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FULL LENGTH PAPER

Sublevel representations of epi-Lipschitz sets and other properties

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Abstract Epi-Lipschitz sets in normed spaces are represented as sublevel sets of Lipschitz functions satisfying a so-called qualification condition. Canonical representations through the signed distance functions associated with the sets are also obtained. New optimality conditions are provided, for optimization problems with epi-Lipschitz set constraints, in terms of the signed distance function.

Keywords Epi-Lipschitz set · Subdifferential · Interior tangent cone · Sublevel set · Signed distance function · Optimality condition

Mathematics Subject Classification 49J27 · 47H10 · 26B25

1 Introduction

Given a subset S of a normed space X, Rockafellar [15,16] showed that the Clarke tangent cone $T(S;\overline{x})$ of the set S at $\overline{x} \in S$ can be described as follows: a vector $v \in T(S;\overline{x})$ provided that, for any neighbourhood V of v in X, there are a real $\varepsilon > 0$ and a neighbourhood U of $\overline{x}$ such that

$$(x + tV) \cap S \neq \emptyset \quad \text{for all } x \in U \cap S, t \in]0, \varepsilon[. \tag{1.1}$$

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In order to establish, for two subsets S_1, S_2 of X with $\overline{x} \in S_1 \cap S_2$, the fundamental inclusion $T(S_1; \overline{x}) \cap T(S_2; \overline{x}) \subset T(S_1 \cap S_2; \overline{x})$, Rockafellar [15,16] introduced in Variational Analysis the concept of epi-Lipschitz sets. It is then proved in [17] that the latter inclusion with the Clarke tangent cones holds true whenever the set S_2 is epi-Lipschitz at $\overline{x}$ in a direction belonging to $T(S_1; \overline{x})$. According to [16] the set S is epi-Lipschitz at $\overline{x}$ in a direction $\overline{v} \in X$ (see the beginning of Sect. 3 for the terminology) provided there exist a real $\varepsilon > 0$ and neigbbourhoods U and V in X of $\overline{x}$ and $\overline{v}$ respectively such that

$$U \cap S +]0, \varepsilon[\, V \subset S. \tag{1.2}$$

The set of such vectors $\overline{v}$, which is obviously an open convex cone, is called the *interior tangent cone* of S at $\overline{x}$ that we will denote by $I(S; \overline{x})$. So, one says that S is epi-Lipschitz at $\overline{x}$ when $I(S; \overline{x}) \neq \emptyset$. If S is epi-Lipschitz at any of its points, or equivalently at any of its boundary points, it is called epi-Lipschitz. If S is closed near $\overline{x}$, [15] shows that it is epi-Lipschitz at $\overline{x}$ if and only if it can be written -near $\overline{x}$- as the epigraph of a Lipschitz function. In fact, this is the justification of the term "epi-Lipschitz set" (see Sect. 3 for details).

Cornet and Czarnecki [5] showed the following representation of closed epi-Lipschitz set.

Theorem 1.1 [5] *A closed set S of $\mathbb{R}^p$ is epi-Lipschitz if and only if there is a locally Lipschitz function $g : \mathbb{R}^p \to \mathbb{R}$ such that*

$$S = \{x \in \mathbb{R}^p : g(x) \leq 0\}$$

and

$$0 \notin \partial g(x) \quad \textit{for all } x \in \operatorname{bdry} S,$$

where bdry S *denotes the boundary of the set S and $\partial g(x)$ is the Clarke generalized gradient of the function g at x.*

This equivalence theorem has been extended to Banach spaces in [9] with the use of a result in [8]. The proofs in [5,8,9] rely on the characterization of epi-Lipschitz sets as epigraphs of Lipschitz functions. The present paper only uses the original definition of Rockafellar with the interior tangent cone, which does not restrict to locally closed sets and thus allows for far more generality. For example, convex sets with nonempty interiors are epi-Lipschitz, see [16, Proposition 3]. This allows to consider epigraphs of convex functions without lower semicontinuity property. Also, open epi-Lipschitz sets commonly occur in PDEs as sets with interior cone property, also called sets with Lipschitz boundary.

On the other hand our approach shows how some arguments in [5] can be modified and adapted to deal in general normed spaces with the above sublevel representation as well as with canonical sublevel representation in terms of the signed distance function. In doing so, some new properties of epi-Lipschitz sets, related to (possibly fixed) interiorly tangent directions, will be proved in Sect. 2. The sublevel representations are established in Sect. 3. Optimality conditions for optimization problems with epi-Lipschitz set constraints are provided in the last section in terms of the signed distance function.

2 Tangential and topological properties of epi-Lipschitz sets

Throughout the paper, unless otherwise stated, $(X, \|\cdot\|)$ will be a (real) normed space and X^* its topological dual. We will denote by $B(x, r)$ (resp. $B[x, r]$) the open (resp. closed) ball centered at $x \in X$ with radius $r > 0$. The topological interior (resp. closure) of the subset S of X will be written as $\mathrm{int}_X S$ (resp. $\mathrm{cl}_X S$). When there is no risk of confusion the subscript X will be omitted.

Let U be an open set in X and $f : X \to \mathbb{R}$ be a locally Lipschitz function. Its *Clarke directional derivative* at $\overline{x}$ is defined by

$$f^o(\overline{x}; v) := \limsup_{t \downarrow 0, x \to \overline{x}} t^{-1}\big(f(x + tv) - f(x)\big), \tag{2.1}$$

and its *Clarke generalized gradient or subdifferential* at $\overline{x}$ is given by

$$\partial f(\overline{x}) := \{x^* \in X^* : \forall v \in X,\ \langle x^*, v\rangle \le f^o(\overline{x}; v)\}. \tag{2.2}$$

The continuous function $f^o(\overline{x}; \cdot)$ being sublinear (that is, convex and positively homogeneous), it is the support function of the weak-$*$ compact convex set $\partial f(\overline{x})$. In the paper, (unless otherwise specified) we will consider only the Clarke generalized gradient *as concept of subdifferential* and only the Clarke tangent cone and the interior tangent cone *as concepts of tangent cones*; so, no confusion will arise with notations $\partial f(\overline{x})$ and $T(S; \overline{x})$.

Given the set S of X containing $\overline{x}$, the Clarke tangent cone $T(S; \overline{x})$ has been defined in [2,3] through the distance function d_S as

$$T(S; \overline{x}) := \{v \in X : d_S^o(\overline{x}; v) \le 0\} = \{v \in X : d_S^o(\overline{x}; v) = 0\}. \tag{2.3}$$

Clearly, $T(S; \overline{x})$ is a closed convex cone in X. Previously to the characterization (1.1) (due to [15,16]) recalled in the introduction, it has been proved in [11][1] that a vector $v \in T(S; \overline{x})$ whenever, for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in S converging to $\overline{x}$, there exists a sequence $(v_n)_n$ converging to v in X such that

$$x_n + t_n v_n \in S \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Combining this with (2.3) one can prove that $v \in T(S; \overline{x})$ if and only if, instead of (2.4), it is required that

$$x_n + t_n v_n \in S \quad \text{for infinitely many } n \in \mathbb{N}; \tag{2.5}$$

[1] Hiriart-Urruty [11] formally states the result in a Banach space. The proof is obviously valid in any normed space.

(see [10,18,19]).

With this at hand, it is easy to see:

Proposition 2.1 *Let S be a set of a normed space $(X, \|\cdot\|)$ and $\overline{x} \in \mathrm{cl}\, S$. Let S' be a subset of X such that $S \cap U \subset S' \cap U \subset (\mathrm{cl}\, S) \cap U$ for some neighbourhood U of $\overline{x}$. Then, a vector $v \in T(S \cup \{\overline{x}\}; \overline{x})$ if and only if, for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in S' converging to $\overline{x}$, there is a sequence $(v_n)_n$ in X converging to v such that $x_n + t_n v_n \in S$ for all $n \in \mathbb{N}$.*

A sequential characterization is also available for the interior tangent cone. Indeed, from (1.2) it easy to see that a vector $v \in I(S; \overline{x})$ if and only if, for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0, any sequence $(x_n)_n$ in S converging to $\overline{x}$, and any sequence $(v_n)_n$ converging to v in X

$$x_n + t_n v_n \in S \quad \text{for all } n \text{ large enough.} \tag{2.6}$$

Combining this with (2.4) it is not difficult to see that

$$I(S; \overline{x}) + T(S; \overline{x}) \subset I(S; \overline{x}).$$

Then, when $I(S; \overline{x}) \neq \emptyset$ (that is, S is epi-Lipschitz at $\overline{x}$), as shown in [16], the equalities

$$T(S; \overline{x}) = \mathrm{cl}_X\, I(S; \overline{x}) \quad \text{and} \quad I(S; \overline{x}) = \mathrm{int}_X\, T(S; \overline{x}), \tag{2.7}$$

hold true.

The next proposition provides a local property concerning the closure of the interior of an epi-Lipschitz set, and this will be used later.

Proposition 2.2 *Let S be a set of a normed space X such that $S \cup \{\overline{x}\}$ is epi-Lipschitz at $\overline{x} \in \overline{S}$ in a direction v, where $\overline{S} := \mathrm{cl}_X\, S$. The following hold.*

(a) *There exists an open neighbourhood U of $\overline{x}$ such that*

$$\overline{\mathrm{int}\, S} \cap U = \overline{\mathrm{int}\, \overline{S}} \cap U = \overline{S} \cap U.$$

(b) *If C is a set containing $\overline{x}$ for which there exits some neighbourhood U_0 of $\overline{x}$ such that*

$$(\mathrm{int}\, \overline{S}) \cap U_0 \subset C \cap U_0 \subset \overline{S} \cap U_0,$$

then C is epi-Lipschitz at $\overline{x}$ in the direction v and $I(C; \overline{x}) = I(S \cup \{\overline{x}\}; \overline{x})$; in particular $\overline{S}$ is epi-Lipschitz at $\overline{x}$ in the direction v and $I(\overline{S}; \overline{x}) = I(S \cup \{\overline{x}\}; \overline{x})$.

Proof To prove (a) we may suppose that $\overline{x} \in \mathrm{bdry}\, S$ since the result is obvious if $\overline{x} \in \mathrm{int}\, S$. From the epi-Lipschitz property of $S \cup \{\overline{x}\}$ at $\overline{x}$ in the direction v, there are by definition a real $\varepsilon > 0$, an open neighbourhood U of $\overline{x}$ and an open neighbourhood V of

v such that $(S \cup \{\overline{x}\}) \cap U +]0, \varepsilon[V \subset S \cup \{\overline{x}\}$, thus in particular $S \cap U +]0, \varepsilon[V \subset S \cup \{\overline{x}\}$. Since the first member of the latter inclusion is open, it ensues that

$$S \cap U +]0, \varepsilon[V \subset \text{int}\,(S \cup \{\overline{x}\}).$$

Fix any $x' \in \overline{S} \cap U$ and choose a sequence $(x_n)_n$ in S converging to x'. For some integer N we have $x_n \in S \cap U$ for all $n \geq N$ since U is a neighbourhood of x'. Therefore, for each $t \in]0, \varepsilon[$ and each $v' \in V$ we have

$$x_n + tv' \in \text{int}\,(S \cup \{\overline{x}\})$$

for all $n \geq N$, thus $x' + tv' \in \overline{\text{int}\,(S \cup \{\overline{x}\})}$. From Lemma 2.1 below we obtain $x' + tv' \in \overline{\text{int}\,S}$, so $x' \in \overline{\text{int}\,S}$. The two latter inclusions ensure $\overline{S} \cap U \subset \overline{\text{int}\,S}$ and

$$\overline{S} \cap U +]0, \varepsilon[V \subset \overline{S}, \quad \text{hence} \quad \overline{S} \cap U +]0, \varepsilon[V \subset \text{int}\,\overline{S}. \tag{2.8}$$

The inclusion $\overline{S} \cap U \subset \overline{\text{int}\,S}$ guarantees the equalities

$$\overline{S} \cap U = \overline{\text{int}\,S} \cap U = \overline{\text{int}\,\overline{S}} \cap U$$

in (a) since we always have $\overline{\text{int}\,S} \subset \overline{\text{int}\,\overline{S}} \subset \overline{S}$.

Let us prove (b). Choose an open neighbourhood $U' \subset U \cap U_0$ of $\overline{x}$, a real $0 < \varepsilon' < \varepsilon$, and a neighbourhood $V' \subset V$ of v such that $U' +]0, \varepsilon'[V' \subset U_0$. This entails by the second inclusion in (2.8)

$$C \cap U' +]0, \varepsilon'[V' = \left(C \cap U' +]0, \varepsilon'[V'\right) \cap U_0 \subset (\text{int}\,\overline{S}) \cap U_0,$$

thus the assumption on C gives $C \cap U' +]0, \varepsilon'[V' \subset C$, which justifies that C is epi-Lipschitz at $\overline{x}$ in the direction v.

From the equalities in (a) and the assumption of (b) it is easily seen that $\overline{S} \cap U' = \overline{C} \cap U'$, or equivalently $\overline{S \cup \{\overline{x}\}} \cap U' = \overline{C} \cap U'$. It ensues that $T(S \cup \{\overline{x}\}; \overline{x}) = T(C; \overline{x})$. The latter equality implies the desired equality $I(S \cup \{\overline{x}\}; \overline{x}) = I(C; \overline{x})$ of (b) since $I(P; x) = \text{int}\big(T(P; x)\big)$ whenever the set P is epi-Lipschitz at x [see (2.7)]. The proof is then finished. □

Lemma 2.1 *Let S be a set of a normed space X, $X \neq \{0\}$, and let $\overline{x} \in X$. Then, denoting by $\overline{S}$ the closure of S one has*

$$\overline{\text{int}\,(S \cup \{\overline{x}\})} = \overline{\text{int}\,S}.$$

Proof We only need to show that $\text{int}\,(S \cup \{\overline{x}\}) \subset \overline{\text{int}\,S}$. Fix any x in the first member that we suppose nonempty. There exists a real $r > 0$ such that $B(x, r) \subset S \cup \{\overline{x}\}$. If $x \neq \overline{x}$, choosing $0 < \rho < \min\{r, \|x - \overline{x}\|\}$ we see that $B(x, \rho) \subset S$, so $x \in \text{int}\,S$.

Now suppose that $x = \overline{x}$, so $B(\overline{x}, r) \subset S \cup \{\overline{x}\}$. Then, considering the nonempty set $U := B(\overline{x}, r) \setminus \{\overline{x}\}$, we derive $U \subset S$, thus $U \subset \text{int}\,S$. Since $\overline{x} \in \overline{U}$, it results that $\overline{x} \in \overline{\text{int}\,S}$, completing the proof. □

In addition to the relevance of the epi-Lipschitz property in the calculus of the Clarke tangent cone of intersection as said in the introduction, Clarke tangent cones of an epi-Lipschitz set and its complement are linked as in the following proposition from [16] (see also [11, page 84]). For completeness and the convenience of the reader, we provide a direct proof.

Proposition 2.3 *Let S be a set of a normed space X, let $\overline{x} \in S \cap \operatorname{bdry} S$, and let $S^c := (X \backslash S)$. The equality*

$$I(S^c \cup \{\overline{x}\}; \overline{x}) = -I(S; \overline{x})$$

holds, thus S is epi-Lipschitz at $\overline{x}$ in a direction $\overline{v}$ if and only if $S^c \cup \{\overline{x}\}$ is epi-Lipschitz at $\overline{x}$ in the opposite direction $-\overline{v}$. So,

$$T(S^c \cup \{\overline{x}\}; \overline{x}) = -T(S; \overline{x})$$

whenever S is epi-Lipschitz at $\overline{x}$.

Proof Suppose that $I(S; \overline{x})$ is nonvoid and, for any fixed $v \in I(S; \overline{x})$, let us prove that $-v \in I(S^c \cup \{\overline{x}\}; \overline{x})$. Take, from the definition of $I(S; \overline{x})$, a real $\varepsilon > 0$, a neighbourhood U of $\overline{x}$, and an open neighbourhood V of v such that $S \cap U +]0, \varepsilon[V \subset S$. Since the set $S \cap U +]0, \varepsilon[V$ is open, we have $S \cap U +]0, \varepsilon[V \subset \operatorname{int} S$. Choose a real $0 < \varepsilon' < \varepsilon$ and neighbourhoods U' and V' of $\overline{x}$ and v such that $U' +]-\varepsilon', \varepsilon'[V' \subset U$. To see that $-v \in I(S^c \cup \{\overline{x}\}; \overline{x})$ it suffices to show that $(S^c \cup \{\overline{x}\}) \cap U' +]0, \varepsilon'[(-V') \subset S^c \cup \{\overline{x}\}$. If the latter inclusion does not hold, there exist a real $0 < t < \varepsilon'$, and elements $x' \in (S^c \cup \{\overline{x}\}) \cap U'$ and $v' \in V'$ such that $u := x' - tv' \in X \backslash (S^c \cup \{\overline{x}\})$, hence $u \in S \cap U$. Writing $x' = u + tv'$ we obtain $x' \in S \cap U + tV \subset \operatorname{int} S$, which is a contradiction since $x' \in S^c \cup \{\overline{x}\} = (X \backslash S) \cup \{\overline{x}\}$ and $\overline{x} \in \operatorname{bdry} S$. Therefore, $-v \in I(S^c \cup \{\overline{x}\}; \overline{x})$ and the inclusion $I(S; \overline{x}) \subset -I(S^c \cup \{\overline{x}\}; \overline{x})$ is justified.

Since $(X \backslash S^c \cup \{\overline{x}\}) \cup \{\overline{x}\} = S$ we may permute S and $S^c \cup \{\overline{x}\}$ to get $I(S^c \cup \{\overline{x}\}; \overline{x}) \subset -I(S; \overline{x})$. The desired equality concerning the interior tangent cone is then established.

The equality above ensures that S is epi-Lipschitz at $\overline{x}$ in a direction $\bar{v}$ if and only if $S^c \cup \{\overline{x}\}$ is epi-Lipschitz at $\overline{x}$ in the direction $-\overline{v}$. Further, under the epi-Lipschitzness assumption the same equality guarantees the equality $T(S; \overline{x}) = -T(S^c \cup \{\overline{x}\}; \overline{x})$ since the Clarke tangent cone is the closure of the interior tangent cone whenever the latter is nonempty [see (2.7)]. This finishes the proof. □

For a convex set with nonempty interior, it is well-known that the interior of the set coincides with the interior of its closure. A first corollary of the previous proposition and Proposition 2.2 shows a similar relationship with an epi-Lipschitz set.

Corollary 2.1 *Let S be a set of a normed space X which is epi-Lipschitz at $\overline{x} \in S \cap \operatorname{bdry} S$. Then, there exists a neighbourhood U of $\overline{x}$ such that, with $\overline{S} := \operatorname{cl} S$, one has*

$$U \cap \operatorname{int} \overline{S} = U \cap \operatorname{int} (\overline{\operatorname{int} S}) = U \cap \operatorname{int} S.$$

Proof Clearly, we have $\overline{x} \in \text{bdry}\,(X\backslash S)$. Further, from the above proposition we know that $(X\backslash S) \cup \{\overline{x}\}$ is epi-Lipschitz at $\overline{x}$. Then, Proposition 2.2 furnishes a neighbourhood U of $\overline{x}$ such that

$$U \cap \text{cl}\left(\text{int}\,(X\backslash S)\right) = U \cap \text{cl}\,(X\backslash S).$$

Writing $\text{cl}\,(X\backslash S) = X\backslash \text{int}\, S$ and

$$\text{cl}\left(\text{int}\,(X\backslash S)\right) = \text{cl}\,(X\backslash \overline{S}) = X\backslash \text{int}\, \overline{S},$$

we deduce the equality $U \cap (X\backslash \text{int}\, \overline{S}) = U \cap (X\backslash \text{int}\, S)$, from which we see (without difficulty) that $U \cap \text{int}\, \overline{S} = U \cap \text{int}\, S$. This combined with the easy double inclusion $\text{int}\, S \subset \text{int}\,(\overline{\text{int}\, S}) \subset \text{int}\, \overline{S}$ yields the desired double equality of the proposition. □

In view of deriving Corollary 2.2 bellow, let us describe the Clarke tangent cone of the boundary of a set. We need first the following proposition which is a direct adaptation of the statement and proof of [12, Theorem 1].[2]

Proposition 2.4 *Let S be a subset of a normed space X and let $\overline{x} \in \text{bdry}\, S$. Let $S^c := X\backslash S$. The following are equivalent, for a vector $v \in X$:*

(a) *The vector $v \in T(S \cup \{\overline{x}\}; \overline{x})$;*
(b) *for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\text{bdry}\, S$ converging to $\overline{x}$, there is a sequence $(v_n)_n$ in X converging to v such that $x_n + t_n v_n \in S$ for infinitely many $n \in \mathbb{N}$;*
(c) *for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\text{bdry}\, S$ converging to $\overline{x}$, there is a sequence $(v_n)_n$ in X converging to v such that $x_n + t_n v_n \in \text{cl}\, S$ for infinitely many $n \in \mathbb{N}$.*

Proof Suppose that $v \in T(S \cup \{\overline{x}\}; \overline{x})$. Proposition 2.1 (applied with $S' = \text{cl}\, S$) says that, for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\text{cl}\, S$, there is a sequence $(v_n)_n$ in X converging to v such that, for each $n \in \mathbb{N}$, we have $x_n + t_n v_n \in S$. Since bdry S $\subset \text{cl}\, S$, we deduce that, for any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\text{bdry}\, S$ converging to $\overline{x}$, there exists a sequence $(v_n)_n$ in X converging to v such that $x_n + t_n v_n \in S$ for all $n \in \mathbb{N}$. This justifies the implication (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) being obvious, suppose (c) is fulfilled, and take any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\text{cl}\,(S \cup \{\overline{x}\}) = \text{cl}\, S$ converging to $\overline{x}$. Consider first the case when $x_n + t_n v \notin \text{cl}\, S$ for large n, say $n \geq N$. For each $n \geq N$, there is some $r_n \in [0, t_n[$ such that $u_n := x_n + r_n v \in \text{bdry}\, S$. Putting $u_n := \overline{x}$ for $n < N$ and noting that $t_n - r_n \to 0$ with $t_n - r_n > 0$, by assumption there exists a sequence $(v'_n)_n$ in X converging to v satisfying $u_n + (t_n - r_n)v'_n \in \text{cl}\, S$ for all $n \in J$, where J is an infinite subset of $\mathbb{N}\backslash\{1, \ldots, N-1\}$. For each $n \in J$, it results that

$$x_n + t_n \left(v + \frac{t_n - r_n}{t_n}(v'_n - v) \right) = u_n + (t_n - r_n)v'_n \in \text{cl}\,(S \cup \{\overline{x}\}),$$

[2] Again, stated in a Banach space, and valid in any normed space.

so with $v_n := v + \frac{t_n - r_n}{t_n}(v'_n - v)$, clearly $v_n \to v$ and $x_n + t_n v_n \in \mathrm{cl}\,(S \cup \{\overline{x}\})$ for all $n \in J$. In the remaining case, there is an infinite set $K \subset \mathbb{N}$ such that, for each $n \in K$, we have $x_n + t_n v \in \mathrm{cl}\, S$. Consequently, in any case there is an increasing function $s : \mathbb{N} \to \mathbb{N}$ and a sequence $(v_n)_n$ converging to v with $x_{s(n)} + t_{s(n)} v_n \in \mathrm{cl}\,(S \cup \{\overline{x}\})$ for all n. This guarantees that $v \in T\big(\mathrm{cl}\,(S \cup \{\overline{x}\}); \overline{x}\big)$ by (2.5). From the definition of Clarke tangent cone [see (2.3)] we obtain that $v \in T(S \cup \{\overline{x}\}; \overline{x})$, that is, the implication (b) $\Rightarrow$ (c) holds true. □

Proposition 2.5 *Let S be a subset of a normed space X and let $\overline{x} \in \mathrm{bdry}\, S$. Then, setting $S^c := X \backslash S$ one has the equality*

$$T(\mathrm{bdry}\, S; \overline{x}) = T\big(S \cup \{\overline{x}\}; \overline{x}\big) \cap T\big(S^c \cup \{\overline{x}\}; \overline{x}\big).$$

Proof Since $\overline{x} \in \mathrm{bdry}\, S = \mathrm{bdry}\,(S^c)$, the inclusion of the first member into the second follows directly from the equivalence between (a) and (c) in the previous proposition. Take now any v in the second member and consider any sequence $(t_n)_n$ in $]0, +\infty[$ tending to 0 and any sequence $(x_n)_n$ in $\mathrm{bdry}\, S$ converging to $\overline{x}$. There exist two sequences $(v'_n)_n$ and $(v_n{}'')_n$ converging to v with $x_n + t_n v'_n \in \mathrm{cl}\, S$ and $x_n + t_n v_n{}'' \in \mathrm{cl}(X \backslash S)$ for all n. Thus there is $\theta_n \in [0, 1]$ such that $x_n + t_n(\theta_n v'_n + (1 - \theta_n) v_n{}'') \in \mathrm{bdry}\, S$. So, putting $v_n := \theta_n v'_n + (1 - \theta_n) v_n{}''$ we obtain $x_n + t_n v_n \in \mathrm{bdry}\, S$ with $v_n \to v$, and this implies that $v \in T(\mathrm{bdry}\, S; \overline{x})$ according to (2.4). □

Remark 2.1 The similar equality with the Bouligand contingent cone $K(\cdot; \cdot)$ in place of the Clarke tangent cone $T(\cdot; \cdot)$ has been established in [14, Corollary 2.4]. It is also worth mentioning that through the equality

$$T(C; x) = \operatorname*{Lim\,inf}_{S \ni u \to x} K(C; u)$$

for any closed set C of a finite dimensional space, the equality in Proposition 2.5 can be derived from the one with the Bouligand contingent cone in [14] when X is finite dimensional. □

Given an epi-Lipschitz set S at $\overline{x} \in S \cap \mathrm{bdry}\, S$, we also note, by Propositions 2.3 and 2.5, that $T(\mathrm{bdry}\, S; \overline{x}) = T(S; \overline{x}) \cap -T(S; \overline{x})$, and this entails that $T(\mathrm{bdry}\, S; \overline{x})$ is a vector space since $T(S; \overline{x})$ is a convex cone. We state those properties in the corollary:

Corollary 2.2 *Let S be a subset of a normed space X which is epi-Lipschitz at $\overline{x} \in S \cap \mathrm{bdry}\, S$. Then, the following equality holds*

$$T(\mathrm{bdry}\, S; \overline{x}) = T(S; \overline{x}) \cap -T(S; \overline{x}),$$

which yields in particular that $T(\mathrm{bdry} S; \overline{x})$ is a closed vector subspace of X.

3 Representations of epi-Lipschitz sets as sublevels

Assume that the subset S of the normed space X is closed near $\overline{x} \in \mathrm{bdry}\, S$, that is, $S \cap U$ is closed in U, for some neighbourhood U of $\overline{x}$. Assume also that S is epi-

Lipschitz at $\overline{x}$ in a direction $\overline{v} \in X$ with $\overline{v} \neq 0$. The main theorem in [15][3] provides a topologically complemented closed vector hyperplane E of $\mathbb{R}\overline{v}$ (so, $X = E \oplus \mathbb{R}\overline{v}$), a neighbourhood W of $\overline{x}$ in X, and a function $f : E \to \mathbb{R}$ locally Lipschitz near $\pi_E \overline{x}$ (with E endowed with the induced norm and $x = \pi_E x + \pi_{\overline{v}} x$ with $\pi_E x \in E$ and $\pi_{\overline{v}} x \in \mathbb{R}$) such that

$$W \cap S = W \cap \{u + r\overline{v} : u \in E, r \in \mathbb{R}, f(u) \leq r\}.$$

Consider the linear isomorphism $\pi : X \to E \times \mathbb{R}$ defined by $\pi(x) := (\pi_E x, \pi_{\overline{v}} x)$ and note that the function $g : X \to \mathbb{R}$, with $g(x) := f(\pi_E x) - \pi_{\overline{v}} x$, is Lipschitz near $\overline{x}$ along with $W \cap S = W \cap \{x \in X : g(x) \leq 0\}$ and

$$g^o(\overline{x}; v) = f^o(\pi_E \overline{x}; \pi_E v) - \pi_{\overline{v}} v.$$

With $\overline{w} := \pi^{-1}(0_E, 1)$ we have $g^o(\overline{x}; \overline{w}) = -1 < 0$, so $0 \notin \partial g(\overline{x})$. Then, we notice that the property says in particular that any set epi-Lipschitz at $\overline{x} \in S$ is locally around $\overline{x}$ the sublevel set of a locally Lipschitz function g with $0 \notin \partial g(\overline{x})$. The next proposition shows that the converse also holds true. It has been proved in [9] through some specific constructions, in order to recover the above characterization. Below we give a simple and direct proof -with the original definition of Rockafellar with the interior tangent cone.

Proposition 3.1 *Let $g : X \to \mathbb{R}$ be a function which is Lipschitz near a point $\overline{x}$ of the normed space X and let $S := \{x \in X : g(x) \leq 0\}$. Assume that $\overline{x} \in$* bdry *S and $0 \notin \partial g(\overline{x})$. Then, the set S is epi-Lipschitz at $\overline{x}$.*

Proof From the definition of the Clarke directional derivative (2.1) of the locally Lipschitz function we see that, for every $v \in X$,

$$g^o(\overline{x}; v) = \limsup_{t \downarrow 0, (x', v') \to (\overline{x}, v)} t^{-1}\big(g(x' + tv') - g(x')\big).$$

Since $0 \notin \partial g(\overline{x})$, there is some $v \in X$ such that $g^o(\overline{x}; v) < 0$, and hence the latter equality furnishes some real $\varepsilon > 0$ such that, for all $x' \in B(\overline{x}, \varepsilon)$, $v' \in B(v, \varepsilon)$, and $t \in]0, \varepsilon[$, we have

$$t^{-1}\big(g(x' + tv') - g(x')\big) < 0, \quad \text{i.e., } g(x' + tv') < g(x').$$

So, for all $x' \in S \cap B(\overline{x}, \varepsilon)$, $v' \in B(v, \varepsilon)$, and $t \in]0, \varepsilon[$, we obtain

$$g(x' + tv') < g(x') \leq 0$$

(the second inequality being due to the inclusion $x' \in S$). This yields that

$$S \cap B(\overline{x}, \varepsilon) +]0, \varepsilon[\, B(v, \varepsilon) \subset S,$$

[3] As the author notices in [16], the argument in [15], written in finite dimensions, remains valid in any normed space.

which means that S is epi-Lipschitz at $\bar{x}$. □

We now use the signed distance function to obtain canonical sublevel representations of epi-Lipschitz sets. Recall that the *signed distance function* Δ_S (also *called oriented distance function*) from a subset S of the normed space X is defined by

$$\Delta_S(x) := d_S(x) - d_{S^c}(x) \quad \text{for all } x \in X,$$

where $S^c := X \backslash S$. It is clear that

$$\mathrm{cl}\, S = \{x \in X : \Delta_S(x) \leq 0\} \quad \text{and} \quad \mathrm{bdry}\, S = \{x \in X : \Delta_S(x) = 0\}. \tag{3.1}$$

When $S \neq \emptyset$ and $S \neq X$, it is known (see, e.g., [11]) that Δ_S is Lipschitz on X with 1 as a Lipschitz constant; further, for such a set S, it is shown in [11, Theorem 3][4] that, for any $x \in \mathrm{bdry}\, S$,

$$T(S \cup \{x\}; x) \cap -T(S^c \cup \{x\}; x) = \{v \in X : \langle x^*, v\rangle \leq 0,\ \forall x^* \in \partial\Delta_S(x)\}. \tag{3.2}$$

We establish first a local sublevel representation.

Theorem 3.1 *Let S be a subset of a normed space $(X, \|\cdot\|)$ and $\bar{x} \in S \cap \mathrm{bdry}\, S$. Assume with $\bar{S} := \mathrm{cl}\, S$ that $(\mathrm{int}\, \bar{S}) \cap U \subset S$ for some neighbourhood U of $\bar{x}$, which holds in particular whenever S is closed at $\bar{x}$. Then, the following assertions hold:*

(a) *The set S is epi-Lipschitz at $\bar{x}$ in a nonzero direction $\bar{v}$ if and only if*

$$(\Delta_S)^o(\bar{x}; \bar{v}) < 0.$$

(b) *The set S is epi-Lipschitz at $\bar{x}$ if and only if* $0 \notin \partial\Delta_S(\bar{x})$.

Proof The function $(\Delta_S)^o(\bar{x}; \cdot)$ being the support function of $\partial\Delta(\bar{x})$, the assertion (b) is a consequence of (a). If $\Delta^o(\bar{x}; \bar{v}) < 0$, by the equality $\bar{S} = \{u \in X : \Delta_S(u) \leq 0\}$ (where $\bar{S} := \mathrm{cl}\, S$) Proposition 3.1 says that $\bar{S}$ is epi-Lipschitz at $\bar{x}$ in the direction $\bar{v}$. Applying Proposition 2.2(b) with $\bar{S}$ in place of S and S in place of C, we derive that S is epi-Lipschitz at $\bar{x}$ in the direction $\bar{v}$.

Now suppose that S is epi-Lipschitz at $\bar{x}$ in the direction $\bar{v}$. Putting $S^c := X \backslash S$, we know that $S^c \cup \{\bar{x}\}$ is epi-Lipschitz at $\bar{x}$ in the direction $-\bar{v}$ (see Proposition 2.3), so by definition of epi-Lipschitz sets there is a real $\varepsilon \in]0, 1[$ such that, with $S' := S^c \cup \{\bar{x}\}$,

$$S \cap B(\bar{x}, 3\varepsilon) +]0, \varepsilon[B(\bar{v}, 2\varepsilon) \subset S \quad \text{and} \quad S' \cap B(\bar{x}, 3\varepsilon) +]0, \varepsilon[B(-\bar{v}, 2\varepsilon) \subset S'. \tag{3.3}$$

By Proposition 2.2(a) applied to the set S, using $\overline{\mathrm{int}\, S} = \overline{X \backslash \overline{S^c}}$, and to the set S^c, using $\overline{\mathrm{int}\, S^c} = \overline{X \backslash \bar{S}}$, we can also choose the above real $\varepsilon > 0$ such that

$$B(\bar{x}, \varepsilon) \cap \bar{S} = B(\bar{x}, \varepsilon) \cap \overline{X \backslash \overline{S^c}} \quad \text{and} \quad B(\bar{x}, \varepsilon) \cap \overline{S^c} = B(\bar{x}, \varepsilon) \cap \overline{X \backslash \bar{S}}. \tag{3.4}$$

[4] In a Banach space, the argument in [11] is valid in any normed space.

Claim 1. $d_S(x) \geq d_S(x+tv)$ for all $t \in]0, \varepsilon[$, $v \in B[\overline{v}, \varepsilon]$ *and* $x \in B(\overline{x}, \varepsilon)$.

Fix any $x \in B(\overline{x}, \varepsilon)$, any $v \in B[\overline{v}, \varepsilon]$, and any $t \in]0, \varepsilon[$. Choose a sequence $(y_n)_n$ in S such that $d_S(x) = \lim_{n\to\infty} \|x - y_n\|$. Since $d_S(x) \leq \|x - \overline{x}\| < \|x - \overline{x}\| + \varepsilon$, we may suppose that $\|x - y_n\| < \|x - \overline{x}\| + \varepsilon$ for all n. Then,

$$\|y_n - \overline{x}\| \leq \|y_n - x\| + \|x - \overline{x}\| < 2\|x - \overline{x}\| + \varepsilon < 3\varepsilon.$$

For every $n \in \mathbb{N}$, it results that $y_n + tv \in S$ by (3.3), hence

$$d_S(x+tv) \leq \|x + tv - (y_n + tv)\| = \|x - y_n\|,$$

so $d_S(x+tv) \leq d_S(x)$ as says the claim.

Claim 2. *For all* $0 < t < \frac{\varepsilon}{2(1+\|\overline{v}\|)}$ *and* $x \in B(\overline{x}, \varepsilon/2) \cap \overline{S^c}$, *one has* $x - t\overline{v} \notin \overline{S}$ *and* $d_S(x - t\overline{v}) \geq d_S(x) + \varepsilon t/2$.

Fix any such real t and fix also any $x \in B(\overline{x}, \varepsilon/2) \cap (X \backslash \overline{S})$. Take any $v \in B[\overline{v}, \varepsilon]$, and note that

$$\|x - tv - \overline{x}\| \leq \|x - \overline{x}\| + t\|v\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(1+\|\overline{v}\|)}(\varepsilon + \|\overline{v}\|) < \varepsilon.$$

Thus, $x - tv \in B(\overline{x}, \varepsilon)$, which gives $d_S(x - tv) \geq d_S(x - tv + tv) = d_S(x)$ by Claim 1. This means that

$$d_S(y) \geq d_S(x) \quad \text{for all } y \in B[x - t\overline{v}, \varepsilon t]. \tag{3.5}$$

Choose $z_t \in S$ such that

$$d_S(x - t\overline{v}) \geq \|x - t\overline{v} - z_t\| - \varepsilon t/2.$$

Noticing that $d_S(x) > 0$ (since $x \notin \overline{S}$), the inclusion $z_t \in S$ and (3.5) imply that $z_t \notin B[x - t\overline{v}, \varepsilon t]$. Then, we can choose $u_t \in [x - t\overline{v}, z_t]$ (the line segment) such that $\|x - t\overline{v} - u_t\| = \varepsilon t$. It ensues that

$$\begin{aligned} d_S(x - t\overline{v}) &\geq \|x - t\overline{v} - z_t\| - \frac{1}{2}\varepsilon t = -\frac{1}{2}\varepsilon t + \|x - t\overline{v} - u_t\| + \|u_t - z_t\| \\ &= \frac{1}{2}\varepsilon t + \|u_t - z_t\| \geq \frac{1}{2}\varepsilon t + d_S(u_t), \end{aligned}$$

hence by (3.5) we get $d_S(x - t\overline{v}) \geq d_S(x) + \varepsilon t/2$. By continuity of the function d_S and by (3.4) we deduce that the latter inequality still holds for all $x \in B(\overline{x}, \varepsilon/2) \cap (X \cap \overline{S^c})$ as stated in the claim. Clearly, the inequality also tells us that $x - t\overline{v} \notin \overline{S}$.

Claim 3. *For all* $0 < t < \frac{\varepsilon}{2(1+\|\overline{v}\|)}$ *and* $x \in B(\overline{x}, \varepsilon/2) \cap \overline{S}$, *one has* $x + t\overline{v} \notin \overline{S^c}$ *and* $d_{S^c}(x + t\overline{v}) \geq d_{S^c}(x) + \varepsilon t/2$.

It suffices to apply Claim 2 with the set $S^c \cup \{\overline{x}\}$ and the vector $-\overline{v}$ in place of S and $\overline{v}$ respectively. Noticing that $\overline{x} \in \text{bdry}\, S$ implies $\text{int}\, S \subset S \backslash \{\overline{x}\} \subset S$, and applying Proposition 2.2, (a), to obtain $\overline{S \backslash \{\overline{x}\}} = \overline{S}$.

Claim 4. *For all* $0 < t < \frac{\varepsilon}{4(1+\|\overline{v}\|)}$ *and* $x \in B(\overline{x}, \varepsilon/4)$,

$$\Delta_S(x + t\overline{v}) - \Delta_S(x) \le -\frac{\varepsilon t}{2}.$$

Fix any such t and any $x \in B(\overline{x}, \varepsilon/4)$.
If $x \in \overline{S}$, by Claim 3, $x + t\overline{v} \notin \overline{S^c}$, i.e., $x + t\overline{v} \in \operatorname{int} S$, and

$$\Delta_S(x + t\overline{v}) - \Delta_S(x) = (-d_{S^c}(x + t\overline{v})) - (-d_{S^c}(x)) \le -\frac{\varepsilon t}{2}.$$

If $x \notin \overline{S}$, i.e., $x \in \operatorname{int} S^c$, let

$$\tau = \sup\{\theta \mid \theta \le t\,,\ [x, x + \theta\overline{v}] \subset X\backslash S\}.$$

Then $x + \tau\overline{v} \in B(\overline{x}, \varepsilon/2) \cap \overline{S^c}$ and by Claim 2,

$$\begin{aligned}\Delta_S(x + \tau\overline{v}) - \Delta_S(x) &= d_S(x + \tau\overline{v}) - d_S(x)\\ &= d_S(x + \tau\overline{v}) - d_S(x + \tau\overline{v} - \tau\overline{v})\\ &\le -\frac{\varepsilon\tau}{2}.\end{aligned} \tag{3.6}$$

In the case $\tau = t$, the latter inequality translates the desired claim. Now, suppose $\tau < t$. Then $x + \tau\overline{v} \in \operatorname{bdry} S$, so writing $x + t\overline{v} = x + \tau\overline{v} + (t - \tau)\overline{v}$, we see by Claim 3 that $x + t\overline{v} \notin \overline{S^c}$ and

$$\Delta_S(x+t\overline{v})-\Delta_S(x+\tau\overline{v}) = (-d_{S^c}(x+\tau\overline{v}+(t-\tau)\overline{v}))-(-d_{S^c}(x+\tau\overline{v})) \le -\frac{\varepsilon(t-\tau)}{2}. \tag{3.7}$$

By adding (3.6) and (3.7), we deduce the claim.
Finally, applying Claim 4 we obtain

$$(\Delta_S)^o(\overline{x}; \overline{v}) \le -\varepsilon/2,$$

which finishes the proof. □

Through the above theorem, the Clarke normal cone of epi-Lipschitz sets can be expressed in terms of the Clarke subdifferential of the signed distance function.

Corollary 3.1 *Let S be a subset of a normed space $(X, \|\cdot\|)$ which is epi-Lipschitz at $\overline{x} \in S \cap \operatorname{bdry} S$. Then, the Clarke normal cone of S at $\overline{x}$ can be described as*

$$N(S; \overline{x}) = \mathbb{R}_+ \partial\Delta_S(\overline{x}),$$

where $\mathbb{R}_+ := [0, +\infty[$.

Proof We recall first that $\overline{S} := \text{cl}\, S = \{u \in X : \Delta_S(u) \leq 0\}$. Since $(\Delta_S)^o(\overline{x}; \overline{v}) < 0$, for some $\overline{v} \in X$, according to Theorem 3.1 above, it results (see, e.g., [4,13]) that the inclusion $N(\overline{S}; \overline{x}) \subset \mathbb{R}_+ \partial \Delta_S(\overline{x})$ holds true.

On the other hand, since $T(S^c \cup \{\overline{x}\}; \overline{x}) = -T(S; \overline{x})$ (see Proposition 2.3), from (3.2) we obtain

$$T(S; \overline{x}) = \{v \in X : \forall x^* \in \partial \Delta_S(\overline{x}),\ \langle x^*, v\rangle \leq 0\},$$

and from this we easily see that $\mathbb{R}_+ \partial \Delta_S(\overline{x}) \subset \big(T(S; \overline{x})\big)^o = N(S; \overline{x})$, where K^o denotes the negative polar cone of a cone K. Since $N(\overline{S}; \overline{x}) = N(S; \overline{x})$, the desired equality of the proposition is justified. □

Putting Proposition 3.1 and Theorem 3.1 together yields the global sublevel representation of epi-Lipschitz sets.

Theorem 3.2 *Let S be a nonempty closed set of a normed space $(X, \|\cdot\|)$ with $S \neq X$. Then, the following are equivalent:*

(a) *The set S is epi-Lipschitz;*
(b) *there exists a Lipschitz function $g : X \to \mathbb{R}$ such that $0 \notin \partial g(x)$ for every $x \in \text{bdry}\, S$ and*

$$S = \{x \in X : g(x) \leq 0\};$$

(c) *the set S enjoys the canonical qualified sublevel representation:*

$$S = \{x \in X : \Delta_S(x) \leq 0\} \quad \textit{with} \quad 0 \notin \partial \Delta_S(u)\ \forall u \in \text{bdry}\, S.$$

Proof The implication (c) ⇒ (b) is obvious and (b) ⇒ (a) follows from Proposition 3.1. Finally, the last implication (a) ⇒ (c) results from Theorem 3.1. □

4 Application in mathematical programming and perspective in mathematical economics

4.1 Optimization problems with epi-Lipschitz set constraints

Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on a Banach space X and S be an epi-Lipschitz set of X. Consider the optimization problems

$$(P_{f,S}) \begin{cases} \text{Minimize } \ f(x) \\ \text{subject to } \ x \in \overline{S} \end{cases} \quad \text{and} \quad (P_{f,\text{bdry}\, S}) \begin{cases} \text{Minimize } \ f(x) \\ \text{subject to } \ x \in \text{bdry}\, S. \end{cases}$$

Let $\overline{x} \in S$ be a local solution of $(P_{f,S})$ (resp. $(P_{f,\text{bdry}\, S})$).

If $\overline{x} \in \text{int}\, S$ is a local solution of $(P_{f,S})$, then $\overline{x}$ is an unconstrained local minimizer of f, so $0 \in \partial f(\overline{x})$.

Suppose now that $\overline{x} \in \operatorname{bdry} S$ is a local solution of $(P_{f,S})$ (resp. $(P_{f,\operatorname{bdry} S})$). By (3.1) the problems can be reformulated in the functional forms

$$\begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & \Delta_S(x) \le 0. \end{cases} \quad \left(\text{resp.} \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & \Delta_S(x) = 0 \end{cases}\right).$$

By [4, Theorem 6.1.1] we know that there exists a non-null pair of reals (ρ, σ) with $\rho \ge 0$ and $\sigma \ge 0$ (resp. $\rho \ge 0$ and $\sigma \in \mathbb{R}$) such that

$$0 \in \partial(\rho f + \sigma \Delta_S)(\overline{x}). \tag{4.1}$$

Since $0 \notin \partial \Delta_S(\overline{x})$ by Theorem 3.2, it results that $\rho > 0$, so multiplying the above condition with $\rho^{-1} > 0$ we obtain with $\lambda := \rho^{-1}\sigma$ that

$$0 \in \partial(f + \lambda \Delta_S)(\overline{x}).$$

Similar arguments hold true either with the approximate subdifferential in Banach space or with the limiting subdifferential in reflexive space.

Proposition 4.1 *Let X be a Banach space, $f : X \to \mathbb{R}$ be a locally Lipschitz function and $\overline{x} \in S$. If $\overline{x}$ is a local solution of $(P_{f,S})$ (resp. $(P_{f,\operatorname{bdry} S})$), there exists a real $\lambda \ge 0$ (resp. $\lambda \in \mathbb{R}$) such that*

$$0 \in \partial(f + \lambda \Delta_S)(\overline{x}),$$

where ∂ denotes here either the Clarke or approximate subdifferential if X is a Banach space or the limiting one if X is reflexive.

An optimality condition in the form

$$0 \in \partial f(\overline{x}) + \mathbb{R}\partial \Delta_S(\overline{x})$$

(encompassed by the above proposition) has been established with a totally different method in [11, (3.18)] for the problem $(P_{f,\operatorname{bdry} S})$ with Clarke subdifferential. Both [11, (3.18)] and Proposition 4.1 require the completeness of the space X. In a general normed space X, an optimality condition of the form

$$0 \in \partial f(\overline{x}) + \lambda \Delta_S(\overline{x}),$$

with $\lambda \ge 0$, for the problem $(P_{f,S})$, is a direct consequence of Corollary 3.1 and the inclusions $0 \in \partial(f + \delta_S)(\overline{x}) \subset \partial f(\overline{x}) + \partial \delta(\overline{x})$, where δ_S is the indicator function of S. Proposition 4.1 gives a more precise condition. It relies on the Lagrangian type optimality condition (4.1), which requires the completeness of the space.

4.2 Equilibria and economics

The representation Theorem 1.1 is at the origin of a wide research on generalized equilibria, in finite dimensions. It allows for smooth normal approximation of epi-Lipschitz sets, see [6], which in turn leads to applications to the existence of Equilibria [7] and in Economics [1]. It opens many perspectives for further research based on our present results, with the goal of extending and adapting the results in [1,6,7] in infinite dimensions.

References

1. Bonnisseau, J.-M., Cornet, B., Czarnecki, M.-O.: The marginal pricing rule revisited. Econ. Theory **33**(3), 579–589 (2007)
2. Clarke, F.H.: Generalized gradients and applications. Trans. Am. Math. Soc. **205**, 247–262 (1975)
3. Clarke, F.H.: A new approach to Lagrange multipliers. Math. Oper. Res. **1**, 165–174 (1976)
4. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley Intersciences, New York (1983). Second Edition: Classics in Applied Mathematics, 5, Society for Industrial and Applied Mathematics, Philadelphia (1990)
5. Cornet, B., Czarnecki, M.-O.: Smooth representations of epi-Lipschitzian subsets. Nonlinear Anal. Theory Methods Appl. **37**, 139–160 (1999)
6. Cornet, B., Czarnecki, M.-O.: Smooth normal approximations of epi-Lipschitz subsets of R^n. SIAM J. Control Optim. **37**(3), 710–730 (1999)
7. Cornet, B., Czarnecki, M.-O.: Existence of generalized equilibria. Nonlinear Anal. Theory Methods Appl. **44**, 555–574 (2001)
8. Cwiszewski, A., Kryszewski, W.: Equilibria of set-valued maps: a variational approach. Nonlinear Anal. Theory Methods Appl. **48**, 707–746 (2002)
9. Czarnecki, M.-O., Gudovich, A.N.: Representation of epi-Lipschitzian sets. Nonlinear Anal. Theory Methods Appl. **73**, 2361–2367 (2010)
10. Hiriart-Urruty, J.-B.: Gradients généralisés de fonctions marginales. SIAM J. Control Optim. **16**, 301–316 (1978)
11. Hiriart-Urruty, J.-B.: Tangent cones, generalized gradients and mathematical programming in Banach spaces. Math. Oper. Res. **4**, 79–97 (1979)
12. Hiriart-Urruty, J.-B.: New concepts in non differentiable programming. Bull. Soc. Math. France Mém. **60**, 57–85 (1979)
13. Penot, J.-P.: Calculus Without Derivatives, Graduate Texts in Mathematics. Spinger, New York (2014)
14. Quincampoix, M.: Differential inclusions and target problems. SIAM J. Control Optim. **30**, 324–335 (1992)
15. Rockafellar, R.T.: Clarke's tangent cone and the boundaries of closed sets in $\mathbb{R}^n$. Nonlinear Anal. Theory Methods Appl. **3**, 145–154 (1979)
16. Rockafellar, R.T.: Generalized directional derivatives and subgradients of nonconvex functions. Can. J. Math. **32**, 157–180 (1980)
17. Rockafellar, R.T.: Directional Lipschitzian functions and subdifferential calculus. Proc. Lond. Math. Soc. **39**, 331–355 (1980)
18. Thibault, L.: Problème de Bolza dans un espace de Banach séparable. C. R. Acad. Sci. Paris Sér. I Math. **282**, 1303–1306 (1976)
19. Thibault, L.: Mathematical programming and optimal control problems defined by compactly Lipschitzian mappings. Sém. Anal. Convexe Montp. Exp. **10** (1978)