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Scattering in the Energy Space for Boussinesq Equations

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Abstract: In this note we show that all small solutions in the energy space of the generalized 1D Boussinesq equation must decay to zero as time tends to infinity, strongly on slightly proper subsets of the space-time light cone. Our result does not require any assumption on the power of the nonlinearity, working even for the supercritical range of scattering. For the proof, we use two new Virial identities in the spirit of works (Kowalczyk et al. in J Am Math Soc 30:769–798, 2017; Kowalczyk et al. in Lett Math Phys 107(5):921–931, 2017). No parity assumption on the initial data is needed.

1. Introduction and Main Results

In this paper we study a class of fourth order nonlinear wave equations appearing as a standard model in Physics. More precisely, we consider the generalized (good) Boussinesq model [6]

$$\partial_t^2 u + \partial_x^4 u - \partial_x^2 u + \partial_x^2 f(u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

Here $u = u(t,x)$ is a real-valued function. This equation appears as a canonical model of shallow water waves as well as the Korteweg-de Vries (KdV) equation, see e.g. [5]. The fundamental works of Bona and Sachs [5], Linares [17], and Liu [20,21], established that (1.1) is locally well-posed (and even globally well-posed for small data [5,17]) in the standard energy space for $(u, \partial_t u) \in H^1 \times L^2$. We assume that the smooth nonlinearity is of power type, in the sense that for some $p > 1$,

$$f(0) = 0, \quad |f'(s)| \lesssim |s|^{p-1}, \quad |s| < 1. \tag{1.2}$$

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The purpose of this paper is to show that the model (1.1) shares important similarities with KdV and (second order) Klein-Gordon equations in their long time decay and behavior. We will prove, using well-chosen Virial identities, that for (1.1), small globally defined solutions (in the energy norm) must decay to zero locally in space. This being said, we prove these results independently of the subcritical, critical or supercritical character of the scattering mechanism for low powers of p (a.k.a. the Strauss exponent).

1.1. Main result. Before stating our result, we need some standard notation. The Boussinesq model (1.1) can be written as follows: if $u_1 := u$, then

$$\begin{cases} \partial_t u_1 = \partial_x u_2, \\ \partial_t u_2 = \partial_x (u_1 - \partial_x^2 u_1 - f(u_1)). \end{cases} \tag{1.3}$$

Theorem 1.1. *There exists an $\varepsilon > 0$ such that if*

$$\|(u_1, u_2)(\cdot, 0)\|_{H^1 \times L^2} < \varepsilon,$$

then one has, for any $C > 0$ arbitrarily large and $I(t) := \left(-\frac{Ct}{\log^2 t}, \frac{Ct}{\log^2 t}\right)$,

$$\lim_{t \to \infty} \|(u_1, u_2)(t)\|_{(H^1 \times L^2)(I(t))} = 0. \tag{1.4}$$

A similar result holds for the case $t \to -\infty$ after a suitable redefinition of $I(t)$.

Remark 1.1. By a result of Linares [17] and Liu [20] (see also [21, Theorems 3.1 and 3.2]), all small $H^1 \times L^2$ solutions are globally defined, thanks to the conservation of the energy

$$E[u_1, u_2](t) := \frac{1}{2} \int (u_2^2 + (\partial_x u_1)^2 + u_1^2)(t, x)dx - \int F(u_1)(t, x)dx,$$

and the smallness assumption on the initial data. More precisely, we have the equivalence $E[u_1, u_2](t) \sim \|(u_1, u_2)(t)\|_{H^1 \times L^2}^2$ with implicit constants independent of time.

Previous results on scattering of small amplitude solutions of (1.3) were proved by Liu [20], Linares-Sialom [18], and Cho-Ozawa [8]. These contributions are mainly based either on the use of weighted Sobolev norms, or mixed $W^{s,p}$ spaces, and the additional condition $p \geq p_c$ (a critical power exponent) is needed to ensure either standard ($p > p_c$) or modified ($p = p_c$) scattering. See also the works by Farah [9] and Farah and Scialom [10] for local well-posedness of (1.3) at low regularities.

Theorem 1.1 shows full scattering to zero in the energy space and in any slightly proper subset of the light cone. It also improves previous decay estimates in [8,18,20] in several directions. First of all, it does not require the assumption $p > p_c$ (the critical exponent for standard scattering results in the literature). Second, Theorem 1.1 describes not only linear but also "nonlinear scattering" on compact sets of space, in the sense that small solitary waves (if any) do "scatter" to infinity following (1.4), see (1.5) for more details. Finally, Theorem 1.1 only needs data in the energy space.

Let us remark that Theorem 1.1 gives no information on the remaining (unbounded) portion of the space, but since small solitary waves seem to persist in time [5,20], it is unlikely to have linear scattering only as reminder term in (1.4) if one works in the energy space. However, a particular integral rate of decay can be obtained for the pair $(u_1, u_2)(t)$: for any $\lambda_0 > 0$ sufficiently large,

$$\int_2^\infty \int \operatorname{sech}^2\left(\frac{x}{\lambda_0}\right)((\partial_x u_1)^2 + u_1^2 + u_2^2)(t,x)dx\,dt \lesssim \lambda_0\varepsilon^2,$$

[see (3.10)] as well as other mild decay estimates depending on time-depending weights. This ensures that both u_1 and u_2 are locally square integrable in time and space.

Theorem 1.1 is also in concordance with the existence of "arbitrary size" solitary waves for (1.3). Indeed, assume that $f(u) = |u|^{p-1}u$, $p > 1$ in (1.3). Let $Q = Q(s)$ be the standard soliton given by

$$Q(s) := \left(\frac{p+1}{2\cosh^2\left(\frac{(p-1)}{2}s\right)}\right)^{\frac{1}{p-1}}.$$

Note that Q solves $Q'' - Q + Q^p = 0$. Then, for any speed $|v| < 1$ and $x_0 \in \mathbb{R}$, the family[1]

$$\begin{aligned} Q_{v,x_0}(t,x) &:= \left(\gamma^{\frac{2}{p-1}} Q(\gamma(x - vt - x_0)), -v\gamma^{\frac{2}{p-1}} Q(\gamma(x - vt - x_0))\right), \\ \gamma &:= \sqrt{1 - v^2}, \end{aligned} \tag{1.5}$$

is a solitary wave for (1.3) [20], [21, p. 52]. Since small energy solitary waves must necessarily have ultra-relativistic speeds ($|v| \sim 1$), Theorem 1.1 must be valid only in the sub-relativistic regime. Note also that slow-speed solitary waves have sufficiently large energy to be ruled out by the hypothesis of Theorem 1.1 (they also are unstable, [20]). For further information about the stability theory of (1.5), see [5,20]. See also [1,3,11,23,24] for other similar stability results in other dispersive or scalar field equations.

Remark 1.2. In particular, Theorem 1.1 precludes the existence of small $H^1 \times L^2$ standing waves or "breathers" [1,3,13] in (1.1) by purely dynamical methods. Even small nonlinear objects moving at speeds below $\sim t\log^{-2} t$ are ruled out by Theorem 1.1.

The proof of Theorem 1.1 follows the recent ideas introduced by Kowalczyk, Martel and the first author [12,13] for the case of second order scalar field equations, which are respectively based in fundamental works by Martel and Merle [22–24], and Merle and Raphaël [25]. In both cases [12,13], decay is showed using well-cooked Virial identities adapted to each model, and under the additional assumption of small odd data perturbations. Here in Theorem 1.1 that condition is no longer needed because of some "KdV dynamics" hidden in the wave equation (1.3) which preserves a particular direction of movement in the dynamics (a "decay of momentum"). In this work we introduce two different Virial identities, one for showing decay of $u_1(t)$, and a second one which shows a smoothing effect hidden in (1.3), as well as decay for $u_2(t)$. For further scattering results around the zero state in scalar field equations, see [4,14–16,19,28,29] and references therein. This list is by no means exhaustive.

Remark 1.3. The proof of Theorem 1.1 also reveals a hidden KdV character of (1.1), probably well-known in the literature, but useful to understand why Theorem 1.1 is valid for all kind of data in the energy space (unlike the results in [13], which needed an oddness assumption). Formally, (1.1) can be written as

$$\partial_x\left(\partial_t(\partial_t\partial_x^{-1}u) + \partial_x\left(\partial_x^2 u - u + f(u)\right)\right) = 0,$$

[1] Note that the "Lorentz boost" is completely different to the one shared by second order scalar field equations; this is because (1.3) does not preserve the standard Lorentz invariance.

so after dropping the ∂_x operator in front, becomes a natural KdV-like equation, with the role of u also played by $\partial_t \partial_x^{-1} u$, just as in (1.3).

Remark 1.4. The interval $I(t)$ in Theorem 1.1 can be slightly improved: $I(t) = \left(-\frac{Ct}{\log^{1+\varepsilon} t}, \frac{Ct}{\log^{1+\varepsilon} t}\right)$, or $I(t) = \left(-\frac{Ct}{\log t \log^{1+\varepsilon} \log t}, \frac{Ct}{\log t \log^{1+\varepsilon} \log t}\right)$, $\varepsilon > 0$ are also completely valid regions for scattering. However, we cannot get the validity of Theorem 1.1 inside the interval $I(t) = \left(-\frac{Ct}{\log t}, \frac{Ct}{\log t}\right)$.

Remark 1.5. We expect that some of the conclusions of Theorem 1.1 could be available for the fourth order nonlinear wave model [7,26]

$$\partial_t^2 u + \partial_x^4 u + mu - f(u) = 0, \quad m \in \mathbb{R}, \tag{1.6}$$

but with harder proofs, because of the lack of particular momentum decay, just as in [13]. Note that this last model and (1.1) are formally related by an homotopy through the *fractional* Laplacian

$$\partial_t^2 u + \partial_x^4 u + m(-\partial_x^2)^{\alpha} u - (-\partial_x^2)^{\alpha} f(u) = 0, \qquad \alpha \in [0,1], \quad m = 1.$$

Also, it is well-known that (1.6) may have solitary wave solutions. Additionally, we believe that our methods can be adapted to more general Boussinesq models, such as those studied in [27]. Finally, see [2] for a recent application of this technique to a quasilinear 1+1 model.

1.2. Organization of this paper. This paper is organized as follows: Sect. 2 deals with a Virial identity needed for the proof of Theorem 1.1. Then, in Sect. 3, we prove a first part of Theorem 1.1. Next, in Sect. 4 we prove new Virial identities and a new smoothing estimate. Finally, Sect. 5 is devoted to the last part of Theorem 1.1, involving the decay of (u_1, u_2).

2. A Virial Identity

We start with the following result (see [12,13] for more details).

Lemma 2.1. *Let $(u_1, u_2) \in H^1 \times L^2$ a solution of (1.3). Consider $\psi = \psi(x)$ a smooth bounded function to be chosen later, and consider $\lambda(t)$ a never zero time scaling. Then for any $t \in \mathbb{R}$ we have*

$$\begin{aligned}\frac{d}{dt}\int \psi\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 = &-\frac{\lambda'(t)}{\lambda(t)}\int \frac{x}{\lambda(t)}\psi'\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 - \frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_2^2\\ &-\frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_1^2 + \frac{1}{2\lambda^3(t)}\int \psi^{(3)}\Big(\frac{x}{\lambda(t)}\Big)u_1^2\\ &-\frac{3}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(\partial_x u_1)^2\\ &+\frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(u_1 f(u_1) - F(u_1)).\end{aligned} \tag{2.1}$$

Here, $F(s)$ stands for $\int_0^s f(r)dr$.

Proof. We compute using (1.3) and integrating by parts:

$$
\begin{aligned}
\frac{d}{dt}\int \psi\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 = & -\frac{\lambda'(t)}{\lambda(t)}\int \frac{x}{\lambda(t)}\psi'\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 + \int \psi\Big(\frac{x}{\lambda(t)}\Big)u_2\partial_x u_2 \\
& + \int \psi\Big(\frac{x}{\lambda(t)}\Big)u_1\partial_t u_2 \\
= & -\frac{\lambda'(t)}{\lambda(t)}\int \frac{x}{\lambda(t)}\psi'\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 - \frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_2^2 \\
& + \int \psi\Big(\frac{x}{\lambda(t)}\Big)u_1\partial_x(u_1 - \partial_x^2 u_1 - f(u_1)) \\
= & \frac{\lambda'(t)}{\lambda(t)}\int \frac{x}{\lambda(t)}\psi'\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 - \frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_2^2 \\
& - \frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_1^2 + \int \partial_x\Big(\psi\Big(\frac{x}{\lambda(t)}\Big)u_1\Big)\partial_x^2 u_1 \\
& + \int \partial_x\Big(\psi\Big(\frac{x}{\lambda(t)}\Big)u_1\Big)f(u_1).
\end{aligned}
$$

The two last terms above can be estimated as follows:

$$
\begin{aligned}
\int \partial_x\Big(\psi\Big(\frac{x}{\lambda(t)}\Big)u_1\Big)\partial_x^2 u_1 = & \frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_1\partial_x^2 u_1 \\
& - \frac{1}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(\partial_x u_1)^2 \\
= & \frac{1}{2\lambda^3(t)}\int \psi^{(3)}\Big(\frac{x}{\lambda(t)}\Big)u_1^2 - \frac{3}{2\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(\partial_x u_1)^2.
\end{aligned}
$$

A further integration by parts gives

$$
\begin{aligned}
\int \partial_x\Big(\psi\Big(\frac{x}{\lambda(t)}\Big)u_1\Big)f(u_1) = & \frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)u_1 f(u_1) + \int \psi\Big(\frac{x}{\lambda(t)}\Big)\partial_x F(u_1) \\
= & \frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(u_1 f(u_1) - F(u_1)).
\end{aligned}
$$

Collecting the last identities, we get (2.1). □

Remark 2.1. Note that (1.3) enjoys an interesting Virial identity. Almost every quadratic term has the correct sign, and bad terms are small compared with good ones. This behavior can be also found in KdV like equations (see [22,23] for instance). The introduction of the $\lambda(t)$ is done in order to encompass almost all of the light cone.

3. Start of Proof of the Theorem 1.1

We only assume the case $t \to +\infty$, the opposite case ($t \to -\infty$) being a direct consequence of a completely similar argument.

3.1. Choice of $\lambda(t)$ *and* $\psi(x)$. Consider (2.1) and assume, without loss of generality, that $t \geq 2$. Given any constant $C > 0$, define

$$\lambda(t) := \frac{Ct}{\log^2 t}, \tag{3.1}$$

and

$$\psi(x) := \tanh(x), \quad \psi'(x) = \operatorname{sech}^2(x). \tag{3.2}$$

Note that

$$\frac{\lambda'(t)}{\lambda(t)} = \frac{1}{t}\Big(1 - \frac{2}{\log t}\Big). \tag{3.3}$$

Lemma 3.1. *Under the assumptions of Theorem 1.1, there exists an increasing sequence of time $t_n \uparrow \infty$ such that*

$$\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t_n)}\Big)(u_1^2 + (\partial_x u_1)^2 + u_2^2)(t_n, x)dx \longrightarrow 0 \text{ as } n \to +\infty. \tag{3.4}$$

Moreover, we have

$$\int_2^\infty \frac{1}{\lambda(t)} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2)(t, x)dx \lesssim \varepsilon^2. \tag{3.5}$$

Proof. Consider (2.1) with the choice of $\lambda(t)$ and $\psi(x)$ given in (3.1) and (3.2). First we estimate the term

$$-\frac{\lambda'(t)}{\lambda(t)} \int \frac{x}{\lambda(t)} \psi'\Big(\frac{x}{\lambda(t)}\Big) u_1 u_2.$$

We claim that for some fixed constant $\tilde{C} > 0$,

$$\Big|\frac{\lambda'(t)}{\lambda(t)} \int \frac{x}{\lambda(t)} \psi'\Big(\frac{x}{\lambda(t)}\Big) u_1 u_2\Big| \leq \frac{1}{4\lambda(t)} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) u_1^2 + \frac{\tilde{C}\varepsilon^2}{t \log^2 t}. \tag{3.6}$$

Indeed, using (3.3),

$$\begin{aligned}
\Big|\frac{\lambda'(t)}{\lambda(t)} \int \frac{x}{\lambda(t)} \psi'\Big(\frac{x}{\lambda(t)}\Big) u_1 u_2\Big| &\leq \frac{1}{t} \int \frac{|x|}{\lambda(t)} \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) |u_1 u_2| \\
&\leq \frac{\log^2 t}{8Ct} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) u_1^2 \\
&\quad + \frac{2C}{t \log^2 t} \int \frac{|x|^2}{\lambda^2(t)} \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) u_2^2 \\
&\leq \frac{1}{8\lambda(t)} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) u_1^2 \\
&\quad + \frac{2C}{t \log^2 t} (\sup_{s \in \mathbb{R}} s^2 \operatorname{sech}^2(s)) \int u_2^2 \\
&\leq \frac{1}{8\lambda(t)} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big) u_1^2 + \frac{\tilde{C}\varepsilon^2}{t \log^2 t}.
\end{aligned}$$

Now we consider the second bad term,

$$\frac{1}{\lambda^3(t)}\int \psi^{(3)}\Big(\frac{x}{\lambda(t)}\Big)u_1^2.$$

For this term clearly we have the estimate (it is enough to take $\lambda(t)$ larger than a fixed constant for all large time)

$$\Big|\frac{1}{2\lambda^3(t)}\int \psi^{(3)}\Big(\frac{x}{\lambda(t)}\Big)u_1^2\Big| \lesssim \frac{1}{2\cdot 8\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_1^2. \tag{3.7}$$

Finally, we consider the nonlinear term

$$\frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(u_1 f(u_1) - F(u_1)).$$

Since by hypothesis (1.2), $|u_1 f(u_1) - F(u_1)| \lesssim |u_1|^{p+1}$, we have

$$\begin{aligned}\Big|\frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(u_1 f(u_1) - F(u_1))\Big| &\lesssim \|u_1(t)\|_{L^\infty}^{p-1} \times \frac{1}{\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_1^2\\ &\lesssim \frac{\varepsilon^{p-1}}{\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_1^2.\end{aligned}$$

By taking $\varepsilon > 0$ small enough, we have

$$\Big|\frac{1}{\lambda(t)}\int \psi'\Big(\frac{x}{\lambda(t)}\Big)(u_1 f(u_1) - F(u_1))\Big| \leq \frac{1}{8\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_1^2. \tag{3.8}$$

Collecting estimates (3.6), (3.7) and (3.8), and replacing in (2.1), we obtain

$$\begin{aligned}\frac{d}{dt}\int \psi\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 \leq &-\frac{1}{2\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_2^2 - \frac{1}{16\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)u_1^2\\ &-\frac{3}{2\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)(\partial_x u_1)^2 + \frac{\tilde{C}\varepsilon^2}{t\log^2 t}.\end{aligned}$$

Note that the two last terms above right are integrable in time. Consequently, we have (3.5):

$$\int_2^\infty \frac{1}{\lambda(t)}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2)(t,x)dx \lesssim \varepsilon^2.$$

Therefore, since $\lambda(t)^{-1}$ is not integrable in $[2,\infty)$, there exists a sequence of time $t_n \to +\infty$ (which can be chosen increasing after taking a subsequence), such that

$$\lim_{n\to\infty}\int \operatorname{sech}^2\Big(\frac{x}{\lambda(t_n)}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2)(t_n,x)dx = 0.$$

The proof is complete. □

Remark 3.1 (On defocusing nonlinearities). If the nonlinearity $f(u_1)$ in (1.3) is a defocusing odd power (i.e. $f(u_1) = -|u_1|^{p-1}u_1$ with $p > 1$), then the smallness condition in Lemma 3.1 is not needed to ensure a finite integral in (3.5). In particular, for defocusing nonlinearities we have complete decay on a sequence no matter the size of the solution.

Let us make a small digression from the main proof. We recall that with a small modification of (3.5), we can already show that $u_1(t)$ decays to zero in H^1 on compact sets of space (but we cannot show that $u_2(t)$ also decays to zero). Using similar arguments as in the previous proof [except that now (3.6) is not necessary], one can prove that

Lemma 3.2. *Let $\lambda_0 > 0$ be a large fixed constant. There exists an increasing sequence of time $t_n \uparrow \infty$ such that*

$$\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)(u_1^2 + (\partial_x u_1)^2 + u_2^2)(t_n, x)dx \longrightarrow 0 \text{ as } n \to +\infty. \tag{3.9}$$

Moreover, one has the estimate equivalent to (3.5)

$$\int_2^\infty \int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2)(t, x)dxdt \lesssim \lambda_0 \varepsilon^2. \tag{3.10}$$

These last estimates, although weaker than (3.4), are enough to conclude the following result.

Lemma 3.3. *For any compact interval $I \subset \mathbb{R}$, we have*

$$\lim_{t\to+\infty} \int_I u_1^2(t, x)dx = 0. \tag{3.11}$$

Proof. We have from (1.3),

$$\frac{d}{dt}\Big(\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2\Big) = \frac{2}{\lambda_0}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1 \partial_x u_2,$$

so that

$$\Big|\frac{d}{dt}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2\Big| \lesssim \frac{1}{\lambda_0}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)(|\partial_x u_1| + |u_1|)|u_2|.$$

Integrating in time, we have

$$\begin{aligned}\Big|\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2(t_n) - \int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2(t)\Big| &\lesssim \\ \lesssim \int_t^{t_n} \frac{1}{\lambda_0}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)&((\partial_x u_1)^2 + u_1^2 + u_2^2)(s, x)dxds.\end{aligned}$$

Sending $n \to \infty$, and using (3.5), we get

$$\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2(t) \lesssim \int_t^\infty \frac{1}{\lambda_0}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2)(s, x)dxds.$$

Finally, sending $t \to \infty$, we get

$$\lim_{t\to+\infty}\int \operatorname{sech}^2\Big(\frac{x}{\lambda_0}\Big)u_1^2(t) = 0,$$

which implies (3.11). □

An easy consequence of this result is the following

Corollary 3.1. *For each bounded interval I one has $\|u_1(t)\|_{H^1(I)} \longrightarrow 0$ as $t \to \infty$.*

Proof. Fix a bounded interval I. Take as sequence $t_n \to +\infty$, and consider the sequence $u_1(t_n)$, bounded in $H^1(\mathbb{R})$. Take any subsequence (still denoted $u_1(t_n)$). Thanks to the compact embedding of $H^1(I)$ into $L^2(I)$ and (3.11), we have $u_1(t_n)$ convergent to zero in $H^1(I)$ (from the uniqueness of the limit). Since every subsequence has a subsequence convergent to the same limit, we conclude. □

In order to show the consequences of Theorem 1.1, in full generality we need additional estimates, part of the next Section.

4. A Second Set of Virial Identities

In order to fully show Theorem 1.1, we need two additional Virial identities that will imply a new smoothing effect in (1.1). For $\lambda(t) > 0$ as in (3.1), define

$$\mathcal{I}_+(t) := \int \phi \partial_x u_1 u_2 dx, \quad \mathcal{I}_-(t) := -\int \partial_x(\phi u_1) u_2 dx, \tag{4.1}$$

where, for the sake of simplicity, we have denoted

$$\phi = \phi(t,x) := \frac{1}{\lambda(t)}\phi_0\Big(\frac{x}{\lambda(t)}\Big), \quad \phi_0 := \operatorname{sech}^2. \tag{4.2}$$

Note that both quantities $\mathcal{I}_+(t)$ and $\mathcal{I}_-(t)$ are well-defined for $H^1 \times L^2$ solutions of (1.3), and we have

$$\sup_{t\in\mathbb{R}} \left(|\mathcal{I}_+(t)| + |\mathcal{I}_-(t)|\right) \lesssim \varepsilon^2. \tag{4.3}$$

Lemma 4.1 (Second Virial identities). *Assume that $(u_1, u_2)(t)$ is a sufficiently smooth and decaying solution of (1.3). Then we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_+(t) = & \int \partial_t\phi \partial_x u_1 u_2 - \int \phi(\partial_x u_2)^2 + \int \phi(\partial_x^2 u_1)^2 + \frac{1}{2}\int \partial_x^2\phi u_2^2 \\ & + \int \Big(\phi - \frac{1}{2}\partial_x^2\phi\Big)(\partial_x u_1)^2 - \int \phi(\partial_x u_1)^2 f'(u_1),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_-(t) = & -\int \partial_x(\partial_t\phi u_1)u_2 + \int \phi(\partial_x u_2)^2 + \int \phi(\partial_x^2 u_1)^2 \\ & - \int (\phi + 2\partial_x^2\phi)(\partial_x u_1)^2 + \frac{1}{2}\int (\partial_x^2\phi + \partial_x^4\phi)u_1^2 \\ & + \int \partial_x\phi u_1 \partial_x u_1 f'(u_1) + \int \phi f'(u_1)(\partial_x u_1)^2.\end{aligned} \tag{4.5}$$

Proof. First we prove (4.4). We have

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_+(t) &= \int \partial_t\phi\partial_x u_1 u_2 + \int \phi\partial_{tx}u_1u_2 + \int \phi\partial_x u_1\partial_t u_2 \\
&= \int \partial_t\phi\partial_x u_1 u_2 + \int \phi\partial_x^2 u_2 u_2 \\
&\quad + \int \phi\partial_x u_1\partial_x(u_1 - \partial_x^2 u_1 - f(u_1)) \\
&= \int \partial_t\phi\partial_x u_1 u_2 - \int \phi(\partial_x u_2)^2 + \frac{1}{2}\int \partial_x^2\phi u_2^2 + \int \phi(\partial_x u_1)^2 \\
&\quad - \frac{1}{2}\int \partial_x^2\phi(\partial_x u_1)^2 + \int \phi(\partial_x^2 u_1)^2 - \int \phi(\partial_x u_1)^2 f'(u_1).
\end{aligned}$$

Rearranging terms, we get (4.4). Now, for the proof of (4.5), we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_-(t) &= \int \partial_t\phi u_1\partial_x u_2 + \int \phi\partial_t(u_1\partial_x u_2) \\
&= -\int \partial_x(\partial_t\phi u_1)u_2 + \int \phi\partial_t u_1\partial_x u_2 + \int \phi u_1\partial_{tx}u_2 \\
&= -\int \partial_x(\partial_t\phi u_1)u_2 + \int \phi(\partial_x u_2)^2 \\
&\quad - \int \partial_x(\phi u_1)\partial_x(u_1 - \partial_x^2 u_1 - f(u_1)) \\
&= -\int \partial_x(\partial_t\phi u_1)u_2 - \int \phi(\partial_x u_2)^2 \\
&\quad - \int (\partial_x\phi u_1 + \phi\partial_x u_1)(\partial_x u_1 - \partial_x^3 u_1 - f'(u_1)\partial_x u_1).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_-(t) = &-\int \partial_x(\partial_t\phi u_1)u_2 + \int \phi(\partial_x u_2)^2 + \frac{1}{2}\int \partial_x^2\phi u_1^2 - \int \phi(\partial_x u_1)^2 \\
&+ \int \partial_x(\partial_x\phi u_1 + \phi\partial_x u_1)\partial_x^2 u_1 + \int \partial_x\phi u_1\partial_x u_1 f'(u_1) \\
&+ \int \phi f'(u_1)(\partial_x u_1)^2.
\end{aligned} \tag{4.6}$$

Finally, the term $\int \partial_x(\partial_x\phi u_1 + \phi\partial_x u_1)\partial_x^2 u_1$ can be reduced to

$$\int \partial_x(\partial_x\phi u_1 + \phi\partial_x u_1)\partial_x^2 u_1 = -2\int \partial_x^2\phi(\partial_x u_1)^2 + \frac{1}{2}\int \partial_x^4\phi u_1^2 + \int \phi(\partial_x^2 u_1)^2.$$

Plugging this identity in (4.6), and rearranging terms, we finally obtain (4.5). □

Lemma 4.1 will be useful to prove a Kato-type local smoothing effect for $H^1 \times L^2$ solutions of (1.3) (note that all computations are easily justified by a standard limiting argument).

Corollary 4.1. *The following smoothing estimate holds*

$$\int_2^\infty \frac{1}{\lambda(t)} \int \operatorname{sech}^2\Big(\frac{x}{\lambda(t)}\Big)((\partial_x^2 u_1)^2 + (\partial_x u_2)^2)(t,x)dxdt < +\infty. \tag{4.7}$$

In particular, there exists an increasing sequence of time $s_n \uparrow \infty$ such that

$$\int \operatorname{sech}^2\Big(\frac{x}{\lambda(s_n)}\Big)((\partial_x^2 u_1)^2 + (\partial_x u_2)^2)(s_n,x)dx \longrightarrow 0 \textit{ as } n \to +\infty. \tag{4.8}$$

Proof. In (4.4), the only complicated term is $\int \partial_t\phi\partial_x u_1 u_2$. For this term, we have

$$\int \partial_t\phi\partial_x u_1 u_2 = -\frac{\lambda'(t)}{\lambda(t)}\int \phi\partial_x u_1 u_2 - \frac{\lambda'(t)}{\lambda^2(t)}\int \frac{x}{\lambda(t)}\phi_0'\Big(\frac{x}{\lambda(t)}\Big)\partial_x u_1 u_2.$$

Using (3.1) and (3.3),

$$\Big|\int \partial_t\phi\partial_x u_1 u_2\Big| \lesssim \int \phi((\partial_x u_1)^2 + u_2^2) + \frac{\varepsilon^2}{t^2}\log^2 t. \tag{4.9}$$

On the other hand, in (4.5) the only complicated term is $-\int \partial_x(\partial_t\phi u_1)u_2$. Here we have

$$\begin{aligned}-\int \partial_x(\partial_t\phi u_1)u_2 = &-\frac{\lambda'(t)}{\lambda(t)}\int \partial_x(\phi u_1)u_2\\ &-\frac{\lambda'(t)}{\lambda^2(t)}\int \partial_x\Big(\frac{x}{\lambda(t)}\phi_0'\Big(\frac{x}{\lambda(t)}\Big)u_1\Big)u_2.\end{aligned}$$

Then, exactly as in the estimate (4.9), we have

$$\Big|\int \partial_x(\partial_t\phi u_1)u_2\Big| \lesssim \int \phi((\partial_x u_1)^2 + u_2^2) + \frac{\varepsilon^2}{t^2}\log^2 t. \tag{4.10}$$

Hence, [using (4.9)–(4.10)] from the addition of (4.4) and (4.5),

$$\Big|\frac{d}{dt}\mathcal{I}_+(t) + \frac{d}{dt}\mathcal{I}_-(t) - 2\int \phi(\partial_x^2 u_1)^2\Big| \lesssim \int \phi((\partial_x u_1)^2 + u_1^2 + u_2^2) + \frac{\varepsilon^2}{t^2}\log^2 t.$$

and from the subtraction of (4.4) and (4.5),

$$\Big|\frac{d}{dt}\mathcal{I}_+(t) - \frac{d}{dt}\mathcal{I}_-(t) + 2\int \phi(\partial_x u_2)^2\Big| \lesssim \int \phi((\partial_x u_1)^2 + u_1^2 + u_2^2) + \frac{\varepsilon^2}{t^2}\log^2 t.$$

Therefore, using (3.5) and (4.3), we have

$$\int_2^\infty \int \phi((\partial_x u_2)^2 + (\partial_x^2 u_1)^2) < \infty.$$

Finally, (4.8) follows by a standard argument [see (3.4)]. □

Remark 4.1. Note that Corollary 4.1 is still valid in the case of large data if $f(u_1)$ is defocusing (see Remark 3.1) and $f'(u_1)$ is just bounded in L^∞ (for instance, if (u_1,u_2) are merely bounded in $H^1 \times L^2$). Under these two assumptions, Theorem 1.1 will be also valid for arbitrary large data in the defocusing case.

5. End of Proof of Theorem 1.1

Now we end the proof of Theorem 1.1. Let

$$\phi_1 := \operatorname{sech}^4 = \phi_0^2. \tag{5.1}$$

The power 4 is necessary because of a slight loss of decay in an estimate below. We will use a third energy estimate:

Lemma 5.1. *Let ϕ_1 be as in (5.1) and F such that $F' = f$ and $F(0) = 0$. Then,*

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \phi_1\Big(\frac{x}{\lambda(t)}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&= \frac{1}{2}\int \partial_t\Big(\phi_1\Big(\frac{x}{\lambda(t)}\Big)\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&\quad + \frac{2}{\lambda(t)}\int \phi_1'\Big(\frac{x}{\lambda(t)}\Big)u_2\partial_x^2 u_1 + \frac{1}{\lambda^2(t)}\int \phi_1''\Big(\frac{x}{\lambda(t)}\Big)\partial_x u_1 u_2\\
&\quad - \frac{1}{\lambda(t)}\int \phi_1'\Big(\frac{x}{\lambda(t)}\Big)u_1u_2 + \frac{1}{\lambda(t)}\int \phi_1'\Big(\frac{x}{\lambda(t)}\Big)u_2 f(u_1).
\end{aligned} \tag{5.2}$$

Proof. We denote $\phi_1 = \phi_1(\frac{x}{\lambda(t)})$ for simplicity, and we compute:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&\quad + \int \phi_1(\partial_{xt}^2 u_1\partial_x u_1 + u_1\partial_t u_1 + u_2\partial_t u_2 - f(u_1)\partial_t u_1).
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&\quad + \int \phi_1(u_1 - \partial_x^2 u_1 - f(u_1))\partial_t u_1 - \int \partial_x\phi_1\partial_x u_1\partial_t u_1 + \int \phi_1 u_2\partial_t u_2\\
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\\
&\quad + \int \phi_1(u_1 - \partial_x^2 u_1 - f(u_1))\partial_x u_2 - \int \partial_x\phi_1\partial_x u_1\partial_x u_2 + \int \phi_1 u_2\partial_t u_2.
\end{aligned}$$

Using (1.3):

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)) \\
&\quad - \int \partial_x\phi_1(u_1 - \partial_x^2 u_1 - f(u_1))u_2 - \int \phi_1\partial_x(u_1 - \partial_x^2 u_1 - f(u_1))u_2 \\
&\quad + \int \phi_1 u_2 \partial_t u_2 - \int \partial_x\phi_1\partial_x u_1\partial_x u_2 \\
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)) \\
&\quad - \int \partial_x\phi_1(u_1 - \partial_x^2 u_1 - f(u_1))u_2 - \int \partial_x\phi_1\partial_x u_1\partial_x u_2.
\end{aligned}$$

Now we integrate by parts to obtain

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))
&= \frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)) \\
&\quad - \int \partial_x\phi_1 u_1 u_2 + 2\int \partial_x\phi_1 u_2\partial_x^2 u_1 + \int \partial_x\phi_1 f(u_1)u_2 + \int \partial_x^2\phi_1\partial_x u_1 u_2.
\end{aligned}$$

Noticing that $\phi_1 = \phi_1(\frac{x}{\lambda(t)})$, we get the result, as desired. □

Now we conclude the proof of Theorem 1.1. First we have

$$\begin{aligned}
&\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)) \\
&= -\frac{\lambda'(t)}{\lambda(t)}\int \frac{x}{\lambda(t)}\phi_1'\Big(\frac{x}{\lambda(t)}\Big)((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)).
\end{aligned}$$

Consequently, using that $\left|\frac{\lambda'(t)}{\lambda(t)}\frac{x}{\lambda(t)}\phi_1'(\frac{x}{\lambda(t)})\right| \lesssim \phi(t,x)$,

$$\left|\frac{1}{2}\int \partial_t\phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\right| \lesssim \int \phi((\partial_x u_1)^2 + u_1^2 + u_2^2).$$

Using this last estimate, we have from (5.2) and the crude estimate $|\partial_x\phi_1| \lesssim \phi$,

$$\left|\frac{d}{dt}\frac{1}{2}\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))\right| \lesssim \int \phi((\partial_x^2 u_1)^2 + (\partial_x u_1)^2 + u_1^2 + u_2^2).$$

From Corollary 4.1 and (3.5) we get for $t < t_n$,

$$\begin{aligned}
&\left|\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))(t_n) - \int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))(t)\right| \\
&\lesssim \int_t^\infty\int \phi((\partial_x^2 u_1)^2 + (\partial_x u_1)^2 + u_1^2 + u_2^2) < \infty
\end{aligned}$$

Sending $t_n \to +\infty$ we have

$$\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))(t) \lesssim \int_t^\infty \int \phi((\partial_x^2 u_1)^2 + (\partial_x u_1)^2 + u_1^2 + u_2^2).$$

Therefore

$$\lim_{t\to+\infty} \int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))(t,x)dx = 0.$$

In particular, from the smallness assumption on the data and the Sobolev inequality,

$$\lim_{t\to+\infty} \int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2)(t,x)dx = 0. \tag{5.3}$$

The conclusion in Theorem 1.1 follows from the fact that $\lambda(t)$ given in (3.1) is such that

$$\int \phi_1((\partial_x u_1)^2 + u_1^2 + u_2^2)(t,x)dx \gtrsim \int_{I(t)} ((\partial_x u_1)^2 + u_1^2 + u_2^2)(t,x)dx,$$

with involved constant independent of time. The proof is complete.

Remark 5.1. Note that the smallness data condition of Theorem 1.1 is not needed in (5.3) if the nonlinearity is defocusing, see Remarks 3.1 and 4.1 for more details. In particular, Theorem 1.1 is valid even if data is large in the defocusing case.

Note added in proof. After the acceptance of this work, we have learnt that Theorem 1.1 also holds in the interval $I(t) := (at - \frac{Ct}{\log^2 t}, at + \frac{Ct}{\log^2 t})$, for any $|a| < 1$. The proof of this result follows easily by changing the variable $\frac{x}{\lambda(t)}$ in ψ in Eq. (2.1) by the new variable $\frac{x-at}{\lambda(t)}$, and performing standard estimates.

References

1. Alejo, M.A., Muñoz, C.: Nonlinear stability of mKdV breathers. Commun. Math. Phys. **324**(1), 233–262 (2013)
2. Alejo, M.A., Muñoz, C.: Almost sharp nonlinear scattering in one-dimensional Born–Infeld equations arising in nonlinear electrodynamics, preprint 2017, to appear in Proc. AMS.
3. Alejo, M.A., Muñoz, C., Palacios, J.M.: On the variational structure of breather solutions I: Sine–Gordon equation. J. Math. Anal. Appl. **453**(2), 1111–1138 (2017)
4. Bambusi, D., Cuccagna, S.: On dispersion of small energy solutions to the nonlinear Klein Gordon equation with a potential. Am. J. Math. **133**(5), 1421–1468 (2011)
5. Bona, J.L., Sachs, R.L.: Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation. Commun. Math. Phys. **118**(1), 15–29 (1988)
6. Boussinesq, J.: Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. J. Math. Pure Appl. **17**(2), 55–108 (1872)
7. Bretherton, F.P.: Resonant interaction between waves: the case of discrete oscillations. J. Fluid Mech. **20**, 457–479 (1964)
8. Cho, Y., Ozawa, T.: On small amplitude solutions to the generalized Boussinesq equations. Discrete Contin. Dyn. Syst. **17**(4), 691–711 (2007)
9. Farah, L.G.: Local solutions in Sobolev spaces with negative indices for the "good" Boussinesq equation. Commun. Partial Differ. Equ. **34**(1–3), 52–73 (2009)

10. Farah, L.G., Scialom, M.: On the periodic "good" Boussinesq equation. Proc. Am. Math. Soc. **138**(3), 953–964 (2010)
11. Henry, D.B., Perez, J.F., Wreszinski, W.F.: Stability theory for solitary-wave solutions of scalar field equations. Commun. Math. Phys. **85**(3), 351–361 (1982)
12. Kowalczyk, M., Martel, Y., Muñoz, C.: Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space. J. Am. Math. Soc. **30**, 769–798 (2017)
13. Kowalczyk, M., Martel, Y., Muñoz, C.: Nonexistence of small, odd breathers for a class of nonlinear wave equations. Lett. Math. Phys. **107**(5), 921–931 (2017)
14. Lindblad, H., Soffer, A.: A Remark on long range scattering for the nonlinear Klein–Gordon equation. J. Hyperbolic Differ. Equ. **2**(1), 77–89 (2005)
15. Lindblad, H., Soffer, A.: A remark on asymptotic completeness for the critical nonlinear Klein–Gordon equation. Lett. Math. Phys. **73**(3), 249–258 (2005)
16. Lindblad, H., Soffer, A.: Scattering for the Klein–Gordon equation with quadratic and variable coefficient cubic nonlinearities. TAMS **367**(12), 8861–8909 (2015)
17. Linares, F.: Global existence of small solutions for a generalized Boussinesq equation. J. Differ. Equ. **106**, 257–293 (1993)
18. Linares, F., Scialom, M.: Asymptotic behavior of solutions of a generalized Boussinesq type equation. Nonlinear Anal. **25**(11), 1147–1158 (1995)
19. Lindbland, H., Tao, T.: Asymptotic decay for a one-dimensional nonlinear wave equation. Anal. PDE **5**(2), 411–422 (2012). https://doi.org/10.2140/apde.2012.5.411
20. Liu, Y.: Instability of solitary waves for generalized Boussinesq equations. J. Dyn. Diff. Equ. **53**, 537–558 (1993)
21. Yue, L.: Decay and scattering of small solutions of a generalized Boussinesq equation. J. Funct. Anal. **147**, 51–68 (1997)
22. Martel, Y., Merle, F.: A Liouville theorem for the critical generalized Korteweg-de Vries equation. J. Math. Pures Appl. (9) **79**(4), 339–425 (2000)
23. Martel, Y., Merle, F.: Asymptotic stability of solitons for subcritical generalized KdV equations. Arch. Ration. Mech. Anal. **157**(3), 219–254 (2001)
24. Martel, Y., Merle, F.: Asymptotic stability of solitons for subcritical gKdV equations revisited. Nonlinearity **18**(1), 55–80 (2005)
25. Merle, F., Raphaël, P.: The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. Ann. Math. (2) **161**(1), 157–222 (2005)
26. Pausader, B.: Scattering and the Levandosky–Strauss conjecture for fourth-order nonlinear wave equations. J. Diff. Equ. **241**(2), 237–278 (2007)
27. Saut, J.C.: Asymptotic models for surface and internal waves, 29th. Coloquio Brasileiro de Matemática, Publicações matemáticas IMPA (2013)
28. Saut, J.C.: Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. Invent. Math. **136**(1), 9–74 (1999)
29. Sterbenz, J.: Dispersive Decay for the 1D Klein–Gordon equation with variable coefficient nonlinearities. Trans. Am. Math. Soc. **368**, 2081–2113 (2016)

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