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SINGULARITY FORMATION FOR THE HARMONIC MAP FLOW FROM A VOLUME INTO S^2

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERA CIVIL MATEMÁTICO

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POR: CATALINA LETICIA PESCE REYES
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FORMACIÓN DE SINGULARIDADES PARA EL FLUJO DEL MAPA ARMÓNICO DESDE UN VOLUMEN HACIA S^2

Consideramos un volumen $V \subset \mathbb{R}^3$ generado al rotar alrededor del eje Z un dominio $\Omega \subset \mathbb{R}^2$ acotado y suave que vive en el plano XZ. En este trabajo se construye una solución del flujo de mapa armónico del volumen V a la esfera S^2 que revienta en tiempo finito, el problema es

$$\begin{aligned} v_t &= \Delta v + |\nabla v|^2 v \text{ in } V \times (0,T) \\ v &= v_{\partial V} \text{ in } \partial V \times (0,T) \\ v(\cdot,0) &= v_0 \text{ in } V, \end{aligned}$$

donde $v : V \times [0,T) \to S^2$, $v_0 : \overline{V} \to S^2$ es suave y $v_{\partial V} = v_0|_{\partial V} : \partial V \to S^2$. Dado un punto $q \in \Omega$ de define la circunferencia $c(q)$ generada al rotar el punto q alrededor del eje Z. Se encuentran datos iniciales y de frontera tales que la solución v revienta exactamente en la curva $c(q)$ en un tiempo finito pequeño. La construcción de la solución se hace reduciendo el problema a 2 dimensiones y usando el método de Dávila, Del Pino y Wei [7] que transforma el problema en un sistema de *inner-outer gluing* que separa el efecto principal de la ecuación cerca y lejos de la singularidad. Se obtiene una solución cuyo orden principal cerca de la singularidad tiene el perfil de un mapa armónico 1-corrotacional escalado.

En la introducción se recuerda la ecuación de flujo de mapa armónico y su origen, se establece el problema y la reducción a 2 dimensiones. En el primer capítulo se enuncian resultados útiles de topología y análisis funcional, y propiedades probadas en [7] para los mapas armónicos 1-corrotacionales y el operador linealizado en torno a ellos. En el segundo capítulo se obtiene un ansatz de la solución y se usa el método de Dávila, Del Pino y Wei [7] para reducir el problema a resolver un sistema de *inner-outer gluing* que después se resuelve usando punto fijo. En el capítulo cuatro se obtienen las hipótesis para el punto fijo mediante estimaciones a priori obtenidas dividiendo el sistema en tres problemas principales: el problema interior, el problema exterior y el problema de los parámetros. En la parte final se concluye con algunas observaciones sobre este trabajo y posibles trabajos futuros en torno a el.

ABSTRACT OF THE THESIS TO QUALIFY TO
THE DEGREE OF MASTER IN ENGINEERING SCIENCES,
MENTION IN APPLIED MATHEMATICS
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BY: CATALINA LETICIA PESCE REYES
DATE: JULY 2018
ADVISOR: MR. MANUEL DEL PINO MANRESA

SINGULARITY FORMATION FOR THE HARMONIC MAP FLOW FROM A VOLUME INTO S^2

Consider a volume $V \subset \mathbb{R}^3$ generated by rotating around the Z axis a bounded smooth domain $\Omega \subset \mathbb{R}^2$ that lives in the XZ plane. We construct a finite time blow-up solution to the harmonic map flow from volume V into the sphere S^2, the problem is

$$\begin{aligned} v_t &= \Delta v + |\nabla v|^2 v \text{ in } V \times (0, T) \\ v &= v_{\partial V} \text{ in } \partial V \times (0, T) \\ v(\cdot, 0) &= v_0 \text{ in } V, \end{aligned}$$

where $v : V \times [0, T) \to S^2$, $v_0 : \overline{V} \to S^2$ is smooth and $v_{\partial V} = v_0|_{\partial V} : \partial V \to S^2$. Given a point $q \in \Omega$ we define the circumference $c(q)$ generated by the rotation of q around Z axis. We find initial and boundary data so that the solution v blows up at exactly the curve $c(q)$ at a finite small time. The construction of the solution is done by reducing the problem to 2 dimensions and using the method of Dávila, Del Pino and Wei [7] that transforms the problem into an *inner-outer gluing* system which separates the main effect of the equation near and far away from the singularity. We obtain a solution that at main order has the profile of a scaled 1-corrotational harmonic map near the singularity.

In the introduction we recall the harmonic map flow equation and its origin, we set the problem and the reduction to 2 dimensions. In the first chapter we recall useful results of topology, functional analysis and properties proved in [7] for 1-corrotational harmonic maps and the linearized operator around them. In the second chapter we obtain an ansatz of the solution and use the method of Dávila, Del Pino and Wei [7] to reduce the problem to solving an *inner-outer gluing* system, which we solve with a fixed point argument. In chapter four obtain the hypothesis for the fixed point through a priori estimates obtained by dividing the system into three main problems, the inner problem, the exterior problem and the parameter problem. In the final part we conclude with some remarks about this work and possible future work related to it.

-Pero yo no quiero andar entre locos- observó Alicia.
-¡Ah!, no podrás evitarlo- dijo el Gato -:
aquí estamos todos locos. Yo estoy loco. Tu estás loca.
-¿Cómo sabes que estoy loca?- dijo Alicia.
-Tienes que estarlo- dijo el Gato, -o no habrías acudido aquí.
- Lewis Carroll, Alicia en el país de las Maravillas

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Contents

Introduction

We start by mentioning the original harmonic map flow equation and part of its history, this will set a relevant context for our work. We define harmonic maps in the same way as Lin, Wang in their book [18]. Let (M, g) be a Rimannian manifold of dimension m with metric g and (N, h) a Rimannian manifold of dimension n with metric h. For any map $u \in C^2(M, N)$ we can define its Dirichlet energy as follows. For any fixed $p \in M$, there exist two normal coordinate charts $U_p \subset M$ of p and $V_q \subset N$ of $q = u(p)$ such that $u(U_p) \subset V_q$. The Dirichlet energy density function $e(u)$ is defined by

$$e(u)(x) = \frac{1}{2} \sum_{\alpha,\beta} \sum_{i,j} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta},$$

where (x_α) and (u^i) are the coordinate system on U_p and V_q respectively. The Dirichlet energy functional is defined by

$$E(u) = \int_M e(u) dv_g.$$

A map $u \in C^2(M, N)$ is a harmonic map if it is a critical point of the Dirichlet energy functional E, which is decreasing along smooth solutions.

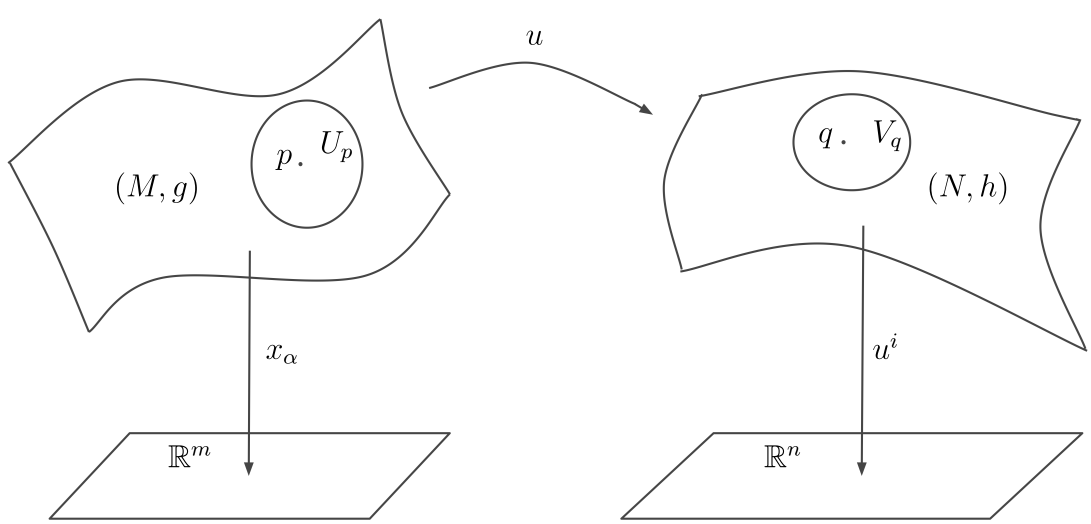

Figure 1: Diagram of (M, g), (N, h) with coordinate system and function $u : M \to N$.

Another characterization of harmonic maps is that $u \in C^2(M, N)$ is a harmonic map if it is a solution of the following partial differential equation:

$$\Delta_g u + A(u)(\nabla u, \nabla u) = 0 \quad \text{ in } M,$$

where

$$A(u)(\nabla u, \nabla u) = \sum_i g^{\alpha\beta} A^i(u)(u_\alpha, u_\beta)\nu_i(u),$$

and $A^i = \nabla \nu_i$ is the second fundamental form of N in the normal direction ν_i.

In 1964 Eells and Sampson [14] proposed to study the evolution equation associated to harmonic maps. The evolution problem, also called harmonic map flow or harmonic map heat flow, can be formulated as follows: for any $u_0 \in C^\infty(M, N)$, find $u : M \times \mathbb{R}_+ \to N$ that solves

$$\begin{aligned} \partial_t u - \Delta_g u &= A(u)(\nabla u, \nabla u) \quad \text{ in } M \times (0, +\infty), & (1)\\ u|_{t=0} &= u_0. & (2) \end{aligned}$$

Notice that (1) is the negative L^2-gradient flow of the Dirichlet energy E.

This harmonic map flow equation has several applications in physics and in mathematics. We outline briefly a few of them. In differential geometry the equation is used to study deformation of Riemannian surfaces in Teichmüller theory, it is also involved in the study of isometric embeddings and has to do with the analysis of many particular surfaces. For more see Lin and Wang [18]. In physics some of the more theoretical uses are mentioned by Misner [19], which are in Gauge field theory, particle theory and Einstein's equations. On the more applied end of things we have that the harmonic map flow is part of a coupled equation with Navier-Stokes equation that arises in the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals, which are used in the screens of most modern electronic devices. See Lin, Lin and Wang [16]. In addition, the harmonic map flow equation is also related to the Ginzburg-Landau model for superconductivity.

Let us now focus on some known properties of the equation. From now on, consider the target manifold $N = S^2$, where S^2 is the standard unit sphere in $\mathbb{R}^3$ with euclidean metric. And consider a bounded open domain $M \subset \mathbb{R}^m$ with euclidean metric, we are especially interested in dimension $m = 2$, where M is a flat domain, and $m = 3$, where M is a volume. In this case the harmonic map flow equation until a time $T > 0$ can be written as

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \text{ in } M \times (0, T), & (3)\\ u &= u_{\partial M} \text{ in } \partial M \times (0, T), & (4)\\ u(\cdot, 0) &= u_0 \text{ in } M, & (5) \end{aligned}$$

where $u : M \to S^2$.

Eells and Sampson [14] established the existence of short time smooth solutions of (1)-(2) until a time $T_{max} > 0$, where loss of smoothness occurs. the authors characterized time T_{max} as

$$\lim_{t \uparrow T_{max}} \|\nabla u(\cdot, t)\|_\infty = +\infty.$$

This phenomenon of regularity loss and explosion is called blow-up and T_{max} is denominated blow-up time. It has been shown that this characterization of the blow-up time is optimal in domain dimension $m = 2$ but is not optimal for $m \geq 3$, Wang [33] has proved that the optimal characterization for higher dimensions is

$$\lim_{t \uparrow T_{max}} \|\nabla u(\cdot, t)\|_{L^m(M)} = +\infty.$$

Struwe [25] proved for $m = 2$ the existence of H^1-weak solutions, where just for a finite number of points in space-time loss of regularity occurs. More precisely, the following fact follows from results by Ding-Tian [6], Lin-Wang [17], Qing [22], Qing-Tian [23], Struwe [25], Topping [28], and Wang [32]: Along a sequence $t_n \to T$ and points $q_1, \ldots, q_k \in M$, not necessarily distinct, $u(x, t_n)$ blow-up occurs at exactly those k points in the form of *bubbling*. Precisely, we have

$$u(x, t_n) - u_*(x) - \sum_{i=1}^{k} \left[U_i \left(\frac{x - q_i}{\lambda_i^n} \right) - U_i(\infty) \right] \to 0 \quad \text{in } H^1(M),$$

where $u_* \in H^1(M)$, $q_i^n \to q_i$, $\lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.$$

Here U_i are solutions $U : \mathbb{R}^2 \to S^2$ of the stationary harmonic map equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

From Topping [28] we have that the energy of these harmonic maps corresponds to the absolute value of their degree $l \in \mathbb{N}$, times the area of the unit sphere,

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi l.$$

In particular, $u(\cdot, t_n) \rightharpoonup u_*$ in $H^1(M)$ and for some positive integers l_i we have

$$|\nabla u(\cdot, t_n)|^2 \rightharpoonup |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi l_i \delta_{q_i},$$

in the measure sense, where δ_q denotes the unit Dirac mass at q.

Struwe [26], [27] also showed that something similar happens for $m \geq 3$, moreover, he proved that the set where the solution loses regularity has locally finite $(m-2)$-dimensional Hausdorff measure with respect to the euclidean metric in $\mathbb{R}^m$ under some assumptions on the initial time. Notice that in $\mathbb{R}^3$ smooth curves of finite length have finite 1-dimensional Hausdorff measure. There are more specific results about the dimension of the set of singularities when N does not support S^2, which is not the case that concerns us. We refer interested readers to chapter 8 of Lin and Wang [18].

In the present work we will construct an example of a blow-up solution of the equation for $m = 3$ that has a 2-dimensional flavour. With this in mind, we refer now to some previously known examples of solutions. For $m = 2$ the few examples of blow-up solutions that exist are concerned with single-point blow-up in radially symmetric corrotational classes. When M is a disk or the entire space, a 1-corrotational solution of (3) is of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(\rho,t) \\ \cos v(\rho,t) \end{pmatrix}, \quad x = \rho e^{i\theta}.$$

Within this class, (3) reduces to

$$v_t = v_{\rho\rho} + \frac{v_\rho}{\rho} - \frac{\sin v \cos v}{\rho^2}. \tag{6}$$

It is known that the function

$$w(\rho) = \pi - 2\arctan(\rho)$$

is a steady state of (6).

Observation 1 Notice that the following function

$$W(x) = \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}, \quad x \in \mathbb{R}^2 \tag{7}$$

is a least energy entire non-trivial harmonic map, which has finite energy,

$$\int_{\mathbb{R}^2} |\nabla W|^2 = 4\pi, \quad W(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that

$$W(x) = \begin{pmatrix} e^{i\theta} \sin w(\rho,t) \\ \cos w(\rho,t) \end{pmatrix}$$

is a 1-corrotational solution of (3). We will refer to this solution as *bubble* for its form.

The first example of blow-up found for $m = 2$ was done by Chang, Ding and Ye [3] and has the following profile

$$u(x,t) = W\left(\frac{x}{\lambda(t)}\right) + O(1),$$

with $O(1)$ bounded in H^1 norm and $0 < \lambda(t) \to 0$ as $t \to T$. Van den Berg, Hulshof and King [29] found that the blow-up rate $\lambda(t)$ for 1-corrotational maps can be generically given by

$$\lambda(t) \approx \kappa \frac{T-t}{|\log(T-t)|^2}, \tag{8}$$

for some $\kappa > 0$. Raphael and Schweyer [20] constructed rigurously an entire 1-corrotational solution with this blow-up rate using methods from dispersive equations. Their method relies heavily on domain symmetry. Recently Dávila, Del Pino and Wei [7] were able to also construct rigorously a 1-corrotational solution with the same blow-up rate, this solution blows up in finite time on any finite set of given points in M. This result is valid for any bounded open smooth domain $M \subset \mathbb{R}^2$, without any symmetry required.

On the counterpart, for $m = 3$ Grotowski [13] constructed weak solutions with finite time blow-up when $M = B^3$ is the unit ball in $\mathbb{R}^3$. This example relies on the symmetry of B^3 and does not give information on the blow-up rate and set. In higher dimensions $m \geq 7$ Biernat [2] constructs solutions with blow-up rates

$$\lambda(t) \approx \frac{-C\sqrt{T-t}}{\log(T-t)+\kappa} \text{ for } m = 7,$$
$$\lambda(t) \approx \kappa(T-t)^{\frac{1}{2}+\beta}, \text{ for } m > 7, \kappa \in \mathbb{R}, \beta > 0.$$

We notice that there seems to be a transition on the blow-up rates between dimensions.

From this review, we see that there are not many examples of solutions of (1)-(2) for higher dimensions, particularly for $m \geq 3$. We hope this work contributes with an interesting example of blow-up on a domain of dimension $m = 3$.

To state our result we first observe the following:

Observation 2 Consider the α-rotation matrix around the z-axis

$$e^{J\alpha} := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define

$$U_{\lambda,q,\alpha}(x) = e^{J\alpha} W\left(\frac{x-q}{\lambda}\right),$$

where W is defined in (7) and $\lambda > 0$, $q \in \mathbb{R}^2, \alpha \in [0, 2\pi]$. These functions solve problem (3)-(5) and satisfy the least energy property

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,\alpha}|^2 = 4\pi.$$

Now, consider the domain $M = V$ a volume in $\mathbb{R}^3$ with euclidean metric, where V is defined as follows: let $\Omega \subset \mathbb{R}^2$ be a bounded domain as a subset of the XZ plane on the three dimensional space and define $V \subset \mathbb{R}^3$ the volume generated by rotating Ω around the Z axis.

Then equation (1)-(2) corresponds to the following problem

$$v_t = \Delta v + |\nabla v|^2 v \text{ in } V \times (0,T), \tag{9}$$
$$v = v_{\partial V} \text{ in } \partial V \times (0,T), \tag{10}$$
$$v(\cdot, 0) = v_0 \text{ in } V, \tag{11}$$

for a function $v : V \times [0,T) \to S^2$. Here $v_0 : \overline{V} \to S^2$ and $v_{\partial V} : \partial V \to S^2$ are given functions.

The main result of this thesis is the following:

Theorem 0.1 *Given a point $q = (q_1, q_2) \in \Omega$, $q \gg 0$, define*

$$c(q) = \{(q_1\cos\theta, q_1\sin\theta, q_2) \in \mathbb{R}^3 : \theta \in [0, 2\pi)\}$$

the circumference generated by rotating q around the Z axis. Given $T > 0$ sufficiently small, there exist v_0 such that the solution $v_{c(q)}(x,t)$ of problem (9)-(11), for $v_{\partial\Omega} = (0,0,1)$, blows-up at the circumference $c(q)$ as $t \uparrow T$. This solution is symmetric with respect to the z axis and can be written as $v_{c(q)}(x,t) = u_q(r,z,t)$, where $r = \sqrt{x^2+y^2}$ and $(r,z) \in \Omega$. More precisely, there exist numbers $\kappa^* > 0$, α^* and a function $u_* \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$u_q(r,z,t) - u_*(r,z) - e^{J\alpha^*}\left[W\left(\frac{(r,z)-q}{\lambda(t)}\right) - W_\infty\right] \to 0 \quad \text{as} \quad t \uparrow T,$$

in the H^1 and uniform senses in Ω, where the blow-up rate is given by

$$\lambda(t) = \kappa^* \frac{T-t}{|\log(T-t)|^2}(1+o(1)) \quad \text{as} \quad t \uparrow T.$$

In particular, we have

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi\delta_q.$$

Observation 3 Notice that the value $v_{\partial\Omega} = (0,0,1)$ corresponds to $W(\infty)$.

Observation 4 Notice that the result is stated for $m = 3$ but the symmetry of the domain V makes the constructed solution exhibit two dimensional phenomena, we can see this in the way the blow-up occurs, matching the reviewed results for dimension 2.

Let us materialize, for didactic purposes, volume V in a simple case when Ω is a circle, here V would be a solid torus as shown in Figure 2.

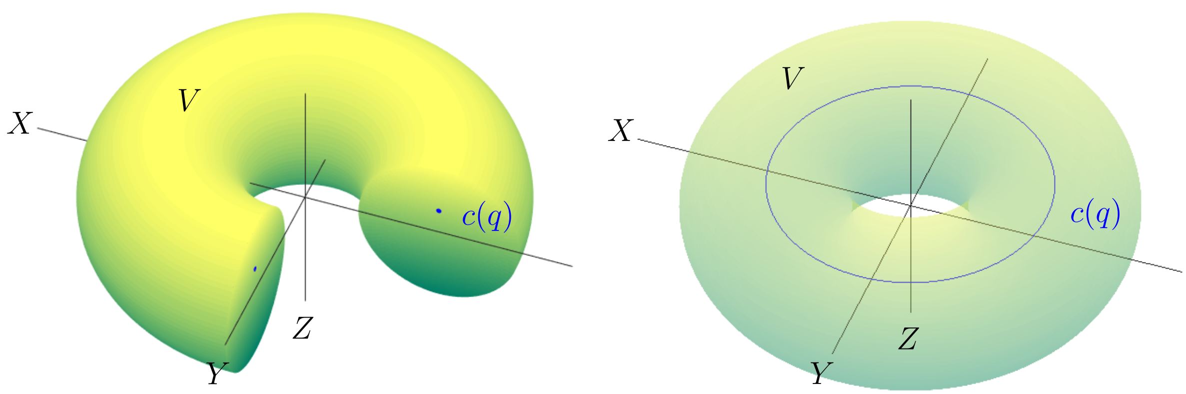

Figure 2: Left: Revolution volume V with cut and points of circumference $c(q)$. Right: Volume V with circumference $c(q)$ inside

The way in which we proceed is the following. We use the axial symmetry of V to transform the three dimensional problem into a two dimensional one. Let $(x,y,z) \in V$, we parametrize the volume V with $\theta \in [0,2\pi)$, $z \in (z_0,z_1)$, $r \in (z_1 - \sqrt{z_1^2 - z^2}, z_1 + \sqrt{z_0^2 - z^2})$, $t \in [0,T)$, the we use a cylindrical change of variables:

$$(x,y,z,t) = (r\cos(\theta), r\sin(\theta), z, t).$$

where $z_0 = \inf_{(x,z)\in\Omega} z$ and $z_1 = \sup_{(x,z)\in\Omega} z$. Since V has axial symmetry with respect to the Z axis then function v does not depend on θ and therefore we can redefine it as a function $u : \Omega \times [0, T) \to S^2$ that satisfies

$$v(x, y, z, t) = u(r, z, t).$$

Then we have that

$$v_t = u_t, \quad \Delta v = u_{rr} + \frac{1}{r} u_r + u_{zz}, \quad \nabla v = u_r \hat{r} + u_z \hat{z}.$$

Replacing this in equation (9) we arrive at

$$u_t = u_{rr} + u_{zz} + |\nabla u|^2 u + \frac{1}{r} u_r.$$

Let us change the names of the variables $x := (x_1, x_2) := (r, z)$, then the problem we have to consider is in two dimensions in the cross section Ω of V. Summarizing, the problem of finding v solution of (9)-(11) that blows up at T at curve $c(q)$ reduces to finding $u : \Omega \times [0, T) \to S^2$ that satisfies

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u + \frac{1}{x_1} u_{x_1} \text{ in } \Omega \times (0, T), && (12) \\ u &= u_{\partial\Omega} \text{ in } \partial\Omega \times (0, T), && (13) \\ u(\cdot, 0) &= u_0, \text{ in } \Omega && (14) \end{aligned}$$

and blows up at time T at point q. Here $u_0 : \overline{\Omega} \to S^2$ and $u_{\partial\Omega} : \partial\Omega \to S^2$ are given functions that correspond to the restrictions on Ω and $\partial\Omega$ of functions v_0 and $v_{\partial\Omega}$, repectively.

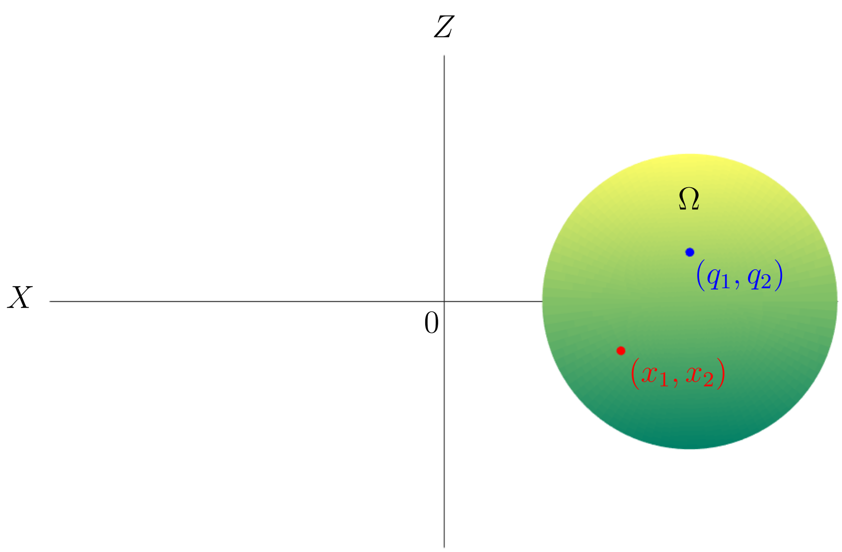

Figure 3: Cross section Ω with point q and variable x. Keep in mind the relationship with the original variables $x_1 = r$ and $x_2 = z$.

Now our problem is similar to the one treated by Dávila, Del Pino and Wei on [7], the only difference is the extra derivative on x_1. We explain the outline of their method. the authors start with a 1-corrotational solution of the form $u = U_{\lambda,\xi,\alpha}$ for λ, ξ, α time dependant parameters to be chosen, and then the authors linearize around this 1-corotational map by adding a small function φ to get $u = U_{\lambda,\xi,\alpha} + \varphi$. the authors compute the linearized operator and separate $\varphi = \varphi^i + \varphi^o$ into two functions that will have a role near and far away the blow-up point, respectively. Replacing this in the equation the authors obtain an *inner-outer gluing* system of equations. At the same time the authors find approximate equations for the parameters, that will be part of a fixed point scheme along with φ^i and φ^o. the authors obtain a priori estimates for the inner and outer parts of φ^i, φ^o and use this in a fixed point argument to obtain the solution in the wanted spaces with the expected blow-up rate. We will see that their method can be applied to problem (12)-(14).

Before we describe the parts of this work we would like to state the importance of the example obtained in Theorem 0.1. As it was mentioned before, there are few examples of blow-up solutions of the harmonic map flow on dimension $m = 3$, but most importantly, there are few examples of blow-up on curves in parabolic equations in general. This has to do with the set where blow-up occurs, which is one of the most important questions in blow-up analysis. When the set where this happens is a single point or a finite union of points the phenomenon is called single-point blow-up or LS-regime, when the set has positive measure and it is not the whole domain we call it regional blow-up or S-regime and when the set is the whole domain we call it global blow-up or HS-regime. More on this in the survey on parabolic blow-up analysis of Galaktionov and Vazquez [11] and the book of Samarskii, Elenin, Zmitrenko, Kurdyumov and Mikhailov [24]. In our case our blow-up region is a curve in $\mathbb{R}^3$ which has zero Lebesgue measure seen as a subset of $\mathbb{R}^3$, but is not a finite union of points, so we are in between single point and regional blow-up.

There are many results of single point blow-up in parabolic equations, we have already mentioned some for the harmonic map flow equation. Regional singularity formation is less common, there are results for some semilinear and nonlinear parabolic equations. Here we mention a few. For the one dimensional equation

$$u_t = \Delta u + u^p, \quad p > 0,$$

there is single point blow-up for $\beta > 2$ and regional blow-up when $m = p$, which in [24] (p.299) is proven to have Lebesgue measure π. In the same book (p.314) the authors study

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^\beta, \quad \sigma, \beta > 0,$$

and get single point blow-up for $\beta > \sigma + 1$ and regional blow-up for $\beta = \sigma + 1$, but no information is given about the measure of the set. In Galaktionov and Vasquez [9], [10] the authors find that

$$u_t = u_{xx} + (1+u)\log^\beta(1+u), \quad \beta > 0,$$

has single point blow-up in 0 for $\beta = 2$ and there is regional blow-up in a set containing the ball of radius π. There are more parabolic equations that have been studied in this sense, but the information given on the set is always divided into single-point, regional (seen as sets with positive Lebesgue measure in the space) and global. See for example Lacey [15], Chaves and Galaktionov [4], Galaktionov [8]. Something closer to our problem is the work of Velazquez [30], [31]. He treats equation

$$\begin{aligned} u_t &= \Delta u + u^p, \quad x \in \mathbb{R}^n, n \geq 1, 1 \leq p < \frac{n+2}{n-2}, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

and proves that the blow-up set has zero Lebesgue measure and that for solutions different from the uniform one the blow-up set has bounded $(n-1)$-dimensional Hausdorff measure on compact sets of $\mathbb{R}^n$. Another wrk that compares to our setting is the one of Del Pino, Musso and Wei [21], where the authors construct a blow-up solution for $\Omega \subset \mathbb{R}^n$ with $n \geq 7$, for the problem

$$\begin{aligned} u_t = \Delta u + |u|^{p-1}u, \quad & x \in \Omega, t \in (0,T), \\ u(x,t) = 0, \quad & x \in \partial\Omega, t \in (0,T), \\ u(x,t) > 0 \quad & x \in \Omega, t \in (0,T). \end{aligned}$$

The blow-up occurs on a circumference that approaches the border of the domain when $t \to T$. In addition, the authors give information on the blow-up rate. As we can see these results are important and help us understand singularity formation, but most of them have a descriptive nature and their intention is to classify blow-ups, instead of constructing them. It seems there is a lack of explicit constructions of solutions with finite time blow-up on curves for parabolic problems in higher dimensions. In this sense, the construction done here has special value.

Now, let us describe how this thesis is organized. In the first chapter, the first and second sections are devoted to recalling some useful results of topology, functional analysis and parabolic regularity, like Arzela-Ascoli theorem, Schauder's fixed point and Schauder estimates for parabolic equations. The third section of chapter 1 mentions properties proved by Dávila, Del Pino and Wei [7] for 1-corrotational harmonic maps and the linearized operator around them.

In the second chapter we obtain an ansatz of the solution that will look at main order like a scaled, translated and rotated 1-corrotational map plus a small function. Then we obtain a first approximation of the scaling, rotation and translation parameters, and use the method of Dávila, Del Pino and Wei [7] to reduce the problem to a final *inner-outer gluing* system. In the last section of this chapter we start by recalling a priori estimates obtained in [7], then we state the final system of equations as a fixed point problem

$$u = \mathcal{F}(u),$$

we provide the necessary results on compactness for $\mathcal{F}$, to be proven in the third chapter, and use Schauder's fixed point theorem to prove Theorem 0.1.

In chapter three we prove that the operator $\mathcal{F}$ meets Schauder's fixed point theorem conditions. We do this by first dividing the analysis of the system of equations into three main problems: the inner problem, the exterior problem and the parameter problem. Then we join the obtained estimates with the ones from [7] to prove that $\mathcal{F}$ goes from a closed ball into itself and is compact. In the final chapter we conclude with some remarks about this work, extensions and possible future work.

Chapter 1

Preliminaries

In this chapter we recall known results that will help us prove Theorem 0.1. In the first section we state some fundamental definitions and results of topology and nonlinear functional analysis. In the second section we state a helpful Schauder type regularity result for solutions of parabolic problems proven in the book by Wu and Yin and Wang [34]. Then, in the third section, we set the class of 1-corrotational functions in which we choose the main part of the solution of (12)-(14) and recall some properties of the linearized operator in that class proved by Dávila, Del Pino and Wei [7].

1.1 General analysis results

Consider a compact metric space X and the space $C(X)$ of real-valued continuous functions on X.

Definition 1.1 *A sequence of functions $(f_n)_{n\in\mathbb{N}} \in C(X)$ is said to be uniformly bounded if there exists $M > 0$ such that*

$$|f_n(x)| \leq M \quad \forall n \in \mathbb{N}, \quad \forall x \in X.$$

Definition 1.2 *A sequence of functions $(f_n)_{n\in\mathbb{N}} \in C(X)$ is said to be equicontinuous if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that*

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n \in \mathbb{N}.$$

Here δ must not depend on n or y.

Observation 5 Note that if a sequence of functions $(f_n)_{n\in\mathbb{N}} \subset C(X)$ is L-Lipschitz or α-Hölder continuous with L and α not depending on n, then $(f_n)_{n\in\mathbb{N}}$ is equicontinuous.

With the last definitions we can state the Arzelà-Ascoli theorem.

Theorem 1.3 *Let $(f_n)_{n\in\mathbb{N}} \in C(X)$ be a sequence of functions that is uniformly bounded and equicontinuous, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges uniformly.*

Let Y, Z be Banach spaces.

Definition 1.4 *An operator $F : Y \to Z$ is said to be compact if the image of any bounded subset of Y has compact closure on Z. In other words, if $W \subseteq Y$ is a bounded subset of Y, then $\overline{F(W)} \subset Z$ is compact.*

Observation 6 We introduce a useful characterization of compactness. An operator $F : Y \to Z$ is compact if for any bounded sequence $(y_n)_{n\in\mathbb{N}}$ in Y, the sequence $(F(y_n))_{n\in\mathbb{N}}$ has a convergent subsequence in Z.

We end this section with Schauder's fixed point theorem.

Theorem 1.5 *Let X be a real Banach space and $B \subset X$ a nonempty, closed, bounded, convex set. Let $F : B \to B$ be a compact operator. Then F has a fixed point in B, that is, there exists $x \in B$ such that*

$$F(x) = x.$$

.

1.2 Schauder estimates for a linear parabolic problem

Consider the initial-boundary value problem of a general linear parabolic equation stated as the following:

$$\begin{aligned} u_t - \sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{x_i x_j} u + \sum_{i=1}^{n} b_i(x,t)\partial_{x_i} u + c(x,t)u &= f(x,t) \text{ in } (x,t) \in Q_T && (1.1) \\ u(x,t) &= \varphi(x,t), \text{ in } (x,t) \in \partial_p Q_T, && (1.2) \end{aligned}$$

where $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $T > 0$, and $\partial_p Q_T = \partial\Omega \times \{0\}$. Here the coefficients a_{ij}, b_i, c satisfy the uniform parabolicity conditions, that is, for some constants $0 < \alpha \le \beta$,

$$\alpha|\xi|^2 \le a_{ij}(x,t)xi_i\xi_j \le \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ (x,t) \in Q_T.$$

Let us denote, for some $\gamma \in (0,1)$, the following continuous and Hölder seminorm:

$$|u|_{\gamma,\frac{\gamma}{2};Q_T} := \sup_{(x,t)\in Q_T} |u(x,t)| + \sup_{(x_1,t_1),(x_2,t_2)\in Q_T} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{|x_1 - x_2|^\gamma + |t_1 - t_2|^{\frac{\gamma}{2}}}.$$

We also define

$$|u|_{2+\gamma,1+\frac{\gamma}{2};Q_T} := \sum_{|s|+2r\le 2} |\partial_x^s \partial_t^r u|_{\gamma,\frac{\gamma}{2};Q_T},$$

and associate the corresponding Hölder spaces $C^{\gamma,\gamma/2}(\overline{Q}_T)$ and $C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$.

In the book by Wu and Yin and Wang [34] the authors use interior and near boundary estimates for the heat equation, and a finite covering technique to establish global Schauder estimates for the solution of (1.1)-(1.2). We write their result as the following theorem.

Theorem 1.6 *Let $\gamma \in (0,1)$, $\partial\Omega \in C^{2,\alpha}$, a_{ij}, b_i, $c \in C^{\gamma,\gamma/2}(\overline{Q}_T)$, $f \in C^{\gamma,\gamma/2}(\overline{Q}_T)$, $\varphi \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$. If $u \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ is the solution of the initial-boundary value problem* (1.1)-(1.2)*, then*

$$|u|_{2+\gamma,1+\frac{\gamma}{2};Q_T} \leq C(|f|_{\gamma,\frac{\gamma}{2};Q_T} + |\varphi|_{2+\gamma,1+\frac{\gamma}{2};Q_T} + \|u\|_{L^\infty(Q_T)}).$$

This theorem gives us Hölder continuity for the solution of linear parabolic problems when the right hand side of the equation and the initial-border condition are Hölder continuous.

1.3 1-corrotational harmonic maps and their linearized operator

Let us recall that the equation to solve is mainly

$$u_t = \Delta u + |\nabla u|^2 u + \frac{1}{x_1} u_{x_1} \text{ in } \Omega \times (0,T).$$

Along this work we will treat the term $\frac{1}{x_1}u_{x_1}$ as part of an error, in some sense it will be a second order term compared to the other ones in the equation. Then it is natural to study the operator associated with $\Delta u + |\nabla u|^2 u$, which we call the harmonic map operator. This section recalls the analysis and formulas obtained in [7] for this operator.

First, consider the harmonic map equation for functions $U : \mathbb{R}^2 \to S^2$,

$$\Delta U + |\nabla U|^2 U = 0 \text{ in } \mathbb{R}^2, \quad |U| = 1. \tag{1.3}$$

Consider, for $\xi \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, $\lambda > 0$, the family of solutions of (1.3) given by the following 1-corrotational harmonic map

$$U_{\lambda,\xi,\omega}(x) := Q_\omega W\left(\frac{x-\xi}{\lambda}\right),$$

where W is

$$W(y) = \frac{1}{1+|y|^2}\begin{pmatrix} 2y \\ |y|^2 - 1 \end{pmatrix}, \quad y \in \mathbb{R}^2, \tag{1.4}$$

and Q_ω is the ω-rotation matrix in the z-axis

$$Q_\omega := \begin{pmatrix} cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that

$$\partial_\omega Q_\omega = Q_\omega J_0, \quad J_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we rename $U = U_{\lambda,\xi,\omega}$, then the linearized operador for (1.3) around U is

$$L_U[\varphi] = \Delta_x\varphi + |\nabla_x U|^2\varphi + 2(\nabla_x\varphi \cdot \nabla_x U)U.$$

We will need expressions for the functions that live in the kernel of L_U, that means that we look for functions φ that satisfy $L_U[\varphi] = 0$. To find these, first we set $y = \frac{x-\xi}{\lambda}$ and using polar coordinates $y = \rho e^{i\theta}$ we obtain that

$$W(y) = \begin{pmatrix} e^{i\theta}\sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho).$$

We notice that

$$w_\rho = -\frac{2}{1+\rho^2}, \quad \sin w = -\rho w_\rho = \frac{2\rho}{1+\rho^2}, \quad \cos w = \frac{\rho^2-1}{1+\rho^2}.$$

Differentiating U with respect to the parameters λ, ξ, ω allows us to obtain expressions that annihilate the operator L_U, which can be written as

$$\begin{aligned} \partial_\lambda U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda}\rho w_\rho(\rho) Q_\omega E_1(y), \\ \partial_\omega U_{\lambda,\xi,\omega}(x) &= \rho w_\rho(\rho) Q_\omega E_2(y), \\ \partial_{\xi_1} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} w_\rho(\rho)[\cos\theta Q_\omega E_1(y) + \sin\theta Q_\omega E_2(y)], \\ \partial_{\xi_2} U_{\lambda,\xi,\omega}(x) &= \frac{1}{\lambda} w_\rho(\rho)[\sin\theta Q_\omega E_1(y) - \cos\theta Q_\omega E_2(y)], \end{aligned} \tag{1.5}$$

where

$$E_1(y) = \begin{pmatrix} e^{i\theta}\cos w(\rho) \\ -\sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix}.$$

Observation 7 $\{E_1(y), E_2(y)\}$ constitutes an orthonormal basis of the tangent space to S^2 at point $W(y)$. Also notice that combining this with the fact that U lives in S^2 we obtain that $E_1(y)$, $E_2(y)$ are pointwise orthonormal to $U(x)$.

It is useful to define

$$\begin{aligned} Z_{01}(y) &= \rho w_\rho E_1, \\ Z_{02}(y) &= \rho w_\rho E_2, \\ Z_{11}(y) &= w_\rho[\cos\theta E_1 + \sin\theta E_2], \\ Z_{12}(y) &= w_\rho[\sin\theta E_1 - \cos\theta E_2]. \end{aligned} \tag{1.6}$$

We also define, for a function $\phi(y)$, the following operator

$$L_W[\phi] = \Delta_y\phi + |\nabla_y W|^2\phi + 2(\nabla_y\phi \cdot \nabla W)W.$$

Because the derivatives of U annihilate L_U we have that $L_W[Z_{ij}] = 0$ for $i = 0, 1$, $j = 1, 2$. In addition to the elements in (1.6) there are two other relevant functions in the kernel of L_W, namely

$$\begin{aligned} Z_{-11}(y) &= \rho^2 w_\rho[\cos\theta E_1 - \sin\theta E_2], \\ Z_{-12}(y) &= \rho^2 w_\rho[\sin\theta E_1 + \cos\theta E_2]. \end{aligned} \tag{1.7}$$

Observation 8 The operators mentioned before satisfy

$$L_U[\varphi] = \frac{1}{\lambda^2} Q_\omega L_W[\phi], \quad \varphi(x) = \phi(y), \quad y = \frac{x-\xi}{\lambda}.$$

It will be important to compute the action of L_U on functions with values pointwise orthogonal to U, so we will cite without proving various formulas derived in chapter 2 of [7].

For a function $\Phi(x)$ with values in $\mathbb{R}^3$ we denote

$$\Pi_{U^\perp}\Phi := \Phi - (\Phi \cdot U)U.$$

Then one has the following formula

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + \tilde{L}_U[\Phi], \tag{1.8}$$

where

$$\tilde{L}_U[\Phi] := |\nabla U|^2 \Pi_{U^\perp}\Phi - 2\nabla(\Phi \cdot U)\nabla U,$$

and

$$\nabla(\Phi \cdot U) = \partial_{x_j}(\Phi \cdot U)\partial_{x_j} U.$$

There is another convenient expression of $\tilde{L}_U[\Phi]$ using polar coordinates. Writing in complex notation

$$\Phi(x) = \Phi(r, \theta), \quad x = \xi + re^{i\theta}.$$

Then

$$\tilde{L}_U[\Phi] = -\frac{2}{\lambda} w_\rho(\rho)\left[(\Phi_r \cdot U)Q_\omega E_1 - \frac{1}{r}(\Phi_\theta \cdot U)Q_\omega E_2\right], \quad \rho = \frac{r}{\lambda}. \tag{1.9}$$

This last formula has an important consequence. Assuming that $\Phi : \Omega \to \mathbb{C} \times \mathbb{R}$ is a C^1 function, which we express in the form

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) \end{pmatrix}.$$

We also denote

$$\varphi = \varphi_1 + i\varphi_2,$$

and define the operators

$$\text{div } \varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \text{curl } \varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

Then

$$\tilde{L}_U[\Phi] = \tilde{L}_U[\Phi]_0 + \tilde{L}_U[\Phi]_1 + \tilde{L}_U[\Phi]_2 \tag{1.10}$$

where

$$
\begin{array}{rcl}
\tilde{L}_U[\Phi]_0 & = & \lambda^{-1}\rho w_\rho^2[\text{div } (e^{-i\omega}\varphi)Q_\omega E_1 + \text{curl } (e^{-i\omega}\varphi)Q_\omega E_2], \\
\tilde{L}_U[\Phi]_1 & = & -2\lambda^{-1}w_\rho \cos w[(\partial_{x_1}\varphi_3)\cos\theta + (\partial_{x_2}\varphi_3)\sin\theta]Q_\omega E_1 \\
 & - & 2\lambda^{-1}w_\rho \cos w[(\partial_{x_1}\varphi_3)\sin\theta + (\partial_{x_2}\varphi_3)\cos\theta]Q_\omega E_1, \\
\tilde{L}_U[\Phi]_2 & = & \lambda^{-1}\rho w_\rho^2[\text{div } (e^{i\omega}\overline{\varphi})\cos 2\theta - \text{curl } (e^{i\omega}\overline{\varphi})\sin 2\theta]Q_\omega E_1 \\
 & + & \lambda^{-1}\rho w_\rho^2[\text{div } (e^{i\omega}\overline{\varphi})\sin 2\theta + \text{curl } (e^{i\omega}\overline{\varphi})\cos 2\theta]Q_\omega E_1.
\end{array}
$$

The above identities will be usefull in obtaining estimates for the linear operator L_U.

Chapter 2

Construction of the solution

In the present chapter we use the properties of the linearized harmonic map flow operator seen in Section 1.3 to arrive to a first system of equations that will separate the effect of the operator near and far away from the blow-up point. In section 2.2 we find a first approximation of the parameters through some simplifications. In section 2.3 we transform the first system of equations into a final coupled system. In section 2.4 we recall results from [7], then we write the problem as a fixed point one with an operator $\mathcal{F}$ and state the propositions that gives us its compactness, which we will prove in the next chapter. Finally, at the end of section 2.4 we use all the mentioned results to prove Theorem 0.1 and find the wanted solution for problem (9)-(11).

2.1 Ansatz for a blowing-up solution

Consider the following parabolic equation for a domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u + \frac{1}{x_1} u_{x_1} \text{ in } \Omega \times (0,T), && (2.1)\\ u &= u_{\partial\Omega} \text{ in } \partial\Omega \times (0,T), && (2.2)\\ u(\cdot,0) &= u_0, \text{ in } \Omega && (2.3)\end{aligned}$$

for a function $u : \overline{\Omega} \times [0,T) \to S^2$. Here $u_0 : \overline{\Omega} \to S^2$ is a given smooth map and

$$u_{\partial\Omega} = u_0|_{\partial\Omega} = \mathbf{e}_3 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $\mathbf{e}_3$ the following canonical vector in $\mathbb{R}^3$,

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Notice that this vector corresponds to $W(\infty)$ where W is the 1-corrotational harmonic map in (1.4).

Given a fixed point $q \in \Omega$, and any sufficiently small number $T > 0$ we look for a solution $u(x,t)$ of problem (2.1)-(2.3) which at main order looks like

$$U(x,t) := U_{\lambda(t),\xi(t),\omega(t)}(x) = Q_{\omega(t)} W\left(\frac{x-\xi(t)}{\lambda(t)}\right),$$

for functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1([0,T])$ that we will determine later. We also need these functions to satisfy

$$\xi(T) = q, \quad \lambda(T) = 0,$$

so that $u(x,t)$ blows up at time T at point q. The solution we look for is of the form $u = U + v$, where $v(x,t)$ is chosen to be small and so that u is a solution of (2.1)-(2.3) with initial condition $u_0(x) = U(x,0) + v(x,0)$.

Let us denote

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u + \frac{1}{x_1} u_{x_1}.$$

Lemma 2.1 *If the constraint $|u| = 1$ is kept at all times and $u = U + v$ with $|v| \le \frac{1}{2}$, then for u to solve (2.1) it suffices that*

$$S(U+v) = b(x,t)U$$

for some scalar function b.

Proof. (Of Lemma 2.1) Indeed, if we take $S(u) = b(x,t)U$, since $|u| \equiv 1$ then

$$b(x,t)U \cdot u = S(u) \cdot u = -\frac{1}{2}\frac{\partial}{\partial t}|u|^2 + \frac{1}{2}\Delta(|u|^2) - \frac{1}{2x_1}\frac{\partial}{\partial x_1}|u|^2 = 0,$$

and because $U \cdot u = (U \cdot U) + (U \cdot v) = 1 + U \cdot v \ge \frac{1}{2}$ we get that $b \equiv 0$. $\square$

Notice that we can parametrize all functions $v(x,t)$ such that $|U + v| = 1$ as

$$v = \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U, \tag{2.5}$$

where φ is a function with values in $\mathbb{R}^3$ and

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U, \quad a(\zeta) := \sqrt{1 - |\zeta|^2} - 1.$$

Indeed, one can compute

$$|U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U|^2 = 1 + 2a(\Pi_{U^\perp}\varphi) + a^2(\Pi_{U^\perp}\varphi) + |\Pi_{U^\perp}\varphi|^2 = 1,$$

since $a^2(\Pi_{U^\perp}\varphi) = 2a(\Pi_{U^\perp}\varphi) + |\Pi_{U^\perp}\varphi|^2$.

Taking v given by (2.5) and using that

$$\Delta U + |\nabla U|^2 = 0$$

we have the following

$$S(U+v) = -U_t + \frac{1}{x_1}U_{x_1} - \partial_t(\Pi_{U^\perp}\varphi) + \frac{1}{x_1}\partial_{x_1}(\Pi_{U^\perp}\varphi) + L_U[\Pi_{U^\perp}\varphi] + N_U(\Pi_{U^\perp}\varphi) + c(\Pi_{U^\perp}\varphi)U,$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a = a(\Pi_{U^\perp}\varphi)$,

$$\begin{aligned} L_U(\zeta) &= \Delta\zeta + |\nabla U|^2\zeta + 2(\nabla U \cdot \nabla\zeta)U, \\ N_U(\zeta) &= \left[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U \cdot \nabla\zeta + |\zeta|^2 + |\nabla(aU)|^2\right]\zeta - aU_t + \frac{1}{x_1}aU_{x_1} + 2\nabla a\nabla U, \\ c(\zeta) &= \Delta a - a_t + \frac{1}{x_1}a_{x_1} + \left(|\nabla(U+\zeta+aU)|^2 - |\nabla U|^2\right)(1+a) - 2\nabla U\cdot\nabla\zeta. \end{aligned}$$

We do not consider the partial derivative in x_1 to define the linear operator L_U because we want to use all the calculations and theorems proved in [7] for this operator.

Since we have Observation 2.1 we only need φ to satisfy

$$-U_t + \frac{1}{x_1}U_{x_1} - \partial_t(\Pi_{U^\perp}\varphi) + \frac{1}{x_1}\partial_{x_1}(\Pi_{U^\perp}\varphi) + L_U[\Pi_{U^\perp}\varphi] + N_U(\Pi_{U^\perp}\varphi) = b(\Pi_{U^\perp}\varphi)U \quad (2.6)$$

for some scalar function b. To achieve this we will decompose φ into the sum of two functions $\varphi = \varphi^i + \varphi^o$, which will be called *inner* and *outer* solutions because their aim is to solve the problem near and far away from the blow-up point.

The inner function $\varphi^i(x,t)$ will be supported near the concentration point $x = \xi(t)$, so it is more convenient to understand φ^i as a function of the scaled space variable

$$y = \frac{x-\xi(t)}{\lambda(t)},$$

and pointwise orthogonal to U, so that $\Pi_{U^\perp}\varphi^i = \varphi^i$. On the other hand, the outer function $\varphi^o(x,t)$ will be constructed to satisfy (2.6) far away from $x = \xi(t)$ and is well defined on the variable x.

Observation 9 Notice that because of Lemma 2.1 when we come across a partial derivative of $\Pi_{U^\perp}\varphi$, for a function φ that is not orthogonal to U, we only need to consider for the equation the part that is orthogonal to U. For example, when we have the following in the equation

$$\partial_t\Pi_{U^\perp}\varphi = \partial_t\varphi - (\partial_t\varphi\cdot U)U - (\varphi\cdot U_t)U - (\varphi\cdot U)U_t,$$

the only relevant part of this is

$$\partial_t\varphi - (\varphi\cdot U)U_t,$$

and we can think of the rest of the derivative as part of the function b on Observation 2.1. In some cases it will be more helpful to consider for the equation the term $\Pi_{U^\perp}\partial_t\varphi - (\varphi\cdot U)U_t$ and leave only $-(\varphi\cdot U_t)U$ as part of b.

To understand the construction of the outer solution we write (2.6) in the following form:

$$\begin{aligned} 0 = \; & - \; \partial_t\varphi^i + L_U[\varphi^i] + \tilde{L}_U[\varphi^o] - \Pi_{U^\perp}[\partial_t\varphi^o - \Delta\varphi^o + U_t - \frac{1}{x_1}U_{x_1}] + \frac{1}{x_1}\partial_{x_1}\varphi^i \\ & + \; \frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\varphi^o - \frac{1}{x_1}(\varphi^o\cdot U)U_{x_1} + (\varphi^o\cdot U)U_t + N_U(\varphi^i + \Pi_{U^\perp}\varphi^o) - bU, \quad (2.7) \end{aligned}$$

where we have used the second decomposition in Observation 9, (1.8) and the fact that $U_t \cdot U = 0$.

We will find a function Φ^0 depending only on the parameters, chosen in such a way that $\Pi_{U^\perp}[\partial_t \varphi^o - \Delta\varphi^o + U_t - \frac{1}{x_1}U_{x_1}]$ concentrates near $\xi(t)$ by eliminating the slower space decay terms in the error $U_t - \frac{1}{x_1}U_{x_1}$, which are the ones associated to the time derivatives of the dilation and rotation parameters. For this reason we write the outer solution as

$$\varphi^o(x,t) = \Phi^0(x,t) + \Psi^*(x,t),$$

where Ψ^* will solve the rest of the outer problem.

For the inner solution, we consider a smooth cut-off function $\eta_0(s)$ with $\eta_0(s) = 1$ for $s < 1$ and $\eta_0(s) = 0$ for $s > \frac{3}{2}$. We also consider a positive, large smooth function $R(t) \to +\infty$ as $t \to T$ that will be specified later as a power of the main order term of the parameter $\lambda(t)$. We define

$$\eta(x,t) := \eta_0\left(\left|\frac{x-\xi(t)}{\lambda(t)R(t)}\right|\right),$$

and we can write the inner solution as the following:

$$\varphi^i(x,t) = \eta(x,t)Q_\omega\phi(y,t),$$

for a function ϕ with initial condition $\phi(\cdot,0) = 0$ that satisfies $\phi(\cdot,t)\cdot W \equiv 0$ for $|y| \le 2R(t)$ and that vanishes as $t \to T$. Then we have

$$\begin{aligned}
\partial_t\varphi^i &= \eta_t Q_\omega\phi + \eta\dot\omega Q_\omega J_0\phi + \eta Q_\omega\phi_t - \frac{1}{\lambda}\eta Q_\omega\dot\xi\cdot\nabla_y\phi - \frac{1}{\lambda}\eta Q_\omega y\cdot\nabla_y\phi,\\
\partial_{x_1}\varphi^i &= \partial_{x_1}\eta Q_\omega\phi + \frac{1}{\lambda}\eta Q_\omega\partial_{y_1}\phi,\\
L_U[\varphi^i] &= \frac{1}{\lambda^2}\eta Q_\omega L_W[\phi] + Q_\omega\Delta\eta\phi + 2Q_\omega\nabla\eta\nabla\phi.
\end{aligned} \tag{2.8}$$

Using (2.8) and the first part of Observation 9 on Ψ^*, equation (2.7) becomes

$$\begin{aligned}
0 &= \frac{1}{\lambda^2}Q_\omega\left[-\lambda^2\phi_t + L_W[\phi] + \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]\right]\\
&+ \eta Q_\omega\left(\frac{1}{\lambda}\dot\xi\cdot\nabla_y\phi + \frac{1}{\lambda}y\cdot\nabla_y\phi - \dot\omega J_0\phi\right)\\
&+ \tilde L_U[\Phi^0] + \Pi_{U^\perp}\left[\partial_t\Phi^0 - \Delta\Phi^0 + U_t - \frac{1}{x_1}U_{x_1}\right]\\
&- \Psi^*_t + \Delta\Psi^* + (1-\eta)\tilde L_U[\Psi^*] + \frac{1}{x_1}\partial_{x_1}\Psi^* + Q_\omega\left[\Delta\eta\phi + 2\nabla\eta\nabla\phi + \frac{1}{x_1}\phi\partial_{x_1}\eta + \frac{1}{\lambda x_1}\eta\partial_{y_1}\phi\right]\\
&+ \frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0 - \frac{1}{x_1}((\Phi^0+\Psi^*)\cdot U)U_{x_1} + ((\Phi^0+\Psi^*)\cdot U)U_t\\
&+ N(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) + bU.
\end{aligned} \tag{2.9}$$

Now we will define Φ^0 to satisfy

$$\partial_t\Phi^0 - \Delta\Phi^0 + U_t - \frac{1}{x_1}U_{x_1} \approx 0,$$

when $|y| \gg 1$.

Invoking formula (1.5) to compute U_t and noticing that $U_{x_1} = U_{\xi_1}$ we get

$$
\begin{aligned}
U_t &= \dot{\lambda}\partial_\lambda U + \dot{\omega}\partial_\omega U + \partial_\xi U \cdot \xi = \mathcal{E}_0 + \mathcal{E}_1,\\
\frac{1}{x_1} U_{x_1} &= \mathcal{E}_2,
\end{aligned}
$$

where, using polar coordinates $y = \frac{x-\xi}{\lambda} = \rho e^{i\theta}$, we obtain

$$
\begin{aligned}
\mathcal{E}_0(x,t) &= -Q_\omega \rho w_\rho(\rho)\left[\frac{\dot{\lambda}}{\lambda}E_1(y) + \dot{\omega}E_2(y)\right],\\
\mathcal{E}_1(x,t) &= -\frac{\dot{\xi}_1}{\lambda} w_\rho(\rho) Q_\omega \left[\cos(\theta)E_1(y) + \sin(\theta)E_2(y)\right]\\
&- \frac{\dot{\xi}_2}{\lambda} w_\rho(\rho) Q_\omega \left[\sin(\theta)E_1(y) - \cos(\theta)E_2(y)\right],\\
\mathcal{E}_2(x,t) &= \frac{w_\rho}{\lambda(\xi_1 + \lambda\rho\cos(\theta))}\left[\cos(\theta)Q_\alpha E_1 + \sin(\theta)Q_\alpha E_2\right].
\end{aligned}
$$

Since $\mathcal{E}_1$ and $\mathcal{E}_2$ have faster space decay in ρ than $\mathcal{E}_0$ we will choose Φ^0 to be an approximate solution of

$$
\Phi^0_t - \Delta_x \Phi^0 + \mathcal{E}_0 = 0. \tag{2.10}
$$

Then we can use the same construction of Φ^0 as in [7]. For $x = \xi + re^{i\theta}$ and

$$
p(t) := \lambda(t)e^{i\omega(t)},
$$

we define

$$
\begin{aligned}
\Phi^0[\omega,\lambda,\xi] : &= \begin{pmatrix} \varphi^0(r,t)e^{i\theta} \\ 0 \end{pmatrix},\\
\varphi^0(r,t) &= -\int_{-T}^{t} \dot{p}(s) r k(z(r), t-s)\,ds, \qquad (2.11)\\
z(r) &= \sqrt{r^2+\lambda^2}, \quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2}.
\end{aligned}
$$

Now we can compute the error produced by Φ^0 on equation (2.10) in the following way:

$$
\Phi^0_t - \Delta\Phi^0 = \tilde{\mathcal{M}}_0 + \tilde{\mathcal{M}}_1 - \tilde{\mathcal{E}}_0, \quad \tilde{\mathcal{R}}_0 = \begin{pmatrix} \mathcal{M}_0 \\ 0 \end{pmatrix}, \quad \tilde{\mathcal{M}}_1 = \begin{pmatrix} \mathcal{M}_1 \\ 0 \end{pmatrix},
$$

where

$$
\tilde{\mathcal{E}}_0(x,t) = -\frac{2r}{r^2+\lambda^2}\begin{pmatrix} \dot{p}(t)e^{i\theta} \\ 0 \end{pmatrix}
$$

is an approximation of $\mathcal{E}_0$ when $r \gg \lambda$, and

$$
\begin{aligned}
\mathcal{M}_0 &= -re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^{t} \dot{p}(s)(zk_z - z^2k_{zz})(z(r), t-s)\,ds,\\
\mathcal{M}_1 &= -e^{i\theta}\mathrm{Re}\,(e^{-i\theta}\dot{\xi}(t))\int_{-T}^{t} \dot{p}(s)k(z(r),t-s)\,ds\\
&+ \frac{r}{z^2}e^{i\theta}(\lambda\dot{\lambda}(t) - \mathrm{Re}\,(re^{i\theta}\dot{\xi}(t)))\int_{-T}^{t} \dot{p}(s) z k_z(z(r),t-s)\,ds.
\end{aligned}
$$

Observe that $\mathcal{M}_1$ is of smaller order than $\mathcal{M}_0$. We write the following expression, derived in [7],

$$\begin{aligned}
\tilde{L}_U[\Phi^0] + \Pi_{U^\perp}\left[\partial_t\Phi^0 - \Delta\Phi^0 + U_t - \frac{1}{x_1}U_{x_1}\right] &= \tilde{L}_U[\Phi^0] - \mathcal{E}_1 + \Pi_{U^\perp}[\tilde{\mathcal{E}}_0] - \mathcal{E}_0 + \Pi_{U^\perp}[\tilde{\mathcal{M}}_0] \\
&\quad + \mathcal{E}_2 + \Pi_{U^\perp}[\tilde{\mathcal{M}}_1], \\
&= \mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_1[x,\xi] + \Pi_{U^\perp}[\tilde{\mathcal{M}}_1],
\end{aligned}$$

where

$$\mathcal{K}_0[p,\xi,\Psi^*] = \mathcal{K}_{01}[p,\xi,\Psi^*] + \mathcal{K}_{02}[p,\xi,\Psi^*], \quad \mathcal{K}_1[p,\xi,\Psi^*] = \mathcal{K}_{11}[p,\xi,\Psi^*] + \mathcal{K}_{12}[p,\xi,\Psi^*],$$

with

$$\begin{aligned}
\mathcal{K}_{01}[p,\xi,\Psi^*] &= -\frac{2}{\lambda}\rho w_\rho^2 \int_{-T}^{t} \left[\mathrm{Re}\ (\dot{p}(s)e^{-i\omega(t)})Q_\omega E_1 + \mathrm{Im}\ (\dot{p}(s)e^{-i\omega(t)})Q_\omega E_2\right] k(z,t-s)ds, \\
\mathcal{K}_{02}[p,\xi,\Psi^*] &= \frac{2}{\lambda}\rho w_\rho^2 \left[\dot{\lambda} - \int_{-T}^{t} \mathrm{Re}\ (\dot{p}(s)e^{-i\omega(t)}) r k_z(z,t-s) z_r ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \cos w \left[\int_{-T}^{t} \mathrm{Re}\ (\dot{p}(s)e^{-i\omega(t)})(z k_z - z^2 k_{zz})(z,t-s)ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \left[\int_{-T}^{t} \mathrm{Re}\ (\dot{p}(s)e^{-i\omega(t)})(z k_z - z^2 k_{zz})(z,t-s)ds\right] Q_\omega E_2, \\
\mathcal{K}_{11}[p,\xi,\Psi^*] &= \frac{1}{\lambda} w_\rho \left[\mathrm{Re}\ ((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta})Q_\omega E_1 + \mathrm{Im}\ ((\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta})Q_\omega E_2\right], \\
\mathcal{K}_{12}[p,\xi,\Psi^*] &= \frac{w_\rho}{\lambda(\xi_1 + \lambda\rho\cos(\theta))}\left[\cos(\theta)Q_\omega E_1 + \sin(\theta)Q_\omega E_2\right].
\end{aligned} \tag{2.12}$$

Using this in equation (2.9) we see that

$$u(x,t) = U + \Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi])U$$

is a solution of equation (2.1) if the pair (ϕ, Ψ^*) solves the following system:

$$\begin{aligned}
\lambda^2 \phi_t &= L_W[\phi] + \lambda^2 Q_{-\omega}\left[\tilde{L}_U[\Psi^*] + \mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_1[p,\xi,\Psi^*]\right] \text{ in } \mathcal{D}_{2R}, \\
\phi \cdot W &= 0 \text{ in } \mathcal{D}_{2R}, \\
\phi(\cdot,0) &= 0 = \phi(\cdot,T),
\end{aligned} \tag{2.13}$$

and

$$\Psi_t^* = \Delta_x \Psi^* + g[p,\xi,\Psi^*,\phi] \quad \text{ in } \Omega \times (0,T), \tag{2.14}$$

where

$$\begin{aligned}
g[p,\xi,\Psi^*,\phi] : &= (1-\eta)\tilde{L}_U[\Psi^*] + \frac{1}{x_1}\partial_{x_1}\Psi^* - \frac{1}{x_1}(\Psi^* \cdot U)U_{x_1} + (\Psi^* \cdot U)U_t \\
&+ Q_\omega\left[\Delta\eta\phi + 2\lambda^{-1}\nabla\eta\nabla_y\phi + \frac{1}{x_1}\phi\partial_{x_1}\eta + \frac{1}{\lambda x_1}\eta\partial_{y_1}\phi - \eta_t\phi\right] \\
&+ \eta Q_\omega\left(\lambda^{-1}\dot{\xi}\cdot\nabla_y\phi + \lambda^{-1}y\cdot\nabla_y\phi - \dot{\omega}J_0\phi\right) \\
&+ (1-\eta)[\mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_1[p,\xi,\Psi^*]] + \Pi_{U^\perp}[\tilde{\mathcal{M}}_1] + \frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0 \\
&- \frac{1}{x_1}(\Phi^0 \cdot U)U_{x_1} + (\Phi^0 \cdot U)U_t + N(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)).
\end{aligned} \tag{2.15}$$

This system of coupled equations is called *inner-outer gluing system* in [7] and we will use that name from now on. Here we denote

$$\mathcal{D}_{\gamma R} = \{(y,t) \in \mathbb{R}^2 \times (0,T) : \ |y| < \gamma R(t)\}.$$

And the boundary condition (2.4) can be expressed as

$$\Pi_{U^\perp}[\Phi^0 + \Psi^*] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^*])U = (\mathbf{e}_3 - U),$$

for which it suffices to have

$$\Psi^*|_{\partial\Omega} = \mathbf{e}_3 - U - \Phi^0. \tag{2.16}$$

We will also ask that

$$\Psi^*(q,T) = 0,$$

because we need the perturbation φ to be small compared to U near the blow-up point when t is near T. To fulfill this condition we will require the following initial condition:

$$\Psi^*(x,0) = Z_0^*(x) + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3,$$

where c_1, c_2, c_3 are constants and Z_0^* is a small function, which will be determined later on.

2.2 Approximate equations for the parameters

In this section we will derive the order of vanishing of the scaling parameter $\lambda(t)$ as $t \to T$ and will obtain a formula for the translation parameter $\xi(t)$.

Consider the first equation of problem (2.13) written in the following form:

$$\lambda^2\phi_t = L_W[\phi] + h[p,\xi,\Psi^*] \quad \text{ in } \mathcal{D}_{2R}, \tag{2.17}$$

where $h(y,t)$ is defined for all $y \in \mathbb{R}^2$, extending the original function. Some terms are extended as 0 outside the disc using $\chi_{\mathcal{D}_{2R}}$, the characteristic function of $\mathcal{D}_{2R}$, and others are not changed. In this way we define

$$h[p,\xi,\Psi^*] := \lambda^2 Q_{-\omega}\left[\tilde{L}_U[\Psi^*] + \mathcal{K}_{12}[p,\xi,\Psi^*]\right]\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\left[\mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_{11}[p,\xi,\Psi^*]\right]. \tag{2.18}$$

If $\lambda(t)$ vanishes relatively smoothly as $t \to T$, which is what we want, then the term $\lambda^2\phi_t$ in equation (2.17) should be of smaller order than the ones on the right hand side. Hence we can approximate equation (2.13) by the elliptic problem

$$L_W[\phi] + h[p,\xi,\Psi^*] = 0, \quad \phi \cdot W = 0 \text{ in } \mathbb{R}^2. \tag{2.19}$$

Applying the $L^2(\mathbb{R}^2)$ product between (2.19) and functions $Z_{lj}(y)$ defined on (1.6) we get

$$\int_{\mathbb{R}^2} L_W[\phi](y,t) \cdot Z_{lj}(y)dy + \int_{\mathbb{R}^2} h[p,\xi,\Psi^*](y,t) \cdot Z_{lj}(y)dy = 0 \quad \text{ for all } t \in (0,T),$$

using integration by parts and recalling that $L_W[Z_{lj}] = 0$ we obtain

$$\int_{\mathbb{R}^2} h[p, \xi, \Psi^*](y, t) \cdot Z_{lj}(y)dy = 0 \quad \text{ for all } t \in (0, T), \tag{2.20}$$

for $l = 0, 1$, $j = 1, 2$. This orthogonality condition between h and the kernel of L_W can be expressed as a system of integro-differential equations. Solving this system will give us the appropiate order of the parameters so that the solution (ϕ, Ψ^*) has the right estimates to obtain a fixed point argument.

We want more useful expressions for both sides of the following equation

$$\begin{aligned} &\lambda^2 \int_{\mathbb{R}^2} Q_{-\omega}\left[\mathcal{K}_0[p, \xi, \Psi^*] + \mathcal{K}_{11}[p, \xi, \Psi^*]\right] \cdot Z_{lj}(y)dy \\ &+ \lambda^2 \int_{\mathbb{R}^2} Q_{-\omega}\mathcal{K}_{12}[p, \xi, \Psi^*]\chi_{\mathcal{D}_{2R}} \cdot Z_{lj}(y)dy = -\lambda^2 \int_{B_{2R}} Q_{-\omega}\tilde{L}_U[\Psi^*] \cdot Z_{lj}(y)dy. \end{aligned}$$

We will define, analogously as in [7], the following terms:

$$\begin{aligned} \mathcal{B}_{lj}[p, \xi, \Psi^*](t) &:= \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}\left[\mathcal{K}_0[p, \xi, \Psi^*] + \mathcal{K}_{11}[p, \xi, \Psi^*]\right] \cdot Z_{lj}(y)dy, \\ \tilde{\mathcal{B}}_{lj}[p, \xi, \Psi^*](t) &:= \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}\mathcal{K}_{12}[p, \xi, \Psi^*]\chi_{\mathcal{D}_{2R}} \cdot Z_{lj}(y)dy, \\ a_{lj}[p, \xi, \Psi^*](t) &:= \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}\tilde{L}_U[\Psi^*] \cdot Z_{lj}(y)dy, \\ a_0[p, \xi, \Psi^*] &:= \frac{1}{2}e^{i\omega(t)}(a_{01}[p, \xi, \Psi^*] + ia_{02}[p, \xi, \Psi^*]), \\ a_1[p, \xi, \Psi^*] &:= -e^{i\omega(t)}(a_{11}[p, \xi, \Psi^*] + ia_{12}[p, \xi, \Psi^*]). \end{aligned} \tag{2.21}$$

Here the expressions differ from the ones we want to compute by a $2\pi\lambda$ factor. This is due to the fact that this factor naturally arises in the calculations and it has been simplified.

In [7] the authors compute the following:

$$\begin{aligned} \mathcal{B}_{01}[p](t) &= 2\int_{-T}^{t} \text{Re } (\dot{p}(s)e^{-i\omega(t)})\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\frac{ds}{t-s}\right) - 2\dot{\lambda}(t), \\ \mathcal{B}_{02}[p](t) &= 2\int_{-T}^{t} \text{Im } (\dot{p}(s)e^{-i\omega(t)})\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\frac{ds}{t-s}\right), \\ \mathcal{B}_{11}[\xi](t) &= 2\dot{\xi}_1(t), \\ \mathcal{B}_{12}[\xi](t) &= 2\dot{\xi}_2(t), \\ a_0[p, \xi, \Psi^*] &= [\text{div } (\psi^*) + i\text{curl } (\psi^*)](\xi, t) + o(1), \\ a_1[p, \xi, \Psi^*] &= o(1), \end{aligned}$$

where $o(1) \to 0$ when $t \to T$ and ψ^* comes from the following decomposition of Ψ^*

$$\Psi^* = \begin{pmatrix} \psi^* \\ \psi_3^* \end{pmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*,$$

and

$$\begin{aligned}
\Gamma_1(\tau) &= -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) + 2\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2} - 4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho,\\
\Gamma_2(\tau) &= -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) - \zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho,
\end{aligned}$$

where

$$K(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta}.$$

Notice that some terms only depend on p and others only on ξ, so we simplify the notation accordingly.

Given the above, we only need to compute the terms associated with $\mathcal{K}_{12}$. We use formula (2.12) and (1.6) to get

$$\begin{aligned}
\tilde{\mathcal{B}}_{01}[p,\xi,\Psi^*](t) &= \frac{1}{2\pi}\int_0^{2R}\int_0^{2\pi} \frac{\cos(\theta)\rho^2 w_\rho^2}{(\xi_1 + \lambda\rho cos(\theta))} d\theta d\rho,\\
&= \frac{1}{2\pi}\int_0^{2R} \rho^2 w_\rho^2 \left(\frac{-2\pi\lambda\rho}{\xi_1^2 - \lambda^2\rho^2 + \xi_1\sqrt{\xi_1^2-\lambda^2\rho^2}}\right) d\rho,\\
&= -\lambda\int_0^{2R} \frac{\rho^3 w_\rho^2}{\xi_1^2 - \lambda^2\rho^2 + \xi_1\sqrt{\xi_1^2-\lambda^2\rho^2}} d\rho,\\
&= \lambda^{1-\tau}\ell(\lambda,\xi),
\end{aligned}$$

for $\tau \in (0,1)$, where we define

$$\ell(\lambda,\xi) := -\int_0^{2R} \frac{\lambda^\tau \rho^3 w_\rho^2}{\xi_1^2 - \lambda^2\rho^2 + \xi_1\sqrt{\xi_1^2-\lambda^2\rho^2}} d\rho.$$

Let us consider for the moment $R(\lambda) = \lambda^{-\beta}$ with $\beta \in (0,1)$. Then we have

$$\lambda(t)\rho \le 2\lambda(t)R(t) \le 2\lambda^{1-\beta}(t) \le 2\lambda^{1-\beta}(0),$$

assuming λ to be decreasing. Choosing $\lambda(0)$ so that $2\lambda^{1-\beta}(0) \le \xi_1(t)\, \forall t \in [0,T]$ we get that $\xi_1^2(t) - \lambda^2(t)\rho^2 > 0$ for all $t \in [0,T]$, hence the denominator of ℓ is never undefined. Here we are assuming that $R(\lambda) = \lambda^{-\beta}$, that λ is decreasing and that ξ_1 has some regularity, we will elaborate on this assumptions on section 2.3.

With all of the above we notice that

$$\begin{aligned}
|\ell(\lambda,\xi)| &\le \frac{4\lambda^\tau}{\xi_1^2 - \lambda^2(2R)^2 + \xi_1\sqrt{\xi_1^2-\lambda^2(2R)^2}} \int_0^{2R} \frac{\rho^3}{(1+\rho^2)^2} d\rho,\\
&= \frac{2\lambda^\tau}{\xi_1^2 - \lambda^2(2R)^2 + \xi_1\sqrt{\xi_1^2-\lambda^2(2R)^2}} \left[\frac{1}{(2R)^2+1} + \log((2R)^2+1) - 1\right].
\end{aligned}$$

To see that ℓ exists it is enough to analyze the higher order term in R, which is $\lambda^\tau \log((2R)^2 + 1)$. Notice that

$$\lambda^\tau \log(4R^2+1) \le C\lambda^\tau \log(\lambda^{-\beta}+1),$$

where $C > 0$ is a constant. Let $n > 0$ arbitrary, then

$$\begin{aligned}\lambda^{\tau}\log(\lambda^{-\beta}+1) &\leq n\lambda^{\tau}((\lambda^{-\beta}+1)^{1/n}-1),\\ &\leq n((\lambda^{n\tau-\beta}+\lambda^{n\tau})^{1/n}-1).\end{aligned}$$

For this to be finite when $t \to T$ we only need to ask $\tau \geq \frac{\beta}{n}$, so fixing $\tau \in (0, \frac{\beta}{2})$ gives us $|\ell(\lambda,\xi)| < +\infty$. Moreover $\lambda^{1-\tau}\ell(\lambda,\xi) = O(\|p\|_{\infty}^{1-\tau})$.

Next we compute the term

$$\begin{aligned}\tilde{\mathcal{B}}_{11}[p,\xi,\Psi^*](t) &= \frac{1}{2\pi}\int_0^{2R}\int_0^{2\pi}\frac{\rho w_\rho^2}{(\xi_1+\lambda\rho cos(\theta))}d\theta d\rho,\\ &= \int_0^{2R}\frac{\rho w_\rho^2}{\sqrt{\xi_1^2-\lambda^2\rho^2}}d\rho.\end{aligned}$$

If we study the last integral when $\lambda \to 0$, noticing that $\lambda\rho \leq \lambda R \leq \lambda^{1-\beta} \to 0$ and using Taylor expansion around 0, like $\sqrt{\xi_1^2 - x^2} = \xi_1 + \frac{1}{2\xi_1}x^2 + o(x^3)$, then we can approximate $\tilde{\mathcal{B}}_{11}$ by

$$\tilde{\mathcal{B}}_{11}[p,\xi,\Psi^*] \approx \frac{1}{\xi_1}\int_0^{2R}\rho w_\rho^2 d\rho = \frac{1}{2\xi_1(1+(2R)^{-2})} \approx \frac{1}{2\xi_1}.$$

We also obtain

$$\begin{aligned}\tilde{\mathcal{B}}_{02}[p,\xi,\Psi^*](t) &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{\sin(\theta)\rho^2 w_\rho^2}{(\xi_1+\lambda\rho cos(\theta))}d\theta d\rho = 0,\\ \tilde{\mathcal{B}}_{12}[p,\xi,\Psi^*](t) &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\frac{(\sin(\theta)\cos(\theta)-\cos(\theta)\sin(\theta))\rho^2 w_\rho^2}{(\xi_1+\lambda\rho cos(\theta))}d\theta d\rho = 0,\end{aligned}$$

where the first one is 0 due to parity arguments.

Observation 10 To compute some of these integrals we used two facts:

- We have that
$$\int_0^{2\pi}\frac{\cos(\theta)}{a+b\cos(\theta)}d\theta d\rho = \frac{-2\pi b}{a^2-b^2+a\sqrt{a^2-b^2}},$$
which comes from applying the Residue Theorem to the complex function
$$f(z) = \frac{z^2+1}{bz\left(z^2+\frac{2a}{b}z+1\right)},$$
in the contour $\{z \in \mathbb{C} : |z| = 1\}$ clockwise oriented.
- We have that
$$\int_0^{2\pi}\frac{1}{a+b\cos(\theta)}d\theta d\rho = \frac{2\pi}{\sqrt{a^2-b^2}},$$
which comes from applying the Residue Theorem to the complex function
$$f(z) = \frac{2}{b\left(z^2+\frac{2a}{b}z+1\right)},$$
in the contour $\{z \in \mathbb{C} : |z| = 1\}$ clockwise oriented.

As in [7] we define

$$\mathcal{B}_1[\xi] := \mathcal{B}_{11}[\xi] + i\mathcal{B}_{12}[\xi], \quad \mathcal{B}_0[p] := \frac{1}{2}e^{i\omega(t)}(\mathcal{B}_{01}[p] + i\mathcal{B}_{02}[p]),$$

and therefore reduce our four orthogonality conditions to a system of two complex equations

$$\mathcal{B}_0[p] + \tilde{\mathcal{B}}_{01}[p, \xi, \Psi^*] = a_0[p, \xi, \Psi^*], \tag{2.22}$$

$$\mathcal{B}_1[\xi] + \tilde{\mathcal{B}}_{11}[p, \xi, \Psi^*] = a_1[p, \xi, \Psi^*]. \tag{2.23}$$

In [7] the authors get, using the decay of the functions Γ_j, $j = 1, 2$, that

$$\mathcal{B}_0[p](t) = \int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds + O(\|\dot{p}\|_\infty), \quad \mathcal{B}_1[p](t) = 2\left(\dot{\xi}_1(t) + i\dot{\xi}_2(t)\right).$$

And from our analysis of the new terms we have that

$$\tilde{\mathcal{B}}_{01}[p, \xi, \Psi^*](t) = O(\|p\|_\infty^{1-\tau}), \quad \tilde{\mathcal{B}}_{11}[p, \xi, \Psi^*](t) = \frac{-1}{2\xi_1(t)}.$$

Then we can write (2.22)-(2.23) in the form

$$\int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds = [\text{div } (\psi^*) + i\text{curl } (\psi^*)](\xi, t) + o(1) + O(\|p\|_\infty^{1-\tau} + \|\dot{p}\|_\infty), \tag{2.24}$$

$$\dot{\xi}_1(t) = -\frac{1}{4\xi_1(t)} + o(1), \tag{2.25}$$

$$\dot{\xi}_2(t) = o(1). \tag{2.26}$$

We make an informal analysis of these equations to derive the main order of the parameters. Notice that

$$\xi_1(t) = \sqrt{q_1^2 + 2(T-t)},$$

is a solution of the ordinary differential equation

$$\dot{\xi}_1(t) = -\frac{1}{\xi_1(t)}, \quad \xi_1(T) = q_1,$$

where q_1 is the first coordinate of the given point $q = (q_1, q_2)$. Hence, if T is small, $\xi_1(t) = q_1$ is a good approximation of the solution of (2.25). The same can be said of $\xi_2(t) = q_2$ as an approximation of the solution of (2.26). We also assume that Ψ^* is sufficiently regular. Taking into account only the higher order terms of equation (2.24) we arrive to

$$\int_{-T}^{t-\lambda^2} \frac{\dot{p}(s)}{t-s} ds = \text{div } \psi^*(q, 0) + i\text{curl } \psi^*(q, 0).$$

We impose the following condition

$$\text{div } \psi^*(q, 0) < 0,$$

that will allow λ to decrease. In [7] the authors arrive to the same equation and obtain the following approximate solution:

$$\dot{\lambda}(t) = -|\text{div } \psi^*(q, 0) + i\text{curl } \psi^*(q, 0)|\dot{\lambda}_*(t),$$

where

$$\dot{\lambda}_*(t) = -\frac{|\log T|}{\log^2(T-t)}. \tag{2.27}$$

And imposing $\lambda_*(T) = 0$ we obtain

$$\lambda_*(t) = \frac{|\log T|}{\log^2(T-t)}(T-t)(1+o(1)), \quad \text{as } t \to T.$$

2.3 The final inner-outer gluing system

Summarizing, for a given point in space $q \in \Omega$ and $T > 0$ a sufficiently small final time, we want to find a solution (ϕ, Ψ^*) of the inner and outer problems (2.13)-(2.14) with boundary condition of the form (2.16) such that the function

$$u(x,t) = U + \Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi])U \tag{2.28}$$

blows up at time T at point q with $U(x,t)$ as main order of blow-up.

The purpose of this section and the next one is to set up all the ingredients to find (ϕ, Ψ^*) and the parameters λ, ω, ξ as solution of a fixed point scheme. In this section we will split the inner and outer problems into a final system of equations, this will allow us to obtain in section 2.4 good estimates on the solutions in relation to the right hand side of the equations. These estimates will allow us to apply Schauder's fixed point theorem.

First, we need to make some assumptions about the parameters $\xi(t)$ and $p(t) = \lambda e^{i\omega(t)}$. We assume there exist constants μ_1, μ_2, μ_3, $\mu_4 > 0$ independent of T such that

$$|\dot{\xi}(t)| \quad \leq \mu_1, \quad \mu_2 \leq |\xi_1(t)| \text{ for all } t \in (0,T), \tag{2.29}$$
$$\mu_3|\dot{\lambda}_*(t)| \leq \quad |\dot{p}(t)| \quad \leq \mu_4|\dot{\lambda}_*(t)| \text{ for all } t \in (0,T). \tag{2.30}$$

Here λ_* is the one defined in (2.27) and $\mu_1 = \frac{1}{q_1}$. We also assume that

$$R(t) = \lambda_*(t)^{-\beta},$$

where $\beta \in (0, \frac{1}{2})$.

As we saw in section 2.2 we can formulate the inner problem as

$$\begin{aligned} \lambda^2 \phi_t &= L_W[\phi] + h[p, \xi, \Psi^*] \quad \text{in } \mathcal{D}_{2R}, \\ \phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}, \\ \phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)}, \end{aligned}$$

where $h[p, \xi, \Psi^*]$ is given by (2.18). To find a nice solution of this problem $h[p, \xi, \Psi^*]$ should satisfy the orthogonality condition (2.20). To use this approach of orthogonality we define, as done in [7], the following weighted projection for any function $h(y,t)$ with sufficient decay,

$$c_{lj}[h](t) := \frac{1}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{lj}|^2} \int_{\mathbb{R}^2} h(y,t) \cdot Z_{lj}(y) dy. \tag{2.31}$$

Observation 11 Notice that if we define

$$I := \langle h(y,t) - \sum_{l=-1}^{1}\sum_{j=1}^{2} c_{lj}[h](t) w_\rho^2 Z_{lj}, Z_{km}\rangle_{L^2(\mathbb{R}^2)},$$

for $k=-1,0,1$, $m=1,2$, then by the definition of c_{ij} we have

$$\begin{aligned} I &= \int_{\mathbb{R}^2} h(y,t)\cdot Z_{km}(y)dy - \sum_{l=-1}^{1}\sum_{j=1}^{2} c_{lj}[h](t)\int_{\mathbb{R}^2} w_\rho^2 Z_{lj}(y)\cdot Z_{km}(y)dy, \\ &= \int_{\mathbb{R}^2} h(y,t)\cdot Z_{km}(y)dy - c_{km}[h](t)\int_{\mathbb{R}^2} w_\rho^2 |Z_{km}|^2 dy = 0. \end{aligned}$$

We split the inner solution ϕ into three functions $\phi = \phi_1 + \phi_2 + \phi_3$, where we will require that ϕ_1 solves

$$\lambda^2 \partial_t \phi_1 = L_W[\phi_1] + h[p,\xi,\Psi^*] - \sum_{l=-1}^{1}\sum_{j=1}^{2} c_{lj}[h[p,\xi,\Psi^*]] w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R}.$$

To have all orthogonality conditions met we should aim to solve $c_{lj}[h[p,\xi,\Psi^*]]=0$, but this is very difficult to achieve since there are no parameters that can guarantee $c_{-1j}[h[p,\xi,\Psi^*]] = 0$ and $c_{0j}[h[p,\xi,\Psi^*]]=0$ involves solving exactly an integro-differential equation. Therefore we will have to handle modes -1 and 0 with a different approach.

We will require that ϕ_3 solves

$$\lambda^2 \partial_t \phi_3 = L_W[\phi_3] + \sum_{j=1}^{2} c_{-1j}[h[p,\xi,\Psi^*]] w_\rho^2 Z_{-1j} \quad \text{in } \mathcal{D}_{2R},$$

where $c_{-1j}[h[p,\xi,\Psi^*]]$ will be kept under control using that it is small compared to $h[p,\xi,\Psi^*]$.

We also need that ϕ_2 solves a similar equation to the one of ϕ_3 for mode 0, but we will include some changes. To better understand c_{0j} notice that

$$c_{0j}[h[p,\xi,\Psi^*]] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2}(\mathcal{B}_{0j}[p] + \tilde{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*]).$$

Then to get the orthogonality we should solve

$$\mathcal{B}_0[p] + \tilde{B}_{01}[p] = a_0[p,\xi,\Psi^*]. \tag{2.32}$$

But since this equation is very difficult to solve exactly we will follow the method in [7] and modify it, so we get a modified equation easier to solve with a the rest, that will turn up to be of smaller order. We will not actually solve this modified equation but instead we will prove that the solution obtained by Dávila, Del Pino and Wei [7] also holds in our setting.

To modify the operator on the left hand side of (2.32) we notice that

$$\mathcal{B}_0[p] + \tilde{B}_{01}[p] = \mathcal{B}_0^*[p] + \mathcal{S}_\alpha[\dot{p}] + \mathcal{R}_\alpha[\dot{p}] + \tilde{B}_{01}[p],$$

where

$$\begin{aligned}
\mathcal{B}_0^*[p] &= \mathcal{B}_0[p] - \int_{-T}^{t-\lambda_*^2} \frac{\dot p(s)}{t-s} ds, \\
\mathcal{S}_\alpha[\dot p] &= \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{\dot p(s)}{t-s} ds + \dot p(t)[2\log\lambda_*(t) - (1-\alpha)\log(T-t)], \\
\mathcal{R}_\alpha[\dot p] &= \int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*^2} \frac{\dot p(s) - \dot p(t)}{t-s} ds.
\end{aligned}$$

Then the modified equation is

$$\mathcal{B}_0^*[p] + \mathcal{S}_\alpha[\dot p] = A \quad \text{ in } [0,T],$$

and $\mathcal{R}_\alpha[\dot p] + \tilde B_{01}[p]$ will be a remainder of smaller order called $\mathcal{R}_0$.

To state the result proved in [7], we define the following norms. Let I denote either the interval $[0,T]$ or $[-T,T]$. For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$ we let

$$\|g\|_{\Theta,l} = \sup_{t\in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|, \tag{2.33}$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, and $l \in \mathbb{R}$ we let

$$[g]_{\gamma,m,l} = \sup (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma},$$

where the supremum is taken over $s \le t$ in I such that $t - s \le \frac{1}{10}(T-t)$.

Dávila, Del Pino and Wei [7] proved the following proposition:

Proposition 2.2 *Let $\alpha, \Theta, \gamma \in (0, \frac{1}{2})$, $m \le \Theta - \gamma$ and $l \in \mathbb{R}$. Let $C_1 > 1$ be a fixed constant and assume that Re $(a(T)) < 0$ with $\frac{1}{C_1} \le$ Re $(a(T)) \le C_1$ and*

$$T^\Theta |\log T|^{1+\sigma-l} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1, \tag{2.34}$$

for some $\sigma > 0$. Then, for $T > 0$ small enough there are two operators $\mathcal{P}$ and $\mathcal{R}_0$ so that $p = \mathcal{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t)$$

with

$$\begin{aligned}
&|\mathcal{R}_0[a](t)| \\
&\le C\Big(T^{\frac{1}{2}+\sigma} + T^\Theta \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big) \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l},
\end{aligned} \tag{2.35}$$

for some $\sigma > 0$.

Observation 12 We select

$$m := \Theta - 2\gamma(1-\beta),$$

and $l < 1 + 2m$, $\alpha > 1 - 2\beta$. With this choice of parameters in [7] the authors obtain

$$\begin{aligned} |\mathcal{R}_0[a](t)| &\le C\Big(T^{\frac{1}{2}+\sigma} + T^{\Theta}\frac{\log|\log T|}{|\log T|}\|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big)\frac{(T-t)^{\Theta+\gamma(\alpha-1+2\beta)}}{|\log(T-t)|^l}, \\ &\le C\lambda_*(t)^{\Theta+\sigma_1} \end{aligned}$$

. for some $\sigma > 0$, $\sigma_1 > 0$. the authors also notice that with these choices one has

$$\begin{aligned} \|a(\cdot) - a(T)\|_{\Theta,l-1} &\le C|\log T|^{l-1-\Theta}, \\ [a]_{\gamma,m,l-1} &\le C|\log T|^{l-1-m}. \end{aligned}$$

Observation 13 Proposition 2.2 applies to our setting for two reasons. First, our function a is the same as the one in [7]. Second, in [7] the authors prove that with $\mathcal{R}_0 = \mathcal{R}_\alpha$ this works. We can prove that adding $\tilde{\mathcal{B}}_{01}$ to $\mathcal{R}_0$ does not change this scheme. Indeed, note that asking for $\tau \le 1 - \Theta - \sigma_1$ we have

$$|\tilde{\mathcal{B}}_{01}[p,\xi]| \le \lambda_*^{1-\tau}|\ell(\lambda,\xi)| \le C\lambda_*^{\Theta+\sigma_1}.$$

As in [7] we write the equation for ϕ_2 leaving out $\mathcal{R}_0$, but then we have to add a new equation that will take care of this term. To obtain these equations we need dome definitions. Using the decomposition of $\tilde{L}_U$ in (1.10) and the definition of a_0 in (2.21) we can decompose a_0 as follows:

$$a_0[p,\xi,\Psi^*] = a_0^{(0)}[p,\xi,\Psi^*] + a_0^{(1)}[p,\xi,\Psi^*] + a_0^{(2)}[p,\xi,\Psi^*],$$

where

$$a_0^{(l)}[p,\xi,\Psi^*] = -\frac{\lambda}{4\pi}e^{i\omega}\int_{B_{2R}}\Big(Q_{-\omega}\tilde{L}_U[\Psi^*]_l\cdot Z_{01} + iQ_{-\omega}\tilde{L}_U[\Psi^*]_l\cdot Z_{02}\Big)\,dy.$$

Then we define

$$\begin{aligned} c_0^*[p,\xi,\Psi^*](t) := \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2|Z_{01}|^2}e^{-i\omega}\Big(&\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*](T)\right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) \\ &+ a_0^{(2)}[p,\xi,\Psi^*](t)\Big), \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} c_{01}^*[p,\xi,\Psi^*] &:= \text{Re }(c_0^*[p,\xi,\Psi^*]), \\ c_{02}^*[p,\xi,\Psi^*] &:= \text{Im }(c_0^*[p,\xi,\Psi^*]). \end{aligned}$$

We leave c_{0j}^* in the system of equations, instead of c_{0j}, and add the reduced equation

$$c_{0j}[h[p,\xi,\Psi^*]] = c_{0j}^*[p,\xi,\Psi^*], \quad j = 1,2,$$

instead of $c_{0j}[h[p,\xi,\Psi^*]] = 0$. Note that this equation is equivalent to

$$\mathcal{B}_0[p] = a_0^{(0)}[p,\xi,\Psi^*](t) + \mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*](T)\right]. \tag{2.37}$$

For more on the description of c_0^* and the reduced equation see [7].

Now, to solve the outer equation (2.14) we decompose

$$\Psi^* = Z^* + \psi,$$

where we choose $Z^* : \Omega \times (0,\infty) \to \mathbb{R}^3$ to satisfy the 2-dimensional heat equation

$$\begin{cases} Z_t^* &= \Delta Z^* \quad \text{in } \Omega \times (0,\infty), \\ Z^*(\cdot,t) &= 0 \quad \text{in } \partial\Omega \times (0,\infty), \\ Z^*(\cdot,0) &= Z_0^* \quad \text{in } \Omega, \end{cases} \tag{2.38}$$

where $Z_0^*(x)$ is a $C^\infty(\Omega)$ function defined as follows:

$$Z_0^*(x) = \begin{pmatrix} z_{01}(x) + iz_{02}(x) \\ z_{03}(x) \end{pmatrix} = \begin{pmatrix} z_0(x) \\ z_{03}(x) \end{pmatrix},$$

and z_0 satisfies

$$\text{div } (z_0)(q) < 0.$$

We also require $\|Z_0^*\|_{L^\infty(\Omega)}$ to be sufficiently small. Notice that gives us some control over Z^*, since

$$\|Z^*\|_{C^\infty(\Omega\times(0,T))} + \|\nabla_x Z^*\|_{C^\infty(\Omega\times(0,T))} \le C\|Z_0^*\|_{L^\infty(\Omega)}$$

because Ω is bounded. We also require that

$$\Psi^*(q,T) = 0,$$

so that the main order of blow-up of $u(x,t)$, defined by (2.28), is given by $U(x,t)$. This constraint will be achieved by three coefficients in the initial datum.

Summarizing all of the above the inner-outer gluing system becomes a new and final system of equations, where we are looking for functions ϕ_1, ϕ_2, ϕ_3, ψ, p, ξ and constants c_1, c_2, c_3 such that:

$$\begin{cases} \psi_t &= \Delta_x\psi + g[p,\xi,Z^* + \psi, \phi_1 + \phi_2 + \phi_3] \quad \text{in } \Omega \times (0,T), \\ \psi &= \mathbf{e}_3 - U - \Phi^0 \quad \text{on } \partial\Omega, \\ \psi(\cdot,0) &= (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3)\chi + (\mathbf{e}_3 - U - \Phi^0)(1-\chi) \quad \text{in } \Omega, \\ \psi(q,T) &= -Z^*(q,T). \end{cases} \tag{2.39}$$

$$\begin{cases} \lambda^2\partial_t\phi_1 &= L_W[\phi_1] + h[p,\xi,\Psi^*] - \displaystyle\sum_{l=-1}^{1}\sum_{j=1}^{2} c_{lj}[h[p,\xi,\Psi^*]]w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R}, \\ \phi_1 \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}, \\ \phi_1(\cdot,0) &= 0 \quad \text{in } B_{2R(0)}. \end{cases} \tag{2.40}$$

$$\begin{cases} \lambda^2\partial_t\phi_2 &= L_W[\phi_2] + \displaystyle\sum_{j=1}^{2} c_{0j}^*[p,\xi,\Psi^*]w_\rho^2 Z_{0j} \quad \text{in } \mathcal{D}_{2R}, \\ \phi_2 \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}, \\ \phi_2(\cdot,t) &= 0 \quad \text{on } \partial B_{2R}, \\ \phi_2(\cdot,0) &= 0 \quad \text{in } B_{2R(0)}. \end{cases} \tag{2.41}$$

$$
\begin{cases}
\lambda^2 \partial_t \phi_3 &= L_W[\phi_3] + \sum_{j=1}^{2} c_{-1j}[h[p,\xi,\Psi^*]] w_\rho^2 Z_{-1j} \quad \text{in } \mathcal{D}_{2R}, \\
\phi_3 \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}, \\
\phi_3(\cdot,t) &= 0 \quad \text{on } \partial B_{2R}, \\
\phi_3(\cdot,0) &= 0 \quad \text{in } B_{2R(0)}.
\end{cases} \tag{2.42}
$$

$$
c_{0j}[h[p,\xi,\Psi^*]](t) - c_{0j}^*[p,\xi,\Psi^*](t) = 0 \quad \text{for all } t \in (0,T), \quad j = 1,2, \tag{2.43}
$$

$$
c_{1j}[h[p,\xi,\Psi^*]](t) = 0 \quad \text{for all } t \in (0,T), \quad j = 1,2. \tag{2.44}
$$

Here χ is a smooth cut-off function with compact support in Ω that is identically 1 on a fixed neighborhood of q independent of T. In addition, g and h are given by (2.15) and (2.18), respectively.

2.4 Solving the inner-outer gluing system

The main idea for solving the final inner-outer gluing system (2.39)-(2.44) is to fix ϕ_1, ϕ_2, ϕ_3, p, ξ and find an operator that given $g[p,\xi,Z^*+\psi,\phi_1+\phi_2+\phi_3]$ returns the solution ψ of problem (2.39), which we will call exterior problem. Then we find operators that given $h[p,\xi,\psi+Z^*]$ return solutions ϕ_1, ϕ_2, ϕ_3 of the interior problem. Finally, we find operators that return p and ξ as solutions of the parameter problem. Next, we define the product of these operators and set the problem as a fixed point in a suitable space.

The objective of this section is to set the basis for the fixed point argument and use it to prove Theorem 0.1. For this we need to use propositions proved by Dávila, Del Pino and Wei [7] that will allow us to construct the operators mentioned before with sufficiently good estimates so we can obtain the necessary compactness to apply Schauder's fixed point theorem. First, we recall the results proved in [7] for the linear equations related to the ones we want to solve. With these propositions we also define the norms and spaces where we will look for the functions ϕ_1, ϕ_2, ϕ_3, ψ, p, ξ.

We formulate the linear problem associated to (2.40) as:

$$
\begin{cases}
\lambda^2 \partial_t \phi &= L_W[\phi] + h - \sum_{l=-1}^{1} \sum_{j=1}^{2} c_{lj}[h] w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R}, \\
\phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R}, \\
\phi(\cdot,0) &= 0 \quad \text{in } B_{2R(0)}.
\end{cases} \tag{2.45}
$$

For a function $h(y,t)$ on the right hand side of (2.45) and $\nu < 1$, $a > 2$ we define the following norm:

$$
\|h\|_{a,\nu} := \sup_{\mathbb{R}^2} (1+|y|^a) \lambda_*(t)^{-\nu} |h(y,t)|, \tag{2.46}
$$

and for $\phi(y,t)$ we define

$$
\|\phi\|_{*,a,\nu} = \sup_{(y,\tau)\in\mathcal{D}_{2R}} \frac{(1+|y|)|\nabla_y \phi(y,t)| + |\phi(y,t)|}{\lambda_*^\nu R^{\frac{5-a}{2}} (1+|y|)^{-1} \min\{1, R^{\frac{5-a}{2}} |y|^{-2}\}}. \tag{2.47}
$$

Then from [7] we have:

Proposition 2.3 *Let $2 < a < 3$, $\nu > 0$. If $\|h\|_{a,\nu} < +\infty$ there exists a solution of problem (2.45) which defines a linear operator $\phi = \mathcal{T}_{\lambda,1}[h]$ and satisfies the estimate*

$$\|\phi\|_{*,a,\nu} \leq C\|h\|_{a,\nu}.$$

To do the same with equations (2.41) and (2.42) consider

$$\begin{cases} \lambda^2 \partial_t \phi & = L_W[\phi] + h(y,t) \quad \text{ in } \mathcal{D}_{2R}, \\ \phi \cdot W & = 0 \quad \text{ in } \mathcal{D}_{2R}, \\ \phi(\cdot, t) & = 0 \quad \text{ on } \partial B_{2R}, \\ \phi(\cdot, 0) & = 0 \quad \text{ in } B_{2R(0)}, \end{cases} \tag{2.48}$$

and define

$$\|\phi\|_{**,\nu_2} = \sup_{(y,t)\in\mathcal{D}_{2R}} \frac{(1+|y|)|\nabla_y \phi(y,t)| + |\phi(y,t)|}{\lambda_*^{\nu_2} R^2 (1+|y|)^{-1}}, \tag{2.49}$$

for $\nu_2 > 0$. Then from [7] we have:

Proposition 2.4 *Let $2 < a < 3$, $\nu_2 > 0$. There exists $C > 0$ such that for h with $\|h\|_{a,\nu_2} < +\infty$, the unique solution $\phi = \mathcal{T}_{\lambda,2}[h]$ of problem (2.48) satisfies the estimate*

$$\|\phi\|_{**,\nu_2} \leq C\|h\|_{a,\nu_2}.$$

The next proposition applies to h of the form

$$h(y,t) = h_{-1,1}(y,t) Z_{-1,1} + h_{-1,2}(y,t) Z_{-1,2}.$$

We introduce the norm

$$\|\phi\|_{***,\nu} = \sup_{(y,t)\in\mathcal{D}_{2R}} \frac{(1+|y|)|\nabla_y \phi(y,t)| + |\phi(y,t)|}{\lambda_*^{\nu} \log(R)}. \tag{2.50}$$

Then from [7] we have:

Proposition 2.5 *Let $2 < a < 3$, $\nu > 0$. There exists $C > 0$ such that for h of the form (2.50) with $\|h\|_{a,\nu} < +\infty$, the unique solution $\phi = \mathcal{T}_{\lambda,3}[h]$ of problem (2.48) satisfies the estimate*

$$\|\phi\|_{***,\nu} \leq C\|h\|_{a,\nu}.$$

Let Θ' and γ' be such that Proposition 2.6 holds. Define Θ, $\gamma > 0$ such that $\gamma < \gamma'$, $\Theta < \Theta'$. Define $a, a' \in (2,3)$, $\nu, \nu', \nu_2, \nu_2' > 0$ such that Propositions 2.3, 2.4 and 2.5 hold for the triplets a', ν', ν_2' and a, ν, ν_2. In addition we need that $a' > a$, $\nu' > \nu$, $\nu_2' > \nu_2$, where the coefficients and their respective primas are close to each other. We also define spaces

$$\begin{aligned} E_1 &= \{\phi \in C(\overline{\mathcal{D}}_{2R}) : \nabla_y \phi \in C(\overline{\mathcal{D}}_{2R}), \ \|\phi\|_{*,a,\nu} < +\infty\}, \\ E_2 &= \{\phi \in C(\overline{\mathcal{D}}_{2R}) : \nabla_y \phi \in C(\overline{\mathcal{D}}_{2R}), \ \|\phi\|_{**,\nu_2} < +\infty\}, \\ E_3 &= \{\phi \in C(\overline{\mathcal{D}}_{2R}) : \nabla_y \phi \in C(\overline{\mathcal{D}}_{2R}), \ \|\phi\|_{***,\nu} < +\infty\}. \end{aligned}$$

Now, we refer to the linear problem associated with (2.39), which can be written in the following way:

$$\begin{cases} \psi_t &= \Delta_x \psi + f(x,t) \quad \text{ in } \Omega \times (0,T), \\ \psi &= 0 \quad \text{ on } \partial\Omega, \\ \psi(\cdot,0) &= (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3)\chi \quad \text{ in } \Omega, \\ \psi(q,T) &= 0. \end{cases} \tag{2.51}$$

Observation 14 Notice that problem (2.51) differs from (2.39) not only in function f, but also that has different border conditions, it is missing $-Z^*$ and $\mathbf{e}_3 - U - \Phi^0$. We omit them for simplicity. This is possible because Z^* and $\mathbf{e}_3 - U - \Phi^0$ behave well enough, and can be added to the solution of (2.51) without changing the results we will mention later.

Let $\Theta > 0$, $\gamma \in (0, 1/2)$ and $\sigma_0 > 0$ small. We define the following weights:

$$\begin{aligned} \varrho_1 &:= \lambda_*^{\Theta}(\lambda_* R)^{-1}\chi_{r<(2+\mu_1)R\lambda_*}, \\ \varrho_2 &:= T^{-\sigma_0}\frac{\lambda_*^{1-\sigma_0}}{r^2}\chi_{r\geq R\lambda_*}, \\ \varrho_3 &:= T^{-\sigma_0}. \end{aligned} \tag{2.52}$$

For a function $f(x,t)$ on the right hand side of (2.51) we use the weights to define

$$\|f\|_{**} := \sup_{\Omega\times(0,T)} \frac{|f(x,t)|}{\left(1 + \sum_{i=1}^{4} \varrho_i(x,t)\right)}. \tag{2.53}$$

and for $\psi(x,t)$ we define

$$\begin{aligned} \|\psi\|_{\#,\Theta,\gamma} &= \frac{\lambda_*(0)^{-\Theta}}{|\log(T)|\lambda_*(0)R(0)} \sup_{\Omega\times(0,T)} |\psi(x,t)| + \lambda_*(0)^{-\Theta} \sup_{\Omega\times(0,T)} |\nabla\psi(x,t)| \\ &\quad + \sup_{\Omega\times(0,T)} \lambda_*^{-\Theta-1} R(0)^{-1} \frac{1}{|\log(T-t)|} |\psi(x,t) - \psi(x,T)| \\ &\quad + \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta} |\nabla\psi(x,t) - \nabla\psi(x,T)| \\ &\quad + \sup \lambda_*(t)^{\Theta}(\lambda_*(t)R(t))^{2\gamma} \frac{|\nabla\psi(x,t) - \nabla\psi(x',t')|}{(|x-x'|^2 + |t-t'|)^{\gamma}}, \end{aligned} \tag{2.54}$$

where the last supremum is taken in the region

$$x, x' \in \Omega, \quad t, t' \in (0,T), \quad |x - x'| \leq 2\lambda_*(t)R(t), \quad |t - t'| < \frac{1}{4}(T-t).$$

Then from [7] we have:

Proposition 2.6 *Let $\beta \in \left(0, \frac{1}{2}\right)$, $\Theta \in (0,\beta)$. For $T > 0$ small there is a linear operator $\mathcal{H}$ that maps a function $f : \Omega \times (0,T) \to \mathbb{R}^3$ with $\|f\|_{**} < \infty$ into ψ, c_1, c_2, c_3 so that (2.51) is satisfied. Moreover the following holds*

$$\|\psi\|_{\#,\Theta,\gamma} + \frac{\lambda_*(0)^{-\Theta}(\lambda_*(0)R(0))^{-1}}{|\log T|}(|c_1| + |c_2| + |c_3|) \leq C\|f\|_{**},$$

where $\gamma \in \left(0, \frac{1}{2}\right)$.

Let Θ' and γ' be such that Proposition 2.6 holds. Define Θ, $\gamma > 0$ such that $\gamma < \gamma'$, $\Theta < \Theta'$. And define the space

$$F = \{\psi \in C(\overline{\Omega} \times [0,T)) : \nabla_x \psi \in C(\overline{\Omega} \times [0,T)),\ \|\psi\|_{\#,\Theta,\gamma} < +\infty\}.$$

Next, we focus on the parameters p, ξ to define spaces and norms for them. In [7] the authors work with a decomposition for p of the form:

$$p = p_{0,\kappa} + p_1 + p_2,$$

for $\kappa \in \mathbb{C}$, p_1 and p_2 are functions smaller than $p_{0,\kappa}$ and

$$p_{0,\kappa}(t) = \kappa |\log T| \int_t^T \frac{1}{|\log(T-t)|^2} ds, \quad t \leq T. \tag{2.55}$$

Then it is more natural to define

$$\begin{aligned} G_1 \ &= \ \mathbb{C} \times \{p_1 \in C^1([-T,T];\mathbb{C}) : \\ &\quad\ \ p_1(T) = 0,\ p_2(T) = 0,\ \|p_1\|_{*,3-\sigma} + \|\dot{p}_2\|_{\Theta,l} < \infty\}, \end{aligned}$$

where $\sigma \in (0,1)$ and the norms are

$$\begin{aligned} \|g\|_{*,3-\sigma} &= \sup_{t \in [-T,T]} |\log(T-t)|^{3-\sigma} |\dot{g}(t)|, \\ \|g\|_{\Theta,l} &= \sup_{t \in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|. \end{aligned}$$

We use that from [7] one can identify p with an element $(\kappa, p_1, p_2) \in G_1$ and write the norm

$$\|p\|_{G_1} = |\kappa| + \|p_1\|_{*,3-\sigma} + \|\dot{p}_2\|_{\Theta,l}.$$

Recall from Section 2.3 that equation (2.43) is equivalent to (2.37),

$$\mathcal{B}_0[p] = a_0^{(0)}[p,\xi,\Psi^*](t) + \mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*](T) \right],$$

which is solvable through the operator $\mathcal{P}$ defined in Proposition 2.2. In [7] the authors prove estimates for $\mathcal{P}$, that we express in the following proposition.

Proposition 2.7 *Let us make the same assumptions as in Proposition 2.2. Then*

$$\mathcal{P}[a] = p_{0,\kappa[a]} + \mathcal{P}_1[a] + \mathcal{P}_2[a],$$

where $p_{0,\kappa}$ is defined in (2.55) and each term

$$\kappa = \kappa[a], \quad p_1 = \mathcal{P}_1[a], \quad p_2 = \mathcal{P}_2[a],$$

has the following bounds:

$$
\begin{aligned}
\kappa &= |a(T)|\left(1+O\left(\frac{1}{|\log T|}\right)\right),\\
|\dot p_1(t)| &\le C\frac{|\log T|^{1-\sigma}\log(|\log T|)^2}{|\log(T-t)|^{3-\sigma}},\\
|\ddot p_1(t)| &\le C\frac{|\log T|}{|\log(T-t)|^3(T-t)},\\
\|\dot p_2\|_{\Theta,l} &\le C\big(T^{\frac12+\sigma-\Theta}+\|a(\cdot)-a(T)\|_{\Theta,l-1}\big),\\
[\dot p_2]_{\gamma,m,l} &\le C(|\log T|^{l-3}T^{\alpha_0-m-\gamma}+T^{\Theta}\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\Theta,l-1}+[a]_{\gamma,m,l-1}),
\end{aligned}
$$

where $\alpha_0>0$ is some fixed some constant and $\sigma>0$ is arbitrary (with C depending on σ).

Observation 15 Notice that due to the selection of parameters done in Observation 12 we can denote

$$
\begin{aligned}
C_0(T) &:= C\big(T^{\frac12+\sigma-\Theta}+\|a(\cdot)-a(T)\|_{\Theta,l-1}\big),\\
C_1(T) &:= C(|\log T|^{l-3}T^{\alpha_0-m-\gamma}+T^{\Theta}\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\Theta,l-1}+[a]_{\gamma,m,l-1}),
\end{aligned}
$$

which are constants that only depend on T and have order $O(T)$.

Let us fix $\sigma'>0$ small, $\Theta'>0$, $l'>0$ that satisfy Proposition 2.7 and choose the parameters σ,Θ,l associated with the norm of G_1 such that $\sigma'<\sigma$, $\Theta'<\Theta$, $l'<l$.

Now, we define

$$
G_2=\{\xi\in C^1([0,T];\mathbb{R}^2):\, \dot\xi_1(T)=\frac{1}{q_1},\, \dot\xi_2(T)=0\},
$$

and

$$
\|\xi\|_{G_2}=\sup_{t\in(0,T)}|\xi(t)|+\sup_{t\in(0,T)}|\dot\xi(t)|.
$$

The problem of parameter ξ, which corresponds to equation (2.44), is equivalent to

$$
\int_{\mathbb{R}^2} h[p,\xi,\Psi^*](y,t)\cdot Z_{lj}(y)dy=0 \quad \text{for all } t\in(0,T),
$$

this is

$$
\lambda\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi^*]\cdot Z_{1j}(y)dy+\lambda\int_{B_{2R}} Q_{-\omega}\mathcal{K}_{12}\cdot Z_{1j}(y)dy=\lambda\int_{B_{2R}} Q_{-\omega}[\mathcal{K}_0+\mathcal{K}_{11}]\cdot Z_{1j}(y)dy.
$$

And using the calculated integrals on Section 2.2 we can formulate equation (2.44) as:

$$
\dot\xi_1=-\frac{1}{\xi_1}-b_1[p,\xi,\Psi^*], \tag{2.56}
$$

$$
\dot\xi_2=-b_2[p,\xi,\Psi^*], \tag{2.57}
$$

where

$$b_j[p,\xi,\Psi^*](t) = \frac{1}{4\pi}(1+(2R)^{-2})\lambda \int_{B_{2R}} Q_{-\omega}\tilde{L}_U[\Psi^*] \cdot Z_{1j}(y)dy.$$

We need to solve these ordinary differential equations to obtain operators for our fixed point argument.

Proposition 2.8 *Define operators $\mathcal{A}_1[p,\xi,\Psi^*]$, $\mathcal{A}_2[p,\xi,\Psi^*]$ that return the solution of equations (2.56) and (2.57) respectively. Then,*

$$\begin{aligned} \|\mathcal{A}_1[p,\xi,\Psi^*]\|_{C^1(0,T)} &\leq \left(q_1 + \frac{1+T}{\mu_2}\right) + (1+T)(\lambda_*(0)^{\Theta}\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}), \\ \|\mathcal{A}_2[p,\xi,\Psi^*]\|_{C^1(0,T)} &\leq q_2 + (1+T)(\lambda_*(0)^{\Theta}\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}), \end{aligned}$$

where q_1 is the first coordenate of the given point $q \in \Omega$ and μ_2 is the below bound of ξ in (2.29).

This proposition is proved in Section 3.3.

We have introduced a few parameters until now, let us summarize their names and uses:

- $\beta \in \left(1, \frac{1}{2}\right)$ participates in the definition of $R(t) = \lambda_*(t)^{-\beta}$.
- $\nu \in (0,1)$, $a \in (2,3)$ are used to estimate $\|h[p,\xi,\Psi^*]\|_{a,\nu}$ for equation (2.40). We can add that the norm $\|\cdot\|_{a,\nu}$ is also used to estimate the right hand side of equation (2.42) and the norm $\|\cdot\|_{a,\nu_2}$ is used to estimate the right hand side of equation (2.41).
- $\alpha_0 > 0$ is the power of the error term in Proposition 2.2.
- $\Theta > 0$ and $\gamma \in \left(0, \frac{1}{2}\right)$ are used to estimate the norm $\|\cdot\|_{\#,\Theta,\gamma}$ of the solution of equation (2.39).
- $\sigma_0 > 0$ is small and will allow us to make parts of the error small in the outer problem.
- $m = \Theta - 2\gamma(1-\beta)$, $l < 1+2m$, $\alpha > 1-2\beta$ allows us to obtain estimates in Proposition 2.2.
- $\sigma \in (0,1)$, l and Θ allow us to control the norm of parameter p and $\sigma', \Theta', l' \in (2,3)$ comes from Proposition 2.7 and helps us achieve compactness.
- $\tau \in (0, \min\{\frac{\beta}{2}, 1-\Theta-\sigma_1\})$ makes integrals involving $\mathcal{K}_{12}$ well defined and allows us to use Proposition 2.2.

From now on we assume that parameters $a, \beta, \nu, \Theta, \nu_2$ satisfy the following additional restrictions:

- $a \in (2,3)$
- $\beta \in (0, \frac{1}{a+2})$
- $\nu \in (\max\{1-\beta, 4\beta\}, \min\{1-\beta(a-2), \frac{\beta}{2}\})$
- $\Theta \in (0, \min\{\beta, \nu-1+\beta, \nu-\beta\frac{(7-a)}{2}, \nu_2-1, \nu_2-3\beta\})$
- $\nu_2 \in (1, 1+\Theta+\sigma_1)$

Observation 16 Notice that since $2 \le \frac{7-a}{2} \le \frac{5}{2}$ this assumptions imply that:

$$\begin{aligned}
&\Theta < \nu + (a-2)\beta \\
&\Theta < \nu - (4-a)\beta \\
&\Theta < \nu - 2\beta \\
&\Theta < \nu_2 - \beta \\
&\Theta < \nu_2 - 3\beta. \\
&\nu < 1 - \beta(a-2) \\
&0 < \Theta + 1 - \nu - \beta(a-2) \\
&\nu < 1 - \frac{\beta}{2}.
\end{aligned}$$

We will use this in sections 3.1 and 3.2.

Recall that we want to find $(\phi_1, \phi_2, \phi_3, \psi, p, \xi)$ solutions of system (2.39)-(2.44). To make notation smoother we define $E = E_1 \times E_2 \times E_3 \times F \times G_1 \times G_2$. The fixed point scheme consists in fixing a suitable $M > 0$ and taking functions $v \in B_M$, where

$$\begin{aligned}
B_M \;=\; & \{v = (\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in E \text{ such that} \\
& \|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} + \|\psi\|_{\#,\Theta,\gamma} + \|p\|_{G_1} + \|\xi\|_{G_2} \le M\}
\end{aligned}$$

Then we write equations (2.39)-(2.44) as a fixed point problem

$$v = \mathcal{F}(v),$$

where we define the following operator

$$\begin{aligned}
\mathcal{F} : B_M \subset E &\to E \\
v &\to \mathcal{F}(v) = (\mathcal{F}_1(v), \mathcal{F}_2(v), \mathcal{F}_3(v), \mathcal{F}_4(v), \mathcal{F}_5(v), \mathcal{F}_6(v)),
\end{aligned}$$

where every component is defined as

$$\begin{aligned}
\mathcal{F}_1(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{T}_{\lambda,1}(h[p, \xi, \Psi^*]), \\
\mathcal{F}_2(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{T}_{\lambda,2}\left(\sum_{j=1}^{2} c^*_{0j}[p, \xi, \Psi^*] w_\rho^2 Z_{0j}\right), \\
\mathcal{F}_3(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{T}_{\lambda,3}\left(\sum_{j=1}^{2} c_{-1j}[h[p, \xi, \Psi^*]] w_\rho^2 Z_{-1j}\right), \\
\mathcal{F}_4(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{H}(g[p, \xi, \psi + Z^*, \phi_1, \phi_2, \phi_3]), \\
\mathcal{F}_5(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{P}\left[a_0^{(0)}[p, \xi, \Psi^*](T)\right], \\
\mathcal{F}_6(\phi_1, \phi_2, \phi_3, \psi, p, \xi) &= \mathcal{A}(p, \xi, \Psi^*, \phi_1, \phi_2, \phi_3),
\end{aligned}$$

where $\mathcal{T}_{\lambda,1}$, $\mathcal{T}_{\lambda,2}$, $\mathcal{T}_{\lambda,3}, \mathcal{H}, \mathcal{P}$ are the operators from Propositions 2.3, 2.4, 2.5, 2.6 and 2.2, and $\mathcal{A}$ is defined as $\mathcal{A} := (\mathcal{A}_1, \mathcal{A}_2)$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the ones from Proposition 2.8.

To solve the fixed point problem we will need to prove the following:

Proposition 2.9 *Assume that p, ξ satisfy (2.30) and (2.29). Let $M > 0$, such that*

$$\frac{M}{6} > \max\left\{ |\kappa|, 2\left(q_1 + q_2 + \frac{1}{\mu_2}\right)\right\}.$$

Then $\mathcal{F}(B_M) \subset B_M$.

Proposition 2.10 *Assume that p, ξ satisfy (2.30) and (2.29). Then $\mathcal{F} : B_M \to B_M$ is a compact operator.*

The proof of these last results is in section 3.4. Now that we have all the ingredients we proceed with the proof of Theorem 0.1.

Proof. Of Theorem 0.1. We have that $\mathcal{F} : B_M \to B_M$ is compact from Propositions 2.9 and 2.10, then by Shauder's fixed point theorem there exists $(\tilde{\phi_1}, \tilde{\phi_2}, \tilde{\phi_3}, \tilde{\psi}, \tilde{p}, \tilde{\xi}) \in B_M$ such that

$$\mathcal{F}(\tilde{\phi_1}, \tilde{\phi_2}, \tilde{\phi_3}, \tilde{\psi}, \tilde{p}, \tilde{\xi}) = (\tilde{\phi_1}, \tilde{\phi_2}, \tilde{\phi_3}, \tilde{\psi}, \tilde{p}, \tilde{\xi}).$$

This gives us functions

$$U(x,t) = U_{\tilde{\lambda},\tilde{\xi},\tilde{\omega}}(x,t)$$

and

$$\tilde{\varphi}(x,t) = \eta(x,t)Q_\omega(\tilde{\phi_1}(y,t) + \tilde{\phi_2}(y,t) + \tilde{\phi_3}(y,t)) + \Phi^0(x,t) + \tilde{\psi}(x,t) + Z^*(x,t).$$

Recalling the form of the candidate solution we constructed we obtain that

$$u_q(x,t) = U(x,t) + \Pi_{U^\perp}\tilde{\varphi} + a(\Pi_{U^\perp}\tilde{\varphi})U$$

is a solution of problem (12)-(14) in Ω. Remembering the reduction to two dimensions that we did in the introduction we define function $v_{c(q)}(x,y,z,t) := u_q(x_1,x_2,t)$ in V, which is the solution of the original 3-dimensional problem (9)-(11). We notice that we have constructed a solution that has exactly the blow-up rate that we wanted and that converges at main order like a bubble U. □

Chapter 3

Results on the fixed point operator

In the present chapter we prove Propositions 2.9 and 2.10, which are the base for the proof of the main theorem, done in the previous chapter. We prove the propositions by first focusing on each set of equations, this means the exterior problem (2.39), the interior problem (2.40)-(2.41)-(2.42) and the parameter problem (2.43)-(2.44). In each of the first three sections we will prove estimates on the right hand side of this problems and use this to get a priori estimates by means of Propositions 2.3, 2.4, 2.5, 2.6 and 2.8. Finally, in the last section we gather these a priori estimates to prove that our fixed point operator $\mathcal{F}$ goes from B_M into B_M (Proposition 2.9) and that it is compact there (Proposition 2.10).

3.1 The exterior problem

In this section we will compute estimates for the exterior problem (2.39) to prove the following proposition:

Proposition 3.1 *Let $p(t) = \lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (2.30), (2.29) and let*

$$(\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M.$$

Then there exists a constant $C > 0$ such that

$$\begin{aligned} \|g[p, \xi, Z^* + \psi, \phi_1 + \phi_2 + \phi_3]\|_{**} \quad &\leq \quad CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} + \|\psi\|_{\#,\Theta,\gamma} \\ &\quad + \|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)} + \|Z_0^*\|_{L^\infty(\Omega)}) \end{aligned} \tag{3.1}$$

where g is defined in (2.15).

Notice that this implies that Proposition 2.6 holds for

$$f = g[p, \xi, Z^* + \psi, \phi_1 + \phi_2 + \phi_3],$$

because $(\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M$.

We write the following useful lemma that will help us in the proof of Proposition 3.1.

Lemma 3.2 *For Φ^0 defined in (2.11) the following holds:*

$$|\Phi^0| \leq Cr(|\dot{\lambda}_*||\log(r^2+\lambda_*^2)|+1),$$

$$\left|\frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0\right| \leq C(|\dot{\lambda}_*||\log(r^2+\lambda_*^2)|+1).$$

PROOF. (Of Proposition 3.1) To get estimate (3.1) we divide g into two parts, called g_1 and g_2. Function g_1 will coincide with the right hand side function of the problem solved in [7], so we will recall their estimates for this part. Function g_2 will contain the terms associated to the partial derivatives $\frac{1}{x_1}\partial_{x_1}$, which we will estimate in detail.

We define

$$\begin{aligned}
g_1 \quad := \quad & (1-\eta)\tilde{L}_U[\Psi^*] + (\Psi^*\cdot U)U_t \\
& + Q_\omega\left[\Delta\eta\phi + 2\lambda^{-1}\nabla\eta\nabla_y\phi - \eta_t\phi\right] \\
& + \eta Q_\omega\left(\lambda^{-1}\dot{\xi}\cdot\nabla_y\phi + \lambda^{-1}y\cdot\nabla_y\phi - \dot{\omega}J_0\phi\right) \\
& + (1-\eta)[\mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_{11}[p,\xi,\Psi^*]] + \Pi_{U^\perp}[\tilde{\mathcal{M}}_1] \\
& + (\Phi^0\cdot U)U_t + \tilde{N}(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) \\
g_2 \quad := \quad & \frac{1}{x_1}\partial_{x_1}\Psi^* - \frac{1}{x_1}(\Psi^*\cdot U)U_{x_1} + Q_\omega\left[\frac{1}{x_1}\phi\partial_{x_1}\eta + \frac{1}{\lambda x_1}\eta\partial_{y_1}\phi\right] \\
& + (1-\eta)\mathcal{K}_{12}[p,\xi,\Psi^*] + \frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0 - \frac{1}{x_1}(\Phi^0\cdot U)U_{x_1} + \frac{1}{x_1}aU_{x_1},
\end{aligned}$$

where

$$\tilde{N}(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) := N(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) - \frac{1}{x_1}aU_{x_1}.$$

In [7] the authors prove that given the following assumptions:

- $a \in (2,3)$
- $\beta \in (0, \frac{1}{a+2})$
- $\nu \in (\max\{1-\beta, 4\beta\}, 1-\beta(a-2))$
- $0 < \Theta < \min\{\beta, \nu-1+\beta, \nu-\beta\frac{7-a}{2}, \nu_2-1, \nu_2-3\beta\}$
- $\nu_2 \in (1, 1+\Theta+\sigma_1)$

and

$$B_{2\lambda_* R}(\xi) \subset B_{(2+\mu_1)\lambda_* R}(q), \tag{3.2}$$

then

$$
\begin{aligned}
\|Q_\omega \left[\Delta\eta\phi + 2\lambda^{-1}\nabla\eta\nabla_y\phi - \eta_t\phi\right]\|_{**} &\le CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu}),\\
\|\eta Q_\omega \left(\lambda^{-1}\dot\xi\cdot\nabla_y\phi + \lambda^{-1}y\cdot\nabla_y\phi - \dot\omega J_0\phi\right)\|_{**} &\le CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu}),\\
\|(1-\eta)\tilde L_U[\Psi^*]\|_{**} &\le T^{\sigma_0}(\|\psi\|_{\#,\Theta,\gamma} + \|\nabla Z^*\|_{L^\infty(\Omega\times(0,T))}),\\
\|(1-\eta)[\mathcal{K}_0[p,\xi,\Psi^*] + \mathcal{K}_{11}[p,\xi,\Psi^*]]\|_{**} &\le CT^{\sigma_0}(\|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}),\\
\|\Pi_{U^\perp}[\tilde{\mathcal{M}}_1] + (\Phi^0\cdot U)U_t\|_{**} &\le CT^{\sigma_0}(\|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}),\\
\|\tilde N(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*))\|_{**} &\le CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu}\\
&\quad + \|\psi\|_{\#,\Theta,\gamma} + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}\\
&\quad + \|Z^*\|_{C^1}).
\end{aligned}
$$

There is only two terms from g_1 left. Since in [7] the authors have weaker restrictions on Z^* and a different estimate on $\dot\xi$ we will redo the estimate for $(\Psi^*\cdot U)U_t$ and $(\Phi^0\cdot U)U_t$ so we can get a simpler expression. Notice that we do not do this for other terms involving Ψ^* and $\dot\xi$ because those estimates only rely on taking the supremum over Z^* and $\dot\xi$.

To estimate $(\Psi^*\cdot U)U_t$ we first note that due to assumptions (2.30) and (2.29) we have that

$$
|U_t| \le \frac{|\dot\lambda|}{\lambda}|\rho w_\rho| + |\dot\omega||\rho w_\rho| + \frac{|\dot\xi|}{\lambda}|w_\rho| \le C\frac{(|\dot\lambda_*| + \|\dot\xi\|_{L^\infty(0,T)})}{r+\lambda_*},
$$

then, remembering that $|U|=1$ and using Cauchy-Schwarz we have

$$
|(\Psi^*\cdot U)U_t| \le C\frac{(|\dot\lambda_*| + \|\dot\xi\|_{L^\infty(0,T)})}{r+\lambda_*}|\Psi^*|.
$$

Since, we only need an estimate for $|\Psi^*|$. Because $\Psi^*(q,T)=0$ we have for $(t,x)\in(0,T)\times\Omega$:

$$
|\Psi^*(x,t)| \le |\Psi^*(x,t) - \Psi^*(x,T)| + |\Psi^*(x,T) - \Psi^*(q,T)|.
$$

Then from the definition of the norm $\|\cdot\|_{\#,\Theta,\gamma}$ we get

$$
\begin{aligned}
|\psi(x,t) - \psi(x,T)| &\le \lambda_*(t)^{\Theta+1}R(t)|\log(T-t)|\|\psi\|_{\#,\Theta,\gamma},\\
|\psi(x,T) - \psi(q,T)| &\le |x-q|\|\psi\|_{\#,\Theta,\gamma}.
\end{aligned}
$$

Note that

$$
|\log(T-t)| = \lambda_*(t)^{-\frac{1}{2}}\sqrt{|\log T|}\sqrt{T-t},
$$

taking $\Theta - 1 - \beta > \frac{1}{2}$ we obtain

$$
|\psi(x,t) - \psi(x,T)| \le \sqrt{|\log T|}\sqrt{T-t}\|\psi\|_{\#,\Theta,\gamma}.
$$

Notice that since Z^* is smooth, in particular it is Lipschitz, then

$$
\begin{aligned}
|Z^*(x,t) - Z^*(x,T)| &\le C|T-t|,\\
|Z^*(x,T) - Z^*(q,T)| &\le C|x-q|.
\end{aligned}
$$

Joining these expressions, naming $r = |x-\xi|$ and using (3.2) we obtain

$$|\Psi^*(x,t)| \leq C(r + \sqrt{|\log T|}\sqrt{|T-t|})\|\psi\|_{\#,\Theta,\gamma} + (r + (T-t)),$$

using that T is small,

$$|\Psi^*(x,t)| \leq C(r + |T-t|)(\|\psi\|_{\#,\Theta,\gamma} + 1). \tag{3.3}$$

Summarizing,

$$|(\Psi^* \cdot U)U_t| \leq C\frac{(|\dot{\lambda}_*| + \|\dot{\xi}\|_{L^\infty(0,T)})}{r+\lambda_*}|\Psi^*| \leq C\frac{(|\dot{\lambda}_*| + \|\dot{\xi}\|_{L^\infty(0,T)})}{r+\lambda_*}(r + |T-t|)(\|\psi\|_{\#,\Theta,\gamma} + 1),$$

and because $|T-t| = \frac{|\lambda_*(t)|}{|\dot{\lambda}_*(t)|}$ we have

$$|(\Psi^* \cdot U)U_t| \leq C\left(\frac{(r|\dot{\lambda}_*| + \lambda_*)}{r+\lambda_*} + \|\dot{\xi}\|_{L^\infty(0,T)} + \frac{T-t}{r+\lambda_*}\|\dot{\xi}\|_{L^\infty(0,T)}\right)(\|\psi\|_{\#,\Theta,\gamma} + 1).$$

We have that

$$\frac{T-t}{r+\lambda_*} \leq CT^{\sigma_0}(\rho_1 + \rho_2 + \rho_3).$$

To see this, we estimate it in the regions $r \geq \lambda_* R$ and $r \leq \lambda_* R$. In region $r \geq \lambda_* R$ we use that $x \leq 1 + x^2$ and

$$\chi_{\{r\geq\lambda_* R\}}\frac{T-t}{r+\lambda_*} \leq \chi_{\{r\geq\lambda_* R\}}\left(1 + \frac{(T-t)^2}{r+\lambda_*}\right) \leq T^{\sigma_0}\rho_3 + \chi_{\{r\geq\lambda_* R\}}\frac{(T-t)^2}{r+\lambda_*}.$$

Notice the following Observation:

Observation 17 The definition of λ_* gives us:

$$\log(\lambda_*) = \log(T-t) + \log(\log T) - \log(\log(T-t)),$$

so the dominant term of $\log(\lambda_*)$ is $\log(T-t)$ and vice versa, therefore there exist positive constants C_1, C_2 such that

$$C_1|\log(\lambda_*)| \leq |\log(T-t)| \leq C_1|\log(\lambda_*)|.$$

Using Observation 17 in our calculations we obtain:

$$\begin{aligned}\frac{(T-t)^2}{r+\lambda_*} &\leq \frac{\lambda_*^2\log^4(T-t)}{r^2},\\ &\leq C\frac{\lambda_*^2\log^4(\lambda_*)}{r^2},\\ &\leq C\frac{\lambda_*}{r^2},\\ &\leq CT^{\sigma_0}\rho_2.\end{aligned}$$

Here we also used the fact that $x_1^{\sigma} \log(x)^{\sigma_2}$ is bounded for $x \in [0,1]$ and $\sigma_1, \sigma_2 > 0$. In region $r \leq \lambda_* R$ we have

$$\begin{aligned}
\chi_{\{r \leq \lambda_* R\}} \frac{T-t}{r+\lambda_*} &\leq \chi_{\{r \leq \lambda_* R\}} \frac{(T-t)}{\lambda_*}, \\
&\leq \chi_{\{r \leq \lambda_* R\}} \frac{\lambda_* \log^2(T-t)}{\lambda_*^{1+\delta}}, \\
&\leq \chi_{\{r \leq \lambda_* R\}} \lambda_*^{-\delta} \log^2(T-t), \\
&\leq C\chi_{\{r \leq \lambda_* R\}} \lambda_*^{-\delta}, \\
&\leq T^{\sigma_0} \rho_1.
\end{aligned}$$

asking $\Theta < 1 - \delta - \beta$. Coming back to the main estimate, since $|\dot{\lambda}_*(t)| \leq |\dot{\lambda}_*(0)|$ and $\|\dot{\xi}\|_{L^\infty(0,T)} \leq C$, we obtain

$$|(\Psi^* \cdot U)U_t| \leq CT^{\sigma_0}(\rho_1 + \rho_2 + \rho_3)(\|\psi\|_{\#,\Theta,\gamma} + 1).$$

Finally, this gives us

$$\|(\Psi^* \cdot U)U_t\|_{**} \leq CT^{\sigma_0}(\|\psi\|_{\#,\Theta,\gamma} + 1).$$

Now, to estimate $(\Phi^0 \cdot U)U_t$:

$$\begin{aligned}
|(\Phi^0 \cdot U)U_t| &\leq Cr(|\dot{\lambda}_*||\log(r^2+\lambda_*^2)| + 1)\frac{(|\dot{\lambda}_*| + \|\dot{\xi}\|_{L^\infty(0,T)})}{r+\lambda_*}, \\
&\leq C(|\dot{\lambda}_*||\log(r^2+\lambda_*^2)| + 1)(|\dot{\lambda}_*| + \|\dot{\xi}\|_{L^\infty(0,T)}).
\end{aligned}$$

Consider the following observation:

Observation 18 Notice that

$$|\dot{\lambda}_*||\log(r^2+\lambda_*^2)| \leq C.$$

Indeed, if $r^2 + \lambda_*^2 \geq 1$ then we use that $|\dot{\lambda}_*| \leq \frac{1}{|\log T|}$, that we are in a bounded domain Ω and that $\log(r^2+\lambda_*^2)$ is continuous, to see that it achieves a maximum in $\overline{\Omega}$ and hence the term is bounded. If $r^2 + \lambda_*^2 < 1$, then $|\log(r^2+\lambda_*^2)| \leq |\log(\lambda_*^2)|$, so

$$\begin{aligned}
|\dot{\lambda}_*||\log(r^2+\lambda_*^2)| &\leq C|\dot{\lambda}_*||\log(\lambda_*)|, \\
&\leq C\frac{|\log T|}{\log^2(T-t)}|\log(\lambda_*)|, \\
&\leq C\frac{|\log T|}{\log^2(\lambda_*)}|\log(\lambda_*)|, \\
&\leq C\frac{|\log T|}{\log(\lambda_*)}, \\
&\leq C\frac{|\log T|}{\log(\lambda_*(0))}.
\end{aligned}$$

Here we have used Observation 17.

And since $|\dot{\lambda}_*|$ is bounded, we obtain:

$$\|(\Phi^0 \cdot U)U_t\|_{**} \leq CT^{\sigma_0}\rho_3(1 + \|\dot{\xi}\|_{L^\infty(0,T)}).$$

Combining this with all the previous estimations and using the estimate on Z^* we obtain

$$\begin{aligned}\|g_1\|_{**} \quad \leq \quad & CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} + \|\psi\|_{\#,\Theta,\gamma} \\ & + \|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)} + \|Z_0^*\|_{L^\infty(\Omega)}).\end{aligned}$$

Observation 19 Notice that we can use the result from [7] because in section 2.4 we have the same assumptions on $a, \beta, \nu, \Theta, \nu_2$, and we can check that (3.2) is also satisfied. In fact, if $|x - \xi| \leq 2\lambda_* R$

$$|x - q| \leq |x - \xi| + |\xi - q| \leq 2\lambda_* R + |\xi - q|,$$

and because of (2.29) and $\xi(T) = q$ we have

$$|\xi - q| = |\xi(t) - \xi(T)| \leq |\dot{\xi}|(T - t) \leq \|\dot{\xi}\|_{L^\infty(0,T)}(T - t) \leq \mu_1(T - t),$$

and since $1 - \beta < 1$:

$$|\xi - q| \leq \mu_1(T - t)^{1-\beta}\frac{|\log T|}{\log^2(T - t)} \sim \lambda_*^{1-\beta} = \mu_1\lambda_* R.$$

Now, we estimate g_2.

- We start with $\frac{1}{x_1}\partial_{x_1}\Psi^*$. Notice that in our setting $|x_1|$ is bounded from below, then there exists a constant $C > 0$ so that $\frac{1}{|x_1|} \leq C$. Therefore,

$$\begin{aligned}\left|\frac{1}{x_1}\partial_{x_1}\Psi^*\right| &\leq C(|\nabla\psi| + |\nabla Z^*|), \\ &\leq C(\lambda_*(0)^\Theta\|\psi\|_{\#,\Theta,\gamma} + \|Z^*\|_{C^1(\Omega\times(0,T))}), \\ &\leq CT^{\sigma_0}\rho_3(\|\psi\|_{\#,\Theta,\gamma} + \|Z^*\|_{C^1(\Omega\times(0,T))}),\end{aligned}$$

  hence

$$\left\|\frac{1}{x_1}\partial_{x_1}\Psi^*\right\|_{**} \leq CT^{\sigma_0}(\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}).$$

- For $\frac{1}{x_1}(\Psi^* \cdot U)U_{x_1}$ we will use the same ideas as in the estimate of $(\Psi^* \cdot U)U_t$. First, recalling that $r = \lambda\rho$ we note that

$$w_\rho = \frac{-2}{1 + \rho^2} = \frac{-2\lambda^2}{\lambda^2 + r^2}.$$

  Then

$$\begin{aligned}|U_{x_1}| &= \left|\frac{w_\rho}{\lambda}\left[\cos(\theta)Q_\alpha E_1 + \sin(\theta)Q_\alpha E_2\right]\right|, \\ &\leq C\frac{\lambda}{\lambda^2 + r^2}(|Q_\alpha E_1| + |Q_\alpha E_2|), \\ &\leq C\frac{\lambda_*}{\lambda_*^2 + r^2}, \\ &\leq C\frac{1}{\lambda_* + r}.\end{aligned}$$

In the last step we used that $Q_\alpha E_1$ and $Q_\alpha E_2$ are unitary vectors and the assumption that the main order of vanishing of λ is λ_*. With this and (3.3) we obtain:

$$\begin{aligned}\left|\frac{1}{x_1}(\Psi^* \cdot U)U_{x_1}\right| &\le C\frac{1}{\lambda_*+r}(r+|T-t|)(\|\psi\|_{\#,\Theta,\gamma}+1),\\ &\le CT^{\sigma_0}(\rho_1+\rho_2+\rho_3)(\|\psi\|_{\#,\Theta,\gamma}+1).\end{aligned}$$

- For the part involving ϕ we shall use the following simple implications from the form of the norms defined in section 2.4:

$$\begin{aligned}|\phi_1| &\le \frac{\lambda_*^\nu}{1+|y|^{a-2}}\|\phi_1\|_{*,a,\nu}, \qquad \text{for } R\le |y|\le 2R,\\ |\phi_2| &\le \frac{\lambda_*^{\nu_2}R^2}{1+|y|}\|\phi_2\|_{**,\nu_2},\\ |\phi_3| &\le \lambda_*^\nu |\log R|\|\phi_3\|_{***,\nu}.\end{aligned}$$

The first term associated with ϕ in g_2 is:

$$|Q_\omega\frac{1}{x_1}\phi\partial_{x_1}\eta| \le C(|\phi_1|+|\phi_2|+|\phi_3|)|\partial_{x_1}\eta|.$$

Recall that $\eta=\eta_0(\frac{|x-\xi|}{\lambda R})$ and $\eta_0(s)$ is a cut-off function that is constant everywhere except in the interval $(1,3/2)$, so

$$|\partial_{x_1}\eta| = \frac{1}{R\lambda}|\partial_s\eta_0| \le C\frac{1}{R\lambda}\chi_{\{R\lambda<r<2R\lambda\}}.$$

Then

$$\begin{aligned}|\phi_1||\partial_{x_1}\eta| &\le C\frac{1}{R\lambda}\frac{\lambda_*^\nu}{1+|y|^{a-2}}\|\phi_1\|_{*,a,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le C\frac{\lambda_*^{\nu-1+\beta}}{1+R^{a-2}}\|\phi_1\|_{*,a,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le C\lambda_*^{\nu-1+\beta(a-1)}\|\phi_1\|_{*,a,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\end{aligned}$$

and using that $\Theta<\nu+\beta(a-2)$ we obtain

$$|\phi_1||\partial_{x_1}\eta| \le CT^{\sigma_0}\rho_1\|\phi_1\|_{*,a,\nu}.$$

Also,

$$\begin{aligned}|\phi_2||\partial_{x_1}\eta| &\le C\frac{1}{R\lambda}\frac{\lambda_*^\nu R^2}{1+|y|}\|\phi_2\|_{**,\nu_2}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le C\lambda_*^{\nu-1}\|\phi_1\|_{*,a,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le CT^{\sigma_0}\rho_1\|\phi_2\|_{**,\nu_2}.\end{aligned}$$

In the last step we used that $\Theta<\nu_2-\beta$. The third one is similar, we use that for R large $|\log R|\le R$, hence

$$\begin{aligned}|\phi_3||\partial_{x_1}\eta| &\le C\frac{1}{R\lambda}\lambda_*^\nu|\log R|\|\phi_3\|_{***,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le C\lambda_*^{\nu-1}\|\phi_3\|_{***,\nu}\chi_{\{R\lambda<r<2R\lambda\}},\\ &\le CT^{\sigma_0}\rho_1\|\phi_3\|_{***,\nu}.\end{aligned}$$

Joining all of the above we obtain

$$\left\| Q_\omega \frac{1}{x_1} \phi \partial_{x_1} \eta \right\|_{**} \leq CT^{\sigma_0} (\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu}).$$

The second term associated with ϕ is $\frac{1}{\lambda x_1} \eta \partial_{y_1} \phi$, we will need the following

$$\begin{aligned}
|\nabla_y \phi_1| &\leq \frac{\lambda_*^\nu R^{3-a}}{(1+|y|)^2} \|\phi_1\|_{*,a,\nu}, \quad \text{for } |y| \leq R, \\
|\nabla_y \phi_2| &\leq \frac{\lambda_*^{\nu_2} R^2}{(1+|y|)^2} \|\phi_2\|_{**,\nu_2}, \quad \text{for } y \in \mathbb{R}^2, \\
|\nabla_y \phi_3| &\leq \frac{\lambda_*^\nu |\log R|}{(1+|y|)} \|\phi_3\|_{***,\nu}, \quad \text{for } y \in \mathbb{R}^2.
\end{aligned}$$

Then

$$\begin{aligned}
\left| \frac{1}{\lambda x_1} \eta \partial_{y_1} \phi \right| &\leq C \frac{1}{\lambda} \chi_{\{r<\lambda R\}} (|\nabla_y \phi_1| + |\nabla_y \phi_2| + |\nabla_y \phi_3|), \\
&\leq C \left(\frac{\lambda_*^{\nu-1} R^{3-a}}{(1+|y|)^2} \|\phi_1\|_{*,a,\nu} + \frac{\lambda_*^{\nu-1} R^2}{(1+|y|)^2} \|\phi_2\|_{**,\nu_2} \right. \\
&\left. + \frac{\lambda_*^{\nu-1} |\log R|}{(1+|y|)} \|\phi_3\|_{***,\nu} \right) \chi_{\{r<\lambda R\}}, \\
&\leq C \left(\lambda_*^{\nu-1-\beta(3-a)} \|\phi_1\|_{*,a,\nu} + \lambda_*^{\nu-1-2\beta} \|\phi_2\|_{**,\nu_2} + \lambda_*^{\nu-1-\beta} \|\phi_3\|_{***,\nu} \right) \chi_{\{r<\lambda R\}},
\end{aligned}$$

using that $\Theta < \nu - (4-a)\beta$, $\Theta < \nu - 3\beta$ and $\Theta < \nu - 2\beta$ we obtain

$$\left| \frac{1}{\lambda x_1} \eta \partial_{y_1} \phi \right| \leq CT^{\sigma_0} \rho_1 \left(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} \right),$$

and therefore

$$\left\| \frac{1}{\lambda x_1} \eta \partial_{y_1} \phi \right\|_{**} \leq CT^{\sigma_0} \left(\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} \right).$$

- The term related to the error $\mathcal{K}_{12}$ is a direct estimation:

$$\begin{aligned}
|(1-\eta)\mathcal{K}_{12}[p,\xi,\Psi^*]| &\leq \left| \frac{w_\rho}{\lambda(\xi_1 + \lambda\rho\cos(\theta))} [\cos(\theta) Q_\alpha E_1 + \sin(\theta) Q_\alpha E_2] \right| \chi_{\{r \geq \lambda R\}}, \\
&\leq C \frac{\lambda_*}{r^2 + \lambda_*^2} \chi_{\{r > \lambda R\}}, \\
&\leq CT^{\sigma_0} \rho_2,
\end{aligned}$$

since $\lambda_* \leq \lambda_*^{1-\sigma_0}$. As a consequence we have,

$$\|(1-\eta)\mathcal{K}_{12}[p,\xi,\Psi^*]\|_{**} \leq CT^{\sigma_0}.$$

- For what comes next is important to remember Lemma 3.2. With this lemma we estimate:

$$\begin{aligned}
\left| \frac{1}{x_1} \Pi_{U^\perp} \partial_{x_1} \Phi^0 \right| &\leq C |\partial_{x_1} \Phi^0|, \\
&\leq C(|\dot{\lambda}_*| |\log(r^2 + \lambda_*^2)| + 1).
\end{aligned}$$

Using Observation 18 we obtain:

$$\left\| \frac{1}{x_1} \Pi_{U^\perp} \partial_{x_1} \Phi^0 \right\|_{**} \leq C T^{\sigma_0}.$$

Now, we estimate the following:

$$\begin{aligned}
\left| \frac{1}{x_1} (\Phi^0 \cdot U) U_{x_1} \right| &\leq C |U_{x_1}| |\Phi^0|, \\
&\leq C \frac{\lambda_*}{r^2 + \lambda_*^2} r (|\dot{\lambda}_*| |\log(r^2 + \lambda_*^2)| + 1), \\
&\leq C \frac{(\lambda_* + r)}{(r + \lambda_*)^2} (r + \lambda_*) (|\dot{\lambda}_*| |\log(r^2 + \lambda_*^2)| + 1), \\
&\leq C (|\dot{\lambda}_*| |\log(r^2 + \lambda_*^2)| + 1), \\
&\leq C T^{\sigma_0} \rho_3.
\end{aligned}$$

Then

$$\left\| \frac{1}{x_1} (\Phi^0 \cdot U) U_{x_1} \right\|_{**} \leq C T^{\sigma_0}.$$

- For the nonlinear part $\frac{1}{x_1} a U_{x_1}$ we write

$$\begin{aligned}
\left| \frac{1}{x_1} a U_{x_1} \right| &\leq C \left| a(\Pi_{U^\perp} \varphi) \right| \frac{\lambda_*}{\lambda_*^2 + r^2}, \\
&\leq C |\varphi|^2 \frac{\lambda_*}{\lambda_*^2 + r^2}, \\
&\leq C \left(|\eta Q_\alpha \phi|^2 + |\Psi^*|^2 + |\Phi^0|^2 \right) \frac{\lambda_*}{\lambda_*^2 + r^2}.
\end{aligned}$$

In the second inequality we have used the Taylor expansion of $\sqrt{1+x}$ near 0, where $x = |\varphi|^2$. Notice that the last right hand side is a multiplication of terms that we have already calculated with the modulus of the functions $\eta Q_\alpha \phi$, Ψ^*, Φ^0, which are already small. Therefore this term is of smaller order than the ones we have already seen. And hence

$$\left\| \frac{1}{x_1} a U_{x_1} \right\|_{**} \leq C T^{\sigma_0} (\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} + \|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}).$$

Combining all the estimates we get

$$\begin{aligned}
\|g_2\|_{**} \leq \; & C T^{\sigma_0} (\|\phi_1\|_{*,a,\nu} + \|\phi_2\|_{**,\nu_2} + \|\phi_3\|_{***,\nu} + \|\psi\|_{\#,\Theta,\gamma} \\
& + \|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)} + \|Z_0^*\|_{L^\infty(\Omega)}).
\end{aligned}$$

Finally, with the estimate on g_1 and g_2 we obtain the result. □

PROOF. (Of Lema 3.2) First, we will estimate $\left| \frac{1}{x_1} \Pi_{U^\perp} \partial_{x_1} \Phi^0 \right|$ and then deduce the one for $|\Phi^0|$.

Recall that in (2.11) we defined

$$
\begin{aligned}
\Phi^0[\omega,\lambda,\xi]: &= \begin{pmatrix} \varphi^0(r,t)e^{i\theta} \\ 0 \end{pmatrix}, \\
\varphi^0(r,t) &= -\int_{-T}^{t} \dot p(s) r k(z(r), t-s)ds, \\
z(r) &= \sqrt{r^2+\lambda^2}, \quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\partial_{x_1}(\varphi^0 e^{i\theta}) &= \left(\partial_r(\varphi^0 e^{i\theta})\partial_{x_1} r + \partial_\theta(\varphi^0 e^{i\theta})\partial_{x_1}\theta\right), \\
&= -\cos(\theta)e^{i\theta}\int_{-T}^{t}\dot p(s)(k(z(r),t-s) + rk_z(z(r),t-s)z_r)ds \\
&\quad + i\frac{\sin(\theta)}{r}e^{i\theta}\int_{-T}^{t}\dot p(s) r k(z(r),t-s)ds, \\
&= \Big[-\cos(\theta)\int_{-T}^{t}\dot p(s)(k(z(r),t-s) + rk_z(z(r),t-s)z_r)ds \\
&\quad + i\sin(\theta)\int_{-T}^{t}\dot p(s) k(z(r),t-s)ds\Big] e^{i\theta}.
\end{aligned}
$$

For a complex valued function $f(r,t)$ one has

$$
\Pi_{U^\perp}\begin{bmatrix} f(r,t)e^{i\theta} \\ 0 \end{bmatrix} = cosw(\rho)\mathrm{Re}\ (f(r,t)e^{-i\omega})Q_\omega E_1 + \mathrm{Im}\ (f(r,t)e^{-i\omega})Q_\omega E_2.
$$

Then we apply the projection of $\partial_{x_1}\Phi^0$ on $U^\perp$ and obtain:

$$
\begin{aligned}
\frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0 = &-\Big[\frac{\cos(w)\cos(\theta)}{\xi_1 + r\cos(\theta)}\int_{-T}^{t} Re(\dot p(s)e^{-i\alpha})(k(z(r),t-s) + rk_z(z(r),t-s)z_r)ds \\
&+ \frac{\cos(w)\sin(\theta)}{\xi_1 + r\cos(\theta)}\int_{-T}^{t} Im(\dot p(s)e^{-i\alpha})k(z(r),t-s)ds\Big] Q_\alpha E_1 \\
&+ \Big[-\frac{\cos(\theta)}{\xi_1 + r\cos(\theta)}\int_{-T}^{t} Im(\dot p(s)e^{-i\alpha})(k(z(r),t-s) + rk_z(z(r),t-s)z_r)ds \\
&+ \frac{\sin(\theta)}{\xi_1 + r\cos(\theta)}\int_{-T}^{t} Re(\dot p(s)e^{-i\alpha})k(z(r),t-s)ds\Big] Q_\alpha E_2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|\frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0\right| &\le C\int_{-T}^{t} |Re(\dot p(s)e^{-i\alpha}) + Im(\dot p(s)e^{-i\alpha})||(k(z(r),t-s) + rk_z(z(r),t-s)z_r)|ds, \\
&\le \underbrace{C\int_{-T}^{t}|\dot p(s)||k(z(r),t-s)|ds}_{I_1} + \underbrace{C\int_{-T}^{t}|\dot p(s)||rk_z(z(r),t-s)z_r|ds}_{I_2}.
\end{aligned}
$$

Consider the first integral:

$$I_1 = \underbrace{\int_{-T}^{t-\frac{z^2}{4}} |\dot{p}(s)||k(z(r), t-s)|ds}_{I_3} + \underbrace{\int_{t-\frac{z^2}{4}}^{t} |\dot{p}(s)||k(z(r), t-s)|ds}_{I_4}.$$

We estimate I_3. Notice that that $s \leq t - \frac{z^2}{4}$ implies that $z^2 \leq 4(t-s)$. We use that $e^{-\frac{x}{4(t-s)}} \sim 1 - \frac{x}{4(t-s)}x + o(x^2)$ and obtain

$$\int_{-T}^{t-\frac{z^2}{4}} |\dot{p}(s)||k(z(r), t-s)|ds \leq C \int_{-T}^{t-\frac{z^2}{4}} \frac{|\dot{p}(s)|}{(t-s)} ds.$$

We analyze two cases. First, when $\frac{z^2}{4} \leq T - t$ we divide the integral into two:

$$\int_{-T}^{t-\frac{z^2}{4}} \frac{|\dot{p}(s)|}{(t-s)} ds = \underbrace{\int_{-T}^{t-(T-t)} \frac{|\dot{p}(s)|}{(t-s)} ds}_{I_5} + \underbrace{\int_{t-(T-t)}^{t-\frac{z^2}{4}} \frac{|\dot{p}(s)|}{(t-s)} ds}_{I_6}.$$

We use that in I_5 we have $T - t \leq t - s$,

$$\begin{aligned}
\int_{-T}^{t-(T-t)} \frac{|\dot{p}(s)|}{(t-s)} ds &\leq \int_{-T}^{t-(T-t)} \frac{|\dot{p}(s)|}{(T-s)} ds, \\
&\leq C|\log T| \int_{-T}^{t-(T-t)} \frac{1}{(T-s)\log^2(T-s)} ds, \\
&= C|\log T| \left[\frac{1}{|\log(T-s)|} \right]_{s=-T}^{s=t-(T-t)}, \\
&= C|\log T| \left[\frac{1}{|\log(2(T-t))|} - \frac{1}{|\log(2T)|} \right], \\
&\leq C.
\end{aligned}$$

In I_6 we have to notice that $t - s \leq T - t$ and since $\frac{1}{\log^2(x)}$ is decreasing, then

$$\begin{aligned}
\int_{t-(T-t)}^{t-\frac{z^2}{4}} \frac{|\dot{p}(s)|}{(t-s)} ds &\leq C \int_{t-(T-t)}^{t-\frac{z^2}{4}} \frac{|\log T|}{\log^2(T-s)(t-s)} ds, \\
&\leq C \frac{|\log T|}{\log^2(T-t)} \int_{t-(T-t)}^{t-\frac{z^2}{4}} \frac{1}{(t-s)} ds, \\
&\leq C \frac{|\log T|}{\log^2(T-t)} \left[|\log(t-s)| \right]_{s=t-(T-t)}^{s=t-\frac{z^2}{4}}, \\
&\leq C \frac{|\log T|}{\log^2(T-t)} \left[|\log\left(\frac{z^2}{4}\right)| - |\log(T-t)| \right], \\
&\leq C \frac{|\log T|}{\log^2(T-t)} \left[|\log(z)| - |\log(T-t)| \right], \\
&\leq C(|\dot{\lambda}_*||\log(r^2 + \lambda_*^2)| + 1).
\end{aligned}$$

The second case is when $\frac{z^2}{4} \geq T-t$, then $t - \frac{z^2}{4} \leq t-(T-t)$ and $T-t \leq t-s$, so

$$\int_{-T}^{t-\frac{z^2}{4}} \frac{|\dot{p}(s)|}{(t-s)} ds \leq \int_{-T}^{t-(T-s)} \frac{|\dot{p}(s)|}{(T-s)} ds \leq C.$$

Now, we estimate I_4. Using that $1-e^{-\frac{z^2}{4(t-s)}} \leq 1$, then

$$\begin{aligned} I_4 = \int_{t-\frac{z^2}{4}}^{t} |\dot{p}(s)||k(z(r),t-s)|ds &\leq \frac{1}{z^2}\int_{t-\frac{z^2}{4}}^{t} |\dot{p}(s)|ds, \\ &\leq C \max_{s\in(t-\frac{z^2}{4},t)} |\dot{\lambda}_*(s)|, \\ &\leq C. \end{aligned}$$

Finally, we compute I_2. Notice that

$$\begin{aligned} |rk_z(z(r),t-s)z_r| &\leq \frac{r^2}{z}|k_z(z(r),t-s)|, \\ &\leq \frac{r^2+\lambda^2}{z}\left| \frac{e^{-\frac{z^2}{4(t-s)}}}{2(t-s)z} - \frac{2(1-e^{-\frac{z^2}{4(t-s)}})}{z^3} \right|, \\ &\leq C\left| \frac{e^{-\frac{z^2}{4(t-s)}}}{(t-s)} \right| + C\left| \frac{(1-e^{-\frac{z^2}{4(t-s)}})}{z^2} \right|, \\ &\leq C\left(\frac{1}{(t-s)} + \frac{(1-e^{-\frac{z^2}{4(t-s)}})}{z^2} \right). \end{aligned}$$

Then

$$I_2 = \int_{-T}^{t} |\dot{p}(s)||rk_z(z(r),t-s)z_r|ds \leq C\int_{-T}^{t} \frac{|\dot{p}(s)|}{(t-s)}ds + \int_{-T}^{t} |\dot{p}(s)|\frac{(1-e^{-\frac{z^2}{4(t-s)}})}{z^2}ds.$$

The second integral is the same as I_1, so we already have an estimate for it. For the first integral we use the same idea of I_3:

$$\begin{aligned} \int_{-T}^{t} \frac{|\dot{p}(s)|}{(t-s)}ds &= \int_{-T}^{t-(T-t)} \frac{|\dot{p}(s)|}{(t-s)}ds + \int_{t-(T-t)}^{t} \frac{|\dot{p}(s)|}{(t-s)}ds, \\ &\leq C\int_{-T}^{t-(T-t)} \frac{|logT|}{\log^2(T-s)(t-s)}ds + C\int_{t-(T-t)}^{t} \frac{|logT|}{\log^2(t-s)(t-s)}ds, \\ &\leq C|\log T|\left(\frac{1}{|\log(2(T-t))|} - \frac{1}{|\log(2T)|} - \frac{1}{|\log(T-t)|} \right), \\ &\leq C. \end{aligned}$$

Replacing all of this in I_1 and I_2 we get

$$\left| \frac{1}{x_1}\Pi_{U^\perp}\partial_{x_1}\Phi^0 \right| \leq C(|\dot{\lambda}_*||\log(r^2+\lambda_*^2)|+1).$$

Notice that

$$|\Phi^0| \le C\int_{-T}^{t} |\dot p(s)||rk(z(r),t-s)|ds \le CrI_1 \le Cr(|\dot\lambda_*||\log(r^2+\lambda_*^2)|+1).$$

$\square$

3.2 The interior problem

In this section we compute estimations for the interior problems (2.40), (2.41) and (2.42) to prove the following propositions:

Proposition 3.3 *Let $p(t)=\lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (2.30), (2.29) and*

$$(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\in B_M.$$

Then there exists a constant $C>0$ such that

$$\begin{aligned}\|h[p,\xi,Z^*,\Psi]\|_{a,\nu} &\le CT^{\sigma_0}(\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)})\\ &\quad +C\lambda_*^{1-\nu-\beta(a-2)}(0)(\|\Psi\|_{\#,\Theta,\gamma}+\|Z_0^*\|_{L^\infty(\Omega)}+1)\end{aligned}$$

where h is defined in (2.18).

Proposition 3.4 *Let $p(t)=\lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (2.30), (2.29) and let*

$$(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\in B_M.$$

Then there exists a constant $C>0$ such that

$$\left\|\sum_{j=1}^{2} c_{0j}^*[p,\xi,\Psi]w_\rho^2 Z_{0j}\right\|_{a,\nu_2} \le C\lambda_*(0)^{1+\alpha_0-\nu_2}.$$

Proposition 3.5 *Let $p(t)=\lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (2.30), (2.29) and let*

$$(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\in B_M.$$

Then there exists a constant $C>0$ such that

$$\left\|\sum_{j=1}^{2} c_{-1j}[h[p,\xi,\Psi]]w_\rho^2 Z_{-1j}\right\|_{a,\nu} \le C\lambda_*^{1-\nu-\tau-\beta(a-2)}(0)(\|\psi\|_{\#,\Theta,\gamma}+\|Z_0^*\|_{L^\infty(\Omega)}+1).$$

Here c_{ij} and c_{0j}^* are defined in (2.31) and (2.36), respectively.

Proof. (Of Proposition 3.3) In [7] the authors prove that

$$\left\|\lambda^2 Q_{-\omega}\left[\mathcal{K}_0[p,\xi]+\mathcal{K}_{11}[p,\xi]\right]\right\|_{a,\nu} \le CT^{\sigma_0}(\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)}).$$

Now, from (1.9) we have that

$$|\lambda^2 Q_{-\omega}\tilde{L}_U[\Psi + Z^*]| \le C\frac{\lambda_*}{(1+|y|^2)}(\|\nabla\Psi\|_{L^\infty} + \|\nabla Z^*\|_{L^\infty}).$$

But we recall that we are solving for $|y| \le 2R$ and hence

$$\begin{aligned}\frac{\lambda_*}{(1+|y|)^2} &= \frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu}(1+|y|)^{a-2},\\ &\le C\frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu}(0)(1+R(0))^{a-2},\end{aligned}$$

because $\nu < 1-\beta(a-2)$. Then

$$\begin{aligned}|\lambda^2 Q_{-\omega}\tilde{L}_U[\Psi]| &\le C\frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu}(0)(1+R(0))^{a-2}(\|\nabla\Psi\|_{L^\infty} + \|\nabla Z^*\|_{L^\infty}),\\ &\le C\frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu}(0)(1+R(0))^{a-2}(\lambda_*^\Theta(0)\|\Psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}),\\ &\le C\frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu+\Theta-\beta(a-2)}(0)\|\Psi\|_{\#,\Theta,\gamma} + \frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*^{1-\nu}(0)\|Z_0^*\|_{L^\infty(\Omega)},\end{aligned}$$

where $1-\nu+\Theta-\beta(a-2) > 0$. Therefore we can form the norm of the left hand side and obtain:

$$\|\lambda^2 Q_{-\omega}\tilde{L}_U[\Psi]\|_{a,\nu} \le \lambda_*^{1-\nu+\Theta}(0)\|\Psi\|_{\#,\Theta,\gamma} + \lambda_*^{1-\nu}(0)\|Z_0^*\|_{L^\infty(\Omega)}).$$

The only part left to estimate is $\lambda^2 Q_{-\omega}\mathcal{K}_{12}[p,\xi,\Psi^*]$.

$$\begin{aligned}|\lambda^2 Q_{-\omega}\mathcal{K}_{12}[p,\xi]| &\le \lambda^2\frac{1}{\lambda(1+\rho^2)},\\ &= \frac{\lambda}{\lambda(1+|y|^2)},\\ &\le C\frac{\lambda_*^\nu}{(1+|y|)^a}\lambda_*(0)^{1-\nu-\beta(a-2)},\end{aligned}$$

then

$$\|\lambda^2 Q_{-\omega}\mathcal{K}_{12}[p,\xi]\|_{a,\nu} \le C\lambda_*(0)^{1-\nu-\beta(a-2)}.$$

Combining the estimates we get

$$\begin{aligned}\|h[p,\xi,Z^*,\Psi]\|_{a,\nu} &\le CT^{\sigma_0}(\|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)})\\ &+ C\lambda_*^{1-\nu+\Theta-\beta(a-2)}(0)\|\Psi\|_{\#,\Theta,\gamma}\\ &+ \lambda_*^{1-\nu}(0)\|Z_0^*\|_{L^\infty(\Omega)}\\ &+ C\lambda_*(0)^{1-\nu-\beta(a-2)}.\end{aligned}$$

Noticing that $\lambda_*(0)$ is small and $1-\nu-\beta(a-2) < 1-\nu+\Theta-\beta(a-2) < 1-\nu$ we obtain the result. $\square$

Proof. (Of Proposition 3.4) Since c_{0j}^* does not depend on h but on $\mathcal{R}_0$, which has the same order of decay as in [7], and a, which is same as the one used in [7], we can use that in [7] the authors prove that

$$\left\| \sum_{j=1}^{2} c_{0j}^*[p,\xi,\Psi] w_\rho^2 Z_{0j} \right\|_{a,\nu_2} \leq \lambda_*(0)^{1+\alpha_0-\nu_2}.$$

□

Proof. (Of Proposition 3.5) In [7] the authors prove:

$$\begin{aligned} |c_{-1j}[\tilde{L}_U[\Psi+Z^*]\chi_{\mathcal{D}_{2R}}]| &\leq C\lambda_* \log(R)(\|\nabla\Psi\|_{L^\infty} + \|\nabla Z^*\|_{L^\infty}), \\ \lambda_*^2|c_{-1j}[\mathcal{K}_0\chi_{\mathcal{D}_{2R}}]| &\leq C\lambda_*, \\ \lambda_*^2|c_{-1j}[\mathcal{K}_{11}\chi_{\mathcal{D}_{2R}}]| &\leq C\lambda_*\|\dot{\xi}\|_{L^\infty}. \end{aligned}$$

We only need to estimate the following:

$$\begin{aligned} \lambda_*^2|c_{-1j}[\mathcal{K}_{12}\chi_{\mathcal{D}_{2R}}]| &\leq C\lambda_*^2 \int_{\mathbb{R}^2} |\mathcal{K}_{12}||Z_{-1j}|\chi_{\mathcal{D}_{2R}}, \\ &\leq C\lambda_*^2 \int_0^{2R} \frac{w_\rho}{\lambda_*}\rho^2 w_\rho \rho d\rho, \\ &\leq C\lambda_* \int_0^{2R} \frac{\rho^3}{(1+\rho^2)^2} d\rho, \\ &\leq C\lambda_* \left[\frac{1}{(2R)^2+1} + \log((2R)^2+1) - 1 \right], \\ &\leq C\lambda_*^{1-\tau}. \end{aligned}$$

The last step is a consequence of the analysis done to function ℓ in section 2.2 with $\tau \in (0, \frac{\beta}{2})$.

Combining all of the estimates above we obtain

$$|c_{-1j}[h[p,\xi,\Psi]]| \leq C\lambda_* \underbrace{\log(R)(\lambda_*(0)^\Theta\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)})}_{A} + C\lambda_*^{1-\tau} + C\lambda_*\|\dot{\xi}\|_{L^\infty}.$$

Then, noticing that $\nu < 1 - \frac{\beta}{2} < 1$ we obtain:

$$\begin{aligned} \left| \sum_{j=1}^{2} c_{-1j}[h[p,\xi,\Psi[p,\xi,\phi_1,\phi_2,\phi_3,Z^*]]] w_\rho^2 Z_{-1j} \right| &\leq C(\lambda_* A + \lambda_*^{1-\tau} + \lambda_*\|\dot{\xi}\|_{L^\infty})\rho^2|w_\rho^3|, \\ &\leq C\frac{\lambda_*^\nu}{(1+|y|^2)}(\lambda_*^{1-\nu}(A + \|\dot{\xi}\|_{L^\infty}) + \lambda_*^{1-\tau-\nu})\rho^2|w_\rho^2|, \\ &\leq C\frac{\lambda_*^\nu}{(1+|y|)^a}(1+R(0))^{a-2}(\lambda_*^{1-\nu}(0)(A + \|\dot{\xi}\|_{L^\infty}), \\ &+ \lambda_*^{1-\tau-\nu}(0))\rho^2|w_\rho^2|. \end{aligned}$$

And since $\rho^2|w_\rho^2| = \frac{\rho^2}{(1+\rho^2)^2} < +\infty$, then

$$\begin{aligned}\left\|\sum_{j=1}^{2} c_{-1j}[h[p,\xi,\Psi]]w_\rho^2 Z_{-1j}\right\|_{a,\nu} &\leq C\lambda_*^{1-\nu+\Theta-\beta(a-2)}(0)(\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}) \\ &\quad + C\lambda_*^{1-\tau-\nu-\beta(a-2)}(0)), \\ &\leq C\lambda_*^{1-\nu-\tau-\beta(a-2)}(0)(\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)} + 1).\end{aligned}$$

Here we have used that $1-\nu-\tau-\beta(a-2) < 1-\nu+\Theta-\beta(a-2)$. This last estimate gives us the result. $\square$

3.3 The parameter problem

In this section we prove Proposition 2.8 and compute estimates on the right hand side of the equations of the parameters.

$$\dot{\xi}_1 = -\frac{1}{\xi_1} - b_1[p,\xi,\Psi^*] \tag{3.4}$$

$$\dot{\xi}_2 = -b_2[p,\xi,\Psi^*], \tag{3.5}$$

Proof. (Of Proposition 2.8) Let us recall that $\mathcal{A}_1[p,\xi,\Psi^*]$ is the operator that returns the solution of (3.4) and $\mathcal{A}_2[p,\xi,\Psi^*]$ returns the solution of (3.5). To obtain the estimate for $\mathcal{A}_1$, we notice that from equation (3.4) we have

$$|\dot{\xi}_1| \leq \frac{1}{\mu_2} + |b_1[p,\xi,\Psi^*]|,$$

and

$$|\xi_1(t)| \leq |\xi_1(t)-\xi_1(T)|+q_1 \leq q_1+\left(\frac{1}{\mu_2}+|b_1[p,\xi,\Psi^*]|\right)(T-t) \leq q_1+T\left(\frac{1}{\mu_2}+|b_1[p,\xi,\Psi^*]|\right),$$

then we get the following estimate:

$$\|\mathcal{A}_1[p,\xi,\Psi^*]\|_{C^1(0,T)} \leq \left(q_1+\frac{1}{\mu_2}+\|b_1[p,\xi,\Psi^*]\|_{L^\infty(0,T)}\right)+T\left(\frac{1}{\mu_2}+\|b_1[p,\xi,\Psi^*]\|_{L^\infty(0,T)}\right). \tag{3.6}$$

For equation (3.5) and $\mathcal{A}_2$ the analysis is simpler, integrating equation (3.5) we get

$$\mathcal{A}_2[p,\xi,\Psi^*](t) = q_2 + \int_t^T b_2[p,\xi,\Psi^*](s)ds,$$

which asserts the existence of the solution and

$$|\mathcal{A}_2[p,\xi,\Psi^*](t)| \leq q_2 + \int_t^T |b_2[p,\xi,\Psi^*](s)|ds \leq q_2 + (T-t)\|b_2[p,\xi,\Psi^*]\|_{L^\infty(0,T)}.$$

Taking supremum over $t \in (0,T)$ we obtain

$$\|\mathcal{A}_2[p,\xi,\Psi^*]\|_{L^\infty(0,T)} \leq q_2 + T\|b_2[p,\xi,\Psi^*]\|_{L^\infty(0,T)}.$$

For the derivative is easier, because

$$||\dot{\mathcal{A}}_2[p,\xi,\Psi^*](t)| \leq |b_2[p,\xi,\Psi^*](t)|,$$

then taking the supremum we obtain:

$$\|\mathcal{A}_2[p,\xi,\Psi^*]\|_{C^1(0,T)} \leq \big(q_2 + \|b_2[p,\xi,\Psi^*]\|_{L^\infty(0,T)}\big) + T\|b_2[p,\xi,\Psi^*]\|_{L^\infty(0,T)}. \tag{3.7}$$

Now we use the following fact:

Lemma 3.6

$$\|b_j[p,\xi]\|_{L^\infty(0,T)} \leq C(\lambda_*(0)^\Theta\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}).$$

and deduce the final result by replacing this estimate in (3.6) and (3.7). □

Proof. (Of Lemma 3.6) Recall that

$$b_j[p,\xi,\Psi^*](t) = \frac{1}{4\pi}(1+(2R)^{-2})\lambda\int_{B_{2R}} Q_{-\omega}\tilde{L}_U[\Psi^*]\cdot Z_{1j}(y)dy.$$

Then, by formula (1.9) we have

$$|\tilde{L}_U[\Psi^*]| \leq \frac{\lambda_*}{r^2+\lambda_*^2}|\nabla\Psi^*| = \frac{1}{\lambda_*(1+\rho^2)}|\nabla\Psi^*|,$$

where $r = \lambda\rho$. Replacing this in b_j we obtain

$$\begin{aligned}
|b_j[p,\xi,\Psi^*](t)| &\leq \frac{1}{4\pi}(1+(2R)^{-2})\lambda\int_{B_{2R}} |\tilde{L}_U[\Psi^*]||Z_{1j}(y)|dy,\\
&\leq \frac{1}{4\pi}(1+(2R)^{-2})\lambda_*\int_0^{2R} \frac{1}{\lambda_*(1+\rho^2)}|\nabla\Psi^*||w_\rho|\rho d\rho,\\
&\leq \frac{1}{4\pi}(1+(2R)^{-2})\int_0^{2R}\frac{\rho}{(1+\rho^2)^2}|\nabla\Psi^*|d\rho,\\
&\leq \frac{1}{4\pi}\|\nabla\Psi^*\|_{L^\infty(\Omega)}(1+(2R)^{-2})\int_0^{2R}\frac{\rho}{(1+\rho^2)^2}d\rho,\\
&\leq \frac{1}{4\pi}\|\nabla\Psi^*\|_{L^\infty(\Omega)},\\
&\leq C(\lambda_*(0)^\Theta\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)}).
\end{aligned}$$

□

3.4 Inclusion and compactness of operator $\mathcal{F}$.

In this section we prove Propositions 2.9 and 2.10 to get that the operator $\mathcal{F}$ defined in Section 2.4 goes from B_M into itself and that is compact there.

First, we prove the inclusion $\mathcal{F}(B_M) \subseteq B_M$.

Proof. (Of Proposition 2.9) We want to prove that given $v := (\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M$ then $\mathcal{F}(\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M$, that is

$$\|\mathcal{F}_1(v)\|_{*,a,\nu} + \|\mathcal{F}_2(v)\|_{**,\nu_2} + \|\mathcal{F}_3(v)\|_{***,\nu} + \|\mathcal{F}_4(v)\|_{\#,\Theta,\gamma} + \|\mathcal{F}_5(v)\|_{G_1} + \|\mathcal{F}_6(v)\|_{G_2} \leq M. \quad (3.8)$$

We will see that each norm on the left hand side is less than $M/6$.

- We have from Proposition 2.3 and Proposition 3.3,

$$\begin{aligned}
\|\mathcal{F}_1(\phi_1, \phi_2, \phi_3, \psi, p, \xi)\|_{*,a,\nu} &= \|\mathcal{T}_{\lambda,1}(h[p, \xi, \Psi^*])\|_{*,a,\nu}, \\
&\leq \|h[p, \xi, Z^*, \Psi\|_{a,\nu}, \\
&\leq CT^{\sigma_0}(\|\dot{p}\|_{L^\infty(-T,T)} + \|\dot{\xi}\|_{L^\infty(0,T)}) \\
&\quad + C\lambda_*^{1-\nu-\beta(a-2)}(0)(\|\Psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)} + 1) \\
&\leq \frac{M}{6},
\end{aligned}$$

  because $(\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M$ and choosing $T > 0$, $\lambda_*(0) > 0$, Z_0^* small enough.
- We have from Proposition 2.4 and Proposition 3.4,

$$\begin{aligned}
\|\mathcal{F}_2(\phi_1, \phi_2, \phi_3, \psi, p, \xi)\|_{**,\nu_2} &= \left\|\mathcal{T}_{\lambda,2}\left(\sum_{j=1}^{2} c_{0j}^*[p, \xi, \Psi^*]w_\rho^2 Z_{0j}\right)\right\|_{**,\nu_2}, \\
&\leq \left\|\sum_{j=1}^{2} c_{0j}^*[p, \xi, \Psi^*]w_\rho^2 Z_{0j}\right\|_{a,\nu_2}, \\
&\leq C\lambda_*(0)^{1+\alpha_0-\nu_2}, \\
&\leq \frac{M}{6},
\end{aligned}$$

  choosing $\lambda_*(0)$ small enough.
- From Proposition 2.5 and Proposition 3.5 we obtain

$$\begin{aligned}
\|\mathcal{F}_3(\phi_1, \phi_2, \phi_3, \psi, p, \xi)\|_{***,\nu} &= \left\|\mathcal{T}_{\lambda,3}\left(\sum_{j=1}^{2} c_{-1j}[h[p, \xi, \Psi^*]]w_\rho^2 Z_{-1j}\right)\right\|_{***,\nu}, \\
&\leq \left\|\sum_{j=1}^{2} c_{-1j}[h[p, \xi, \Psi^*]]w_\rho^2 Z_{-1j}\right\|_{a,\nu}, \\
&\leq C\lambda_*^{1-\nu-\tau-\beta(a-2)}(0)(\|\psi\|_{\#,\Theta,\gamma} + \|Z_0^*\|_{L^\infty(\Omega)} + 1) \\
&\leq \frac{M}{6},
\end{aligned}$$

  because $(\phi_1, \phi_2, \phi_3, \psi, p, \xi) \in B_M$ and choosing $\lambda_*(0)$, Z_0^* small enough.

- From Proposition 2.6 and Proposition 3.1 we obtain

$$\begin{aligned}\|\mathcal{F}_4(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\|_{\#,\Theta,\gamma} &= \|\mathcal{H}(g[p,\xi,\psi+Z^*,\phi_1,\phi_2,\phi_3])\|_{\#,\Theta,\gamma},\\ &\le \|\mathcal{H}(g[p,\xi,\psi+Z^*,\phi_1,\phi_2,\phi_3])\|_{\#,\Theta',\gamma'},\\ &\le \|g[p,\xi,\psi+Z^*,\phi_1,\phi_2,\phi_3]\|_{**},\\ &- \frac{\lambda_*(0)^{-\Theta}(\lambda_*(0)R(0))^{-1}}{|\log T|}(|c_1|+|c_2|+|c_3|),\\ &\le CT^{\sigma_0}(\|\phi_1\|_{*,a,\nu}+\|\phi_2\|_{**,\nu_2}+\|\phi_3\|_{***,\nu}+\|\psi\|_{\#,\Theta,\gamma},\\ &+\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)}+\|Z_0^*\|_{L^\infty(\Omega)}),\\ &-\frac{\lambda_*(0)^{\beta-\Theta-1}}{|\log T|}(|c_1|+|c_2|+|c_3|),\\ &\le \frac{M}{6},\end{aligned}$$

  choosing $\lambda_*(0)$, T and Z_0^* small enough.
- From Proposition 2.7 we have that

$$\begin{aligned}\|\mathcal{F}_5(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\|_{G_1} &= |\kappa|+\|p_1\|_{*,3-\sigma}+\|p_2\|_{\Theta,l},\\ &\le |\kappa|+\|p_1\|_{*,3-\sigma'}+\|p_2\|_{\Theta',l'},\\ &\le |\kappa|+C|\log T|^{1-\sigma'}\log(|\log T|)^2+C_0(T),\\ &\le \frac{M}{6},\end{aligned}$$

  by definition of M and choosing T small enough.
- From Proposition 2.8 we have

$$\begin{aligned}\|\mathcal{F}_6(\phi_1,\phi_2,\phi_3,\psi,p,\xi)\|_{G_2} &\le q_1+q_2+\frac{(1+T)}{\mu_2}+2(1+T)(\lambda_*(0)^{\Theta}\|\psi\|_{\#,\Theta,\gamma}+\|Z_0^*\|_{L^\infty(\Omega)}),\\ &\le \frac{M}{6},\end{aligned}$$

  by definition of M and choosing $\lambda_*(0)$ and $\|Z_0^*\|_{L^\infty(\Omega)}$ sufficiently small so that

$$2(1+T)(\lambda_*(0)^{\Theta}\|\psi\|_{\#,\Theta,\gamma}+\|Z_0^*\|_{L^\infty(\Omega)}) \le \frac{M}{12}.$$

Using all the above estimates on the norm of $\mathcal{F}$ we obtain (3.8). □

Next, we prove the compactness of the operator $\mathcal{F}$.

Proof. (Of Proposition 2.10) To obtain the compactness of $\mathcal{F}$ we will divide the proof into proving the compactness of every $\mathcal{F}_i$ for $i=1,\dots,6$. We will do the compactness of $\mathcal{F}_5$ and $\mathcal{F}_6$ in detail, because the authors are simple to write, for the other operators we give the main ideas, without all the technical calculations.

Let $v_n := (\phi_{1_n},\phi_{2_n},\phi_{3_n},\psi_n,p_n,\xi_n)$ for $n\in\mathbb{N}$ such that $(v_n)_{n\in\mathbb{N}}\subset B_M$. We need a convergent subsequence of

$$(\mathcal{F}_1(v_n),\mathcal{F}_2(v_n),\mathcal{F}_3(v_n),\mathcal{F}_4(v_n),\mathcal{F}_5(v_n),\mathcal{F}_6(v_n))\subset B_M.$$

We will treat each component of this sequence separately by providing the existence of convergent subsequences of $\mathcal{F}_i(v_n)$ for all $i = 1, \ldots, 6$ in their respective spaces and norms.

Compactness of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. We check only the compactness of $\mathcal{F}_1$, because for $\mathcal{F}_2$ and $\mathcal{F}_3$ the idea is similar. Notice that problem (2.40) becomes singular when $t \to T$, so we will find convergent subsequences for any time smaller than T and find a subsequence that satisfies this and that also converges in the interval between the smaller time and T. Let us name for simplicity $\phi_n := \mathcal{T}_{\lambda,1}(h(p_n, \xi_n, \Psi_n^*))$ and define

$$f_n(y,t) := \frac{|\phi_n(y,t)|}{\lambda_*^\nu R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}},$$
$$g_n(y,t) := \frac{|\nabla_y \phi_n(y,t)|}{\lambda_*^\nu R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}},$$

which are the parts inside the supremum of the norm $\|\phi_n\|_{*,a,\nu}$. Then the problem of finding a convergent subsequence of $(\mathcal{F}_1(v_n))_{n\in\mathbb{N}}$ in $(E_1, \|\cdot\|)_{*,a,\nu}$ reduces to finding a convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $(C(\overline{\Omega}, [0,T]), \|\cdot\|_{L^\infty(\Omega\times[0,T])})$. Notice that since both convergences are in the space of continuous functions the limit of $\nabla_y\phi_n$ will be the derivative of the limit of ϕ_n.

We define the sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of $\varepsilon_m > 0$ for all $m \in \mathbb{N}$ such that $\varepsilon_m \to 0$ when $m \to \infty$ and $\varepsilon_m < T$ for all $m \in \mathbb{N}$. First, we will find, using the Arzela-Ascoli theorem, convergent subsequences of f_n and g_n on $(C(\overline{\Omega}, [0,T]), \|\cdot\|_{L^\infty(\Omega\times[0,T-\varepsilon_m])})$ for all $m \in \mathbb{N}$. Notice that both f_n and g_n are uniformly bounded since the operator $\mathcal{F}_1$ goes from the ball B_M into itself. So to use Arzela-Ascoli we only need to prove that both sequences are equicontinuous. We do this by proving that f_n and g_n are c-Hölder continuous for all $n \in \mathbb{N}$, where c only depends on $T - \varepsilon_m$.

We will do as an example the Hölder continuity of f_n, for g_n is similar. We denote

$$\varrho(y,t) := \lambda_*^\nu(t) R^{\frac{5-a}{2}}(t)(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}(t)|y|^{-2}\}.$$

Let $t_1, t_2 \in [0, T-\varepsilon_m]$ and $y_1, y_2 \in \Omega$, then a straight forward calculation using Mean Value Theorem on ϕ_n gives us:

$$\begin{aligned}
|f_n(y_1,t_1) - f_n(y_2,t_2)| &\le |f_n(y_1,t_1) - f_n(y_2,t_1)| + |f_n(y_2,t_1) - f_n(y_2,t_2)| \\
&\le \frac{1}{\varrho(y_1,t_1)}|\phi_n(y_1,t_1) - \phi_n(y_2,t_1)| + |\phi_n(y_2,t_1)|\left|\frac{1}{\varrho(y_1,t_1)} - \frac{1}{\varrho(y_2,t_1)}\right| \\
&+ \frac{1}{\varrho(y_2,t_2)}|\phi_n(y_2,t_1) - \phi_n(y_2,t_2)| + |\phi_n(y_2,t_1)|\left|\frac{1}{\varrho(y_2,t_1)} - \frac{1}{\varrho(y_2,t_2)}\right| \\
&\le \frac{1}{\varrho(y_1,t_1)}|\nabla_y\phi_n(\xi,t_1)||y_1-y_2| + \frac{1}{\varrho(y_2,t_2)}|\partial_t\phi_n(y_2,s)||t_1-t_2| \\
&+ \frac{|\phi_n(y_2,t_1)|}{\varrho(y_2,t_1)}\left(\left|\frac{\varrho(y_2,t_1)-\varrho(y_1,t_1)}{\varrho(y_1,t_1)}\right| + \left|\frac{\varrho(y_2,t_2)-\varrho(y_2,t_1)}{\varrho(y_2,t_2)}\right|\right).
\end{aligned}$$

Now we notice an important fact: $h(p,\xi,\Psi^*)(y,t)$ is Hölder in space and time. This can be proved by using the Hölder continuity of Ψ^*, which comes from the norm of Ψ^* and the regularity of Z^*. We can use this on the parabolic estimate in Theorem 1.6 to obtain estimates on the uniform norms of $\nabla_y \phi_n$, $\nabla_y^2 \phi_n$ and $\partial_t \phi_n$. Using this and the Lipschitz continuity of λ_* we obtain

$$\begin{aligned}|f_n(y_1,t_1) - f_n(y_2,t_2)| \leq{}& C_1(T-\varepsilon_m)|y_1-y_2| + C_2(T-\varepsilon_m)|t_1-t_2| \\ &+ M\left(C_3(T-\varepsilon_m)|y_2-y_1| + C_4(T-\varepsilon_m)|t_2-t_1|^{\alpha}\right),\end{aligned}$$

for some $\alpha \in (0,1)$ and C_1, C_2, C_3, C_4 are positive constants only depending on $T-\varepsilon_m$. This means that for each $m \in \mathbb{N}$, the functions f_n are Hölder continuous on space and time on $[0, T-\varepsilon_m]$. Applying Arzela-Ascoli we obtain for each $m \in \mathbb{N}$ subsequences ϕ_n^m that converge to some ϕ^m in the norm $\|\cdot\|_{*,a,\nu}$ restricted to the time $[0, T-\varepsilon_m]$. We take the diagonal subsequence and name it $\phi_n := \phi_n^n$ that converges to $\phi := \lim_{m\to\infty} \phi^m$.

Let $\delta > 0$, we will prove that ϕ_n converges in the whole interval of time to ϕ. Since we have convergence on intervals $[0, T-\varepsilon_m]$ we choose $m_0 \in \mathbb{N}$ sufficiently big such that:

$$\sup_{(y,\tau)\in\Omega\times[0,T-\varepsilon_m]} \frac{(1+|y|)|\nabla_y(\phi_n(y,t)-\phi(y,t))| + |\phi_n(y,t)-\phi(y,t)|}{\lambda_*^{\nu} R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}} \leq \frac{\delta}{2},$$

and such that

$$2M\lambda(T-\varepsilon_{m_0})^{\nu'-\nu-\beta\left(\frac{a-a'}{2}\right)} \leq \frac{\delta}{2}.$$

Then

$$\begin{aligned}\|\phi_n-\phi\|_{*,a,\nu} ={}& \sup_{(y,\tau)\in\Omega\times[0,T-\varepsilon_m]} \frac{(1+|y|)|\nabla_y(\phi_n(y,t)-\phi(y,t))| + |\phi_n(y,t)-\phi(y,t)|}{\lambda_*^{\nu} R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}} \\ &+ \sup_{(y,\tau)\in\Omega\times[T-\varepsilon_m,T]} \frac{(1+|y|)|\nabla_y(\phi_n(y,t)-\phi(y,t))| + |\phi_n(y,t)-\phi(y,t)|}{\lambda_*^{\nu} R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}} \\ \leq{}& \frac{\delta}{2} + \sup_{(y,\tau)\in\Omega\times[T-\varepsilon_m,T]} \frac{(1+|y|)|\nabla_y(\phi_n(y,t)-\phi(y,t))| + |\phi_n(y,t)-\phi(y,t)|}{\lambda_*^{\nu} R^{\frac{5-a}{2}}(1+|y|)^{-1}\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}} \\ \leq{}& \frac{\delta}{2} + 2M\lambda(T-\varepsilon_{m_0})^{\nu'-\nu-\beta\left(\frac{a-a'}{2}\right)} \\ \leq{}& \delta.\end{aligned}$$

This is due to that $\nu' > \nu$ and $a' > a$. For more details on this part we refer the reader to the proof of the compactness of $\mathcal{F}_5$, the calculations and arguments are very similar, but since the functions are simpler we wrote it in more detail.

We have proved that $(\phi_n)_{n\in\mathbb{N}}$ is the subsequence that we looked for. Therefore $\mathcal{F}_1$ is compact.

Compactness of $\mathcal{F}_4$. Notice that this operator is basically a heat operator composed with function g. We use standard parabolic estimates to get the hypothesis to use Arzelà-Ascoli. With Arzelà-Ascoli and the compactness of the Hölder space inclusion $C^{2\gamma',\gamma',1}(\Omega) \hookrightarrow C^{2\gamma,\gamma,1}(\Omega)$ we obtain the result.

Compactness of $\mathcal{F}_5$. Let $(p_n)_{n\in\mathbb{N}} \subset G_1$ be the fifth coordinate of v_n, then by Proposition 2.7 we have that

$$\mathcal{P}[a_0^{(0)}[p_n, \xi, \Psi^*](t)] = p_{0,\kappa_n} + p_{1_n} + p_{2_n},$$

where for p_{1_n} we have

$$\|p_{1_n}\|_{*,3-\sigma'} \leq C|\log T|^{1-\sigma'} \log(\log T)^2 \tag{3.9}$$

and

$$|\ddot{p}_{1_n}(t)| \leq C\frac{|\log T|}{|\log(T-t)|^3(T-t)} \qquad \text{for } t \in [-T, T).$$

For p_{2_n} we have

$$\|\dot{p}_2\|_{\Theta,l} \leq C_0(T), \quad [\dot{p}_2]_{\gamma,m,l} \leq C_1(T),$$

We want to find a subsequence of $(\kappa_n, p_{1_n}, p_{2_n})_{n\in\mathbb{N}}$ that converges in G_1.

Since κ_n are constants that are bounded then by Bolzano-Weierstrass theorem there is a subsequence of them that converges, we will call it κ_n as well. Then we only need to find a subsequence of p_{1_n} and p_{2_n} that converge in norms $\|\cdot\|_{*,3-\sigma}$ and $\|\cdot\|_{\Theta,l}$, respectively.

First, we find the subsequence for p_{1_n}.Note that the sequence $|\log(T-t)|^{3-\sigma}|\dot{p}_{1_n}(t)|$ is uniformly bounded, because of $\|p_{1_n}\|_{*,3-\sigma} \leq \|p_{1_n}\|_{*,3-\sigma'}$. Also notice that there is some loss of compactness for $t \to T$ due to the logarithmic main order of the parameter p_1. We approach this issue by first taking a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of $\varepsilon_m > 0$ for all $m \in \mathbb{N}$ such that $\varepsilon_m \to 0$ when $m \to \infty$ and $\varepsilon < T$ for all $m \in \mathbb{N}$. We aim to find convergent subsequences on intervals $[-T, T-\varepsilon_m]$ and see that for some $m_0 \in \mathbb{N}$ the remaining part of the norm in $[T-\varepsilon_{m_0}, T]$ is small.

Let $m \in \mathbb{N}$ and $t_1, t_2 \in [-T, T-\varepsilon_m]$, then using mean value theorem on $|\log(T-\cdot)|^\sigma$ and $\dot{p}_{1_n}$ and the estimates given by Proposition 2.7, we obtain the following:

$$\begin{aligned}
||\log(T-t_1)|^{3-\sigma}\dot{p}_{1_n}(t_1) - |\log(T-t_2)|^{3-\sigma}\dot{p}_{1_n}(t_2)| &\leq ||\log(T-t_1)|^{3-\sigma} - |\log(T-t_2)|^{3-\sigma}| \\
&\quad \cdot |\dot{p}_{1_n}(t_1)| + |\log(T-t_2)|^{3-\sigma}|\dot{p}_{1_n}(t_1) - \dot{p}_{1_n}(t_2)| \\
&\leq C\left(\frac{1}{\varepsilon_m|\log(\varepsilon_m)|^{1+\sigma-\sigma'}} + \frac{1}{\varepsilon_m|\log(\varepsilon_m)|^\sigma}\right)|t_1 - t_2| \\
&\leq C(\varepsilon_m)|t_1 - t_2|
\end{aligned}$$

which means that $|\log(T-t)|^{3-\sigma}\dot{p}_{1_n}(t)$ Lipschitz continuous in $[-T, T-\varepsilon_m]$ and hence it is equicontinuous in that interval. Also notice that $|\log(T-t)|^{3-\sigma}\dot{p}_{1_n}(t)$ is uniformly bounded due to the fact that $\mathcal{F}$ lives in B_M. Then, by Arzelà-Ascoli theorem there exists a subsequence which we denote $p^m_{1_n}$ such that $p^m_{1_n} \to p^m_1$ in the norm $\|\cdot\|_{*,3-\sigma}$ restricted to $[-T, T-\varepsilon_m]$, and $p^m_1 \in G_1$. Since this is for m arbitrary, we can take the diagonal subsequence $p^n_1 := p^n_{1_n}$, that converges to $p_1 \in G$ in $[-T, T-\varepsilon_m]$, $\forall m \in N$.

So our candidate for convergent subsequence is p^n_1 and it expected limit on $[-T, T]$ is p_1. Let $\delta > 0$. There exists $m_0 \in M$ such that for all $m \geq m_0$

$$\sup_{[-T, T-\varepsilon_m]} |\log(T-t)|^{3-\sigma}|\dot{p}^n_1(t) - \dot{p}_1(t)| \leq \frac{\delta}{2}.$$

Then, choosing $m \geq m_0$ such that

$$2|\log \varepsilon_m|^{\sigma'-\sigma}C_0|\log T|^{1-\sigma'}\log(\log T)^2 < \frac{\delta}{2},$$

we can estimate the norm of the difference between our candidates.

$$
\begin{aligned}
\|p_1^n - p_1\|_{*,3-\sigma} &\leq \sup_{[-T,T-\varepsilon_m]} |\log(T-t)|^{3-\sigma}|\dot{p}_1^n(t) - \dot{p}_1(t)| + \sup_{[T-\varepsilon_m,T]} |\log(T-t)|^{3-\sigma}|\dot{p}_1^n(t) - \dot{p}_1(t)| \\
&\leq \frac{\delta}{2} + \sup_{[T-\varepsilon_m,T]} |\log(T-t)|^{3-\sigma'}|\dot{p}_1^n(t) - \dot{p}_1(t)||\log(T-t)|^{\sigma'-\sigma} \\
&\leq \frac{\delta}{2} + |\log\varepsilon_m|^{\sigma'-\sigma} \sup_{[T-\varepsilon_m,T]} |\log(T-t)|^{3-\sigma'}|\dot{p}_1^n(t) - \dot{p}_1(t)| \\
&\leq \frac{\delta}{2} + |\log\varepsilon_m|^{\sigma'-\sigma} \sup_{[T-\varepsilon_m,T]} \Big(|\log(T-t)|^{3-\sigma'}|\dot{p}_1^n(t)| + |\log(T-t)|^{3-\sigma'}|\dot{p}_1(t)|\Big) \\
&\leq \frac{\delta}{2} + 2|\log\varepsilon_m|^{\sigma'-\sigma} C_0 |\log T|^{1-\sigma'} \log(\log T)^2 \\
&\leq \delta
\end{aligned}
$$

Here we have used that p_1^n and p_1 satisfy estimate (3.9). Therefore p_1^n is the convergent subsequence that we were looking for.

We can use a simpler approach to get a convergent subsequence for p_{2_n}, since we already have Hölder estimates for p_{2_n} for $t \in [0,T]$.

Compactness of $\mathcal{F}_6$. Let $(\xi_n)_{n\in\mathbb{N}} \subset G_2$ be the sixth coordinate of v_n. We are looking for a subsequence of $\mathcal{A}[p,\xi_n]$ that converges in G_2. Notice that this sequence of functions is uniformly bounded, we will prove that it is equicontinous and use Arzelà-Ascoli theroem to obtain a subsequence that converges.

Let t_1, $t_2 \in [0,T]$, since $\mathcal{A}[p,\xi_n] \in C^1([0,T];\mathbb{R}^2)$ we can apply the Mean Value Theorem for each $n \in \mathbb{N}$ and obtain

$$
|\mathcal{A}[p,\xi_n](t_1) - \mathcal{A}[p,\xi_n](t_2)| \leq |\dot{\mathcal{A}}[p,\xi_n](\bar{t})||t_1 - t_2| \leq C(M)|t_1 - t_2|,
$$

where $C(M)$ is a constant (independent of n) that appears from using the estimates on Section 3.3 and that only depends on μ_2, $\lambda_*(0)$, ψ, Z_0^* and ψ is bounded in its norm by M and the rest are constants. Then $\mathcal{A}[p,\xi_n]$ is Lipschitz continuous for each $n \in \mathbb{N}$, this gives us that ξ_n is equicontinuous and therefore, due to Arzelà-Ascoli theorem, there exists a subsequence, that we will call by the same name, that converges uniformly to some $\bar{\xi} \in C([0,T];\mathbb{R}^2)$.

Now we focus on the derivatives to obtain the convergence in G_2. Let us analyze $\mathcal{A}_1$ and $\mathcal{A}_2$ separately. Let t_1, $t_2 \in [0,T]$, since $\mathcal{A}_1$ and $\mathcal{A}_2$ give us the solutions of differential equations (2.56) and (2.57) we have

$$
\begin{aligned}
|\dot{\mathcal{A}}_1[p,\xi_n](t_1) - \dot{\mathcal{A}}_1[p,\xi_n](t_2)| &\leq \left|\frac{1}{\xi_{n_1}(t_2)} - \frac{1}{\xi_{n_1}(t_1)}\right| + |b_1[p,\xi_n](t_2) - b_1[p,\xi_n](t_1)|, \\
&\leq \frac{1}{\mu_2^2}|\xi_{n_1}(t_1) - \xi_{n_1}(t_2)| + |b_1[p,\xi_n](t_2) - b_1[p,\xi_n](t_1)|, \\
&\leq C|t_1 - t_2| + |b_1[p,\xi_n](t_2) - b_1[p,\xi_n](t_1)|, \\
|\dot{\mathcal{A}}_2[p,\xi_n](t_1) - \dot{\mathcal{A}}_2[p,\xi_n](t_2)| &\leq |b_2[p,\xi_n](t_2) - b_2[p,\xi_n](t_1)|.
\end{aligned}
$$

Then it all reduces to find a Hölder estimate of the difference of b_j. Let us name the following integral as

$$I(t) = \lambda(t) \int_{B_{2R(t)}} Q_{-\omega(t)} \tilde{L}_U[\Psi^*](t) \cdot Z_{1j}(y) dy.$$

Let $j = 1, 2$, we estimate

$$\begin{aligned}
|b_j[p, \xi_n](t_1) - b_j[p, \xi_n](t_2)| \leq C \bigg| &(1 + (2R(t_1))^{-2})\lambda(t_1) \int_{B_{2R(t_1)}} Q_{-\omega(t_1)} \tilde{L}_U[\Psi^*](t_1) \cdot Z_{1j}(y) dy \\
&- (1 + (2R(t_2))^{-2})\lambda(t_2) \int_{B_{2R(t_2)}} Q_{-\omega(t_2)} \tilde{L}_U[\Psi^*](t_2) \cdot Z_{1j}(y) dy \bigg|, \\
&\leq C|(1 + (2R(t_1))^{-2}) - (1 + (2R(t_2))^{-2})| \, |I(t_1)| \\
&+ C|(1 + (2R(t_2))^{-2})| \, |I(t_1) - I(t_2)| \, .
\end{aligned}$$

We can compute for $0 < 2\beta < 1$ the following:

$$\begin{aligned}
|(1 + (2R(t_1))^{-2}) - (1 + (2R(t_2))^{-2})| &\leq \frac{1}{4} |\lambda(t_1)^{2\beta} - \lambda(t_2)^{2\beta}|, \\
&\leq \frac{1}{4} |\lambda(t_1) - \lambda(t_2)|^{2\beta}, \\
&\leq \frac{1}{4} |t_1 - t_2|^{2\beta}.
\end{aligned}$$

Note that as in the proof of Lemma 3.6 we obtain

$$\begin{aligned}
I(t_1) = \left| \lambda \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j}(y) dy \right| &\leq \lambda(t_1) \int_{B_{2R(t_1)}} |\tilde{L}_U[\Psi^*](t_1)| |Z_{1j}| dy, \\
&\leq C \|\nabla_x \Psi^*\|_{L^\infty} \int_0^{2R(t_1)} \frac{\rho}{(1 + \rho^2)^2} dy, \\
&\leq C \frac{1}{(1 + (2R(t_1))^{-2})} \|\nabla_x \Psi^*\|_{L^\infty}, \\
&\leq C \|\nabla_x \Psi^*\|_{L^\infty},
\end{aligned}$$

and

$$|(1 + (2R(t_2))^{-2})| \leq |1 + \frac{1}{4} \lambda(t_2)^{2\beta}| \leq 1 + \frac{1}{4} \lambda(0)^{2\beta} \leq C.$$

Then we are only missing the difference of the integrals $I(t_1)$ and $I(t_2)$, which we estimate as follows:

$$
\begin{aligned}
|I(t_1)-I(t_2)| \leq & \int_{\mathbb{R}^2} \Big| \mathbb{1}_{B_{2R(t_1)}} \lambda(t_1) Q_{-\omega(t_1)} \tilde{L}_U[\Psi^*](t_1) \cdot Z_{1j}(y) \\
& - \mathbb{1}_{B_{2R(t_2)}} \lambda(t_2) Q_{-\omega(t_2)} \tilde{L}_U[\Psi^*](t_2) \cdot Z_{1j}(y) \Big| \, dy, \\
\leq & \int_{B_{2R(t_1)}} |\lambda(t_1) Q_{-\omega(t_1)} \tilde{L}_U[\Psi^*](t_1) - \lambda(t_2) Q_{-\omega(t_2)} \tilde{L}_U[\Psi^*](t_2)||Z_{1j}(y)| dy \\
& + |\mathbb{1}_{B_{2R(t_1)}} - \mathbb{1}_{B_{2R(t_2)}}||\lambda(t_2) Q_{-\omega(t_2)} \tilde{L}_U[\Psi^*](t_2)||Z_{1j}(y)| dy, \\
\leq & C \frac{1}{(1+(2R(t_1))^{-2})} |\nabla \Psi^*(t_1) - \nabla \Psi^*(t_2)| \\
& + \left| \frac{1}{(1+(2R(t_2))^2)} - \frac{1}{(1+(2R(t_2))^2)} \right| \|\nabla_x \Psi^*\|_{L^\infty}, \\
\leq & C |\nabla \Psi^*(t_1) - \nabla \Psi^*(t_2)| \\
& + \frac{|R(t_1)^{-2} - R(t_2)^{-2}|}{(1+(2R(t_1))^{-2})(1+(2R(t_2))^{-2})} \|\nabla_x \Psi^*\|_{L^\infty}, \\
\leq & C |\nabla \Psi^*(t_1) - \nabla \Psi^*(t_2)| + C|t_1 - t_2|^{2\beta} \|\nabla_x \Psi^*\|_{L^\infty},
\end{aligned}
$$

Then

$$
\begin{aligned}
|b_j[p,\xi_n](t_1) - b_j[p,\xi_n](t_2)| \leq & C \left(|t_1 - t_2|^{2\beta} \|\nabla_x \Psi^*\|_{L^\infty} + |\nabla \Psi^*(t_1) - \nabla \Psi^*(t_2)| \right), \\
\leq & C \left(|t_1 - t_2|^{2\beta} [\|\nabla_x \psi\|_{L^\infty} + \|\nabla Z_0^*\|_{L^\infty}] \right. \\
& \left. + |\nabla \Psi^*(t_1) - \nabla \Psi^*(t_2)| + |\nabla Z^*(t_1) - \nabla Z^*(t_2)| \right), \\
\leq & C \left(|t_1 - t_2|^{2\beta} [\lambda_*(0)^{\Theta} \|\psi\|_{\#,\Theta,\gamma} + \|\nabla Z_0^*\|_{L^\infty}] \right. \\
& \left. + \lambda_*(t_1)^{\Theta+2\gamma-2\gamma\beta} |t_1 - t_2|^{\gamma} \|\psi\|_{\#,\Theta,\gamma} + |t_1 - t_2| \right), \\
\leq & C(M) \left(|t_1 - t_2|^{2\beta} + \lambda_*(t_1)^{\Theta+2\gamma-2\gamma\beta} |t_1 - t_2|^{\gamma} + |t_1 - t_2| \right).
\end{aligned}
$$

Here we have used that Z^* is smooth and therefore Lipschitz. Moreover, since $\Theta+2\gamma-2\beta\gamma > 0$ the term with λ_* is bounded by its value at 0 and then we obtain

$$
|\dot{\mathcal{A}}_j[p,\xi_n](t_1) - \dot{\mathcal{A}}_j[p,\xi_n](t_2)| \leq C|t_1 - t_2|^{\min\{2\beta,\gamma\}}.
$$

Then $\dot{\mathcal{A}}[p,\xi_n]$ is equicontinuous and there exists a subsequence that converges uniformly. Then the subsequence that we have obtaind converges in G_2 and $\mathcal{F}_6$ is compact. $\square$

Conclusion

We finish this work with some brief remarks.

First, since the result proved by Dávila, Del Pino and Wei in [7] is for a finite set of blow-up points $q_1, q_2, \ldots, q_k$ in the domain, we could extend Theorem 0.1 for a finite number of circumferences, which could be an interesting result as an example of finite time blow-up for a higher dimensional parabolic nonlinear equation on a set of curves.

Second, we have seen in work the method developed by Dávila, Del Pino and Wei in [7] of separating the effect of the equation near and far away from the blow-up point by means of a coupled system called the inner-outer gluing system. During chapters 2 and 3 we can see that the method has some space to introduce errors of order $\frac{r}{\lambda^2+r^2}$, which is exactly the error associated with first partial derivatives of U in space. This means that the same method could be applied for other volumes in $\mathbb{R}^3$ that have different domain symmetry that reduce the 3-dimensional harmonic map flow to a 2-dimensional problem, mainly,

$$u_t = \Delta u + |\nabla u|^2 u + f(u_{x_1}, u_{x_2}, x), \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{3.10}$$

where f corresponds to the extra terms associated with the symmetry of the domain. We could use this approach to do the construction on a ball $B^3 \subset \mathbb{R}^3$, like in [1] and [13], or on other symmetric domain.

Third, we do not give any results on stability of the solution when we change the initial data. From the results in [7] it seems likely that one can construct in a similar way a stable solution. Following the example of [7], proving stability requires redoing most of our estimates to obtain Lipschitz ones. This could be a future continuation of the present work.

Fourth, the most important unanswered question here is the totally non symmetric case in $\mathbb{R}^3$. The main question would be: Which conditions on a domain $\Lambda \subset \mathbb{R}^3$ and a curve Γ do we need to get blow-up exactly on Γ? To be more specific, for $T > 0$, $\Lambda \subset \mathbb{R}^3$, $\Gamma \subset \Lambda$ a curve, and u a solution of the problem

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \text{ in } \Lambda \times (0,T), \\ u &= u_{\partial\Lambda} \text{ in } \partial\Lambda \times (0,T), \\ u(\cdot, 0) &= u_0, \text{ in } \Lambda. \end{aligned}$$

We look for conditions on Λ and Γ so that u blows up at time T exactly at the curve Γ. Here we have seen that for $\Lambda = V$ and $\Gamma = c(q)$ we have blow-up, but there is nothing said for other settings. In addition, one could guess that this phenomenon is unstable with respect to Γ, that is, if we change a bit the curve Γ the blow-up may not occur. This is a difficult question to answer completely and could be good starting point for future work.

Finally, we expect the method of Dávila, Del Pino and Wei [7] can be applied to more settings to extend our knowledge on singularity formation in parabolic equations. We hope our construction can be a way of understanding better this method and that it becomes a relevant example of finite time blow-up on curves for nonlinear parabolic equations.

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