Math. Program., Ser. B (2018) 168:631–643
https://doi.org/10.1007/s10107-017-1110-2

FULL LENGTH PAPER

On Brøndsted–Rockafellar's Theorem for convex lower semicontinuous epi-pointed functions in locally convex spaces

Rafael Correa[1,2,3] · **Abderrahim Hantoute**[4] · **Pedro Pérez-Aros**[1,2]

Received: 25 February 2016 / Accepted: 6 January 2017 / Published online: 2 February 2017

Abstract In this work we give an extension of the Brøndsted–Rockafellar Theorem, and some of its important consequences, to proper convex lower-semicontinuous epi-pointed functions defined in locally convex spaces. We use a new approach based on a simple variational principle, which also allows recovering the classical results in a natural way.

Keywords Convex and epi-pointed functions · Locally convex spaces · Fenchel subdifferential · Brøndsted–Rockafellar's Theorem

Mathematics Subject Classification Primary 49J53 · 52A41 · 46N10

Dedicated to Professor R. Tyrrell Rockafellar on the occasion of his 80th birthday.

This work is partially supported by CONICYT Grants: Fondecyt 1151003, Fondecyt 1150909, Basal PFB -03 and Basal FB0003, CONICYT-PCHA/doctorado Nacional/2014-21140621.

✉ Rafael Correa
rcorrea@dim.uchile.cl

Abderrahim Hantoute
ahantoute@dim.uchile.cl

Pedro Pérez-Aros
pperez@dim.uchile.cl

1 Center for Mathematical Modeling and Department of Mathematical Engineering, Universidad de Chile, Santiago, Chile

2 Universidad de Chile, Santiago, Chile

3 Universidad de O'Higgins, Rancagua, Chile

4 Center for Mathematical Modeling, Universidad de Chile, Av. Beauchef 851, Edificio norte, Piso 7, Santiago, Chile

1 Introduction

It is known that the Brøndsted–Rockafellar Theorem is not valid outside Banach spaces (see [4]) for all lower semi-continuous (lsc) proper convex functions. This observation motivates the work to provide a suitable family of lsc proper convex functions defined on locally convex spaces, which satisfies the Brøonsted–Rockafellar Theorem.

The main features of this work are (1) to show that epi-pointed lsc convex functions, defined on any locally convex space, satisfy the Brøndsted–Rockafellar theorem (2) to provide a different proof of Brøndsted–Rockafellar's theorem for this class of epi-pointed functions, in the sense that it is based on a very simple variational principle, which is valid in locally convex spaces, without requiring such tools as Ekeland's or Bishop-Phelps' variational principles (3) Since every convex function in Banach spaces can be adequately perturbed to obtain an epi-pointed function, we recover in the Banach setting the usual Brøndsted–Rockafellar theorem (4) we also obtain other important results in the same spirit, as the maximal monotonicity of the Fenchel subdifferential operator of proper lsc convex functions, and the subdifferential limiting calculus rules for convex functions.

The class of epi-pointed functions has been successfully utilized recently with the purpose of extending results, which were known exclusively for Banach spaces or convex functions, to locally convex spaces and nonconvex functions. For some of these results we refer to [1,5–10,14] among others.

2 Notation and preliminary results

In the following, X and X^* will be two (separated) locally convex spaces (lcs) in duality by the bilinear form $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$. In X and X^* the weak topology is denoted by $w(X, X^*)$ and $w(X^*, X)$ respectively (w and w^* for short) and the Mackey topology is denoted by $\tau(X, X^*)$ and $\tau(X^*, X)$ respectively, the space X will be endowed with a compatible initial topology τ (i.e $w(X, X^*) \subset \tau \subset \tau(X, X^*)$). Only in Sect. 5, when X is a Banach space, will X^{**} refer to the bidual of X, that is, $X^{**} := (X^*, \|\cdot\|_*)^*$. We will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.

For a given function $f : X \to \overline{\mathbb{R}}$, the (effective) *domain* and the *epigraph* of f are $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \leq \lambda\}$, respectively. We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f > -\infty$, and *inf-compact* if for every $\lambda \in \mathbb{R}$ the set $[f \leq \lambda] := \{x \in X \mid f(x) \leq \lambda\}$ is compact. We denote $\Gamma_0(X)$ the class of proper lower semicontinuous (lsc) convex functions on X. The *conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\},$$

and the *biconjugate* of f is $f^{**} := (f^*)^* : X^{**} \to \overline{\mathbb{R}}$. For $\varepsilon \geq 0$, the ε-*subdifferential* of f at a point $x \in X$, where it is finite, is the set

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon,\ \forall y \in X\};$$

if $f(x)$ is not finite, we set $\partial f(x) := \emptyset$. The special case $\varepsilon = 0$ is the *Fenchel Subdifferential* and it is denoted by $\partial f(x)$. The domain and the image of the subdifferential mapping are defined by $\operatorname{dom} \partial f := \{x \in X \mid \partial f(x) \neq \emptyset\}$ and $\operatorname{Im} \partial f := \bigcup_{x \in X} \partial f(x)$ respectively.

The *indicator* and the *support* functions of a set A ($\subset X, X^*$) are, respectively,

$$I_A(x) := \begin{cases} 0 & x \in A \\ +\infty & x \notin A, \end{cases} \qquad \sigma_A := I_A^*.$$

The *inf-convolution* of $f, g : X \to \overline{\mathbb{R}}$ is the function $f \Box g := \inf_{z \in X}\{f(z) + g(\cdot - z)\}$.

We denote by $\operatorname{Int}(A)$, $\overline{A}$, $\operatorname{conv}(A)$ and $\overline{\operatorname{conv}}(A)$, the interior, the closure, the *convex hull* and the *convex closed hull* of A, respectively. The *polar* of A is the set $A^o := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 1, \forall x \in A\}$. Given a seminorm $\rho : X \to \mathbb{R}$, $x \in X$ and $r \geq 0$ we denote $B_\rho(x, r) := \{y \in X : \rho(x - z) \leq r\}$.

In the following propositions we recall well-known results in convex analysis, which are used in the proof of our main results (Theorems 4.2, 4.4, 4.6 and 4.7).

Proposition 2.1 *(a) ([13], Theorem 6.6.7) Given two proper convex lsc functions $g, h : X \to \overline{\mathbb{R}}$ such that g is continuous at some point of* dom *h, for all $x \in X$*

$$\partial(g + h)(x) = \partial g(x) + \partial h(x)$$

(b) ([13], Theorem 6.3.9) A proper lsc convex function $g : X^* \to \overline{\mathbb{R}}$ is $\tau(X^*, X)$-continuous at $x^* \in$ dom *g if and only if $g^* - x^*$ is $w(X, X^*)$-inf-compact.*

(c) ([13], Theorem 6.5.8) Given two proper convex lsc functions $g, h : X \to \overline{\mathbb{R}}$ such that g is continuous at some point of dom *h we have $(g + h)^* = g^* \Box h^*$.*

(d) ([13], Theorem 6.5.4) Given two proper functions $g, h : X \to \overline{\mathbb{R}}$ we have $(g \Box h)^* = g^* + h^*$.

Proposition 2.2 [12, Theorem 3.1] *Let X, Z be two lcs, $f \in \Gamma_0(Z)$ and $A \in \mathcal{L}(X, Z)$. Then*

$$\partial (f \circ A)(x) = \bigcap_{\eta > 0} \overline{A^*\left(\partial_\eta f(Ax)\right)}^{w^*}, \quad \text{for all } x \in X.$$

3 The class of epi-pointed functions

The class of epi-pointed functions was introduced in the finite dimension by Hiriart-Urruty and Benoist [2] but the original definition goes back to the Nobel laureate mathematical economist Gérard Debreu in the fifties [11].

To introduce this class of functions, the authors used the notion of asymptotic function f^∞, for a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, defined by

$$f^\infty(x) := \liminf_{\substack{t \to 0^+ \\ y \to x}} t f(t^{-1} y).$$

So, f is said to be epi-pointed if the epigraph of its asymptotic function (which is clearly a cone) is pointed (that is, $\xi_1, \ldots, \xi_p \in \text{epi } f^\infty$ and $\xi_1 + \ldots + \xi_p = 0 \Rightarrow \xi_i = 0\ i = 1, \ldots, p$).

The following proposition allows us to better appreciate the definition of the epi-pointed function.

Proposition 3.1 *Consider $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ lsc and proper. Then the following statements are equivalent:*

(a) f is epi-pointed.
(b) There exist $\bar{u} \in \mathbb{R}^n$, $\alpha > 0$ and $r \in \mathbb{R}$ such that:

$$f(x) \geq \langle \bar{u}, x\rangle + \alpha\|x\| + r \quad \forall x \in \mathbb{R}^n$$

(c) There exists $\bar{u} \in \mathbb{R}^n$ such that f^* is bounded from above on a neighbourhood of $\bar{u}$.

Proof First we will prove that

$$\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \inf_{\|x\|=1} f^\infty(x). \tag{1}$$

Indeed, consider sequences (x_k), (t_k), (y_k) and points w_0, w_1 of norm one such that:

- $\|x_k\| \to +\infty$, $\liminf\limits_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \lim \frac{f(x_k)}{\|x_k\|}$ and $\frac{x_k}{\|x_k\|} \to w_0$.
- $t_k \to 0^+$, $y_k \to w_1$ and $\inf\limits_{\|x\|=1} f^\infty(x) = f^\infty(w_1) = \lim t_k f(\frac{y_k}{t_k})$ (f^∞ is lsc).

Then

$$\begin{aligned}\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|} &= \lim \frac{f(x_k)}{\|x_k\|} = \lim \frac{f(\frac{x_k}{\|x_k\|}\|x_k\|)}{\|x_k\|} \geq f^\infty(w_0) \geq f^\infty(w_1)\\ &= \lim t_k f(\frac{y_k}{t_k}) = \lim \|y_k\| \frac{t_k}{\|y_k\|} f(\frac{y_k}{t_k}) \geq \liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|}.\end{aligned}$$

Now we will prove the proposition:

- $(a) \Rightarrow (b)$ Suppose that f is epi-pointed. Then $\inf\limits_{\|x\|=1} f^\infty(x) =: \gamma > -\infty$ (if this does not happen, $\{0\} \times \mathbb{R} \subset \text{epi } f^\infty$). Now we will show that there exists $\bar{u} \in \mathbb{R}^n$ such that for every $\|x\| = 1$, $f^\infty(x) > \langle \bar{u}, x\rangle$. In fact, consider the set $K := \overline{\text{conv}}\{(x, \alpha) \in \text{epi } f^\infty : \|x\| = 1\}$. We have $(0, 0) \notin K$, because if this is not true, then there exists $(x_k, \alpha_k) = \sum_{i=0}^n \lambda_i^k (x_i^k, \beta_i^k)$ (n is the dimension of the underlying space, by Carathéodory's Theorem) with $(x_i^k, \beta_i^k) \in K$, $\lambda_i^k \geq 0$ and $\sum_{i=0}^n \lambda_i^k = 1$ with $(x_k, \alpha_k) \to (0, 0)$. By taking a subsequence we can suppose that for every i, $x_i^k \to x_i$ and $\lambda_i^k \to \lambda_i$; moreover, because $\beta_i^k \geq \gamma$, we conclude that $\lambda_i^k \beta_i^k \to \beta_i$ (taking another subsequence). From the fact that f^∞ is lsc and positively homogeneous we get $(\lambda_i x_i, \beta_i) \in \operatorname{epi} f^\infty$, that is to say, there are $(w_i, \alpha_i) \in \operatorname{epi} f^\infty$ (not all identically zero) such that $(w_0, \alpha_0)+\ldots+(w_n, \alpha_n) = 0$, which contradicts the epi-pointed assumption. Now we apply the Hahn-Banach Theorem (if $K = \emptyset$ the result is trivial) to conclude the existence of $w \in \mathbb{R}^n$, $d \in \mathbb{R}$ and η such that $d\alpha + \langle w, x\rangle > \eta > 0$ for every $(x, \alpha) \in K$, then necessarily $d \geq 0$. So, taking $\bar{u} = -d^{-1}w$ if $d > 0$ or $\bar{u} = -2\frac{|\gamma|}{\eta}w$ if $d = 0$, we get $\inf_{\|x\|=1}(f - \bar{u})^\infty(x) = \inf_{\|x\|=1} f^\infty(x) - \langle \bar{u}, x\rangle > 0$. Now using Eq. 1 we conclude

$$\liminf_{\|x\|\to\infty} \frac{f(x) - \langle \bar{u}, x\rangle}{\|x\|} > 0.$$

Then there are $\alpha > 0$ and $M > 0$ such that $f(x) \geq \langle \bar{u}, x\rangle + \alpha\|x\|$ for every $\|x\| \geq M$. So, taking $r := \min_{w\in B(0,M)}\{f(w) - \langle \bar{u}, w\rangle - \alpha\|w\|, 0\}$ (because f is lsc and proper) we conclude (b).

- $(b) \Rightarrow (a)$ Suppose there exists $\{(x_i, \alpha_i)\}_{i=0}^p \subset \operatorname{epi} f^\infty$ (not all identically zero) such that $(x_0, \alpha_0) + \cdots + (x_p, \alpha_p) = 0$. Then, taking $h(x) = f(x) - \langle \bar{u}, x\rangle$ we have

$$\liminf_{\|x\|\to\infty} \frac{h(x)}{\|x\|} > 0,$$

and so we get

$$\inf_{\|x\|=1} h^\infty(x) = \inf_{\|x\|=1} f^\infty(x) - \langle \bar{u}, x\rangle > 0.$$

Let us compute the sum $\sum_{i=0}^p\{f^\infty(x_i) - \langle \bar{u}, x_i\rangle\}$; as $\sum_{i=0}^p\{f^\infty(x_i) - \langle \bar{u}, x_i\rangle\} \leq \sum_{i=0}^p \alpha_i - \sum_{i=0}^p\langle \bar{u}, x_i\rangle$, and $\sum_{i=0}^p \alpha_i - \sum_{i=0}^p\langle \bar{u}, x_i\rangle = 0$, we deduce that there exists some $x_i \neq 0$ such that $f^\infty(x_i) - \langle \bar{u}, x_i\rangle \leq 0$, and then $h^\infty(\frac{x_i}{\|x_i\|}) \leq 0$, which is a contradiction.
- $(b) \Rightarrow (c)$ Take an arbitrary $w \in B(0, \alpha)$. Then for every $x \in \mathbb{R}^n$, $\langle w, x\rangle - \alpha\|x\| - r \geq \langle \bar{u} + w, x\rangle - f(x)$. Therefore, $f^*(\bar{u} + w) \leq -r$ for every $w \in B(0, \alpha)$.
- $(c) \Rightarrow (b)$ Let $M \geq 0$ and $\alpha > 0$ such that $f^*(\bar{u} + w) \leq M$ for all $w \in B(0, \alpha)$. Then $\langle \bar{u} + w, x\rangle - f(x) \leq M$ for every $w \in B(0, \alpha)$ and every $x \in \mathbb{R}^n$. Then taking $w := \alpha\frac{x}{\|x\|}$ we conclude (b).

This equivalence shows that the property of epi-pointedness is characterized by the continuity of f^* at some point $\bar{u}$ and it justifies the definition for functions defined on a locally convex space X, that we will adopt in this work.

Definition 3.2 A function $f : X \to \overline{\mathbb{R}}$ is said to be epi-pointed if f^* is proper and $\tau(X^*, X)$-continuous at some point of its domain.

Next, we give typical examples that illustrate the amplitude of this class of functions. For simplicity, we consider a reflexive Banach space $(X, \|\cdot\|)$, because in this class of spaces the Mackey topology coincide with the norm topology.

Example 3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 0$ and the closed convex envelope of f is positive; that is to say, $\overline{\text{conv}} f(x) > 0$ for all $x \in \mathbb{R}\backslash\{0\}$. Then f is an epi-pointed function.

Example 3.4 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a 1-coercive function; that is, $\|x\|^{-1} f(x) \to +\infty$ as $\|x\| \to +\infty$. Then, f is epi-pointed.

Example 3.5 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then, for every $\alpha > 0$, the function $x \to f(x) + \alpha\|x\|^2$ is epi-pointed.

Example 3.6 Let $f : X \to \mathbb{R} \cup \{+\infty\}$ proper and prox-bounded function; that is, $f \geq -\mu\|x\|^2$ for some $\mu \geq 0$. Then for every $\varepsilon > 0$ the function $f + (\mu + \varepsilon)\| \cdot \|^2$ is epi-pointed.

Example 3.7 For every function $f : X \to \mathbb{R} \cup \{+\infty\}$ and every bounded set $C \subset X$ such that $C \cap \text{dom}\, f \neq \emptyset$ and f is minorized on C by a continuous affine form, the function $f + I_C$ is epi-pointed.

Example 3.8 If X is a lcs and $f : X \to \overline{\mathbb{R}}$ is convex and continuous at some point of its domain, then $f^* : X^* \to \overline{\mathbb{R}}$ is epi-pointed.

4 Brøndsted–Rockafellar Theorem and consequences

First we give in Lemma 4.1 a simple variational principle, for convex functions defined on lcs, that is the key tool in the proof of our main results.

Lemma 4.1 *Fix $x_0 \in X$ and let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\rho : X \to [0, \infty)$ be two convex lsc functions such that $\rho(0) = 0$ and the function $f(\cdot) + \rho(\cdot - x_0)$ is epi-pointed. For any $\varepsilon \geq 0$ and $x_0^* \in \partial_\varepsilon f(x_0)$ with $x_0^* \in \text{Int}(\text{dom}(f + \rho(\cdot - x_0))^*)$, there exists $x_\varepsilon \in X$ such that:*

(a) $\rho(x_0 - x_\varepsilon) \leq \varepsilon$,
(b) $x_0^* \in \partial(f + \rho(\cdot - x_0))(x_\varepsilon)$,
(c) $|f(x_0) - f(x_\varepsilon)| \leq |\langle x_0^*, x_0 - x_\varepsilon \rangle| + \varepsilon$.

Proof Define $g := f + \rho(\cdot - x_0) - x_0^*$. By Proposition 2.1 (b) we see that the (proper lsc convex) function g is inf-compact, and there exists $x_\varepsilon \in \text{argmin}\, g$ (the minima of g), such that

$$f(x_\varepsilon) + \rho(x_\varepsilon - x_0) - x_0^*(x_\varepsilon) \leq f(x_0) - x_0^*(x_0) \leq f(x_\varepsilon) - x_0^*(x_\varepsilon) + \varepsilon.$$

Hence, $\rho(x_\varepsilon - x_0) \leq \varepsilon$. Now, since $0 \in \partial g(x_\varepsilon)$, by Proposition 2.1 (a) we have

$$x_0^* \in \partial(f + \rho(\cdot - x_0))(x_\varepsilon).$$

Finally, $\langle x_0^*, x_0 - x_\varepsilon \rangle \leq f(x_0) - (f(x_\varepsilon) + \rho(x_\varepsilon - x_0)) \leq f(x_0) - f(x_\varepsilon) \leq \langle x_0^*, x_0 - x_\varepsilon \rangle + \varepsilon$, gives us the last statement.

The following result gives the counterpart of Brøndsted–Rockafellar-Borwein Theorem (e.g., [3]) for convex lsc epi-pointed functions defined in locally convex spaces.

Theorem 4.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex, lsc and epi-pointed function. Consider $\varepsilon \geq 0$, $\beta \in [0, \infty)$, a continuous seminorm p, $\lambda > 0$ and $x_0 \in X$. If $x_0^* \in \partial_\varepsilon f(x_0) \cap \operatorname{Int}(\operatorname{dom} f^*)$, then there are $x_\varepsilon \in X$, $y_\varepsilon^* \in B_p(0,1)^\circ$ and $\lambda_\varepsilon \in [-1, 1]$ such that:*

(a) $p(x_0 - x_\varepsilon) + \beta|\langle x_0^*, x_0 - x_\varepsilon\rangle| \leq \lambda$,
(b) $x_\varepsilon^* := x_0^* + \frac{\varepsilon}{\lambda}\left(y_\varepsilon^* + \beta\lambda_\varepsilon x_0^*\right) \in \partial f(x_\varepsilon)$,
(c) $|\langle x_\varepsilon^*, x_0 - x_\varepsilon\rangle| \leq \varepsilon + \frac{\lambda}{\beta}$,
(d) $|f(x_0) - f(x_\varepsilon)| \leq \varepsilon + \frac{\lambda}{\beta}$,
(e) $x_\varepsilon^* \in \partial_{2\varepsilon} f(x_0)$,
With the convention $\frac{1}{0} = +\infty$.

Proof Define $\rho(x) = \frac{\varepsilon}{\lambda}\left(p(x) + \beta|\langle x_0^*, x\rangle|\right)$. We apply Lemma 4.1 to f and ρ to conclude the existence of $x_\varepsilon \in X$ such that $\rho(x_0 - x_\varepsilon) \leq \varepsilon$, $x_0^* \in \partial(f + \rho(\cdot - x_0))(x_\varepsilon)$ and $|f(x_0) - f(x_\varepsilon)| \leq |\langle x_0^*, x_0 - x_\varepsilon\rangle| + \varepsilon$. So, x_ε verifies (a) and (d). Now we use Proposition 2.1 (a) to obtain $0 \in \partial f(x_\varepsilon) - x_0^* + \frac{\varepsilon}{\lambda} B_p(0,1)^\circ + \frac{\varepsilon}{\lambda}\beta \cdot [-1, 1] \cdot x_0^*$, from which we find $y_\varepsilon^* \in B_p(0,1)^\circ$ and $\lambda_\varepsilon \in [-1, 1]$ such that $x_\varepsilon^* := x_0^* + \frac{\varepsilon}{\lambda}\left(y_\varepsilon^* + \beta\lambda_\varepsilon x_0^*\right) \in \partial f(x_\varepsilon)$. Then

$$\begin{aligned}|\langle x_\varepsilon^* - x_0^*, x_0 - x_\varepsilon\rangle| &\leq \tfrac{\varepsilon}{\lambda}|\langle y_\varepsilon^* + \beta\lambda_\varepsilon x_0^*, x_0 - x_\varepsilon\rangle| \\ &\leq \tfrac{\varepsilon}{\lambda}\left(|\langle y_\varepsilon^*, x_0 - x_\varepsilon\rangle| + \beta|\langle x_0^*, x_0 - x_\varepsilon\rangle|\right) \\ &\leq \tfrac{\varepsilon}{\lambda}\left(p(x_0 - x_\varepsilon) + \beta|\langle x_0^*, x_0 - x_\varepsilon\rangle|\right) \leq \varepsilon,\end{aligned}$$

and (c) follows (using (a)) $|\langle x_\varepsilon^*, x_0 - x_\varepsilon\rangle| \leq |\langle x_\varepsilon^* - x_0^*, x_0 - x_\varepsilon\rangle| + |\langle x_0^*, x_0 - x_\varepsilon\rangle| \leq \varepsilon + \frac{\lambda}{\beta}$. Finally, since $x_0^* \in \partial_\varepsilon f(x_0)$ and $x_\varepsilon^* \in \partial f(x_\varepsilon)$ we get, for every $x \in X$,

$$\begin{aligned}\langle x_\varepsilon^*, x - x_0\rangle &= \langle x_\varepsilon^*, x - x_\varepsilon\rangle + \langle x_\varepsilon^* - x_0^*, x_\varepsilon - x_0\rangle + \langle x_0^*, x_\varepsilon - x_0\rangle \\ &\leq f(x) - f(x_\varepsilon) + \varepsilon + f(x_\varepsilon) - f(x_0) + \varepsilon \\ &= f(x) - f(x_0) + 2\varepsilon;\end{aligned}$$

that is, $x_\varepsilon^* \in \partial_{2\varepsilon} f(x_0)$.

This Theorem allows us to obtain the counterpart of the classical statement of Brøndsted–Rockafellar's Theorem for lsc convex epi-pointed functions in locally convex spaces.

Corollary 4.3 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex lsc epi-pointed function. Then for every $x \in \operatorname{dom} f$ there exist nets $\{(x_\alpha), (x_\alpha^*)\}_{\alpha \in \mathbb{A}}$ such that $x_\alpha^* \in \partial f(x_\alpha)$, $x_\alpha \to x$ and $f(x_\alpha) \to f(x)$.*

Proof Consider a filtered family of seminorms $\mathcal{N}$ (ordered by $\rho_1 \leq \rho_2$ if and only if $\rho_1(x) \leq \rho_2(x)$ for all $x \in X$), which defines the topology on X. We also define the index set $\mathbb{A} := \mathcal{N} \times (0, 1)$ associated with the partial order

$$\alpha_1 = (\rho_1, \varepsilon_1) \leq \alpha_2 = (\rho_2, \varepsilon_2) \quad \text{if and only if } \rho_1 \leq \rho_2 \text{ and } \varepsilon_1 \geq \varepsilon_2.$$

It is easy to see that for every $\varepsilon \in (0,1)$, $\partial_\varepsilon f(x) \cap \operatorname{Int}(\operatorname{dom} f^*) \neq \emptyset$. Therefore, for every $\varepsilon \in (0,1)$ and for every continuous seminorm $\rho \in \mathcal{N}$ we can apply Theorem 4.2 to f with $\beta = 1$ and $\lambda = \sqrt{\varepsilon}$. We get that there exists $(x_{\varepsilon,\rho}, x^*_{\varepsilon,\rho})$ in the graph of the subdifferential of f such that $\rho(x - x_{\varepsilon,\rho}) \leq \sqrt{\varepsilon}$ and $|f(x) - f(x_{\varepsilon,p})| \leq \varepsilon + \sqrt{\varepsilon}$. To prove the convergence of this net, take a neighborhood V of zero and $\delta > 0$ and let seminorm $\rho_0 \in \mathcal{N}$ and $\varepsilon_0 \in (0,1)$ such that $B_{\rho_0}(0, \sqrt{\varepsilon_0}) \subseteq V$ and $\varepsilon_0 + \sqrt{\varepsilon_0} \leq \delta$. Therefore, for every $(\rho, \varepsilon) \geq (\rho_0, \varepsilon_0)$ we have that $\rho_0(x - x_{\varepsilon,\rho}) \leq \rho(x - x_{\varepsilon,\rho}) \leq \sqrt{\varepsilon} \leq \delta$ and $|f(x) - f(x_{\varepsilon,p})| \leq \varepsilon + \sqrt{\varepsilon} \leq \varepsilon_0 + \sqrt{\varepsilon_0} \leq \delta$, which implies that $x_{\varepsilon,\rho} \in x + V$ and $|f(x) - f(x_{\varepsilon,p})| \leq \delta$.

Now, we apply Theorem 4.2 to obtain the maximal monotonicity of the subdifferential operator of proper lsc convex epi-pointed functions in locally convex spaces.[1] For proper lsc convex functions defined in Banach spaces, this corresponds to the famous theorem by Rockafellar, [15, Theorem A].

Theorem 4.4 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex lsc function. If either f or f^* is epi-pointed, then ∂f and ∂f^* are maximal monotone operators.*

Proof Without lost of generality we consider that f is an epi-pointed function, that $f(0) = 0$ and that $0 \notin \partial f(0)$. We pick an $x \in X$ such that $f(2x) < f(x) < 0$ (such an element exists, since by supposing the contrary, one deduces that $f(x) \geq f(\frac{x}{2})$ for any x such that $f(x) < 0$, and we get $f(x) \geq f(\frac{x}{2^n})$ for any $n \in \mathbb{N}$, which leads us to the contradiction $f(x) \geq \liminf_{n \to +\infty} f(\frac{x}{2^n}) \geq f(0)$). If $a := f(x) - f(2x)$ and $\delta \in (0, \frac{a}{a+3})$, we choose an $x^* \in \partial_{\delta^2} f(x) \cap \operatorname{Int}(\operatorname{dom} f^*)$ (this choice is possible since $f(x) = f^{**}(x) = \sup\{\langle y^*, x\rangle - f^*(y^*) : y^* \in \operatorname{Int}(\operatorname{dom} f^*)\}$). Then

$$\langle x^*, x\rangle = \langle x^*, 2x - x\rangle \leq f(2x) - f(x) + \delta^2 \leq \delta^2 - a < 0.$$

Define $p(x) = |\langle x^*, x\rangle|$ a continuous seminorm so that $B_p(0,1) = \{x \in X : |\langle x^*, x\rangle| \leq 1\}$ and $(B_p(0,1))^\circ = [-1,1] \cdot x^*$. Then Theorem 4.2 (with $\varepsilon = \delta^2$, $\lambda = \delta$ and $\beta = 0$) ensures the existence of $x_0 \in X$ and $x_0^* \in \partial f(x_0)$ such that $x - x_0 \in \delta B_p(0,1)$ and $x^* - x_0^* \in \delta (B_p(0,1))^\circ$. Thus, $|\langle x^* - x_0^*, x - x_0\rangle| \leq \delta^2$, $|\langle x^* - x_0^*, x\rangle| \leq \delta |\langle x^*, x\rangle| = \delta(a - \delta^2)$ and $|\langle x^*, x - x_0\rangle| \leq \delta$. In conclusion, we get

$$\begin{aligned}\langle x_0^*, x_0\rangle &= \langle x^*, x\rangle + \langle x^* - x_0^*, x - x_0\rangle + \langle x_0^* - x^*, x\rangle + \langle x^*, x_0 - x\rangle \\ &\leq \delta^2 - a + \delta^2 + \delta(a - \delta^2) + \delta < -a + \delta(a+3) < 0.\end{aligned}$$

We finish this section by applying Theorem 4.2 to get limiting calculus rules for the subdifferential mapping of the composition with a linear mapping and the sum of convex functions. Our proof is an adaptation of [16, Theorem 3] that uses the following lemma.

[1] The arguments used in the proof of this result (Theorem 4.4) follows the suggestion made by one of the referees.

Lemma 4.5 *Let X, Z be two lcs, $f \in \Gamma_0(Z)$ be an epi-pointed function and $A \in \mathcal{L}(X, Z)$ (linear and continuous mapping). Then for every $x \in \operatorname{dom}(f \circ A)$*

$$\partial (f \circ A)(x) = \bigcap_{\eta > 0} \overline{A^* \left[\partial_\eta f(Ax) \cap \operatorname{Int}(\operatorname{dom} f^*)\right]}^{w^*}.$$

Proof By Proposition 2.2 we only need to prove that for every $\eta > 0$

$$A^* \left(\partial_\eta f(Ax)\right) \subset \overline{\left[A^* \left(\partial_\eta f(Ax) \cap \operatorname{Int}(\operatorname{dom} f^*)\right)\right]}^{w^*}.$$

Indeed, if $z \in \operatorname{dom} f$, then

$$f(z) = \sup\{\langle x^*, z\rangle - f(x^*) : x^* \in \operatorname{Int}(\operatorname{dom} f^*)\}$$

and $\partial_\eta f(z) \cap \operatorname{Int}(\operatorname{dom} f^*) \neq \emptyset$ for every $\eta > 0$. Because $\operatorname{Int}(\operatorname{dom} f^*)$ is open and dense in $\operatorname{dom} f^*$ we have that $\overline{\partial_\eta f(z) \cap \operatorname{Int}(\operatorname{dom} f^*)} = \overline{\partial_\eta f(z)} = \partial_\eta f(z)$. Hence, since A^* is w^* to w^* continuous we conclude the lemma.

Theorem 4.6 *Let X, Z be two lcs, $A \in \mathcal{L}(X, Z)$, $g \in \Gamma_0(Z)$ be an epi-pointed function, $f := g \circ A$ and $x \in \operatorname{dom} f$. Then $x^* \in \partial f(x)$ if and only if there exists a net $(z_i, z_i^*)_{i \in I} \in Z \times Z^*$ such that $z_i^* \in \partial g(z_i)$, $z_i \to y = Ax$, $g(z_i) \to g(y)$, $\langle z_i - z, z_i^*\rangle \to 0$ and $A^*(z_i^*) \overset{w^*}{\to} x^*$.*

Proof Take $x_0^* \in \partial f(x)$. Consider a filtered family of seminorms $\mathcal{N}_1$ (ordered by $\rho_Z^1 \le \rho_Z^2$ if and only if $\rho_Z^1(z) \le \rho_Z^2(z)$ for all $z \in Z$), which defines the topology on Z and a filtered family of seminorms $\mathcal{N}_2$ (ordered in a similar way as $\mathcal{N}_1$: $\rho_{X^*}^1 \le \rho_{X^*}^2$ if and only if $\rho_{X^*}^1(x^*) \le \rho_{X^*}^2(x^*)$ for all $x^* \in X^*$), which defines the weak* topology on X^*. We also define the index set $I := \mathcal{N}_1 \times \mathcal{N}_2 \times (0, 1)$ ordered by $i_1 = (\rho_Z^1, \rho_{X^*}^1, \varepsilon_1) \le i_2 = (\rho_Z^2, \rho_{X^*}^2, \varepsilon_2)$ if and only if $\rho_Z^1 \le \rho_Z^2$, $\rho_{X^*}^1 \le \rho_{X^*}^2$ and $\varepsilon_1 \ge \varepsilon_2$. Now, we take $i = (\rho_Z, \rho_{X^*}, \varepsilon) \in I$ and set $U := \{z \in Z : p_Z(z) \le 1\}$. Choose an $\eta > 0$ such that $\sqrt{\eta} + \eta \le \frac{\varepsilon}{2}$ and $2\sqrt{\eta} \le \varepsilon \left(\max_{U^\circ} p_{X^*}(A^*(y^*)) + p_{X^*}(x_0^*) + 1\right)^{-1}$. Then by Lemma 4.5 we take $z_0^* \in \partial_\eta g(z) \cap \operatorname{Int}(\operatorname{dom} g^*)$ such that $p_{X^*}(A^* z_0^* - x_0^*) \le \eta$. By Theorem 4.2 (with $\beta = 1$, $\lambda = \sqrt{\eta}$) there exists $z_i^* \in \partial g(z_i)$ such that $p_Z(z_i - z) \le \sqrt{\eta}$, $z_i^* = z_0^* + \sqrt{\eta}\left(u_i^* + \lambda_\eta z_0^*\right) \in \partial g(z_i)$, $u_\eta^* \in U^\circ$, $|\langle z_i^*, z - z_i\rangle| \le \eta + \sqrt{\eta}$ and $|g(z) - g(z_i)| \le \eta + \sqrt{\eta}$. Therefore, $p_Z(z_i - z) \le \varepsilon$, $|\langle z_i^*, z - z_i\rangle| \le \varepsilon$ and $|g(z) - g(z_i)| \le \varepsilon$. Finally,

$$\begin{aligned} p_{X^*}(A^* z_i^* - x_0^*) &\le p_{X^*}(A^* z_0^* - x_0^*) + p_{X^*}(A^* z_i^* - A^* z_0^*) \\ &\le \frac{\varepsilon}{2} + \sqrt{\eta}\left(p_{X^*}(A^* u_i^*) + p_{X^*}(A^* z_0^* - x_0^*) + p_{X^*}(x_0^*)\right) \\ &\le \frac{\varepsilon}{2} + \sqrt{\eta}\left(p_{X^*}(A^* u_i^*) + \eta + p_{X^*}(x_0^*)\right) \\ &\le \frac{\varepsilon}{2} + \sqrt{\eta}\left(\max_{U^\circ} p_{X^*}(A^*(y^*)) + p_{X^*}(x_0^*) + 1\right) \le \varepsilon. \end{aligned}$$

To prove the necessity part, let $(y_i, y_i^*)_{i\in I} \subset \text{gph}\,\partial g$ a net such that $(y_i) \to y = Ax$, $g(y_i) \to g(y)$, $\langle y_i - y, y_i^*\rangle \to 0$ and $A^*(y_i^*) \overset{w^*}{\to} x^*$. Then $\langle y - y_i, y_i^*\rangle \leq g(y) - g(y_i)$ for every $i \in I$ and $y \in X$. It follows that

$$\langle z - x, A^* y_i^*\rangle + \langle y - y_i, y_i^*\rangle = \langle Az - y_i, y_i^*\rangle \leq f(z) - g(y_i) \quad \forall i \in I,\ \forall z \in X.$$

Taking the limits gives $\langle z - x, x^*\rangle \leq f(z) - f(x)$, for all $z \in X$, and so $x^* \in \partial f(x)$.

Remark 1 We give a simple example which shows that the above subdifferential calculus rule is not valid without the epi-pointedness assumption. Let $g : X \to \overline{\mathbb{R}}$ be a convex lsc and proper function with empty subdifferential everywhere (see [4]). Then take a continuous linear function $A : \mathbb{R} \to X$ such that $\text{dom}\, g \cap A(\mathbb{R}) \neq \emptyset$. We easily check that the function $f = g \circ A$ is proper, convex, and lsc in $\mathbb{R}$. This implies that there exists a point $x_0 \in \mathbb{R}$ such that $\partial f(x_0) \neq \emptyset$ and Theorem 4.6 does not hold.

From the last theorem we deduce the subdifferential limiting calculus rule for the sum of convex epi-pointed functions.

Theorem 4.7 *Let $f_1, f_2 \in \Gamma_0(X)$ be epi-ponted functions and $x \in \text{dom}(f_1 + f_2)$. Then $x^* \in \partial(f_1 + f_2)(x)$ if and only if there exist two nets $(x_{k,i}, x_{k,i}^*)_{i\in I} \subset X \times X^*$ such that $x_{k,i}^* \in \partial f_k(x_{k,i})$ $k = 1, 2$, $x_{k,i} \to x$, $f_k(x_{k,i}) \to f_k(x)$, $\langle x_{k,i} - x, x_{k,i}^*\rangle \to 0$, for $k = 1, 2$, and $(x_{1,i}^* + x_{2,i}^*) \overset{w^*}{\to} x^*$.*

Proof We apply Theorem 4.6 with $Z = X \times X$, $A : X \to Z$ defined by $Ax = (x, x)$ and $g(x_1, x_2) = f_1(x_1) + f_2(x_2)$. We have $f := f_1 + f_2 = g \circ A$. The sufficiency part is immediate, taking $y = (x, x) = Ax$, $y_i = (x_{1,i}, x_{2,i})$ and $y_i^* = (x_{1,i}^*, x_{2,i}^*)$, for $i \in I$. For the necessity part, we take $x^* \in \partial f(x)$. By Theorem 4.6 there exists $(y_i, y_i^*)_{i\in I} \subset Z \times Z^*$ such that $y_i^* \in \partial g(y_i)$, $y_i \to y = Ax$, $g(y_i) \to g(y)$, $\langle y_i - y, y_i^*\rangle \to 0$ and $A^*(y_i^*) \overset{w^*}{\to} x^*$. Taking $y_i = (x_{1,i}, x_{2,k})$ and $y_i^* = (x_{1,i}^*, x_{2,i}^*)$, by the formula $\partial g(x_1, x_2) = \partial f_1(x_1) \times \partial f_2(x_2)$, we get $x_{k,i}^* \in \partial f_k(x_{k,i})$ and $x_{k,i} \to x$ for $k = 1, 2$. Suppose that $\limsup f_1(x_{1,i}) - f_1(x) > \delta > 0$. Then $J := \{i \in I : f_1(x_{1,i}) - f_1(x) > \delta\}$ is a co-final set in I. It follows that $g(y_i) - g(y) \geq \delta + f_2(x_{2,i}) - f_2(x)$ for every $i \in J$. Then taking the lower limits we get $0 \geq \delta$ and, hence, $f_k(x_{k,i}) \to f_k(x)$, for $k = 1, 2$. Finally, using the following inequalities

$$\begin{aligned} f_1(x_{1,i}) - f_1(x) &\leq \langle x_{1,i} - x, x_{1,i}^*\rangle \leq \langle x_{1,i} - x, x_{1,i}^*\rangle \\ &\quad + \langle x_{2,i} - x, x_{2,i}^*\rangle + f_2(x) - f_2(x_{2,i})\ \forall i \in I, \end{aligned}$$

and taking the limits, we infer that $\langle x_{1,i} - x, x_{1,i}^*\rangle \to 0$.

5 Banach spaces

In this last section, we show how to recover from our previous results the classical Brøndsted–Rockafellar Theorem in the context of Banach spaces, for any proper lsc convex functions which are not necessarily epi-pointed. In the case of reflexive spaces, it is an easy exercise, using adequate perturbations of the convex function in order to obtain an epi-pointed function, in the line of Examples 3.5 and 3.7.

Proposition 5.1 *Let X be a reflexive Banach space and $f \in \Gamma_0(X)$. Consider $x_0 \in$* dom f*, $\varepsilon \geq 0$ and $x_0^* \in \partial_\varepsilon f(x_0)$. Then there exist $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that $x_\varepsilon^* \in \partial f(x_\varepsilon)$, $\|x_\varepsilon - x_0\| \leq \sqrt{\varepsilon}$ and $\|x_\varepsilon^* - x_0^*\| \leq \sqrt{\varepsilon}$. In particular,* $\operatorname{dom} f \subset \overline{\operatorname{dom} \partial f}$ *and* $\operatorname{dom} f^* \subset \overline{\operatorname{Im} \partial f}$.

Proof Consider $x_0^* \in \partial_\varepsilon f(x_0)$. We define the function $g(w) := f(w) + I_{B(0,M)}$, where $M \geq \|x_0\| + \varepsilon + \sqrt{\varepsilon}$. It is easy to see that g is epi-pointed (recall Example 3.7), $\operatorname{dom}(g^*) = X^*$ and $x_0^* \in \partial_\varepsilon g(x_0)$, hence we apply Theorem 4.2 with $\lambda = \sqrt{\varepsilon}$, and $\beta = 0$, to get the existence of $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that $x_\varepsilon^* \in \partial g(x_\varepsilon)$), $\|x_0 - x_\varepsilon\| \leq \sqrt{\varepsilon}$ and $\|x_\varepsilon^* - x_0^*\| \leq \sqrt{\varepsilon}$. From the fact that $x_\varepsilon \in \operatorname{Int} B(0, M)$, we conclude that $x_\varepsilon^* \in \partial f(x_\varepsilon)$.

Proposition 5.2 *Let X be a reflexive Banach space and $f \in \Gamma_0(X)$. Then ∂f is a maximal monotone operator.*

Proof Let $f \in \Gamma_0(X)$ such that for every $(x, x^*) \in \operatorname{gph} \partial f$ we have $\langle x^*, x\rangle \geq 0$. We take the function $g(w) := f(w) + \frac{1}{2}\|x\|^2$. Clearly, g is epi-pointed (recall Example 3.5), and so by Theorem 4.4 ∂g is a maximal monotone operator. Moreover, for every $x \in X$, $\partial g(x) = \partial f(x) + \partial \frac{1}{2}\|\cdot\|^2(x)$, where $\partial \frac{1}{2}\|\cdot\|^2(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$. Then, for every $w^* \in \partial g(x)$, $\langle w^*, x\rangle \geq 0$ and by the maximality we get $0 \in \partial g(0)$. Finally, because $\partial f(0) = \partial g(0)$ we obtain $0 \in \partial f(0)$.

Proposition 5.3 *Let X be an lcs, Z be a reflexive Banach space, $A \in \mathcal{L}(X, Z)$, $g \in \Gamma_0(Z)$, $f := g \circ A$ and $x \in$* dom f*. Then $x^* \in \partial f(x)$ if and only if there exists a net $(y_i, y_i^*)_{i \in I} \subset Z \times Z^*$ such that $y_i^* \in \partial g(y_i)$, $y_i \to y = Ax$, $g(y_i) \to g(y)$, $\langle y_i - y, y_i^*\rangle \to 0$ and $A^*(y_i^*) \overset{w^*}{\to} x^*$.*

Proof Consider the functions $\tilde{g} := g + I_{B(A(x),1)}$ and $\tilde{f} := \tilde{g} \circ A$. We apply Theorem 4.6 to $\tilde{f}$ and obtain the existence of a net $(y_i, y_i^*)_{i \in I} \in Z \times Z^*$ such that $y_i^* \in \partial \tilde{g}(y_i)$, $(y_i) \to y = Ax$, $(g(y_i)) \to g(y)$, $\langle y_i - y, y_i^*\rangle \to 0$ and $A^*(y_i^*) \overset{w^*}{\to} x^*$. Because $(y_i) \to y = Ax$ we can suppose that $(y_i)? \subset \operatorname{Int} B(A(x), 1)$, so we have $\tilde{g}(y_i) = g(y_i)$ and by Proposition 2.1 (a) we have $y_i^* \in \partial g(y_i)$.

Finally we show that Brøndsted–Rockafellar's Theorem can also be obtained from Lemma 4.1.

Theorem 5.4 *Let X be a Banach space and $h : X \to \overline{\mathbb{R}}$ a convex lsc function. If $x_0^* \in \partial_{\varepsilon^2} h(x_0)$, then for every $\delta > 0$ there exist $x_\varepsilon \in X$ and $x_\varepsilon^* \in X^*$ such that $\|x_0 - x_\varepsilon\| < \varepsilon + \delta$, $\|x_0^* - x_\varepsilon^*\| < \varepsilon + \delta$ and $x_\varepsilon^* \in \partial h(x_\varepsilon)$.*

Proof Take a sequence of positive numbers $\{\delta_n\}_{n \geq 1}$, such that $\sum_{n=1}^\infty \delta_n < \delta$ and $\delta_0 := \varepsilon$. We claim that if $x_n^* \in \partial_{\delta_n^2} h(x_n)$, then there exists $(x_{n+1}, x_{n+1}^*) \in X \times X^*$ such that $\|x_n - x_{n+1}\| \leq \delta_n$, $\|x_n^* - x_{n+1}^*\| \leq \delta_n$ and $x_{n+1}^* \in \partial_{\delta_{n+1}^2} h(x_{n+1})$. Take $f := h^*$ and $\rho := \delta_n \|\cdot\|$ and consider the duality pair $(X^*, w^*, X, \|\cdot\|)$. Since $x_n \in$

$\partial_{\delta^2_n} f(x_n^*)\cap \mathrm{Int}(\mathrm{dom}(f+\rho(\cdot - x_n^*))^*)$ $(\mathrm{dom}(f+\rho(\cdot - x_n^*))^* \supseteq \mathrm{dom}\, h+\delta_n B_X(0,1))$, we apply Lemma 4.1 and we conclude that there exists $x_{n+1}^* \in X^*$ such that $\|x_{n+1}^* - x_n^*\| \leq \delta_n$ and $x_n \in \partial(f+\rho(\cdot - x_n^*))(x_{n+1}^*)$. By applying Proposition 2.1 (a) in X^{**}, $x_n \in \partial f^*(x_{n+1}^*) + \delta_n B_{X^{**}}$, and so there exists $x^{**} \in X^{**}$ such that $x_{n+1}^* \in \partial f^*(x^{**})$ and $\|x_n - x^{**}\| \leq \delta_n$. Finally, we apply Proposition 2.1 (c) to h and $I_{B_X(x_n,\delta_n)}$ with the duality pair $(X, \|\cdot\|, X^*, w^*)$ to get $h^* \square \sigma_{B_X(x_n,\delta_n)} = (h + I_{B_X(x_n,\delta_n)})^*$. Next, we apply Proposition 2.1 (d) to h^* and $\sigma_{B_X(x_n,\delta_n)} = \sigma_{B_{X^{**}}(x_n,\delta_n)}$ with the duality pair $(X^*, \|\cdot\|_*, X^{**}, w^*)$ to get

$$\mathrm{epi}(h^{**} + I_{B_{X^{**}}(x_n,\delta_n)}) = \mathrm{epi}(h + I_{B_X(x_n,\delta_n)})^{**} = \overline{\mathrm{epi}}^{w^*}(h + I_{B(x_n,\delta_n)}).$$

Therefore there exist a net $(x_i, \alpha_i)_{i\in I} \in \mathrm{epi}(h+I_{B(x_n,\delta_n)}) \subseteq X\times\mathbb{R}$ such that $x_i \overset{w^*}{\to} x^{**}$ and $\alpha_i \to h^*(x^{**})$, which implies the existence of an element $i_0 \in I$ such that $x_{i_0} \in B(x_n,\delta_n)$, $\frac{\delta_{n+1}^2}{2} + h^{**}(x^{**}) > \alpha_{i_0} \geq h(x_{i_0})$ and $\frac{\delta_{n+1}^2}{2} + \langle x_{n+1}^*, x_{i_0}\rangle > \langle x_{n+1}^*, x^{**}\rangle$. Set $x_{n+1} := x_{i_0}$. Then we get

$$h(x_{n+1}) + h^*(x_{n+1}^*) \leq h^{**}(x^{**}) + h^*(x_{n+1}^*) + \frac{\delta_{n+1}^2}{2} \leq \langle x^{**}, x_{n+1}^*\rangle + \frac{\delta_{n+1}^2}{2} \leq \langle x_{n+1}^*, x_{n+1}^*\rangle + \delta_{n+1}^2,$$

and the construction of the sequences x_n and x_n^* is done. From the facts that $\|x_n - x_{n+1}\| \leq \delta_n$, $\|x_n^* - x_{n+1}^*\| \leq \delta_n$ and $\sum_{i=1}^{\infty} \delta_i < +\infty$, it follows that (x_n, x_n^*) is a Cauchy sequence. By the completeness of X and X^*, we conclude that $(x_n, x_n^*) \overset{\|\cdot\|}{\to} (x_\varepsilon, x_\varepsilon^*)$, and so $\|x_0 - x_\varepsilon\| < \varepsilon + \delta$, $\|x_0^* - x_\varepsilon^*\| < \varepsilon + \delta$. Hence, $x_\varepsilon^* \in \partial h(x_\varepsilon)$.

Acknowledgements We are very grateful to both referees for their suggestions and comments. We also thank one of them for drawing our attention to the new proof of Theorem 4.4.

References

1. Benoist, J., Daniilidis, A.: Integration of Fenchel subdifferentials of epi-pointed functions. SIAM J. Optim. **12**(3), 575–582 (2002)
2. Benoist, J., Hiriart-Urruty, J.-B.: What is the subdifferential of the closed convex hull of a function? SIAM J. Math. Anal. **27**(6), 1661–1679 (1996)
3. Borwein, J.M.: A note on ε-subgradients and maximal monotonicity. Pacif. J. Math. **103**(2), 307–314 (1982)
4. Brøndsted, A., Rockafellar, R.T.: On the subdifferentiability of convex functions. Proc. Am. Math. Soc. **16**, 605–611 (1965)
5. Correa, R., García, Y., Hantoute, A.: Integration formulas via the Fenchel subdifferential of nonconvex functions. Nonlinear Anal. **75**(3), 1188–1201 (2012)
6. Correa, R., Hantoute, A.: New formulas for the Fenchel subdifferential of the conjugate function. Set-Valued Var. Anal. **18**(3–4), 405–422 (2010)
7. Correa, R., Hantoute, A.: Subdifferential of the conjugate function in general Banach spaces. Top **20**(2), 328–346 (2012)
8. Correa, R., Hantoute, A.: Lower semicontinuous convex relaxation in optimization. SIAM J. Optim. **23**(1), 54–73 (2013)

9. Correa, R., Hantoute, A., Pérez-Aros, P.: On the Klee–Saint Raymond's characterization of convexity. SIAM J. Optim. **26**(2), 1312–1321 (2016)
10. Correa, R., Hantoute, A., Salas, D.: Integration of nonconvex epi-pointed functions in locally convex spaces. J. Convex Anal. **23**(2), 511–530 (2016)
11. Debreu, Gerard: Theory of value: an axiomatic analysis of economic equilibrium. Cowles Foundation for Research in Economics at Yale University, Monograph 17. John Wiley, Inc., New York (1959)
12. Hiriart-Urruty, J.-B., Phelps, R.R.: Subdifferential calculus using ε-subdifferentials. J. Funct. Anal. **118**(1), 154–166 (1993)
13. Laurent, Pierre-Jean: Approximation et optimisation. Hermann, Paris (1972)
14. Pérez-Aros, P.: Propiedades variacionales de funciones convexas desde el análisis epi-puntado. Engineering thesis, Universidad de Chile (2014)
15. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
16. Thibault, Lionel: A generalized sequential formula for subdifferentials of sums of convex functions defined on Banach spaces. In: Recent developments in optimization (Dijon, 1994), vol. 429 of Lecture Notes in Econom. and Math. Systems, pp. 340–345. Springer, Berlin, (1995)