_{s,j})_{s\in S}, C_k]_{(k,j)\in K\times J, \mathbb{A}\subset K}$.

Definition 1. An equilibrium for the economy $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$ is given by prices and rates of payment

$$[\bar p, \bar q, \bar r] \in \mathbb{P} := \mathbb{R}_+^{L\times S*} \times \mathbb{R}_+^J \times \mathbb{R}_+^K \times [0,1]^{S\times J},$$

and allocations

$$[\bar x^h, \bar\varphi^h, \bar\delta^h, \bar\theta^h] \in \mathbb{X} := X \times \mathbb{R}_+^K \times \mathbb{R}_+^{K\times S} \times \mathbb{R}_+^J,$$

for each agent $h \in H$, such that

(A) For each agent $h \in H$, $(\bar x^h, \bar\varphi^h, \bar\delta^h, \bar\theta^h) \in B^h(\bar p, \bar q, \bar r)$.
(B) Physical Markets are cleared

$$\sum_{h\in H}\bar x_0^h = W_0, \qquad \sum_{h\in H}\bar x_s^h = W_s + Y_s W_0 \quad \forall\ s\in S.$$

(C) Agents make optimal choices, i.e. for each $h \in H$ and $(x^h, \varphi^h, \delta^h, \theta^h) \in B^h(\bar p, \bar q, \bar r)$:

$$V^h(\bar p_{-0}; (\bar x^h, \bar\varphi^h, \bar\delta^h, \bar\theta^h)) \ge V^h(\bar p_{-0}; (x^h, \varphi^h, \delta^h, \theta^h)).$$

(D) For each $\mathbb{A} \subset K$, the value of derivatives aggregate purchases must match the value of the aggregate short sales

$$\sum_{h \in H} \sum_{j \in J(\mathbb{A})} \bar{q}_j \bar{\theta}^h_j = \sum_{h \in H} \sum_{k \in \mathbb{A}} \bar{q}_k \bar{\varphi}^h_k.$$

(E) At each state $s \in S$ and for each class $\mathbb{A} \subset K$, the total payments of the derivatives must be equal to the total deliveries made by the borrowers

$$\sum_{h \in H} \sum_{j \in J(\mathbb{A})} \bar{r}_{s,j} \bar{p}_s A_{s,j} \bar{\theta}^h_j = \sum_{h \in H} \sum_{k \in \mathbb{A}} \bar{\delta}^h_{s,k}.$$

(F) At each state $s \in S$, payment rates must be consistent with the financial structure

$$(\bar{r}_{s,j})_{j \in J(\mathbb{A})} \in \mathcal{R}(\mathbb{A}) \quad \forall \ \mathbb{A} \subset K.$$

Conditions (E) and (F) imply that, for a given tranche $j^m(\mathbb{A})$ which is negotiated and has strictly positive promises values, its equilibrium rate of payment $\bar{r}_{s,j^m(\mathbb{A})}$ takes into account the payments made to the previous tranches, in the sense that

$$\bar{r}_{s,j^m(\mathbb{A})} = \max \left\{ 0; \min \left\{ \frac{\sum_{h \in H} \sum_{k \in \mathbb{A}} \bar{\delta}^h_{s,k} - \sum_{i=1}^{m-1} \bar{p}_s A_{s,j^i(\mathbb{A})} \sum_{h \in H} \bar{\theta}^h_{j^i(\mathbb{A})}}{\bar{p}_s A_{s,j^m(\mathbb{A})} \sum_{h \in H} \bar{\theta}^h_{j^m(\mathbb{A})}}; 1 \right\} \right\}.$$

Finally, it is important to remark that the existence of utility penalties can allow borrowers to raise more capital than the collateral value. However, if extra-economic penalties do not exist, the value of the unitary physical collateral will be, at equilibrium, strictly greater than the value of the asset. Otherwise, when an agent makes the joint operation of buying the collateral and selling the promises, he has an arbitrage opportunity, since he raises non-negative transfers today, receives non-negative returns tomorrow and has the right to consume the collateral requirements.

3. Collateral avoids over-pessimistic beliefs

Our definition of equilibrium could generate misleading results. When agents are allowed to have pessimistic beliefs about the derivatives rates of payment it is always possible to trivially guarantee the existence of an equilibrium.

In fact, suppose that the price of primitives and the rates of payment of derivatives are equal to zero, i.e. $(\bar{q}_k, \bar{r}_{s,j})_{(s,j,k) \in S \times J \times K} = 0$. Since an agent h does not expect to receive any payment if he buys a derivative, he has no incentive to do it, so the allocation $\bar{\theta}^h = 0$ is optimal. Similarly, since primitive assets have zero price, $\bar{\varphi}^h = 0$ is optimal for each agent $h \in H$. Furthermore, as agents will not have any promise to pay at the second period, $\bar{\delta}^h_{s,k} = 0$, for each $(s,k) \in S \times K$, is also optimal. Therefore, the model becomes equivalent to a general equilibrium model with

durable goods and without financial markets. Existence of a *pure spot market equilibrium* in this framework is not difficult to prove.

Note that, when over-pessimistic beliefs are allowed, the proof mentioned above would be as good as any other. Thus, it would not be satisfactory to guarantee the existence of equilibrium without excluding this possibility.

It is worth to note that this problem is not idiosyncratic to our model. In fact, it should be considered at every model in which agents take the payment rates of assets promises as given. Although the expected rates of payment are determined endogenously in equilibrium, if derivatives are not traded, any rate of payment is consistent with equilibrium. Thus, agents could be extremely pessimistic, believing that no deliveries would be made in any state, for any asset, which in turn leads to non-negotiation of derivatives.

In their seminal paper, Dubey et al. (2005) address this topic proposing a refined equilibrium concept in order to avoid these over-pessimistic beliefs. They define a ε-boosted equilibrium as an equilibrium of an abstract economy, in which exists an external agent who buys and sells ε units of each asset (that may be interpreted as a government that guarantees an infinitesimal minimum delivery rate), and always delivers the total promises, injecting new commodities in the economy. Therefore, lenders are not over-pessimistic and the rates of payment at each ε-boosted equilibrium are strictly positive. When ε goes to zero, they obtain a refined equilibrium.

In their refinement, Dubey et al. (2005) use the touch of optimism introduced by the ε-agent to banish extremely pessimistic beliefs about the future rates of default. In our model, however, physical collateral requirements introduce a new dimension: it is natural to suppose lenders will expect to receive positive payments when the depreciated collateral bundles of the underling primitives are different from zero. In this sense, collateral avoids over-pessimistic belief without having to use an external agent.

More formally, we propose another refinement concept in which we guarantee that, at each state of nature, when primitives associated with a CLO give positive returns, independent of extra-economic enforcements, the most senior tranche, which made non-zero promises at this state, has a non-zero payment rate. Also, when some derivative has a positive rate of payment, at least one of the primitives that backs it has positive price.

Definition 2. An equilibrium $[(\bar{p}, \bar{q}, \bar{r}); (\bar{x}^h, \bar{\varphi}^h, \bar{\delta}^h, \bar{\theta}^h)_{h \in H}]$ is non-trivial if the expected payment rates are not over-pessimistic. That is, at each state $s \in S$ and for each class $\mathbb{A} \subset K$, if

$$\min_{k \in \mathbb{A}} \{\bar{p}_s A_{s,k}; \bar{p}_s Y_s C_k\} > 0,$$

then

$$[\bar{r}_{s, j^m(\mathbb{A})} > 0 \ \forall \ m \leq m_s^*(\mathbb{A})] \wedge [\exists k' \in \mathbb{A}, \bar{q}_{k'} > 0],$$

where $m_s^*(\mathbb{A}) := \min\{m : \|A_{s, j^m(\mathbb{A})}\|_1 \neq 0\}$ when $\sum_{j \in J(\mathbb{A})} A_{s,j} \neq 0$, and $m_s^*(\mathbb{A}) = n(\mathbb{A})$ otherwise.

Note that it would not be reasonable to ask agents to expect more optimistic rates of payment, since they do not know what is the total amount of primitives that was sold by the borrowers. In fact, rates of payment depend, in equilibrium, on both the total units of primitives sold and the total units of derivatives bought.

Finally, note that even with our refinement concept, it is possible that, at equilibrium, it does not exists a class of primitives that satisfies the conditions stated in Definition 2. In this case, a

pure spot market equilibrium can be assured in a trivial manner and, as we said above, our proof is superfluous. Hence, we discuss below, after the statement of the assumptions, the characteristics over the financial structure that guarantee that a family of derivatives has an equilibrium with non-trivial rates of payment.

4. Equilibrium existence

In order to guarantee the existence of a refined equilibrium we will make the following assumptions.

Assumption 1. For each agent $h \in H$, the function $U^h : X \to \mathbb{R}_+$ is continuous, concave and strictly increasing.

Assumption 2. For each primitive $k \in K$, collateral requirements C_k are different from zero.

Assumption 3. Given $x = (x_l; l \in L) \in \mathbb{R}^L_+$, let $\|x\|_{\max} = \max_{l \in L} x_l$. We assume that, for each $h \in H$:

$$\lim_{x \in \mathbb{R}^L_{++}; \|x\|_{\max} \to +\infty} U^h(x) = +\infty.$$

Assumption 4. For each $(k, j) \in K \times J$, real promises $A_k = (A_{s,k})_{s \in S}$ and $A_j = (A_{s,j})_{s \in S}$ are different from zero. Moreover, for each class of primitives $\mathbb{A} \subset K$:

$$\sum_{k \in \mathbb{A}} (A_{s,k})_l \neq 0 \Leftrightarrow \sum_{j \in J(\mathbb{A})} (A_{s,j})_l \neq 0 \quad \forall\ (s, l) \in S \times L.$$

It is important to remark that physical collateral requirements, which back promises in case of default, will guarantee that at equilibrium short sales of primitives are bounded (see Assumption 2). Hypothesis 3 will guarantees that all equilibrium commodity prices are uniformly bounded from below at each state of nature $s \in S^*$ (see Lemma 4 in Appendix A). This property on prices is sufficient to assure that it exists an equilibrium with non-trivial rates of payment.

In the other hand, Assumption 4 assure that, independently of the price level, one derivative has positive real promises if and only if at least one primitive also has it.

Theorem 1. *Under Assumptions 1–4 our economy $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$ has a non-trivial equilibrium.*

Note that, as we suppose that preferences are strictly monotonic on consumption, equilibrium commodity prices (if they exist) will be strictly positive, $\bar{p} \gg 0$, which implies that for each class of primitives $\mathbb{A}$ a necessary and sufficient condition to guarantee that

$$\min_{k \in \mathbb{A}} \{\bar{p}_s A_{s,k}; \bar{p}_s Y_s C_k\} > 0$$

is that $\min_{k \in \mathbb{A}} \{\|A_{s,k}\|_1; \|Y_s C_k\|_1\} > 0$.

It follows that, a family of derivatives will have positive rates of payment at $s \in S$ if both $\min_{k \in \mathbb{A}} \|A_{s,k}\|_1 > 0$ and $\min_{k \in \mathbb{A}} \|Y_s C_k\|_1 > 0$. Thus, under our assumptions, the requirement that guarantees that a family of derivatives has a non-zero vector of rates of payment is independent of equilibrium prices.

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Appendix A. Proof of the theorem

Let $\Delta^n_+ := \{z \in \mathbb{R}^n_+ : \sum_{i=1}^n z_i = 1\}$. In order to guarantee the equilibrium existence, we consider prices (p_0, q_K, q_J) that belong to the convex-compact set

$$\Xi = \left\{ (p_0, q_K, q_J) \in \mathbb{R}^L_+ \times [0,1]^K \times \mathbb{R}^J_+ : (p_0, q_J) \in \Delta^{\#L+\#J}_+, \right.$$

$$\left. \sum_{j \in J(\mathbb{A})} q_j \leq \sum_{k \in \mathbb{A}} q_k \ \forall \ \mathbb{A} \subset K \right\},$$

where $q_K = (q_k)_{k \in K}$, $q_J = (q_j)_{j \in J}$. Moreover, for each state of nature $s \in S$, we assume that commodity prices $p_s \in \Delta^L_+$. Note that, given a finite set A, $\Delta^A_+ := \Delta^{\#A}_+$.

For convenience of notation, let $\pi = (p, q, r)$ be a generic vector of prices and rates of payment. A generic allocation for an agent h will be denoted by $\eta^h = (x^h, \varphi^h, \delta^h, \theta^h)$, and $\eta = (\eta^h)_{h \in H}$ denotes a generic vector of allocations.

Also, we will define for each $h \in H$ the decision correspondence Ψ^h as follow:

$$\Psi^h(p_{-0}, \eta^h) := \{\hat{\eta}^h \in \mathbb{X} : V^h(p_{-0}; \hat{\eta}^h) > V^h(p_{-0}; \eta^h)\}.$$

Remark 1. Suppose that there are non-trivial equilibria for any economy $\mathcal{E}$ that satisfies, at each $s \in S$, $\sum_{k \in \mathbb{A}} A_{s,k} \leq \sum_{j \in J(\mathbb{A})} A_{s,j}$, for every $\mathbb{A} \subset K$. Then, it is possible to find a non-trivial equilibrium for any economy $\mathcal{E}'$ in which primitive and derivative promises only satisfy Assumption 4.

In fact, for such $\mathcal{E}'$ we have that

$$\left[\sum_{k \in \mathbb{A}} (A'_{s,k})_l \neq 0 \right] \Leftrightarrow \left[\sum_{j \in J(\mathbb{A})} (A'_{s,j})_l \neq 0 \right] \quad \forall \ (s,l) \in S \times L, \tag{A.1}$$

and consequently there exists $\lambda \in \mathbb{R}_{++}$ such that $\sum_{k \in \mathbb{A}} A'_{s,k} \leq \sum_{j \in J(\mathbb{A})} \lambda A'_{s,j}$, for each $\mathbb{A} \subset K$, for each $s \in S$.

Let $(\bar{\pi}, \bar{\eta})$ be an equilibrium for the economy $\mathcal{E}$, which is equal to $\mathcal{E}'$ except for the derivatives promises, that are given by $A_{s,j} = \lambda A'_{s,j}$.

Consider the allocation $(\bar{\pi}', \bar{\eta}')$ given by $(\bar{p}', \bar{q}'_K, \bar{r}'; \bar{x}', \bar{\varphi}', \bar{\delta}') = (\bar{p}, \bar{q}_K, \bar{r}; \bar{x}, \bar{\varphi}, \bar{\delta})$, $\bar{\theta}' = \lambda \bar{\theta}$ and $\bar{q}'_J = (1/\lambda)\bar{q}_J$. One can easily verify that the allocation $(\bar{\pi}', \bar{\eta}')$ is an equilibrium for the economy $\mathcal{E}'$.

It follows that, without loss of generality, we can assume that $\sum_{k \in \mathbb{A}} A_{s,k} \leq \sum_{j \in J(\mathbb{A})} A_{s,j}$ for each $\mathbb{A} \subset K$, for each $s \in S$.

Lemma 1. *Given $\pi = (p, q, r)$, if allocations $(\eta^h)_{h\in H} = (x^h, \varphi^h, \delta^h, \theta^h)_{h\in H}$ satisfies equilibrium conditions (A)–(C) then, for each agent $h \in H$, the vector $(x^h, \varphi^h, \delta^h)$ is uniformly bounded.*

Proof. Condition (B) implies that, at the first period, $\sum_{h\in H} x_0^h = W_0$. Thus, it follows that, for each commodity $l \in L$ and for each agent $h \in H$, the consumption allocation satisfies $x_{0,l}^h \leq W_{0,l}$.

Moreover, it follows from Condition (A) that $\eta^h \in B^h(\pi)$ and, therefore

$$\sum_{k\in K} C_{k,l}\varphi_k^h \leq x_{0,l}^h \leq W_{0,l}. \tag{A.2}$$

Then, summing over $l \in L$, we obtain that

$$\varphi_k^h \leq \Omega_k := \frac{1}{\sum_{l\in L} C_{k,l}} \sum_{l\in L} W_{0,l}, \tag{A.3}$$

which implies that short sales are bounded. Now, equilibrium condition (B) guarantees that, at each state $s \in S$, $\sum_{h\in H} x_s^h = W_s + Y_s W_0$. Since each term on left hand side, in the last equation, is non-negative, it follows that bundles x_s^h are bounded.

Finally, as commodity prices, at each state of nature $s \in S$, belong to the simplex Δ_+^L, the value of primitive promises, $p_s A_{s,k}\varphi_k^h$, is bounded for each $(h, k) \in H \times K$. Thus, payments δ^h are bounded from above, node by node, primitive by primitive. □

Now, we will truncate endogenous variables in order to find an optimal allocation for the economy. Our goal is to prove that, given upper and lower bounds on allocations, there exists an equilibrium for a truncated economy (as defined below). Furthermore, we show that this truncated equilibria allocations converge, when the appropriated limit is taken, to an equilibrium allocation of our original economy $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$.

The truncated economy $\mathcal{E}_M$. We define, for each

$$M \in \mathcal{M} := \{(M_1, M_2) \in \mathbb{R}_{++}^2 : M_1 < M_2\},$$

a truncated economy $\mathcal{E}_M$ in which the structure of uncertainty and the physical markets are the same as in $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$.

Each agent $h \in H$ can demand commodities, can sell primitives $k \in K$ and can buy derivatives $j \in J$ restricted to the space of allocations $\mathbb{X}_M$, which is given by the set of vectors $\eta^h \in \mathbb{X}$ that satisfies

$$\|x^h\|_\infty \leq M_1, \qquad \|\varphi^h\|_\infty \leq 2\Omega, \qquad \|\theta^h\|_\infty \leq 2(\#H)\Omega,$$
$$\|\delta^h\|_\infty \leq 2\Omega \max_{(s,k)\in S\times K} \|A_{s,k}\|_1,$$

where $\|\,\|_\infty$ denotes the sup-norm and $\Omega := \max_{k\in K}\Omega_k$ is the maximum of upper bounds on short sales defined on Lemma 1.

Moreover, in order to guarantee the existence of a non-trivial equilibrium, we need to find a lower bound away from zero for the anonymous rates of payment of the derivatives. To attempt this, given a class $\mathbb{A} \subset K$, we define the *truncated space of CLO payment rates* as the set of vectors $(r_{s,j^m(\mathbb{A})}) \in \mathbb{R}_+^{n(\mathbb{A})}$ that belongs to

$$\Upsilon_M^s(\mathbb{A}) := \prod_{m=1}^{n(\mathbb{A})} [\beta_M^{s,m}(\mathbb{A}), 1],$$

where, for each $m \in \{1, \dots, n(\mathbb{A})\}$:

$$\beta_M^{s,m}(\mathbb{A}) = \begin{cases} \dfrac{1}{M_1} & \text{if } \min\limits_{k \in \mathbb{A}} \{\|A_{s,k}\|_1, \|Y_s C_k\|_1\} > 0 \wedge m \leq m_s^*(\mathbb{A}), \\ \dfrac{1}{M_2} & \text{in other case.} \end{cases} \tag{A.4}$$

Therefore, in the economy $\mathcal{E}_M$, the space of prices and rates of payment $\pi = (p, q, r)$ is given by

$$\mathbb{P}_M := \Xi \times (\Delta_+^L)^S \times \prod_{\mathbb{A} \subset K s \in S} \Upsilon_M^s(\mathbb{A}).$$

For a given vector of prices and anonymous payment rates $\pi \in \mathbb{P}_M$, let $B_M^h(\pi) = B^h(\pi) \cap \mathbb{X}_M$ be the *truncated* budget set. For each agent h, we define the *truncated decision correspondence* $\Psi^{h,M} : (\Delta_+^L)^S \times \mathbb{X}_M \twoheadrightarrow \mathbb{X}_M$ as the restriction of the correspondence Ψ^h to the space $\mathbb{X}_M$.

Now, associated to each $h \in H$, we define a reaction correspondence $\psi_M^h : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow \mathbb{X}_M$ by

$$\psi_M^h(\pi, \eta) = \begin{cases} \dot{B}_M^h(\pi) & \text{if } \eta^h \notin B_M^h(\pi), \\ \dot{B}_M^h(\pi) \cap \Psi^{h,M}(p_{-0}, \eta^h) & \text{if } \eta^h \in B_M^h(\pi), \end{cases}$$

where $\dot{B}_M^h(\pi)$ denotes the interior of $B_M^h(\pi)$ relative to $\mathbb{X}_M$. Reaction correspondences are also defined for each state $s \in S^*$. Let $\psi_M^0 : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow \Xi$ be

$$\psi_M^0(\pi, \eta) = \left\{ (p_0', q_K', q_J') : p_0' \left[\sum_{h \in H} x_0^h - W_0 \right] + \sum_{j \in J} q_j' \sum_{h \in H} \theta_j^h - \sum_{k \in K} q_k' \sum_{h \in H} \varphi_k^h > 0 \right\},$$

and, for each $s \in S$, let $\psi_M^s : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow \Delta_+^L$ be

$$\psi_M^s(\pi, \eta) = \left\{ p_s' \in \Delta_+^L : p_s' \left(\sum_{h \in H} [x_s^h - Y_s x_0^h] - W_s \right) > p_s \left(\sum_{h \in H} [x_s^h - Y_s x_0^h] - W_s \right) \right\}.$$

Given a class $\mathbb{A}$, for each $s \in S$, and for each $m \in \{1, 2, \dots, n(\mathbb{A})\}$, we define the reaction correspondence

$$\psi_M^{s, j^m(\mathbb{A})} : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow [\beta_M^{s,m}(\mathbb{A}), 1],$$

as the set function that associates, to each vector $(\pi, \eta) \in \mathbb{P}_M \times \mathbb{X}_M^H$, the set of numbers $r' \in [\beta_M^{s,m}(\mathbb{A}), 1]$ that satisfies

$$\left(r' p_s A_{s, j^m(\mathbb{A})} \sum_{h \in H} \theta_{j^m(\mathbb{A})}^h + \sum_{i=1}^{m-1} r_{s, j^i(\mathbb{A})} p_s A_{s, j^i(\mathbb{A})} \sum_{h \in H} \theta_{j^i(\mathbb{A})}^h - \sum_{k \in \mathbb{A}} \sum_{h \in H} \delta_{s,k}^h \right)^2$$
$$< \left(\sum_{i=1}^{m} r_{s, j^i(\mathbb{A})} p_s A_{s, j^i(\mathbb{A})} \sum_{h \in H} \theta_{j^i(\mathbb{A})}^h - \sum_{k \in \mathbb{A}} \sum_{h \in H} \delta_{s,k}^h \right)^2.$$

Definition 3. Given $M \in \mathcal{M}$, an equilibrium for the truncated economy $\mathcal{E}_M$ is a vector

$$(\bar{\pi}, \bar{\eta}) = ((\bar{p}_M, \bar{q}_M, \bar{r}_M), (\bar{x}_M^h, \bar{\varphi}_M^h, \bar{\delta}_M^h, \bar{\theta}_M^h)_{h \in H}) \in \mathbb{P}_M \times \mathbb{X}_M^H,$$

at which all the reaction correspondences defined above have an empty value.

Lemma 2. *Given a vector $M \in \mathcal{M}$, if Assumptions 1 and 2 hold, there exists an equilibrium for the truncated economy $\mathcal{E}_M$.*

Proof. Observe that from Assumption 2, $\dot{B}_M^h(\pi)$ has non-empty values and has open graph. Then, it follows from Assumption 1, that the reaction correspondences $(\psi_M^s)_{s \in S^*}$, $(\psi_M^h)_{h \in H}$, and $(\psi_M^{s,j})_{\{(s,j) \in S \times J(\mathbb{A}), \mathbb{A} \subset K\}}$ satisfy the assumptions of the Gale-Mas-Colell Fixed Point Theorem (see Gale and Mas-Colell, 1975, 1979), that is, all correspondences are lower hemicontinuous with convex and open values, have the same domain, and the product of the *image spaces* coincides with theirs domains.

Thus, there exists a vector $(\bar{\pi}_M, \bar{\eta}_M) \in \mathbb{P}_M \times \mathbb{X}_M^H$ such as

- $\psi_M^h(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$ or $\bar{\eta}_M^h \in \psi_M^h(\bar{\pi}_M, \bar{\eta}_M)$, for each agent $h \in H$;
- $\psi_M^0(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$ or $((\bar{p}_M)_0, \bar{q}_M) \in \psi_M^0(\bar{\pi}_M, \bar{\eta}_M)$;
- $\psi_M^s(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$ or $(\bar{p}_M)_s \in \psi_M^s(\bar{\pi}_M, \bar{\eta}_M)$, for each state of nature $s \in S$;
- $\psi_M^{s,j^m(\mathbb{A})}(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$ or $(\bar{r}_M)_{s,j^m(\mathbb{A})} \in \psi_M^{s,j^m(\mathbb{A})}(\bar{\pi}_M, \bar{\eta}_M)$, for each state of nature $s \in S$, for every $\mathbb{A} \subset K$ and for each $m \in \{1, 2, \ldots, n(\mathbb{A})\}$.

Clearly it is not possible to $\bar{\eta}_M^h \notin B_M^h(\bar{\pi}_M)$, because in this case it would neither be a fixed point, nor an empty value. Moreover, we can not have $\bar{\eta}_M^h \in \psi_M^h(\bar{\pi}_M, \bar{\eta}_M)$ because it contradicts the fact that $\bar{\eta}_M^h \notin \Psi^{h,M}((\bar{p}_{-0})_M, \bar{\eta}_M^h)$. Thus, we must have $\psi_M^h(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$, for each agent $h \in H$.

As noted above, $\bar{\eta}_M^h \in B_M^h(\bar{\pi}_M)$. Adding over the agents, it follows that

$$(\bar{p}_M)_0 \left[\sum_{h \in H} (\bar{x}_M^h)_0 - W_0 \right] + \sum_{\mathbb{A} \subset K} \left(\sum_{j \in J(\mathbb{A})} (\bar{q}_M)_j \sum_{h \in H} (\bar{\theta}_M^h)_j - \sum_{k \in \mathbb{A}} (\bar{q}_M)_k \sum_{h \in H} (\bar{\varphi}_M^h)_k \right) \leq 0.$$

Thus, $((\bar{p}_M)_0, \bar{q}_M) \notin \psi_M^0(\bar{\pi}_M, \bar{\eta}_M)$ and, therefore, $\psi_M^0(\bar{\pi}_M, \bar{\eta}_M)$ is an empty set. Finally, one can easily see that, from its definition, correspondences ψ_M^s and $\psi_M^{s,j^m(\mathbb{A})}$ can not have a fixed point. Then, we have that $\psi_M^s(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$ and $\psi_M^{s,j^m(\mathbb{A})}(\bar{\pi}_M, \bar{\eta}_M) = \emptyset$. □

Now, when mistakes are not possible, we suppress the subscript of the allocations $(\bar{\pi}_M, \bar{\eta}_M)$. So, with $M \in \mathcal{M}$ fixed, we already know that an equilibrium allocation for the truncated economy, $(\bar{\pi}, \bar{\eta})$, satisfies $\bar{\eta}^h \in B_M^h(\bar{\pi})$ and $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \cap \dot{B}_M^h(\bar{\pi}) = \emptyset$.

Also, as $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h)$ and $\dot{B}_M^h(\bar{\pi})$ are open sets, it follows that $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \cap$ closure $[\dot{B}_M^h(\bar{\pi})] = \emptyset$. As $\dot{B}_M^h(\bar{\pi})$ is a non-empty and convex set, we conclude that $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \cap B_M^h(\bar{\pi}) = \emptyset$. Furthermore, since $\psi^0(\bar{\pi}, \bar{\eta}) = \emptyset$, for any $(p', q'_K, q'_J) \in \Xi$ we have that

$$p'_0 \left[\sum_{h \in H} \bar{x}_0^h - W_0 \right] + \sum_{\mathbb{A} \subset K} \left(\sum_{j \in J(\mathbb{A})} q'_j \sum_{h \in H} \bar{\theta}_j^h - \sum_{k \in \mathbb{A}} q'_k \sum_{h \in H} \bar{\varphi}_k^h \right) \leq 0. \tag{A.5}$$

Thus, suppose that $\sum_h \bar{x}^h_{0,l} - W_{0,l} > 0$ for some $l \in L$. Then, setting $p'_{0,l} = 1$, $p'_{0,l'} = 0$ for all $l' \neq l$, $q_J = 0$ and $q_K = 0$, we obtain a contradiction. In the other side, suppose that $\sum_h \bar{\theta}^h_j > \sum_h \bar{\varphi}^h_k$ for some pair $(k, j) \in \mathbb{A} \times J(\mathbb{A})$. Thus, letting $p_0 = 0$, $q_j = 1$, and $q_{j'} = 0$ for all $j' \neq j$, $q_k = 1$, and $q_{k'} = 0$ for all $k' \neq k$, we obtain a contradiction with Eq. (A.5). It follows that $\sum_h \bar{x}^h_{0,l} - W_{0,l} \leq 0$ for $l \in L$, and $\sum_h \bar{\theta}^h_j \leq \sum_h \bar{\varphi}^h_k$ for each pair $(k, j) \in \mathbb{A} \times J(\mathbb{A})$, and for all $\mathbb{A} \in K$.

Lemma 3. *There exists $M^*_1 > 0$ such that, for each $M_1 > M^*_1$, if Assumptions 1–3 hold, each equilibrium allocations $(\bar{\pi}, \bar{\eta})$ for $\mathcal{E}_M$, with $M = (M_1, M_2) \in \mathcal{M}$, satisfies*

(3.1) *For each agent $h \in H$, $\bar{\eta}^h \in B^h_M(\bar{\pi})$;*
(3.2) $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \cap B^h_M(\bar{\pi}) = \emptyset \ \forall \ h \in H$;
(3.3) $\sum_{h \in H} \bar{x}^h_0 = W_0$;
(3.4) $\sum_{j \in J(\mathbb{A})} \bar{q}_j \sum_{h \in H} \bar{\theta}^h_j = \sum_{k \in \mathbb{A}} \bar{q}_k \sum_{h \in H} \bar{\varphi}^h_k$, *for each class $\mathbb{A} \subset K$;*
(3.5) *For each $s \in S$, $\mathbb{A} \subset K$, $j^m(\mathbb{A}) \in J(\mathbb{A})$, the payment rate $\bar{r}_{s, j^m(\mathbb{A})}$ minimizes the function*

$$\left(rF^{s,m}_{\mathbb{A}}(\bar{\pi}, \bar{\eta}) + \sum_{i=1}^{m-1} \bar{r}_{s, j^i(\mathbb{A})} F^{s,i}_{\mathbb{A}}(\bar{\pi}, \bar{\eta}) - \sum_{k \in \mathbb{A}} \sum_{h \in H} \bar{\delta}^h_{s,k} \right)^2,$$

subject to $r \in [\beta^{s,m}_M(\mathbb{A}), 1]$, where

$$F^{s,i}_{\mathbb{A}}(\bar{\pi}, \bar{\eta}) := \bar{p}_s A_{s, j^i(\mathbb{A})} \sum_{h \in H} \bar{\theta}^h_{j^i(\mathbb{A})};$$

(3.6) *For each $(s, l) \in S^* \times L$, the consumption allocations $(\bar{x}^h_{s,l})_{h \in H}$ satisfy, $\bar{x}^h_{s,l} < M^*_1$;*
(3.7) *For each $s \in S$ and $l \in L$*

$$\sum_{h \in H} \bar{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq \sum_{j \in J} \bar{r}_{s,j} \bar{p}_s A_{s,j} \sum_{h \in H} \bar{\theta}^h_j - \sum_{k \in K} \sum_{h \in H} \bar{\delta}^h_{s,k};$$

(3.8) $\sum_h \bar{\theta}^h_j \leq \sum_h \bar{\varphi}^h_k$ *for each pair $(k, j) \in \mathbb{A} \times J(\mathbb{A})$, for all $\mathbb{A} \subset K$.*

Proof. As discussed above, items (3.1), (3.2) and (3.8) hold for each $M = (M_1, M_2) \in \mathcal{M}$. Now, as $\sum_h \bar{x}^h_{0,l} - W_{0,l} \leq 0$ for $l \in L$, there exists M'_1 such that, for each $M_1 > M'_1$, an equilibrium consumption allocation of the economy $\mathcal{E}_M$ satisfies $\bar{x}^h_{0,l} < M_1$. Thus, given $M_1 > M'_1$, suppose that agent h equilibrium allocation satisfies

$$\bar{p}_0 \bar{x}^h_0 + \sum_{j \in J} \bar{q}_j \bar{\theta}^h_j - \sum_{k \in K} \bar{q}_k \bar{\varphi}^h_k < \bar{p}_0 w^h_0.$$

As $\bar{x}^h_0$ is interior, there exists $\hat{x}^h_0 \gg \bar{x}^h_0$ such that $\hat{\eta}^h = (\hat{x}^h_0, \bar{\varphi}^h, \bar{\delta}^h, \bar{\theta}^h) \in B^h_M(\bar{\pi})$. From the strict monotonicity of $\Psi^{h,M}$ on x_0, we have that $\Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \cap B^h_M(\bar{\pi}) \neq \emptyset$, which contradicts item (3.2). Thus, for each agent h, first period budget constraint must hold with equality. Summing

over the agents, it follows that

$$\bar{p}_0 \left[\sum_{h \in H} \bar{x}_0^h - W_0 \right] + \sum_{j \in J} \bar{q}_j \sum_{h \in H} \bar{\theta}_j^h - \sum_{k \in K} \bar{q}_k \sum_{h \in H} \bar{\varphi}_k^h = 0. \tag{A.6}$$

Now, given $\mathbb{A}$, defining k' as

$$\sum_{h \in H} \varphi_{k'}^h = \min_{k \in \mathbb{A}} \sum_{h \in H} \varphi_k^h,$$

it follows from $\sum_{h \in H} \bar{\theta}_j^h \leq \sum_{h \in H} \bar{\varphi}_k^h$, for all $(j,k) \in J(\mathbb{A}) \times \mathbb{A}$, that

$$\begin{aligned} \sum_{j \in J(\mathbb{A})} \bar{q}_j \sum_{h \in H} \bar{\theta}_j^h - \sum_{k \in \mathbb{A}} \bar{q}_k \sum_{h \in H} \bar{\varphi}_k^h &\leq \sum_{j \in J(\mathbb{A})} \bar{q}_j \sum_{h \in H} \bar{\varphi}_{k'}^h - \sum_{k \in \mathbb{A}} \bar{q}_k \sum_{h \in H} \bar{\varphi}_{k'}^h \\ &= \sum_{h \in H} \bar{\varphi}_{k'}^h \left(\sum_{j \in J(\mathbb{A})} \bar{q}_j - \sum_{k \in \mathbb{A}} \bar{q}_k \right) \leq 0, \end{aligned} \tag{A.7}$$

where the last inequality is a consequence of $\sum_{j \in J(\mathbb{A})} \bar{q}_j \leq \sum_{k \in \mathbb{A}} \bar{q}_k$.

It follows from (A.7) and from the inequality $\sum_h \bar{x}_0^h \leq W_0$ that the left hand side of Eq. (A.6) is a sum of non-positive terms. Thus, each term must be zero, and condition (3.4) hold, i.e.

$$\sum_{j \in J(\mathbb{A})} \bar{q}_j \sum_{h \in H} \bar{\theta}_j^h - \sum_{k \in \mathbb{A}} \bar{q}_k \sum_{h \in H} \bar{\varphi}_k^h = 0,$$

for each class $\mathbb{A} \subset K$.

Furthermore, suppose that there exists a commodity $l \in L$ such that, $\sum_{h \in H} \bar{x}_{0,l}^h < W_{0,l}$. From Eq. (A.6), we must have $\bar{p}_{0,l} = 0$. But it follows from the strict monotonicity of $\Psi^{h,M}$ on $x_{0,l}$ that $B_M^h(\bar{\pi}) \cap \Psi^{h,M}(\bar{p}_{-0}, \bar{\eta}^h) \neq \emptyset$, which is a contradiction. Therefore, item (3.3) holds.

Now, for a given CLO $j^m(\mathbb{A})$, state $s \in S$, and $M \gg 0$, it follows from Lemma 2 that for all $r \in [\beta_M^{s,m}(\mathbb{A}), 1]$:

$$\begin{aligned} &\left(r F_{\mathbb{A}}^{s,m}(\bar{\pi}, \bar{\eta}) + \sum_{i=1}^{m-1} \bar{r}_{s,j^i(\mathbb{A})} F_{\mathbb{A}}^{s,i}(\bar{\pi}, \bar{\eta}) - \sum_{k \in \mathbb{A}} \sum_{h \in H} \bar{\delta}_{s,k}^h \right)^2 \\ &\quad \geq \left(\sum_{i=1}^{m} \bar{r}_{s,j^i(\mathbb{A})} F_{\mathbb{A}}^{s,i}(\bar{\pi}, \bar{\eta}) - \sum_{k \in \mathbb{A}} \sum_{h \in H} \bar{\delta}_{s,k}^h \right)^2. \end{aligned}$$

Therefore,

$$\bar{r}_{s,j^m(\mathbb{A})} \in \operatorname{argmax}_{r \in [\beta_M^{s,m}(\mathbb{A}),1]} - \left(r F^{s,m}(\bar{\pi}, \bar{\eta}) + \sum_{i=1}^{m-1} \bar{r}_{s,j^i(\mathbb{A})} F^{s,i}(\bar{\pi}, \bar{\eta}) - \sum_{k \in \mathbb{A}} \sum_{h \in H} \bar{\delta}_{s,k}^h \right)^2,$$

and item (3.5) holds.

Now, given an equilibrium $(\bar{\pi}, \bar{\eta})$ for the abstract economy $\mathcal{E}_M$, with $M = (M_1, M_2)$ and $M_1 > M_1'$, we know that $\psi_M^s(\bar{\pi}, \bar{\eta}) = \emptyset$ for each state of nature $s \in S$. Then, for all prices $p_s' \in \Delta_+^L$

we have

$$p'_s\left(\sum_{h\in H}[\bar{x}^h_s - Y_s\bar{x}^h_0] - W_s\right) \leq \bar{p}_s\left(\sum_{h\in H}[\bar{x}^h_s - Y_s\bar{x}^h_0] - W_s\right). \tag{A.8}$$

Moreover, it follows from item (3.1) that $\bar{\eta}^h \in B^h_M(\bar{\pi})$ for each agent $h \in H$. Thus, given an state of nature $s \in S$:

$$\bar{p}_s\left(\sum_{h\in H}[\bar{x}^h_s - Y_s\bar{x}^h_0] - W_s\right) \leq \sum_{j\in J}\bar{r}_{s,j}\bar{p}_s A_{s,j}\sum_{h\in H}\bar{\theta}^h_j - \sum_{k\in K}\sum_{h\in H}\bar{\delta}^h_{s,k}. \tag{A.9}$$

Letting, at Eq. (A.8), $p'_{s,l} = 1$ and $p'_{s,l'} = 0$ for each $l' \neq l$, we have from (A.8), (A.9) and item (3.3) that

$$\sum_{h\in H}\bar{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq \sum_{j\in J}\bar{r}_{s,j}\bar{p}_s A_{s,j}\sum_{h\in H}\bar{\theta}^h_j - \sum_{k\in K}\sum_{h\in H}\bar{\delta}^h_{s,k}, \tag{A.10}$$

which proofs (3.7). As in the economy $\mathcal{E}_M$, (a) the positions on primitives, $\bar{\varphi}^h_j$, are bounded by above by 2Ω and (b) the aggregated purchase of each derivative, $\sum_{h\in H}\bar{\theta}^h_j$, is bounded by the total short position on primitives; it follows from Eq. (A.10) that

$$\sum_{h\in H}\bar{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq 2\sum_{j\in J}\|A_{s,j}\|_1(\#H)\Omega. \tag{A.11}$$

Then, for each $l \in L$:

$$\bar{x}^h_{s,l} \leq \max_{(s,l)\in S\times L}\left\{W_{s,l} + (Y_s W_0)_l + 2\sum_{j\in J}\|A_{s,j}\|_1(\#H)\Omega\right\} \quad \forall\ h \in H, \tag{A.12}$$

which guarantees that consumption allocations $\bar{x}^h_s$, $s \in S$, are uniformly bounded from above, independently of the value of $M_1 > M'_1$. Moreover, item (3.3) guarantees that first period consumption allocations, $\bar{x}^h_0$, are also uniformly bounded, independent of $M = (M_1, M_2)$. Therefore, there exists $M^*_1 > M'_1$ such that, $\bar{x}^h_{s,l} < M^*_1$, for any $(s, l) \in S^* \times L$, which proofs item (3.6). □

Definition 4. Given $M \in \mathcal{M}$, a M-semi-equilibrium is an allocation $(\tilde{\pi}_M, \tilde{\eta}_M) \in \mathbb{P}_M \times \mathbb{X}^H_M$ which satisfies items (3.1)–(3.7).

It is important to remark that item (3.8) does not enter into the definition of M-semi-equilibrium. Note that, given $M = (M_1, M_2)$, it follows from Lemma 3 that, for a given $M_1 > M^*_1$, an M-semi-equilibrium always exists. We will suppress the subscript M on M-semi-equilibrium allocations when mistakes are not possible.

Lemma 4. *Under* Assumptions 1 and 2, *there exists* $M^{**}_1 > 0$ *such that, for each* M*-semi-equilibrium* $(\tilde{\pi}, \tilde{\eta})$, *with* $M_1 > M^{**}_1$, *the commodity prices* $\tilde{p}_{s,l}$, *with* $(s, l) \in S^* \times L$, *have a uniform lower bound* $\underline{p} > 0$ *independent of* $M = (M_1, M_2)$.

Proof. It follows from Assumption 3 that, given an agent $h \in H$ and a scalar $\varepsilon > 0$, there exists, for each $x^h \in X$ and for each pair $(s, l) \in S^* \times L$, a constant $Z^h_{s,l}(x^h, \epsilon) \in \mathbb{R}_{++}$ such that the allocation y^h, with

$$y^h_{s',l'} = \begin{cases} \varepsilon & \text{if } (s', l') \neq (s, l), \\ Z^h_{s,l}(x^h, \epsilon) & \text{if } (s', l') = (s, l) \end{cases}$$

is strictly preferred to x^h by agent h. Moreover, we can always suppose that functions $Z^h_{s,l} : X \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ are non-decreasing in x.

Fix $M = (M_1, M_2)$ with $M_1 > M_1^*$. It follows from (3.6) that a M-semi-equilibrium allocation $(\tilde{\pi}, \tilde{\eta})$ satisfies $\tilde{x}^h_{s,l} < M_1^*$, for all $(s, l) \in S^* \times L$, which guarantees that $\tilde{p}_{s,l} > 0$.[4] Moreover, for each $(s, h) \in S \times H$, $\underline{m}^h_s := \min_{p_s \in \Delta^L_+} p_s w^h_s > 0$, because Δ^L_+ is compact.

Define, for each pair (l, l'), the compact set

$$G(l, l') = \left\{ p_0 \in \mathbb{R}^L_+ : \left(p_{0,l'} \geq \frac{1 - p_{0,l}}{\#L + \#J - 1} \right) \wedge (\exists (q_K, q_J), (p_0, q_K, q_J) \in \Xi) \right\}.$$

Since $0 \notin G(l, l')$, we have that

$$\underline{m}^h_0 := \min_{l \in L} \min_{l' \neq l} \min_{p_0 \in G(l,l')} p_0 w^h_0 > 0 \quad \forall\ h \in H.$$

As for any M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ the vector $(\tilde{p}_0, \tilde{q}_J) \in \Delta^{\#L+\#J}_+$, in order to guarantee that, for a given $l \in L$, $\tilde{p}_{0,l}$ is uniformly bounded (independent of M), we have to consider two possibilities.

Case I. There exists a commodity $l' \neq l$ for which $\tilde{p}_{0,l'} \geq (1 - \tilde{p}_{0,l})/(\#L + \#J - 1)$.

In this case, $\tilde{p}_0 \in G(l, l')$, which implies that $\tilde{p}_0 w^h_0 \geq \underline{m}^h_0$. Thus, any agent h can choose the allocation $(\hat{x}^h, 0, 0, 0)$, defined by

$$\hat{x}^h_{s'',l''} = \begin{cases} \varepsilon & \text{if } (s'', l'') \neq (0, l), \\ \min \left\{ \dfrac{\underline{m}^h_0}{2\tilde{p}_{0,l}}, M_1 \right\} & \text{if } (s'', l'') = (0, l), \end{cases}$$

where $\varepsilon := \min_{h \in H} \{\underline{m}^h_0/2; \min_{s \in S} \underline{m}^h_s/2\} > 0$.

On the other hand, since each allocation $(\tilde{x}^h)_{h \in H}$ is uniformly bounded

$$Z^h_{0,l}(\tilde{x}, \varepsilon) \leq \tilde{Z}_\varepsilon := \max_{h \in H} \max_{(s'',l'') \in S^* \times L} Z^h_{s'',l''}((M_1^*, \ldots, M_1^*), \varepsilon). \tag{A.13}$$

As the right hand side in the inequality above does not depends on M, there exists $(M_1^*)' \geq M_1^*$ such that, if $M_1 > (M_1^*)'$, $\tilde{Z}_\epsilon < M_1$. Thus, it follows from Assumption 3 and from the optimality condition (3.2) that, for each M-semi-equilibrium with $M_1 > (M_1^*)'$, $\tilde{Z}_\varepsilon > \underline{m}^h_0/2\tilde{p}_{0,l}$, which

[4] In other case, as preferences are strictly monotonic on consumption, each agent $h \in H$ could increase the consumption of a zero-price commodity, choosing another allocation $\hat{\eta}^h$ that improves their situation and still belongs to the budget set $B^h_M(\tilde{\pi})$, which contradicts item (3.2).

implies that

$$\tilde{p}_{0,l} \geq \underline{p}_0^I := \max_{h \in H} \frac{m_0^h}{2\tilde{Z}_\epsilon} > 0. \tag{A.14}$$

Case II. There exists an asset $j \in J$ for which $\tilde{q}_j \geq (1 - \tilde{p}_{0,l})/(\#L + \#J - 1)$.

Define $\underline{W}_0 = \min_{l \in L} W_{0,l}$. Note that there always exists an agent $h(\tilde{p}_0) \in H$ that can demand $\underline{W}_0/\#H$ units of each good at the first period, without making any financial transaction. In fact, suppose that such agent does not exist. Then, it follows from the first period budget constraint that $\tilde{p}_0 w_0^h < \|\tilde{p}_0\|_1(\underline{W}_0/\#H)$ for all $h \in H$. Assumption 1, however, implies that $\sum_{h \in H} \tilde{p}_0 w_0^h \geq \|\tilde{p}_0\|_1 \underline{W}_0$, which is a contradiction.

Moreover, since we are restricting $(p_0, q_K, q_J) \in \Xi$, it follows that there exists $k \in K$ for which $\tilde{q}_k \geq (1 - \tilde{p}_{0,l})/((\#L + \#J - 1)\#K)$.

Now, $h(\tilde{p}_0)$ can demand the bundle $\hat{x}^{h(\tilde{p}_0)}$, defined as

$$\hat{x}_{s',l'}^{h(\tilde{p}_0)} = \begin{cases} \varepsilon' & \text{if } (s', l') \neq (0, l), \\ \min\left\{\varepsilon' + \dfrac{q_k \gamma}{\tilde{p}_{0,l}}, M_1\right\} & \text{if } (s', l') = (0, l), \end{cases}$$

where $\varepsilon' := \min_{h \in H}\{\underline{W}_0/2\#H; \min_{s \in S} \underline{m}_s^h/2\} > 0$, selling γ units of the primitive k, without making any other financial transaction, and paying all his promises at the second period, where γ satisfy

$$\gamma \|C_k\|_1 \leq \frac{W_0}{2\#H}; \gamma \leq 2\Omega(\#H); \gamma \|A_{s,k}\|_1 \leq \varepsilon' \quad \forall\ s \in S.$$

Therefore, this allocation belongs to the budget set of agent $h(\tilde{p}_0)$ and γ is independent on prices. Hence, it follows from Assumption 3 and from optimality condition (3.2) that there exists $(M_1^*)_0 > (M_1^*)'$ such that, for each M-semi-equilibrium, with $M = (M_1, M_2)$, if $M_1 > (M_1^*)_0$ then $\varepsilon' + (\tilde{q}_k \gamma/\tilde{p}_{0,l}) \leq \tilde{Z}_{\varepsilon'}$, which implies that

$$\tilde{p}_{0,l} \geq \underline{p}_0^{II} := \frac{\gamma}{\gamma + (\#L + \#J - 1)\#K \tilde{Z}_{\varepsilon'}} > 0. \tag{A.15}$$

Therefore, first period M-semi-equilibrium commodity prices (where $M_1 > (M_1^*)_0$) satisfies $\tilde{p}_{0,l} \geq \underline{p}_0 := \min\{\underline{p}_0^I; \underline{p}_0^{II}\}$.

Now, since $\tilde{p}_{0,l} \geq \underline{p}_0$, define ϵ_S as

$$\varepsilon_S := \min_{h \in H}\left\{\min_{p_0 \in \Xi_1} p_0 w_0^h; \min_{s \in S} \frac{1}{2}\underline{m}_s^h\right\} > 0,$$

where Ξ_1 denotes the set of prices $p_0 \geq \underline{p}_0(1, 1, \ldots, 1)$ such that there exists prices (q_K, q_J) for which $(p_0, q_K, q_J) \in \Xi$.

Thus, for a given M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$, with $M_1 > (M_1^*)_0$, and for a fixed pair $(s, l) \in S \times L$, any agent can demand an allocation $(\hat{x}^h, 0, 0, 0)$, defined as

$$\hat{x}^h_{s',l'} = \begin{cases} \varepsilon_S & \text{if } (s', l') \neq (s, l), \\ \min\left\{ \dfrac{\underset{s}{\overset{h}{m}}}{2\tilde{p}_{s',l'}}, M_1 \right\} & \text{if } (s', l') = (s, l). \end{cases}$$

Then, there exist $M_1^{**} > \max\{\tilde{Z}_{\varepsilon_S}, (M_1^*)_0\}$ such that, if $M_1 > M_1^{**}$, then $\tilde{Z}_{\varepsilon_S} > \underline{m}^h_s/2\tilde{p}_{s',l'}$. This implies that the commodity M-semi-equilibrium prices at the second period are uniformly bounded from below by

$$\tilde{p}_{s,l} \geq \underline{p}_s := \max_{h \in H} \frac{\underline{m}^h_s}{2\tilde{Z}_{\epsilon_S}} > 0. \tag{A.16}$$

Therefore, we conclude that, for each M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ with $M_1 > M_1^{**}$, the commodity prices $(\tilde{p}_s)_{s \in S^*}$ satisfy $\tilde{p}_{s,l} \geq \underline{p} := \min_{s \in S^*} \underline{p}_s \ \forall \ (s, l) \in S^* \times L$. □

Now, take $M = (M_1, M_2) \in \mathcal{M}$ such that $M_1 > M_1^{**}$. Fix an M-semi-equilibrium allocation $(\check{\pi}, \check{\eta})$ that also satisfies item (3.8) (it is sufficient to take an equilibrium of the truncated economy $\mathcal{E}_M$).

Given a class $\mathbb{A}$, it follows from items (3.4) and (3.8) that, if there exists a primitive $k \in \mathbb{A}$ that satisfies $\sum_{h \in H} \check{\varphi}^h_k > \min_{k' \in \mathbb{A}} \sum_{h \in H} \check{\varphi}^h_{k'}$, then $\check{q}_k = 0$.

Analogously, if there exists $j' \in J(\mathbb{A})$ such that $\sum_{h \in H} \check{\theta}^h_{j'} < \max_{j \in J(\mathbb{A})} \sum_{h \in H} \check{\theta}^h_j$, then $\check{q}_{j'} = 0$. Moreover, optimality conditions on agents allocations (item (3.2)) implies that, for such j', $\check{r}_{s,j'} \check{p}_s A_{s,j'} = 0$ for all $s \in S$. However, as (i) the payment rate of j' is bounded from below by $(1/M_2) > 0$, and (ii) the commodity prices, at each state $s \in S$, are strictly positive; we must have that $\|A_{s,j'}\|_1 = 0$ for all $s \in S$, which is a contradiction with Assumption 4. Therefore, $\check{q}_J \gg 0$ and $\sum_{h \in H} \check{\theta}^h_{j'} = \sum_{h \in H} \check{\theta}^h_j$ for all $j, j' \in J(\mathbb{A})$, $\mathbb{A} \subset K$.

Thus, as it follows from item (3.4) that

$$\sum_{j \in J(\mathbb{A})} \check{q}_j \sum_{h \in H} \check{\theta}^h_j = \sum_{k \in \mathbb{A}} \check{q}_k \min_{k' \in \mathbb{A}} \sum_{h \in H} \check{\varphi}^h_{k'},$$

using (3.8), we have that, $\sum_{h \in H} \check{\theta}^h_j = \min_{k' \in \mathbb{A}} \sum_{h \in H} \check{\varphi}^h_{k'}$ for all j in $J(\mathbb{A})$.

In conclusion, $\sum_{h \in H} \check{\theta}^h_j = \sum_{h \in H} \check{\varphi}^h_k$, for all pair $(k, j) \in \mathbb{A} \times J(\mathbb{A})$ such that the M-semi-equilibrium price $\check{q}_k$ is strictly positive.

Define a new allocation $(\tilde{\pi}, \tilde{\eta}) \in \mathbb{P}_M \times \mathbb{X}^H_M$ as

$$(\tilde{\pi}; \tilde{x}^h, \tilde{\delta}^h, \tilde{\theta}^h_j) = (\check{\pi}; \check{x}^h, \check{\delta}^h, \check{\theta}^h_j) \quad \forall \ j \in J,$$

$$\tilde{\varphi}^h_k = \begin{cases} \check{\varphi}^h_k & \text{if } \check{q}_k > 0, \\ 0 & \text{if } \check{q}_k = 0 \end{cases} \quad \forall \ h \in H, \forall \ k \in K.$$

It follows that the allocation $(\tilde{\pi}, \tilde{\eta})$ is still a M-semi-equilibrium. Moreover, for a given $\mathbb{A}$, the following conditions are satisfied

$$\sum_{h\in H}\tilde{\theta}^h_j = \sum_{h\in H}\tilde{\varphi}^h_k, \forall\ (k,\ j)\in \mathbb{A}\times J(\mathbb{A}) \quad \text{for which } \tilde{q}_k > 0; \tag{A.17}$$

$$\sum_{h\in H}\tilde{\varphi}^h_k = 0 \quad \forall\ k\in K \ \text{ for which } \tilde{q}_k = 0. \tag{A.18}$$

In the other hand, given a M-semi-equilibrium $(\breve{\pi}, \breve{\eta})$, consider any allocation $(\breve{\pi}', \breve{\eta}')$ with $(\breve{p}', \breve{q}', \breve{\eta}') = (\breve{p}, \breve{q}, \breve{\eta})$ and, for each derivative $j^m(\mathbb{A})\in J(\mathbb{A})$:

$$\breve{r}'_{s,\,j^m(\mathbb{A})} = \begin{cases} 1 & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} = 0 \wedge m < m^*_s(\mathbb{A}),\\ \beta^{s,m}_M(\mathbb{A}) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} = 0 \wedge m \geq m^*_s(\mathbb{A}),\\ \alpha(j^m(\mathbb{A})) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} \neq 0 \wedge \|A_{s,\,j^m(\mathbb{A})}\|_1 = 0,\\ \breve{r}_{s,\,j^m(\mathbb{A})} & \text{otherwise,}\end{cases} \tag{A.19}$$

where $\alpha(j^m(\mathbb{A}))\in[\beta^{s,m}_M(\mathbb{A}), 1]$. It follows that $B_M(\breve{\pi}')\subset B_M(\breve{\pi})$. Thus, as $\breve{r}_{s,\,j}$ appears multiplied by $A_{s,\,j}$ and $\sum_{h\in H}\breve{\theta}_j$ at item (3.5), we conclude that $(\breve{\pi}', \breve{\eta}')$ is also a M-semi-equilibrium.

Lemma 5. *There exists $M^*_1 > 0$ such that for each $M = (M_1, M_2)\in\mathcal{M}$, with $M_1 > M^*_1$, there exists a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ in which for every class $\mathbb{A}\subset K$ we have*

(5.1) *The rates of payment $(\tilde{r}_{s,\,j})_{j\in J(\mathbb{A})}\in\mathcal{R}^s_M(\mathbb{A})$, for all $s\in S$, where* [5]

$$\mathcal{R}^s_M(\mathbb{A}) := \{r\in\Upsilon^s_M(\mathbb{A}) : \exists r'\in\mathcal{R}(\mathbb{A}),\ r_m = \max\{r'_m, \beta^{s,m}_M(\mathbb{A})\}\}.$$

(5.2) $$0 \leq \sum_{j\in J(\mathbb{A})}\tilde{r}_{s,\,j}\tilde{p}_s A_{s,\,j}\sum_{h\in H}\tilde{\theta}^h_j - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} \leq \frac{2}{M_2}\sum_{j\in J(\mathbb{A})}\|A_{s,\,j}\|_1(\#H)^2\Omega \quad \forall\ s\in S.$$

(5.3) *For any $s\in S$, if $\tilde{r}_{s,\,j^m(\mathbb{A})} = \beta^{s,m}_M(\mathbb{A})$ then either $m = m^*_s(\mathbb{A})$ or $\beta^{s,m}_M(\mathbb{A}) = 1/M_2$.*

(5.4) *Given a state of nature $s\in S$, for each $m < m^*_s(\mathbb{A})$, $\tilde{r}_{s,\,j^m(\mathbb{A})} = 1$.*

Proof. Given $M = (M_1, M_2)\in\mathcal{M}$, with $M_1 > M^{**}_1$, take a M-semi-equilibrium $(\breve{\pi}, \breve{\eta})$ that satisfies Eqs. (A.17) and (A.18). From Lemma 4 and previous comments, we know that such a M-semi-equilibrium exists.

[5] Equivalently, the set $\mathcal{R}^s_M(\mathbb{A})$ can be defined as

$$\mathcal{R}^s_M(\mathbb{A}) := \{r\in\Upsilon^s_M(\mathbb{A}) : \exists m,\ 1\leq m\leq n(\mathbb{A}),\ (r_{m'} = 1,\ \forall\ m' < m)\wedge(r_{m'} = \beta^{s,m}(\mathbb{A}),\ \forall\ m' > m)\}.$$

Thus, consider a different allocation $(\tilde{\pi}, \tilde{\eta})$ with $(\tilde{p}, \tilde{q}, \tilde{\eta}) = (\check{p}, \check{q}, \check{\eta})$ and, for each derivative $j^m(\mathbb{A})$, $\tilde{r}_{s,j^m(\mathbb{A})}$ defined by

$$\begin{cases} 1 & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} = 0 \wedge m < m^*_s(\mathbb{A}), \\ \beta^{s,m}_M(\mathbb{A}) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} = 0 \wedge m \geq m^*_s(\mathbb{A}), \\ \check{r}_{s,j^m(\mathbb{A})} & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} \neq 0 \wedge \|A_{s,j^m(\mathbb{A})}\|_1 \neq 0, \\ 1 & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} \neq 0 \wedge \|A_{s,j^m(\mathbb{A})}\|_1 = 0 \wedge m = 1, \\ \tilde{r}_{s,j^{m-1}(\mathbb{A})} & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} \neq 0 \wedge \|A_{s,j^m(\mathbb{A})}\|_1 = 0 \wedge m \neq 1 \wedge \tilde{r}_{s,j^{m-1}(\mathbb{A})} = 1, \\ \beta^{s,m}_M(\mathbb{A}) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A})} \neq 0 \wedge \|A_{s,j^m(\mathbb{A})}\|_1 = 0 \wedge m \neq 1 \wedge \tilde{r}_{s,j^{m-1}(\mathbb{A})} \neq 1. \end{cases} \tag{A.20}$$

Since $(\tilde{\pi}, \tilde{\eta})$ respects Eq. (A.19), $(\tilde{\pi}, \tilde{\eta})$ is a M-semi-equilibrium and it still satisfies Eqs. (A.17) and (A.18). We will show that $(\tilde{\pi}, \tilde{\eta})$ satisfies all conditions of this lemma. □

Fix a class $\mathbb{A} \subset K$. It follows from (24) that condition (5.4) holds. To guarantee that items (5.1)–(5.3) hold, we will analyze two cases.

Case I. Suppose that $\sum_{h\in H}\tilde{\varphi}^h_k = 0$ for all $k \in \mathbb{A}$.

It follows from item (3.4), and from the fact that $\tilde{q}_j > 0$ for $j \in J$, that $\sum_{h\in H}\tilde{\theta}_j = 0$ for all $j \in J(\mathbb{A})$ (see discussion before Lemma 5). Thus, for any $(r_{s,j})_{j\in J(\mathbb{A})} \in \Upsilon^s_M(\mathbb{A})$, we have that

$$\sum_{j\in J(\mathbb{A})} r_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} = 0 \quad \forall\ s \in S,$$

because $\tilde{\delta}^h_{s,k} = 0$, for each pair $(s,k) \in S \times K$. Consequently, whenever the primitives are not negotiated, the property (5.2) holds.

Moreover, since $\sum_{h\in H}\tilde{\theta}^h_{j^m(\mathbb{A})} = 0$ for all $m \in \{1, 2, \ldots, n(\mathbb{A})\}$, it follows from (A.20) that $(\tilde{r}_{s,j^m(\mathbb{A})})_{m=1}^{n(\mathbb{A})}$ belongs to $\mathcal{R}^s_M(\mathbb{A})$ and, therefore, item (5.1) holds. Moreover, item (5.3) also holds.

Case II. Suppose that $\sum_{h\in H}\tilde{\varphi}^h_k > 0$ for some $k \in \mathbb{A}$.

It follows from (A.17) that $\sum_{h\in H}\tilde{\theta}^h_j > 0$ for all $j \in J(\mathbb{A})$. Since $(\tilde{\pi}, \tilde{\eta})$ is a M-semi-equilibrium, we know that $\tilde{r}_{s,j^m(\mathbb{A})}$ belongs to

$$\operatorname{argmax}_{r\in[\beta^{s,m}_M(\mathbb{A}),1]} - \left(rF^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) + \sum_{i=1}^{m-1}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k}\right)^2, \tag{A.21}$$

where $F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) := \tilde{p}_s A_{s,j^i(\mathbb{A})}\sum_{h\in H}\tilde{\theta}^h_{j^i(\mathbb{A})}$.

As $\sum_{h\in H}\tilde{\theta}^h_j > 0$ for all $j \in J(\mathbb{A})$, $F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) = 0$ if and only if $\|A_{s,j^i(\mathbb{A})}\|_1 = 0$. Now, define for each state of nature $s \in S$ the set

$$I^s_{\mathbb{A}} = \{m : \|A_{s,j^m(\mathbb{A})}\|_1 \neq 0\}.$$

If $I^s_{\mathbb{A}}$ is empty, then it follows from (A.20) that $\tilde{r}_{s,j^m(\mathbb{A})} = 1$, for all $m \in \{1, 2, \ldots, n(\mathbb{A})\}$. Thus, item (5.1) holds in this case. Otherwise, suppose that $I^s_{\mathbb{A}} \neq \emptyset$ and consider the following claims.

Claim 1. *Given $m \in I^s_{\mathbb{A}}$, if*

$$F^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) \leq \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} - \sum_{i=1}^{m-1}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) \tag{A.22}$$

holds, then $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$, for each $m' \leq m$ that belongs to $I^s_{\mathbb{A}}$.

Proof. As $m \in I^s_{\mathbb{A}}$, if (A.22) holds, then $r_{s,j^m(\mathbb{A})} = 1$ is the unique maximizer of the objective function in (A.21), and consequently $\tilde{r}_{s,j^m(\mathbb{A})} = 1$. Now, suppose that there exists $m' \in I^s_{\mathbb{A}}$ such that $\tilde{r}_{s,j^{m'}(\mathbb{A})} < 1$ and $m' < m$. Since (A.22) holds for m

$$\sum_{i=1}^{m'}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) < \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k}, \tag{A.23}$$

which is a contradiction with condition (3.5). Therefore, $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$. □

Claim 2. *Given $m \in I^s_{\mathbb{A}}$, if*

$$\beta^{s,m}_M(\mathbb{A})F^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) < \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} - \sum_{i=1}^{m-1}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) < F^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) \tag{A.24}$$

holds, then $\tilde{r}_{s,j^m(\mathbb{A})} \in (\beta^{s,m}_M(\mathbb{A}), 1)$:

$$\tilde{r}_{s,j^m(\mathbb{A})}F^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) = \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} - \sum_{i=1}^{m-1}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}), \tag{A.25}$$

and $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$ for each $m' < m$ with $m' \in I^s_{\mathbb{A}}$.

Proof. If (A.24) is satisfied, the global maximum of the function in (A.21) is attainable and, therefore, (A.25) holds. Moreover, we have that $\tilde{r}_{s,j^m(\mathbb{A})} \in (\beta^{s,m}_M(\mathbb{A}), 1)$. Now, suppose that there exists $m' < m$ such that $\tilde{r}_{s,j^{m'}(\mathbb{A})} < 1$ and $m' \in I^s_{\mathbb{A}}$. Since (A.24) holds for m, we have that

$$0 < \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} - \sum_{i=1}^{m'}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}), \tag{A.26}$$

which is a contradiction with (3.5). Therefore, $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$. □

Claim 3. *Given $m \in I^s_{\mathbb{A}}$, if*

$$\sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}^h_{s,k} - \sum_{i=1}^{m-1}\tilde{r}_{s,j^i(\mathbb{A})}F^{s,i}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) \leq \beta^{s,m}_M(\mathbb{A})F^{s,m}_{\mathbb{A}}(\tilde{\pi},\tilde{\eta}) \tag{A.27}$$

holds, then $\tilde{r}_{s,j^{m'}(\mathbb{A})} = \beta^{s,m'}_M(\mathbb{A})$ for each $m' \geq m$ that belongs to $I^s_{\mathbb{A}}$.

Proof. If (A.27) holds, then $\beta_M^{s,m}(\mathbb{A})$ is the unique maximizer of function in (A.21) and, therefore, $\tilde{r}_{s,j^m(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A})$. Since (A.27) is valid for each $m' > m$, if $m' \in I_{\mathbb{A}}^s$, then $\tilde{r}_{s,j^{m'}(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A})$.

It is easy to see that each $m \in I_{\mathbb{A}}^s$ satisfies the conditions of one and only one of the claims above. Additionally, the set of $m \in I_{\mathbb{A}}^s$ that satisfies the conditions of a specific claim may be empty.

Furthermore, the following facts are valid:

- There exists *at most one* $m \in I_{\mathbb{A}}^s$ for which conditions of Claim 2 holds.
- If $m \in I_{\mathbb{A}}^s$ satisfies the condition of Claims 1 and 2, then each $m' < m$, with $m' \in I_{\mathbb{A}}^s$, satisfies the condition of Claim 1.
- If $m \in I_{\mathbb{A}}^s$ satisfies the condition of Claims 2 and 3, then each $m'' > m$, with $m'' \in I_{\mathbb{A}}^s$, satisfies the condition of Claim 3.

Suppose that there exists $m \in I_{\mathbb{A}}^s$ that satisfies the condition of Claim 2. Then, it follows from (A.20) that (i) $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$, for all $m' < m$; (ii) $\tilde{r}_{s,j^m(\mathbb{A})} \in (\beta_M^{s,m}(\mathbb{A}_C), 1)$; and (iii) $\tilde{r}_{s,j^{m''}(\mathbb{A})} = \beta_M^{s,m''}(\mathbb{A})$, for all $m'' > m$. This guarantees that condition (5.1) holds in this case.

If there exists no $m \in I_{\mathbb{A}}^s$ that satisfies condition of Claim 2, we have two possibilities:

- There exists $m \in I_{\mathbb{A}}^s$ such that $\tilde{r}_{s,j^m(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A})$. In this case, define $\tilde{m} = \min\{m' \in I_{\mathbb{A}}^s : \tilde{r}_{s,j^{m'}(\mathbb{A})} = \beta_M^{s,m'}(\mathbb{A})\}$. Items above guarantee that $\tilde{m}$ satisfies the condition of Claim 3. This implies, using (A.20), that

$$\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1 \quad \forall\ m' < \tilde{m}; \qquad \tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = \beta_M^{s,m'}(\mathbb{A}) \quad \forall\ m' > \tilde{m}.$$

- All $m \in I_{\mathbb{A}}^s$ satisfy $\tilde{r}_{s,j^m(\mathbb{A})} = 1$. Thus, $\tilde{r}_{s,j^{m'}(\mathbb{A})} = 1$, for all $m' \in \{1, 2, \ldots, n(\mathbb{A})\}$.

We conclude that (5.1) always holds.

We will now prove that (5.2) and (5.3) always hold when $\sum_{h\in H}\tilde{\theta}_j^h > 0$ for all $j \in J(\mathbb{A})$.

It follows from Eqs. (A.17) and (A.18), jointly with Assumption 4 that

$$\begin{aligned}\sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h &= \sum_{\{k\in\mathbb{A}:\tilde{q}_k\neq 0\}}\sum_{h\in H}\tilde{\delta}_{s,k}^h \le \sum_{\{k\in\mathbb{A}:\tilde{q}_k\neq 0\}}\tilde{p}_s A_{s,k}\sum_{h\in H}\tilde{\varphi}_k^h \\ &= \tilde{p}_s\sum_{\{k\in\mathbb{A}:\tilde{q}_k\neq 0\}}A_{s,k}\max_{j\in J(\mathbb{A})}\sum_{h\in H}\tilde{\theta}_j^h \le \tilde{p}_s\sum_{j\in J(\mathbb{A})}A_{s,j}\sum_{h\in H}\tilde{\theta}_j^h. \end{aligned} \tag{A.28}$$

Thus, it follows from (3.5) and (A.28) that having

$$\sum_{j\in J(\mathbb{A})}\tilde{r}_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}_j^h < \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h,$$

would lead us to a contradiction. Therefore

$$\sum_{j\in J(\mathbb{A})}\tilde{r}_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}_j^h \ge \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h.$$

Now, as we assume that $\sum_{k\in\mathbb{A}}\tilde{\varphi}_k^h > 0$, for some $k\in\mathbb{A}$, budget feasibility and Lemma 3 assure that

$$\sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h \geq (\#\mathbb{A})\underline{p}\min_{k\in\mathbb{A}}\{\|A_{s,k}\|_1, \|Y_sC_k\|_1\}\max_{j\in J(\mathbb{A})}\sum_{h\in H}\tilde{\theta}_j^h. \tag{A.29}$$

Assume that $\min_{k\in\mathbb{A}}\{\|A_{s,k}\|_1, \|Y_sC_k\|_1\} > 0$. Thus

$$\varsigma^s(\mathbb{A}) := (\#\mathbb{A})\underline{p}\min_{k\in\mathbb{A}}\{\|A_{s,k}\|_1, \|Y_sC_k\|_1\}$$

is strictly positive. Moreover, by the definition of $\Upsilon_M^s(\mathbb{A})$, we have $\beta_M^{s,m}(\mathbb{A}) = 1/M_1$, for all $m \leq m_s^*(\mathbb{A})$. Thus, there exists $M_1^*(\mathbb{A}) > M_1^{**}$ such that for each M-semi-equilibrium, with $M_1 > M_1^*(\mathbb{A})$, we can assure that $(1/M_1)\|A_{s,j^{m^*(\mathbb{A})}(\mathbb{A})}\|_1 < \varsigma^s(\mathbb{A})$. It implies

$$\frac{1}{M_1}\tilde{p}_sA_{s,j^{m_s^*(\mathbb{A})}(\mathbb{A})}\sum_{h\in H}\tilde{\theta}^h_{j^{m_s^*(\mathbb{A})}(\mathbb{A})} < \varsigma^s(\mathbb{A})\sum_{h\in H}\tilde{\theta}^h_{j^{m_s^*(\mathbb{A})}(\mathbb{A})} \leq \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h, \tag{A.30}$$

and it follows from the fact that, for any $m < m_s^*(\mathbb{A})$, $\|A_{s,j^m(\mathbb{A})}\|_1 = 0$ and from (3.5) that both

$$\sum_{m=1}^{m_s^*(\mathbb{A})}\tilde{r}_{s,j^m(\mathbb{A})}\tilde{p}_sA_{s,j^m(\mathbb{A})}\sum_{h\in H}\tilde{\theta}^h_{j^m(\mathbb{A})} - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h \leq 0, \tag{A.31}$$

and $\tilde{r}_{s,j^{m_s^*(\mathbb{A})}(\mathbb{A})} > \beta_M^{s,m_s^*(\mathbb{A})}(\mathbb{A}) = 1/M_1$.

Therefore, when $\min_{k\in\mathbb{A}}\{\|A_{s,k}\|_1, \|Y_sC_k\|_1\} > 0$, if $\tilde{r}_{s,j^m(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A})$ then $\tilde{r}_{s,j^m(\mathbb{A})} = 1/M_2$. In fact, otherwise $\tilde{r}_{s,j^m(\mathbb{A})} = 1/M_1$, which implies that $m \leq m_s^*(\mathbb{A})$, contradicting (A.30) and the fact that $\tilde{r}_{s,j^{m_s^*(\mathbb{A})}(\mathbb{A})} > 1/M_1$.

In the other side, if $\min_{k\in\mathbb{A}}\{\|A_{s,k}\|_1, \|Y_sC_k\|_1\} = 0$, from definition we have that $\beta_M^{s,m}(\mathbb{A}) = 1/M_2$ for each $m\in\{1,2,\ldots,n(\mathbb{A})\}$ and, consequently, if $\tilde{r}_{s,j^m(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A})$, then $\tilde{r}_{s,j^m(\mathbb{A})} = 1/M_2$.

Thus, item (5.3) holds.

Now, arguments above imply that $\sum_{i=1}^m\tilde{r}_{s,j^i(\mathbb{A})}\tilde{p}_sA_{s,j^i(\mathbb{A})}\sum_{h\in H}\tilde{\theta}^h_{j^i(\mathbb{A})} - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h$ is greater than zero if and only if $\tilde{r}_{s,j^m(\mathbb{A})} = \beta_M^{s,m}(\mathbb{A}) = 1/M_2$. Thus, define $m^{**} = \min\{m : \tilde{r}_{s,j^m(\mathbb{A})} = 1/M_2\}$. Note that $m^* < m^{**}$.

Finally,

$$\begin{aligned}
0 &\leq \sum_{j\in J(\mathbb{A})}\tilde{r}_{s,j}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}_j^h - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h \\
&= \sum_{m=1}^{m^{**}-1}\tilde{r}_{s,j^m(\mathbb{A})}F_{\mathbb{A}}^{s,m}(\tilde{\pi},\tilde{\eta}) + \sum_{m=m^{**}}^{n(\mathbb{A})}\tilde{r}_{s,j^m(\mathbb{A})}F_{\mathbb{A}}^{s,m}(\tilde{\pi},\tilde{\eta}) - \sum_{k\in\mathbb{A}}\sum_{h\in H}\tilde{\delta}_{s,k}^h \\
&\leq \sum_{m=m^{**}}^{n(\mathbb{A})}\frac{1}{M_2}\tilde{p}_sA_{s,j^m(\mathbb{A})}\sum_{h\in H}\tilde{\theta}^h_{j^m(\mathbb{A})} \leq \frac{2}{M_2}\sum_{j\in J(\mathbb{A})}\|A_{s,j}\|_1(\#H)^2\Omega,
\end{aligned}$$

which guarantees that item (5.2) always holds. Therefore, lemma holds taking $M_1^* = \max_{\mathbb{A}\subset K}M_1^*(\mathbb{A})$. □

Lemma 6. *For each $M = (M_1, M_2) \in \mathcal{M}$ with $M_1 > M_1^*$, there exists a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ in which conditions* (5.1), (5.2) *hold and the following properties are satisfied*:

(6.1) *For each $(s, l) \in S \times L$*

$$\sum_{h \in H} \tilde{x}_{s,l}^h - (Y_s W_0)_l - W_{s,l} \leq \frac{2}{M_2} \sum_{\mathbb{A} \subset K} \sum_{j \in J(\mathbb{A})} \|A_{s,j}\|_1 (\#H)^2 \Omega;$$

(6.2) *For each $h \in H$, $\Psi^h(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h(\tilde{\pi}) = \emptyset$.*

Proof. We know from Lemma 5 that there exists, for each $M \in \mathcal{M}$ with $M_1 > M_1^*$, a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ that satisfies conditions (5.1), (5.2). Therefore, fix $(\tilde{\pi}, \tilde{\eta})$ in which all the above properties hold. Item (6.1) follows directly from items (3.7) and (5.2).

Suppose that it exists $y = (x, \varphi, \delta, \theta) \in \Psi^h(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h(\tilde{\pi})$. It follows from Assumption 2 that there exists $\lambda \in (0, 1]$ (sufficiently small) such that $z := \lambda y + (1 - \lambda)\tilde{\eta}^h \in \Psi^h(\tilde{p}_{-0}, \tilde{\eta}^h)$ and is an interior point of $\mathbb{X}_M$ relative to $\mathbb{X}$ (this is a consequence of the fact that $\tilde{\eta}^h$ has this property). Therefore, as $z \in B_M^h(\tilde{\pi})$ we have a contradiction with item (3.2), which assure that $\Psi^{h,M}(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B_M^h(\hat{\pi}) = \emptyset$. □

Finally, the proof of Theorem 1 is a direct consequence of the lemma below.

Lemma 7. *There exists a non-trivial equilibrium for the economy $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$, which can be obtained as the limit of a sequence of M-semi-equilibriums when M_2 goes to infinity and $M_1 > M_1^*$.*

Proof. We know from Lemma 6 that there exists, for each $M \in \mathcal{M}$ with $M_1 > M_1^*$, a M-semi-equilibrium $(\tilde{\pi}_M, \tilde{\eta}_M)$ that satisfies conditions (5.1), (5.2), (6.1), (6.2).

Fix a $M_1 > M_1^*$ and construct a sequence of M-semi-equilibriums $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2})$, indexed only by M_2, which satisfy the above conditions for all M_2. It follows from the fact that $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2})$ belongs to a compact set, independent of M_2, that there exists a convergent subsequence. We will denote the limit of this subsequence as $(\hat{\pi}, \hat{\eta})$.

It is straightforward that items (3.3) and (3.4) still hold for the limit allocation $(\hat{\pi}, \hat{\eta})$. Moreover, one can easily see that at the limit items (3.1), (5.2), (6.1) become, respectively,

(3.1*) For each $h \in H$, $\hat{\eta} \in B^h(\hat{\pi})$;
(5.2*) For each class $\mathbb{A} \subset K$ and each $s \in S$

$$\sum_{j \in J(\mathbb{A})} \hat{r}_{s,j} \hat{p}_s A_{s,j} \sum_{h \in H} \hat{\theta}_j^h - \sum_{k \in \mathbb{A}} \sum_{h \in H} \hat{\delta}_{s,k}^h = 0;$$

(6.1*) For each $(s, l) \in S \times L$

$$\sum_{h \in H} \hat{x}_{s,l}^h - (Y_s W_0)_l - W_{s,l} \leq 0;$$

where item (3.1*) follows from the closed graph of the budget set correspondence B^h.

We know that, for each M_2, the second-periods budget constraints are satisfied with equality. Then, the limit second period budget constraints still hold with equality. This fact, jointly with

item (6.1*) above, imply that, for each $(s, l) \in S \times L$

$$\sum_{h \in H} \hat{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} = 0. \tag{A.32}$$

Therefore, in order to assure that $(\hat{\pi}, \hat{\eta})$ is a non-trivial equilibrium we need to show that the properties of Definition 2 hold, jointly with items (C) and (F) of Definition 1.

First, we will assure, for each $h \in H$, the individual optimality of the allocation $\hat{\eta}^h$ at prices $\hat{\pi}$. We affirm that, for every $h \in H$, there is nothing in the interior of the budget set that is strictly preferred that $\hat{\eta}^h$. Suppose, by contradiction, that there is an allocation y such that $y \in \Psi^h(\hat{p}_{-0}, \hat{\eta}^h) \cap \dot{B}^h(\hat{\pi})$. Since Ψ^h is lower hemicontinuous correspondence and $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2}) \to (\hat{\pi}, \hat{\eta})$, there exists $y_{M_2} \in \Psi^h((\tilde{p}_{-0})_{M_2}, \tilde{\eta}_{M_2})$ such that $y_{M_2} \to y$. Since $\dot{B}_M$ has open values, for M_2 sufficiently large, $y_{M_2} \in \Psi^h((\tilde{p}_{-0})_{M_2}, \tilde{\eta}^h_{M_2}) \cap \dot{B}^h(\tilde{\pi}_{M_2})$, which is a contradiction with (6.2). Thus, $\Psi^h(\hat{p}_{-0}, \hat{\eta}^h) \cap \dot{B}^h(\hat{\pi}) = \emptyset$.

It follows from Lemma 4 that the M-semi-equilibrium commodity prices are uniformly bounded away from zero. Therefore, the limit allocation has prices strictly greater than zero. This implies that the interior of the limit budget set has non-empty values, $\dot{B}^h(\hat{\pi}) \neq \emptyset$. Now, as B^h also has convex values, we have that the closure of $\dot{B}^h(\hat{\pi})$ is equal to the original budget set, $B^h(\hat{\pi})$. Then, it follows that $\Psi^h(\hat{p}_{-0}, \hat{\eta}^h) \cap B^h(\hat{\pi}) = \emptyset$. That completes the proof of optimality.

Second, take as given $\mathbb{A} \subset K$ and $s \in S$. Note that, every convergent sequence of elements belonging to $\mathcal{R}^s_M(\mathbb{A})$ for each M_2 has a limit at $\mathcal{R}(\mathbb{A})$. Indeed, as $\mathcal{R}(\mathbb{A})$ is a compact set we can assume, without loss of generality, that there exists a subsequence $((r_s)_{M_2})_{M_2 > M_1} \subset \mathcal{R}(\mathbb{A})$ such that

(a) $(r_s)_{M_2} = ((r_{s,m})_{M_2}; m \in \{1, \ldots, n(\mathbb{A})\})$.
(b) There exists $\hat{r} = (\hat{r}_{s,m}; m \in \{1, \ldots, n(\mathbb{A})\}) \in \mathcal{R}(\mathbb{A})$ such that

$$\lim_{M_2 \to +\infty} (r_s)_{M_2} = \hat{r}.$$

(c) For each M_2 (in the subsequence)

$$(\tilde{r}_{s,j^m(\mathbb{A})})_{M_2} = \max\{(r_{s,m})_{M_2}, \beta^{s,m}_M(\mathbb{A})\} \quad \forall\ m \in \{1, \ldots, n(\mathbb{A})\},$$

where $M = (M_1, M_2)$.

Note that, taking the limit as M_2 goes to infinity, item (c) above implies that, for each $m \in \{1, \ldots, n(\mathbb{A}\}$, $\hat{r}_{s,j^m(\mathbb{A})} \geq \hat{r}_{s,m}$. As our objective is to prove that $(\hat{r}_{s,j})_{j \in J(\mathbb{A})} \in \mathcal{R}(\mathbb{A})$, assume that there exists $m \in \{1, \ldots, n(\mathbb{A})\}$ such that $\hat{r}_{s,j^m(\mathbb{A})} > \hat{r}_{s,m}$.

It follows that, for M_2 sufficiently high, $(\tilde{r}_{s,j^m(\mathbb{A})})_{M_2} = \beta^{s,m}_M(\mathbb{A})$, with $M = (M_1, M_2)$.

If $\min_{k \in \mathbb{A}}\{\|A_{s,k}\|_1, \|Y_s C_k\|_1\} = 0$ then condition (5.3) implies that $\hat{r}_{s,j^m(\mathbb{A})} = 0$, a contradiction. In other case, if $\min_{k \in \mathbb{A}}\{\|A_{s,k}\|_1, \|Y_s C_k\|_1\} > 0$, then item (5.3) guarantee that $m = m^*_s(\mathbb{A})$ and $\hat{r}_{s,m^*_s(\mathbb{A})} < 1/M_1$. Moreover, $\hat{r}_{s,j^{m'}(\mathbb{A})} = \hat{r}_{s,m'}$ for any $m' \neq m^*_s(\mathbb{A})$. Thus, $(\hat{r}_{s,j})_{j \in J(\mathbb{A})} \in \mathcal{R}(\mathbb{A})$.

Finally, fix a class $\mathbb{A} \subset K$ and state $s \in S$. As the parameter M_1 is fixed, the definition of $\mathcal{R}^s_M(\mathbb{A})$ assure that, when $\min_{k \in \mathbb{A}}\{\|A_{s,k}\|_1, \|Y_s C_k\|_1\} > 0$, then for any $M_2 > M_1$ we have that $(\tilde{r}_{s,j^m(\mathbb{A})})_{M_2} \geq 1/M_1$ for each $m \leq m^*_s(\mathbb{A})$. Thus, limit rates of payment satisfy

$$\left[\min_{k \in \mathbb{A}}\{\|A_{s,k}\|_1, \|Y_s C_k\|_1\} > 0\right] \Rightarrow \hat{r}_{s,j^m(\mathbb{A})} \geq \frac{1}{M_1} \quad \forall\ m \leq m^*_s(\mathbb{A}). \tag{A.33}$$

Using the fact that $\hat{p}_{s,l} \geq \underline{p} > 0$, for all $l \in L$, we conclude that

$$\left[\min_{k \in \mathbb{A}} \{\hat{p}_s Y_s C_k; \hat{p}_s A_{s,k}\} > 0\right] \Rightarrow \hat{r}_{s,j^m(\mathbb{A})} > 0 \quad \forall\ m \leq m_s^*(\mathbb{A}). \tag{A.34}$$

Also, as $\hat{r}_{s,j_{\mathbb{A}}^{m_s^*(\mathbb{A})}} > 0$, optimality conditions on agents allocations and Assumption 4 guarantee that $(\hat{q}_j; j \in J(\mathbb{A})) \neq 0$. Thus, as $(\hat{p}, \hat{q}_K, \hat{q}_J) \in \Xi$, there exists at least one primitive $k \in \mathbb{A}$ for which $\hat{q}_k > 0$. That conclude the proof that $(\hat{\pi}, \hat{\eta})$ is a non-trivial equilibrium. □

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