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Alumno: Daniel Jaar Michea

Profesor Guía: Juan Pablo Torres-Martínez

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SUPER-REPLICATION IN ECONOMIES WITH IMPERFECT FINANCIAL MARKETS

DANIEL JAAR

Abstract. This article addresses equilibrium existence in an infinite horizon economy with incomplete markets and endogenous credit segmentation. We do not restrict consumption allocation to be bounded and we do not impose uniform impatience on preferences. Our equilibrium results consider a general framework of positive net supply assets, and introduce short-lived securities in zero net supply through a super-replication property proposed in the literature of two-period economies with financial segmentation.

Keywords: Infinite horizon, incomplete markets, financial segmentation, competitive equilibrium.

1. Introduction

Several difficulties arise when analyzing equilibrium existence in infinite horizon economies with incomplete markets. First, the open-ended nature of the consumers' problem may lead to agents postponing indefinitely the payment of their financial obligations, entering into Ponzi schemes. Thus, arbitrary debt constraints or transversality conditions have been required to limit the growth rate of agents' indebtedness (Kehoe, 1989). On the other hand, available equilibrium results require assumptions on the structure of financial markets, with the aim of ensuring that finite asset prices are compatible with non-arbitrage conditions. The latter does not hold whenever agents may attain arbitrarily large utility levels through a particular security, as exemplified by Hernández and Santos (1996, Example 3.9 p. 118). For this reason, the literature has systematically restricted the nature of the financial securities available for trade, by limiting assets' lifespan and/or deliveries (Kehoe and Levine, 1993; Magill and Quinzii, 1994, 1996), assuming that financial markets are collateralized (Araujo, Páscoa, and Torres-Martínez, 2002, 2011; Kubler and Schmedders, 2003; Iraola, Sepúlveda, and Torres-Martínez, 2017), or imposing positive net supply assumptions (Moreno-García and Torres-Martínez, 2012).

This paper attempts to study equilibrium existence in economies with a more general financial structure than those previously considered in the literature of non-collateralized financial markets. In particular, it incorporates endogenous financial segmentation to infinite horizon economies (cf., Seghir and Torres-Martínez, 2011, Cea-Echenique and Torres-Martínez 2016; Faias and Torres-Martínez, 2017).[1] Our equilibrium result also incorporates the presence of short-lived assets in zero net supply. To do so, we adapt a *super-replication* property from the literature of financial

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[1]See Iraola, Sepúlveda and Torres-Martínez (2017) for the case of collateralized asset markets.

segmentation in two-period economies (Cea-Echenique and Torres-Martínez, 2016). That is, when the deliveries of assets in zero net supply may be fully hedged by those of portfolios of commodities and assets in positive net supply, the former have finite prices in equilibrium.

Magill and Quinzii (1994) and Hernández and Santos (1996) proved that equilibrium exists for economies with short-lived numeraire securities by requiring agents' preferences to comply with a uniform impatience property, and restricting allocations to the space of bounded sequences. Hernández and Santos (1996) also obtained a competitive equilibrium in an economy where only one real asset was available for trade. When assets are long-lived, endogenous bounds on short sales may cease to exist, if debt constraints target portfolio value instead of the amount of borrowing.[2] In this context, Magill and Quinzii (1996) proved that equilibrium existence for dense subsets of economies with long-lived real securities, by requiring uniform impatience and restricting consumption bundles and assets' deliveries to the space of bounded sequences.

Araujo, Páscoa, and Torres-Martínez (2011) showed that when financial markets are collateralized and the seizure of physical guarantees is the only enforcement in case of default, no additional assumptions are required on the financial structure to ensure equilibrium existence. Moreover, their model considers both finite and infinitely lived assets, and they neither need uniform impatience properties nor require allocations/deliveries to be bounded. Collateralized markets with strategic default are special in the sense that they provide natural solutions to several of the underlying challenges of modeling financial trade economies with incomplete markets. First, short-sales are limited by the availability of collateral goods. Thus, collateral rules out Ponzi schemes at the same time it provides endogenous bounds on portfolios. Moreover, and due to strategic default, security payments are endogenously bounded by the value of collateral bundles, which limits the utility attainable through financial promises. Hence, finite prices are always compatible with non-arbitrage conditions in equilibrium. Iraola, Sepúlveda and Torres-Martínez (2017) presented equilibrium results using a more general asset structure of collateralized assets. Indeed, and building from Iraola and Torres-Martínez (2014), they include financial segmentation and model credit contracts as functions determining collateral requirements, coupon payments and prepayment rules, which provide a context compatible with heterogeneous payment strategies from agents borrowing in a particular contract.

Nevertheless, collateralized markets do not consider a broad range of promises that are widely traded in financial markets, such as shares, insurance, or derivatives. Importantly, the different assumptions we impose on financial structures are determinant on the behavior of asset prices in equilibrium. In particular, available theoretical results assert that positive net supply securities cannot give rise to rational price bubbles under uniform impatience and whenever endowments are uniformly bounded away from zero (Magill and Quinzii, 1996). The latter is also true for collateralized securities (Araujo, Pascoa and Torres-Martínez, 2011), but does not holds for non-collateralized

[2]When assets are real, the rank of return matrices becomes price dependent, which threatens the existence of endogenous bounds of short sales whenever agents may increase their access to credit by investing more in financial securities.

assets in zero net supply (Magill and Quinzii, 1996). Thus, attaining a greater generality in financial structures compatible with equilibrium existence is relevant for understanding the behavior of financial markets.

Crucially, this is also true when considering the perspective of financial markets begin segmented. Unequal access to financial markets may emerge as a consequence of informational frictions, or due to regulatory considerations (Cea-Echenique and Torres-Martínez, 2016). Nonetheless, a broad range of restrictions are observed in financial markets, such as income-based access to funding, differential investment opportunities, and collateral requirements. Moreover, financial segmentation may be relevant to understand the prevalence of a wide range of phenomena in financial markets, such as negative equity loans (Iraola and Torres-Martínez, 2014), asset pricing puzzles (Guvenen, 2009; Gromb and Vayanos, 2017) and may even play an important role in determining the impact of macroprudential policies (Vayanos and Vila, 2009; Chen, Cúrdia, and Ferrero, 2012; He and Krishnamurthy, 2013).

This article proves equilibrium existence in an economy where agents are subject to price-dependent credit constraints limiting attainable allocations. Importantly, we do not require any sort of *financial survival* assumptions.[3] Altough investment segmentation is not allowed, we show that equilibrium existence is compatible with broad forms of credit segmentation. Following Moreno-García and Torres-Martínez (2012), we work with a general real asset structure and fairly weak assumptions on preferences; we neither require uniform impatience, nor allocations are restricted to bounded spaces. Precisely, competitive equilibrium is shown to exist in economies in which agents made trade infinitely lived assets in positive net supply and short-lived zero net supply securities, and where borrowing constraints prevent agents from engaging into Ponzi schemes.

The rest of the article is organized as follows. Sections 2 and 3 introduce our model, describe our standard assumptions over agents' preferences and the structure of the financial segmentation. Section 4 presents the super-replication property and our equilibrium result. Section 5 concludes. All proofs are relegated to the Appendix.

2. Model

Uncertainty. Let $\mathcal{E}$ represent a discrete time, infinite horizon economy. There is a set S of states of nature characterizing uncertainty, which is homogeneous among agents and represented by a finite partition $\mathcal{F}_t$ of S at each period t. There is no information available at $t = 0$, i.e., $\mathcal{F}_0 = S$. Additionally, $\mathcal{F}_t$ is at least as fine as $\mathcal{F}_{t-1}$ at every period $t \geq 0$. Thus, there is no loss of information throughout time.

A *node* ξ is characterized by a pair (t, σ), where $t \in \mathbb{N}$ and $\sigma \in \mathcal{F}_t$. Accordingly, $t(\xi)$ and $\sigma(\xi)$ denote the date and the information set associated to ξ. Let ξ^-, and ξ^+ be, respectively, node ξ's unique predecessor and the set of all immediate successors. We also say $\mu \geq \xi$ whenever $t(\mu) \geq t(\xi)$ and $\sigma(\mu) \subseteq \sigma(\xi)$, so $\xi^+ = \{\mu \geq \xi | t(\mu) = t(\xi) + 1\}$. Analogously, $\mu > \xi$ indicates that $\mu \geq \xi$ and $\mu \neq \xi$.

[3]For example, requiring there exists a $\varepsilon > 0$ such that all agents may obtain ε through borrowing.

There is a unique initial node ξ_0, marking the beginning of the *event-tree* D, formed by the set of nodes in our economy. The *subtree* starting at node ξ is denoted by $D(\xi)$ and corresponds to the set $\{\mu \in D : \mu \geq \xi\}$. Moreover, define $D^t(\xi) = \{\mu \in D(\xi) | t(\mu) \leq t(\xi) + t\}$ as the branch of $D(\xi)$ spanning until date t. Similarly, $D_t(\xi) = \{\mu \in D(\xi) : t(\mu) = t + t(\xi)\}$ is the set of nodes in $D(\xi)$ whose dates coincide with $t + t(\xi)$.

Markets. At every $\xi \in D$ there is a finite and ordered set $\mathcal{L}$ of perfectly divisible commodities. Commodities may experience transformations through nodes, which are modeled as $\mathcal{L} \times \mathcal{L}$ matrices with non-negative entries. In particular, a commodity bundle $x(\xi) \in \mathbb{R}^{\mathcal{L}}_+$ is transformed into bundle $Y_\mu x(\xi) \in \mathbb{R}^{\mathcal{L}}_+$ at any successor $\mu \in \xi^+$. We denote as $p(\xi) = (p_l(\xi))_{l \in \mathcal{L}}$ the vector of commodity spot prices at ξ, and $p = (p(\xi))_{\xi \in D}$ as the commodity price process along D.

There is an ordered set $J := J_+ \cup J_0$ of financial securities. Every asset $j \in J$ is characterized by an issuing node $\xi_j \in D$, and a payment stream that depends on whether j belongs to J_+ or J_0. Precisely, if $j \in J_+$ its payment stream consists of the spot-market value of a non-trivial process of commodity bundles $(A_j(\mu))_{\mu > \xi_j} \in \mathbb{R}^{\mathcal{L} \times D(\xi_j) \setminus \xi_j}$. Alternatively, if $j \in J_0$ its payment stream is given by $\{R_j(\mu, .)| \ \mu > \xi_j\} \in \mathbb{R}_+^{D(\xi_j) \setminus \xi_j}$, where $R_j(\xi, .) : \mathbb{R}^{\mathcal{L}}_+ \to \mathbb{R}_+$ is a continuous function of commodity spot prices at ξ. Additionally, for each $j \in J_0$ there exists node μ such that $R_j(\mu, p(\mu)) > 0$ for any $p(\mu) \gg 0$.

Assets may be finite or infinitely lived. Precisely, asset $j \in J_+$ is finitely lived if there exists $T_j \in \mathbb{N}$ such that the set $\{\xi \in D | (t(\xi) > T_j) \wedge (A_j(\xi) > 0)\}$ is empty. Respectively, security $k \in J_0$ is finitely lived if the set $\{\xi \in D | (t(\xi) > T_k) \wedge (R_k(\xi, p(\xi)) > 0)\}$ is empty for some $T_k \in \mathbb{N}$ and for any price process $p \in \mathbb{R}_+^{D \times \mathcal{L}}$. A security $j \in J$ is infinitely lived if one of the above does not hold for any $T \in \mathbb{N}$, and it is said to be short-lived whenever $T_j = \xi_j + 1$

There is a finite set $J(\xi) := J_+(\xi) \cup J_0(\xi)$ of assets available for trade at $\xi \in D$, where $J_+(\xi) := \{j \in J_+ | \exists \mu > \xi : A_j(\mu) \neq 0\}$ and $J_0(\xi) := \{k \in J_0 | \exists (\mu, p) \in D(\xi) \times \mathbb{R}_+^{\mathcal{L} \times D} : R_k(\mu, p) > 0\}$.[4] Let $q(\xi) = (q_j(\xi))_{j \in J(\xi)}$ and $q = (q(\xi))_{\xi \in D}$ be node ξ's asset prices and the asset price process respectively. Let $D(J) = \{(\xi, j) \in D \times J : j \in J(\xi)\}$, and let the space of commodity and asset prices be $\mathbb{P} := \mathbb{R}_+^{D \times \mathcal{L}} \times \mathbb{R}_+^{D(J)}$. Define $D(J_+)$ and $D(J_0)$ analogously.

Agents. There is a finite set H of agents participating in the economy. Each agent h is characterized by an utility function $U^h : \mathbb{R}_+^{D \times \mathcal{L}} \to \mathbb{R}_+ \cup \{+\infty\}$, and endowments consisting in both commodities and real assets $(w^h(\xi), e^h(\xi))_{\xi \in D} \in \mathbb{R}_+^{D \times \mathcal{L}} \times \mathbb{R}_+^{D(J_+)}$, which may be understood as generated by the unmodeled productive processes of our abstract economy $\mathcal{E}$. Let $\bar{e}^h(\xi)$ stand for agent h's cumulative financial endowments up to node ξ; accordingly, at ξ agent h receives aggregate endowments $W^h(\xi) = w^h(\xi) + Y_\xi W^h(\xi^-) + \sum_{j \in J_+(\xi^-)} A_j(\xi) \bar{e}^h_j(\xi^-)$.[5] Aggregate wealth at node ξ is therefore $W(\xi) = \sum_{h \in H} W^h(\xi)$. We assume that assets in $J_+(\xi)$ are in *positive net supply*. That is, $j \in J_+(\xi)$ implies that $\sum_{h \in H} \bar{e}^h_j(\xi) > 0$. In turn, securities $k \in J_0(\xi)$ are always in *zero net supply*: $\sum_{h \in H} \bar{e}^h_k(\xi) = 0$.

[4]Note that we are ruling out the existence of fiat money.

[5]$A(\xi_0, j) = 0$ for every $j \in J_+(\xi_0)$.

Each $h \in H$ must choose an allocation $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi)) \in \mathbb{E}_\xi := \mathbb{R}^{\mathcal{L}} \times \mathbb{R}_+^{J(\xi)} \times \mathbb{R}_+^{J(\xi)}$ for every $\xi \in D$, composed by commodity bundles and long and short positions in financial securities. Accordingly, $(x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in D} \in \mathbb{E} := \prod_{\xi \in D} \mathbb{E}_\xi$. Moreover, each agent h is subject to trading constraints $\Phi^h : \mathbb{P} \twoheadrightarrow \mathbb{E}$, restricting her admissible allocations throughout event-tree D. Hence, given prices $(p, q) \in \mathbb{P}$, agent h's *choice set correspondence* $C^h(p, q)$ is defined as all allocations complying with both budget feasibility and trading constraints. In particular, budget feasibility at node ξ implies that:

$$
\begin{aligned}
&p(\xi)\left(x^h(\xi) - w^h(\xi) - Y_\xi x^h(\xi^-)\right) + q(\xi)\left(\theta^h(\xi) - \varphi^h(\xi) - e^h(\xi)\right) \\
&\le \sum_{j \in J_+(\xi^-)} \left(p(\xi)A_j(\xi) + q_j(\xi)\right)\left(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)\right) + \sum_{k \in J_0(\xi^-)} \left(R_k(\xi, p(\xi)) + q_k(\xi)\right)\left(\theta_k^h(\xi^-) - \varphi_k^h(\xi^-)\right).
\end{aligned}
$$

In turn, compatibility with trading constraints is equivalent to:

$$(x^h, \theta^h, \varphi^h) \in \Phi^h(p, q),$$

where $(x^h, \theta^h, \varphi^h) = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in D}$ and $(\theta^h(\xi_0^-), \varphi^h(\xi_0^-)) = (0, 0)$. We also use $y^h(\xi) = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))$, and $y^h = (y(\xi))_{\xi \in D}$, to shorten notation.

Definition 1. A *competitive equilibrium* for the economy $\mathcal{E}$ is composed by a pair of price processes $(p, q) \in \mathbb{P}$ and a set of allocations $(x^h, \theta^h, \varphi^h)_{h \in H} \in \mathbb{E}^H$ such that

(1) For every $h \in H$, $(x^h, \theta^h, \varphi^h) \in \operatorname{argmax}_{(x,\theta,\varphi) \in C^h(p,q)} U^h(x)$.
(2) Commodity and financial markets clear, i.e.,

$$\sum_{h \in H} x^h(\xi) = W(\xi), \quad \sum_{h \in H} \theta^h(\xi) = \sum_{h \in H} (\varphi^h(\xi) + \bar{e}^h(\xi)), \ \forall \xi \in D.$$

3. Standard Economies

The following assumptions characterize the fundamentals of economy $\mathcal{E}$. Any economy $\mathcal{E}$ satisfying them is called *standard*.

Assumption A1. *For each agent $h \in H$, $U^h(x) = \sum_{\xi \in D} u^h(\xi, x(\xi))$, where $u^h(\xi, .) : \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}_+$ is continuous, concave, strictly increasing and unbounded.*

Assumption A2. *At every $(\xi, h) \in D \times H, w^h(\xi) \gg 0$. Additionally, $U^h(W) < +\infty$.*

Following Moreno-García and Torres-Martínez (2012), we require node-by-node separable utility functions in order to obtain equilibrium existence in the infinite horizon case through asymptotic techniques reliant on equilibrium existence for finite-horizon truncated economies. Unboundedness of $(u^h(\xi, .))_{(\xi,h) \in D \times H}$ allows us to determine upper bounds on prices of positive net supply securities and, jointly with bounded utility on aggregate wealth, lower bounds on asset prices as well.

The following assumption characterizes the credit segmentation considered in our model.

Assumption A3. *Trading constraints $(\Phi^h)_{h \in H}$ comply with the following properties:*

a. Every correspondence Φ^h is lower hemicontinuous in the product topology, has a closed graph, convex values, and satisfies $0 \in \Phi^h(p, q), \ \forall (p, q) \in \mathbb{P}$.

b. For any $(p,q)\in\mathbb{P}$, and for every pair $y,\ y'\in\Phi^h(p,q)$,

$$\lambda\diamond y+(1-\lambda)\diamond y'\in\Phi^h(p,q),\ \text{for any}\ \lambda\in\prod_{\xi\in D}[0,1],$$

where $\lambda\diamond y=(\lambda_\xi y_\xi)_{\xi\in D}$.

c. Agents may always sell their financial endowments $(e^h(\xi))_{\xi\in D}$. Also, for every $h\in H$, $\Phi^h(p,q)+\left(\mathbb{R}_+^{D\times\mathcal{L}}\times\mathbb{R}_+^{D(J)}\times\{0\}\right)\subseteq\Phi^h(p,q)$.

d. For any feasible allocation $(x,\theta,\varphi)\in\Phi^h(p,q)$ we have that $(x,\theta,\varphi)-(0,\bar\theta,0)\in\Phi^h(p,q)$, for any $\bar\theta\in\mathbb{R}_+^{D(J)}$ such that $\bar\theta_j(\xi)\in[0,\theta_j(\xi)]$ if $j\in J_0(\xi)$, and zero otherwise.

Closed graph of trading constraints implies that for every convergent sequence of prices and feasible allocations there is a feasible limit allocation. Convexity of Φ^h_ξ ensures that linear combinations of feasible allocations are feasible as well.

Introducing financial market segmentation without incorporating any *financial survival* assumptions may lead to empty interiors of choice sets if there is no access to credit and no physical wealth or agents are prevented from consuming their endowments due to binding portfolio constraints (Seghir and Torres-Martínez, 2011); Example 1 illustrates this point. An empty interior, in turn, menaces the choice set's continuity, a requisite for determining equilibrium existence. Hence, Assumption A3.a, plus the interiority of commodity endowments, solve this issue.

Example 1. Suppose that agent h_1 must choose allocations that comply with the following constraint:

$$(x,\theta,\varphi)\in\Phi^{h_1}(p,q)\Rightarrow q(\xi_0)\varphi(\xi_0)\leq p(\xi_0)w^{h_1}(\xi_0).$$

If agent h_1 has no physical wealth at ξ_0 (i.e., $p(\xi_0)w^{h_1}(\xi_0)=0$), the interior of her choice set is empty. Also, assume that agent h_2 is subject to the following constraints:

$$(x,\theta,\varphi)\in\Phi^{h_2}(p,q)\Rightarrow\begin{cases}p(\xi_0)x(\xi_0)\geq M,\\ q(\xi_0)\varphi(\xi_0)\leq q(\xi_0)\theta(\xi_0),\end{cases}$$

for some $M>0$. If agent h is not rich enough at ξ_0 (i.e., $M\geq p(\xi_0)w^h(\xi_0)+q(\xi_0)e^h(\xi_0)$), her inability to obtain resources from financial markets to purchase commodities implies the interior of her choice set is empty as well.

Assumption A3.b is a convexity requirement, stronger than convexity of Φ^h, as the values of $(\lambda(\xi))_{\xi\in D}$ may vary across nodes. It is necessary in order to prove the individual optimality of the allocation obtained as the limit of equilibrium allocations of finite horizon economies. Importantly, it holds when imposing node-by-node separability of trading constraints.

Remark 1. Assumption A3 holds if we assume that trading constraints $(\Phi^h)_{h\in H}$ are of the type $\Phi^h=\prod_{\xi\in D}\Phi^h_\xi$, where each correspondence $\Phi^h_\xi:\mathbb{P}\twoheadrightarrow\mathbb{E}_\xi$ complies with A3.a, A3.c and A3.d.

Assumption A3.c simply states agents are not obliged to keep any financial endowments they receive, and that both commodity consumption and investment may always increase independently of the allocation. To determine upper and lower bounds on asset prices, we construct a series of arguments based on the fact that agents should not be able to purchase arbitrarily large consumption bundles/investment portfolios, as they may lead to utility levels greater than those provided by aggregate wealth. These arguments require that agents may freely determine both their consumption and investment portfolio processes, and hence, we need to rule out both commodity and investment segmentation.

In order to assert that zero net supply assets have well behaved prices, we build a non-arbitrage argument that relies on the capacity of agents of reducing their investments in a zero net supply security, and purchasing its respective super-replicating portfolio. Importantly, these arguments are not valid if agents are using investments in zero net supply assets to obtain access to credit, as reducing their positions in those assets could compromise the feasibility of their chosen allocations. Thus, and in a manner akin to Cea-Echenique and Torres-Martínez (2016), we need to rule out the possibility that zero net supply assets may be used as financial collateral. Assumption A3.d simply states that long positions in assets in $J_0(\xi)$ may be always reduced without compromising the feasibility of the allocation.

Although several requirements are needed as to make financial segmentation compatible with equilibrium existence, our model still allows for broad forms of credit segmentation. Note that we do not need to rule out margin calls, nor we require agents to maintain some level of liquidity throughout the event tree D.

Example 2. The following constraints are examples of restrictions that are included in our framework:

$$(x,\theta,\varphi) \in \Phi_h(p,q) \Rightarrow \begin{cases} \varphi_k(\xi) = 0, \text{ for any } k \in \bar{J}(\xi). \\ q(\xi)\varphi_j(\xi) \geq \alpha_\xi q(\mu), \text{ for some } (\mu, j) \in D \times J(\xi) \setminus \bar{J}(\xi). \\ \varphi_s(\xi) \leq \max\{\theta_1(\xi), \theta_2(\xi)\}, \text{ for } s \in J(\xi) \setminus \bar{J}(\xi). \end{cases}$$

Note that we do not require to impose any *financial survival* assumptions; that is, agents may not have access to credit throughout event-tree D (Aouani and Cornet, 2009). Moreover, access to credit may depend on past or future prices, as illustrated by the second restriction. Finally, the third constraint represents a typical case of financial collateral.

Example 3. The following constraints are examples of restrictions that are ruled out in our framework:

$$(x,\theta,\varphi) \in \Phi_h(p,q) \Rightarrow \begin{cases} p(\xi)x(\xi) + q(\xi)\theta(\xi) \leq A, \text{ for } A > 0. \\ \theta_j(\xi) \in [1,3], \text{ for some } j \in J(\xi). \\ \varphi(\xi) \geq \alpha\varphi(\xi^-), \text{ for } \alpha > 0. \end{cases}$$

The first restriction violates Assumptions A3.c, as it curtails hypothetical agents from freely consuming commodities and investing in financial securities. The second constraint obliges agents to maintain a positive and bounded investment in security j; this scenario is explicitly ruled out by Assumption A3.c as well. Lastly, the third restriction forces long positions in financial markets to be dependent on those at the previous node, which may contravene Assumption A3.b.

We follow Moreno-García and Torres-Martínez (2012) and assume that trading constraints $(\Phi^h(p,q))_{h\in H}$ incorporate a borrowing constraint preventing agents from engaging into Ponzi schemes.

Assumption A4. *For every $h \in H$,*

$$(x^h, \theta^h, \varphi^h) \in \Phi^h(p,q) \Rightarrow q(\xi)\varphi^h(\xi) \leq \kappa p(\xi) w^h(\xi) \quad \forall \xi \in D.$$

The literature of two-period economies with incomplete and segmented financial markets has required additional assumptions in order to ensure that equilibrium portfolios are bounded, such as non-redundancy conditions (Siconolfi, 1989) or boundedness of the set of aggregated attainable allocations (Cea-Echenique and Torres-Martínez, 2016; Faias and Torres-Martínez, 2017). In our model, these bounds are ensured by both the presence of borrowing constraints and endogenously determined lower bounds on asset prices.

It is important to highlight that our framework is compatible with a wide array of financial promises. Indeed, it encompasses nominal securities, collateralized debt contracts and derivatives, as illustrated by the following example.

Example 4. *Nominal securities* are all those $r \in J_0$ whose payment streams may be characterized by a process $(N_r(\mu))_{\mu > \xi_r} \in \mathbb{R}_+^{D(\xi_r)}$. A *collateralized debt contract* subject to strategic default may be modeled by a security $j \in J_0$ whose payment stream is $(\min\{p(\mu)C(\mu), A(\mu)\})_{\mu > \xi_j}$, where each $C(\mu) \in \mathbb{R}_+^{\mathcal{L}}$ corresponds to the transformed collateral bundle $C(\xi_j)$ up to node μ, and that is subject to a conventional collateral constraint (see Araujo, Páscoa and Torres-Martínez, 2011):

$$C(\mu)\varphi_j^h(\mu) \leq x^h(\mu), \quad \forall (h,\mu) \in H \times D(\xi_j).$$

Note that any agent h obtaining credit through j doest not pay a coupon greater than the market value of the collateral bundle. *Derivatives* may be also included in this framework. In particular, let $d_s \in J_0$ be an asset that makes positive deliveries only if asset $s \in J$ fails to deliver an amount greater than certain threshold $\bar{R}$ up to time T. That is, $R_{d_s}(\xi, p) = \alpha \max\{\bar{R} - R_s(\xi, p), 0\}$, with $\alpha > 0$, for nodes $\xi > \xi_s$ such that $t(\xi) \leq T$. Analogously, we may model asset $f \in J_0$ as a "forward" issued at node ξ_f and that pays its owner the difference between the spot price of commodity $l \in \mathcal{L}$ (i.e., copper) and a pre-established price $\bar{p}_l$ at time T_f:

$$R_f(\mu, p) = \max\{\bar{p}_l - p_l(\mu), 0\} \text{ if } t(\mu) = T_f, \text{ and 0 otherwise.}$$

4. Main Result

In this section, we present our equilibrium existence result. In particular, we prove that when zero net supply securities are short-lived[6] and when financial and commodity markets comply with a *super-replication* property (detailed below), a competitive equilibrium always exists.

As noted by Hernández and Santos (1996), and Moreno-García and Torres-Martínez (2012), the inclusion of assets in zero net supply is problematic as non-arbitrage conditions may be incompatible with finite prices in equilibrium (see Hernández and Santos 1996, example 3.9, p.118). Thus, we require an additional mechanism to ensure that prices of asset in zero net supply have endogenous upper bounds.

The following super-replication property was first introduced by Cea-Echenique and Torres-Martínez (2016), in the context of a two-period economy with incomplete markets and credit segmentation. Super-replication of an asset's payment stream demands the existence of portfolio of commodities and/or financial securities whose deliveries are greater than those of the super-replicated asset at every successor. Cea-Echenique and Torres-Martínez (2016) used it to determine

[6] A security $j \in J$ is short-lived if $T_j = t(\xi_j) + 1$.

equilibrium existence in an economy where segmented assets could be super-replicated by commodities and assets that every agent could short-sale[7]. We formulate the hypothesis as to allow zero net supply assets to be super-replicated by commodities and/or assets in positive net supply.

Definition 2. At node $\xi \in D$, the payments of asset $j \in J_0(\xi)$ may be *super-replicated* by commodities and assets in positive net supply if for every (p, q) in a compact set $\widehat{\mathbb{P}} \subset \mathbb{P}$ there exists a pair $(\hat{x}(\xi, j), \hat{\theta}(\xi, j)) \in \mathbb{R}^{\mathcal{L}}_{+} \times \mathbb{R}^{J_+(\xi)}_{+}$ such that:

$$R_j(\mu, p(\mu)) \leq p(\mu) Y_\mu \hat{x}(\xi, j) + \sum_{k \in J_+(\xi)} p(\mu) A_k(\mu) \hat{\theta}(\xi, j), \quad \text{at every } \mu \in \xi^+.$$

Intuitively, and without additional frictions, the equilibrium price of an asset should be smaller than the price of a portfolio that yields greater returns at every state of nature. Therefore, if the price of the super-replicating portfolio is correctly defined, this property allows to assert that the prices of the super-replicated securities are bounded in equilibrium. Thus, we are now ready to present our first equilibrium result.

Theorem 1. *Equilibrium exists for any standard economy $\mathcal{E}$ in which:*

(1) Zero net supplies assets are short-lived: $T_j = t(\xi_j) + 1$.

(2) Agent $h \in H$ may super-replicate the payments of zero net supply securities with commodities and assets in positive net supply.

It is relevant to remark that the super-replication property is *ex-ante* verifiable for any economy $\mathcal{E}$ by observing both transformation matrices and assets' deliveries.

Remark 2. Assume a context of real assets in which deliveries are of the sort $R_j(\xi, p(\xi)) = p(\xi) A_j(\xi)$, with $A_j(\xi) \in \mathbb{R}^{\mathcal{L}}_{+}$, for all (ξ, j) in $D \times J_0(\xi)$. Then, super-replication of zero net supply securities holds if, at every ξ there is a portfolio of positive net supply securities $\tilde{\theta}(\xi)$ such that:

$$\sum_{j \in J_+(\xi)} A_j(\mu) \bar{\theta}_j(\xi) \in \mathbb{R}^{\mathcal{L}}_{++}, \quad \forall \mu \in \xi^+.$$

Equivalently, the super-replication property holds in any economy in which zero net supply securities deliveries consist of durable commodities (see Cea-Echenique and Torres-Martínez, 2016).

Moreover, as noted by Iraola, Sepúlveda and Torres-Martínez (2017), the super-replication property trivially holds in a context of collateralized assets with strategic default, as asset deliveries are systematically bounded from above by the market value of collateral bundles.

Super-replication does not allow to incorporate zero net supply assets with a longer lifespan, as agents may resell them in posterior nodes. Hence, although super-replicating portfolios may completely hedge deliveries, they do not necessarily do the same with the reselling prices. This issue remains a pending challenge for future research.

[7]In particular, as in their model the prices of unsegmented assets may be incorporated in the unitary simplex, there is no uncertainty with respect to the boundedness of those prices in equilibrium.

5. Conclusion

This article introduced endogenous financial segmentation to an infinite horizon model with financial markets that include real assets in positive net supply and short-lived zero net supply securities. We show that equilibrium existence is compatible with broad forms of credit segmentation, without any *financial survival* or uniform impatience requirements on preferences. We rely on a super-replication property proposed by Cea-Echenique and Torres-Martínez (2016) to incorporate short-lived assets in zero net supply. More research has to be done as to determine both the implications of financial segmentation for equilibrium existence and additional conditions under which finitely and infinitely lived zero net supply assets may be incorporated to financial trade economies.

6. Appendix

The proof of our theorem is analogous to the one developed by Moreno-García and Torres-Martínez (2012), where the main innovations appear when determining upper bounds for prices in zero net supply. Nevertheless, the presence of durable commodities and financial segmentation introduces slight variations that justify the repetition of the whole proof. The following outline sketches the proof, explaining the main ideas and highlighting the relevance of our Assumptions.

(1) We prove equilibrium existence in $\mathcal{E}$ by using asymptotic arguments that derive from the existence of equilibrium for finite horizon economies. In particular, truncated economy $\mathcal{E}^T$ circumscribes agents' original problem into a finite horizon economy with $T \in \mathbb{N}$ periods. The structure of uncertainty is given by sub-tree $D^T(\xi_0)$, and endowments, preferences, trading constraints and financial assets available for trade are redefined in order to fit this particular setting.

(2) Equilibrium in each truncated economy $\mathcal{E}^T$ is determined in four steps:

(a) We prove the existence of a non-empty set of fixed points in a generalized game resembling economy $\mathcal{E}^T$. Here, players representing agents $h \in H$ choose allocations in compact and convex strategy sets given by correspondences which, thanks to Assumptions A3, are continuous functions of prices. Moreover, fictitious auctioneers choose commodity prices (normalized into the unitary simplex) and asset prices belonging to arbitrarily bounded spaces (see Lemma 6.1).

(b) We rely on Assumptions A1 and A2 to assert that bounds imposed on prices of positive net supply securities, when large enough, are not binding (see Lemma 2). This results is analogous to Lemma 6.2 of Moreno-García and Torres-Martínez (2012).

(c) We establish conditions under which the prices of zero net supply securities are endogenously bounded in equilibrium, by means of both the super-replicaton property and the limited lifespan of assets in zero net supply; Assumption 3.c plays a key role (see Lemma 6.3).

(d) We then rely on Assumptions A1-A4 to show that bounds imposed on allocations are not binding as well, as long as they are large enough. Therefore, Lemma 4 proves that the fixed points found in Lemma 1 are effectively competitive equilibria of economy $\mathcal{E}^T$; it is also equivalent to its counterpart in Moreno-García and Torres-Martínez (2012). The presence of borrowing constraints (due to Assumption A4) is key to the result (see Lemma 6.4).

(3) As we have equilibrium existence for every truncated economy $\{\mathcal{E}^T\}_{T\in\mathbb{N}}$, Assumption A2 allows us to assert that the series formed by equilibrium prices, allocations and the Lagrange multipliers associated to agents' problem are bounded, for every $\{\mathcal{E}^T\}_{T\in\mathbb{N}}$. Thus, we use Tychonoff's Theorem to ensure the existence of a limit of a sub-sequence of equilibria for truncated economies $\{\mathcal{E}^{T_k}\}_{k\geq 0}$, whose allocations comply with market clearing. We treat this limit's prices and allocation as our equilibrium candidate.

(4) Lastly, we prove that the former limit allocations are optimal given the limit's prices, and thus, a competitive equilibrium exists for $\mathcal{E}$; here, the convexity requirement on trading constraints (Assumption A3.b) is crucial (see Lemma 6.5).

6.1. **Finite horizon economies.** Let $\mathcal{E}^T$ be the truncated version of standard economy $\mathcal{E}$ up to time $T \in \mathbb{N}$. That is, $\mathcal{E}^T$ is a finite horizon economy starting at node ξ_0 and circumscribed to event-tree $D^T(\xi_0)$. Now, at every node $\xi \in D^{T-1}(\xi_0)$ there is a set $J^T(\xi) := J_+^T(\xi) \cap J_0^T(\xi)$ of assets available for trade, where $J_+^T(\xi) = \{j \in J_+(\xi) | \exists \mu > \xi : t(\mu) < T, A_j(\mu) \neq 0\}$ and $J_0^T(\xi) = \{j \in J_0(\xi) | \exists \mu > \xi : t(\mu) < T, R_j(\mu, p(\mu)) > 0 \text{ for } p(\mu) \gg 0\}$. By assumption, $J^T(\xi) = \emptyset$ for all $\xi \in D_T(\xi)$. Importantly, given $\xi \in D^{T-1}(\xi_0)$, $J^T(\xi) = J(\xi)$ for T large enough. Let $D^T(J) = \{(\xi, j) \in D^T(\xi_0) \times J^T(\xi) : j \in J^T(\xi)\}$; the sets $D^T(J_+)$ and $D^T(J_0)$ are defined analogously.

We consider prices (p, q) belonging to space

$$\mathbb{P}^T = \prod_{\xi \in D^{T-1}(\xi_0)} \left(\Delta_+^{\mathcal{L}} \times \mathbb{R}_+^{J^T(\xi)}\right) \times \prod_{\xi \in D_T(\xi_0)} \Delta_+^{\mathcal{L}},$$

where $\Delta_+^{\mathcal{L}} := \{p \in \mathbb{R}_+^{\mathcal{L}} : \|p\|_\Sigma = 1\}$.

Agents' problem is reformulated to fit event-tree $D^T(\xi_0)$. In particular, agent $h \in H$ is characterized by endowments $(w^h(\xi), e^h(\xi))_{\xi \in D^T(\xi_0)}$, and a modified utility functional over consumption streams, $U^{h,T} = \sum_{\xi \in D^T(\xi_0)} u^h(\xi, x(\xi))$. Accordingly, she must choose an allocation $(y^h(\xi))_{\xi \in D^T(\xi_0)} = (x^h(\xi), \theta^h(\xi), \varphi^h(\xi))_{\xi \in D^T(\xi_0)}$ belonging to space $\mathbb{E}^T := \mathbb{R}_+^{D^T(\xi_0) \times \mathcal{L}} \times \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)}$. We denote $y^h = (y^h(\xi))_{\xi \in D^T(\xi_0)}$.

Every agent $h \in H$ is also subject to trading constraints $\Phi^{h,T}(p, q) : \mathbb{P}^T \twoheadrightarrow \mathbb{E}^T$, where $\Phi^{h,T}$ is defined as the *projection* of Φ^h onto subtree $D^T(\xi_0)$. That is, for prices $(p, q) \in \mathbb{P}^T$, $\Phi^{h,T}$ is defined as all allocations $y \in \mathbb{E}^T$ for which we may find a pair $(\tilde{y}, (\tilde{p}, \tilde{q})) \in \mathbb{E}$ such that:

$$\left(\tilde{y} \in \Phi^h(\tilde{p}, \tilde{q})\right) \wedge \left((y(\xi), (p(\xi), q(\xi)))_{\xi \in D^{T-1}(\xi_0)} = (\tilde{y}(\xi), (\tilde{p}(\xi), \tilde{q}(\xi)))_{\xi \in D^{T-1}(\xi_0)}\right).$$

Thus, for prices $(p, q) \in \mathbb{P}^T$, agent h's truncated choice set correspondence $C^{h,T}(p, q)$ considers allocations $y^h \in \mathbb{E}^T$ complying with constraints:

$$g^{h,T}(\xi, y^h(\xi), y^h(\xi^-); p, q) \leq 0, \quad \forall \xi \in D^T(\xi_0)$$
$$(y^h(\xi))_{\xi \in D^T(\xi_0)} \in \Phi^{h,T}(p, q),$$

where $(\theta^h(\xi_0^-), \varphi^h(\xi_0^-)) = (0, 0)$ and for every $\xi \in D^T(\xi_0)$ the function $g^{h,T}(\xi, .)$ is given by:

$$\begin{aligned} g^{h,T}(\xi, y(\xi), y(\xi^-); p, q) := {} & p(\xi)\left(x^h(\xi) - w^h(\xi) - Y_\xi x^h(\xi^-)\right) + q(\xi)(\theta^h(\xi) - \varphi^h(\xi) - e^h(\xi)) \\ & - \sum_{j \in J_+(\xi^-)} (p(\xi) A_j(\xi) + q_j(\xi))(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)) \\ & - \sum_{k \in J_0(\xi^-)} (R_k(\xi, p(\xi)) + q_k(\xi))(\theta_j^h(\xi^-) - \varphi_j^h(\xi^-)). \end{aligned}$$

Definition 3. A *competitive equilibrium* for economy $\mathcal{E}^T$ is composed by a price process $(p, q) \in \mathbb{P}^T$ and allocations $(y^h)_{h \in H} \in (\mathbb{E}^T)^H$ such that

(1) For every $h \in H$, $y^h \in \operatorname{argmax}_{y \in C^{h,T}(p,q)} U^{h,T}(x)$.
(2) Physical and financial markets clear, i.e.,

$$\sum_{h \in H} x^h(\xi) = W(\xi) \;\; \forall \xi \in D^T(\xi_0), \quad \text{and} \quad \sum_{h \in H} \theta^h(\xi) = \sum_{h \in H} (\varphi^h(\xi) + \bar{e}^h(\xi)) \; \forall \xi \in D^{T-1}(\xi_0).$$

6.2. **Equilibrium in finite horizon economies.** Let $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ be a generalized game in which fictitious players representing agents $h\in H$ and auctioneers at every node $\xi\in D^T(\xi_0)$, seek to maximize objective functions by choosing strategy profiles in truncated strategy sets. More precisely, for $(\mathcal{X},\Theta,\Psi,M)\in\mathbb{E}^T\times\mathbb{R}_{++}^{D(J)}$ given, define the truncated allocation and price spaces as

$$\mathcal{K}(\mathcal{X},\Theta,\Psi):=[0,\mathcal{X}]\times[0,\Theta]\times[0,\Psi],$$
$$\text{and } \mathbb{P}_M^T:=\prod_{\xi\in D^{T-1}(\xi_0)}\left(\Delta_+^{\mathcal{L}}\times[0,M_\xi]\right)\times\prod_{\xi\in D_T(\xi_0)}\Delta_+^{\mathcal{L}}$$

respectively. Consequently, and given prices $(p,q)\in\mathbb{P}_M^T$, fictitious player $h\in H$ (representing agent $h\in H$) chooses an allocation lying within the truncated choice set $C^{h,T}(p,q)\cap\mathcal{K}(\mathcal{X},\Theta,\Psi)$ as to maximize $U^{h,T}$. Additionally, at every node $\xi\in D^{T-1}(\xi_0)$ (resp. $\xi\in D_T(\xi_0)$) there is an auctioneer maximizing $\sum_{h\in H}g^{h,T}(\xi,y^h(\xi),y^h(\xi^-);p,q)$ by choosing both asset and commodity prices $(p(\xi),q(\xi))\in\Delta_+^{\mathcal{L}}\times[0,M_\xi]$ (resp. commodity prices only, $p(\xi)\in\Delta_+^{\mathcal{L}}$).

Definition 4. A *Nash equilibrium* of game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ consists of a strategy profile

$$[(p(\xi),q(\xi));\ (y^h(\xi))_{h\in H}]_{\xi\in D^T(\xi_0)}\in\mathbb{P}_M^T\times(\mathcal{K}(\mathcal{X},\Theta,\Psi))^H$$

such that each player maximizes her utility given the strategies chosen by the rest of the players (i.e. no one has incentives to deviate).

Lemma 6.1. *The set of Cournot Nash equilibria of the game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is non-empty.*

Proof. Note that, at every node $\xi\in D^T(\xi_0)$, the corresponding auctioneer maximizes an objective function that is continuous and quasi-concave in his own strategy, whereas his strategy space is a non-empty, compact, and convex set. Assumption A1 implies players $h\in H$ are also endowed with continuous utility functions, which are quasi-concave in their own strategy. Moreover, we assure that their strategy spaces, $(C^{h,T}(p,q)\cap\mathcal{K}(\mathcal{X},\Theta,\Psi))_{h\in H}$ are determined by continuous correspondences. Berge's Maximum theorem allows us to assert that the maximizing arguments of the correspondence formed by the best-response functions of players of game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ is non-empty, compact-valued and upper-hemicontinuous. Consequently, Kakutani's Fixed Point Theorem ensures game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$ has a non-empty set of fixed points. Hence, to prove Lemma 1 we need to show that the correspondence $\hat{C}^{h,T}:=C^{h,T}\cap\mathcal{K}(\mathcal{X},\Theta,\Psi)$ is continuous.

Lower hemicontinuity. We prove first the lower hemicontinuity of the trading constraints correspondence $\Phi^{h,T}$. Fix $(y^T,(p^T,q^T))\in\mathbb{E}^T\times\mathbb{P}^T$ such that $y^T\in\Phi^{h,T}(p^T,q^T)$ and let there be a sequence $\{(p_n^T,q_n^T)\}_{n\in\mathbb{N}}\subset\mathbb{P}^T$ whose limit is (p^T,q^T). The definition of $\Phi^{h,T}$ implies there exist $(y,(p,q))\in\mathbb{E}\times\mathbb{P}$ complying with both $y\in\Phi^h(p,q)$ and $(y^T,(p^T,q^T))=(y(\xi),(p(\xi),q(\xi)))_{\xi\in D^{T-1}(\xi_0)}$. Consider the following sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}}\subset\mathbb{P}$:

$$(p_n(\xi),q_n(\xi))=\begin{cases}(p_n^T(\xi),q_n^T(\xi)) & \text{if } \xi\in D^{T-1}(\xi_0)\\ (p(\xi),q(\xi)) & \text{if } \xi\notin D^{T-1}(\xi_0)\end{cases},$$

Clearly, $(p_n(\xi),q_n(\xi))\to(p,q)$. Therefore, Assumption A3.a and the sequential characterization of lower hemicontinuity ensure there exists $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{E}:\left(y_n\in\Phi^h(p_n,q_n)\ \forall n\in\mathbb{N}\right)\wedge(y_n\to y)$.

Moreover, the construction of $\{y_n\}_{n\in\mathbb{N}}$ allows us to assert that the sequence $\{y_n^T\}_{n\in\mathbb{N}} \subset \mathbb{E}^T$ defined as

$$y_n^T(\xi) = \begin{cases} y_n(\xi) & \text{for } \xi \in D^{T-1}(\xi_0) \\ y^T(\xi) & \text{for } \xi \in D^T(\xi_0) \end{cases} \quad \text{for every } n \in \mathbb{N},$$

complies with both $y_n^T \in \Phi^{h,T}(p_n^T, q_n^T)\ \forall n \in \mathbb{N}$ and $y_n^T \to y^T$, which implies that $\Phi^{h,T}$ is lower hemicontinuous.

Now, consider the correspondence defined as $\dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$, where $\dot{C}^{h,T}(p,q)$ corresponds to all allocations $y \in C^{h,T}(p,q)$ complying with budget constraints with strict inequalities. Assumptions A2 and A3 ensure that $\dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ has non-empty values, as commodity endowments are strictly positive, commodity spot prices always belong to the unitary simplex and there are no trading constraints curtailing h from consuming a fraction of her endowments. Moreover, given a pair $(y,(p,q)) \in \mathbb{E}^T \times \mathbb{P}^T$ such that $y \in \dot{C}^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ and a sequence $\{(p_n,q_n)\}_{n\in\mathbb{N}}$ converging to (p,q), the lower hemicontinuity of $\Phi^{h,T}$ allows us to assert that there exists $\{y_n\}_{n\in\mathbb{N}}$ complying with $y_n \in \Phi^{h,T}(p_n,q_n)\ \forall n \in \mathbb{N}$ and converging to y. Thus, and for $n \in \mathbb{N}$ sufficiently large, $y_n \in \dot{C}^{h,T}(p_n,q_n) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ as well; $\dot{C}^{h,T} \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ is lower hemicontinuous.

Since $\dot{C}^{h,T}$ has non-empty and convex values, its closure coincides with $C^{h,T}$, and so we learn that $C^{h,T} \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ is lower hemicontinuous.

Upper hemicontinuity. As $\mathcal{K}(\mathcal{X},\Theta,\Psi)$ is a closed, convex and compact set containing 0, it follows from Assumptions A2 and A3.a that $C^{h,T}(p,q) \cap \mathcal{K}(\mathcal{X},\Theta,\Psi)$ has non-empty, convex and compact values and a closed graph, which are sufficient conditions for upper hemicontinuity. □

Consider a Nash equilibrium $\left[(\bar{p}(\xi),\bar{q}(\xi));(\bar{y}^h(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ of game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$, with allocations $\bar{y}^h(\xi) = (\bar{x}^h(\xi),\bar{\theta}^h(\xi),\bar{\varphi}^h(\xi))$. By aggregating agents' budget set constraints at any node in $D^T(\xi_0)$ retrieves the auctioneer's objective function at that node.[8] As equilibrium allocations comply with budget feasibility, we learn the latter is non-positive at every ξ. This implies, jointly with the construction of strategy sets in game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$, and the optimality of the auctioneer's behavior, that there exists a bound on aggregate consumption at node ξ in equilibrium, defined recursively by

$$\sum_{h\in H} \bar{x}^h(\xi) \le \mathcal{Y}^T(\Theta,\xi) := Y_\xi \mathcal{Y}^T(\Theta,\xi^-) + \sum_{h\in H}\left(w^h(\xi) + \sum_{j\in J_+^T(\xi^-)} A_j(\xi)\Theta(\xi^-,j)\right),$$

where $\mathcal{Y}^T(\Theta,\xi_0) = W(\xi_0)$.

Indeed, if it were not the case, the auctioneer could choose prices as to make her objective function positive at ξ. Define $\mathcal{Y}^T(\Theta) = (\mathcal{Y}^T(\Theta,\xi))_{\xi\in D^T(\xi_0)}$, and fix an agent $h \in H$. Assumption A1 ensures that at every $\xi \in D^T(\xi_0)$ there exists a consumption bundle $a_\Theta^{h,T}(\xi) \in \mathbb{R}_{++}^{\mathcal{L}}$, that allows h to obtain greater utility than the one attainable through aggregate feasible consumption in any equilibrium of game $\mathcal{G}^T(\mathcal{X},\Theta,\Psi,M)$:

$$u^h(\xi, a_\Theta^{h,T}(\xi)) > U^{h,T}(\mathcal{Y}^T(\Theta)).$$

[8] Note that the auctioneer's objective function used here differs from the ones traditionally used in the literature, but it allows us to assert that this particular argument holds.

Due to A1, it is straightforward that bundle $a_\Theta^T(\xi) = \sum_{h\in H} a_\Theta^h(\xi)$ also complies with the condition stated above. Let $a_\Theta^T = (a_\Theta^T(\xi))_{\xi\in D^T(\xi_0)}$, and define $\mathcal{X}_\Theta := a_\Theta^T$.

Lemma 6.2. *Fix $\xi \in D^T(\xi_0)$. Given $\mathcal{X} > \mathcal{X}_\Theta$, there exists $M_\Theta(\xi)$ such that $\bar{q}_j(\xi) \le M_\Theta(\xi)$ for every $j \in J_+^T(\xi)$.*

Proof. Assumption A3.b ensures that agents may sell without restrictions their financial endowments, and choose consumption bundles regardless of their portfolio choices. From the individual optimality of allocation $\bar{y}^h$, it follows that the liquidation value of h's accumulated financial endowments at ξ is not enough to purchase bundle $a_\Theta^T(\xi)$, so $\bar{p}(\xi)w^h(\xi) + \bar{q}(\xi)\bar{e}^h(\xi) < \bar{p}(\xi)a_\Theta^T(\xi)$. In particular, for asset $j \in J_+^T(\xi)$ we have that $\bar{q}(\xi)\bar{e}_j^h(\xi) < \bar{p}(\xi)a_\Theta^T(\xi)$, which, by aggregating in H and due to the fact that $\|\bar{p}(\xi)\|_\Sigma = 1$, allows us to obtain the following upper bounds for prices of assets in $J_+(\xi)$:

$$\bar{q}_j(\xi) < M_\Theta(\xi, j) := \frac{\#H a_\Theta^T(\xi)}{\sum_{h\in H} \bar{e}_j^h(\xi)}.$$

As the election of $j \in J_+^T(\xi)$ was arbitrary, we learn that at node ξ the prices of assets in positive net supply are bounded by $M_\Theta(\xi) := \max_{j\in J_+^T(\xi)}\{M_\Theta(\xi, j)\}$ in equilibrium. □

Define $M_\Theta = (M_\Theta(\xi))_{\xi\in D^T(\xi_0)}$. As implied by Lemma 2, and as the election of ξ was arbitrary, if $(M, \mathcal{X}) \ge (M_\Theta, \mathcal{X}_\Theta)$, bounds imposed on positive net supply securities are not binding in equilibrium. We assume the former holds from now on.

Lemma 6.3. *Fix $\xi \in D^T(\xi_0)$. If $j \in J_0(\xi)$ is in excess demand, there exists $\widehat{M}(\xi)$ such that $\bar{q}_j(\xi) \le \widehat{M}(\xi)$, in any equilibrium where $\sum_{k\in J_+^T(\xi)} \sum_{h\in H}(\bar{\theta}_k^h(\xi) - \bar{e}_k^h(\xi) - \bar{\varphi}_k^h(\xi)) \le 0$, whenever $\Theta \gg \bar{\Theta} := \#H\Psi$.*

Proof. Fix $j \in J_0^T(\xi)$. We assert that

$$\sum_{k\in J_+^T(\xi)} \sum_{h\in H} (\bar{\theta}_k^h(\xi) - \bar{e}_k^h(\xi) - \bar{\varphi}_k^h(\xi)) \le 0 \Rightarrow \bar{q}_j(\xi) \le \widehat{M}(\xi, j),$$

for some $\widehat{M}(\xi, j) \in \mathbb{R}_{++}$. The property of super-replication ensures there exist a portfolio $(\hat{x}(\xi, j), \hat{\theta}(\xi, j)) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{J_+^T(\xi)}$ such that

$$R_j(\mu, \bar{p}) \le \bar{p}(\mu)Y_\mu \hat{x}(\xi, j) + \sum_{k\in J_+(\xi)} p(\mu)A_k(\mu)\hat{\theta}_k(\xi, j), \quad \text{at every } \mu \in \xi^+.$$

Now, suppose that the following inequality holds:

$$\bar{q}_j(\xi) > \widehat{M}(\xi, j) := \bar{p}(\xi)\hat{x}(\xi, j) + \sum_{k\in J_+^T(\xi)} \bar{q}_k(\xi)\hat{\theta}_k(\xi, j).$$

Assumption A3.d ensures that for agent $h(j) \in H$ investing in j there exists $\varepsilon > 0$ such that $h(j)$ may reduce her position in j by ε. Because there is no excess demand for assets in $J_+^T(\xi)$, plus the fact that $\Theta \gg \bar{\Theta}$ we know that $h(j)$ may invest those resources into purchasing ε units of $(\hat{x}(\xi, j), \hat{\theta}(\xi, j))$ and additionally increasing her consumption at ξ by an amount equivalent to

$\left[\bar{q}_j(\xi) - \widehat{M}(\xi, j)\right] \varepsilon > 0$. By doing so, player $h(j)$ would be increasing her consumption at ξ, whilst leaving the rest of her consumption stream unaltered. This would contradict the optimality of $\bar{y}^{h(j)}$. Therefore, it must be that $\bar{q}_j(\xi) \leq \widehat{M}(\xi, j)$ in equilibrium.

As the election of $j \in J_0^T(\xi)$ was arbitrary, we learn that there exists $\widehat{M}(\xi) := \max_{j\in J_0^T(\xi)} \left\{\widehat{M}(\xi, j)\right\}$ such that $\bar{q}_j(\xi) \leq \widehat{M}(\xi)$ for every $j \in J_0^T(\xi)$ in excess demand, which ends the proof. □

As the election of ξ was arbitrary as well, we know that this result holds throughout all event-tree $D^T(\xi_0)$. That is, there exist $(\widehat{M}(\xi))_{\xi\in D^T(\xi_0)}$ bounding the prices of zero net supply securities whenever the requirements of Lemma 6.3 hold. Now, define

$$\widehat{M}_\Theta(\xi) = \max\left\{\widehat{M}(\xi), M_\Theta(\xi)\right\} \quad \text{for every } \xi \in D^T(\xi_0),$$

and $\widehat{M}_\Theta = (\widehat{M}_\Theta(\xi))_{\xi\in D^T(\xi_0)}$. We assume that $M \gg \widehat{M}_\Theta$ from now on.

Lemma 6.4. *There exist $(\Theta^*, \Psi^*) \in \mathbb{R}_+^{D^T(J)} \times \mathbb{R}_+^{D^T(J)}$ such that, for $(\Theta, \Psi) \geq (\Theta^*, \Psi^*)$ the fixed points in $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ are competitive equilibria of economy $\mathcal{E}^T$.*

Proof. **Step 1: Commodity markets clear.**
Choose any equilibrium $\left[(\bar{p}(\xi), \bar{q}(\xi)); (\bar{y}^h(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ of game $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$. Aggregating agents' budget constraints at node ξ_0 retrieves the auctioneer's objective function at ξ_0, which we learn is non-positive. This implies $\sum_{h\in H}(\bar{x}^h(\xi_0) - w^h(\xi_0)) \leq 0$; indeed, if $\sum_{h\in H}(\bar{x}_l^h(\xi_0) - w_l^h(\xi_0)) > 0$ for some $l \in \mathcal{L}$, the auctioneer could choose $\tilde{q}(\xi_0) = 0$ and $\tilde{p}_l(\xi_0) = 1$ and obtain positive values in his objective function at ξ_0.

As $M \geq \widehat{M}_\Theta$, Lemma 6.2 ensures bounds imposed on asset prices are non-binding for securities in $J_+^T(\xi_0)$. This implies that $\sum_{h\in H}\sum_{j\in J_+^T(\xi_0)}(\bar{\theta}_j^h(\xi_0) - \bar{e}_j^h(\xi_0) - \bar{\varphi}_j^h(\xi_0)) \leq 0$; otherwise, the auctioneer at ξ_0 would have incentives to deviate by raising the price of some asset in positive net supply to $M_j(\xi)$, contradicting Lemma 2. Thus, and due to the optimal behavior of the auctioneer, we learn that there cannot be excess demand for assets in $J_0^T(\xi_0)$ either, or it would contradict Lemma 6.3. Moreover, we learn that the value of excess demand in financial markets is zero as well:

$$\bar{q}(\xi_0) \sum_{j\in J^T(\xi_0)} (\bar{\theta}^h(\xi_0) - \bar{\varphi}^h(\xi_0) - e^h(\xi_0)) = 0.$$

Otherwise, the auctioneer would trivially improve by setting the price of assets in excess supply to zero. As $\mathcal{X} > \mathcal{X}_\Theta$, bounds imposed on consumption at ξ_0 are not binding. Because there is no excess demand in financial markets at ξ_0, the monotonicity of preferences implies both $\bar{p}(\xi_0) \gg 0$ (or agents would optimally set $\bar{x}_l^h(\xi_0) = \mathcal{X}_l(\xi_0)$) and that commodity markets clear at ξ_0 (or some h would increase her utility by spending her non-spent resources at ξ_0).

Now, fix any node $\xi \in \xi_0^+$. The aggregation of individual budget constraints again retrieves the auctioneer's objective function, and because there is no excess demand in financial markets at ξ_0 we learn that its value is non-positive. Therefore, because $\mathcal{X}(\xi) \geq a_\Theta^T(\xi)$, and by repeating the exact same arguments exposed above, we learn that the value of excess demand in financial markets is zero, and that commodity markets clear at ξ.

The consecutive iteration of these arguments ensures these results holds for every $\xi \in D^T(\xi_0)$. That is, commodity markets clear at every $\xi \in D^T(\xi_0)$, $p(\xi) \gg 0$, there is no excess demand in financial markets, and the value of excess demand in financial markets is zero at any node in $D^{T-1}(\xi_0)$.

Step 2: Lower bounds for asset prices. As commodity markets clear throughout sub-tree $D^T(\xi_0)$, we know that equilibrium allocations never exceed aggregate wealth. Hence, we know that $U^{h,T}(W)$ is an upper bound for consumer h's utility in any equilibrium of $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$.

Now, fix a pair $(\xi, j) \in D^{T-1}(\xi_0) \times J_0^T(\xi)$ and consider any successor $\mu \in D^T(\xi)$ such that $R_j(\mu, \bar{p}) > 0$. The definition of $J_0^T(\xi)$, plus the fact that $\bar{p} \gg 0$, implies μ exists. Assumptions A1 and A2 ensure there exists number $b(\xi, j) \in (0,1)$, independent of $T > t(\mu)+1$, such that for every $h \in H$,

$$\frac{R_j(\mu, \bar{p}) \min_{l\in\mathcal{L}} w_l^h(\xi)}{b(\xi, j)} > \|a_\Theta^T(\mu)\|.$$

Also, assume $\Theta(\xi, j) > \widehat{\Theta}(\xi, j) := \max_{h\in H} \frac{\min_{l\in\mathcal{L}} w^h(\xi)}{b(\xi,j)}$. We claim that $\bar{q}_j(\xi) > b(\xi, j)$. If it were not true, Assumptions A1 and A2 imply any player could obtain an utility greater than the one attainable through aggregate wealth, by purchasing portfolio $\theta_j(\xi) = \left(\frac{\min_{l\in\mathcal{L}} w_l^h(\xi)}{b(\xi,j)}\right)$ at ξ and consuming bundle $a_\Theta^T(\mu)$ at μ. Moreover, this allocation is compatible with the structure given to trading constraints by Assumption A3. Hence, it follows from each player's optimal behavior that $\bar{q}_j(\xi) > b(\xi, j)$, for each $(j, \xi) \in J_0^T(\xi) \times D^T(\xi_0)$ such that $\Theta(\xi, j) > \widehat{\Theta}(\xi, j)$ for all $j \in J_0^T(\xi)$.

Importantly, note that the same argument holds for any tuple $(\xi, k) \in D^{T-1}(\xi_0) \times J_+^T(\xi)$, where the asset's payment at some node μ is given by $\bar{p}(\mu)A_k(\mu) > 0$.

Step 3: Non-binding short-sale constraints.

Define $\widehat{\Theta} = (\widehat{\Theta}(\eta, j))_{(\eta,j)\in D^T(J)}$. Then, whenever $\Theta \gg \widehat{\Theta}$, asset prices are bounded from below and away from zero. This implies borrowing constraints induce upper bounds on short-sales, for every $h \in H$:

$$\varphi_j^h(\xi) < \Psi_j^*(\xi) := \kappa \frac{\max_{(h,l)\in H\times\mathcal{L}} w_l^h(\xi)}{b(\xi, j)}, \ \forall (\xi, j) \in D^T(J).$$

Define $\Psi^* = (\Psi_j^*(\xi))_{(\xi,j)\in D^T(J)}$. Then, whenever $\Psi \gg \Psi^*$, short-sale constraints are non-binding in equilibrium.

Step 4: Financial markets clear and non-binding constraints on long positions

Assume $(\Theta, \Psi) \gg (\widehat{\Theta}, \Psi^*)$ and that $\mathcal{X} \gg \mathcal{X}_\Theta$. From Step 1, there is no excess demand in financial markets, and the value of excess demand in financial markets is zero at each $\xi \in D^{T-1}(\xi_0)$. If there were excess supply for any asset $j \in J^T(\xi)$, the optimal behavior of the auctioneer at ξ would imply that $q_j(\xi) = 0$, which would contradict the lower bound on that asset's price at ξ, $b(\xi, j)$. Therefore, financial markets clear in equilibrium.

Additionally, at every $\xi \in D^{T-1}(\xi_0)$, $\sum_{h\in H} \bar{\varphi}^h(\xi)$ is bounded. Hence, no excess demand in financial markets implies that $\sum_{h\in H} \bar{\theta}^h(\xi)$ is bounded as well. Thus, there exists $\Theta^* \geq \widehat{\Theta}$ such that, whenever $\Theta \gg \Theta^*$, bounds imposed on long positions are non-binding.

Step 5: Individual optimality.

Whenever $(\Theta, \Psi) \gg (\Theta^*, \Psi^*)$, equilibrium allocations $(\hat{y}^h)_{h\in H}$ belong to the interior of $(C^{h,T}(\hat{p}, \hat{q}) \cap \mathcal{K}(\mathcal{X}, \Theta, \Psi, M)$ (relative to $\mathbb{E}^{\mathbb{T}}$) for every $h \in H$. Hence, Assumption A1 ensures $(\hat{y}^h)_{h\in H}$ are optimal relative to the original truncated choice sets. Therefore, any equilibrium $\left[(\bar{p}(\xi), \bar{q}(\xi)); (\bar{y}^h(\xi))_{h\in H}\right]_{\xi\in D^T(\xi_0)}$ of game $\mathcal{G}^T(\mathcal{X}, \Theta, \Psi, M)$ is a competitive equilibrium of economy $\mathcal{E}^T$. $\square$

6.3. **Asymptotic equilibrium.** We know that competitive equilibrium exists for any finite-horizon economy $\mathcal{E}^T$ starting at ξ_0. Furthermore, equilibrium allocations are bounded, node by node, by a set of bounds which are *independent* of the truncation horizon T (provided T is large enough). Indeed, as equilibrium commodity allocations are bounded by aggregate wealth, prices of assets in positive net supply at node ξ are bounded due to similar arguments than those exposed in Lemma 6.2, by bundle $a(\xi) \in \mathbb{R}^{\mathcal{L}}_{+}$ complying with:

$$\min_{h\in H}\left\{u^h(\xi, a(\xi))\right\} > \max_{h\in H}\left\{U^{h,T}(W)\right\},$$

which is independent of $T > t(\xi)$. In turn, this implies that the prices of the super-replicating portfolios used to determine upper bounds of prices of assets in $J_0(\xi)$ are also independent of the truncation horizon, provided it is large enough. Therefore, and although we are not able to ensure that the prices of zero net supply securities are bounded by those of their super-replicating portfolios, the bounds on asset prices imposed on the generalized game become independent of T.

To prove equilibrium existence for the infinite horizon economy, we will use a series of sequential arguments which rely on the existence of an equilibrium for any economy $\mathcal{E}^T$ with $T \in \mathbb{N}$. It is particularly important for this objective that we bound the Lagrange multipliers characterizing each agent's problem, which is what we do now.

Let $[p^T, q^T;\ (y^{h,T})_{h\in H}]$ be a competitive equilibrium of economy $\mathcal{E}^T$, with $T \in \mathbb{N}$. Then, for every $h \in H$ there exist Lagrange multipliers $(\gamma_\xi^{h,T})_{\xi\in D^T(\xi_0}$ such that

$$\gamma_\xi^{h,T} g_\xi^{h,T}(y^{h,T}(\xi), y^{h,T}(\xi^-); p^T, q^T) = 0,\ \forall \xi \in D^T(\xi_0).$$

Also, for every allocation $y' \in \mathbb{E}^T \cap \Phi^{h,T}(p^T, q^T)$, the following saddle point property holds (Rockafellar, 1997):

$$U^{h,T}(x') - \sum_{\xi\in D^T(\xi_0)} \gamma_\xi^{h,T} g_\xi^{h,T}(y'(\xi), y'(\xi^-); p^T, q^T) \leq U^{h,T}(x^{h,T}). \tag{1}$$

In particular, for allocation $y' = (0,0,0)$ inequality (1) states that

$$\sum_{\xi\in D^T(\xi_0)} \gamma_\xi^{h,T} p^T(\xi) w^h(\xi) \leq U^h(W) < +\infty, \tag{2}$$

which in turn implies that the multipliers $(\gamma_\xi^{h,T})_{\xi\in D^T(\xi_0}$ are bounded:

$$\gamma_\xi^{h,T} \leq \frac{U^h(W)}{\underline{w}^h(\xi)},\ \forall \xi \in D,$$

with $\underline{w}^h(\xi) = \min_{l\in\mathcal{L}} w_l^h(\xi) > 0$. Recall that Assumption A2 ensures the former expression is correctly defined throughout the event tree D. Therefore, for every economy $\mathcal{E}^T$, equilibrium allocations, Lagrange multipliers and asset prices are, node by node, uniformly bounded from above, whereas commodity prices belong to the unitary simplex. Thus, Tychonoff's Theorem ensures there exists a common subsequence $(T_k)_{k\in\mathbb{N}}$ such that, for each $\xi \in D$,

$$\lim_{k\to+\infty}[p^{T_k}(\xi), q^{T_k}(\xi);\ (y^{h,T_k}(\xi), \gamma^{h,T_k}(\xi))_{h\in H}] = [\bar{p}(\xi), \bar{q}(\xi), (\bar{y}^h(\xi), \bar{\gamma}^h(\xi))_{h\in H}].$$

From the definition of Φ^h and $\Phi^{h,T}$, for every $h \in H$ we have that $(\bar{y}^h(\xi))_{\xi \in D} \in C^h(\bar{p}, \bar{q})$. Additionally, as limit allocations are cluster points, node by node, of equilibria in finite-horizon economies, market clearing follows. Thus, to prove that $[\bar{p}(\xi), \bar{q}(\xi), (\bar{y}^h(\xi)_{h\in H}]_{\xi \in D}$ is a competitive equilibrium of economy $\mathcal{E}$, we must show solely that for every agent $h \in H$ the allocation $\bar{y}^h = (\bar{y}^h(\xi))_{\xi\in D}$ is optimal with respect to $C^h(\bar{p}, \bar{q})$.

Lemma 6.5. *For every $h \in H$, $U^h(\bar{x}) \geq U^h(\tilde{x})$ for every $\tilde{y} \in C^h(\bar{p}, \bar{q})$.*

Proof. Consider the allocation $\hat{y}$ defined as

$$\hat{y}(\mu) = \begin{cases} \bar{y}^h(\mu) & \text{if } \mu \neq \xi \\ \tilde{y}(\xi) & \text{if } \mu = \xi \end{cases}.$$

Assumption A3.b allows us to assert that $\hat{y} \in C^h(\bar{p}, \bar{q})$. Moreover, and due to very similar arguments to those exposed in the proof of Lemma 1, given the sequence $\{(p^{T_k}, q^{T_k})\} \to (\bar{p}, \bar{q})$ and for $k^* \in \mathbb{N}$ sufficiently large, there exists a sequence $\{\hat{y}^{T_k}\}_{k \geq k^*}$ complying with $\hat{y}^{T_k} \in C^{h,T_k}(p^{T_k}, q^{T_k})$ and $\hat{y}^{T_k} \to \hat{y}$.

Thus, for a node $\xi \in D$ and $T \in \mathbb{N}$ large enough as to ensure that $J^T(\mu) = J(\mu)$ at every $\mu \leq \xi$, inequality (1) allows us to assert that

$$u^h(\xi, \hat{x}^T(\xi)) - u^h(\xi, x^{h,T}(\xi)) \leq \gamma_\xi^{h,T} g_\xi^h(\hat{y}^T(\xi), y^{h,T}(\xi^-); p^T, q^T) + \sum_{\mu \in \xi^+} \gamma_\mu^{h,T} g_\mu^h(y^{h,T}(\mu), \hat{y}^T(\xi); p^T, q^T)$$

Taking the limit as $T = T_k$ goes to infinity retrieves the following expression:

$$u^h(\xi, \tilde{x}(\xi)) - u^h(\xi, \bar{x}^h(\xi)) \leq \bar{\gamma}_\xi^h g_\xi^h(\tilde{y}(\xi), \bar{y}^h(\xi^-); \bar{p}, \bar{q}) + \sum_{\mu \in \xi^+} \bar{\gamma}_\mu^h g_\mu^h(\bar{y}^h(\mu), \tilde{y}(\xi); \bar{p}, \bar{q}).$$

As both $\bar{y}^h$ and $\hat{y}$ are budget feasible, adding up this inequality over sub-tree $D^N(\xi_0)$ for $N \in \mathbb{N}$ leads to:

$$U^{h,N}(\tilde{x}) - U^{h,N}(\bar{x}^h) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \bar{\gamma}_\mu^h g_\mu^h(\bar{y}^h(\mu), \tilde{y}(\mu^-); \bar{p}, \bar{q}).$$

From the fact that $\tilde{y}$ complies with the short sales constraint at every $\xi \in D$ it follows that

$$U^{h,N}(\tilde{x}) - U^{h,N}(\bar{x}^h) \leq \sum_{\mu \in D_{N+1}(\xi_0)} \bar{\gamma}_\mu^h \left(\bar{p}(\mu)\bar{x}^h(\mu) + \bar{q}(\mu)\left(\bar{\theta}(\mu) - \bar{\varphi}(\mu)\right) + \kappa \bar{p}(\mu) w^h(\mu)\right). \tag{3}$$

Now, for node $\xi \in D^T(\xi_0)$ consider allocation $\hat{y}^T \in \Phi^{h,T}(p^T, q^T)$ defined as

$$\hat{y}^T(\mu) = \begin{cases} y^{h,T}(\mu) & \text{if } \mu \neq \xi \\ (0,0,0) & \text{if } \mu = \xi \end{cases}.$$

Assumption A3.b ensures $\hat{y}^T \in \Phi^{h,T}(p^T, q^T)$, whereas inequality (1) implies both

$$\begin{aligned} &\gamma_\xi^{h,T}\left(p^T(\xi)x^{h,T}(\xi) + q^T(\xi)\left(\theta^{h,T}(\xi) - \varphi^{h,T}(\xi)\right)\right) \leq \\ u^h(\xi, x^{h,T}(\xi)) + \sum_{\mu \in D^{T-1}(\xi_0)} &\gamma_\mu^{h,T}\left(p^T(\mu)x^{h,T}(\mu) + q^T(\mu)\left(\theta^{h,T}(\mu) - \varphi^{h,T}(\mu)\right)\right) \end{aligned}$$

and

$$\gamma_\xi^{h,T}\left(p^T(\xi)x^{h,T}(\xi) + q^T(\xi)\left(\theta^{h,T}(\xi) - \varphi^{h,T}(\xi)\right)\right) \leq u^h(\xi, x^{h,T}(\xi))$$

for nodes $\xi \in D^{T-1}(\xi_0)$ and $\xi \in D_T(\xi_0)$ respectively. Thus, Assumption A1 lets us assert that:

$$\sum_{\xi \in D_{N+1}(\xi_0)} \gamma_\xi^{h,T} \left(p^T(\xi)x^{h,T}(\mu) + q^T(\xi)\left(\theta^{h,T}(\xi) - \varphi^{h,T}(\xi)\right)\right) \leq \sum_{\mu \in D^T(\xi_0) \setminus D^N(\xi_0)} u^h(\mu, W(\mu)), \;\; \forall T > N+1$$

Taking the limit for $T = T_k$ leads to

$$\sum_{\mu \in D_{N+1}(\xi_0)} \bar{\gamma}_\mu^h \left(\bar{p}(\mu)\bar{x}^h(\mu) + \bar{q}(\mu)\left(\bar{\theta}(\mu) - \bar{\varphi}(\mu)\right)\right) \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)),$$

which in turn implies that

$$U^{h,N}(\tilde{x}) - U^{h,N}(\bar{x}^h) \leq \sum_{\mu \in D \setminus D^N(\xi_0)} u^h(\mu, W(\mu)) + \kappa \sum_{\mu \in D_{N+1}(\xi_0)} \bar{\gamma}_\mu^h \bar{p}(\mu) w^h(\mu). \tag{4}$$

Assumption A1 implies that the first term on the right hand side of inequality (4) tends to zero as N goes to infinity, whereas we claim equation (2) ensures that the second term tends to zero with N as well. Indeed, for any $\overline{T} \in \mathbb{N}$ the following holds:

$$\lim_{T_k \to +\infty} \left[\sum_{\xi \in D^{\overline{T}}(\xi_0)} \gamma_\xi^{h,T} p^T(\xi) w^h(\xi)\right] = \sum_{\xi \in D^{\overline{T}}(\xi_0)} \bar{\gamma}_\xi^h \bar{p}(\xi) w^h(\xi) \leq U^h(W).$$

As this inequality holds for any $\overline{T} \in \mathbb{N}$, it must also holds in the limit, and thus, we learn that $\sum_{\xi \in D(\xi_0)} \bar{\gamma}_\xi^h \bar{p}(\xi) w^h(\xi) \leq U^h(W)$. This implies our assertion (otherwise, the previous sum would not converge). Therefore, the sequence $\{U^{h,N}(\tilde{x})\}_{N \in \mathbb{N}}$ converges, as it is both non-decreasing and bounded by $U^h(\bar{x}^h) + 1$, for all $N > N^*$ and $N^* \in \mathbb{N}$ large enough.

Hence, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $U^{h,N}(\tilde{x}) \leq U^{h,N}(\bar{x}^h) + \varepsilon, \;\; \forall N \geq N_\varepsilon$. In turn, this implies that

$$U^h(\tilde{x}) \leq U^h(\bar{x}^h) + \varepsilon, \;\; \forall \varepsilon > 0,$$

which ends the proof. □

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