Quantization of the gravitational constant in odd dimensions

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It is pointed out that the action recently proposed by Bañados, Teitelboim, and Zanelli for the study of black holes in odd dimensions higher (and lower) than four provides a natural quantization for the gravitational constant. These theories possess no dimensionful parameters and hence they may be power counting renormalizable.

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Gravitation in dimensions greater than 2 is best described by the so-called Lovelock action [1]. This is a linear combination of the dimensional continuations to D dimensions of all the Euler classes of dimensions $2p < D$ [2,3]. The Lovelock Lagrangian could be defined by the previous statement, but it can also be derived in three other seemingly independent ways: (i) It is the most general invariant constructed out of the metric and curvature that yields second-order covariant field equations [1]; (ii) it is the most general local D-form invariant under tangent space rotations, constructed out of the vielbein, the spin connection, and their exterior derivatives without using the Hodge* dual [4]; (iii) it is the most general low-energy effective theory of gravity that can be derived from string theory [5].

In a recent article, Bañados *et al.* [6] have considered a particular form of the Lovelock action which results from embedding the group of tangent space rotations,[1] SO(D), into SO($D+1$). For odd dimensions there is a particular choice that makes the action invariant under SO($D+1$), whereas for even D that possibility does not exist.

The proposed Lagrangian for $D = 2n-1$ dimensions reads

$$L_{2n-1} = \kappa \sum_{p=0}^{n-1} \frac{1}{D-2p} \begin{pmatrix} n-1 \\ p \end{pmatrix} l^{2p-D} \epsilon_{a_1 \cdots a_{2n-1}} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \cdots \wedge e^{a_{2n-1}}. \tag{1}$$

Here κ is the gravitational constant analogous to Newton's in $D = 4$, l is a constant with dimension of length, R^{ab} is the curvature two-form, and e^a are the vielbein. As is shown in Ref. [6], this is the Euler-Chern-Simons density. That is, L_{2n-1} is a $(2n-1)$-form whose exterior derivative is the Euler class $\mathcal{E}_{2n}$ for $(2n)$-dimensional manifolds:

$$d \wedge L_{2n-1} = \kappa \mathcal{E}_{2n}, \tag{2}$$

with

$$\mathcal{E}_{2n} = \epsilon_{A_1 \cdots A_{2n}} \bar{R}^{A_1 A_2} \wedge \cdots \wedge \bar{R}^{A_{2n-1} A_{2n}}, \tag{3}$$

where $A, B, C, ... = 1, ..., 2n$. Here $\bar{R}^{AB}$ is the $2n$-dimensional curvature two-form associated with the (anti−) de Sitter group:

$$\bar{R}^A_B = \begin{bmatrix} R^a_b - \epsilon l^{-2} e^a_b & \epsilon l^{-1} T^a \\ -l^{-1} T_b & 0 \end{bmatrix}, \tag{4}$$

where $T^a = de^a + \omega^a_b \wedge e^b$ is the torsion two-form.

L_{2n-1} is the analogue of the Pontryagin Chern-Simons form encountered in gauge theories. In gravity there is also a Pontryagin form, often called the Hirzebruch class, in $D = 4n$ and a corresponding Chern-Simons density in $D = 4n-1$ (see, e.g., [7]), but we will not consider them here as they cannot be dimensionally continued.

For a non-Abelian gauge theory in 2+1 dimensions with the gauge group G the existence of large gauge transformations within a nontrivial homotopy class implies the quantization of the coupling constant g that multiplies the Chern-Simons action [8,9]. Roughly speaking, if $\pi_3(G) \neq 0$, g must be quantized.

The same argument applied to asymptotically flat gravitation theory in 2+1 dimensions does not lead to the quantization of Newton's constant because in that case the relevant group is ISO(2,1), and $\pi_3(\text{ISO}(2,1)) = 0$ [9].

On the other hand, the theories considered in [6] have a cosmological constant; their solutions are not asymptotically flat and hence the relevant groups are anti−de Sitter SO($2n-2,2$); so the nonquantization argument does not apply here.

Our argument for the quantization of κ, however, does not rely on the existence of large gauge transformations but on the possibility of rewriting the $(2n-1)$-dimensional gravity action as a topological theory on a

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[1]For brevity here SO(D) will denote the group of rotations in D dimensions or any of its complex extensions SO(p,q), with $p+q = D$. We shall make the distinction below when commenting on the spacetime signature and the Wick rotation.

$2n$-dimensional manifold whose boundary is the spacetime one wants to describe. This is similar to the standard discussion leading to the quantization of the coupling constant in the Wess-Zumino-Witten (WZW) theory [10].

Let us consider the particular case of a compact $(2n-1)$-dimensional, simply connected manifold M that is the boundary of some $2n$-dimensional orientable manifold Ω (for definiteness, M could be taken to be S^{2n-1}). Then, by Stokes' theorem, the action for M can be expressed as

$$S_\Omega[M] = \kappa \int_\Omega \mathcal{E}_{2n}. \tag{5}$$

Obviously, there is a large freedom in the choice of Ω, as there are infinitely many ways to extend M. However, since the action $S_\Omega[M]$ is to describe the dynamical properties of M, it is reasonable to demand that the observables of the system should be insensitive to changing in Eq. (5) Ω by a different $2n$-manifold, Ω', with the same boundary (M) [11]:

$$\partial\Omega = M = \partial\Omega'. \tag{6}$$

Also,

$$\begin{aligned} S_\Omega[M] &= \kappa \int_\Omega \mathcal{E} = \kappa \left(\int_\Omega \mathcal{E} - \int_{\Omega'} \mathcal{E} \right) + \kappa \int_{\Omega'} \mathcal{E} \\ &= \kappa \left(\int_\Omega \mathcal{E} + \int_{-\Omega'} \mathcal{E} \right) + S_{\Omega'}[M]. \end{aligned}$$

The first term on the right hand side of the last equality is κ times the Euler class of the manifold formed by joining Ω and $-\Omega'$ smoothly along M. The minus sign accounts for the fact that the orientation of one of the two halves must be reversed in order for their union to possess a well-defined orientation throughout. Thus, we finally have

$$S_\Omega[M] = \kappa\chi[\Omega \cup -\Omega'] + S_{\Omega'}[M]. \tag{7}$$

Although the action of any classical system is defined modulo an arbitrary additive constant, quantum mechanically this constant must be an integer multiple of Planck's constant h so that the path integral of the system will be unaffected. This implies, in particular, that under continuous transformations of the fields the additive constant cannot change.

On the other hand, the replacement $\Omega \to \Omega'$ could not be attainable through a continuous transformation if Ω and Ω' have different topologies. This means that the difference $S_{\Omega'}[M] - S_\Omega[M]$ must be an integer multiple of h, and one concludes from (7) that since χ is an integer, κ must be quantized. [The same argument does not hold for gravity in $2n$ dimensions because there is no analogue of Eq. (2) in that case and hence the even dimensional action cannot be written as the integral of an exact $(2n+1)$-form.]

Our point rests on the assumption that there exists a manifold of the form $\Omega \cup -\Omega'$ with a nonzero Euler characteristic. It is actually not difficult to envision many examples of this type; e.g., Ω and Ω' can be two halves of a $2n$-sphere with any number of "handles" attached to each hemisphere. A different question is whether $\Omega \cup -\Omega'$ could be a classical solution for a $2n$-dimensional theory. This question is related to the existence of instantons such as the D'Auria-Regge solution [12] in four-dimensional Euclidean gravity (that instanton, however, is not a solution of pure gravity but requires torsion and matter fields). The existence of such solutions is not required for the validity of our argument and does not concern us here since we are taking the point of view that the fundamental theory is the one defined in $2n-1$ dimensions.

The quantization argument is valid regardless of the signature of spacetime. This is because the Euler characteristic is a topological invariant and hence insensitive to changes in the signature of the metric. The action is constructed entirely out of form fields which are independent of the coordinates and therefore invariant under coordinate changes, including Wick rotations. Thus, both in the hyperbolic and the Euclidean signature the integrand of the path integral is $\exp(\frac{-i}{\hbar}S[M])$, where $S[M]$ is the (real) action constructed as in (1).

It might seem paradoxical that the action is insensitive to the signature of the metric, especially in view of the explicit dependence of L_{2n-1} on the vielbein. The paradox is resolved by noting that the Wick rotation must be performed simultaneously on the spacetime coordinates and on the tangent space, as this is the only consistent way to maintain the relationship between the spacetime metric and that of its tangent space (soldering), $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$. Since in the end all (spacetime and tangent space) indices are contracted in the Lagrangian, no extra factors of i appear in the Euclidean action.

Actions whose construction require the Hodge*-dual (such as $\partial_\mu\phi\partial^\mu\phi\sqrt{|g|}d^4x$ or $F^{\mu\nu}F_{\mu\nu}\sqrt{|g|}d^4x$) do change under Wick rotations because they explicitly involve the ϵ symbol, which is a pseudoscalar and hence transforms with an additional factor of i. The usual Euclidean action for gravity, $I_E = i\int d^4x_E\sqrt{g_E}R$, carries an extra i that can be viewed, in the language of forms, as resulting from the substitution $\epsilon_{abcd} \to i\epsilon_{abcd}$ when going from SO(3,1) to SO(4). Our point of view here is that this is not necessary (in odd dimensional spacetimes), but if one insists on introducing an i when the group is changed, the definition of the theory should be such that the Euclidean sector gives an imaginary phase for the path integral. This is consistent with the requirement that time reversal be equivalent to conjugation in the path integral (see, e.g., [13]).

The proposal of Ref. [6] for a gravitation theory in odd dimensions possesses a number of interesting features and, as we have shown here, its coupling constant is quantized under standard assumptions. An interesting consequence of the quantization of the gravitational constant is that the Hilbert space for 2+1 gravity with cosmological constant has finite-dimensional unitary representations [14].

In addition, the action (1), written in terms of the rescaled vielbein $e^a \to le^a$, has no dimensionful constants and all fields have canonical dimension 1. Furthermore, the fields ω^a_b and e^a are different components of the connection and the action describes a bona fide SO$(D+1)$ gauge system. The corresponding quantum theory would be renormalizable by power counting and possibly finite. Witten has shown this to be the case in three dimensions using the fact that the diffeomorphism constraints can be solved classically for $D = 3$, leaving only a discrete set degrees of freedom to be quantized [15].

For $D > 3$, however, the construction of a quantum theory might be a formidable task. In fact, a construction analogous to that of Ref. [15] may not exist at all.

Note added. After this paper was submitted, it was pointed out to the author that the Lagrangian (1) was also independently studied by Chamseddine [16], who concluded that the gravitational constant κ was not necessarily quantized. The argument there was based on the nonexistence of topologically nontrivial (large) gauge transformations in M. That argument clearly does not contradict our results, based on the independence of the action under changes of Ω.

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