Ramanujan J (2011) 26:155–183
DOI 10.1007/s11139-010-9258-x

On the analogue of Weil's converse theorem for Jacobi forms and their lift to half-integral weight modular forms

Yves Martin · Denis Osses

Received: 14 April 2010 / Accepted: 15 July 2010 / Published online: 2 April 2011

Abstract We generalize Weil's converse theorem to Jacobi cusp forms of weight k, index m and Dirichlet character χ over the group $\Gamma_0(N) \ltimes \mathbb{Z}^2$. Then two applications of this result are given; we generalize a construction of Jacobi forms due to Skogman and present a new proof for several known lifts of such Jacobi forms to half-integral weight modular forms.

Keywords Jacobi forms · Dirichlet series · Functional equations

Mathematics Subject Classification (2000) Primary 11F50 · Secondary 11F37 · 11F66

1 Introduction

A converse theorem in the theory of automorphic forms always refers to the equivalence of Dirichlet series satisfying certain analytic properties, on the one hand, and automorphic forms over some group, on the other. Most familiar is the converse theorem due to E. Hecke, which says that a sequence of complex numbers $\{c(n)\}_{n\geq 1}$ with $c(n) = O(n^{\sigma})$ for some $\sigma > 0$ defines a cuspidal modular form $f(\tau) = \sum_{n\geq 1} c(n) \exp(2\pi i n\tau)$ of weight $2k$ over the group $SL_2(\mathbb{Z})$ if and only if the completed Dirichlet series $\Lambda(f;s) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} c(n)n^{-s}$ admits a holomorphic continuation to $\mathbb{C}$ which is bounded on any vertical strip and satisfies

Y. Martin was partially supported by Fondecyt grant 1090094.

Y. Martin (✉) · D. Osses
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile
e-mail: ymartin@uchile.cl

D. Osses
e-mail: doosses@gmail.com

$\Lambda(f;s) = (-1)^k \Lambda(f;k-s)$. (Hecke's result is more general, but this statement suffices for our purposes.)

Weil's converse theorem is the analytic characterization all Dirichlet series constructed with the Fourier coefficients of modular forms over congruence subgroups $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$. It was obtained by André Weil [20] in 1967 and is a very significant generalization of the corresponding result for $N=1$, proved by Hecke [7] in 1936. Among the applications of Weil's theorem one finds Shimura's proof [16] of a remarkable correspondence between elliptic cusp forms of half-integral weight $k+1/2$ and integral even weight $2k$ cusp forms.

Analogues of these converse theorems have been proved in other contexts; for Maass wave forms by H. Maass [14], for certain Hilbert modular forms by K. Doi and H. Naganuma [4], and for automorphic integrals with rational period functions by J. Hawkins and M. Knopp [6]. G. Shimura mentions in [16] the corresponding converse theorem for half-integral weight modular forms (a detailed proof of the later can be found in J. Bruinier's work "Modulformen halbganzen Gewichts und Beziehungen zu Dirichletreihen", www.mathematik.tu-darmstadt.de/b̃ruinier/). The converse theorem for $GL(n)$ automorphic representations is a great achievement of several authors: H. Jacquet and R. Langlands [9] when $n=2$, H. Jacquet, I. Piatetski-Shapiro and J. Shalika [10] if $n=3$, J. Cogdell and I. Piatetski-Shapiro [3] for general n. The first named author found in [13] the analogue of the classical Hecke converse theorem for Jacobi cusp forms. The same result for Hilbert–Jacobi forms was established by K. Bringmann and S. Hayashida [2].

The purpose of this article is to find the generalization of Weil's converse theorem for Jacobi cusp forms over subgroups $\Gamma_0(N) \ltimes \mathbb{Z}^2$ of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and show that, as a simple consequence of this result, one gets several known lifts from Jacobi forms to elliptic half-integral weight modular forms.

Jacobi forms have been studied systematically in the last 25 years or so, even though particular examples of them have been used in mathematics and physics for about two centuries. They are automorphic forms of two complex variables which combine properties of modular forms and elliptic functions. Typical examples are theta series and Fourier–Jacobi coefficients of Siegel modular forms of degree two. Mostly, the existing literature is on Jacobi forms over the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. In this article, we study Jacobi forms over groups $\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle \exp(2\pi i/\beta)\rangle$, where N and β are positive integers.

In order to state our results precisely, fix positive integers k, m, β, N with $\beta | N$ and a Dirichlet character χ mod N. For each integer $1 \le \mu \le 2m$, consider a sequence of complex numbers $\{c_\mu(D)\}$ (resp., $\{d_\mu(D)\}$) indexed by the set of positive integers D such that $-D \equiv \beta\mu^2 \pmod{4m}$ (resp., $-D \equiv N\mu^2 \pmod{4m}$). Then put

$$f_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D) \exp\left(\frac{\pi i D}{2m}\tau\right), \qquad g_\mu(\tau) = \sum_{D=1}^{\infty} d_\mu(D) \exp\left(\frac{\pi i D}{2m\beta}\tau\right),$$

and build with them series of two variables

$$f(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\beta\tau, \beta z), \qquad g(\tau, z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\Theta_{m,\mu}\left(\frac{N}{\beta}\tau, Nz\right), \quad (1)$$

where $\Theta_{m,\mu}(\tau/2m, z/2m)$ is the theta function associated to the lattice $\mathbb{Z}$ translated by $\mu/2m$ (see Definition 2).

As in the classical case, it is possible to twist the Fourier series $f(\tau, z)$ and $g(\tau, z)$ by a Dirichlet character ψ mod M in a sensible way. Namely,

$$f_\psi(\tau, z) = \sum_{\mu=1}^{2mM} f_{\psi,\mu}(\tau)\Theta_{mM,\mu}(M\beta\tau, \beta z) \tag{2}$$

with $f_{\psi,\mu}(\tau) = \sum_{D=1}^{\infty} \psi(\frac{D+\beta\mu^2}{4m})c_\mu(D)\exp(\frac{\pi i D}{2m}\tau)$, and

$$g_\psi(\tau, z) = \sum_{\mu=1}^{2mM} g_{\psi,\mu}(\tau)\Theta_{mM,\mu}\left(\frac{MN}{\beta}\tau, Nz\right) \tag{3}$$

with $g_{\psi,\mu}(\tau) = \sum_{D=1}^{\infty} \psi(\frac{D+N\mu^2}{4m})d_\mu(D)\exp(\frac{\pi i D}{2m\beta}\tau)$.

We observe here that neither $f_{\psi,\mu}(\tau)$ nor $g_{\psi,\mu}(\tau)$ is the standard twist of the Fourier series $f_\mu(\tau)$ or $g_\mu(\tau)$ by the character ψ. Anyhow, using this definition we associate to $h_\psi(\tau, z) \in \{f_\psi(\tau, z), g_\psi(\tau, z)\}$ and each $1 \le \mu \le 2mM$ a completed Dirichlet series $\Lambda_{N,\mu}(h_\psi; s) = (\frac{2\pi}{M\sqrt{N}})^{-s}\Gamma(s)L_\mu(h_\psi; s)$ with

$$L_\mu(f_\psi; s) = \sum_{D=1}^{\infty} \psi\left(\frac{D+\beta\mu^2}{4m}\right)c_\mu(D)\left(\frac{D}{4m}\right)^{-s} \tag{4}$$

and

$$L_\mu(g_\psi; s) = \sum_{D=1}^{\infty} \psi\left(\frac{D+N\mu^2}{4m}\right)d_\mu(D)\left(\frac{D}{4m\beta}\right)^{-s}. \tag{5}$$

Finally, we denote by $\mathcal{M}$ any set of odd prime numbers or 4 which are relatively prime to N whose intersection with every arithmetic progression is not empty, and consider an analogue of the Fricke involution for Jacobi forms. The latter is a particular linear function denoted as $\widetilde{W}_N$ that takes Jacobi forms to Jacobi forms, preserves the weight but changes the index (see Sect. 3 for more details on these concepts).

The main theorem of this paper combines all these elements and characterize the Dirichlet series associated to Jacobi cusp forms.

Theorem 1 *Let*

$$\{c_\mu(D)\}_{D=1,-D\equiv\beta\mu^2\ (4m)}^{\infty} \quad \textit{and} \quad \{d_\mu(D)\}_{D=1,-D\equiv N\mu^2\ (4m)}^{\infty}$$

be two collections of sequences in $\mathbb{C}$ *indexed by* $1 \le \mu \le 2m$, *which satisfy the estimates* $c_\mu(D) = O(D^\sigma)$ *and* $d_\mu(D) = O(D^\sigma)$ *for some real number* $\sigma > 0$ *plus the compatibility condition* $(-1)^k d_{2m-\mu}(D) = \chi(-1)d_\mu(D)$ *for all* $D \ge 1$, $0 \le \mu \le m$ (*by definition*, $d_0(D) = d_{2m}(D)$ *for all* D). *Then the following two statements are equivalent*:

(i) *The Fourier series* $f(\tau, z)$ *is a Jacobi cusp form of weight* k, *index* $m\beta$ *and character* χ *over the group* $\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle \exp(2\pi i/\beta)\rangle$. *The function* $g(\tau, z)$ *is the image of* $f(\tau, z)$ *under the Fricke involution* $\widetilde{W}_N$.

(ii) *If* ψ *denotes the trivial character of conductor* $M = 1$ *or any primitive Dirichlet character of conductor* M *in* $\mathcal{M}$ *and* $1 \le \mu \le 2mM$, *each one of the series* $\Lambda_{N,\mu}(f_\psi, s)$, $\Lambda_{N,\mu}(g_{\overline{\psi}}, s)$ *admits a holomorphic continuation to the whole* s-*plane, they are bounded on any vertical strip, and satisfy the system of* $2mM$ *functional equations*

$$\Lambda_{N,a}(f_\psi; s) = i^k C_\psi \sqrt{\frac{\beta}{2mM\sqrt{N}}} \sum_{\mu=1}^{2mM} e\left(\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)$$

where $1 \le a \le 2mM$, $C_\psi = \chi(M)\psi(-N)\mathcal{G}_\psi \mathcal{G}_{\overline{\psi}}^{-1}$ *and* $\mathcal{G}_\psi$ *is the Gauss sum associated to* ψ. *Moreover, each* $L_\mu(f; s)$ $(1 \le \mu \le 2m)$ *converges absolutely at* $s = k - 1 - \epsilon$ *for some* $\epsilon > 0$.

Essentially, our proof is modeled after the one given by Weil. The key new idea is that the relevant characteristic twists are not the standard ones on each of the $2m$ Dirichlet series associated to a Jacobi form of index m introduced in [1] (see also [13]), but the twists of a Fourier series in two variables mentioned above (see also Definition 3). These objects have been investigated in [8] as they play a role in the study of certain Rankin–Selberg convolution of Siegel modular forms. The rest of the proof is an adaptation of Weil's argument which presents some technical difficulties due to the nature of Jacobi forms. For example, one has to deal with two different integral transforms (as opposed to one in the elliptic case) in order to represent the two sides of the functional equations.

Then we give, as applications of this theorem, very simple proofs of certain statements known otherwise by some complicated computations. Indeed, by elementary arguments we derive from our theorem and basic properties of theta functions the following: (i) a new proof for the construction of distinct Jacobi forms from the theta decomposition of a given one due to H. Skogman [17], (ii) its generalization to higher levels, (iii) a new proof of the lift from Jacobi forms to half-integral weight modular forms used in the argument for the Saito–Kurokawa conjecture presented in [5], and (iv) a family of similar lifts labeled by the divisors of the index of a Jacobi cusp form. More precisely, we get

Corollary 1 *Let* $f(\tau, z)$ *be a Jacobi cusp form as in part* (i) *of Theorem* 1 *with theta series expansion* (1). *Let* δ *be a positive divisor of* m. *Then*

$$F(\tau, z) = \sum_{\substack{\mu=1 \\ \delta|\mu}}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\beta\tau, \beta z)$$

is a Jacobi cusp form of weight k, *index* $m\beta$ *and character* χ *over the group* $\Gamma_0(N') \ltimes (\mathbb{Z} \times (\delta\beta)^{-1}\mathbb{Z})\langle \exp(2\pi i/\delta\beta)\rangle$, *where* $N' = N\delta/\gcd(\delta, N/\beta)$.

This result for $N = 1$ and m square-free was established by Skogman in [17]. His proof is based on explicit formulas for the $\Gamma_0(2m)$-action on the space of theta series $\bigoplus_\mu \mathbb{C}\Theta_{m,\mu}(\tau, z)$. Our approach makes those formulas unnecessary as we work with the set of functional equations determined in Theorem 1.

A particular case of the previous corollary (namely, $\delta = m$) can be used to give a new proof of a classical lift from Jacobi cusp forms to elliptic half-integral weight cusp forms.

Corollary 2 *Let $f(\tau, z)$ be a Jacobi cusp form as in the previous corollary subject to the condition $\beta = 1$. Then*

$$F_{2m}(\tau) = \sum_{\mu=1}^{2m} f_\mu(4m\tau)$$

is a cusp form of weight $k - \frac{1}{2}$ and character $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \mapsto \chi(\gamma)(\frac{-1}{d})^k(\frac{Nm}{d})$ over $\Gamma_0(4Nm)$.

Here $(\frac{*}{d})$ denotes the Legendre symbol as defined, for example, in [15, p. 82]. This result for $N = 1$ is discussed in [5], and plays an important part in the proof of the Saito–Kurokawa lift. In a very similar way, we can derive from Corollary 1 a different, less known, lift.

Corollary 3 *Let $f(\tau, z)$ be a Jacobi cusp form as above and assume $4|Nm$. Then both*

$$F_m(\tau) = \sum_{\substack{\mu=1 \\ \mu\equiv 0\ (2)}}^{2m} f_\mu(4m\tau) \quad \text{and} \quad F_{2m}(\tau) - F_m(\tau) = \sum_{\substack{\mu=1 \\ \mu\equiv 1\ (2)}}^{2m} f_\mu(4m\tau)$$

are cusp forms of weight $k - \frac{1}{2}$ and character $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \mapsto \chi(\gamma)(\frac{-1}{d})^k(\frac{Nm}{d})$ over $\Gamma_0(4Nm)$.

At the end of this work, we indicate how to get a similar statement for the function $F_\delta(\tau) = \sum_{\mu=1,\ \delta|\mu}^{2m} f_\mu(4m\tau)$ where δ is any positive divisor of m. Moreover, we write a few words about the compatibility of the lift $F_m(\tau)$ with the action of Hecke operators. The maps $f(\tau, z) \mapsto F_\delta(\tau)$ mentioned above have been investigated thoroughly by N.-P. Skoruppa [18] in the case $N = 1$ using a different approach. Also, there are several statements on this topic in [11] and in Sect. 4 of [12].

This article is organized as follows: In the next section, we recall basic concepts from the theory of Jacobi forms with level, in particular, the theta decomposition at infinity of a Jacobi cusp form. Then we collect some technical estimates which are necessary for later manipulations of integrals and infinite series. In Sect. 3, we study the character twists and Fricke involution of Jacobi forms. In Sect. 4, we set the stage for the statement and proof of the converse theorem. Namely, we associate a finite set of Dirichlet series to each Fourier series in two variables of a certain type and to their character twists. Then we exhibit an integral representation of such Dirichlet series. Our main theorem is proved in Sect. 5. In Section 6, we obtain the applications mentioned above.

Notation Throughout the article, we write I_2 for the 2 by 2 identity matrix, $\det Y$ for the determinant of a matrix Y and $e(z)$ for the exponential function $e^{2\pi i z}$ $(z \in \mathbb{C})$. We often write $e^a(z)$ instead of $e(az)$ when a is a rational number. If t is a positive integer, ζ_t denotes a primitive tth root of unity. In this article, the symbol χ denotes both a Dirichlet character mod N and the linear representation $\chi : \Gamma_0(N) \to \mathbb{C}^*$ given by $\chi(\gamma) := \chi(d)$ for the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We write $a|b$ for positive integers a, b, whenever a divides b.

2 Basic definitions

The real Jacobi group G^J is the set of triples $[\gamma, Y, \zeta]$ where

$$\begin{aligned} \gamma &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \qquad Y = (\lambda, \nu) \in \mathbb{R}^2, \\ \zeta &\in S^1 := \left\{ w \in \mathbb{C} \mid |w| = 1 \right\}. \end{aligned} \tag{6}$$

Its product is

$$[\gamma_1, Y_1, \zeta_1][\gamma_2, Y_2, \zeta_2] := \left[\gamma_1\gamma_2, Y_1\gamma_2 + Y_2, \zeta_1\zeta_2 e\left(\det \begin{pmatrix} Y_1\gamma_2 \\ Y_2 \end{pmatrix} \right) \right].$$

There is a group action of G^J on $\mathcal{H} \times \mathbb{C}$, where $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$. Namely, any group element $h = [\gamma, Y, \zeta]$ as in (6) sends the pair $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ to

$$h(\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \nu}{c\tau + d} \right).$$

Now fix positive integers k and m. We consider a map $j_{k,m} : G^J \times \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ given by

$$j_{k,m}(h, \tau, z) := \zeta^m (c\tau + d)^{-k} e^m \left(\frac{-c(z + \lambda\tau + \nu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\nu \right).$$

This is a 1-cocycle, and it is used to define a group action of G^J on the set of holomorphic functions on $\mathcal{H} \times \mathbb{C}$. Indeed, any such $f(\tau, z) : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is send by $h \in G^J$ to $f|_{k,m}[h](\tau, z) := j_{k,m}(h, \tau, z) f(h(\tau, z))$.

For positive integers α, N, we write

$$\Gamma_0(\alpha, N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid N|c,\ \alpha|b \right\}.$$

As usual, $\Gamma_0(N) = \Gamma_0(1, N)$. Clearly, $SL_2(\mathbb{Z})$ acts on $\mathbb{Z} \times \mathbb{Z}$ via multiplication on the right, and we may consider the semidirect product $\Gamma_0(N) \ltimes \mathbb{Z}^2$. In this article, we are mainly interested in Jacobi forms over such a group, but for technical reasons it is better to study Jacobi forms over more general discrete subgroups of G^J.

Let α, β, η and N be positive integers such that $\eta\beta|N$. We consider the groups $\Gamma = \Gamma_0(\alpha, N)$, $T = \eta\alpha^{-1}\mathbb{Z} \times \beta^{-1}\mathbb{Z} \subseteq \mathbb{Q} \times \mathbb{Q}$ and $\langle\zeta_{\alpha\beta}\rangle \subseteq S^1$, where the latter is the cyclic group generated by a primitive $\alpha\beta$th root of unity $\zeta_{\alpha\beta}$. Then $\Gamma \ltimes (T \cdot \langle\zeta_{\alpha\beta}\rangle)$ is a discrete subgroup of the real Jacobi group G^J which we simply denote as $\Gamma \ltimes T\langle\zeta_{\alpha\beta}\rangle$.

Definition 1 Let $k, m, \alpha, \beta, \eta$ and N be positive integers such that $\eta\beta|N$. Let Γ, T be as above and χ a Dirichlet character mod N. A Jacobi form of weight k, index $m\alpha\beta$ and character χ over the discrete group $\Gamma \ltimes T\langle\zeta_{\alpha\beta}\rangle$ is any holomorphic function $f(\tau, z) : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following conditions:

(i) If $h = [\gamma, Y, \zeta] \in \Gamma \ltimes T\langle\zeta_{\alpha\beta}\rangle$ then

$$f|_{k,m\alpha\beta}[h](\tau, z) = \chi(\gamma) f(\tau, z).$$

(ii) For every σ in $SL_2(\mathbb{Q})$ the function $f|_{k,m\alpha\beta}[\sigma^{-1}, 0, 0, 1](\tau, z)$ has a series representation

$$f|_{k,m\alpha\beta}[\sigma^{-1}, 0, 0, 1](\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4m\alpha\beta n t_\sigma \geq r^2}} c_\sigma(n, r) e\left(\frac{n}{t_\sigma}\tau\right) e\left(\frac{r}{t_\sigma}z\right)$$

for some complex numbers $c_\sigma(n, r)$, where t_σ is a positive integer which depends on $\sigma^{-1}(\infty) \in \mathbb{Q} \cup \{\infty\}$.

The set of all these functions is a finite dimensional $\mathbb{C}$-vector space (see, for example, [5, p. 10]) which we denote as $J_{k,m\alpha\beta,\chi}(\Gamma \ltimes T\langle\zeta_{\alpha\beta}\rangle)$. If a Jacobi form as above has all its series representations (ii) indexed by integers n, r such that $4m\alpha\beta n t_\sigma > r^2$, we call it a Jacobi cusp form. The set of all of them is a subspace which we denote as $J^{\text{cusp}}_{k,m\alpha\beta,\chi}(\Gamma \ltimes T\langle\zeta_{\alpha\beta}\rangle)$. In the particular case $\alpha\beta = 1$, we just write $J^{\text{cusp}}_{k,m,\chi}(\Gamma \ltimes T)$.

Any Jacobi form $f(\tau, z)$ is a finite sum of theta series multiplied by certain functions on the upper half-plane. Since such decomposition plays an important role in the following, we recall it with some detail.

Definition 2 Let L be a positive integer and μ any integer. The theta function $\Theta_{L,\mu}(\tau, z)$ is

$$\Theta_{L,\mu}(\tau, z) := \sum_{\substack{l\in\mathbb{Z} \\ l\equiv\mu\ (2L)}} e\left(\frac{l^2}{4L}\tau\right) e(lz).$$

Next we observe that any Jacobi form $f(\tau, z)$ as in Definition 1 is invariant under translations $(\tau, z) \to (\tau + \alpha, z)$ and $(\tau, z) \to (\tau, z + 1/\beta)$. Hence

$$f(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r) e\left(\frac{n}{\alpha}\tau\right) e(r\beta z) \tag{7}$$

for some complex numbers $c(n,r)$. By part (ii) in Definition 1, one has $c(n,r)=0$ whenever $4mn<\beta r^2$. On the other hand, the functional equation

$$f(\tau,z)=e^{m\beta}\big(\eta^2\alpha^{-1}\lambda^2\tau+2\eta\lambda z\big)f\big(\tau,z+\eta\alpha^{-1}\lambda\tau\big),$$

valid for any $\lambda\in\mathbb{Z}$ by (i) of the same definition, yields the relation $c(n,r)=c(n+r\beta\eta\lambda+m\beta\eta^2\lambda^2,r+2m\eta\lambda)$. Therefore, if we set $c_r(D):=c(n,r)$ whenever $D=4mn-\beta r^2$, the previous equation is equivalent to the identity $c_r(D)=c_{r'}(D)$ for any integers r,r' such that $r\equiv r' \pmod{2m\eta}$. If we use this equation in (7) we can write

$$f(\tau,z)=\sum_{\mu=1}^{2m\eta}f_\mu(\tau)\Theta_{m\eta,\mu}\left(\frac{\eta\beta}{\alpha}\tau,\beta z\right)\quad\text{where } f_\mu(\tau)=\sum_{D=0}^{\infty}c_\mu(D)e\left(\frac{D}{4m\alpha}\tau\right)$$

(see [5, p. 58] for the explicit argument). This is called the theta decomposition of $f(\tau,z)$ (at infinity).

An important technical ingredient of this work is the following parameterization of $\mathcal{H}\times\mathbb{C}$ by real coordinates; a pair (τ,z) is identified with the 4-tuple of real numbers (x,y,p,q) determined by $\tau=x+iy$ and $z=p\tau+q$.

A straightforward computation yields that any Jacobi cusp form $f(\tau,z)$ in $J^{\text{cusp}}_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ defines a function $y^{k/2}|e^{m\beta}(pz)f(\tau,z)|$ on $\mathcal{H}\times\mathbb{C}$, invariant under the action of the group. Since $f(\tau,z)$ is cuspidal, such a real-valued continuous function is well-defined on the compactification of the quotient space $\mathcal{H}\times\mathbb{C}/\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle$. Thus $y^{k/2}|e^{m\beta}(pz)f(\tau,z)|$ is bounded. In complete analogy with the case of elliptic cusp forms, this bound yields an estimate on the size of the Fourier coefficients of $f(\tau,z)$.

Lemma 1 *Let $f(\tau,z)\in J^{\text{cusp}}_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ with Fourier series representation (7). Then there exists a real constant K which depends on $f(\tau,z)$ such that*

$$\big|c(n,r)\big|=\big|c_r(D)\big|\le K\,D^{k/2}\quad\textit{for all } n,r.$$

For technical reasons, we recall some analytic consequences of an estimate like this.

Lemma 2 *For any $1\le\mu\le 2m$, let $\{c_\mu(D)\}$ be a sequence in $\mathbb{C}$ indexed by the set of positive integers D such that $-D\equiv\beta\mu^2 \pmod{2m}$. To each one of these sequences associate the series*

$$f_\mu(\tau)=\sum_{D=1}^{\infty}c_\mu(D)e\left(\frac{D}{4m}\tau\right)\quad(\tau\in\mathcal{H}).$$

If $c_\mu(D)=O(D^\sigma)$ for some $\sigma>0$ then:

(i) *Each series converges absolutely and uniformly on any compact subset of $\mathcal{H}$. In particular, every $f_\mu(\tau)$ defines a holomorphic function on $\mathcal{H}$.*

(ii) *Each series*

$$f_\mu(\tau)\Theta_{m,\mu}(\beta\tau,\beta z)=\sum_{\substack{D,r\in\mathbb{Z}\\ r\equiv\mu\ (2m)\\ D=4mn-\beta r^2>0}} c_\mu(D)e(n\tau)e(r\beta z)$$

is absolutely and uniformly convergent on any compact subset of $\mathcal{H}\times\mathbb{C}$. *Hence the function* $f(\tau,z)=\sum_{\mu=1}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\beta\tau,\beta z)$ *is holomorphic on* $\mathcal{H}\times\mathbb{C}$.

(iii) *The estimates*

$$e^{m\beta}(pz)f_\mu(\tau)\Theta_{m,\mu}(\beta\tau,\beta z)=O\big(y^{-\sigma-3/2}\big)\quad \text{as } y\to 0,$$

$$e^{m\beta}(pz)f_\mu(\tau)\Theta_{m,\mu}(\beta\tau,\beta z)=O\left(e\left(\frac{iy}{4m\alpha\beta}\right)\right)\quad \text{as } y\to\infty$$

hold uniformly on $x=\mathrm{Re}(\tau)$.

Proof This lemma is a straightforward generalization of a similar statement proved in [15, p. 117] for elliptic modular forms; see also [13, p. 186]. □

Lemma 3 *Let* $f(\tau,z):\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ *be a holomorphic function satisfying part* (i) *of Definition* 1 *over the group* $\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle$. *If the estimate* $e^{m\beta}(pz)f(\tau,z)=O(y^{-\nu})$ *as* $y\to 0$ *holds uniformly with respect to* $\mathrm{Re}(\tau)$ *for some positive real number* ν, *then* $f(\tau,z)$ *is a Jacobi form in* $J_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$. *Moreover, if* $\nu<k-1/2$ *then* $f(\tau,z)$ *is a cusp form.*

Sketch of the Proof Let σ be any matrix in $SL_2(\mathbb{Q})$. As above, the holomorphicity and symmetries of $f(\tau,z)$ under $\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle$ yield a theta decomposition for $f|_{k,m\beta}[\sigma^{-1},0,0,1](\tau,z)$. Namely,

$$f|_{k,m\beta}\big[\sigma^{-1},0,0,1\big](\tau,z)=\sum_{\mu=1}^{2m\beta t_\sigma N_\sigma} f_{\sigma,\mu}\left(\frac{\tau}{t_\sigma}\right)\Theta_{m\beta t_\sigma N_\sigma,\mu}\left(\frac{N_\sigma\tau}{t_\sigma},\frac{z}{t_\sigma}\right)$$

for some positive integers N_σ, t_σ depending on σ, and component functions

$$f_{\sigma,\mu}(\tau)=\sum_{D\in\mathbb{Z}} c_{\sigma,\mu}(D)e\left(\frac{D}{4m\beta t_\sigma}\tau\right)$$

determined by certain Fourier coefficients $c_{\sigma,\mu}(D)$. Using that for any fixed τ in $\mathcal{H}$ one has

$$\int_{p=0}^{N_\sigma}\int_{q=0}^{t_\sigma}\Theta_{m\beta t_\sigma N_\sigma,\mu}\left(\frac{N_\sigma\tau}{t_\sigma},\frac{z}{t_\sigma}\right)\overline{\Theta_{m\beta t_\sigma N_\sigma,\nu}\left(\frac{N_\sigma\tau}{t_\sigma},\frac{z}{t_\sigma}\right)}e\big(2m\beta p^2 iy\big)\,dp\,dq$$
$$=\frac{t_\sigma}{\sqrt{4m\beta y}}\delta_{\mu,\nu}$$

(where $\delta_{\mu,\nu}=1$ if $\mu=\nu$ and zero otherwise), we get an integral representation for the coefficients of $f_{\sigma,\mu}(\tau)$. Indeed, for any τ_0 in $\mathcal{H}$

$$\begin{aligned} c_{\sigma,\mu}(D) = & \frac{1}{2\sqrt{m\beta}t_\sigma^3}\int_{\tau_0}^{\tau_0+4m\beta t_\sigma^2}\int_{p=0}^{N_\sigma}\int_{q=0}^{t_\sigma} f|_{k,m\beta}\big[\sigma^{-1},0,0,1\big](\tau,z) \\ & \times \overline{\Theta_{m\beta t_\sigma N_\sigma,\mu}\left(\frac{N_\sigma\tau}{t_\sigma},\frac{z}{t_\sigma}\right)e\big(2m\beta p^2 iy\big)} \\ & \times e\left(-\frac{D}{4m\beta t_\sigma^2}\tau\right)y^{1/2}\,dp\,dq\,d\tau. \end{aligned} \tag{8}$$

Using Definition 2, one can prove that $|\Theta_{m\beta t_\sigma N_\sigma,\mu}(\frac{N_\sigma\tau}{t_\sigma},\frac{z}{t_\sigma})|e(m\beta p^2 iy)$ is bounded as $y\to\infty$. On the other hand, for any $\sigma^{-1}=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ with $c\neq 0$, an argument like the one in [15, p. 42] yields $e^{m\beta}(pz)f|_{k,m\beta}[\sigma^{-1},0,0,1](\tau,z)=O(y^{\nu-k})$ as $y\to\infty$ uniformly on $\mathrm{Re}(\tau)$. Therefore, (8) and these estimates imply the existence of positive real constants K and y_K such that

$$\big|c_{\sigma,\mu}(D)\big|\leq \big(2\sqrt{m\beta}KN_\sigma\big)y_0^{\nu-k+\frac{1}{2}}e^{\pi Dy_0/2m\beta t_\sigma^2}$$

whenever τ_0 is chosen with $y_0=\mathrm{Im}(\tau_0)>y_K$. If $D<0$, the right hand side of this inequality goes to 0 as $y_0\to\infty$. Thus $c_{\sigma,\mu}(D)=0$ for every $D<0$. This shows that $f(\tau,z)$ satisfies Condition (ii) in Definition 1, and therefore $f(\tau,z)\in J_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$. If $\nu<k-1/2$, the previous inequality also yields $c_{\sigma,\mu}(0)=0$ for every σ and μ. Hence $f(\tau,z)$ is a Jacobi cusp form. □

3 Characteristic twists of Jacobi forms and a generalization of the Fricke involution

As in the previous section, we fix positive integers k,m,α,β,η and N such that $\eta\beta|N$. Also we fix a Dirichlet character χ mod N.

Definition 3 Let $f(\tau,z)\in J_{k,m\alpha\beta,\chi}(\Gamma_0(\alpha,N)\ltimes(\alpha^{-1}\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_{\alpha\beta}\rangle)$ with Fourier series representation (7). Let ψ be a primitive Dirichlet character mod M with $\gcd(N,M)=1$. We associate to $f(\tau,z)$ and ψ the holomorphic function on $\mathcal{H}\times\mathbb{C}$ defined by the series

$$f_\psi(\tau,z):=\sum_{n,r\in\mathbb{Z}}\psi(n)c(n,r)e\left(\frac{n}{\alpha}\tau\right)e(r\beta z).$$

Lemma 4 *Consider $f(\tau,z)$ and ψ as in Definition 3. Then*

$$f_\psi(\tau,z)\in J_{k,m\alpha\beta,\chi\psi^2}\big(\Gamma_0\big(\alpha,NM^2\big)\ltimes\big(M\alpha^{-1}\mathbb{Z}\times\beta^{-1}\mathbb{Z}\big)\langle\zeta_{\alpha\beta}\rangle\big).$$

If $f(\tau,z)$ is a cuspidal Jacobi form, then so is $f_\psi(\tau,z)$.

Proof For any x in $\mathbb{R}$, set $\theta(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and $\widetilde{\theta}(x) = [\theta(x), 0, 0, 1]$. Then $\widetilde{\theta}(x) \in G^J$ and, clearly,

$$f|_{k,m\alpha\beta}\left[\widetilde{\theta}\left(\frac{\alpha a}{M}\right)\right](\tau, z) = f\left(\tau + \frac{\alpha a}{M}, z\right) = \sum_{n,r\in\mathbb{Z}} e\left(\frac{an}{M}\right)c(n,r)e\left(\frac{n}{\alpha}\tau\right)e(r\beta z)$$

for any $a \in \mathbb{Z}$. Therefore,

$$\sum_{a=0}^{M-1} \overline{\psi}(a) f|_{k,m\alpha\beta}\left[\widetilde{\theta}\left(\frac{\alpha a}{M}\right)\right](\tau, z) = \mathcal{G}_{\overline{\psi}} f_\psi(\tau, z), \tag{9}$$

where $\mathcal{G}_\psi = \sum_{a=0}^{M-1} \psi(a)e(\frac{a}{M})$ is the Gauss sum associated to the primitive Dirichlet character ψ.

For convenience set $L = NM^2$ and consider any $\gamma = \left(\begin{smallmatrix} a & b \\ cL & d \end{smallmatrix}\right) \in \Gamma_0(\alpha, L)$. It is easy to see that $\gamma' = \theta(\frac{\alpha u}{M})\gamma\theta(\frac{\alpha d^2 u}{M})^{-1}$ is in $\Gamma_0(\alpha, L) \subseteq \Gamma_0(\alpha, N)$ and $\chi(\gamma') = \chi(\gamma) = \chi(d)$. Thus, for any group element $[\gamma, (\lambda, \nu), \zeta_{\alpha\beta}^j]$ in $\Gamma_0(\alpha, L) \ltimes (M\alpha^{-1}\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_{\alpha\beta}\rangle$ one has

$$\begin{aligned} &f|_{k,m\alpha\beta}\left[\widetilde{\theta}\left(\frac{\alpha u}{M}\right)\right][\gamma, (\lambda, \nu), \zeta_{\alpha\beta}^j](\tau, z) \\ &= f|_{k,m\alpha\beta}\left[\gamma', \left(\lambda, \nu - \frac{\lambda\alpha\, d^2 u}{M}\right), \zeta_{\alpha\beta}^j\right]\left[\theta\left(\frac{\alpha\, d^2 u}{M}\right), (0,0), 1\right](\tau, z) \\ &= \chi(\gamma) f|_{k,m\alpha\beta}\left[\widetilde{\theta}\left(\frac{\alpha\, d^2 u}{M}\right)\right](\tau, z). \end{aligned}$$

This equation and (9) yield $f_\psi|_{k,m\alpha\beta}[\gamma, (\lambda, \nu), \zeta_{\alpha\beta}^j](\tau, z) = \chi\psi^2(\gamma) f_\psi(\tau, z)$. Thus $f_\psi(\tau, z)$ is a holomorphic function on $\mathcal{H} \times \mathbb{C}$ which satisfies part (i) of Definition 1. Observe next, for $\sigma \in SL_2(\mathbb{Q})$ and $0 \le a \le M - 1$, that

$$f|_{k,m\alpha\beta}\left[\widetilde{\theta}\left(\frac{\alpha a}{M}\right)\right]\Big|_{k,m\alpha\beta}[\sigma, 0, 0, 1](\tau, z) = f|_{k,m\alpha\beta}\left[\theta\left(\frac{\alpha a}{M}\right)\sigma, 0, 0, 1\right](\tau, z)$$

must have a Fourier series representation like the one in part (ii) of Definition 1. Hence the same holds for the function $f_\psi|_{k,m\alpha\beta}[\sigma^{-1}, 0, 0, 1](\tau, z)$. This proves the first half of the lemma. The statement about the cuspidal form $f(\tau, z)$ is handled similarly. □

Definition 4 For any Jacobi form $f(\tau, z)$ and positive real number L define

$$U_L f(\tau, z) := f(\tau, Lz).$$

Notice that the operator U_L maps any Jacobi (cusp) form of index m onto a Jacobi (cusp) form of index mL^2. The weight remains the same and the corresponding groups change a bit.

Definition 5 Any matrix $W_L = \left(\begin{smallmatrix} 0 & -1 \\ L & 0 \end{smallmatrix}\right)$ in $GL_2^+(\mathbb{R})$ with L a positive integer is called a Fricke involution. It normalizes $\Gamma_0(L)$. We define the linear operator $|_{k,m}[\widetilde{W}_L]$ on Jacobi forms of weight k and index m by

$$f|_{k,m}[\widetilde{W}_L](\tau, z) := L^{-k/2}\tau^{-k}e^{mL}\left(-\frac{z^2}{\tau}\right)f\left(\frac{-1}{L\tau}, \frac{z}{\tau}\right).$$

It is immediate to check that

$$f|_{k,m}[\widetilde{W}_L](\tau, z) = (U_{\sqrt{L}}f)|_{k,mL}\left[\begin{pmatrix} 0 & -1/\sqrt{L} \\ \sqrt{L} & 0 \end{pmatrix}, 0, 0, 1\right](\tau, z). \tag{10}$$

Lemma 5 *Let β, η, L be positive integers such that $\eta\beta|L$. Let ρ be a Dirichlet character mod L. If $f(\tau, z) \in J_{k,m\beta,\rho}(\Gamma_0(L) \ltimes (\eta\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ then*

$$f|_{k,m\beta}[\widetilde{W}_L](\tau, z) \in J_{k,m\beta L,\overline{\rho}}\big(\Gamma_0(\beta, L) \ltimes \big(\beta^{-1}\mathbb{Z} \times (L/\eta)^{-1}\mathbb{Z}\big)\langle\zeta_{\beta L/\eta}\rangle\big).$$

If $f(\tau, z)$ is a cuspidal Jacobi form, then so is $f|_{k,m\beta}[\widetilde{W}_L](\tau, z)$.

Proof Clearly, $f|_{k,m\beta}[\widetilde{W}_L](\tau, z)$ is a holomorphic function on $\mathcal{H} \times \mathbb{C}$. Using direct computations, it is easy to see for $\left(\begin{smallmatrix} a & b \\ cL & d \end{smallmatrix}\right)$ in $\Gamma_0(\beta, L) \subseteq \Gamma_0(L)$ that

$$\begin{aligned}
&f|_{k,m\beta}[\widetilde{W}_L]|_{k,m\beta L}\left[\begin{pmatrix} a & b \\ cL & d \end{pmatrix}, 0, 0, 1\right](\tau, z)\\
&\quad= (U_{\sqrt{L}}f)|_{k,m\beta L}\left[\begin{pmatrix} d & -c \\ -bL & a \end{pmatrix}, 0, 0, 1\right]\left[\begin{pmatrix} 0 & -1/\sqrt{L} \\ \sqrt{L} & 0 \end{pmatrix}, 0, 0, 1\right](\tau, z)\\
&\quad= U_{\sqrt{L}}\left(f|_{k,m\beta}\left[\begin{pmatrix} d & -c \\ -bL & a \end{pmatrix}, 0, 0, 1\right]\right)\Bigg|_{k,m\beta L}\left[\begin{pmatrix} 0 & -1/\sqrt{L} \\ \sqrt{L} & 0 \end{pmatrix}, 0, 0, 1\right](\tau, z)\\
&\quad= \overline{\rho}(d)f|_{k,m\beta}[\widetilde{W}_L](\tau, z).
\end{aligned} \tag{11}$$

On the other hand, $f|_{k,m\beta}[\widetilde{W}_L]|_{k,m\beta L}[I_2, \lambda, \nu, \zeta_{\beta L}^j](\tau, z)$ is equal to

$$(U_{\sqrt{L}}f)|_{k,m\beta L}\left[I_2, -\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}, \zeta_{\beta L}^j\right]\left[\begin{pmatrix} 0 & -\frac{1}{\sqrt{L}} \\ \sqrt{L} & 0 \end{pmatrix}, 0, 0, 1\right](\tau, z),$$

and for $\lambda \in \beta^{-1}\mathbb{Z}$, $L\nu \in \eta\mathbb{Z}$, we have

$$\begin{aligned}
&(U_{\sqrt{L}}f)|_{k,m\beta L}\left[I_2, -\sqrt{L}\nu, \frac{\lambda}{\sqrt{L}}, \zeta_{\beta L}^j\right](\tau, z)\\
&\quad= e^{m\beta}\big((-L\nu)^2\tau - 2L\nu\sqrt{L}z - L\nu\lambda\big)f(\tau, \sqrt{L}z - L\nu\tau + \lambda) = (U_{\sqrt{L}}f)(\tau, z).
\end{aligned}$$

These two identities plus (10) yield

$$f|_{k,m\beta}[\widetilde{W}_L]|_{k,m\beta L}\big[I_2, \lambda, \nu, \zeta_{\beta L}^j\big](\tau, z) = f|_{k,m\beta}[\widetilde{W}_L](\tau, z) \tag{12}$$

for any $(\lambda,\nu)\in\beta^{-1}\mathbb{Z}\times(L/\eta)^{-1}\mathbb{Z}$. In order to check that $f|_{k,mB}[\widetilde{W}_L](\tau,z)$ satisfies Condition (ii) of Definition 1, notice that for any $\sigma^{-1}=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ in $SL_2(\mathbb{Q})$ one has

$$\begin{aligned}
&f|_{k,m\beta}[\widetilde{W}_L]|_{k,m\beta L}\big[\sigma^{-1},0,0,1\big](\tau,z)\\
&\quad=(U_{\sqrt{L}}f)|_{k,m\beta L}\left[\begin{pmatrix}-c/L&-d\\a&bL\end{pmatrix}\begin{pmatrix}\sqrt{L}&0\\0&1/\sqrt{L}\end{pmatrix},0,0,1\right](\tau,z)\\
&\quad=L^{k/2}\left(f|_{k,m\beta}\left[\begin{pmatrix}-c/L&-d\\a&bL\end{pmatrix},0,0,1\right]\right)(L\tau,Lz).
\end{aligned}$$

Therefore, $f|_{k,m\beta}[\widetilde{W}_L]|_{k,m\beta L}[\sigma^{-1},0,0,1](\tau,z)$ has a Fourier series representation as required. This fact plus (11) and (12) imply that the function $f|_{k,m\beta}[\widetilde{W}_L](\tau,z)$ is a Jacobi form as stated. The second part of the lemma is proved in a similar way. □

Lemma 6 *Given any $f(\tau,z)$ in $J_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\eta\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ and a primitive Dirichlet character ψ mod M with* $\gcd(N,M)=1$, *set $g(\tau,z):=f|_{k,m\beta}[\widetilde{W}_N](\tau,z)$. Then*

$$f_\psi|_{k,m\beta}[\widetilde{W}_{NM^2}](\tau,z)=C_\psi g_{\overline{\psi}}(\tau,Mz)\quad \textit{with } C_\psi=C_{N,\psi}=\chi(M)\psi(-\beta N)\frac{\mathcal{G}_\psi}{\mathcal{G}_{\overline{\psi}}}.$$

Proof For an integer u with $\gcd(u,M)=1$, pick integers x and y such that $xM-yu\beta N=1$. Then $\left(\begin{smallmatrix}M&-y\beta\\-uN&x\end{smallmatrix}\right)\in\Gamma_0(\beta,N)$ and

$$\theta\left(\frac{u}{M}\right)\begin{pmatrix}0&-1/M\sqrt{N}\\M\sqrt{N}&0\end{pmatrix}=\begin{pmatrix}0&-1/\sqrt{N}\\\sqrt{N}&0\end{pmatrix}\begin{pmatrix}M&-y\beta\\-uN&x\end{pmatrix}\theta\left(\frac{\beta y}{M}\right).$$

Since $U_{M\sqrt{N}}(f|_{k,m\beta}[\widetilde{\theta}(\frac{u}{M})])(\tau,z)=(U_{M\sqrt{N}}f)|_{k,m\beta NM^2}[\widetilde{\theta}(\frac{u}{M})](\tau,z)$, the previous identity yields

$$\begin{aligned}
&f|_{k,m\beta}\left[\widetilde{\theta}\left(\frac{u}{M}\right)\widetilde{W}_{NM^2}\right](\tau,z)\\
&\quad=(U_{M\sqrt{N}}f)|_{k,m\beta NM^2}\left[\begin{pmatrix}0&-1/\sqrt{N}\\\sqrt{N}&0\end{pmatrix}\begin{pmatrix}M&-y\beta\\-uN&x\end{pmatrix}\theta\left(\frac{\beta y}{M}\right),0,0,1\right](\tau,z)\\
&\quad=U_M\left(f|_{k,m\beta}[\widetilde{W}_N]|_{k,m\beta N}\left[\begin{pmatrix}M&-y\beta\\-uN&x\end{pmatrix},0,0,1\right]\Big|_{k,m\beta N}\left[\widetilde{\theta}\left(\frac{\beta y}{M}\right)\right]\right)(\tau,z)\\
&\quad=U_M\left(g|_{k,m\beta N}\left[\begin{pmatrix}M&-y\beta\\-uN&x\end{pmatrix},0,0,1\right]\Big|_{k,m\beta N}\left[\widetilde{\theta}\left(\frac{\beta y}{M}\right)\right]\right)(\tau,z).
\end{aligned}$$

On the other hand, by Lemma 5, we know that $g(\tau,z)=f|_{k,m\beta}[\widetilde{W}_N](\tau,z)$ is in $J_{k,m\beta N,\overline{\chi}}(\Gamma_0(\beta,N)\ltimes(\beta^{-1}\mathbb{Z}\times(N/\eta)^{-1}\mathbb{Z})\langle\zeta_{\beta N/\eta}\rangle)$. Hence

$$f|_{k,m\beta}\left[\widetilde{\theta}\left(\frac{u}{M}\right)\widetilde{W}_{NM^2}\right](\tau,z)=\chi(M)g|_{k,m\beta N}\left[\widetilde{\theta}\left(\frac{\beta y}{M}\right)\right](\tau,Mz),$$

using that $\overline{\chi}(x) = \chi(M)$. From this relation and (9), one gets

$$f_\psi|_{k,m\beta}[\widetilde{W}_{NM^2}](\tau,z) = \frac{1}{\mathcal{G}_{\overline{\psi}}}\sum_{u=0}^{M-1}\overline{\psi}(u) f|_{k,m\beta}\left[\widetilde{\theta}\left(\frac{u}{M}\right)\right]\Big|_{k,m\beta}[\widetilde{W}_{NM^2}](\tau,z)$$
$$= \chi(M)\psi(-\beta N)\frac{\mathcal{G}_\psi}{\mathcal{G}_{\overline{\psi}}} g_{\overline{\psi}}(\tau, Mz).$$

□

For completeness sake, we observe that the function $g(\tau,z)$ introduced in Lemma 6 is in $J_{k,m\beta NM^2,\overline{\chi}\overline{\psi}^2}(\Gamma_0(\beta,NM^2)\ltimes(\beta^{-1}\mathbb{Z}\times(NM)^{-1}\mathbb{Z})\langle\zeta_{\beta NM}\rangle)$ whenever $f(\tau,z)$ is in $J_{k,m\beta,\chi}(\Gamma_0(N)\ltimes(\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ (Lemmas 4, 5).

4 Dirichlet series

Previously we have studied some basic properties of Jacobi forms, their characteristic twists and their images under a linear operator based on a Fricke involution. In this section, we put ourselves in a more general situation and consider instead Fourier series in two variables of a particular kind. We associate to them a set of Dirichlet series builded with their Fourier coefficients and show that such Dirichlet series have integral representations.

Definition 6 Fix positive integers m and β. We say that a Fourier series $f(\tau,z)$ in the variables $\tau\in\mathcal{H}$ and $z\in\mathbb{C}$ is of type J if

$$f(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>\beta r^2}} c(n,r)e(n\tau)e(r\beta z) \tag{13}$$

for some $c(n,r)\in\mathbb{C}$ and the following properties hold:

(I) The series $f(\tau,z)$ converges absolutely and uniformly on every compact subset of $\mathcal{H}\times\mathbb{C}$.
(II) There are positive real numbers C, σ such that $|c(n,r)| < C(4mn-\beta r^2)^\sigma$ for all n, r.
(III) The Fourier coefficients in (13) satisfy $c(n,r) = c(n+\lambda r\beta+\lambda^2 m\beta, r+2m\lambda)$ for all $\lambda\in\mathbb{Z}$.

Observe that Condition I is equivalent to the fact that the series (13) defines a holomorphic function $f(\tau,z):\mathcal{H}\times\mathbb{C}\to\mathbb{C}$. Moreover, it is possible to deduce from I and III (as we did for Jacobi forms in a previous section) that (13) can be written as a finite combination of theta functions:

$$f(\tau,z) = \sum_{\mu=1}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\beta\tau,\beta z) \quad \text{where } f_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D)e\left(\frac{D}{4m}\tau\right) \tag{14}$$

and $c_r(D) = c(n,r)$ whenever $D = 4mn-\beta r^2$. We call (14) the theta decomposition of $f(\tau,z)$.

Any cuspidal Jacobi form $f(\tau, z)$ in $J^{\text{cusp}}_{k,m\beta,\chi}(\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ is represented by a Fourier series of type J (with $\sigma = k/2$).

Definition 7 Let $f(\tau, z)$ be a series of type J, M any positive integer and ψ a primitive Dirichlet character mod M. The twist of $f(\tau, z)$ by ψ is defined as the series

$$f_\psi(\tau, z) := \sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>\beta r^2}} \psi(n)c(n, r)e(n\tau)e(r\beta z).$$

It is easy to see that the Fourier series $f_\psi(\tau, z)$ satisfies Conditions I and II of Definition 6. Also it satisfies III with $\lambda \in M\mathbb{Z}$ instead of $\lambda \in \mathbb{Z}$. As before, these properties yield the theta decomposition (2) for $f_\psi(\tau, z)$.

Of course, $f(\tau, z) = f_\psi(\tau, z)$ whenever ψ is the trivial character with conductor $M = 1$.

Definition 8 To any series $f(\tau, z)$ of type J and to any primitive Dirichlet character ψ mod M we associate $2mM$ Dirichlet series related to the theta decomposition (2) of $f_\psi(\tau, z)$. Namely, $L_\mu(f_\psi; s)$ for $\mu = 1, 2, \ldots, 2mM$ is defined as the series in (4). Furthermore, in analogy with the elliptic case, we complete these series with some exponential and gamma factors. For a positive integer N with $\beta|N$ and $\gcd(N, M) = 1$, we set

$$\Lambda_{N,\mu}(f_\psi; s) := \left(\frac{2\pi}{M\sqrt{N}}\right)^{-s} \Gamma(s)L_\mu(f_\psi; s). \tag{15}$$

Notice that II yields that all these series are uniformly convergent on the complex half-plane $\text{Re}(s) > 1 + \sigma$.

Our next task is to find an integral representation for the series (15). With that goal in mind we recall the real coordinates x, y, p, q of $\mathcal{H} \times \mathbb{C}$ and consider

$$\mathcal{I}_a(s) := \int_{y=0}^{\infty}\int_{p=0}^{M} e^m\left(p^2\frac{iy}{M\sqrt{N}}\right) f_\psi\left(\frac{iy}{\beta M\sqrt{N}}, p\frac{iy}{\beta M\sqrt{N}} - \frac{a}{2m\beta M}\right) \times\ y^{s-1/2}\, dp\, dy. \tag{16}$$

Lemma 7 *Let $f(\tau, z)$ and ψ be as in Definition 8. For any $a = 1, 2, \ldots, 2mM$ and any $s \in \mathbb{C}$ with* $\text{Re}(s) > 0$, *one has*

$$\mathcal{I}_a(s) = \left(\frac{2m}{M\sqrt{N}}\right)^{-1/2} \beta^s \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right)\Lambda_{N,\mu}(f_\psi; s).$$

Proof Using the theta representation of $f_\psi(\tau, z)$, we can write the double integral $\mathcal{I}_a(s)$ in (16) as

$$\int_{y=0}^{\infty}\int_{p=0}^{M} \sum_{\mu=1}^{2mM} f_{\psi,\mu}\left(\frac{iy}{\beta M\sqrt{N}}\right)\Theta_{mM,\mu}\left(\frac{iy}{\sqrt{N}}, p\frac{iy}{M\sqrt{N}} - \frac{a}{2mM}\right)$$

$$\times e^m\left(p^2\frac{iy}{M\sqrt{N}}\right)y^{s-1/2}\,dp\,dy.$$

Then we plug the Fourier series representations of $f_{\psi,\mu}(\tau,z)$ and $\Theta_{mM,\mu}(\tau,z)$ in this expression and get

$$\mathcal{I}_a(s)=\sum_{\mu=1}^{2mM}e\left(-\frac{a\mu}{2mM}\right)\int_{y=0}^{\infty}\sum_{D=1}^{\infty}\psi\left(\frac{D+\beta\mu^2}{4m}\right)c_\mu(D)e\left(\frac{D}{4m}\frac{iy}{\beta M\sqrt{N}}\right)$$
$$\times\, y^{s-1/2}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu\ (2mM)}}\int_{p=0}^{M}e\left(\frac{(r+2mp)^2}{4m}\frac{iy}{M\sqrt{N}}\right)dp\,dy.$$

Next we make the change of variable $\widetilde{p}=p+\frac{r}{2m}$ in the inner integral and obtain

$$\mathcal{I}_a(s)=\sum_{\mu=1}^{2mM}e\left(-\frac{a\mu}{2mM}\right)\int_{y=0}^{\infty}\sum_{D=1}^{\infty}\psi\left(\frac{D+\beta\mu^2}{4m}\right)c_\mu(D)e\left(\frac{D}{4m}\frac{iy}{\beta M\sqrt{N}}\right)$$
$$\times\left(\int_{\widetilde{p}=-\infty}^{\infty}e^m\left(\widetilde{p}^2\frac{iy}{M\sqrt{N}}\right)d\widetilde{p}\right)y^{s-1/2}\,dy$$
$$=\left(\frac{2m}{M\sqrt{N}}\right)^{-1/2}\sum_{\mu=1}^{2mM}e\left(-\frac{a\mu}{2mM}\right)\sum_{D=1}^{\infty}\psi\left(\frac{D+\beta\mu^2}{4m}\right)c_\mu(D)$$
$$\times\int_{y=0}^{\infty}e\left(\frac{D}{4m}\frac{iy}{\beta M\sqrt{N}}\right)y^{s-1}\,dy.$$

The last integral is equal to $(\frac{\pi D}{2m\beta M\sqrt{N}})^{-s}\Gamma(s)$ whenever $s\in\mathbb{C}$ is in the right half-plane $\mathrm{Re}(s)>0$, and the lemma follows. □

Definition 9 Fix positive integers m, β and N with $\beta|N$. We say that a Fourier series $g(\tau,z)$ in the variables $\tau\in\mathcal{H}$ and $z\in\mathbb{C}$ is of type J_N if

$$g(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z}\\ 4mn>Nr^2}}d(n,r)e\left(\frac{n}{\beta}\tau\right)e(rNz) \tag{17}$$

for some $c(n,r)\in\mathbb{C}$ and the properties I, II, III given in Definition 6 hold for $g(\tau,z)$ with a parameter N instead of β. (In other words, Conditions II and III satisfied by (17) are $|d(n,r)|<C(4mn-Nr^2)^\sigma$ for all n,r and $d(n,r)=d(n+\lambda rN+\lambda^2mN,r+2m\lambda)$ for all $\lambda\in\mathbb{Z}$.)

As in the case of series of type J, Conditions I and III yield a theta decomposition for $g(\tau,z)$:

$$g(\tau,z)=\sum_{\mu=1}^{2m}g_\mu(\tau)\Theta_{m,\mu}\left(\frac{N}{\beta}\tau,Nz\right)$$

$$\text{where } g_\mu(\tau) = \sum_{D=1}^{\infty} d_\mu(D) e\left(\frac{D}{4m\beta}\tau\right), \tag{18}$$

and $d_r(D) = d(n,r)$ whenever $D = 4mn - Nr^2$.

If $f(\tau,z) \in J^{\text{cusp}}_{k,m\beta,\chi}(\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ then the map $g(\tau,z) = f|_{k,m\beta} \times [\widetilde{W}_N](\tau,z)$ is in $J^{\text{cusp}}_{k,m\beta N,\overline{\chi}}(\Gamma_0(\beta,N) \ltimes (\beta^{-1}\mathbb{Z} \times N^{-1}\mathbb{Z})\langle\zeta_{\beta N}\rangle)$ (see Lemma 5). Hence $g(\tau,z)$ is represented by a series of type J_N (with $\sigma = k/2$).

The obvious generalization of Definition 7 indicate us how to twist the series $g(\tau,z)$ in (17) with a primitive Dirichlet character ψ mod M. In this way, we get a Fourier series $g_\psi(\tau,z)$ which satisfies I and II. It also satisfies III as above with $\lambda \in M\mathbb{Z}$ instead of $\lambda \in \mathbb{Z}$. At any rate, these properties yield the theta decomposition (3) for $g_\psi(\tau,z)$.

As in Definition 8, we associate to $g_\psi(\tau,z)$ the collection of $2mM$ Dirichlet series $L_\mu(g_\psi; s)$ where $\mu = 1, 2, \dots, 2mM$ given in (5), and the corresponding completed Dirichlet series $\Lambda_{N,\mu}(g_\psi; s)$.

Next we consider the conjugate character $\overline{\psi}$ of ψ and the integral

$$\mathcal{J}_a(s) := \int_{y=0}^{\infty}\int_{p=0}^{M} e\left(\frac{ap}{M} + \frac{a^2\sqrt{N}iy}{4mM}\right) g_{\overline{\psi}}\left(\frac{\beta iy}{M\sqrt{N}}, -\frac{aiy}{2mM\sqrt{N}} - \frac{p}{NM}\right) \times y^{s-1}\,dp\,dy. \tag{19}$$

Lemma 8 *Let $g(\tau,z)$ and $\overline{\psi}$ be as above. For any $a = 1, 2, \dots, 2mM$ and any $s \in \mathbb{C}$ with* $\operatorname{Re}(s) > 0$, *one has*

$$\mathcal{J}_a(s) = M\beta^{-s}\Lambda_{N,a}(g_{\overline{\psi}}; s).$$

Proof If the theta representation of $g_{\overline{\psi}}(\tau,z)$ is used on the right hand side of (19), we get

$$\mathcal{J}_a(s) = \int_{y=0}^{\infty}\int_{p=0}^{M}\sum_{\mu=1}^{2mM} g_{\overline{\psi},\mu}\left(\frac{\beta iy}{M\sqrt{N}}\right)\Theta_{mM,\mu}\left(\sqrt{N}iy, -\frac{a\sqrt{N}iy}{2mM} - \frac{p}{M}\right) \times e\left(\frac{ap}{M} + \frac{a^2\sqrt{N}iy}{4mM}\right) y^{s-1}\,dp\,dy.$$

Hence

$$\mathcal{J}_a(s) = \int_{y=0}^{\infty}\sum_{\mu=1}^{2mM}\sum_{D=1}^{\infty}\overline{\psi}\left(\frac{D+N\mu^2}{4m}\right) d_\mu(D) e\left(\frac{D}{4m}\frac{iy}{M\sqrt{N}}\right) \times \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu\ (2mM)}} e\left(\frac{r^2-2ar+a^2}{4mM}\sqrt{N}iy\right)\left(\int_{p=0}^{M} e\left((a-r)\frac{p}{M}\right)dp\right) y^{s-1}\,dy.$$

Since the inner integral is zero whenever $a \neq r$ and M otherwise, we obtain

$$\begin{aligned}\mathcal{J}_a(s) &= M \int_{y=0}^{\infty} \sum_{D=1}^{\infty} \overline{\psi}\left(\frac{D+Na^2}{4m}\right) d_a(D) e\left(\frac{D}{4m}\frac{iy}{M\sqrt{N}}\right) y^{s-1}\, dy \\ &= M \sum_{D=1}^{\infty} \overline{\psi}\left(\frac{D+Na^2}{4m}\right) d_a(D) \left(\frac{\pi D}{2mM\sqrt{N}}\right)^{-s} \Gamma(s) \\ &= M\beta^{-s} \Lambda_{N,a}(g_{\overline{\psi}}; s). \end{aligned}$$

□

5 Proof of the main theorem

Now that we have integral representations for $\sum_{\mu} e(-a\mu/2mM)\Lambda_{N,\mu}(f_{\psi}; s)$ and $\Lambda_{N,a}(g_{\overline{\psi}}; s)$, we adapt the classical argument of A. Weil for the proof of our converse theorem. We break the argument into a proposition, its corollary, two lemmas, and another proposition. In each proof, we only sketch steps which are completely analogous to some in the elliptic case (see [15, pp. 125–127] and [20] for details). The corollary and the last proposition of this section constitute our main result as they together yield Theorem 1.

Proposition 1 *Fix positive integers m, β and N with $\beta | N$. Let $f(\tau, z)$ be a Fourier series of type J and $g(\tau, z)$ a Fourier series of type J_N. Also, let ψ be a primitive Dirichlet character mod M with* gcd$(N, M) = 1$. *Then, for any positive integer k, the following two statements are equivalent*:

(A) *There exists a complex number C_{ψ} such that*

$$f_{\psi}|_{k,m\beta}[\widetilde{W}_{NM^2}](\tau, z) = C_{\psi} g_{\overline{\psi}}(\tau, Mz).$$

(B) *Each series $\Lambda_{N,\mu}(f_{\psi}, s)$ and $\Lambda_{N,\mu}(g_{\overline{\psi}}, s)$ $(1 \le \mu \le 2mM)$ admits a holomorphic continuation to the whole s-plane, they are bounded on any vertical strip, and satisfy the functional equations*

$$\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}(f_{\psi}; s) = i^k C_{\psi} \left(\frac{2m\beta M}{\sqrt{N}}\right)^{1/2} \Lambda_{N,a}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)$$

for $1 \le a \le 2mM$.

Proof First, we prove that (A) implies (B).

The integral (16) can be written as a sum $\mathcal{I}_a(s) = \mathcal{I}_a^0(s) + \mathcal{I}_a^{\infty}(s)$, where the terms on the right hand side are the integrals $\mathcal{I}_a^0(s) := \int_{y=0}^{1} \int_{p=0}^{M} \ldots dp\, dy$ and $\mathcal{I}_a^{\infty}(s) := \int_{y=1}^{\infty} \int_{p=0}^{M} \ldots dp\, dy$ (the integrand in each of them is the same as the one in $\mathcal{I}_a(s)$). From Condition II, one deduces

$$e^{m\beta}(pz) f_{\psi,\mu}(\tau) \Theta_{mM,\mu}(M\beta\tau, \beta z) = O\left(e\left(\frac{iy}{4m}\right)\right) \quad \text{as } y \to \infty$$

for $\mu = 1, 2, \ldots, 2mM$ (see, for example, [13, p. 186] or [15, p. 117]). Hence

$$e^{m\beta}(pz) f_\psi(\tau, z) = O\left(e\left(\frac{iy}{4m}\right)\right) \quad \text{as } y \to \infty.$$

This estimate applied to the integrand of $\mathcal{I}_a^\infty(s)$ yields the existence of a real constant T such that

$$\left|\mathcal{I}_a^\infty(s)\right| < T \int_{y=1}^{\infty} \int_{p=0}^{M} e\left(\frac{iy}{4m\beta M\sqrt{N}}\right) y^{\mathrm{Re}(s)-1/2}\, dp\, dy.$$

Therefore, $\mathcal{I}_a^\infty(s)$ is a well-defined, entire function of s, bounded on any vertical strip of the s-plane.

Analogously, $\mathcal{J}_a(s) = \mathcal{J}_a^0(s) + \mathcal{J}_a^\infty(s)$ with $\mathcal{J}_a^0(s) := \int_{y=0}^1 \int_{p=0}^M \ldots dp\, dy$ and $\mathcal{J}_a^\infty(s) := \int_{y=1}^\infty \int_{p=0}^M \ldots dp\, dy$. Arguing as before, one can prove that $\mathcal{J}_a^\infty(s)$ defines an entire function of s, bounded on any vertical strip.

In order to get similar properties for $\mathcal{I}_a^0(s)$ and $\mathcal{J}_a^0(s)$, we proceed in a different way. A straightforward computation allow us to deduce from (A) the identity

$$\begin{aligned} &e^m\left(p^2 \frac{iy}{M\sqrt{N}}\right) f_\psi\left(\frac{iy}{\beta M\sqrt{N}}, p\frac{iy}{\beta M\sqrt{N}} - \frac{a}{2m\beta M}\right) \\ &= C_\psi \left(-\frac{iy}{\beta}\right)^{-k} e\left(\frac{ap}{M} + \frac{a^2\sqrt{N}i}{4mMy}\right) \\ &\quad \times g_{\overline{\psi}}\left(\frac{\beta i}{M\sqrt{N}y}, -\frac{p}{NM} - \frac{ai}{2mM\sqrt{N}y}\right). \end{aligned} \tag{20}$$

Using this in $\mathcal{I}_a^0(s)$ and the change of variable $\widetilde{y} = y^{-1}$, we get $\mathcal{I}_a^0(s) = (i\beta)^k C_\psi \mathcal{J}_a^\infty(k - s - \frac{1}{2})$. Consequently, we can write the integral (16) as

$$\mathcal{I}_a(s) = \mathcal{I}_a^\infty(s) + (i\beta)^k C_\psi \mathcal{J}_a^\infty\left(k - s - \frac{1}{2}\right). \tag{21}$$

From this expression, Lemma 7 and the properties of $\mathcal{I}_a^\infty(s)$, $\mathcal{J}_a^\infty(s)$ obtained above, we conclude that

$$\left(\frac{2m}{M\sqrt{N}}\right)^{-1/2} \beta^s \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}(f_\psi; s)$$

has a holomorphic continuation to the whole complex plane, which is bounded on any vertical strip.

Observe that the change of variables used in $\mathcal{I}_a^0(s)$ can also be applied to $\mathcal{J}_a^0(k - s - \frac{1}{2})$. More precisely, we can write the integrand of the latter in terms of $f_\psi(\tau, z)$ using (20). Then we make the change of variable $\widetilde{y} = y^{-1}$ and get $\mathcal{J}_a^0(k - s - \frac{1}{2}) = (i\beta)^{-k} C_\psi^{-1} \mathcal{I}_a^\infty(s)$. This allows us to write (19) as

$$i^k \beta^k C_\psi \mathcal{J}_a\left(k - s - \frac{1}{2}\right) = \mathcal{I}_a^\infty(s) + (i\beta)^k C_\psi \mathcal{J}_a^\infty\left(k - s - \frac{1}{2}\right). \tag{22}$$

Comparing now the right hand side of (21) and (22) (expressions which are holomorphic functions on $\mathbb{C}$) we obtain $\mathcal{I}_a(s) = i^k\beta^k C_\psi \mathcal{J}_a(k-s-\frac{1}{2})$. Thus

$$\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right)\Lambda_{N,\mu}(f_\psi;s) = i^k C_\psi\left(\frac{2m\beta M}{\sqrt{N}}\right)^{1/2}\Lambda_{N,a}\left(g_{\overline{\psi}};k-s-\frac{1}{2}\right).$$

One consequence of this relation is that $\Lambda_{N,a}(g_{\overline{\psi}};s)$ admits a holomorphic continuation to the whole s-plane bounded on vertical strips. In order to get the same conclusion for each series $\Lambda_{N,\mu}(f_\psi;s)$, it suffices to observe that $(e(-\frac{ar}{2mM}))_{1\le a,r\le 2mM}$ is an invertible matrix, and therefore every $\Lambda_{N,\mu}(f_\psi;s)$ is a combination of the $2mM$ series $\Lambda_{N,a}(g_{\overline{\psi}};s)$. This proves (B).

For the proof of the converse implication, we first rewrite (A) as

$$\left(NM^2\right)^{-k/2}\tau^{-k}e^{m\beta NM^2}\left(-\frac{z^2}{\tau}\right)f_\psi\left(\frac{-1}{NM^2\tau},\frac{z}{\tau}\right) = C_\psi g_{\overline{\psi}}(\tau, Mz).$$

Then we replace τ by $-1/NM^2\tau$ and z by $-z/NM^2\tau$, obtaining

$$f_\psi(\tau,z) = (-1)^k C_\psi\left(NM^2\right)^{-k/2}\tau^{-k}e^{m\beta}\left(-\frac{z^2}{\tau}\right)g_{\overline{\psi}}\left(\frac{-1}{NM^2\tau},\frac{-z}{NM\tau}\right). \qquad (23)$$

Next we use the theta decompositions of $f_\psi(\tau,z)$ and $g_{\overline{\psi}}(\tau,z)$ plus the functional equation of theta functions

$$\Theta_{L,\mu}\left(\frac{-1}{\tau},\frac{z}{\tau}\right) = \sqrt{\frac{\tau}{2Li}}e^L\left(\frac{z^2}{\tau}\right)\sum_{\nu\ (2L)} e\left(-\frac{\mu\nu}{2L}\right)\Theta_{L,\nu}(\tau,z)$$

in order to write (23) as

$$\begin{aligned}
&\sum_{\mu=1}^{2mM} f_{\psi,\mu}(\tau)\Theta_{mM,\mu}(M\beta\tau,\beta z)\\
&\quad= (-1)^k C_\psi\left(NM^2\right)^{-k/2}\tau^{-k}e^{m\beta}\left(-\frac{z^2}{\tau}\right)\sum_{\mu=1}^{2mM} g_{\overline{\psi},\mu}\left(\frac{-1}{NM^2\tau}\right)\\
&\qquad\times\Theta_{mM,\mu}\left(\frac{-1}{\beta M\tau},\frac{-z}{M\tau}\right)\\
&\quad= (-1)^k C_\psi\left(NM^2\right)^{-k/2}\tau^{-k}\sqrt{\frac{\beta\tau}{2mi}}\sum_{\mu=1}^{2mM} g_{\overline{\psi},\mu}\left(\frac{-1}{NM^2\tau}\right)\sum_{\nu=1}^{2mM} e\left(-\frac{\mu\nu}{2mM}\right)\\
&\qquad\times\Theta_{mM,\nu}(M\beta\tau,-\beta z)\\
&\quad= \sum_{\nu=1}^{2mM}\left((-1)^k C_\psi\left(NM^2\right)^{-k/2}\tau^{-k}\sqrt{\frac{\beta\tau}{2mi}}\sum_{\mu=1}^{2mM} e\left(\frac{\mu\nu}{2mM}\right)g_{\overline{\psi},\mu}\left(\frac{-1}{NM^2\tau}\right)\right)
\end{aligned}$$

$$\times \Theta_{mM,\nu}(M\beta\tau, \beta z).$$

Consequently (23), and therefore (A), is equivalent to the system of equations

$$f_{\psi,r_0}(\tau) = (-1)^k C_\psi \left(NM^2\right)^{-k/2} \tau^{-k} \sqrt{\frac{\beta\tau}{2mi}} \sum_{\mu=1}^{2mM} e\left(\frac{\mu r_0}{2mM}\right) g_{\overline{\psi},\mu}\left(\frac{-1}{NM^2\tau}\right)$$

indexed by $r_0 = 1, 2, \ldots, 2mM$. In the following, we show how to get such a set of identities from (B). Since $f_{\psi,\mu}(\tau)$ and $g_{\overline{\psi},\mu}(\tau)$ are holomorphic functions on $\mathcal{H}$ (by Condition II), it suffices to prove that (B) yields

$$\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) f_{\psi,\mu}\left(\frac{iy}{\beta M\sqrt{N}}\right) = i^k \beta^k C_\psi y^{-k} \sqrt{\frac{2mMy}{\sqrt{N}}} g_{\overline{\psi},a}\left(\frac{-\beta}{M\sqrt{N}iy}\right) \tag{24}$$

for every $y \in \mathbb{R}$, $y > 0$. Using the expression for $f_{\psi,\mu}(\tau)$ in (2) and the inverse Mellin transform of the exponential function, we have

$$f_{\psi,\mu}\left(\frac{iy}{\beta M\sqrt{N}}\right) = \frac{1}{2\pi i} \sum_{D=1}^{\infty} \psi\left(\frac{D+\beta\mu^2}{4m}\right) c_\mu(D) \int_{\mathrm{Re}(s)=r} \left(\pi \frac{D}{2m} \frac{y}{\beta M\sqrt{N}}\right)^{-s} \Gamma(s)\, ds$$

for any real $r > 0$. If $r > \sigma + 1$, the Dirichlet series $L_\mu(f_\psi; s)$ is uniformly convergent and bounded on $\mathrm{Re}(s) = r$. Hence Stirling's formula for $\Gamma(s)$ implies that $\Lambda_{N,\mu}(f_\psi; s)$ is absolutely integrable. Thus we can exchange the order of summation and integration in the previous identity and get

$$\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) f_{\psi,\mu}\left(\frac{iy}{\beta M\sqrt{N}}\right) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=r} y^{-s} \beta^s \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}(f_\psi; s)\, ds. \tag{25}$$

The Dirichlet series $L_\mu(f_\psi; s)$ is bounded on $\mathrm{Re}(s) = r$. Thus the estimate $|\Lambda_{N,\mu}(f_\psi; s)| = O(|\mathrm{Im}(s)|^{-t})$ as $|\mathrm{Im}(s)| \to \infty$ is valid for any $t > 0$ on the vertical line $\mathrm{Re}(s) = r$. Henceforth,

$$\left| \beta^s \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}(f_\psi; s) \right| = O\left(\left|\mathrm{Im}(s)\right|^{-t}\right) \tag{26}$$

as $|\mathrm{Im}(s)| \to \infty$ on the line $\mathrm{Re}(s) = r$. Next take r' in $\mathbb{R}$ such that $k - r' > \sigma + \frac{3}{2}$. Then the Dirichlet series $L_a(g_{\overline{\psi}}; k - s - \frac{1}{2})$ is bounded on $\mathrm{Re}(s) = r'$ and we may conclude

$$\left(\frac{2mM}{\sqrt{N}}\right)^{1/2} \left|\beta^k C_\psi \Lambda_{N,a}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)\right| = O\left(\left|\mathrm{Im}(s)\right|^{-t}\right) \tag{27}$$

as $|\mathrm{Im}(s)| \to \infty$ for any $t > 0$ on the vertical line $\mathrm{Re}(s) = r'$. As the left hand sides of (26) and (27) are equal and bounded on the strip $r' \le \mathrm{Re}(s) \le r$ by hypothesis, these two estimates and the Phragmen–Lindelof theorem yield

$$\begin{aligned}
&\left|\beta^s \sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}(f_\psi; s)\right| \\
&\quad = \left(\frac{2mM}{\sqrt{N}}\right)^{1/2} \beta^k \left|C_\psi \Lambda_{N,a}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)\right| = O\left(\left|\mathrm{Im}(s)\right|^{-t}\right)
\end{aligned}$$

as $|\mathrm{Im}(s)| \to \infty$, uniformly on the vertical strip $\{s \in \mathbb{C} \mid r' \le \mathrm{Re}(s) \le r\}$. This fact and the holomorphic continuation to $\mathbb{C}$ of the completed Dirichlet series $\Lambda_{N,\mu}(f_\psi; s)$ and $\Lambda_{N,a}(g_{\overline{\psi}}; s)$ allow us to change the path of integration in (25) from $\mathrm{Re}(s) = r$ to $\mathrm{Re}(s) = r'$. Thus

$$\begin{aligned}
&\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right) f_{\psi,\mu}\left(\frac{iy}{\beta M \sqrt{N}}\right) \\
&\quad = i^k C_\psi \left(\frac{2m\beta M}{\sqrt{N}}\right)^{1/2} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=r'} y^{-s} \beta^s \Lambda_{N,a}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right) ds \\
&\quad = i^k \beta^k C_\psi y^{-k} \sqrt{\frac{2mMy}{\sqrt{N}}} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k-r'-\frac{1}{2}} y^s \beta^{-s} \Lambda_{N,a}(g_{\overline{\psi}}; s)\, ds.
\end{aligned}$$

Finally, we use the argument prior to (25) and the absolute integrability of $\Lambda_{N,a}(g_{\overline{\psi}}; s)$ to get the following:

$$g_{\overline{\psi},a}\left(\frac{-\beta}{M\sqrt{N}iy}\right) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=k-r'-\frac{1}{2}} y^s \beta^{-s} \Lambda_{N,a}(g_{\overline{\psi}}; s)\, ds.$$

From the last two sets of identities, we obtain (24), as desired. □

The first half of our main result is an immediate consequence of Lemma 6, the remark prior to Definition 7, the remark below Definition 9 and Proposition 1, e.g.,

Corollary 4 *Let m, β, N and M be positive integers with $\beta | N$, $\gcd(M, N) = 1$. Let χ be a Dirichlet character mod N and ψ a primitive Dirichlet character mod M. If $f(\tau, z)$ is a Jacobi form in $J^{\mathrm{cusp}}_{k,m\beta,\chi}(\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle)$ then the completed*

Dirichlet series $\Lambda_{N,\mu}(f_\psi; s)$, *for* $\mu = 1, 2, \ldots, 2mM$, *admits a holomorphic continuation to the whole s-plane. They are all bounded on vertical strips and satisfy the system of functional equations*

$$\sum_{\mu=1}^{2mM} e\left(-\frac{a\mu}{2mM}\right)\Lambda_{N,\mu}(f_\psi; s) = i^k C_\psi \left(\frac{2m\beta M}{\sqrt{N}}\right)^{1/2} \Lambda_{N,a}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)$$

for $1 \le a \le 2mM$, $g(\tau, z) = f|_{k,m\beta}[\widetilde{W}_N](\tau, z)$, $C_\psi = \chi(M)\psi(-\beta N)\mathcal{G}_\psi \mathcal{G}_{\overline{\psi}}^{-1}$.

The rest of this section is devoted to prove a converse of this corollary.

For two integers a, b such that $\gcd(a, bN) = 1$, consider integers c, d so that $ad - bc\beta N = 1$ (recall that we always assume $\beta|N$) and define $\gamma(a, b) = \left(\begin{smallmatrix} a & -b\beta \\ -cN & d \end{smallmatrix}\right) \in \Gamma_0(\beta, N)$. Though $\gamma(a, b)$ is not uniquely determined, the value of $c \pmod a$ is uniquely determined, and we have the key identity

$$\theta\left(\frac{c}{a}\right) W_{Na^2} = a W_N \gamma(a, b) \theta\left(\frac{b\beta}{a}\right). \tag{28}$$

Lemma 9 *Let* m, β, N *and* k *be positive integers such that* $\beta|N$. *Consider a Fourier series* $f(\tau, z)$ *of type* J *and a Fourier series* $g(\tau, z)$ *of type* J_N. *Let* a *be* 1, 4 *or an odd prime number with* $\gcd(a, N) = 1$. *If*

$$f_\psi|_{k,m\beta}[\widetilde{W}_{Na^2}](\tau, z) = C_\psi g_{\overline{\psi}}(\tau, az) \quad \textit{with } C_\psi = \chi(a)\psi(-\beta N)\mathcal{G}_\psi \mathcal{G}_{\overline{\psi}}^{-1}$$

for all primitive Dirichlet characters ψ *mod* a, *then*

$$g|_{k,m\beta}\big(\chi(a)[I_2, 0, 0, 1] - [\gamma(a, c), 0, 0, 1]\big)\big|_{k,m\beta}\left[\widetilde{\theta}\left(\frac{c\beta}{a}\right)\right](\tau, z)$$
$$= g|_{k,m\beta}\big(\chi(a)[I_2, 0, 0, 1] - [\gamma(a, b), 0, 0, 1]\big)\big|_{k,m\beta}\left[\widetilde{\theta}\left(\frac{b\beta}{a}\right)\right](\tau, z)$$

for all integers b, c *relatively prime to* a.

Sketch of the Proof This is a straightforward adaptation of the proof given for the corresponding statement in the case of elliptic cusp forms (see, for example, [15, p. 126]). Here one uses (28). □

Lemma 10 *Let* m, β, N *and* k *be positive integers such that* $\beta|N$. *Consider a Fourier series* $f(\tau, z)$ *of type* J *and a Fourier series* $g(\tau, z)$ *of type* J_N. *Let* a, d *be odd prime numbers or* 4 *with* $\gcd(a, N) = \gcd(d, N) = 1$. *If*

$$f_\psi|_{k,m\beta}[\widetilde{W}_{Na^2}](\tau, z) = C_\psi g_{\overline{\psi}}(\tau, az) \quad \textit{with } C_\psi = \chi(a)\psi(-\beta N)\mathcal{G}_\psi \mathcal{G}_{\overline{\psi}}^{-1}$$

for the trivial primitive character and for all primitive Dirichlet characters ψ *mod* a *and mod* d, *then* $g|_{k,m\beta}[\gamma, 0, 0, 1](\tau, z) = \overline{\chi}(\gamma)g(\tau, z)$ *whenever* γ *is a matrix of the form* $\gamma = \left(\begin{smallmatrix} a & -b\beta \\ -cN & d \end{smallmatrix}\right)$ *in* $\Gamma_0(\beta, N)$.

Proof Again, this is a direct adaptation of a proof given for the corresponding statement in the case of elliptic cusp forms (see, for example, [15, pp. 126–127]). □

In order to have a complete analogy with Weil's theorem, we denote by $\mathcal{M}$ any set of odd prime numbers or 4 such that (i) every element of $\mathcal{M}$ is prime to N and (ii) $\mathcal{M} \cap P(a,b) \neq \emptyset$ for any $P(a,b) = \{a + lb \mid l \in \mathbb{Z}\}$.

Proposition 2 *Let m, β, N and k be positive integers such that $\beta|N$ and χ a Dirichlet character mod N. Let $\{c(n,r)\}$ and $\{d(n,r)\}$ be two sequences of complex numbers such that*

$$f(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>\beta r^2}} c(n,r)e(n\tau)e(r\beta z)$$

and

$$g(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn>Nr^2}} d(n,r)e\left(\frac{n}{\beta}\tau\right)e(rNz)$$

are series of type J and J_N respectively, with $(-1)^k g(\tau,-z) = \overline{\chi}(-1)g(\tau,z)$. Assume that for every primitive character ψ of conductor M in $\mathcal{M}\cup\{1\}$ and all $1 \le \mu \le 2mM$ each one of the series $\Lambda_{N,\mu}(f_\psi,s)$, $\Lambda_{N,\mu}(g_{\overline{\psi}},s)$ satisfies the conditions in (B) *of Proposition 1 with $C_\psi = \chi(M)\psi(-\beta N)\mathcal{G}_\psi\mathcal{G}_{\overline{\psi}}^{-1}$. Assume also that all series $L_\mu(f;s)$ $(1\le\mu\le 2m)$ converge absolutely at $s = k-1-\epsilon$ for some $\epsilon>0$. Then*

$$f(\tau,z) \in J^{\text{cusp}}_{k,m\beta,\chi}\big(\Gamma_0(N) \ltimes (\mathbb{Z}\times\beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle\big)$$

and

$$g(\tau,z) = f|_{k,m\beta}[\widetilde{W}_N](\tau,z).$$

Proof First, we observe that $f(\tau,z)$ and $g(\tau,z)$ are holomorphic functions on $\mathcal{H}\times\mathbb{C}$ by Lemma 2. Second, we notice that the hypothesis with ψ equal to the trivial character plus Proposition 1 yield $g(\tau,z) = f|_{k,m\beta}[\widetilde{W}_N](\tau,z)$.

The invariance of $g(\tau,z)$ under the group $(\beta^{-1}\mathbb{Z}\times N^{-1}\mathbb{Z})\langle\zeta_{\beta N}\rangle$ follows immediately from the theta decomposition (18) and the transformation law satisfied by $\Theta_{m,\mu}(\frac{N}{\beta\tau}, Nz)$ whenever z is replaced by $z + \frac{\lambda\tau}{\beta} + \frac{\nu}{N}$ with $\lambda,\nu\in\mathbb{Z}$. Consequently, if we want to show $g|_{k,m\beta M}[h](\tau,z) = \overline{\chi}(\gamma)g(\tau,z)$ for every $h = [\gamma,*,*,*]$ in $\Gamma_0(\beta,N)\ltimes(\beta^{-1}\mathbb{Z}\times N^{-1}\mathbb{Z})\langle\zeta_{\beta N}\rangle$, it suffices to check

$$g|_{k,m\beta M}[\gamma,0,0,1](\tau,z) = \overline{\chi}(\gamma)g(\tau,z) \quad \text{for any } \gamma = \begin{pmatrix} a & b\beta \\ cN & d \end{pmatrix} \in \Gamma_0(\beta,N). \tag{29}$$

Notice that now we are in a situation where the classical argument works perfectly well (see [15, p. 128]). For the reader's convenience, we repeat it here.

If $c=0$ then (29) is clear. Assume next $c\neq 0$. Since $\gcd(a,cN) = \gcd(d,cN) = 1$ and $\beta|N$, there exist integers s,t such that $a + tc\beta N\in\mathcal{M}$ and $d + sc\beta N \in \mathcal{M}$. Put

$a' = a + tc\beta N$, $d' = d + sc\beta N$, $c' = -c$ and $b' = -(b + as + stc\beta N + dt)$. Then

$$\begin{pmatrix} a & b\beta \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & -t\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & -b'\beta \\ -c'N & d' \end{pmatrix} \begin{pmatrix} 1 & -s\beta \\ 0 & 1 \end{pmatrix}.$$

By Lemma 10, one has

$$g|_{k,mM}\left[\begin{pmatrix} a' & -b'\beta \\ -c'N & d' \end{pmatrix}, 0, 0, 1\right](\tau, z) = \overline{\chi}(d')g(\tau, z) = \overline{\chi}(\gamma)g(\tau, z).$$

Equation (29) follows from these identities and the invariance of $g(\tau, z)$ under the translation $\tau \to \tau + \beta$. Equations (29) and $g(\tau, z) = f|_{k,m\beta}[\widetilde{W}_N](\tau, z)$ imply that $f(\tau, z)$ is invariant (up to χ) under $\Gamma_0(N) \ltimes (\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_\beta\rangle$.

Finally, the behavior of $f(\tau, z)$ at the cusps of $\Gamma_0(N)$ can be determined as in the case of elliptic modular forms. Indeed, for every $1 \le \mu \le 2m$ one can show the estimate $e^{m\beta}(pz) f_\mu(\tau)\Theta_{m,\mu}(\beta\tau, \beta z) = O(y^{-k+\epsilon+1/2})$ as $y \to 0$ uniformly on $x = \mathrm{Re}(\tau)$ using, for example, the argument in [15, p. 129]. This implies the estimate $e^{m\beta}(pz) f(\tau, z) = O(y^{-(k-\epsilon-1/2)})$ as $y \to 0$ uniformly on $x = \mathrm{Re}(\tau)$. Now we apply Lemma 3 to $f(\tau, z)$ and the proposition follows. □

As we already mentioned, Corollary 4 and Proposition 2 constitute the main results of this work. Indeed, if we use the relation

$$\sum_{a=1}^{2mM} e\left(\frac{a(b-\mu)}{2mM}\right) = \begin{cases} 2mM & \text{if } \mu = b, \\ 0 & \text{otherwise,} \end{cases}$$

Lemmas 1 and 2 plus (14) and (18), both results are summarized in the equivalence stated in Theorem 1.

Remark This result can be easily extended to a converse theorem for Jacobi cusp forms $f(\tau, z)$ in $J^{\mathrm{cusp}}_{k,m\alpha\beta,\chi}(\Gamma_0(\alpha, N) \ltimes (\alpha^{-1}\mathbb{Z} \times \beta^{-1}\mathbb{Z})\langle\zeta_{\alpha\beta}\rangle)$ using essentially the same proof. In such a case, $f(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\frac{\beta}{\alpha}\tau, \beta z)$ with $f_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D) e(\frac{D}{4m\alpha}\tau)$, and the functional equations of the corresponding Dirichlet series are

$$\Lambda_{N,a}(f_\psi; s) = i^k \widetilde{C}_\psi \sqrt{\frac{\beta}{2m\alpha M\sqrt{N}}} \sum_{\mu=1}^{2mM} e\left(\frac{a\mu}{2mM}\right) \Lambda_{N,\mu}\left(g_{\overline{\psi}}; k - s - \frac{1}{2}\right)$$

where $1 \le a \le 2mM$ and $\widetilde{C}_\psi = \chi(M)\psi(-\alpha\beta N)\mathcal{G}_\psi \mathcal{G}_{\overline{\psi}}^{-1}$. The precise statement and its proof are left to the interested reader.

6 Applications

Our first application is both a new proof and a generalization of a theorem established by H. Skogman in [17]. We are referring to Corollary 1, given in the introduction.

Proof of Corollary 1 Without loss of generality, N can be changed by N' in the hypothesis, so we assume that $\delta\beta|N$. Next we consider the theta decomposition (14) of $f(\tau,z)$ and define $F(\tau,z) := \sum_{\mu'=1}^{2m'} F_{\mu'}(\tau)\Theta_{m',\mu'}(\delta\beta\tau,\delta\beta z)$ with $F_{\mu'}(\tau) := f_{\mu'\delta}(\tau) = \sum_{D=1}^{\infty} c_{\mu'\delta}(D)e(\frac{D'}{4m'}\tau)$, where $D = D'\delta$. This makes sense because $c_{\mu'\delta}(D) \neq 0$ only if $4m|D+\beta(\mu'\delta)^2$. It is easy to check that the character twist of $F(\tau,z)$ by any primitive Dirichlet character ψ mod M has a theta representation $F_\psi(\tau,z) = \sum_{\mu'=1}^{2m'M} F_{\psi,\mu'}(\tau)\Theta_{m'M,\mu'}(M\delta\beta\tau,\delta\beta z)$ where

$$F_{\psi,\mu'}(\tau) = \sum_{D'=1}^{\infty} \psi\left(\frac{D'+\delta\beta(\mu')^2}{4m'}\right)c_{\mu'\delta}(D'\delta)e\left(\frac{D'}{4m'}\tau\right).$$

Notice that $F_{\psi,\mu'}(\tau) = f_{\psi,\mu'\delta}(\tau)$ for all $1 \le \mu' \le 2m'M$. It is straightforward to check that $L_{\mu'}(F_\psi;s) = L_{\mu'\delta}(f_\psi;s)$ and $\Lambda_{N,\mu'}(F_\psi;s) = \Lambda_{N,\mu'\delta}(f_\psi;s)$.

On the other hand, we consider $g(\tau,z) = f|_{k,m\beta}[\widetilde{W}_N](\tau,z)$, which has a theta decomposition $g(\tau,z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\Theta_{m,\mu}(\frac{N}{\beta}\tau, Nz)$. From it we define

$$G(\tau,z) = \sum_{\mu'=1}^{2m'} G_{\mu'}(\tau)\Theta_{m',\mu'}\left(\frac{N}{\delta\beta}\tau, Nz\right) \quad \text{with } G_{\mu'}(\tau) = \sum_{\substack{\mu=1\\ \mu\equiv\mu'\ (2m')}}^{2m} g_\mu(\tau).$$

As before, one observes that for any primitive Dirichlet character ψ mod M one has $G_{\overline{\psi}}(\tau,z) = \sum_{\mu'=1}^{2m'M} G_{\overline{\psi},\mu'}(\tau)\Theta_{m'M,\mu'}(\frac{MN}{\delta\beta}\tau, Nz)$ with

$$G_{\overline{\psi},\mu'}(\tau) = \overline{\psi}(\delta) \sum_{\substack{\mu=1\\ \mu\equiv\mu'\ (2m'M)}}^{2mM} g_{\overline{\psi},\mu}(\tau).$$

In turn, these identities yield $L_{\mu'}(G_{\overline{\psi}};s) = \overline{\psi}(\delta)\sum_{\substack{\mu\ (2mM)\\ \mu\equiv\mu'\ (2m'M)}} L_\mu(g_{\overline{\psi}};s)$ and $\Lambda_{N,\mu'}(G_{\overline{\psi}};s) = \overline{\psi}(\delta)\sum_{\substack{\mu\ (2mM)\\ \mu\equiv\mu'\ (2m'M)}} \Lambda_{N,\mu}(g_{\overline{\psi}};s)$, for all $1 \le \mu' \le 2m'M$.

At this point, we use that part (ii) of Theorem 1 holds for $f(\tau,z)$. Hence for any primitive ψ of conductor $M \in \mathcal{M}\cup\{1\}$ and every $1 \le \mu' \le 2m'M$, the series $\Lambda_{N,\mu'}(F_\psi,s)$, $\Lambda_{N,\mu'}(G_{\overline{\psi}},s)$ satisfy the functional equations

$$\begin{aligned}\Lambda_{N,a'}(F_\psi;s) &= i^k C_\psi \sqrt{\frac{\beta}{2mM\sqrt{N}}} \sum_{\mu=1}^{2mM} e\left(\frac{a'\delta\mu}{2mM}\right)\Lambda_{N,\mu}\left(g_{\overline{\psi}};k-s-\frac{1}{2}\right)\\ &= i^k \widetilde{C}_\psi \sqrt{\frac{\delta\beta}{2m'M\sqrt{N}}} \sum_{\mu'=1}^{2m'M} e\left(\frac{a'\mu'}{2m'M}\right)\Lambda_{N,\mu'}\left(\delta^{-1}G_{\overline{\psi}};k-s-\frac{1}{2}\right),\end{aligned}$$

where $C_\psi = \chi(M)\psi(-\beta N)\mathcal{G}_\psi\mathcal{G}_{\overline{\psi}}^{-1}$ and $\widetilde{C}_\psi = \psi(\delta)C_\psi$ for every $1 \le a' \le 2m'M$. In other words, part (ii) of the previous theorem holds for the $4m'$ sequences of Fourier coefficients determined by $F(\tau, z)$ and $\delta^{-1}G(\tau, z)$. The equivalence stated in Theorem 1 proves the corollary. □

As a consequence of the previous result, we can give a simple proof for the lifts described in Corollaries 2 and 3 from $J^{\text{cusp}}_{k,m,\chi}(\Gamma_0(N) \ltimes \mathbb{Z}^2)$ to the space $\mathfrak{S}_{k-\frac{1}{2}}(4Nm, (\frac{Nm}{\cdot})(\frac{-1}{\cdot})^k\chi)$ of elliptic cusp forms of weight $k-1/2$ and character $(\frac{Nm}{\cdot})(\frac{-1}{\cdot})^k\chi$ over the group $\Gamma_0(4Nm)$. Since the arguments are very similar, we just prove Corollary 3.

Proof of Corollary 3 Let $f(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau)\Theta_{m,\mu}(\tau, z)$ be a Jacobi form in $J^{\text{cusp}}_{k,m,\chi}(\Gamma_0(N) \ltimes \mathbb{Z}^2)$ and $g(\tau, z) = f|_{k,m}[\widetilde{W}_N](\tau, z)$ with theta decomposition $g(\tau, z) = \sum_{\mu=1}^{2m} g_\mu(\tau)\Theta_{m,\mu}(N\tau, Nz)$. If we apply Corollary 1 to $g(\tau, z)$ with $\delta = m$, we obtain that

$$G(\tau, z) = g_m(\tau)\Theta_{m,m}(N\tau, Nz) + g_{2m}(\tau)\Theta_{m,2m}(N\tau, Nz)$$

is a Jacobi form in $J^{\text{cusp}}_{k,mN,\overline{\chi}}(\Gamma_0(N') \ltimes (\mathbb{Z} \times (mN)^{-1}\mathbb{Z})\langle\zeta_{mN}\rangle)$ where $N' = Nm/\gcd(N, m)$.

As $\Theta_{m,m}(\tau, z) = \vartheta_2(2m\tau, 2mz)$ and $\Theta_{m,2m}(\tau, z) = \vartheta_3(2m\tau, 2mz)$, where $\vartheta_2(\tau, z) = \sum_{l\in\mathbb{Z}} e^{\pi i(l+1/2)^2\tau} e^{2\pi i(l+1/2)z}$ and $\vartheta_3(\tau, z) = \sum_{l\in\mathbb{Z}} e^{\pi i l^2\tau} e^{2\pi i l z}$ are the classical Jacobi theta series, we can write

$$G(\tau, z) = g_m(\tau)\vartheta_2(2mN\tau, 2mNz) + g_{2m}(\tau)\vartheta_3(2mN\tau, 2mNz), \quad (30)$$

$$G\left(\tau, z + \frac{1}{2mN}\right) = -g_m(\tau)\vartheta_2(2mN\tau, 2mNz) + g_{2m}(\tau)\vartheta_3(2mN\tau, 2mNz). \quad (31)$$

From these equations and $\vartheta_2(2\tau, \tau) = e(-\tau/4)\vartheta_3(2\tau, 0)$, we deduce

$$G\left(\tau, \frac{\tau}{2}\right) - G\left(\tau, \frac{\tau}{2} + \frac{1}{2mN}\right) = 2g_m(\tau)e^{mN}\left(-\frac{\tau}{4}\right)\vartheta_3(2mN\tau, 0). \quad (32)$$

On the other hand, from $e^{mN}(\frac{\tau}{4})G(\tau, \frac{\tau}{2}) = G|_{k,mN}[I_2, \frac{1}{2}, 0, 1](\tau, 0)$ and $e^{mN}(\frac{\tau}{4} + \frac{1}{4mN})G(\tau, \frac{\tau}{2} + \frac{1}{2mN}) = G|_{k,mN}[I_2, \frac{1}{2}, \frac{1}{2mN}, 1](\tau, 0)$, one gets that $e^{mN}(\frac{\tau}{4})G(\tau, \frac{\tau}{2}) + ie^{mN}(\frac{\tau}{4} + \frac{1}{4mN})G(\tau, \frac{\tau}{2} + \frac{1}{2mN})$ is an elliptic cusp form of weight k and character $\overline{\chi}$ over the group $\Gamma_0(4Nm)$ (in this argument we use the hypothesis $4|mN$). Next we recall that $\vartheta_3(2\tau, 0)^2$ is a modular form of weight 1 and character $(\frac{-1}{\cdot})$ over $\Gamma_0(4)$. Hence $\vartheta_3(2mN\tau, 0)^{2k}$ is a modular form of weight k and character $(\frac{-1}{\cdot})^k$ over $\Gamma_0(4Nm)$. Putting together these facts and (32), one gets that $g_m(\tau)$ is a cusp form in $\mathfrak{S}_{k-\frac{1}{2}}(4Nm, (\frac{-1}{\cdot})^k\overline{\chi})$.

Next we consider

$$\sum_{\mu=1}^{2m} e\left(-\frac{\mu}{2}\right)\Lambda_{N,\mu}(f; s) = i^k\left(\frac{2m}{\sqrt{N}}\right)^{1/2}\Lambda_{N,m}\left(g; k - s - \frac{1}{2}\right).$$

This is one of the functional equations satisfied by $f(\tau, z)$ according to Theorem 1 (or Corollary 4), and it is equivalent to

$$\left(\frac{2\pi}{\sqrt{4Nm}}\right)^{-s}\Gamma(s)L(F;s) = \left(\frac{2\pi}{\sqrt{4mN}}\right)^{s-k+\frac{1}{2}}\Gamma\left(k-\frac{1}{2}-s\right)L\left(\widehat{g}_m;k-\frac{1}{2}-s\right), \tag{33}$$

where $F(\tau)=\sum_{\mu=1}^{2m}(-1)^\mu f_\mu(4m\tau)$, $\widehat{g}_m(\tau)=i^k(\sqrt{4m})^{-k+3/2}(4N)^{-1/4}g_m(\tau)$ and $L(F;s)$ (resp., $L(g_m;s)$) denotes the usual Dirichlet series associated to the Fourier expansion of $F(\tau)$ (resp., $g_m(\tau)$). This is the other place where we use that $4|Nm$, since then $L(g_m;s)=L_m(g;s)$. At any rate, (33) yields that $F(\tau)$ is the image of $\widehat{g}_m(\tau)$ under the Fricke involution of half-integral weight modular forms. Using now a basic property of such an involution [16, p. 448], we deduce that $F(\tau)$ is in $\mathfrak{S}_{k-\frac{1}{2}}(4Nm,(\frac{Nm}{\cdot})(\frac{-1}{\cdot})^k\chi)$. Corollary 3 follows from this and Corollary 2. □

Two final remarks One can generalize Corollaries 2 and 3 a bit. If $f(\tau,z)=\sum_{\mu=1}^{2m}f_\mu(\tau)\Theta_{m,\mu}(\tau,z)$ is a Jacobi form in $J_{k,m,\chi}^{\text{cusp}}(\Gamma_0(N)\ltimes\mathbb{Z}^2)$ and δ is a positive divisor of m, say $m=m'\delta$, then Corollary 1 yields

$$F(\tau,z)=\sum_{\mu'=1}^{2m'}f_{\mu'\delta}(\tau)\Theta_{m',\mu'}(\delta\tau,\delta z)\in J_{k,m'\delta,\chi}^{\text{cusp}}\big(\Gamma_0(N')\ltimes\big(\mathbb{Z}\times\delta^{-1}\mathbb{Z}\big)\langle\zeta_\delta\rangle\big)$$

where $N'=N\delta/\gcd(\delta,N)$. Clearly, we cannot apply Corollary 2 to $F(\tau,z)$ as it is. However, it is possible to extend Corollary 2 using the final remark of Sect. 5 and then apply it to $F(\tau,z)$. In order to keep this short, we leave the details to the reader and just state the conclusion obtained in this way.

$$F_\delta(\tau)=\sum_{\substack{\mu=1\\ \delta|\mu}}^{2m}f_\mu(4m'\tau)\in\mathfrak{S}_{k-\frac{1}{2}}\left(4N'm,\left(\frac{N'm'}{\cdot}\right)\left(\frac{-1}{\cdot}\right)^k\chi\right).$$

As a final comment, we note that the lift $f(\tau,z)\mapsto F_m(\tau)$ established in Corollary 3 is compatible with the action of the Hecke algebra whenever k is even, $\chi=1$ is the trivial character and Nm is a square divisible by 4.

More precisely, if $f(\tau,z)\in J_{k,m,1}^{\text{cusp}}(\Gamma_0(N)\ltimes\mathbb{Z}^2)$, T_l^J denotes the lth Hecke operator of Jacobi forms for any prime l with $\gcd(l,2mN)=1$ (see the definition in [5]) and $h(\tau,z)=T_l^J f(\tau,z)$, then the corresponding half-integral weight modular forms $F_m(\tau)$ and $H_m(\tau)$ associated to $f(\tau,z)$ and $h(\tau,z)$ as indicated in Corollary 3 satisfy $H_m(\tau)=T_lF_m(\tau)$ where T_l is the lth Hecke operator of half-integral weight modular forms (denoted as $T_{2k-1,1}^{4Nm}(l^2)$ in [16]). This statement follows from comparing the Fourier coefficients of $T_l^J f(\tau,z)$ (see [19, p. 119], for example) with those of $T_lF_m(\tau)$ (in [16, p. 450]) and observing in them the lift $h(\tau,z)\mapsto H_m(\tau)$ whenever the quadratic symbol $(\frac{Nm}{l})$ is 1.

References

1. Berndt, R.: L-functions for Jacobi forms á la Hecke. Manuscr. Math. **84**, 101–112 (1994)
2. Bringmann, K., Hayashida, S.: A converse theorem for Hilbert–Jacobi forms. Rocky Mt. J. Math. **39**, 423–435 (2009)
3. Cogdell, J., Piatetski-Shapiro, I.: Converse theorem for GL_n. J. Reine Angew. Math. **507**, 165–188 (1999)
4. Doi, K., Naganuma, H.: On the functional equation of certain Dirichlet series. Invent. Math. **9**, 1–14 (1969)
5. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Birkhäuser, Boston (1985)
6. Hawkins, J., Knopp, M.: A Hecke–Weil correspondence theorem for automorphic integrals on $\Gamma_0(N)$, with arbitrary rational period functions. Contemp. Math. **143**, 451–475 (1993)
7. Hecke, E.: Über die Bestimmung Dirichletscher Reichen durch ihre Funktionalgleichung. Math. Ann. **112**, 664–699 (1936) (Werke 33)
8. Imamoglu, O., Martin, Y.: On characteristic twists of multiple Dirichlet series associated to Siegel cusp forms. Math. Z. **263**, 345–368 (2009)
9. Jacquet, H., Langlands, R.: Automorphic Forms on $GL(2)$. Springer Lecture Notes in Math., vol. 278. Springer, Berlin (1972)
10. Jacquet, H., Piatetski-Shapiro, I., Shalika, J.: Automorphic forms on $GL(3)$. Ann. Math. **109**, 169–212 (1979)
11. Manickam, M., Ramakrishnan, B., Vasudevan, T.: On Saito–Kurokawa descent for congruence subgroups. Manuscr. Math. **81**, 161–182 (1993)
12. Manickam, M., Ramakrishnan, B.: On Shimura, Shintani and Eichler–Zagier correspondences. Trans. Am. Math. Soc. **352**, 2601–2617 (2000)
13. Martin, Y.: A converse theorem for Jacobi forms. J. Number Theory **61**, 181–193 (1996)
14. Maass, H.: Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. Math. Ann. **121**, 141–183 (1949)
15. Miyake, T.: Modular Forms. Springer, Berlin (1989)
16. Shimura, G.: On modular forms of half-integral weight. Ann. Math. **97**, 440–481 (1973)
17. Skogman, H.: Generating Jacobi forms. Ramanujan J. **10**, 325–339 (2005)
18. Skoruppa, N.-P.: Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts. Bonner Math. Schriften, vol. 159, Bonn (1985)
19. Skoruppa, N.-P., Zagier, D.: Jacobi forms and a certain space of modular forms. Invent. Math. **94**, 113–146 (1988)
20. Weil, A.: Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. Math. Ann. **168**, 149–156 (1967)