SUPERSYMMETRIC NON-RELATIVISTIC QUANTUM MECHANICS

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We generalize Gozzi's analysis for one-dimensional quantum mechanics to three dimensions. The hidden supersymmetry of quantum mechanics requires a particular spin–orbit coupling to manifest itself in three dimensions.

Recently, Gozzi [1] has shown that a one-dimensional quantum mechanical system is supersymmetric, provided it has a normalizable ground state. The proof uses the ground state wavefunction representation for the hamiltonian to show that the vacuum–vacuum correlation functions are the same as those obtained from the supersymmetric extension of the system.

In this note we extend the result of ref. [1] to the three-dimensional central force quantum mechanical system. Consider the hamiltonian

$$H = -\tfrac{1}{2}\nabla^2 + U(r) . \qquad (1)$$

We assume the existence of a normalizable ground state wave function ψ_0 with energy E_0:

$$H\psi_0 = E_0\psi_0 . \qquad (2)$$

The lowest energy state ψ_0 can not have nodes anywhere for finite $|\boldsymbol{r}|$ [2], therefore it can be written as

$$\psi_0 = \exp[-V(r)] , \qquad (3)$$

where we have ignored a normalization constant that we assume positive and finite. Substituting (3) in (2) one obtains the ground state wave function representation for U

$$U(r) = \tfrac{1}{2}([\nabla V]^2 - \nabla^2 V) + E_0 , \qquad (4)$$

and for the hamiltonian,

$$H = -\tfrac{1}{2}\nabla^2 + \tfrac{1}{2}([\nabla V]^2 - \nabla^2 V) + E_0 . \qquad (5)$$

One can renormalize the hamiltonian, $H \to \widetilde{H} = H - E_0$, and the resulting $\widetilde{H}$ gives rise to the same quantum mechanics as the original one. In the spirit of ref. [1], we write

$$\widetilde{H} \equiv H - E_0 = \tfrac{1}{2}\, \boldsymbol{Q}^\dagger \cdot \boldsymbol{Q} , \qquad (6)$$

where

$$\boldsymbol{Q} = \boldsymbol{\nabla} + \boldsymbol{\nabla} V , \quad \boldsymbol{Q}^\dagger = -\boldsymbol{\nabla} + \boldsymbol{\nabla} V . \qquad (7)$$

The supersymmetric extension of $\widetilde{H}$ is obtained by the substitution [3]

$$\widetilde{H} \to H_{\mathrm{ss}} = \tfrac{1}{2}\{Q, Q^\dagger\} , \qquad (8)$$

where the generators of supersymmetry Q are obtained by linear combinations of the Q_i, $Q \equiv Q_i J_i = \boldsymbol{Q}\cdot\boldsymbol{J}$. The coefficients of this linear combination must be nilpotent in order that $Q^2 = Q^{\dagger 2} = 0$, and since the simplest supersymmetric extension of a point particle is a spinning particle [4], it is natural to identify these coefficients with the spin degrees of freedom of the quantum particle [‡1]. Thus, we propose the ansatz [‡2] $\boldsymbol{J} = \boldsymbol{\sigma} \otimes \tau_-$,

$$Q = Q_i \sigma^i \otimes \tau_- , \quad Q^\dagger = Q_i^\dagger \sigma^i \otimes \tau_+ , \qquad (9)$$

where $i = 1,2,3$, σ^i are the Pauli spin matrices and

‡1 For an interesting alternative interpretation, see ref. [5].

‡2 After the completion of this work we have been informed about the article by Ui [6] where this ansatz is discussed in the context of Witten's supersymmetric quantum mechanics.

$$\tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \tau_-^\dagger \,. \tag{10}$$

Substituting (9) into (8) we get

$$H_{ss} = \tfrac{1}{2}(\tau_+\tau_-) \otimes [\delta^{ij} \mathbf{1}_{2\times 2} + i\epsilon^{ijk}\sigma^k] Q_i^\dagger Q_j$$
$$+ \tfrac{1}{2}(\tau_-\tau_+) \otimes [\delta^{ij} \mathbf{1}_{2\times 2} + i\epsilon^{ijk}\sigma^k] Q_i Q_j^\dagger \,. \tag{11}$$

The first and the third term in (11) give the naive generalization to three dimensions of the form found by Gozzi,

$$\frac{1}{2}\begin{pmatrix} Q^\dagger \cdot Q & \\ & Q\cdot Q^\dagger \end{pmatrix}. \tag{12}$$

We see, however, that there appear two new terms in H_{ss}, that have no analogue in the one-dimensional case. The expression

$$\tfrac{1}{2} i\epsilon^{ijk}\sigma^k Q_i^\dagger Q_j \tag{13}$$

is

$$i(\nabla V \times \nabla)\cdot \boldsymbol{\sigma}, \tag{14}$$

and thus the complete supersymmetric hamiltonian becomes

$$H_{ss} = \tfrac{1}{2}\{Q^\dagger, Q\}$$
$$= \frac{1}{2}\begin{pmatrix} Q^\dagger \cdot Q & \\ & Q\cdot Q^\dagger \end{pmatrix}$$
$$-2\begin{pmatrix} \boldsymbol{s}\cdot(\nabla V \times \boldsymbol{P}) & \\ & -\boldsymbol{s}\cdot(\nabla V\times \boldsymbol{P}) \end{pmatrix}, \tag{15}$$

where we have identified the spin operator $\boldsymbol{s}$ as

$$\boldsymbol{s} = \tfrac{1}{2}\boldsymbol{\sigma} \,. \tag{16}$$

In this case of central forces, $U = U(r)$ – and therefore $V = V(r)$ – the last term in (15) can be identified with the spin–orbit coupling,

$$H_{ss} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\tfrac{1}{2} Q^\dagger \cdot Q - \frac{2}{r}\frac{dV}{dr} \boldsymbol{L}\cdot\boldsymbol{s}\right)$$
$$+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left(\tfrac{1}{2} Q\cdot Q^\dagger + \frac{2}{r}\frac{dV}{dr}\boldsymbol{L}\cdot\boldsymbol{s}\right) . \tag{17}$$

The state

$$\psi_0 = \begin{pmatrix} e^{-V} \\ 0 \end{pmatrix} \tag{18}$$

is by construction annihilated by Q and $Q^\dagger$,

$$Q\psi_0 = \boldsymbol{\sigma}\cdot(\nabla + \nabla V)\begin{pmatrix} 0 \\ e^{-V} \end{pmatrix} \equiv 0 \,, \tag{19}$$

$$Q^\dagger \psi_0 = \boldsymbol{\sigma}\cdot(-\nabla + \nabla V)\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

Therefore ψ_0 is also annihilated by $H_{ss} = \tfrac{1}{2}\{Q, Q^\dagger\}$, and H_{ss} being a positive semi-definite operator by construction, ψ_0 is necessarily the ground state of the system. Since the ground state energy is zero, the supersymmetry can not be spontaneously broken.

According to Gozzi [1], the generating function for the hamiltonian (1) in one dimension is the same as that of its supersymmetric extension H_{ss}, and therefore the resulting quantum theories are indistinguishable. In our case, the result would be that the generating function for the supersymmetric system (17)

$$Z_{ss}[J] = \int \mathcal{D}J\mathcal{D}\boldsymbol{x} \exp\left(\int_{-\infty}^{\infty} [H_{ss}(\boldsymbol{x}, \nabla_x, J) + \boldsymbol{J}\cdot\boldsymbol{x}]\, dt\right) \tag{21}$$

is equal to the generating function of the system described by the hamiltonian

$$H_{-\sigma} = \tfrac{1}{2} Q^\dagger \cdot Q - (2/r)(dV/dr)\boldsymbol{L}\cdot\boldsymbol{s} \,. \tag{22}$$

Thus, the supersymmetric system (17) is indistinguishable from the one given by (22), but differs from the original H – eq. (1) – due to the presence of the spin–orbit coupling term

$$W_{L-S} = -(2/r)(dV/dr)\boldsymbol{L}\cdot\boldsymbol{s} \,. \tag{23}$$

From this point of view, Gozzi's result can be seen to be a consequence of the fact that in one dimension spin and angular momentum do not exist.

The standard spin–orbit coupling in quantum mechanics due to an electrostatic potential $U(r)$ takes the form [7]

$$H_{\mathrm{L-S}} = (\hbar^2/2m^2c^2)(1/r)(\mathrm{d}U/\mathrm{d}r)\boldsymbol{L}\cdot\boldsymbol{s} \ . \quad (24)$$

Introducing back the mass $m, \hbar$, etc. which were equal to one above, the spin–orbit coupling required by our supersymmetry is

$$W_{\mathrm{L-S}} = (-\hbar^2/m)(2/r)(\mathrm{d}V/\mathrm{d}r)\boldsymbol{L}\cdot\boldsymbol{s} \ . \quad (25)$$

The origin of $H_{\mathrm{L-S}}$ is the interaction of a magnetic moment $\boldsymbol{\mu} = (e/mc)\boldsymbol{s}$ with the magnetic field $\boldsymbol{B} = (-\boldsymbol{v}/c) \times (1/e)\nabla U$ felt in the rest frame of the charged spinning particle. In contrast with this, $W_{\mathrm{L-S}}$ nowhere required U to have electromagnetic origin and no reference is made to the rest frame of the spinning particle – which, by the way, is not necessarily charged. Substituting for U and V the corresponding expressions for the hydrogen atom,

$$U = -e^2/r \ , \quad V = (me/\hbar^2)r \ , \quad (26)$$

we find

$$\langle H_{\mathrm{L-S}} \rangle = \alpha^2 (k_l/4n) \langle W_{\mathrm{L-S}} \rangle \ , \quad (27)$$

where $\alpha = e^2/hc$ is the fine structure constant, $k_l = [l(l+1)(l+\frac{1}{2})]^{-1}$ and n is the principal quantum number, $n = 1,2,3, \ldots$. This shows that our particular supersymmetry ‡3 can not be accommodated by the standard spin–orbit coupling of the hydrogen atom. The disagreement between this supersymmetry and the hydrogen atom is, in this case for the lowest lying excited states, some four orders of magnitude larger than the observed fine structure splitting. This mismatch between our supersymmetric spectrum and that observed in the hydrogen atom does not mean that there is no supersymmetry left in the latter. In fact, there is a mixture of two supersymmetries here: one corresponds to that of the radial equation (see e.g. ref. [8]), and the other being the supersymmetry spin algebra $\mathrm{OSp}(\frac{1}{2})$ discussed by Balantekin [9]. That these are two separate and distinct supersymmetries has been shown by Kostelecky et al. [10].

For the spherical harmonic oscillator,

$$U = (m\omega^2/2)r^2 \ , \quad V = (m\omega/2\hbar)r^2 \ , \quad (28)$$

so that in this case

$$\langle H_{\mathrm{L-S}} \rangle = (\tfrac{1}{2}\hbar\omega/2mc^2)\langle W_{\mathrm{L-S}} \rangle \ . \quad (29)$$

It is curious to note that a charged spinning particle in a spherical harmonic potential would be supersymmetric if the ground state energy is sufficient to create a pair.

In ref. [1] Gozzi has proved the following theorem: Any quantum mechanical system in one dimension is supersymmetric provided it possesses a normalizable ground state. Here we find that the simplest physical extension to three dimensions of Gozzi's construction does not yield a similar theorem when there is spin. In fact, not all three-dimensional quantum mechanical systems possess a hidden supersymmetry of the form proposed in ref. [1]. Only those having a particular spin–orbit coupling do. Supersymmetry restricts the form of the spin–orbit coupling term much in the same way as it restricts the possible coupling between different fields in supersymmetric field theories. This makes us conjecture that the theorem might in fact not be valid in more than one dimension. There exists nevertheless a supersymmetry of a different kind as discussed in refs. [9,10].

‡3 One could for example consider the supersymmetric extension of the radial equation only, which by the argument of ref. [1] must exist and reproduce the spectrum of the hydrogen atom. This extension however is not truly three-dimensional and can not account for the spin–orbit coupling, c.f. ref. [8].

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