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ESTABILIDAD DE SOLUCIONES TIPO SOLITON PARA CIERTAS ECUACIONES DISPERSIVAS NO LINEALES

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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Este trabajo consiste principalmente en dos resultados matemáticos, basados en el estudio de ecuaciones dispersivas no lineales, la estabilidad de ciertas soluciones de las mismas, como así también la posible explosión en tiempo finito.

En una primera parte, Capítulo 1, presentamos una breve introducción a los tópicos tratados en esta memoria. Se hace especial énfasis en la descripción de los conceptos de ecuación dispersiva, buen colocamiento, 2-solitones, estabilidad y explosión.

En el Capítulo 2 probaremos que las soluciones de tipo 2-soliton de la ecuación de sine-Gordon (SG) son orbitalmente estables en el espacio de energía, el espacio natural para resolver este problema. Las soluciones que estudiamos son los 2-*kink*, *kink-antikink* y *breather* de SG. Con el objetivo de probar este resultado, utilizaremos las transformaciones de Bäcklund implementadas gracias al Teorema de la Función Implícita. Estas transformaciones nos permitirán reducir el problema de estabilidad para cada una de la soluciones, al caso de la solución cero. Probaremos estos resultados siguiendo el espíritu de un paper de M. A. Alejo y C. Muñoz, que trata el caso de la ecuación de Korteweg-de Vries modificada. Sin embargo, más adelante veremos que el caso de la ecuación de SG presenta varias nuevas dificultades dado el carácter vectorial de sus soluciones. Este resultado mejora los anteriores probados por M. A. Alejo et al., y entrega una primera demostración rigurosa de la estabilidad de los 2-solitones de la ecuación de SG en el espacio de energía.

En el Capítulo 3 nuestro principal objetivo será estudiar nuevas propiedades de blow-up dispersivo para el sistema de Schrödinger-Korteweg-de Vries. Más precisamente, probaremos explosión para datos iniciales en $H^{2-}(\mathbb{R}) \times H^{3/2-}(\mathbb{R})$, como consecuencia de mostrar previamente una nueva propiedad de persistencia del flujo asociado al sistema, establecida sobre ciertos espacios de Sobolev con pesos fraccionarios cuidadosamente escogidos.

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This work consists mainly of two mathematical results based on the study of non-linear dispersive equations, the stability of certain solutions of them, as well as the possible finite time blow-up.

In a first part, Chapter 1, we present a brief introduction to the topics discussed in this thesis. Special emphasis is placed on the description of the concepts of dispersive equation, well-posedness, 2-solitons, stability and blow-up.

In Chapter 2 we prove that the 2-soliton solutions of the sine-Gordon equation (SG) are orbitally stable in the energy space, the natural space to solve this problem. The solutions that we studied are the 2-*kink*, *kink-antikink* and *breather* of SG. In order to prove this result, we will use Bäcklund transformations implemented by the Implicit Function Theorem. These transformations will allow us to reduce the stability problem for each of the solutions to the case of the zero solution. We shall prove these results in the same spirit of a paper done by M. A. Alejo and C. Muñoz, which deals with the case of the modified Korteweg-de Vries equation. However, we shall see that SG presents several new difficulties due to the vectorial character of its solutions. This result improves the previous ones proved by Alejo et al., and provides a first rigorous demonstration of the stability of the 2-solitons of the SG equation in the energy space.

In chapter 3 our main goal is to study new properties of dispersive blow-up for the Schrödinger-Korteweg-de Vries system. More precisely, we prove dispersive blow-up for initial data in $H^{2-}(\mathbb{R}) \times H^{3/2^-}(\mathbb{R})$, as a consequence of previously proving a new persistence property of the flow associated to the system. This persistence property is established on certain Sobolev spaces with well-chosen fractional weights.

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Chapter 1

Introduction

1.1 Linear Dispersive Equations

A constant-coefficient linear dispersive PDE is an equation of the form

$$\partial_t u(t,x) = Lu(t,x), \quad u(0,x) = u_0(x), \tag{1.1.1}$$

where the field $u : \mathbb{R} \times \mathbb{R}^d \to V$ takes values in a finite-dimensional Hilbert space V, and L is a skew-adjoint constant coefficient differential operator in space, taking the form

$$Lu(x) := \sum_{|\alpha| \leq k} c_\alpha \partial_x^\alpha u(x),$$

where $k \geq 1$ is an integer (the order of the differential operator), $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+^n$ ranges over all multi-indices with $|\alpha| := \alpha_1 + ... + \alpha_d$ less than or equal to k, ∂_x^α is the partial derivative

$$\partial_x^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d},$$

and $c_\alpha \in \mathrm{End}(V)$ are coefficients that do not depend on x, where $\mathrm{End}(V)$ is the set of linear transformation $V \to V$. This operator is classically only defined on k-times continuously differentiable functions, but we may extend it to distributions or functions in other function spaces in the usual manner, thus we can talk about both classical and weak (distributional) solutions to the equation.

In order to give a complete notion of what a dispersive partial differential equation is, consider the one-dimensional frame. We look for *plane wave solutions* of the form

$$u(t,x) = Ae^{i(kx-\omega t)},$$

where A, k and ω are constants representing the amplitude, the wavenumber, and the frequency, respectively. Hence u will be a solution of (1.1.1) if and only if

$$\omega + \sum_{\alpha \leq k} c_\alpha i^{\alpha-1} k^\alpha = 0.$$

This equation for ω is called the *dispersion relation*. A commonly used defining criteria for dispersive equations is that $\omega(\xi)$ is a real valued function of ξ and $\frac{d^2\omega}{d\xi^2} \neq 0$. In the physical context this means that different frequencies in this equation will tend to propagate at different velocities, thus when time evolves, the different waves disperse in the medium, with the result that a single hump breaks into wave-trains. Under this notion the transport equation

$$\partial_t u = -v \cdot \nabla u, \quad u(0,x) = u_0(x),$$

and the wave equation

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x),$$

are not dispersive. This is due to the fact that the first one moves all frequencies with the same velocity (and is thus a degenerate case of a dispersive equation), while the last one is such that the frequency of a wave determines the direction of propagation, but not the speed.

The relation between ω and ξ also characterizes the plane wave motion. Consider, for example, the linear Schrödinger equation

$$iu_t + u_{xx} = 0,$$

and the plane wave $u(t,x) = e^{i(\xi x - \omega t)}$. Then, $u(t,x)$ is a solution of the equation if and only if the dispersion relation $\omega = \xi^2$ holds. Note that in this case ω is a real valued function of the frequency. An interesting notion is the phase velocity of the waves which is defined by

$$\nu_p(\xi) := \frac{\omega}{\xi}.$$

With this definition one can re-write the solution as:

$$u(t,x) = e^{i\xi(x-\nu_p(\xi)t)} = u(0, x - \nu_p(\xi)t),$$

and conclude that the wave travels with velocity $\nu_p(\xi)$. In particular, large frequency data travel faster than smaller ones. Another related notion is the group velocity,

$$\nu_g(\xi) := \omega'(\xi) = \frac{d\omega}{d\xi},$$

which describes how a frequency localized bump function around ξ moves. To see why the group velocity is different than the phase velocity, let $g(x) := u(0,x)$ be concentrated around a frequency ξ_0. Using the Fourier transform, we can write the solution as

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-it\xi^2} \widehat{g}(\xi) d\xi.$$

Approximating $\omega(\xi) = \xi^2$ around ξ_0 we have,

$$\begin{aligned}\xi^2 &= \omega(\xi_0) + (\xi - \xi_0)\omega'(\xi_0) + O\big((\xi - \xi_0)^2\big) \\ &= 2\xi_0\xi - \xi_0^2 + O\big((\xi - \xi_0)^2\big),\end{aligned}$$

and ignoring the error term, we obtain

$$\begin{aligned}u(t,x) &\approx \frac{e^{it\xi_0^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-2it\xi_0\xi} \widehat{g}(\xi) d\xi \\ &= e^{it\xi_0^2} g(x - 2\xi_0 t).\end{aligned}$$

This suggests that the bump function moves with group velocity $\omega'(\xi_0) = 2\xi_0$, which is twice the phase velocity at ξ_0.

Comparing the plain wave solutions above with the plain wave solutions of the heat equation $u_t - u_{xx} = 0$, we see that $\omega = -i\xi^2$ is complex valued. Therefore, each nonzero mode decays exponentially in time. Spatially localized solutions of dispersive equations also decay in time (at a slower rate) due to a more involved reason. When no boundary conditions are imposed, different frequency components of the data evolve with different velocities, and hence spread out in space as time increases. This causes the solution to decay in time at a polynomial rate. This can also be understood by noting the L^2 conservation $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$ for the solutions of the linear Schrödinger equation. A spatially localized smooth bump of height $\sqrt{k}$ and base $1/k$ would spread out to an interval of length kt at time t and hence by the L^2 conservation one expects the solution to decay to the height $\frac{1}{\sqrt{kt}}$.

In contrast, on bounded domains such as the torus $\mathbb{T}$ the solution cannot spread out, but instead different frequency components rotate around the torus with different velocities. This makes time averages smoother because of a subtle cancellation between different frequency components.

For a more detailed discussion about these concepts see [20, 54, 82].

1.2 Semilinear equations and Well-Posedness Theory

In partial differential equations theory, the fact that one has to work in infinite dimensional spaces leads to difficulties even in the definition of a *well-posed* problem. This fact was first put forward by J. Hadamard in the beginning of the twentieth century and we recall his classical example of an *ill-posed problem* (see [76] for a more detailed explanation of these examples).

We aim to solving the Cauchy problem in the upper half-plane for the Laplace equation:

$$\begin{cases} u_{tt} + u_{xx} = 0, & \text{in } D = \mathbb{R} \times (0, \infty) \\ u(x,0) = 0, & u_t(x,0) = f(x). \end{cases} \tag{1.2.1}$$

By Schwarz reflexion principle, the data f has to be analytic if u is required to be continuous on $\overline{D}$. We consider the sequence of initial data ϕ_n, $n \in \mathbb{N}$:

$$\phi_n(x) = e^{-\sqrt{n}} n \sin(nx), \quad \phi_0(x) \equiv 0.$$

It is easily checked that for any $k \geq 0$,

$$\phi_n \to 0$$

in the C^k-norm. In fact, for any $\varepsilon > 0$, there exist $N_{\varepsilon,k} \in \mathbb{N}$ such that

$$\sup_x \sum_{j \leq k} |\phi_n^{(j)}(x)| \leq \varepsilon \quad \text{if } n \geq N_{\varepsilon,k}.$$

Note that ϕ_n oscillates more and more as $n \to +\infty$. On the other hand one finds by separation of variables that for any $n \in \mathbb{N}$, the Cauchy problem (1.2.1) with $f = \phi_n$ has the unique solution

$$\nu_n(x,t) = e^{-\sqrt{n}} \sin(nx) \sinh(nt),$$

and of course $\nu_0(x,t) \equiv 0$. Things seem going well but for any $t_0 > 0$ (even arbitrary small), and any $k \in \mathbb{N}$,

$$\sup_x |\nu_n^{(k)}(x, t_0)| = n^k e^{-\sqrt{n}} \sinh(nt_0) \to +\infty$$

as $n \to +\infty$. In other words, the map $T : \varphi_n \to \nu_n(\cdot, t_0)\cdot$ is not continuous in any C^k-topology. This catastrophic instability to short waves is called *Hadamard instability.* It is totally different from instability phenomenon one encounters in ODE problems, for example, the exponential growth in time of solutions.

On the other hand, the fact that norms in an infinite dimensional space are not equivalent implies of course that the asymptotic behavior of solutions to PDE's depends on the topology as shows the following example.

Let consider the Cauchy problem

$$\begin{cases} u_t + x u_x = 0, \\ u(x,0) = u_0, \end{cases}$$

where $u_0 \in \mathcal{S}(\mathbb{R})$ is in the Scharwtz class. The L^p norms of the space derivatives $u_x^{(k)} = \partial_x^k u$ of the solution $u(x,t) = u_0(xe^{-t})$ are

$$\|u_x^{(k)}\|_{L^p} = e^{t(\frac{1}{p} - k)} \|u_0^{(k)}\|_{L^p}.$$

For instance, the L^∞ norm of u is constant while the L^p norms, $1 \le p < \infty$ grow exponentially. On the other hand all the homogeneous Sobolev norms $\dot{W}^{k,p}$, $k \ge 1$, $p > 1$ decay exponentially to zero.

One can state a general concept of a *well-posed problem* for any PDE problem $(\mathcal{P})$. Let be given three topological vector spaces (most often Banach spaces) U, V and F, with $U \subset V$. Let f be a vector of data (initial conditions, boundary data, forcing terms, etc) and u be the solution of $(\mathcal{P})$. One says that $(\mathcal{P})$ is *well-posed* (in the considered functional framework) if the three following conditions are fullfilled:

1. For any $f \in F$, there exists a solution $u \in U$ of $\mathcal{P}$.
2. This solution is unique in U.
3. The mapping $f \in F \mapsto u \in V$ is continuous from F into V.

To be more specific, consider for instance scalar Cauchy problems of type:

$$\begin{cases} \partial_t u = iLu(t) + F(u(t)), \\ u(0) = u_0. \end{cases} \tag{1.2.2}$$

Here $u = u(t,x)$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The operator L is a skew-adjoint operator defined in Fourier variables by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where the symbol p is a real function (not necessary a polynomial) and F is a nonlinear term depending on u and possibly on its space derivatives. The linear part of (1.2.2) thus generates an unitary group $S(t)$ in $L^2(\mathbb{R}^n)$ (and in all Sobolev spaces) which is unitary equivalent to $\hat{u}_0 \mapsto e^{itp(\xi)}\hat{u}_0$.

Classical examples of equations in the form (1.2.2) involve the nonlinear Schrödinger equation (NLS), where u is complex-valued

$$iu_t + \Delta u + |u|^p u = 0,$$

the *generalized* Korteweg-de Vries equation

$$u_t + u^p u_x + u_{xxx} = 0,$$

or the Benjamin-Ono equation

$$u_t + uu_x - \mathcal{H}u_{xx} = 0,$$

where $\mathcal{H}$ is the Hilbert transform, and many of the classical semilinear dispersive equations.

Definition 1.2.1. *The Cauchy problem* (1.2.2) *is said to be locally well-posed in X, where X is a function space, if for every $u_0 \in X$ there exist $T > 0$ and a unique solution of* (1.2.2)*:*

$$u \in C([0,T); X).$$

Moreover, the map data solution, $u_0 \mapsto u(t,\cdot)$, locally defined from X into $C([0,T);X)$, is continuous.

Therefore, our notion of well-posedness includes existence, uniqueness and persistence (the solution $u(t)$ belongs to the same space as the initial data and its time trajectory describes a curve on it). Thus, the solution flow of the equation defines a dynamical system in X. In the case that T can be taken arbitrarily large, we shall say that the equation is *globally well-posed* in X. In some *critical cases* T does not depend only on the X-norm of u_0, but on u_0 itself in a more complicated way.

For a more detailed discussion about well-posedness see [76] and [82].

1.3 Nonlinear Stability of Solutions

The study of perturbations of solitons or solitary waves leads to the introduction of the concepts of *orbital* and *asymptotic stability.* In order to give a complete notion of these concepts, consider for instance the *sine-Gordon* (SG) equation:

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \sin u(t,x) = 0, \qquad t,x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R} \vee \mathbb{C}. \tag{1.3.1}$$

This equation has *soliton* (and *multisoliton*) solutions given by the following family of real functions: Let $x_0 \in \mathbb{R}$ be a shift parameter and $\beta \in (-1,1)$ a velocity parameter. Then, the soliton solutions of the SG equation are given by

$$Q(t,x) := Q_\gamma(x - \beta t - x_0), \quad Q_\gamma(x) := 4\arctan\left(e^{\gamma(x+x_0)}\right), \quad \gamma := \frac{1}{\sqrt{1-\beta^2}}.$$

The following definition gives us the precise statement of *nonlinear orbital stability* for the standard Sobolev spaces $H^s(\mathbb{R})$.

Definition 1.3.1 (Orbital Stability)**.** *Let $s \in \mathbb{R}$ fixed. We say that Q_γ is nonlinearly stable in H^s if for any $u_0 \in H^s$ such that $\|u_0 - Q_\gamma\|_{H^s} < \alpha$, then one has that the corresponding solution of the equation* (1.3.1) *satisfies*

$$\sup_{t\in\mathbb{R}} \|(u,\partial_t u)(t) - (Q_\gamma(\cdot - \rho(t)), (\partial_t Q_\gamma)(\cdot - \rho(t)))\|_{H^s \times H^{s-1}} \lesssim \alpha, \tag{1.3.2}$$

for some $\rho(t) \in \mathbb{R}$. Otherwise, we say that Q_γ is unstable.

In other words, a small perturbation of a soliton solution stays close enough to a soliton with a corrected translation parameter. Note that the constant involved in (1.3.2) does not depende on t and α. The parameter $\rho(t)$ is absolutely necessary since if $\gamma \sim \gamma'$, with $|\gamma - \gamma'| = \alpha \ll 1$, one has $\|Q_\gamma - Q_{\gamma'}\|_{H^s} \sim \alpha \ll 1$, but the corresponding solution $\|Q_\gamma(\cdot - \beta t) - Q_{\gamma'}(\cdot - \beta' t)\|_{H^s} \sim 1$, as $t \to \infty$. In that sense, we say that (1.3.2) is a sort of *orbital* stability.

Orbital stability fo the SG kink was proved in $H^1 \times L^2$ by Henry-Perez-Wreszinski [37].

On the other hand, asymptotic stability concerns with the small residue given by the stability result above mentioned. It is legitimate to wonder whether $u(t)$ should actually converge to a soliton in some sense. Consider for example the generalized Korteweg-de Vries equation:

$$u_t + (u_{xx} + u^m)_x = 0, \qquad t, x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R}.$$

This equation has soliton solutions of the form

$$u(t,x) := Q_c(x - ct), \quad Q_c(s) = c^{\frac{1}{m-1}} Q(\sqrt{c}s), \quad Q(x) = \left(\frac{m+1}{2\cosh^2\left(\frac{(m-1)x}{2}\right)} \right)^{\frac{1}{m-1}}.$$

Thus, two solitons of similar velocities $Q_c(x - \rho(t))$ and $Q_{\tilde{c}}(x - \rho(t))$, with $c \sim c'$ but different, are always at a positive distance in any Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. In addition, a standard classification result tell us that if $\|u(t) - Q_c(\cdot - x(t))\|_{H^1(\mathbb{R})} \to 0$ as $t \to +\infty$, for some $x(t)$, then $u(t)$ is a *pure soliton solution*. These two arguments suggest the non existence of a completely and general H^1-convergence to a soliton solution of a small perturbation of a soliton.

As Muñoz explain in [68], in order to solve this problem one needs to reformulate the asymptotic stability property, either by introducing some suitable weighted spaces [62], or by considering only local norms [49]. In this last formulation, one has the existence of $\beta > 0$ depending of α small enough and $c^+ > 0$ with $|c^+ - c| \lesssim \alpha$ such that

$$\|u(t) - Q_{c^+}(\cdot - \rho(t))\|_{H^1(x > \beta t)} \to 0,$$

as $t \to +\infty$. Moreover, $\lim_{t \to +\infty} \rho'(t) = c^+$. In other words, there is strong H^1-convergence near the soliton.

For a more detailed discusion about stability and asymptotic stability of solitons see [68, 5].

1.4 Why studying breathers

In some dispersive PDEs there is a special family of solutions called *Breathers*. **Breathers are oscillatory bound states**. They are periodic in time (after a suitable space shift) and localized in space real functions. Solution like breathers have become a canonical example of complexity in nonlinear *integrable* systems. Moreover, their surprising mixed behavior, combining oscillatory and soliton character has focused the attention of many researchers since thirty years ago (see [78, 15, 26]).

From the physical point of view, breather solutions seem to be relevant to localization-type phenomena in optics, condensed matter physics and biological processes [9]. They also play an important role in the modeling of freak and rogue waves events on surface gravity waves and also of internal waves in the stratified ocean, in Josephson junctions and even in nonlinear optics. See [1, 34, 33, 29] for a representative set of these examples.

From a mathematical point of view, breather solutions arise in different contexts. For instance, if we consider the modified Korteweg-de Vries (mKdV) equation:

$$u_t + (u_{xx} + u^3)_x = 0, \qquad t, x \in \mathbb{R}, \quad u(t,x) \in \mathbb{R},$$

in a geometrical setting, mKdV breathers appear in the evolution of closed planar curves, playing the role of smooth localized deformations traveling along the closed curve [5]. Moreover, it is interesting to point out that mKdV breather solutions have also been considered by Kenig, Ponce and Vega in their proof of the non-uniform continuity of the mKdV flow in the Sobolev spaces H^s for $s < \frac{1}{4}$ [46]. Furthermore, they should play an important role in the *soliton-resolution* conjecture, according to the analysis developed by Schuur in [78]. On the other hand, consider for example the sine-Gordon (sG) equation:

$$u_{tt} + u_{xx} + \sin u = 0, \qquad t, x \in \mathbb{R}, \quad u(t,x) \in \mathbb{K} = \mathbb{R} \vee \mathbb{C}.$$

In this case, SG breathers play an important role in the so-called asymptotic stability problem for the *kink* solution (see [79, 50, 51]).

It should be said that there is no universal definition for breathers. Although, there are definitions for some particular equations, for instance, in [7] you can find a definition that match the sine-Gordon case.

For a more detailed discusion of the relevance of breather solution see [7, 6, 5, 3].

1.5 Dispersive Blow-Up

Dispersive blow up is a phenomenon of focusing of smooth initial disturbances with finite mass (or finite energy, depending on the physical context) that relies upon the dispersion relation guaranteeing that, in the linear regime, different wavelengths propagate at different speeds. This is especially the case for models wherein the linear dispersion is unbounded, so that energy can be moved around at arbitrarily high speeds, but even bounded dispersion can exhibit this type of singularity formation [17].

Roughly speaking, dispersive blow up refers to the fact that the loss of some type of smoothness presented by some solutions is carried out by the linear part and not by the nonlinear term.

To be more concrete, consider the Cauchy problema for the linear (free) Schrödinger equation

$$i\partial_t u + \Delta u = 0, \qquad u(t=0) = u_0(x), \tag{1.5.1}$$

where $x \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. For $u_0 \in L^2(\mathbb{R}^n)$, elementary Fourier analysis shows the solution to this initial-value problem is

$$u(t,x) = e^{it\Delta}u_0(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-it|\xi|^2}\hat{u}_0(\xi)e^{i\xi\cdot x}d\xi. \tag{1.5.2}$$

Here, $\hat{u}_0$ denotes the Fourier transformed initial data,

$$\mathcal{F}u_0(\xi) \equiv \hat{u}_0(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}u_0(x)dx.$$

The corresponding inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$. From (1.5.2), it is immediately inferred that for any $s \in \mathbb{R}$, solutions lie in $C(\mathbb{R}, H^s)$ whenever u_0 lies in the L^2-based Sobolev space H^s. Moreover, the evolution preserves all these Sobolev-space norms, which is to say

$$\|u(t,\cdot)\|_{H^s(\mathbb{R}^n)} = \|u_0\|_{H^s(\mathbb{R}^n)},$$

for $t \in \mathbb{R}$. In certain applications of this model, the case $s = 0$ in the last formula corresponds to conservation of total mass in the underlying physical system.

However, in Theorem 2.1 of [18], it was shown that for any given point $(t_*, x_*) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists initial data $u_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the solution $u(t,x)$ of the corresponding initial-value problem for the free Schrödinger equation:

$$i\partial_t u + \Delta u = 0, \qquad u(t=0) = u_0(x), \tag{1.5.3}$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{(t_*, x_*)\}$, but

$$\lim_{(t,x)\to(t_*,x_*)} |u(t,x)| = +\infty.$$

This fact is referred to as **(finite-time) dispersive blow up**. The analogous phenomena also appears in other linear dispersive equations, such as the linear Korteweg-de Vries equation [14] and the free surface water waves system linearized around the rest state [18].

As Bona et al. explain in [17], at first sight, one would expect that nonlinear terms would destroy dispersive blow up. What is a little surprising is that even the inclusion of physically relevant nonlinearities in various models of wave propagation does not prevent dispersive blow up. Indeed, theory shows in some important cases that initial data leading to this focusing singularity under the linear evolution continues to blow up in exactly the same way when nonlinear terms are included. In [16], this was shown to be true for the Korteweg-de Vries equation, a model for shallow water waves and other simple wave phenomena. This result and analogous dispersive blow up theory in [18] for solutions of the one-dimensional nonlinear Schrödinger equations,

$$i\partial_t u + \partial_x^2 u \pm |u|^p u = 0, \quad u(x,0) = u_0(x),$$

where $x \in \mathbb{R}$ and $p \in (0,3)$, lead to the speculation that such focusing might be one road to the formulation of rogue waves in shallow and deep water and nonlinear optics (see [27, 28, 47, 80]).

The analysis put forward in [16] and [18] revolves around providing bounds on the nonlinear terms in a Duhamel representation of the evolution. Because the phenomenon is due to the linear terms in the equation, data of arbitrarily small size will still exhibit dispersive blow up, and indeed it can be organized to happen arbitrarily quickly. This emphasizes the linear aspect of these singularities and differentiates it from the blow up that occurs for some of the same models when the nonlinear term is focusing and sufficiently strong (see [81] for a general overview of this aspect of Schrödinger equations). Dispersive blow up thereby also serves to demonstrate ill-posedness of the considered models in L^∞-spaces.

1.6 Results of this Thesis

This thesis contains essentially two main results, which are part of the following two articles:

- C. Muñoz, and J. M. Palacios, *Nonlinear stability of 2-solitons of the Sine-Gordon equation in the energy space*, preprint arXiv:1801.09933, preprint sent for publication [69]. (Chapter 2.)
- F. Linares, and J. M. Palacios, *On the persistence properties of solutions to the Schrödinger-Korteweg-de Vries system, and applications to dispersive blow-up*, preprint 2018 [55]. (Chapter 3.)

1.6.1 First part: stability of Sine-Gordon 2-solitons

Consider the sine-Gordon equation in physical coordinates for a scalar field ϕ:

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0. \tag{1.6.1}$$

Here, $\phi = \phi(t,x)$ is a real or complex-valued function, and $(t,x) \in \mathbb{R}^2$. This equation has the following families of explicit solutions: For $x_1, x_2 \in \mathbb{R}$ shift parameters, $\beta \in (-1,1)$ a scaling parameter, and $\gamma = (1-\beta^2)^{-1/2}$ the Lorentz factor. We will study:

1. The sine-Gordon *breather* $B = B(t,x) = B(t,x;\beta,x_1,x_2)$ given by

$$B(t,x;\beta,x_1,x_2) = 4\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha(t+x_1))}{\cosh(\beta(x+x_2))}\right), \quad \alpha = \sqrt{1-\beta^2}, \quad \beta \neq 0,$$

   which represents a solution which is localized in space and oscillatory in time because of the parameter α.

2. The sine-Gordon *2-kink* $R = R(t,x)$, given by

$$R(t,x;\beta,x_1,x_2) = 4\arctan\left(\beta\frac{\sinh(\gamma(x+x_2))}{\cosh(\gamma(t+x_1))}\right), \quad \beta \neq 0,$$

   which represents the interaction of two SG kinks with speeds $\pm\beta$.

3. Finally, we shall consider the sine-Gordon *kink-antikink* $A = A(t,x)$, which is given by:

$$A(t,x;\beta,x_1,x_2) = 4\arctan\left(\frac{1}{\beta}\frac{\sinh(\gamma(t+x_1))}{\cosh(\gamma(x+x_2))}\right), \quad \beta \neq 0,$$

   which represents the elastic collision between a sine-Gordon kink and an anti-kink, with speeds $\pm\beta$.

The aim of this first part is to prove the following nonlinear stability of the 2-soliton solutions of the SG equation defined above.

Theorem 1.6.1 (Stability of 2-solitons in the energy space)**.** *The 2-solitons of SG* (1.6.1) *are nonlinearly stable under perturbations in the energy space $H^1 \times L^2$. More precisely, there exist $C_0 > 0$ and $\eta_0 > 0$ such that the following holds. Let (ϕ, ϕ_t) be a solution of* (1.6.1)*, with initial data (ϕ_0, ϕ_1) such that*

$$\|(\phi_0,\phi_1) - (D,D_t)(0,\ \cdot\ ;\beta,0,0)\|_{H^1\times L^2} < \eta, \tag{1.6.2}$$

for some $0 < \eta < \eta_0$ sufficiently small, and where $(D,D_t)(t,\ \cdot\ ;\beta,0,0)$ is a 2-soliton (breather, 2-kink or kink-antikink). Then, there are shifts $x_1(t), x_2(t) \in \mathbb{R}$ well-defined and differentiable such that

$$\sup_{t\in\mathbb{R}} \|(\phi(t),\phi_t(t)) - (D,D_t)(t,\ \cdot\ ;\beta,x_1(t),x_2(t))\|_{H^1\times L^2} < C_0\eta. \tag{1.6.3}$$

Moreover, we have

$$\sup_{t\in\mathbb{R}} |x_1'(t)| + |x_2'(t)| \lesssim C_0\eta.$$

This theorem generalizes to the case of 2-solitons solutions the previous result of Henry-Perez-Wreszinski concerning the nonlinear stability of the 1-soliton solution of the sine-Gordon equation (see [37]).

1.6.2 Second part: Dispersive blow-up and persistence properties for the Schrödinger-Korteweg de Vries system

Consider the Initial Value Problem associated to the Schrödinger-Korteweg-de Vries system in $\mathbb{R}_t \times \mathbb{R}_x$,

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = \alpha uv, \qquad t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.6.4}$$

where $u = u(t,x)$ is a complex-valued function and $v(t,x)$ is a real-valued function. The aim of this second part is to prove the following two results concerning the solutions of the IVP (1.6.4). First, we prove persistence of regularity and decay.

Theorem 1.6.2 (Persistence)**.** *Let $s \in \mathbb{R}$, $r_1, r_2 \geq 0$ be fixed parameters. Let*

$$(u_0, v_0) \in \left(H^{s+1/2}(\mathbb{R}) \cap L^2(|x|^{r_1}dx)\right) \times \left(H^s(\mathbb{R}) \cap L^2(|x|^{r_2}dx)\right),$$

with

$$s > \frac{3}{4}, \quad s + \frac{1}{2} > r_1, \quad \text{and} \quad s > 2r_2.$$

Then there exist $T = T(\|u_0\|_{s+\frac{1}{2}} + \|v_0\|_s) > 0$ and a unique solution $(u(t), v(t))$ of the IVP (1.6.4) satisfying

$$u \in C([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{r_1}dx)), \quad v \in C([0,T]; H^s(\mathbb{R}) \cap L^2(|x|^{r_2}dx)).$$

Furthermore,

$$\begin{aligned} \|D_x^s \partial_x u\|_{L_x^\infty L_T^2} + \|D_x^{s-\frac{1}{2}} \partial_x v\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} &< \infty, \\ \|u\|_{L_x^2 L_T^\infty} + \|v\|_{L_x^2 \| L_T^\infty} &< \infty, \\ \|\partial_x u\|_{L_T^4 L_x^\infty} + \|\partial_x v\|_{L_T^4 L_x^\infty} &< \infty. \end{aligned}$$

Moreover, given $T' \in (0,T)$, the map data solution is Lipschitz continuous.

The proof of this theorem is based on the contracting mapping principle. Moreover, as a consequence of Theorem 1.6.2, we are able to prove the existence of *dispersive blow-up* for the Schrödinger-Korteweg-de Vries system.[1]

Theorem 1.6.3 (Dispersive blow-up)**.** *There exists initial data*

$$u_0 \in C^\infty(\mathbb{R}) \cap H^{2^-}(\mathbb{R}), \quad v_0 \in C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}),$$

[1] Here we adopt the following notation, for $s \geq 0$

$$H^{s^+}(\mathbb{R}) = \bigcup_{s'>s} H^{s'}(\mathbb{R}), \qquad H^{s^-}(\mathbb{R}) = \bigcup_{0\leq s'<s} H^{s'}(\mathbb{R}).$$

such that the following holds: there exists $t^* \in [0,T]$ *such that the corresponding solution* $(u,v)(\cdot,\cdot)$ *of the IVP 1.6.4:*

$$u \in C([0,T] : H^{2^-}(\mathbb{R})), \qquad v \in C([0,T] : H^{3/2^-}(\mathbb{R})),$$

provided by Theorem 1.6.2 is such that

$$u(t^*,\cdot) \notin H^2(\mathbb{R}), \quad v(t^*,\cdot) \notin C^1(\mathbb{R}).$$

This theorem generalizes the results for each of the components itself, this time proved for the coupled NLS-KdV system (see [17, 56]).

It should be said that in Theorem 1.6.3 there is no dispersive blow-up for component of the solution related to the nonlinear Schrödinger equation, since neither the solution nor any of its derivatives is loosing regularity in terms of the L^∞-norm (at least we were not able to prove that). Nevertheless, we shall prove a smoothing effect by a quarter of derivative associated to its corresponding nonlinear term. The main issue to establish the dispersive blow-up for the NLS solution is to deal with the **coupled term** (in KdV), which has both worse regularity and worse persistence in weighted Sobolev spaces than the NLS part. More explicitly, to show that the NLS solution has dispersive blow-up we must prove that the solution $u(t,x)$ is in $H^{\frac{5}{2}+\varepsilon}$, even when it has a term which is only in $H^{\frac{3}{2}^-}(\mathbb{R})$.

Organization of this work

In what follows we shall divide the manuscript into two chapters, each corresponding to a different article.

Chapter 2 is organized as follows: In Section 2.1 the problem and some previous literature are introduced. In Section 2.2 we introduce the *Bäcklund Transformation* and prove some preliminary Lemmas. In this section we also state and prove some basic properties of the sine-Gordon equation. Sections 2.3 and 2.4 are devoted to study the relationship between 1-soliton and 2-solitons solutions in terms of the Bäcklund transformation. Sections 2.6, 2.7, 2.8, 2.9 and 2.10 are devoted to study invertibility properties of the relationship between solitons solutions stablished in the previous sections. Finally, in Section 2.11 we prove the Main Theorem 2.1.1.

Chapter 3 is organized as follows: Section 3.1 we introduce the problem and previous literature. In Section 3.2 we state a series of results needed in the remainder of this chapter. The dispersive blow-up for each of the linear equation is established in Section 3.3. In this section we show how to construct the initial data which shall develop dispersive blow-up. Section 3.4 is devoted to prove the Main Theorem 3.1.2. This last result is used to complete the analysis in Section 3.3. The dispersive blow-up for the coupled system (Theorem 3.1.3) is then proved in the last Section 3.5.

Chapter 2

Nonlinear stability of 2-solitons of the sine-Gordon in the energy space

In this chapter we prove that 2-soliton solutions of the sine-Gordon equation (SG) are orbitally stable in the natural energy space of the problem. The solutions that we study are the *2-kink, kink-antikink and breather* of SG. In order to prove this result, we will use Bäcklund transformations implemented by the Implicit Function Theorem. These transformations will allow us to reduce the stability of the three solutions to the case of the vacuum solution, in the spirit of previous results by Alejo and the first author, which was done for the case of the scalar modified Korteweg-de Vries equation. However, we will see that SG presents several difficulties because of its vector valued character. Our results improve those in Alejo et al., and give a first rigorous proof of the stability in the energy space of SG 2-solitons.

This chapter is part of the article

- C. Muñoz, and J. M. Palacios, *Nonlinear stability of 2-solitons of the Sine-Gordon equation in the energy space*, preprint arXiv:1801.09933, preprint enviado a referato [69].

2.1 Introduction and Main results

2.1.1 The model

This chapter considers the sine-Gordon (SG) equation in physical coordinates for a scalar field ϕ:

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0. \tag{2.1.1}$$

Here, $\phi = \phi(t,x)$ is a real or complex-valued function, and $(t,x) \in \mathbb{R}^2$. SG has been extensively studied in differential geometry (constant negative curvature surfaces), as well as relativistic field theory and soliton integrable systems. The interested reader may consult the monograph by Lamb [53, section 5.2], and for more details about the Physics of SG, see e.g. Dauxois and Peyrard [24].

Using the standard notation $\vec{\phi} := (\phi, \phi_t)$, corresponding to a wave-like dynamics, and given data $\vec{\phi}(t=0)$, a natural energy space for (2.1.1) is $(H^1 \times L^2)(\mathbb{R};\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), as it is revealed by the conservation laws Energy and Momentum, respectively:

$$E[\vec{\phi}](t) = \frac{1}{2}\int_{\mathbb{R}} (\phi_x^2 + \phi_t^2)(t,x)dx + \int_{\mathbb{R}} (1 - \cos\phi(t,x))dx = E[\vec{\phi}](0), \tag{2.1.2}$$

and

$$P[\vec{\phi}](t) = \frac{1}{2}\int_{\mathbb{R}} \phi_t(t,x)\phi_x(t,x)dx = P[\vec{\phi}](0), \tag{2.1.3}$$

although spaces slightly different may be considered, using the fact that $\vec{\phi}$ need not be necessarily zero at infinity for E and P being well-defined. However, real-valued solutions of (2.1.1) that initially are in $H^1 \times L^2$ are preserved for all time. Additionally, they are globally well-defined thanks to standard Strichartz estimates and the fact that $\sin(\cdot)$ is a smooth bounded function. In what follows, we will assume that we have a real-valued solution of (2.1.1) (in vector form) $\vec{\phi} \in C(\mathbb{R}; H^1 \times L^2)$, although complex-valued solutions, or solutions with nonzero values at infinity will be also considered in some places of this paper.

Solutions of (2.1.1) are known for satisfying several symmetry properties: shifts in space and time, as well as *Lorentz boosts*: for each $\beta \in (-1,1)$, given $\vec{\phi}(t,x) = (\phi,\phi_t)(t,x)$ solution, then

$$(\phi,\phi_t)_\beta(t,x) := (\phi,\phi_t)\big(\gamma(t-\beta x), \gamma(x-\beta t)\big), \quad \gamma := (1-\beta^2)^{-1/2}, \tag{2.1.4}$$

is another solution of (2.1.1). The parameter γ is called Lorentz scaling factor, having an important role in what follows.

2.1.2 2-soliton solutions

In this chapter we will show stability of a certain class of particular solutions of 2-soliton type for (2.1.1). In order to explain better the 2-solitons forms that we will study, first we need to understand the notion of 1-soliton. This is an exact solution of (2.1.1) usually referred as the *kink* [53]:

$$Q(x) := 4\arctan(e^{x+x_0}), \quad x_0 \in \mathbb{R}.$$

Thanks to (2.1.4), it is possible to define a kink of arbitrary speed $\beta \in (-1,1)$. From the integrability of SG, interactions between kinks are elastic, i.e. they are solitons [53]. Also, $-Q(x)$ is another stationary solution of SG, usually called *anti-kink*. It is well-known that $(Q,0)$ is stable under small perturbations in the energy space $(H^1 \times L^2)(\mathbb{R})$, see Henry-Perez-Wreszinski [37].

These kinks are also locally asymptotically stable in the energy space under odd perturbations, a property that follows from the proofs in [50], as well as some of the methods exposed in this chapter.

A 2-soliton is formally a solution that behaves as the elastic interaction between two forms of 1-soliton, and under different scalings (or speeds, real or complex-valued). This structure remains valid for all time. The 2-solitons considered in this paper are the following (see Lamb [53, pp. 145–149]):

Notation: Let $x_1, x_2 \in \mathbb{R}$ be shift parameters, $\beta \in (-1,1)$ be a scaling parameter, and $\gamma = (1-\beta^2)^{-1/2}$ be the Lorentz factor. We will study

1. First of all, the SG *breather* $B = B(t,x) = B(t,x;\beta,x_1,x_2)$ given by
$$B(t,x;\beta,x_1,x_2) = 4\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha(t+x_1))}{\cosh(\beta(x+x_2))}\right), \quad \alpha = \sqrt{1-\beta^2}, \quad \beta \neq 0, \tag{2.1.5}$$
which represents a solution (even in $x+x_2$) which is localized in space and oscillatory in time because of the parameter α. This solution can be made arbitrarily small provided β is small, and has energy $E[B,B_t] = 16\beta$, see [53, 7]. Additionally, B is a counterexample to the asymptotic stability property of the vacuum solution under small perturbations (except if perturbations are odd), as was discussed in [51] (see Fig. 2.2). Similarly, in [7] it was conjectured, thanks to numerical evidence, that this solution is stable.

2. Second, the stability of the *2-kink* $R = R(t,x)$, given by
$$R(t,x;\beta,x_1,x_2) = 4\arctan\left(\beta\frac{\sinh(\gamma(x+x_2))}{\cosh(\gamma(t+x_1))}\right), \quad \beta \neq 0, \tag{2.1.6}$$
which represents the interaction of two SG kinks with speeds $\pm\beta$, with limits as $x \to \pm\infty$ equal to -2π and 2π respectively[1] (i.e., R do not decay to zero). Note that R is *odd wrt the axis* $x = -x_2$. See Fig. 2.3 for more details.

3. Finally, we shall consider the *kink-antikink* $A = A(t,x)$:
$$A(t,x;\beta,x_1,x_2) = 4\arctan\left(\frac{1}{\beta}\frac{\sinh(\gamma(t+x_1))}{\cosh(\gamma(x+x_2))}\right), \quad \beta \neq 0, \tag{2.1.7}$$

[1]Note that the classic 2-kink should connect the states 0 and 4π, but the substraction of 2π to a solution of SG is still a solution.

which represents the elastic collision between a SG kink and an anti-kink, with speeds $\pm\beta$. This solution decays to zero at infinity, and it is even wrt $x + x_2$. See Fig. 2.4.

These three time depending functions are exact solutions of SG that have two modes of independent variables, in contrast with the kink Q which has only one. Another type of degenerate solitons, not treated in this paper, can be found in [21].

2.1.3 Main results

The purpose of this paper is to give a first proof of the fact that the three 2-solitons of SG are stable under perturbations well-defined in the natural *energy space* associated to the problem, this without *any additional decay assumption*, and no use of the Inverse Scattering methods. Consecuently, our results extends those of Henry-Perez-Wreszinski [37] to the case of SG 2-solitons, and allow possible extensions to the case of three or more soltions. Our main theorem is the following:

Theorem 2.1.1 (Stability of 2-solitons in the energy space)**.** *The 2-solitons of SG* (2.1.1) *are nonlinearly stable under perturbations in the energy space $H^1 \times L^2$. More precisely, there exist $C_0 > 0$ and $\eta_0 > 0$ such that the following holds. Let (ϕ, ϕ_t) be a solution of* (2.1.1)*, with initial data (ϕ_0, ϕ_1) such that*

$$\|(\phi_0, \phi_1) - (D, D_t)(0, \ \cdot\ ; \beta, 0, 0)\|_{H^1\times L^2} < \eta, \tag{2.1.8}$$

for some $0 < \eta < \eta_0$ sufficiently small, and where $(D, D_t)(t, \ \cdot\ ; \beta, 0, 0)$ is a 2-soliton (breather (2.1.5)*, 2-kink* (2.1.6) *or kink-antikink* (2.1.7)*). Then, there are shifts $x_1(t), x_2(t) \in \mathbb{R}$ well-defined and differentiable such that*

$$\sup_{t\in\mathbb{R}} \|(\phi(t), \phi_t(t)) - (D, D_t)(t, \ \cdot\ ; \beta, x_1(t), x_2(t))\|_{H^1\times L^2} < C_0\eta. \tag{2.1.9}$$

Moreover, we have

$$\sup_{t\in\mathbb{R}} |x_1'(t)| + |x_2'(t)| \lesssim C_0\eta.$$

Remark 2.1.1. Note that in Theorem 2.1.1 we do not specify the space where (ϕ, ϕ_t) are posed, this because $(R, R_t)(t)$ in (2.1.6) does not belong to $H^1 \times L^2$. However, it is possible to show local well-posedness (LWP) in each of the three cases involved in this chapter, such that $H^1 \times L^2$ perturbations are naturally allowed.

Rigorous proofs of stability of SG 2-solitons are not known in the literature, as far as we can understand. Formal descriptions of the dynamics can be found in [30], and in [78], under additional assumptions of rapid decay for the initial data. These last two results are strongly based on the Inverse Scattering Theory (IST), therefore the extra decay is essential. Theorem 2.1.1 do not require this assumptions, only perturbation data in the energy space (and probably even less regular).

A first result on conditional stability (only for the SG breather case) can be found in Alejo et. al. [7]. In this work it was shown that, under certain spectral conditions, breathers are stable under $H^2 \times H^1$ perturbations. This result follows some of the ideas in [3, 4], works dealing with the modified KdV case, a simpler breather. Additionally, in the same work, the spectral conditions required in [7] where numerically verified in a large set of parameters for the problem. Theorem 2.1.1 improves the results in [7] in two senses: first, it establishes the stability of 2-solitons for SG in a rigorous form; and second, the proof works in the energy space of the problem, without any additional assumption.

Although 2-solitons are stable, it is known that breathers should disappear under perturbations of the equation itself. In that sense, the literature is huge, from the physical and mathematical point of view. Nonexistence results for breathers can be found in [15, 48, 23, 26, 52, 86], under different conditions on the nonlinearity. Recently, Kowalczyk, Martel and C. Muñoz [51] showed nonexistence of odd breathers for scalar field equations with odd nonlinearities, with no other assumptions on the nonlinearity, except being C^1. However, in [10] it was shown existence of breathers in scalar field equations with non-homogeneous coefficients. Finally, [67] considers in a rigorous way the stability question for Peregrine and Ma breathers, showing that they are indeed unstable, even if the equation is locally well-posed.

On the other hand, stability and asymptotic stability results for N-solitons of several dispersive nonlinear equations, are largely available in the literature. Concerning the NLS equation, see [41, 74]. We also refer to the works [72, 58, 59, 60, 61] for the case of solitons and 2-solitons in gKdV equations. The works [79, 50] are deeply concerned with scalar field equations, and [71] deals with the Benjamin-Ono equation and its 2-solitons. See also [73] for the study of 2-solitons in Dirac type equations. Finally, Alejo et al. [6] worked the case of periodic mKdV breathers.

In this work we extend the ideas introduced in [5] to the SG case. More precisely, we will study the Bäcklund Transformations (BT) between two solutions (ϕ, φ) for SG, and fixed parameter a:

$$\begin{aligned} \varphi_x - \phi_t &= \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) + a\sin\left(\frac{\varphi-\phi}{2}\right), \\ \varphi_t - \phi_x &= \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) - a\sin\left(\frac{\varphi-\phi}{2}\right). \end{aligned}$$

These two equations allow to describe the dynamics of 2-solitons using the reduction of complexity induced by the BT. These ideas have been successfully implemented in several contexts: Hoffman and Wayne [38] used BT to show abstract results of stability. Next, Mizumachi and Pelinovsky [64] showed L^2 stability of the NLS soliton using this approach. The case in [5] was the first where a BT was used in the case of breathers.

In the case of SG 2-solitons, the dynamics is more complex than usual, because, unlike mKdV in [5], here we will work with a system for (ϕ, ϕ_t), and not only scalar equations. This fact makes proofs more involved, in the sense that we must work with systems at every step of the proof.

In order to fix ideas, let us consider the case of the SG breather (2.1.5). First of all, we will need to work with complex-valued solutions. We will introduce the kink function (K, K_t):

$$(K, K_t)(t, x) := \left(4\arctan\left(e^{\beta x + i\alpha t}\right), \frac{4i\alpha e^{\beta x+i\alpha t}}{1+e^{2(\beta x+i\alpha t)}}\right).$$

This complex-valued SG solution is connected to zero via a BT of parameter $\beta - i\alpha$. We have (Lemma 2.3.5):

$$\begin{aligned} K_x &= \frac{1}{\beta - i\alpha}\sin\left(\frac{K}{2}\right) + (\beta - i\alpha)\sin\left(\frac{K}{2}\right), \\ K_t &= \frac{1}{\beta - i\alpha}\sin\left(\frac{K}{2}\right) - (\beta - i\alpha)\sin\left(\frac{K}{2}\right). \end{aligned} \tag{2.1.10}$$

On the other hand, the complex-valued kink is a singular solution to SG, in the sense that it blows up (in L^∞ norm) in a sequence of times t_k, without accumulation point (Remark 2.3.3). Even under this problem, it is possible to define a dynamics for perturbations of (K, K_t), for times $t \neq \tilde{t}_k \sim t_k$, and proving a kind of manifold stability:

Corollary 2.1.2. *Let $(K, K_t)(t)$ be a complex-valued kink profile such that at time $t = 0$ does not blow up. For each $(u_0, s_0) \in (H^1 \times L^2)(\mathbb{R};\mathbb{C})$ sufficiently small and such that Corollary 2.8.3 holds, there is a unique solution of SG*

$$(\phi, \phi_t)(t) = (\widetilde{K}, \widetilde{K}_t)(t) + (u, s)(t), \quad (u, s)(t) \in (H^1 \times L^2)(\mathbb{R};\mathbb{C}),$$

where $(\widetilde{K}, \widetilde{K}_t)(t)$ is a complex-valued profile suitably modified via modulations in time. This solution is well-defined for each $t \neq \tilde{t}_k$, a sequence of times unbounded and without accumulation points, close to each t_k. Similarly, this solution blows-up at time $t = \tilde{t}_k$.

The advantage of introducing the profiles (K, K_t) in Theorem 2.1.1 is the following: this profile is connected to the breather (B, B_t) via a new BT of parameter $\beta + i\alpha$ (Proposition 2.4.4):

$$\begin{aligned} B_x - K_t &= \frac{1}{\beta + i\alpha}\sin\left(\frac{B+K}{2}\right) + (\beta + i\alpha)\sin\left(\frac{B-K}{2}\right), \\ B_t - K_x &= \frac{1}{\beta + i\alpha}\sin\left(\frac{B+K}{2}\right) - (\beta + i\alpha)\sin\left(\frac{B-K}{2}\right). \end{aligned} \tag{2.1.11}$$

An important portion of this chapter deals with the generalization of these two identities, (2.1.10) and (2.1.11), to the case of time-dependent perturbations of the breather (B, B_t). However, this procedure presents several difficulties. First, a correct connection between neighborhoods of the breather and the zero solution. (Proposition 2.6.1). The obtained function near zero must be real-valued, otherwise our method does not work (see Theorem 2.1.3 below). Next, we need to come back to the original solution for any possible time. This step presents several difficulties since in general the BT are not invertible for free and we need to impose additional conditions, in order to find the correct dynamics (Proposition 2.7.4). Another problem comes from the fact that the method falls down when the time t approaches $\tilde{t}_k$. We need another method for proving stability at those times, based in energy estimates (subsection 2.11). Some of these problems were already solved in [5] for the mKdV case, however here we propose another method, more intuitive and based in the uniqueness returned by the modulation in time (Corollary 2.5.3). Through this chapter, we will give a rigorous meaning to the diagram of Fig. 2.1 which describes the proof of Theorem 2.1.1, based in two "descents" and two "ascents" from perturbations of the breather (or any 2-soliton), to the zero solution, which is orbitally stable thanks to a respective Cauchy theory.

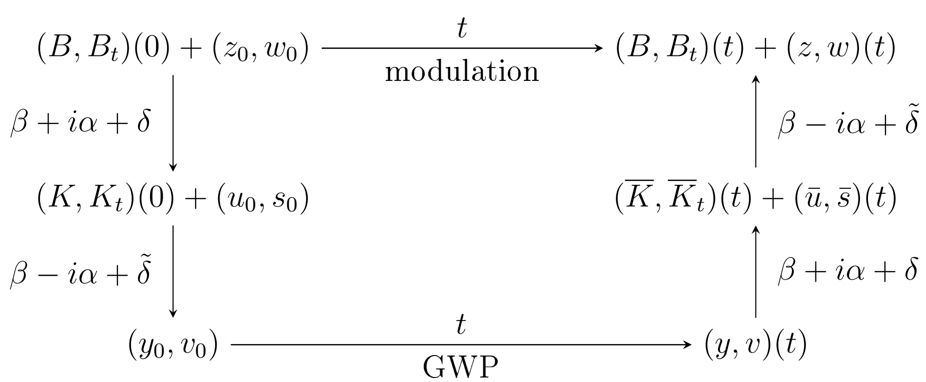

Figure 2.1: Diagram of proof of Theorem 2.1.1 in the breather case (B,B_t), for times different to $\tilde{t}_k$. Here, $(\overline{K},\overline{K}_t)(t)$ represents the complex conjugate of the function $(K,K_t)(t)$ at time t.

A first consequence of the (rigorous) methods associated to Fig. 2.1 is the following:

Theorem 2.1.3 (Real-valued character of the double BT)**.** *Under hypotheses from Theorem 2.1.1 in the breather case (B,B_t), the LHS of the diagram in Fig. 2.1 is well-defined and the functions $(y_0,v_0)\in H^1\times L^2$ obtained are necessarily real-valued, even if (u_0,s_0) are not.*

For more details about this result, see section 2.8 and Corollary 2.8.4. Another consequence of the same diagram in Fig. 2.1 is the following method of computing the energy and momentum of each involved perturbation of a 2-soliton:

Corollary 2.1.4 (Energy and momentum identities)**.** *Under the consequences of Theorem 2.1.1, and according to the diagram in Fig. 2.1, the following identities are satisfied for* *each time $t \in \mathbb{R}$:*

$$E[B+z, B_t+w] = E[y,v] + 8(\beta + \operatorname{Re}\delta)\left(1 + \frac{1}{1+2\beta\operatorname{Re}\delta + 2\alpha\operatorname{Im}\delta + |\delta|^2}\right), \quad (2.1.12)$$

$$P[B+z, B_t+w] = P[y,v] + 4(\beta + \operatorname{Re}\delta)\left(\frac{1}{1+2\beta\operatorname{Re}\delta + 2\alpha\operatorname{Im}\delta + |\delta|^2} - 1\right). \quad (2.1.13)$$

Completely similar identities are satisfied by the other 2 cases: $D = A$ or $D = R$, after suitable modifications.

Finally, but not least, let us mention the fundamental work by Merle and Vega [63], who introduced the idea of the nonlinear Miura transformation for the KdV soliton, proving L^2 stability. See also [8, 65, 66] for other generalizations of this idea to other contexts.

Organization of this chapter

section 2.2 presents preliminaries that we will need along this paper. section 2.3 introduces the basic notions of complex-valued kink profile, and section 2.4 describes in detail the 2-soliton profiles. section 2.5 deals with modulation of 2-solitons, and section 2.6 is devoted to the connection between breathers and the zero solution. In section 2.7 we study the corresponding inverse dynamics, while in section 2.8 we prove Theorem 2.1.3. section 2.9 and 2.10 study the 2-kink and kink-antikink cases, and section 2.11 is devoted to the proof of Theorem 2.1.1 and Corollary 2.1.4.

2.2 Preliminaries

The purpose of this section is to announce a set of simple but fundamental properties that we will need through this chapter. Proofs are not difficult to establish or being checked in the literature.

2.2.1 Bäcklund Transformation

As a first step, let us write (2.1.1) in matrix form, that is $\vec{\phi} = (\phi, \phi_t) = (\phi_1, \phi_2)$, in such a form that (2.1.1) reads now

$$\begin{cases} \partial_t \phi_1 = \phi_2 \\ \partial_t \phi_2 = \partial_x^2 \phi_1 - \sin\phi_1. \end{cases} \quad (2.2.1)$$

Formally speaking, we will say that a *profile* is a function of the form $(\phi_1, \phi_2)(x)$, independent of time, which under a particular time-dependent transformation, may be exact or approximate solution of (2.2.1) described above. Although not a rigorous definition, this one will allow us to understand in a better form the concepts described below. Now we introduce the Bäcklund transformation that we will use in this chapter. Recall that $\dot{H}^1$ represents the closure of C_0^∞ under the norm $\|\partial_x \cdot\|_{L^2}$.

Definition 2.2.1 (Bäcklund Transformation)**.** *Let $a \in \mathbb{C}$ be fixed. Let $\vec{\phi} = (\phi, \phi_t)(x)$ be a function defined in $\dot{H}^1(\mathbb{C}) \times L^2(\mathbb{C})$. We will say that $\vec{\varphi}$ in $\dot{H}^1(\mathbb{C}) \times L^2(\mathbb{C})$ is a* **Bäcklund transformation** *(BT) of $\vec{\phi}$ by the parameter a, denoted*

$$\mathbb{B}(\vec{\phi}) \xrightarrow{a} \vec{\varphi}, \tag{2.2.2}$$

if the triple $(\vec{\phi}, \vec{\varphi}, a)$ satisfies the following equations, for all $x \in \mathbb{R}$:

$$\varphi_x - \phi_t = \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) + a\sin\left(\frac{\varphi-\phi}{2}\right), \tag{2.2.3}$$

$$\varphi_t - \phi_x = \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) - a\sin\left(\frac{\varphi-\phi}{2}\right). \tag{2.2.4}$$

Remark 2.2.1. Note that if the triple $(\vec{\phi}, \vec{\varphi}, a)$ satisfies Definition 2.2.1, then so $(\vec{\varphi}, \vec{\phi}, -a)$ does, and we have $\mathbb{B}(\vec{\varphi}) \xrightarrow{-a} \vec{\phi}$. In that sense, the order between ϕ and φ will not play an important role.

Remark 2.2.2. Note also that we do not ask for uniqueness for φ in Definition 2.2.1. However, in this articule we will construct functions φ which are uniquely defined as BT (with fixed parameter) of a unique ϕ.

Remark 2.2.3 (Different BT for SG). In [53] (2.1.1) is written in "laboratory coordinates" (u, v) given by

$$u := \frac{x-t}{2}, \quad v := \frac{x+t}{2} \iff x = u + v, \quad t = v - u.$$

Under these new variables SG (2.1.1) reads $\sigma_{uv} = \sin\sigma$, where $\sigma(u, v) := \phi(t, x)$. It is not difficult to show that in this case, (2.2.3)-(2.2.4) are equivalent to the equations

$$\frac{1}{2}(\sigma_u + \widetilde{\sigma}_u) = a\sin\left(\frac{\sigma-\widetilde{\sigma}}{2}\right), \quad \frac{1}{2}(\sigma_v - \widetilde{\sigma}_v) = \frac{1}{a}\sin\left(\frac{\sigma+\widetilde{\sigma}}{2}\right),$$

which are precisely the BT appearing in [53].

The following result is standard in the literature, justifying the introduction of the BT (2.2.3)-(2.2.4).

Lemma 2.2.2. *If $(\vec{\phi}, \vec{\varphi})$ are $C^2_{t,x}$ functions related via a BT* (2.2.3)-(2.2.4)*, then both solve* (2.2.1).

Proof. By smoothness, it is enough to check that both solve (2.1.1). Now, we prove that φ solves SG. We take derivative in (2.2.3) and (2.2.4), so that

$$\varphi_{tt} - \varphi_{xx} = \frac{1}{2a}(\varphi_t - \varphi_x + \phi_t - \phi_x)\cos\left(\frac{\varphi+\phi}{2}\right)$$

$$+\frac{a}{2}(\phi_t+\phi_x-\varphi_t-\varphi_x)\cos\left(\frac{\varphi-\phi}{2}\right)$$
$$=-\sin\left(\frac{\varphi-\phi}{2}\right)\cos\left(\frac{\varphi+\psi}{2}\right)-\sin\left(\frac{\varphi+\phi}{2}\right)\cos\left(\frac{\varphi-\phi}{2}\right)=-\sin(\varphi).$$

Similarly, one easily proves that ϕ satisfies SG. □

Using a standard density argument, the previous result can be extended to solutions defined in the energy space, and satisfying the Duhamel formulation for SG. Now, we will need the following notion, generalization of Definition 2.2.1.

Definition 2.2.3 (Bäcklund Functionals)**.** *Let $(\varphi_0,\,\varphi_1,\,\phi_0,\,\phi_1,\,a)$ be data in a space $X(\mathbb{K})$ to be chosen later, with $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$. Let us define the functional with vector values $\mathcal{F}:=(\mathcal{F}_1,\mathcal{F}_2)$, where $\mathcal{F}=\mathcal{F}(\varphi_0,\,\varphi_1,\,\phi_0,\,\phi_1,\,a)\in L^2(\mathbb{K})\times L^2(\mathbb{K})$, given by the system:*

$$\mathcal{F}_1\big(\varphi_0,\,\varphi_1,\,\phi_0,\,\phi_1,\,a\big):=\varphi_{0,x}-\phi_1-\frac{1}{a}\sin\left(\frac{\varphi_0+\phi_0}{2}\right)-a\sin\left(\frac{\varphi_0-\phi_0}{2}\right),\tag{2.2.5}$$
$$\mathcal{F}_2\big(\varphi_0,\,\varphi_1,\,\phi_0,\,\phi_1,\,a\big):=\varphi_1-\phi_{0,x}-\frac{1}{a}\sin\left(\frac{\varphi_0+\phi_0}{2}\right)+a\sin\left(\frac{\varphi_0-\phi_0}{2}\right).\tag{2.2.6}$$

2.2.2 Conserved quantities

The following result establishes a direct relation between the BT (2.2.3)-(2.2.4) and the conserved quantities (2.1.2)-(2.1.3), without using the original equation (2.2.1).

Lemma 2.2.4 (BT and conserved quantities)**.** *Let*[2] (ϕ,ϕ_t), $(\varphi,\varphi_t)\in(L^\infty\cap\dot H^1)(\mathbb{R};\mathbb{C})\times L^2(\mathbb{R};\mathbb{C})$ *be functions related by a BT with parameter a, i.e., such that*

$$\mathbb{B}(\phi,\phi_t)\xrightarrow{a}(\varphi,\varphi_t).$$

Let us additionally assume that

$$\ell^+_\pm(t):=\lim_{x\to\pm\infty}\left(1-\cos\left(\frac{\varphi+\phi}{2}\right)\right),\quad \ell^-_\pm(t):=\lim_{x\to\pm\infty}\left(1-\cos\left(\frac{\varphi-\phi}{2}\right)\right),\tag{2.2.7}$$

are well-defined and finte. Then we have

$$E[\vec\varphi]=E[\vec\phi]+\frac{2}{a}(\ell^+_+-\ell^+_-)(t)+2a(\ell^-_+-\ell^-_-)(t),\tag{2.2.8}$$
$$P[\vec\varphi]=P[\vec\phi]+\frac{1}{a}(\ell^+_+-\ell^+_-)(t)-a(\ell^-_+-\ell^-_-)(t),\tag{2.2.9}$$

where E and P are the corresponding energy and momentum defined in (2.1.2)-(2.1.3).

[2]Note that not necessarily ϕ,φ belong to L^2.

A simple consequence of the previous result is the following:

Corollary 2.2.5 (Parametric rigidity of BT versus Energy and Momentum)**.** *Under the hypotheses from previous lemma, let us assume in addition that ϕ, φ are such that $E[\vec{\varphi}]$, $E[\vec{\phi}]$ and $P[\vec{\varphi}]$ and $P[\vec{\phi}]$ are conserved in time $t \in \mathbb{R}$ (see subsection 2.2.3 below for details). Then, if both $(\ell_+^+ - \ell_-^+)(t)$ and $(\ell_+^- - \ell_-^-)(t)$ do not depend on time, the parameter "a" in the BT cannot depend on time.*

Remark 2.2.4. In general, all solutions considered in this chapter do satisfy the hypotheses in Corollary 2.2.5. Even more, if the corresponding limits in (2.2.7) are constant (our case), then the BT parameter a cannot depend on time.

Proof of Lemma 2.2.4. First we prove that (2.2.8) holds. For that, adding the squares of equations (2.2.3) and (2.2.4), we have

$$\varphi_x^2 + \varphi_t^2 + \phi_x^2 + \phi_t^2 - 2(\varphi_x\phi_t + \varphi_t\phi_x) = \frac{2}{a^2}\sin^2\left(\frac{\varphi+\phi}{2}\right) + 2a^2\sin^2\left(\frac{\varphi-\phi}{2}\right).$$

Now, replacing the values of φ_x and φ_t given by equations (2.2.3) and (2.2.4),

$$\begin{aligned}\varphi_x^2 + \varphi_t^2 + \phi_x^2 + \phi_t^2 = {} & 2\phi_t\left(\phi_t + \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) + a\sin\left(\frac{\varphi-\phi}{2}\right)\right)\\ & + 2\phi_x\left(\phi_x + \frac{1}{a}\sin\left(\frac{\varphi+\phi}{2}\right) - a\sin\left(\frac{\varphi-\phi}{2}\right)\right)\\ & + \frac{2}{a^2}\sin^2\left(\frac{\varphi+\phi}{2}\right) + 2a^2\sin^2\left(\frac{\varphi-\phi}{2}\right).\end{aligned}$$

Simplifying and gathering similar terms,

$$\begin{aligned}\varphi_x^2 + \varphi_t^2 - \phi_x^2 - \phi_t^2 = {} & \frac{2}{a}(\phi_t + \phi_x)\sin\left(\frac{\varphi+\phi}{2}\right) + 2a(\phi_t - \phi_x)\sin\left(\frac{\varphi-\phi}{2}\right)\\ & + \frac{2}{a^2}\sin^2\left(\frac{\varphi+\phi}{2}\right) + 2a^2\sin^2\left(\frac{\varphi-\phi}{2}\right).\end{aligned} \tag{2.2.10}$$

Now, adding and sustracting φ_x in the RHS of (2.2.10), and integrating

$$\begin{aligned}&\int_{\mathbb{R}} \varphi_x^2 + \varphi_t^2 - \phi_x^2 - \phi_t^2\\ &= \frac{2}{a}\int_{\mathbb{R}}(\phi_t - \varphi_x)\sin\left(\frac{\varphi+\phi}{2}\right) + 2a\int_{\mathbb{R}}(\phi_t - \varphi_x)\sin\left(\frac{\varphi-\phi}{2}\right)\\ &\quad + \frac{2}{a^2}\int_{\mathbb{R}}\sin^2\left(\frac{\varphi+\phi}{2}\right) + 2a^2\int_{\mathbb{R}}\sin^2\left(\frac{\varphi-\phi}{2}\right)\\ &\quad + \frac{4}{a}\int_{\mathbb{R}}\partial_x\left(1 - \cos\left(\frac{\varphi+\phi}{2}\right)\right) + 4a\int_{\mathbb{R}}\partial_x\left(1 - \cos\left(\frac{\varphi-\phi}{2}\right)\right).\end{aligned}$$

Recall that $\mathbb{B}(\phi,\phi_t) \xrightarrow{a} (\varphi,\varphi_t)$. Using (2.2.7), we conclude

$$\begin{aligned}&\int_{\mathbb{R}} \varphi_x^2+\varphi_t^2-\phi_x^2-\phi_t^2\\ &= -4\int_{\mathbb{R}} \sin\left(\frac{\varphi+\phi}{2}\right)\sin\left(\frac{\varphi-\phi}{2}\right)+\frac{4}{a}(\ell_+^+-\ell_-^+)(t)+4a(\ell_+^- -\ell_-^-)(t). \end{aligned}\tag{2.2.11}$$

Lastly, multiplying (2.2.11) by $\frac{1}{2}$ and using that $\cos\varphi-\cos\phi=-2\sin(\frac{\varphi+\phi}{2})\sin(\frac{\varphi-\phi}{2})$, we arrive to the identity

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}} \varphi_x^2+\varphi_t^2+\int_{\mathbb{R}}\big(1-\cos\varphi\big)\\ &=\frac{1}{2}\int_{\mathbb{R}} \phi_x^2+\phi_t^2+\int_{\mathbb{R}}\big(1-\cos\phi\big)+\frac{2}{a}(\ell_+^+-\ell_-^+)(t)+2a(\ell_+^- -\ell_-^-)(t),\end{aligned}$$

which finally proves (2.2.8). Similarly, we will show (2.2.9). Multiplying (2.2.3) and (2.2.4) we have

$$\varphi_x\varphi_t+\phi_x\phi_t-\varphi_x\phi_x-\varphi_t\phi_t=\frac{1}{a^2}\sin^2\left(\frac{\varphi+\phi}{2}\right)-a^2\sin^2\left(\frac{\varphi-\phi}{2}\right).$$

Replacing φ_x and φ_t given by (2.2.3) and (2.2.4) we obtain

$$\begin{aligned}\varphi_x\varphi_t=\phi_x\phi_t&+\frac{1}{a}\big(\phi_x+\phi_t\big)\sin\left(\frac{\varphi+\phi}{2}\right)+a\big(\phi_x-\phi_t\big)\sin\left(\frac{\varphi-\phi}{2}\right)\\ &+\frac{1}{a^2}\sin^2\left(\frac{\varphi+\phi}{2}\right)-a^2\sin^2\left(\frac{\varphi-\phi}{2}\right).\end{aligned}\tag{2.2.12}$$

Finally, using once again that $\mathbb{B}(\phi,\phi_t)\xrightarrow{a}(\varphi,\varphi_t)$, multiplying (2.2.12) by $\frac{1}{2}$ and integrating, we get

$$\frac{1}{2}\int_{\mathbb{R}}\varphi_x\varphi_t=\frac{1}{2}\int_{\mathbb{R}}\phi_x\phi_t+\frac{1}{a}(\ell_+^+-\ell_-^+)(t)-a(\ell_+^- -\ell_-^-)(t),$$

which finally ends the proof. □

2.2.3 Local well-posedness

The purpose of this paragraph is to announce the LWP results that we will need through this chapter. First of all, note that the energy (2.1.2) can be written as

$$E[\vec{\phi}](t)=\frac{1}{2}\int_{\mathbb{R}}(\phi_x^2+\phi_t^2)(t,x)dx+\int_{\mathbb{R}}\sin^2\left(\frac{\phi}{2}\right)(t,x)dx. \tag{2.2.13}$$

Then, naturally the largest energy space for SG is $H^1_{\sin}\times L^2$ [25], where

$$H^1_{\sin}:=\{\phi_0\in\dot{H}^1\ :\ \sin\phi_0\in L^2\}.$$

Since we will consider small perturbations in this paper, $\phi_0 \in H^1$ small enough implies $\phi_0 \in H^1_{\sin}$.

Theorem 2.2.6 (GWP for real-valued data)**.** *Let* $(\phi_0, \phi_1) \in (H^1 \times L^2)(\mathbb{R})$ *be initial data. Then there exists a unique solution* $\vec{\phi} \in C(\mathbb{R}, (H^1 \times L^2)(\mathbb{R}))$ *(in the Duhamel sense) of* (2.2.1)*. Moreover, both the momentum* P *in* (2.1.3) *and the energy* E *in* (2.1.2) *are conserved by the flow, and we have*

$$\sup_{t\in\mathbb{R}} \|(\phi, \phi_t)(t)\|_{H^1\times L^2} \lesssim \|(\phi_0, \phi_1)\|_{H^1\times L^2}, \tag{2.2.14}$$

with involved constants independent of time.

Proof. This result is direct from the Duhamel formulation for (2.2.1), the conservation of energy, plus the fact that $\sin(\cdot)$ is smooth and bounded if the argument is real-valued. □

We will also need a LWP result for complex-valued initial data.

Theorem 2.2.7 (LWP for complex-valued data)**.** *Let* $(\phi_0, \phi_1) \in (H^1 \times L^2)(\mathbb{C})$ *be complex-valued initial data. Then there exists* $T = T((\phi_0, \phi_1)) > 0$ *and a unique solution* $\vec{\phi} \in C((-T, T), (H^1 \times L^2)(\mathbb{C}))$ *(in the Duhamel sense) of* (2.2.1)*. Moreover, both the momentum* P *in* (2.1.3) *as well as the energy* E *in* (2.1.2) *are conserved by the flow during* $(-T, T)$*.*

Remark 2.2.5. Note that SG with complex-valued data do have finite time blow-up solutions. See Lemma 2.3.3 for more details on this problem.

Proof. The same proof for the real-valued case works for the complex-valued one. Only global existence is not satisfied. □

Finally, we will need a last result for the case of nontrivial values at infinity, more precisely for the case of the 2-kink R in (2.1.6).

Theorem 2.2.8 (Global well-posedness for real valued data with nontrivial values at infinity, see e.g. [63, 25])**.** *Let* (ϕ_0, ϕ_1) *be initial data such that for* $R = R(t, x; \beta, x_1, x_2)$ *fixed 2-kink as in* (2.1.6)*, and* R_t *its corresponding time derivative, one has*

$$\|(\phi_0, \phi_1) - (R, R_t)(t = 0)\|_{(H^1\times L^2)(\mathbb{R})} < +\infty.$$

Then there exists a unique real-valued solution (ϕ, ϕ_t) *for SG such that* $(\phi, \phi_t) - (R, R_t)(t) \in C(\mathbb{R}, (H^1 \times L^2)(\mathbb{R}))$ *(in the Duhamel sense). Moreover, the momentum* P *in* (2.1.3) *as well as the energy* E *in* (2.1.2) *are conserved by the flow.*

2.3 Real and complex valued kink profiles

2.3.1 Definitions

The following concept is standard in the literature.

Definition 2.3.1 (Real-valued kink profile). *Let $\beta \in (-1,1)$, $\beta \neq 0$, and $x_0 \in \mathbb{R}$ be fixed parameters. we define the real-valued kink profile $\vec{Q} := (Q, Q_t)$ with speed β as*

$$Q(x) := Q(x;\beta,x_0) = 4\arctan\left(e^{\gamma(x+x_0)}\right), \qquad \gamma := (1-\beta^2)^{-1/2}, \tag{2.3.1}$$

and

$$Q_t(x) := Q_t(x;\beta,x_0) = \frac{-4\beta\gamma e^{\gamma(x+x_0)}}{1+e^{2\gamma(x+x_0)}} = \frac{-2\beta\gamma}{\cosh(\gamma(x+x_0))}. \tag{2.3.2}$$

Remark 2.3.1. This profile (Q, Q_t), although not an exact solution of (2.2.1), can be understood as follows: for each $(t,x) \in \mathbb{R}^2$, $(t,x) \mapsto (Q,Q_t)(x;\beta,x_0-\beta t)$ is an exact solution of (2.2.1), moving with speed β.

With small but essential modifications, we introduce a complex-valued version of the previous kink profile.

Definition 2.3.2 (Complex-valued kink profile). *Let $\beta \in (-1,1)\setminus\{0\}$, $\alpha = \sqrt{1-\beta^2}$, be fixed, and consider shift parameters $x_1, x_2 \in \mathbb{R}$. We define the complex-valued kink profile (K, K_t) with zero speed as*

$$K(x) := K(x;\beta,x_1,x_2) = 4\arctan\left(e^{\beta(x+x_2)+i\alpha x_1}\right), \tag{2.3.3}$$

and

$$K_t(x) := K_t(x;\beta,x_1,x_2) = \partial_{x_1}K(x;\beta,x_1,x_2) = \frac{4i\alpha e^{\beta(x+x_2)+i\alpha x_1}}{1+e^{2(\beta(x+x_2)+i\alpha x_1)}}. \tag{2.3.4}$$

Remark 2.3.2 (Multi-valued profiles). Note that K is well-defined for all $x \in \mathbb{R}$ as a univalued function with complex values, provided we choose a particular Riemann surface for the $\arctan z$ function. In this chapter we will assume that arctan possesses two branch cuts in $C := (-i\infty, -i] \cup [i, i\infty)$, in such a way that it remains univalued and analytic in $\mathbb{C} - C$. However, in this paper this bad behavior will be of no importance, since we will work with functions of type sin, cos, or similar, for which all computations will remain well-defined. See [5] for a similar phenomenon.

Remark 2.3.3 (Singular profile). Note now that K_t is a function that may be singular for certain values of x. More precisely, whenever the condition

$$e^{2(\beta(x+x_2)+i\alpha x_1)} = -1,$$

(i.e., $2(\beta(x+x_2)+i\alpha x_1) = i(\pi + 2k\pi)$, for some $k \in \mathbb{Z}$), is satisfied. In this case, one has

$$x_1 = \frac{\pi}{\alpha}\left(\frac{1}{2}+k\right), \quad \text{for some} \quad k \in \mathbb{Z}, \tag{2.3.5}$$

and if $x = -x_2$, then K_t is singular. See [5] for a similar phenomenon in the mKdV case.

Lemma 2.3.3 (Blow-up)**.** *Under the notation in Definition 2.3.2, the function*

$$(K, K_t)(t) := (K(x;\beta, t + x_1, x_2), K_t(x;\beta, t + x_1, x_2))$$

is a smooth solution of SG (2.1.1) *for all* (t, x_1) *such that* (2.3.5) *is not satisfied; i.e., outside the countable set of points with no accumulation point:*

$$t_k = -x_1 + \frac{\pi}{\alpha}\left(\frac{1}{2} + k\right), \quad k \in \mathbb{Z}. \tag{2.3.6}$$

Note that, at each of the points t_k, $K_t(t)$ leaves the Schwartz class. Consequently, $K_t(t)$ *blows up in finite time* (in L^∞ norm), as t approaches some t_k.

Proof. Direct, see Remarks 2.3.1 and 2.3.3. □

2.3.2 Kink profiles and BT

In what follows, we prove connections between kink profiles and the zero solution in SG. Although some of this results are standard, recall that we prove below not only for exact solutions, but also for profiles which are not exact solutions of SG.

Lemma 2.3.4 (Kink as BT of zero)**.** *Let* (Q, Q_t) *be a SG kink profile with scaling parameter* $\beta \in (-1, 1)$, $\beta \neq 0$, *and shift* x_0, *see Definition 2.3.1. Then,*

1. *We have the identities*

$$\sin\left(\frac{Q}{2}\right) = \operatorname{sech}(\gamma(x + x_0)), \quad \cos\left(\frac{Q}{2}\right) = \tanh(\gamma(x + x_0)). \tag{2.3.7}$$

2. *For each* $x \in \mathbb{R}$, (Q, Q_t) *is a BT of the origin* $(0, 0)$ *with parameter*

$$a = a(\beta) := \left(\frac{1+\beta}{1-\beta}\right)^{1/2}. \tag{2.3.8}$$

That is,

$$Q_x = \frac{1}{a}\sin\left(\frac{Q}{2}\right) + a\sin\left(\frac{Q}{2}\right), \quad Q_t = \frac{1}{a}\sin\left(\frac{Q}{2}\right) - a\sin\left(\frac{Q}{2}\right).$$

Proof. Direct. □

Remark 2.3.4 (Antikink and kink with opposite speeds). Note that, thanks to Lemma 2.3.4, both

$$(Q, Q_t)(x; -\beta, x_0) \quad \text{and} \quad (Q, Q_t)(-x; -\beta, x_0),$$

obey respective BT with properly chosen parameters. Indeed, for

$$a_2 := a(-\beta) = \frac{(1-\beta)^{1/2}}{(1+\beta)^{1/2}}, \qquad a_3 := \, -a(\beta) = -\frac{(1+\beta)^{1/2}}{(1-\beta)^{1/2}}, \tag{2.3.9}$$

we obtain

$$\mathbb{B}(0,0) \xrightarrow{a_2} (Q,Q_t)(x;-\beta,x_0), \quad \mathbb{B}(0,0) \xrightarrow{a_3} (Q,Q_t)(-x;-\beta,x_0). \tag{2.3.10}$$

These two profiles will be important in the next sections, when studying the dynamics of the kink-antikink and 2-kink respectively.

Now we deal with the case of complex-valued profiles. Here, we need additional conditions in order to ensure smooth functions in space.

Lemma 2.3.5. *Let (K,K_t) be a complex-valued kink profile, with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts x_1,x_2, just as in Definition 2.3.2, and such that* (2.3.5) *does not hold. Then,*

1. *We have the identities*

$$\sin\left(\frac{K}{2}\right) = \operatorname{sech}(\beta(x+x_2)+i\alpha x_1)), \quad \cos\left(\frac{K}{2}\right) = \tanh(\beta(x+x_2)+i\alpha x_1)). \tag{2.3.11}$$

2. *For each $x\in\mathbb{R}$, (K,K_t) is a BT of the origen $(0,0)$, with parameter $\beta - i\alpha$ (and where $\alpha^2+\beta^2=1$). That is to say,*

$$K_x = \frac{1}{\beta - i\alpha}\sin\left(\frac{K}{2}\right) + (\beta - i\alpha)\sin\left(\frac{K}{2}\right), \tag{2.3.12}$$

$$K_t = \frac{1}{\beta - i\alpha}\sin\left(\frac{K}{2}\right) - (\beta - i\alpha)\sin\left(\frac{K}{2}\right), \tag{2.3.13}$$

where $\sin z$ and $\cos z$ are defined in the complex place as usual.

3. *Moreover, K_x, iK_t, $\sin(K/2)$ and $i\cos(K/2)$ posses even real part and odd imaginary part, with respect to the axis $x=-x_2$.*

Proof of Lemma 2.3.5. We prove first that K satisfies (2.3.12). Indeed, from (2.3.3) we have

$$K_x = \frac{4\beta e^{\beta(x+x_2)+i\alpha x_1}}{1+e^{2\beta(x+x_2)+2i\alpha x_1}} = \frac{2\beta}{\cosh(\beta(x+x_2)+i\alpha x_1)}. \tag{2.3.14}$$

Using that $\cosh(a+ib) = \cosh(a)\cos(b) + i\sinh(a)\sin(b)$, we obtain

$$\begin{aligned} K_x &= \frac{2\beta}{\cosh(\beta(x+x_2))\cos(\alpha x_1) + i\sinh(\beta(x+x_2))\sin(\alpha x_1)} \\ &= \frac{2\beta(\cosh(\beta(x+x_2))\cos(\alpha x_1) - i\sinh(\beta(x+x_2))\sin(\alpha x_1))}{\cosh^2(\beta(x+x_2))\cos^2(\alpha x_1) + \sinh^2(\beta(x+x_2))\sin^2(\alpha x_1)}. \end{aligned} \tag{2.3.15}$$

Therefore, $\operatorname{Re} K_x$ is even wrt $-x_2$ and $\operatorname{Im} K_x$ is odd wrt $-x_2$.

On the other hand, since $\alpha^2+\beta^2=1$, we have $\frac{1}{\beta-i\alpha}+\beta-i\alpha=\beta+i\alpha+\beta-i\alpha=2\beta$, and the RHS of (2.3.12) reads

$$\begin{aligned}\text{RHS}((2.3.12)) =&\ 2\beta\sin\left(\frac{K}{2}\right) = 4\beta\frac{\sin(\arctan e^{\beta(x+x_2)+i\alpha x_1})}{\cos(\arctan e^{\beta(x+x_2)+i\alpha x_1})}\cos^2(\arctan e^{\beta(x+x_2)+i\alpha x_1})\\ =&\ \frac{4\beta e^{\beta(x+x_2)+i\alpha x_1}}{1+e^{2(\beta(x+x_2)+i\alpha x_1)}} = \frac{2\beta}{\cosh(\beta(x+x_2)+i\alpha x_1)}.\end{aligned}$$

Similar to (2.3.15), we can conclude that $\sin(K/2)$ has even real part and odd imaginary part wrt to $x=-x_2$. Finally, note that

$$\begin{aligned}\cos\left(\frac{K}{2}\right) =&\ \tanh(\beta(x+x_2)+i\alpha x_1) = \frac{\tanh(\beta(x+x_2))+i\tan(\alpha x_1)}{1+i\tanh(\beta(x+x_2))\tan(\alpha x_1)}\\ =&\ \frac{\tanh(\beta(x+x_2))\operatorname{sech}^2(\alpha x_1)+i\operatorname{sech}^2(\beta(x+x_2))\tan(\alpha x_1)}{1+\tanh^2(\beta(x+x_2))\tan^2(\alpha x_1)}.\end{aligned}$$

Therefore, $\cos(\frac{K}{2})$ has odd real part and even imaginary part (wrt $-x_2$). This ends the proof of (2.3.12).

Now, in order to show that (2.3.13) is satisfied, it is enough to see that from the definition in (2.3.4),

$$K_t = \frac{4i\alpha e^{\beta(x+x_2)+i\alpha x_1}}{1+e^{2(\beta(x+x_2)+i\alpha x_1)}} = \frac{i\alpha}{\beta}K_x = 2i\alpha\sin\left(\frac{K}{2}\right),$$

which proves the result, since $\frac{1}{\beta-i\alpha}-(\beta-i\alpha)=\beta+i\alpha-\beta+i\alpha=2i\alpha$. The parity of K_t is direct from that of K_x. □

Let $(\overline{K},\overline{K}_t)$ denote the complex-valued kink profile of parameters β and $-\alpha$, i.e.,

$$\begin{aligned}\overline{K}(x) = \overline{K}(x;\beta,x_1,x_2) :=&\ 4\arctan\left(e^{\beta(x+x_2)-i\alpha x_1}\right), \qquad \text{and} \qquad (2.3.16)\\ \overline{K}_t(x) = \overline{K}_t(x;\beta,x_1,x_2) :=&\ -\frac{4i\alpha e^{\beta(x+x_2)-i\alpha x_1}}{1+e^{2(\beta(x+x_2)-i\alpha x_1)}}.\end{aligned}$$

Corollary 2.3.6. *Let $(\overline{K},\overline{K}_t)$ be a SG conjugate kink profile, with scaling parameter $\beta\in(-1,1)\setminus\{0\}$ and shifts x_1,x_2, as in (2.3.16), and such that (2.3.5) do not hold. Then, for each $x\in\mathbb{R}$, $(\overline{K},\overline{K}_t)$ is a BT of the origen $(0,0)$ with parameter $\beta+i\alpha$:*

$$\begin{aligned}\overline{K}_x =&\ \frac{1}{\beta+i\alpha}\sin\left(\frac{\overline{K}}{2}\right)+(\beta+i\alpha)\sin\left(\frac{\overline{K}}{2}\right),\\ \overline{K}_t =&\ \frac{1}{\beta+i\alpha}\sin\left(\frac{\overline{K}}{2}\right)-(\beta+i\alpha)\sin\left(\frac{\overline{K}}{2}\right).\end{aligned}$$

Proof. Direct from Lemma 2.3.5 after conjugation of (2.3.12) and (2.3.13). □

2.4 2-soliton profiles

2.4.1 Definitions

With a small abuse of notation (wrt the exact solutions of SG (2.1.5)-(2.1.6)-(2.1.7), denoted in the same form), we will introduce profiles of 2-soliton solutions. The following definition is standard, see e.g. [7].

Definition 2.4.1 (Static breather profile)**.** *Let $\beta \in (-1,1)$, $\beta \neq 0$, and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. We define the static breather profile as*

$$B := B(x;\beta,x_1,x_2) := 4\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)}{\cosh(\beta(x+x_2))}\right), \quad \alpha := \sqrt{1-\beta^2}. \tag{2.4.1}$$

We also define the "time-derivative profile" as

$$B_t := B_t(x;\beta,x_1,x_2) := \frac{4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2)) + \beta^2\sin^2(\alpha x_1)}. \tag{2.4.2}$$

Finally, note that B_t vanishes only if x_1 satisfies (2.3.5)*.*

Remark 2.4.1. Note that from the previous definition we can recover the standing SG breather [53, 7] if we put $t + x_1$ instead of x_1:

$$B(t,x) = 4\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha(t+x_1))}{\cosh(\beta(x+x_2))}\right), \quad \alpha := \sqrt{1-\beta^2}, \tag{2.4.3}$$

and similar for $B_t(t,x)$ (see Fig. 2.2).

In what follows, we want to study the remaining two SG 2-solitons. Recall that $R(t,x)$ and $A(t,x)$ represent the 2-kink and kink-antikink, respectively, see (2.1.6) and (2.1.7). Once again, with a small abuse of notation, we define first the generalized associated profile for the 2-kink.

Definition 2.4.2 (2-kink profile)**.** *Let $\beta \in (-1,1)$, $\beta \neq 0$, and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. We define the 2-kink profile with speed β as*

$$R := R(x;\beta,x_1,x_2) := 4\arctan\left(\frac{\beta\sinh(\gamma(x+x_2))}{\cosh(\gamma x_1)}\right), \quad \gamma := (1-\beta^2)^{-1/2}. \tag{2.4.4}$$

We also define the "time derivative profile" R_t by

$$R_t := R_t(x;\beta,x_1,x_2) := -\frac{4\beta^2\gamma\sinh(\gamma(x+x_2))\sinh(\gamma x_1)}{\cosh^2(\gamma x_1) + \beta^2\sinh^2(\gamma(x+x_2))}. \tag{2.4.5}$$

Note that (R, R_t) is odd wrt $x = -x_2$.

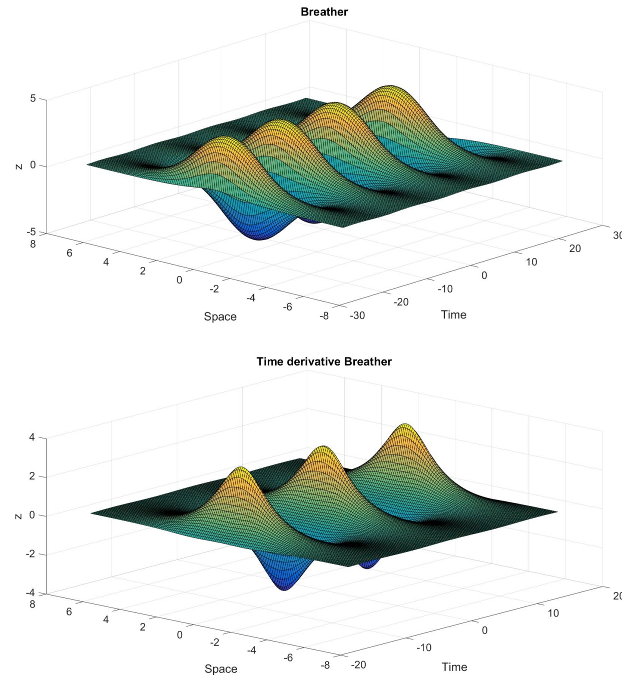

Figure 2.2: Static breather profile (B, B_t), defined in (2.4.1) with $\alpha = \frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$ and $x_1 = t$. Above, B, and below, B_t. Under these parameters, (B, B_t) is an exact solution for SG as in (2.1.5).

Remark 2.4.2. The SG 2-kink solution $R(t,x)$ [53] written in (2.1.6) can be recovered if x_1 is replaced by $x_1 + \beta t$ in (2.4.4). Fig. 2.3 shows the evolution of this exact SG solution in time.

Finally, with a slight abuse of notation wrt (2.1.7), we define the kink-antikink profile.

Definition 2.4.3 (kink-antikink profile)**.** *Let $\beta \in (-1,1)$, $\beta \neq 0$ and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. We define the kink-antikink profile with speed β by*

$$A := A(x;\beta,x_1,x_2) := 4\arctan\left(\frac{\sinh(\gamma x_1)}{\beta\cosh(\gamma(x+x_2))}\right), \quad \gamma := (1-\beta^2)^{-1/2}. \tag{2.4.6}$$

We also define the "time derivative profile" A_t as follows:

$$A_t := A_t(x;\beta,x_1,x_2) := \frac{4\beta^2\gamma\cosh(\gamma(x+x_2))\cosh(\gamma x_1)}{\beta^2\cosh^2(\gamma(x+x_2)) + \sinh^2(\gamma x_1)}. \tag{2.4.7}$$

Note that (A, A_t) are even wrt $x = -x_2$.

Remark 2.4.3. Similarly to the previous case, the kink-antikink solution $A(t,x)$ [53] mentioned in the Introduction (see (2.1.7)) can be recovered by replacing x_1 by $x_1 + \beta t$ in (2.4.6). Figure 2.4 shows this exact SG solution.

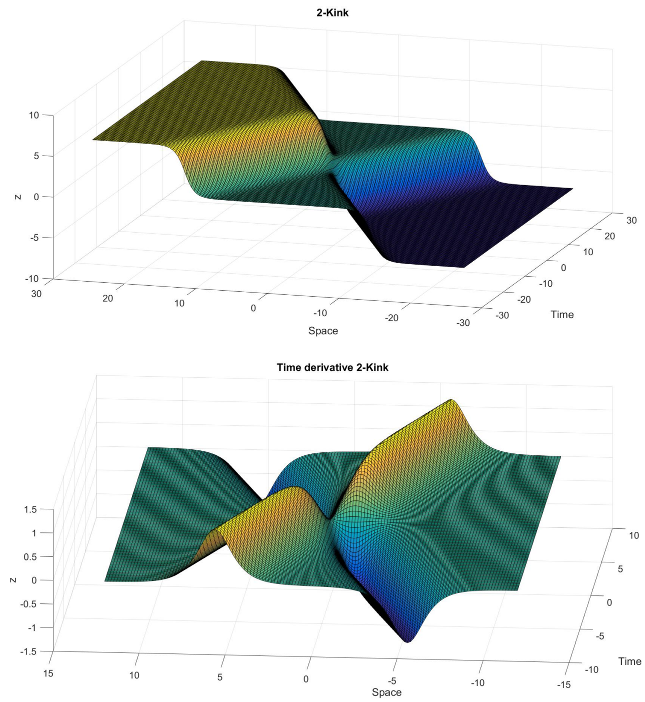

Figure 2.3: Above: space-time evolution of a 2-kink R with parameters $\beta = \frac{1}{2}$, $x_2 = 0$ and $x_1 = \beta t$; below: its corresponding time derivative R_t. Here (R, R_t) is an exact solution of SG (2.1.1), see (2.1.6).

2.4.2 2-soliton profiles and BT

In what follows we will study how to connect breathers and complex-valued kinks, by means of a BT.

Proposition 2.4.4. *Let (B, B_t) and (K, K_t) be SG breather and complex-valued kink profiles respectively, both with parameters $\beta \in (-1, 1) \setminus \{0\}$ and x_1, x_2, as in Definitions 2.4.1 and 2.3.2, and such that condition (2.3.5) is not satisfied. Then,*

1. *We have the limits*

$$\lim_{x\to\pm\infty} \cos\left(\frac{B+K}{2}\right) = \lim_{x\to\pm\infty} \cos\left(\frac{B-K}{2}\right) = \mp 1. \tag{2.4.8}$$

2. *For each $x \in \mathbb{R}$, (B, B_t) is a BT of (K, K_t) with complex-valued parameter $\beta + i\alpha$. That is,*

$$B_x - K_t = \frac{1}{\beta + i\alpha} \sin\left(\frac{B+K}{2}\right) + (\beta + i\alpha) \sin\left(\frac{B-K}{2}\right), \tag{2.4.9}$$

$$B_t - K_x = \frac{1}{\beta + i\alpha} \sin\left(\frac{B+K}{2}\right) - (\beta + i\alpha) \sin\left(\frac{B-K}{2}\right). \tag{2.4.10}$$

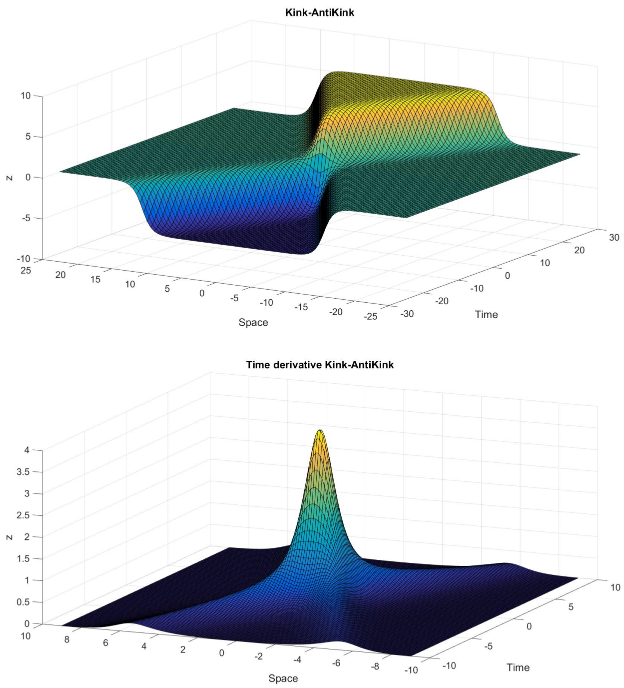

Figure 2.4: Above: representation of the kink-antikink solution (as the collision of kink and antikink), with speed $\beta = \frac{1}{2}$, and parameters $x_2 = 0$, $x_1 = \beta t$. Below: the corresponding time derivative A_t. Here, (A, A_t) is an exact solution of SG (2.1.1), just like $A(t,x)$ in (2.1.7).

Proof of Proposition 2.4.4. For proving (2.4.8), we simply use the values of B and K at infinity, and the fact that cos is analytic in $\mathbb{C}$.

Let us show now (2.4.9) and (2.4.10). Let us start by proving (2.4.9). Taking derivative of B in (2.4.1) wrt to x and simplifying, we have

$$
\begin{aligned}
B_x &= 4\partial_x \arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)}{\cosh(\beta(x+x_2))}\right) \\
&= \frac{4\alpha^2\cosh^2(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha x_1)}\frac{-\beta\sin(\alpha x_1)}{\alpha\cosh^2(\beta(x+x_2))}\beta\sinh(\beta(x+x_2)) \\
&= \frac{-4\alpha\beta^2\sin(\alpha x_1)\sinh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha(t+x_1))}. \qquad (2.4.11)
\end{aligned}
$$

On the other hand, basic trigonometric identities show that

$$
\begin{aligned}
\sin\left(\frac{B\pm K}{2}\right) &= 2\sin\left(\frac{B\pm K}{4}\right)\cos\left(\frac{B\pm K}{4}\right) = 2\tan\left(\frac{B\pm K}{4}\right)\cos^2\left(\frac{B\pm K}{4}\right) \\
&= 2\tan\left(\frac{B\pm K}{4}\right)\left(1+\tan^2\left(\frac{B\pm K}{4}\right)\right)^{-1}
\end{aligned}
$$

$$= \frac{2\tan\left(\arctan\left(\frac{\beta}{\alpha}\frac{\sin\alpha x_1}{\cosh\beta(x+x_2)}\right) \pm \arctan\left(e^{\beta(x+x_2)+i\alpha x_1}\right)\right)}{1+\tan^2\left(\arctan\left(\frac{\beta}{\alpha}\frac{\sin\alpha x_1}{\cosh\beta(x+x_2)}\right) \pm \arctan\left(e^{\beta(x+x_2)+i\alpha x_1}\right)\right)}. \tag{2.4.12}$$

For the sake of notation, let $\theta := \beta(x+x_2)+i\alpha x_1$. Then, using that $\tan(a\pm b) = \frac{\tan a \pm \tan b}{1\mp \tan a \tan b}$, we obtain that (2.4.12) reads now

$$\sin\left(\frac{B\pm K}{2}\right) = \frac{2\left(\frac{\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)}{\cosh\beta(x+x_2)}\pm e^\theta}{1\mp\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)e^\theta}{\cosh\beta(x+x_2)}}\right)}{1+\left(\frac{\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)}{\cosh\beta(x+x_2)}\pm e^\theta}{1\mp\frac{\beta}{\alpha}\frac{\sin(\alpha x_1)e^\theta}{\cosh\beta(x+x_2)}}\right)^2} = \frac{2\left(\frac{\beta\sin(\alpha x_1)\pm\alpha e^\theta\cosh\beta(x+x_2)}{\alpha\cosh\beta(x+x_2)\mp\beta\sin(\alpha x_1)e^\theta}\right)}{1+\left(\frac{\beta\sin(\alpha x_1)\pm\alpha e^\theta\cosh\beta(x+x_2)}{\alpha\cosh\beta(x+x_2)\mp\beta\sin(\alpha x_1)e^\theta}\right)^2}$$
$$= \frac{2(\beta\sin(\alpha x_1)\pm\alpha e^\theta\cosh\beta(x+x_2))(\alpha\cosh\beta(x+x_2)\mp\beta\sin(\alpha x_1)e^\theta)}{(\alpha\cosh(\beta(x+x_2))\mp\beta\sin(\alpha x_1)e^\theta)^2+(\beta\sin(\alpha x_1)\pm\alpha e^\theta\cosh(\beta(x+x_2)))^2},$$

and simplifying,

$$\sin\left(\frac{B\pm K}{2}\right) = \frac{2f_1(x)}{\left(1+e^{2\theta}\right)\left(\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha x_1)\right)}, \tag{2.4.13}$$

where $f_1(x) = f_1(x;\beta,x_1,x_2)$ is such that

$$\begin{aligned} f_1(x) := {}& \alpha\beta\cosh(\beta(x+x_2))\sin(\alpha x_1) \mp \beta^2 e^\theta \sin^2(\alpha x_1) \\ & \pm \alpha^2 e^\theta\cosh^2(\beta(x+x_2)) - \alpha\beta e^{2\theta}\cosh(\beta(x+x_2))\sin(\alpha x_1). \end{aligned}$$

Now we show (2.4.9). Substracting (2.3.4) from (2.4.11), we get

$$B_x - K_t = \frac{-4\alpha\beta^2\sin(\alpha x_1)\cdot\sinh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha(t+x_1))} - \frac{4i\alpha e^\theta}{1+e^{2\theta}} = \frac{\widetilde{A}}{\widetilde{C}},$$

where

$$\widetilde{C} = \left(1+e^{2\theta}\right)\left(\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2\alpha x_1\right), \tag{2.4.14}$$

$$\begin{aligned} \widetilde{A} = {}& -4\alpha\beta^2(1+e^{2\theta})\sin\alpha x_1\sinh(\beta(x+x_2)) \\ & -4i\alpha e^\theta\left(\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2\alpha x_1\right). \end{aligned}$$

On the other hand, recalling that $\alpha^2+\beta^2=1$, from (2.4.13) we obtain

$$(\beta+i\alpha)\sin\left(\frac{B-K}{2}\right)+\frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right) = \frac{\widetilde{B}}{\widetilde{C}}, \tag{2.4.15}$$

where $\widetilde{C}$ is given by (2.4.14) and

$$\widetilde{B} = 4\alpha\beta^2\left(1-e^{2\theta}\right)\sin\alpha x_1\cosh(\beta(x+x_2)) + 4i\alpha\beta^2 e^\theta\sin^2\alpha x_1$$

$$- 4i\alpha^3 e^{\theta}\cosh^2(\beta(x+x_2)).$$

Therefore, (2.4.9) reduces to prove $\widetilde{A} - \widetilde{B} \equiv 0$. Indeed,

$$\begin{aligned}\widetilde{A} - \widetilde{B} = &-4\alpha\beta^2\big((1+e^{2\theta})\sin\alpha x_1 \sinh(\beta(x+x_2)) + 2ie^{\theta}\sin^2\alpha x_1\big)\\ &- 4\alpha\beta^2(1-e^{2\theta})\sin\alpha x_1\cosh(\beta(x+x_2)) = 0.\end{aligned}$$

This proves (2.4.9). Finally, we prove that (2.4.10) is satisfied. We follow the same idea as before. From (2.3.15) and (2.4.2) we obtain

$$B_t - K_x = \frac{4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2)) + \beta^2\sin^2(\alpha x_1)} - \frac{4\beta e^{\theta}}{1+e^{2\theta}} = \frac{\widetilde{A}_2}{\widetilde{C}},$$

where $\widetilde{C}$ is given by (2.4.14) and

$$\begin{aligned}\widetilde{A}_2 = &\, 4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))\big(1+e^{2\theta}\big)\\ &- 4\beta e^{\theta}\big(\alpha^2\cosh^2(\beta(x+x_2)) + \beta^2\sin^2(\alpha x_1)\big).\end{aligned}$$

On the other hand, recalling that $\alpha^2+\beta^2 = 1$ and making similar simplifications as for (2.4.15), we have

$$\frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right) - (\beta+i\alpha)\sin\left(\frac{B-K}{2}\right) = \frac{\widetilde{B}_2}{\widetilde{C}},$$

where $\widetilde{C}$ is given by (2.4.14) and

$$\begin{aligned}\widetilde{B}_2 = &\, 4\big(\alpha^2\beta e^{\theta}\cosh^2(\beta(x+x_2)) - \beta^3 e^{\theta}\sin^2(\alpha x_1)\\ &+ i\alpha^2\beta e^{2\theta}\cosh(\beta(x+x_2))\sin(\alpha x_1) - i\alpha^2\beta\cosh^2(\beta(x+x_2))\sin(\alpha x_1)\big).\end{aligned}$$

Hence, (2.4.10) is reduced to show that $\widetilde{A}_2 - \widetilde{B}_2 \equiv 0$. Indeed, simplifying,

$$\begin{aligned}&\widetilde{A}_2 - \widetilde{B}_2\\ &= 4\alpha^2\beta\cosh(\beta(x+x_2))\,\big(\cos\alpha x_1 + i\sin\alpha x_1 + e^{2\theta}(\cos\alpha x_1 - i\sin\alpha x_1)\big)\\ &\quad - 8\alpha^2\beta e^{\theta}\cosh^2(\beta(x+x_2))\\ &= 8\alpha^2\beta e^{\theta}\cosh^2(\beta(x+x_2)) - 8\alpha^2\beta e^{\theta}\cosh^2(\beta(x+x_2)) = 0.\end{aligned}$$

□

The following corollary shows that there is also a relationship between the breather and the conjugate of the complex-valued kink profile.

Corollary 2.4.5. *Let (B, B_t) and $(\overline{K}, \overline{K}_t)$ be SG breather and complex-valued kink profiles respectively, both with scaling parameters $\beta \in (-1,1)\backslash\{0\}$ and shifts x_1, x_2 such that* (2.3.5) *do not satisfy. Then, for each $x \in \mathbb{R}$, (B, B_t) is a BT of $(\overline{K}, \overline{K}_t)$ with parameter $\beta - i\alpha$:*

$$B_x - \overline{K}_t = \frac{1}{\beta - i\alpha} \sin\left(\frac{B + \overline{K}}{2}\right) + (\beta - i\alpha) \sin\left(\frac{B - \overline{K}}{2}\right), \qquad (2.4.16)$$

$$B_t - \overline{K}_x = \frac{1}{\beta - i\alpha} \sin\left(\frac{B + \overline{K}}{2}\right) - (\beta - i\alpha) \sin\left(\frac{B - \overline{K}}{2}\right). \qquad (2.4.17)$$

Proof. Direct from previous result. □

When working with multiple profiles it is convenient to introduce a schematic representation of the BT, see [53]. Figure 2.5 shows a diagram where each arrow represents the BT of the SG solution $(\phi_i, \phi_{i,t})$ towards another solution $(\phi_j, \phi_{j,t})$ with parameter a_k, and given in Definition 2.2.1. The fact that both BT arrive to the same solution is not a coincidence and it is called in the literature as *Permutability Theorem.* In this chapter we will present a rigorous proof of this result for solutions of SG which are perturbations of the profiles showed in the previous section.

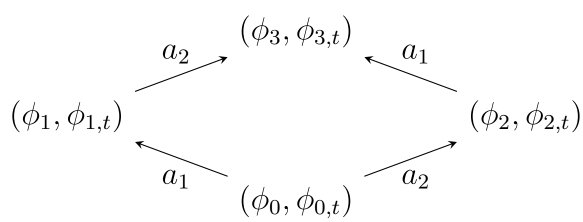

Figure 2.5: A diagram representing two consecutive applications of the BT with inverse parameters a_1 y a_2. The permutability property says that $(\phi_3, \phi_{3,t})$ is the unique final function, independently of the two considered paths.

We remark that Proposition 2.4.4, together with Corollary 2.4.5 show the validity of the diagram in Fig. 2.6 for SG profiles, and not only solutions of the equation itself. This diagram is valid as soon as x_1 does not satisfy (2.3.5), in order to avoid the lack of good definition for K and $\overline{K}$.

Now we want to study the conection between the SG kink and kink-antikink.

Proposition 2.4.6 (Kink-Antikink connection)**.** *Let (A, A_t) be a SG kink-antikink profile, with speed parameter $\beta \in (-1,1) \setminus \{0\}$ and shifts x_1, x_2, as was introduced in Definition 2.4.3. Let also*

$$\vec{Q} := (Q, Q_t) := (Q, Q_t)(x; -\beta, x_1 + x_2), \qquad (2.4.18)$$

be a real-valued kink profile (see Definition 2.3.1 and Observation 2.3.4), with speed parameter $-\beta \in (-1,1)\backslash\{0\}$ and shift $(x_1 + x_2)$.[3] *Then, the following is satisfied:*

[3]Note the specific character of the choice in the shift parameter.

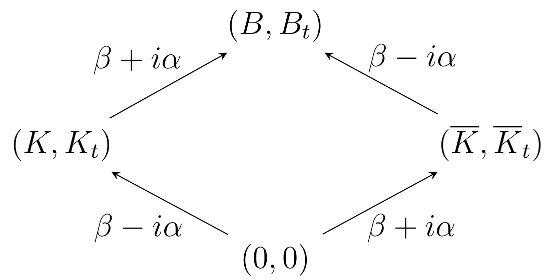

Figure 2.6: Diagram for the breather B in Proposition 2.4.4. Note that (B, B_t) is obtained independently of the chosen path [53].

1. *We have the identities*

$$\lim_{x\to\pm\infty} \cos\left(\frac{A\pm Q}{2}\right) = \begin{cases} -1, & x\to+\infty \\ 1, & x\to-\infty \end{cases}. \tag{2.4.19}$$

2. *For each $x \in \mathbb{R}$, (A, A_t) is a BT of (Q, Q_t) with real-valued parameter $a = a(\beta)$ (see* (2.3.8)*). That is,*

$$A_x - Q_t = \frac{1}{a}\sin\left(\frac{A+Q}{2}\right) + a\sin\left(\frac{A-Q}{2}\right), \tag{2.4.20}$$

$$A_t - Q_x = \frac{1}{a}\sin\left(\frac{A+Q}{2}\right) - a\sin\left(\frac{A-Q}{2}\right). \tag{2.4.21}$$

Remark 2.4.4. Generally speaking, we have the validity of the diagram in Fig. 2.7 (above), as soon as we choose kink profiles of parameters $(Q, Q_t)(x, \beta, -x_1+x_2)$ and $(Q, Q_t)(x; -\beta, x_1+x_2)$. In this sense, the reconstruction of (A, A_t) requires a different rigidity than that of the breather. In this paper, we will only use the RHS connection via $(Q, Q_t)(x; -\beta, x_1 + x_2)$.

Proof of Proposition 2.4.6. The proof of this result is very similar to that of Proposition 2.4.4. See Appendix A. □

In order to conclude this section we will study the relationship between real-valued kinks and 2-kinks of SG.

Corollary 2.4.7 (2-kink connection)**.** *Let (R, R_t) be a SG 2-kink profile, with speed parameter $\beta \in (-1, 1) \setminus \{0\}$ and shifts x_1, x_2. Let $\vec{Q}$ denote the kink defined in* (2.4.18)*, with speed parameter $-\beta \in (-1, 1)$ and shift $(x_1 + x_2)$. Then,*

1. *We have the limits*

$$\lim_{x\to\pm\infty} \cos\left(\frac{R\pm Q}{2}\right) = \begin{cases} 1, & x\to+\infty \\ -1, & x\to-\infty \end{cases}. \tag{2.4.22}$$

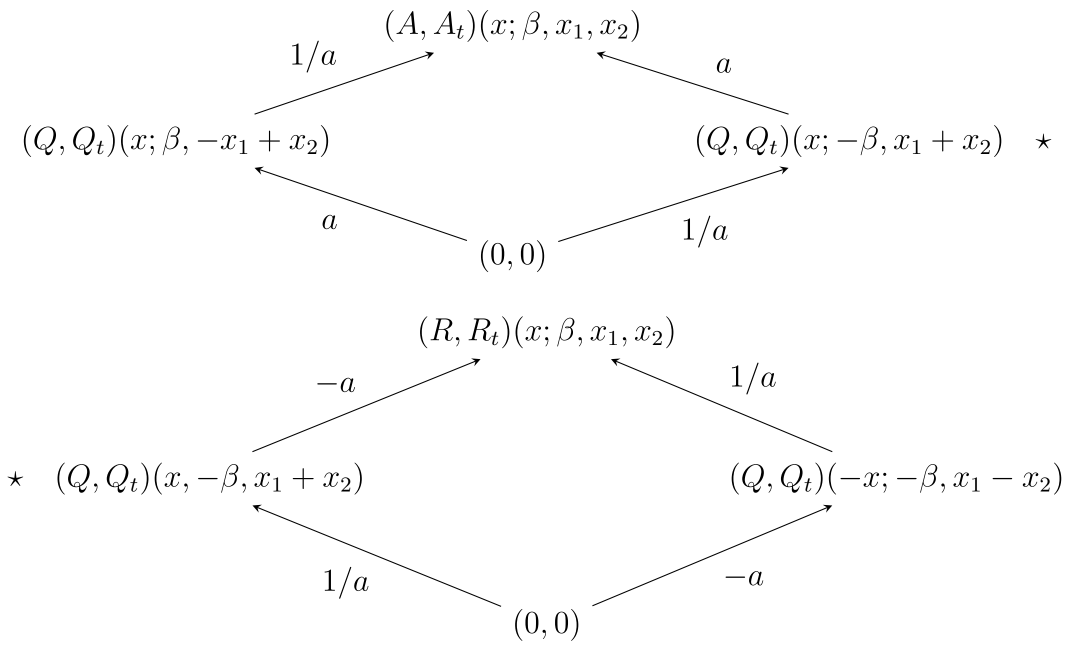

Figure 2.7: Schematic diagram for the kink-antikink pair (A, A_t) (above), and the 2-kink (R, R_t) (below). In this paper, we will follow the paths refereed with $\star$.

2. *For each $x \in \mathbb{R}$, (R, R_t) is a BT of (Q, Q_t) with parameter $a_3 = -a(\beta)$ (see (2.3.9)):*

$$R_x - Q_t = \frac{1}{a_3} \sin\left(\frac{R+Q}{2}\right) + a_3 \sin\left(\frac{R-Q}{2}\right), \tag{2.4.23}$$

$$R_t - Q_x = \frac{1}{a_3} \sin\left(\frac{R+Q}{2}\right) - a_3 \sin\left(\frac{R-Q}{2}\right). \tag{2.4.24}$$

Remark 2.4.5. We have in general the validity of the diagram in Fig. 2.7 (below), but we will only use its left side component.

Proof. Direct from Proposition 2.4.6, it is enough to change the roles of $x + x_2$ and x_1, and $a(\beta)$ by $-a(\beta)$. □

2.5 Modulation of 2-solitons

In order to prove Theorem 2.1.1, we will show first some modulation lemmas. Here we will follow the ideas in [59] and [7].

2.5.1 Static modulation

We will consider three pair of objects to deal with:

1. (B, B_t) a SG breather profile with scaling parameter $\beta \in (-1, 1)$, $\beta \neq 0$ fixed, and shifts $x_1, x_2 \in \mathbb{R}$, as in Definition 2.4.1.

2. (R, R_t) a SG 2-kink profile with speed $\beta \in (-1, 1)$, $\beta \neq 0$ fixed, and shifts $x_1, x_2 \in \mathbb{R}$, as in Definition 2.4.2.

3. (A, A_t) a SG kink-antikink profile with speed $\beta \in (-1, 1)$, $\beta \neq 0$ fixed, and shifts $x_1, x_2 \in \mathbb{R}$, as in Definition 2.4.3.

Let D denote any of the capital letters A, B or R. We will use subindexes 1 and 2 to denote derivatives of A, B and R wrt the shifts x_1 and x_2 respectively, namely for $j = 1, 2$

$$D_j(x; \beta, x_1, x_2) := \partial_{x_j} D(x; \beta, x_1, x_2), \tag{2.5.1}$$

$$(D_t)_j(x; \beta, x_1, x_2) := \partial_{x_j} D_t(x; \beta, x_1, x_2). \tag{2.5.2}$$

Remark 2.5.1. In Appendix B we can find an explicit description of the derivatives above mentioned in the cases $D = A$ and $D = R$, showing clearly that these are localized functions (see subsection B.2).

Let $\nu > 0$ be a small real number. Let us also consider the following tubular neighborhood of a 2-soliton (D, D_t) of radius ν:

$$\mathcal{U}(\nu) := \Big\{ (\phi, \phi_t) \ : \ \inf_{x_1, x_2 \in \mathbb{R}} \big\| (\phi, \phi_t) - (D, D_t)(\cdot; \beta, x_1, x_2) \big\|_{H^1 \times L^2} < \nu \Big\}.$$

It is important to mention that this set has no temporal dependence. Since (ϕ, ϕ_t) does not necessarily decay to zero (e.g. 2-kink case), the key is the difference with (D, D_t). However in the case of kink-antikink or breather, $(\phi, \phi_t) \in H^1 \times L^2$. For the proof of next result, see Appendix C.

Lemma 2.5.1 (Static Modulation)**.** *There exists $\nu_0 > 0$ such that for each $0 < \nu < \nu_0$, the following is satisfied. For each pair $(\phi, \phi_t) \in \mathcal{U}(\nu)$, there exists a unique couple of C^1 functions $\tilde{x}_1, \tilde{x}_2 : \mathcal{U}(\nu) \to \mathbb{R}$ such that , if we consider $z = z(x)$ and $w = w(x)$ defined as*

$$z(x) := \phi(x) - D(x; \beta, \tilde{x}_1, \tilde{x}_2), \quad w(x) := \phi_t(x) - D_t(x; \beta, \tilde{x}_1, \tilde{x}_2),$$

then, the following orthogonality conditions hold:

$$\int_{\mathbb{R}} (z, w) \cdot \big(D_1, (D_1)_t\big) dx = \int_{\mathbb{R}} (z, w) \cdot \big(D_2, (D_2)_t\big) dx = 0.$$

2.5.2 Dynamical modulation

We need now a dynamical version of the previous lemma. Let (ϕ, ϕ_t) be a solution of (2.1.1), with initial data (ϕ_0, ϕ_1) such that

$$\|(\phi_0, \phi_1) - (D, D_t)(\cdot; \beta, 0, 0)\|_{H^1\times L^2} < \eta, \tag{2.5.3}$$

for some $0 < \eta < \eta_0$ small enough, with η_0 given by Theorem 2.1.1.

Definition 2.5.2 (Recurrence Time)**.** *Let $C^* > 1$ be a large parameter (to be chosen later), and let $(\phi, \phi_t)(t)$ be the unique globally defined solution of SG with initial data (ϕ_0, ϕ_1), and satisfying* (2.5.3)*. We define $T^* := T^*(C^*) > 0$ as the maximal time for which there are parameters $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ such that*

$$\sup_{t\in[0,T^*]} \|(\phi, \phi_t)(t) - (D, D_t)(\cdot; \beta, \tilde{x}_1(t), \tilde{x}_2(t))\|_{H^1\times L^2} \leq C^*\eta. \tag{2.5.4}$$

Note that T^* is well-defined thanks to continuity of the SG flow, (2.5.3) and the fact that $C^* > 1$. Later we will prove that T^* can be taken infinity for all C^* large enough. Even more,

$$\text{In what follows we will assume that } T^* \text{ is } \textbf{finite}. \tag{2.5.5}$$

By choosing η_0 sufficiently small if necessary, we will have $C^*\eta < \nu_0$ in Lemma 2.5.1, and the following result will be valid:

Corollary 2.5.3 (Dynamical modulation)**.** *Under the assumptions of Definition 2.5.2, there are C^1 functions $x_1, x_2 : [0, T^*] \to \mathbb{R}$ such that, if*

$$\begin{aligned} z(t,x) &:= \phi(t,x) - D(x; \beta, x_1(t), x_2(t)), \\ w(t,x) &:= \phi_t(t,x) - D_t(x; \beta, x_1(t), x_2(t)), \end{aligned} \tag{2.5.6}$$

then, for each $t \in [0, T^*]$,

$$\int_{\mathbb{R}} (z, w) \cdot \big(D_1, (D_1)_t\big)(t,x)dx = \int_{\mathbb{R}} (z, w) \cdot \big(D_2, (D_2)_t\big)(t,x)dx = 0, \tag{2.5.7}$$

and moreover

$$\sup_{t\in[0,T^*]} \|(z, w)(t)\|_{H^1\times L^2} \lesssim C^*\eta, \tag{2.5.8}$$

$$\|(z, w)(0)\|_{H^1\times L^2} + |x_1(0)| + |x_2(0)| \lesssim \eta, \tag{2.5.9}$$

and

$$\sup_{t\in[0,T^*]} (|x_1'(t)| + |x_2'(t)|) \lesssim \sup_{t\in[0,T^*]} \|(z, w)(t)\|_{H^1\times L^2} \lesssim C^*\eta. \tag{2.5.10}$$

Moreover, if $D = R$ and (z_0, w_0) are odd, or if $D = B, A$ and (z_0, w_0) are even, then we can choose $x_2(t) \equiv 0$, and the parity property on (z, w) is preserved in time.

Proof. Direct from Lemma 2.5.1 and (2.5.4). □

2.6 Perturbations of breathers

2.6.1 Statement

In this section we will assume $\mathbb{K} = \mathbb{C}$ in Definition 2.2.3. Our goal will be to show the following result.

Proposition 2.6.1 (Descent to the zero solution). *Let (B, B_t) be a SG breather profile, as in Definition 2.4.1, with scaling parameter $\beta \in (-1,1)\backslash\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$, such that x_1 does not satisfy* (2.3.5)*. Let also (K, K_t) be the complex-valued kink profile associated to (B, B_t), that is with same parameters as (B, B_t). Then, there are constants $\eta_0 > 0$ and $C > 0$ such that, for all $0 < \eta < \eta_0$ and all $(z_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that*[4]

$$\|(z_0, w_0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \eta,$$

then the following is satisfied:

1. *There are unique (u_0, s_0, δ) defined in an open subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ such that the Bäcklund functional* (2.2.3) *satisfies*

$$\mathcal{F}(B + z_0, B_t + w_0, K + u_0, K_t + s_0, \beta + i\alpha + \delta) = (0, 0),$$

   and where

$$\|(u_0, s_0)\|_{H^1\times L^2} + |\delta| < C\eta.$$

2. *Making η_0 even smaller if necessary, there are unique $(y_0, v_0, \tilde{\delta})$, defined in an open subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$, and such that*

$$\mathcal{F}(K + u_0, K_t + s_0, y_0, v_0, \beta - i\alpha + \tilde{\delta}) = (0, 0),$$

   and

$$\|(y_0, v_0)\|_{H^1\times L^2} + |\tilde{\delta}| < C\eta.$$

The rest of the section will be devoted to the proof of this result, for which we will need some auxiliary lemmas.

2.6.2 Integrant Factor

Let us start with an auxiliary result on existence of integrant factors for some ODEs appearing naturally when studying breathers and BT.

[4] Note that both (z_0, w_0) are real-valued.

Lemma 2.6.2 (Existence of Integrant Factor)**.** *Let (B, B_t) and (K, K_t) be breather and complex-valued kink profiles, both with scaling parameter $\beta \in (-1,1)$, $\beta \neq 0$, and shifts $x_1, x_2 \in \mathbb{R}$. Let us consider*

$$\mu_K(x) := \frac{1}{\cosh(\beta(x+x_2)+i\alpha x_1)} = \frac{K_x(x)}{2\beta}, \qquad (\textit{see } (2.3.14)), \tag{2.6.1}$$

and

$$\mu_B(x) := \frac{\cosh(\beta(x+x_2)+i\alpha x_1)}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha x_1)} = \frac{1}{4\alpha^2\beta^2}(\beta B_t - i\alpha B_x)(x). \tag{2.6.2}$$

Then the following holds:

1. *(Local and global behavior)*

   (a) $\mu_K(x)$ is well-defined and smooth for any $\beta \in (-1,1)\setminus\{0\}$, and $x_1, x_2 \in \mathbb{R}$, provided x_1 does not satisfy (2.3.5)*. Additionally, it decays exponentially fast in space as $x \to \pm\infty$.*

   (b) $\mu_B(x)$ is well-defined and smooth for any $\beta \in (-1,1)\setminus\{0\}$, and $x_1, x_2 \in \mathbb{R}$. Additionally, it decays exponentially fast in space as $x \to \pm\infty$. Finally, μ_B is not zero if (2.3.5) *is not satisfied.*

2. *(ODEs) We have that $\mu_K(x)$ satisfies the ODE*

$$\mu_x - \beta\cos\left(\frac{K}{2}\right)\mu = 0, \tag{2.6.3}$$

   and $\mu_B(x)$ solves the ODE

$$\mu_x - \left(\frac{(\beta - i\alpha)}{2}\cos\left(\frac{B+K}{2}\right) + \frac{(\beta+i\alpha)}{2}\cos\left(\frac{B-K}{2}\right)\right)\mu = 0. \tag{2.6.4}$$

3. *(Non orthogonality) For each x_1 such that* (2.3.5) *is not satisfied, we have*

$$\int_{\mathbb{R}} \mu_K \sin\left(\frac{K}{2}\right) = \frac{2}{\beta}, \tag{2.6.5}$$

   and μ_B is not orthogonal to $(B_x - K_t)$, that is:

$$\int_{\mathbb{R}} \mu_B\,(B_x - K_t) = -\frac{4i}{\alpha\beta}. \tag{2.6.6}$$

   Finally, these identities can be extended by continuity to all $x_1 \in \mathbb{R}$.

Proof. The proof of this result is direct but cumbersome, see Appendix D for the proof. □

2.6.3 Proof of Proposition 2.6.1

Using Lemma 2.6.2, the first item in Proposition 2.6.1 will be a consequence of the following result.

Lemma 2.6.3. *Let (B, B_t) and (K, K_t) be breather and complex-valued kink profiles, both with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$, and such that* (2.3.5) *is not satisfied. Then, there are constants $\eta_0 > 0$ and $C > 0$ such that for all $0 < \eta < \eta_0$ and for all $(z_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that*

$$\|(z_0, w_0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \eta,$$

there are unique (u_0, s_0, δ) defined in an open subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ and such that $\mathcal{F}$ in (2.2.3) *satisfies*

$$\mathcal{F}(B + z_0, B_t + w_0, K + u_0, K_t + s_0, \beta + i\alpha + \delta) = (0,0), \tag{2.6.7}$$

and

$$\|(u_0, s_0)\|_{H^1\times L^2} + |\delta| \leq C\eta. \tag{2.6.8}$$

Proof. Let $(z_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ be given, with a size to be defined below. Consider the system of equations given by the Bäcklund functionals (2.2.5)-(2.2.6) in the variables $(u_0, s_0, \delta) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ (note that this space and $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ define the space $X(\mathbb{K})$ for $\mathcal{F}$):

$$\begin{aligned}
&\mathcal{F}_1\big(B + z_0, B_t + w_0, K + u_0, K_t + s_0, \beta + i\alpha + \delta\big) = \\
&\quad = B_x + z_{0,x} - K_t - s_0 - \frac{1}{\beta + i\alpha + \delta}\sin\left(\frac{B + z_0 + K + u_0}{2}\right) \\
&\qquad - (\beta + i\alpha + \delta)\sin\left(\frac{B + z_0 - K - u_0}{2}\right),
\end{aligned} \tag{2.6.9}$$

$$\begin{aligned}
&\mathcal{F}_2\big(B + z_0, B_t + w_0, K + u_0, K_t + s_0, \beta + i\alpha + \delta\big) = \\
&\quad = B_t + w_0 - K_x - u_{0,x} - \frac{1}{\beta + i\alpha + \delta}\sin\left(\frac{B + z_0 + K + u_0}{2}\right) \\
&\qquad + (\beta + i\alpha + \delta)\sin\left(\frac{B + z_0 - K - u_0}{2}\right).
\end{aligned} \tag{2.6.10}$$

We look for a unique choice of (u_0, s_0, a) such that

$$\mathcal{F}(B + z_0, B_t + w_0, K + u_0, K_t + s_0, \beta + i\alpha + \delta) = (0,0).$$

We will use the Implicit Function Theorem for $(\mathcal{F}_1, \mathcal{F}_2)$. Note that from (2.6.9) that once $(u_0,\ \delta)$ are defined, s_0 gets completely determined from (2.6.9). Hence, we will only solve (2.6.10) for (u_0, δ). Thanks to the indentity $\mathcal{F}(B, B_t, K, K_t, \beta + i\alpha) = (0,0)$, through a rearrangement of (2.6.9) and (2.6.10) we have that these equations can be written as

$$\widetilde{\mathcal{F}}_1\big(z_0, w_0, u_0, s_0, \delta\big)$$

$$
\begin{aligned}
:= \; & z_{0,x} - s_0 - \frac{1}{\beta + i\alpha + \delta}\sin\left(\frac{B+K+z_0+u_0}{2}\right) + \frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right) \\
& - (\beta+i\alpha+\delta)\sin\left(\frac{B-K+z_0-u_0}{2}\right) + (\beta+i\alpha)\sin\left(\frac{B-K}{2}\right) = 0, \qquad (2.6.11)
\end{aligned}
$$

$$
\begin{aligned}
& \widetilde{\mathcal{F}}_2\big(z_0, w_0, u_0, s_0, \delta\big) \\
:= \; & w_0 - u_{0,x} - \frac{1}{\beta + i\alpha + \delta}\sin\left(\frac{B+K+z_0+u_0}{2}\right) + \frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right) \\
& + (\beta+i\alpha+\delta)\sin\left(\frac{B-K+z_0-u_0}{2}\right) - (\beta+i\alpha)\sin\left(\frac{B-K}{2}\right) = 0. \qquad (2.6.12)
\end{aligned}
$$

Clearly $\widetilde{\mathcal{F}}_2$ defines a $\mathcal{C}^1$ functional in the vicinity of zero, and $\widetilde{\mathcal{F}}_2\big(0,0,0,0,0\big) = 0$. Then, we must verify that the partial derivative of $\widetilde{\mathcal{F}}_2$ at $(0,0,0,0,0)$ defines a bounded linear operator, invertible with continuous inverse. From (2.6.12) we must check that the ODE

$$
\begin{aligned}
-u_{0,x} + & \frac{\delta}{(\beta+i\alpha)^2}\sin\left(\frac{B+K}{2}\right) - \frac{u_0}{2(\beta+i\alpha)}\cos\left(\frac{B+K}{2}\right) \\
& + \delta\sin\left(\frac{B-K}{2}\right) - \frac{(\beta+i\alpha)u_0}{2}\cos\left(\frac{B-K}{2}\right) \; = \; f, \qquad (2.6.13)
\end{aligned}
$$

has a unique solution (u_0, δ) such that $u_0 \in H^1(\mathbb{R};\mathbb{C})$, $\delta \in \mathbb{C}$, for each $f \in H^1(\mathbb{R};\mathbb{C})$. Rewriting (2.6.13), calling $f \mapsto -f$, and using that $(\beta+i\alpha)^{-1} = \beta - i\alpha$, we have

$$
\begin{aligned}
u_{0,x} + & \left(\frac{(\beta-i\alpha)}{2}\cos\left(\frac{B+K}{2}\right) + \frac{(\beta+i\alpha)}{2}\cos\left(\frac{B-K}{2}\right)\right)u_0 \\
& = \; f + \frac{\delta}{(\beta+i\alpha)^2}\sin\left(\frac{B+K}{2}\right) + \delta\sin\left(\frac{B-K}{2}\right). \qquad (2.6.14)
\end{aligned}
$$

Consider $\mu_B = \mu_B(x)$ defined in Lemma 2.6.2, see (2.6.2). Thanks to (2.6.4), we have

$$
u_0 \; = \; \frac{1}{\mu_B}\int_{-\infty}^{x} \mu_B\left(f + \delta(\beta-i\alpha)^2\sin\left(\frac{B+K}{2}\right) + \delta\sin\left(\frac{B-K}{2}\right)\right).
$$

Recalling that (B, B_t) and (K, K_t) satisfy (2.4.9), and since $\alpha^2+\beta^2 = 1$, we arrive to the simplified expression

$$
u_0 = \frac{1}{\mu_B}\int_{-\infty}^{x} \mu_B\left(f + \delta(\beta-i\alpha)\left(B_x - K_t\right)\right).
$$

From (2.6.6), we know that $\int_{\mathbb{R}} \mu_B \cdot (B_x - K_t) \neq 0$. Consequently, we can choose $\delta \in \mathbb{C}$ in a unique fashion and such that

$$
\int_{\mathbb{R}} \mu_B\left(f + \delta(\beta-i\alpha)\left(B_x - K_t\right)\right) = 0. \qquad (2.6.15)
$$

Note that from this choice we have $|\delta| \leq C\|f\|_{L^2(\mathbb{R})}$, where C is a constant depending on β and $\|\mu_B\|_{L^2(\mathbb{R};\mathbb{C})}$. Let us prove that $u_0 \in H^1(\mathbb{R};\mathbb{C})$. Indeed, from

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \mu_B(x) = \lim_{x\to\pm\infty} B_x = \lim_{x\to\pm\infty} K_t = 0,$$

(see (2.6.2), (2.4.11) and (2.3.4)), we obtain

$$\lim_{x\to\pm\infty} u_0 = \lim_{x\to\pm\infty} \frac{\mu_B}{(\mu_B)_x}(f + \delta(\beta - i\alpha)(B_x - K_t)) = 0.$$

Lastly, note that if $s \leq x \ll -1$, then we have that

$$\left|\frac{\mu_B(s)}{\mu_B(x)}\right| \;\leq\; C\left|\frac{\cosh(\beta(x+x_2))}{\cosh(\beta(s+x_2))}\right| \;\leq\; C\left|\exp\left(\beta(s-x)\right)\right|.$$

Hence, for $x \ll -1$ we get

$$\begin{aligned} |u_0(x)| \;&\leq\; C\int_{-\infty}^{x} e^{-\beta(x-s)}\;|f + \delta(\beta - i\alpha)\,(B_x - K_t)|\,ds \\ &\leq\; Ce^{-\beta x} \star \left(|f(\cdot) + \delta(\beta - i\alpha)\,(B_x - K_t)|\;\mathbb{1}_{(-\infty,\,x]}(\cdot)\right). \end{aligned}$$

On the other hand, if $x \gg 1$, using (2.6.15) we have

$$u_0(x) = -\frac{1}{\mu_B}\int_x^\infty \mu_B\left(f + \delta(\beta - i\alpha)\,(B_x - K_t)\right).$$

From this last result, it is not difficult to show decay estimates for $x \gg 1$, changing $e^{-\beta x}$ by $e^{\beta x}$. In consequence, from Young's inequality,

$$\|u_0\|_{L^2(\mathbb{R};\mathbb{C})} \lesssim \|f + \delta(\beta - i\alpha)\,(B_x - K_t)\|_{L^2(\mathbb{R};\mathbb{C})}.$$

Finally, in order to prove $u_0 \in H^1$ we only must check that $u_{0,x} \in L^2(\mathbb{R};\mathbb{C})$, which is direct if we recall that $f \in H^1$ and $(\mu_B)_x/\mu_B$ is bounded. Therefore, $u_0 \in H^1(\mathbb{R};\mathbb{C})$. The Implicit Function Theorem guaranties (2.6.7). The proof of (2.6.8) is direct from the smallness of the data. □

Finally, the second item in Proposition 2.6.1 is consequence of the following:

Lemma 2.6.4. *Let (K, K_t) be a complex-valued kink profile with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$, and such that x_1 does not satisfy* (2.3.5)*. Then, there are constants $\nu_0 > 0$ and $C > 0$ such that for all $0 < \nu < \nu_0$ and for all $(u_0, s_0) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$ such that*

$$\|u_0\|_{H^1(\mathbb{R};\mathbb{C})} + \|s_0\|_{L^2(\mathbb{R};\mathbb{C})} < \nu,$$

there are unique $(y_0, v_0, \widetilde{\delta})$ defined in an open subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ and such that

$$\mathcal{F}(K + u_0, K_t + s_0, y_0, v_0, \beta - i\alpha + \tilde{\delta}) = (0,0), \tag{2.6.16}$$

and moreover,

$$\|(y_0, v_0)\|_{H^1\times L^2} + |\tilde{\delta}| < C\nu. \tag{2.6.17}$$

Idea of proof. The proof is very similar to that of Lemma 2.6.3, so we will only sketch the main steps.

Let $(u_0, s_0) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$ be given. Consider the rescaled BT functionals (see (2.2.5)-(2.2.6) and Lemma 2.3.5),

$$\begin{aligned}
&\widetilde{\mathcal{F}}_1(u_0,\ s_0,\ y_0,\ v_0,\ \tilde{\delta}) \\
&= u_{0,x} - v_0 - \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{K + u_0 + y_0}{2}\right) + \frac{1}{\beta - i\alpha} \sin\left(\frac{K}{2}\right) \\
&\quad - (\beta - i\alpha + \tilde{\delta}) \sin\left(\frac{K + u_0 - y_0}{2}\right) + (\beta - i\alpha) \sin\left(\frac{K}{2}\right), \qquad (2.6.18)\\
&\widetilde{\mathcal{F}}_2(u_0,\ s_0,\ y_0,\ v_0,\ \tilde{\delta}) \\
&= s_0 - y_{0,x} - \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{K + u_0 + y_0}{2}\right) + \frac{1}{\beta - i\alpha} \sin\left(\frac{K}{2}\right) \\
&\quad + (\beta - i\alpha + \tilde{\delta}) \sin\left(\frac{K + u_0 - y_0}{2}\right) - (\beta - i\alpha) \sin\left(\frac{K}{2}\right), \qquad (2.6.19)
\end{aligned}$$

for some $(y_0, v_0, a) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$. We will use the Implicit Function Theorem on the previous system. Note that once we find $(y_0, \tilde{\delta})$, v_0 rests completely determined from (2.6.18), so that we only need to solve for (2.6.19) and $(y_0, \tilde{\delta})$.

A simple computation in (2.6.19) reveals that the problem is reduced to prove that the equation

$$\begin{aligned}
&- y_{0,x} + \frac{\tilde{\delta}}{(\beta - i\alpha)^2} \sin\left(\frac{K}{2}\right) - \frac{y_0}{2(\beta - i\alpha)} \cos\left(\frac{K}{2}\right) \\
&\quad + \tilde{\delta} \sin\left(\frac{K}{2}\right) - \frac{(\beta - i\alpha) y_0}{2} \cos\left(\frac{K}{2}\right) = f,
\end{aligned} \qquad (2.6.20)$$

has a unique solution $(y_0,\ a)$ such that $y_0 \in H^1(\mathbb{R};\mathbb{C})$, for each $f \in H^1(\mathbb{R};\mathbb{C})$, continuous in function of the parameters of the problem. Simplifying (2.6.20), we obtain the equation

$$y_{0,x} + \beta \cos\left(\frac{K}{2}\right) y_0 = f + \tilde{\delta}(1 + (\beta + i\alpha)^2) \sin\left(\frac{K}{2}\right). \qquad (2.6.21)$$

Recalling that $\alpha^2 + \beta^2 = 1$, and that μ_K in (2.6.1) is integrant factor for the last ODE, we obtain

$$y_0 = \frac{1}{\mu_K} \int_{-\infty}^{x} \mu_K \left(f + \frac{2\tilde{\delta}\beta}{\beta - i\alpha} \sin\left(\frac{K}{2}\right) \right).$$

On the other hand, from (2.6.5) we conclude that we can choose $\tilde{\delta} \in \mathbb{C}$ in a unique form and such that

$$\int_{\mathbb{R}} \mu_K \left(f + \frac{2\tilde{\delta}\beta}{\beta - i\alpha} \sin\left(\frac{K}{2}\right) \right) = 0. \qquad (2.6.22)$$

We also have $|\tilde{\delta}| \leq C\|f\|_{L^2(\mathbb{R})}$. Finally, note that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \sin\left(\frac{K}{2}\right) = 0,$$

and that from (2.3.11) $\lim_{x\to\pm\infty} \beta\cos(\frac{K}{2}) = \mp\beta$. The rest of the proof is very similar to the proof of Lemma 2.6.3. □

2.7 Perturbations of breathers: inverse dynamics

2.7.1 Preliminaries

In this section we will continue assuming $\mathbb{K} = \mathbb{C}$ in Definition 2.2.3. Proposition 2.6.1 showed us the connection between a vicinity of (B, B_t) with another vicinity of the vacuum solution. Our objetive now will be the proof of an inverse result. Important differences will appear in this case, in particular we will need the orthogonality conditions (2.5.7) in the case of the breather:

$$\int_{\mathbb{R}} (z,w)\cdot\big(B_1,(B_1)_t\big)(t,x)dx = \int_{\mathbb{R}} (z,w)\cdot\big(B_2,(B_2)_t\big)(t,x)dx = 0. \tag{2.7.1}$$

Recall that B_1 and B_2, defined in general in (2.5.1)-(2.5.2), are given explicitly in (B.1.1).

Lemma 2.7.1 (Nondegenerate profile $\widetilde{B}_0$)**.** *Let us define the function*

$$\begin{aligned}\widetilde{B}_0 := B_{xxt} &+ \frac{1}{2}(\beta - i\alpha)(B - B_{t,x})\cos\left(\frac{B+K}{2}\right)\\ &- \frac{1}{2}(\beta + i\alpha)(B + B_{t,x})\cos\left(\frac{B-K}{2}\right).\end{aligned} \tag{2.7.2}$$

Then $\widetilde{B}_0$ is in the Schwartz class, provided x_1 does not satisfy (2.3.5)*. Additionally, we have the nondegeneracy condition*

$$\int_{\mathbb{R}} \widetilde{B}_0 K_x \in \mathbb{R}\backslash\{0\}. \tag{2.7.3}$$

Proof of Lemma 2.7.1. The fact that $\widetilde{B}_0$ belongs to the Schwartz class is direct, provided that K_t or K_x are well-defined, which is the case if x_1 does not satisfy (2.3.5). An explicit computation of (2.7.3) has escaped to us. For the numerical computation of this constant, see Appendix F. □

In next result, we will translate one of the orthogonality conditions in (2.7.1) to the case of a pair of functions $(u, s)(t)$ already unknown.

Lemma 2.7.2 (A priori almost orthogonality conditions). *Let $t \in [0, T^*]$ be fixed as in Definition 2.5.2. Let $(z, w)(t)$ be $H^1 \times L^2$ functions, and $x_1(t), x_2(t)$ modulational parameters given by Corollary 2.5.3, such that the second condition in* (2.7.1) *and the bound* (2.5.8) *are satisfied, and where $x_1(t)$ does not satisfy* (2.3.5). *Finally, let $\delta \in \mathbb{C}$ be a small fixed parameter, independent of time. Let us assume also that, for all $\eta > 0$ small, there are functions $(u, s)(t)$, defined in $H^1(\mathbb{R}; \mathbb{C}) \times L^2(\mathbb{R}; \mathbb{C})$, and such that*

$$\sup_{t \in [0, T^*]} \|(u, s)(t)\|_{H^1 \times L^2} \lesssim \eta, \tag{2.7.4}$$

and satisfy, for each $t \in [0, T^*]$:

$$\mathcal{F}(B + z, B_t + w, K + u, K_t + s, \beta + i\alpha + \delta) = (0, 0). \tag{2.7.5}$$

Then, necessarily we have the almost orthogonality condition

$$\int_{\mathbb{R}} (u, s) \cdot (\widetilde{B}_0, B) = \mathcal{N}(\delta, u, z), \tag{2.7.6}$$

where $\mathcal{N}$ satisfies $\mathcal{N}(0, 0, z) = O(z^2)$ (see (2.7.10)*), and $\widetilde{B}_0$ is given by* (2.7.2).

Remark 2.7.1. Condition (2.7.6) can be recast as a necessary condition for (u, s) close to zero, for being candidate to solution in (2.7.5). This condition, motivated by (2.7.1), implies that no every pair of functions (u, s) is allowed at the time of solving the inverse dynamics of Bäcklund equations. This new condition will be essential to get uniqueness when applying the Implicit Function Theorem. See [73] for another approach to this method, involving the Lyapunov-Schmidt reduction.

Proof. Explicitly writing (2.7.5), and using (2.2.5)-(2.2.6), we get the equations

$$\begin{aligned}
&B_x + z_x - K_t - s \\
&\qquad - \frac{1}{\beta + i\alpha + \delta} \sin\left(\frac{B + z + K + u}{2}\right) - (\beta + i\alpha + \delta) \sin\left(\frac{B + z - K - u}{2}\right) = 0, \\
&B_t + w - K_x - u_x \\
&\qquad - \frac{1}{\beta + i\alpha + \delta} \sin\left(\frac{B + z + K + u}{2}\right) + (\beta + i\alpha + \delta) \sin\left(\frac{B + z - K - u}{2}\right) = 0.
\end{aligned}$$

Let us try to use the second orthogonality condition in (2.5.7) with $D = B$, so that $D_2 = B_2 = \partial_{x_2} B$ (see (2.5.1)-(2.5.2)). Since $B_2 = B_x$ and $B_{2,t} = B_{t,x}$ (see (2.4.1)), we have that multiplying the first equation above by B, and the second by $B_{t,x}$, and integrating on x, we will get (after some simple cancelations, see the end of Lemma 2.3.5)

$$\begin{aligned}
&\int B_2 z + i \operatorname{Im} \int B K_t + \int B s + \frac{1}{\beta + i\alpha + \delta} \int B \sin\left(\frac{B + z + K + u}{2}\right) \\
&\qquad + (\beta + i\alpha + \delta) \int B \sin\left(\frac{B + z - K - u}{2}\right) = 0,
\end{aligned}$$

$$\int B_{2,t}w - i\,\mathrm{Im}\int B_{t,x}K_x - \int B_{2,t}u_x - \frac{1}{\beta+i\alpha+\delta}\int B_{t,x}\sin\left(\frac{B+z+K+u}{2}\right)$$
$$+ (\beta+i\alpha+\delta)\int B_{t,x}\sin\left(\frac{B+z-K-u}{2}\right) = 0.$$

Adding both equations, and using (2.5.7), we have

$$\begin{aligned}
&\int B_{xxt}u + \int Bs + \frac{1}{\beta+i\alpha+\delta}\int (B-B_{t,x})\sin\left(\frac{B+z+K+u}{2}\right)\\
&\quad + (\beta+i\alpha+\delta)\int (B+B_{t,x})\sin\left(\frac{B+z-K-u}{2}\right)\\
&= i\,\mathrm{Im}\int B_{t,x}K_x - i\,\mathrm{Im}\int BK_t.
\end{aligned} \tag{2.7.7}$$

The term $\sin\left(\frac{B+z\pm K\pm u}{2}\right)$ can be expanded as

$$\sin\left(\frac{B+z\pm K\pm u}{2}\right) = \sin\left(\frac{B\pm K}{2}\right) + \frac{1}{2}\cos\left(\frac{B\pm K}{2}\right)(z\pm u) + \mathcal{N}_{2,\pm}(x,z,u).$$

Here, $\mathcal{N}_{2,\pm}$ are nonlinear functions in (x,z,u), quadratic in (z,u). Hence, replacing in (2.7.7) we get

$$\begin{aligned}
&\int B_{xxt}u + \int Bs + \frac{1}{2(\beta+i\alpha+\delta)}\int (B-B_{t,x})\cos\left(\frac{B+K}{2}\right)u\\
&\quad - \frac{1}{2}(\beta+i\alpha+\delta)\int (B+B_{t,x})\cos\left(\frac{B-K}{2}\right)u\\
&= i\,\mathrm{Im}\int B_{t,x}K_x - i\,\mathrm{Im}\int BK_t - \frac{1}{\beta+i\alpha+\delta}\int (B-B_{t,x})\sin\left(\frac{B+K}{2}\right)\\
&\quad - (\beta+i\alpha+\delta)\int (B+B_{t,x})\sin\left(\frac{B-K}{2}\right)\\
&\quad - \frac{1}{2(\beta+i\alpha+\delta)}\int (B-B_{t,x})\cos\left(\frac{B+K}{2}\right)z\\
&\quad - \frac{1}{2}(\beta+i\alpha+\delta)\int (B+B_{t,x})\cos\left(\frac{B-K}{2}\right)z + \mathcal{N}_2(z,u).
\end{aligned}$$

Here, $\mathcal{N}_2$ is a nonlinear term of second order in (z,u). Let us define

$$\begin{aligned}
\widetilde{B}_\delta := B_{txx} &+ \frac{1}{2(\beta+i\alpha+\delta)}(B-B_{t,x})\cos\left(\frac{B+K}{2}\right)\\
&- \frac{1}{2}(\beta+i\alpha+\delta)(B+B_{t,x})\cos\left(\frac{B-K}{2}\right).
\end{aligned}$$

Thanks to Lemma 2.7.1, $\widetilde{B}_\delta = \widetilde{B}_0 + O_S(\delta)$, where $O_S(\delta)$ represents a function in the Schwartz class, bounded by δ, uniformly in space. Then,

$$\begin{aligned}\int \widetilde{B}_0 u + \int Bs = {} & i\,\mathrm{Im}\int B_{t,x}K_x - i\,\mathrm{Im}\int BK_t \\ & - \frac{1}{\beta + i\alpha + \delta}\int (B - B_{t,x})\sin\left(\frac{B+K}{2}\right) \\ & - (\beta + i\alpha + \delta)\int (B + B_{t,x})\sin\left(\frac{B-K}{2}\right) \\ & + \mathcal{N}_{1,1}(\delta, z) + \mathcal{N}_2(z,u). \qquad (2.7.8)\end{aligned}$$

Here, $\mathcal{N}_{1,1}(\delta, z)$ represents a quadratic term in δ, z, with $\mathcal{N}_{1,1}(0,z) = \mathcal{N}_{1,1}(\delta, 0) = 0$. Lastly, we will use the following result:

Lemma 2.7.3. *For each $\beta > 0$, and x_1, x_2 shifts such that x_1 does not satisfy* (2.3.5), *we have*

$$\begin{aligned}& i\,\mathrm{Im}\int B_{tx}K_x - i\,\mathrm{Im}\int BK_t - \frac{1}{\beta + i\alpha}\int (B - B_{t,x})\sin\left(\frac{B+K}{2}\right) \\ & \quad - (\beta + i\alpha)\int (B + B_{t,x})\sin\left(\frac{B-K}{2}\right) = 0. \qquad (2.7.9)\end{aligned}$$

Assuming this result, we have

$$\begin{aligned}& i\,\mathrm{Im}\int B_{tx}K_x - i\,\mathrm{Im}\int BK_t - \frac{1}{\beta + i\alpha + \delta}\int (B - B_{t,x})\sin\left(\frac{B+K}{2}\right) \\ & \quad - (\beta + i\alpha + \delta)\int (B + B_{t,x})\sin\left(\frac{B-K}{2}\right) = \mathcal{N}_{1,2}(\delta),\end{aligned}$$

where $\mathcal{N}_{1,2}$ is a term of first order in δ, with $\mathcal{N}_{1,2}(0) = 0$. Therefore, coming back to (2.7.8), we can conclude that

$$\int \widetilde{B}_0 u + \int Bs = \mathcal{N}_{1,2}(\delta) + \mathcal{N}_{1,1}(\delta, z) + \mathcal{N}_2(z,u),$$

which shows (2.7.6). For further references, $\mathcal{N}$ is given by

$$\mathcal{N}(\delta, u, z) := \mathcal{N}_{1,2}(\delta) + \mathcal{N}_{1,1}(\delta, z) + \mathcal{N}_2(z,u). \qquad (2.7.10)$$

Clearly, $\mathcal{N}(0,0,z) = O(z^2)$. □

Proof of Lemma 2.7.3. From (2.4.9)-(2.4.10), we have

$$\begin{aligned}& \text{RHS of (2.7.9)} \\ & = i\,\mathrm{Im}\int B_{tx}K_x - i\,\mathrm{Im}\int BK_t - \int B(B_x - K_t) + \int B_{tx}(B_t - K_x) \\ & = i\,\mathrm{Im}\int B_{tx}K_x - i\,\mathrm{Im}\int BK_t + \int BK_t - \int B_{tx}K_x = 0.\end{aligned}$$

Last cancelations are coming from the parity properties of K_x and K_t, see Lemma 2.3.5. $\square$

Our second result is the following (compare with Proposition 2.6.1):

Proposition 2.7.4 (Ascent to the perturbed breather profile)**.** *Let (B, B_t) be a breather profile as in Definition 2.4.1, with scaling parameter $\beta \in (-1,1)$ and shifts $x_1, x_2 \in \mathbb{R}$, and such that x_1 does not satisfy* (2.3.5)*. Let also (K, K_t) denote the complex-valued kink profile associated to (B, B_t), that is, with same parameters as (B, B_t). Then, there exist constants $\eta_1 > 0$ and $C > 0$ such that for all $0 < \eta < \eta_1$ and for all $(y, v, \tilde{\delta}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \in \mathbb{R}$ such that*[5]

$$\|(y,v)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} + |\tilde{\delta}| \leq \eta,$$

then the following is satisfied:

1. *There are unique (u, s) defined in a subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$ such that*

$$\mathcal{F}(K+u, K_t+s, y, v, \beta - i\alpha + \tilde{\delta}) = (0,0),$$

   (2.7.6) *is satisfied, and*

$$\|(u,s)\|_{H^1\times L^2} \leq C\eta.$$

2. *For all $\delta > 0$ small enough, making η_1 smaller if necessary, there are unique (z, w), defined in a subset of $H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$, and such that*

$$\mathcal{F}(B+z, B_t+w, K+u, K_t+s, \beta + i\alpha + \delta) = (0,0),$$

   (2.7.1) *is satisfied for B_1, and also,*

$$\|(z,w)\|_{H^1\times L^2} + |\delta| \leq C\eta.$$

For the proof of this result we will use several auxiliary results. The first item in Proposition 2.7.4 is consequence of the following result.

Proposition 2.7.5. *Let (K, K_t) be a complex-valued kink profile, with scaling parameter $\beta \in (-1,1)$, $\beta \neq 0$, and shifts $x_1, x_2 \in \mathbb{R}$. Then, there are constants $\nu_1 > 0$ and $C > 0$ such that for all $0 < \nu < \nu_1$ and for all $(y, v, \tilde{\delta}) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ such that*

$$\|y\|_{H^1(\mathbb{R};\mathbb{C})} + |\tilde{\delta}| < \nu,$$

there are unique $(u,s) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$ such that

1. *Smallness. We have*

$$\|(u,s)\|_{H^1\times L^2} \leq C\nu,$$

[5]Note that $(y, v, \tilde{\delta})$ are real-valued.

2. The BT are satisfied, in the sense that (u,s) *solve (see* (2.7.6)*):*

$$\mathcal{F}(K+u,\ K_t+s,\ y,\ v,\ \beta-i\alpha+\tilde{\delta}) \equiv (0,0), \tag{2.7.11}$$

$$\int_{\mathbb{R}} (u,s)\cdot(\widetilde{B}_0, B) = \mathcal{N}(\delta,u,z), \tag{2.7.12}$$

where $\mathcal{N}$ *was defined in* (2.7.10).

Remark 2.7.2. Note that (2.7.12) is a necessary condition to get

$$\int_{\mathbb{R}} (z,w)\cdot\big(B_2,(B_2)_t\big)(t,x)dx = 0,$$

obtained via modulation theory. Additionally, (2.7.12) ensures existence and uniqueness for the solution constructed via Implicit Function.

Proof of Proposition 2.7.5. Let $(y,v,\tilde{\delta}) \in H^1(\mathbb{R};\mathbb{C})\times L^2(\mathbb{R};\mathbb{C})\times\mathbb{C}$ be given and small. Let us consider the BT functionals equal zero:

$$\begin{aligned}\mathcal{F}_1\big(K+u,K_t+s,y,v,\beta-i\alpha+\tilde{\delta}\big) =&\ K_x+u_{0,x}-v\\ -\frac{1}{\beta-i\alpha+\tilde{\delta}}\sin\left(\frac{K+u+y}{2}\right) &-(\beta-i\alpha+\tilde{\delta})\sin\left(\frac{K+u-y}{2}\right)=0,\end{aligned} \tag{2.7.13}$$

$$\begin{aligned}\mathcal{F}_2\big(K+u,K_t+s,y,v,\beta-i\alpha+\tilde{\delta}\big) =&\ s-y_{0,x}\\ -\frac{1}{\beta-i\alpha+\tilde{\delta}}\sin\left(\frac{K+u+y}{2}\right) &+(\beta-i\alpha+\tilde{\delta})\sin\left(\frac{K+u-y}{2}\right)=0,\end{aligned} \tag{2.7.14}$$

plus the almost orthogonality condition (2.7.12), for some $(u,s)\in H^1(\mathbb{R};\mathbb{C})\times L^2(\mathbb{R};\mathbb{C})$. Here, z in (2.7.12) is given by a modulation (in a fixed time t far enough from the times t_k in (2.3.6)) on the breather profile. We look for a unique choice of (u,s) such that (2.7.13)-(2.7.14) are satisfied.

For simplicity, we shall redefine variables. Using $\mathcal{F}(K,K_t,0,0,\beta-i\alpha)=(0,0)$ (Lemma 2.3.5), we have

$$\begin{aligned}\widetilde{\mathcal{F}}_1(u,s,y,v,\tilde{\delta}) = u_x - v &- \frac{1}{\beta-i\alpha+\tilde{\delta}}\sin\left(\frac{K+u+y}{2}\right)+\frac{1}{\beta-i\alpha}\sin\left(\frac{K}{2}\right)\\ &-(\beta-i\alpha+\tilde{\delta})\sin\left(\frac{K+u-y}{2}\right)+(\beta-i\alpha)\sin\left(\frac{K}{2}\right),\end{aligned} \tag{2.7.15}$$

$$\begin{aligned}\widetilde{\mathcal{F}}_2\big(u,s,y,v,\tilde{\delta}\big) = s - y_x &- \frac{1}{\beta-i\alpha+\tilde{\delta}}\sin\left(\frac{K+u+y}{2}\right)+\frac{1}{\beta-i\alpha}\sin\left(\frac{K}{2}\right)\\ &+(\beta-i\alpha+\tilde{\delta})\sin\left(\frac{K+u-y}{2}\right)-(\beta-i\alpha)\sin\left(\frac{K}{2}\right).\end{aligned} \tag{2.7.16}$$

Recall that y, v and $\tilde{\delta}$ are data of the problem. We must then solve $\widetilde{\mathcal{F}}_1 = \widetilde{\mathcal{F}}_2 = 0$ and (2.7.12), for the unknown (u, s). First of all, note that once we know u, the value of s is evident from (2.7.16). Therefore, we only solve (2.7.15), for u.

Clearly $\widetilde{\mathcal{F}}_1$ defines a $\mathcal{C}^1$ functional in a neighborhood of the origin. Even more, using Lemma 2.3.5, we have $\mathcal{F}(K, K_t, 0, 0, \beta - i\alpha) = (0, 0)$ and then, $\widetilde{\mathcal{F}}_1(0, 0, 0, 0) = 0$. In order to apply Implicit Function, we must verify that the Gateaux derivative of $\widetilde{\mathcal{F}}_1$ defines a linear continuous functional, and a homeomorphism between the considered spaces. A simple checking in (2.7.15) reveals that the problem is reduced to show that the equations

$$u_x - \frac{u}{2(\beta - i\alpha)} \cos\left(\frac{K}{2}\right) - \frac{(\beta - i\alpha)}{2} \cos\left(\frac{K}{2}\right) u = f, \tag{2.7.17}$$

$$\int_{\mathbb{R}} (u, s) \cdot (\widetilde{B}_0, B) = c, \tag{2.7.18}$$

have a unique solution $u \in H^1(\mathbb{R}; \mathbb{C})$, for all $f \in H^1(\mathbb{R}; \mathbb{C})$ and $c \in \mathbb{C}$ given, continuous wrt the parameters of the problem. Simplifying (2.7.17) we get

$$u_x - \beta \cos\left(\frac{K}{2}\right) u = f.$$

Recall that $\lim_{x \to \pm\infty} \cos\left(\frac{K}{2}\right) = \mp 1$ (see (2.3.11)). From μ_K in (2.6.1), we have

$$u = \frac{\mu_K}{\mu_K(0)} u(x = 0) + \mu_K \int_0^x \frac{f}{\mu_K}.$$

In what follows, (2.7.18) will help us to find u in a unique form. Indeed, it is enough to show that

$$\int \widetilde{B}_0 \mu_K \sim \int \widetilde{B}_0(x) \operatorname{sech}(\beta(x + x_2) + i\alpha x_1)) dx \sim \int \widetilde{B}_0 K_x \neq 0,$$

which holds thanks to (2.7.3). The rest of the proof is similar to the one for Lemma 2.6.4. □

The second item in Proposition 2.7.4 requires the following previous result.

Lemma 2.7.6. *Let (B, B_t) and (K, K_t) breather and complex-valued kink profiles respectively, both with parameters $\beta \in (-1, 1) \setminus \{0\}$, shifts $x_1, x_2 \in \mathbb{R}$ and such that (2.3.5) is not satisfied. Let us consider*

$$\mu^B(x) = \frac{1}{\mu_B}(x) := \frac{\alpha^2 \cosh^2(\beta(x + x_2)) + \beta^2 \sin^2(\alpha x_1)}{\cosh(\beta(x + x_2) + i\alpha x_1)}$$

Then, $\mu^B(x)$ solves the ODE

$$\mu_x + \left(\frac{(\beta - i\alpha)}{2} \cos\left(\frac{B + K}{2}\right) + \frac{(\beta + i\alpha)}{2} \cos\left(\frac{B - K}{2}\right)\right) \mu = 0. \tag{2.7.19}$$

Proof. Direct from Lemma 2.6.2. □

Finally, the second item in Proposition 2.7.4 is a consequence of the following result.

Proposition 2.7.7. *Let (B, B_t) and (K, K_t) denote breather and complex-valued kink profiles respectively, both with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$, with x_1 not satisfying (2.3.5). Then, there are constants $\eta_1 > 0$ and $C > 0$ such that for all $0 < \eta < \eta_1$ and for all $(u, s, \delta) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ such that*

$$\|u\|_{H^1(\mathbb{R};\mathbb{C})} + |\delta| < \eta,$$

there are unique $(z, w) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$ with

$$\|(z,w)\|_{H^1\times L^2} \leq C\eta,$$

$$\mathcal{F}(B+z,\, B_t+w,\, K+u,\, K_t+s,\, \beta+i\alpha+\delta) \equiv (0,0),$$

and

$$\int_{\mathbb{R}} (z,w)\cdot\big(B_1,(B_1)_t\big)(t,x)dx = 0. \tag{2.7.20}$$

Proof. Let $(u, s, \delta) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ be given. Let us consider the system of equations for the BT (2.2.5)-(2.2.6):

$$\begin{aligned}
&\mathcal{F}_1\big(B+z,\, B_t+w,\, K+u,\, K_t+s,\, \beta+i\alpha+\delta\big)\\
&= B_x + z_x - K_t - s - \frac{1}{\beta+i\alpha+\delta}\sin\left(\frac{B+z+K+u}{2}\right)\\
&\quad - (\beta+i\alpha+\delta)\sin\left(\frac{B+z-K-u}{2}\right) = 0,
\end{aligned} \tag{2.7.21}$$

$$\begin{aligned}
&\mathcal{F}_2\big(B+z,\, B_t+w,\, K+u,\, K_t+s,\, \beta+i\alpha+\delta\big)\\
&= B_t + w - K_x - u_x - \frac{1}{\beta+i\alpha+\delta}\sin\left(\frac{B+z+K+u}{2}\right)\\
&\quad + (\beta+i\alpha+\delta)\sin\left(\frac{B+z-K-u}{2}\right) = 0,
\end{aligned} \tag{2.7.22}$$

for some $(z, w) \in H^1(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$. We will use the Implicit Function Theorem in $(\mathcal{F}_1, \mathcal{F}_2)$. Note that once defined z_0, w_0 gets completely defined from (2.7.22), therefore we just need to solve (2.7.21) for z_0. Thanks to the identity $\mathcal{F}(B, B_t, K, K_t, \beta+i\alpha) = (0,0)$, rearranging (2.7.21) and (2.7.22) we have

$$\begin{aligned}
&\widetilde{\mathcal{F}}_1\big(z,w,u,s,\delta\big)\\
&:= \; z_x - s - \frac{1}{\beta+i\alpha+\delta}\sin\left(\frac{B+K+z+u}{2}\right) + \frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right)\\
&\quad - (\beta+i\alpha+\delta)\sin\left(\frac{B-K+z-u}{2}\right) + (\beta+i\alpha)\sin\left(\frac{B-K}{2}\right) = 0,
\end{aligned} \tag{2.7.23}$$

$$\begin{aligned}
&\widetilde{\mathcal{F}}_2\big(z,w,u,s,\delta\big)\\
&:= \; w_0 - u_x - \frac{1}{\beta+i\alpha+\delta}\sin\left(\frac{B+K+z+u}{2}\right) + \frac{1}{\beta+i\alpha}\sin\left(\frac{B+K}{2}\right)\\
&\quad + (\beta+i\alpha+\delta)\sin\left(\frac{B-K+z-u}{2}\right) - (\beta+i\alpha)\sin\left(\frac{B-K}{2}\right) = 0. \qquad (2.7.24)
\end{aligned}$$

Clearly $\widetilde{\mathcal{F}}_1$ defines a $\mathcal{C}^1$ functional near zero, moreover, we have $\widetilde{\mathcal{F}}_2(0,0,0,0,0) = 0$. Then, from (2.7.23) we obtain that the problem is reduced to show that the equation

$$z_x - \frac{z_0}{2(\beta+i\alpha)}\cos\left(\frac{B+K}{2}\right) - \frac{(\beta+i\alpha)z}{2}\cos\left(\frac{B-K}{2}\right) \; = \; f,$$

possesses a unique solution $z \in H^1(\mathbb{R};\mathbb{C})$ for all $f \in H^1(\mathbb{R};\mathbb{C})$. Rearranging terms,

$$z_x - \left(\frac{\beta-i\alpha}{2}\cos\left(\frac{B+K}{2}\right) + \frac{\beta+i\alpha}{2}\cos\left(\frac{B-K}{2}\right)\right) z \; = \; f.$$

Thanks to Lemma 2.6.2, we can use the integrant factor $1/\mu_B$ (exponentially increasing) defined in (2.6.2) and (2.3.15) to obtain

$$z \; = \; \frac{\mu_B}{\mu_B(x=0)}\, z(x=0) + \mu_B \int_0^x \frac{f}{\mu_B}. \qquad (2.7.25)$$

Note that μ_B is zero only if x_1 satisfies (2.3.5), which is not the case. On the other hand, z is well-defined from condition (2.7.20), which holds true because of

$$\int_{\mathbb{R}} \mu_B B_1 dx \sim \int_{\mathbb{R}} \mu_B B_t dx \neq 0.$$

In fact, thanks to (2.6.2) and Corollary (B.1.2), and that B_t is not zero,

$$\int_{\mathbb{R}} \mu_B B_t dx \sim \int B_t(\beta B_t - i\alpha B_x) \sim \int B_t^2.$$

The rest of the proof is very similar to the one in Lemma 2.6.3. □

2.8 Permutability

2.8.1 Preliminaries

In this section we want to answer the following question: are (y_0, v_0), the functions obtained in Proposition 2.6.1, real-valued? We will show here that, if (z_0, w_0) in Proposition 2.6.1 are real-valued, then (y_0, v_0) will also be real-valued. *This fact shows Theorem 2.1.3.*

This result will hold true because of two main ingredients: (*i*) Propositions 2.4.4 and 2.4.5 combined, and (*ii*) the uniqueness property of perturbations as a consequence of the Implicit Function Theorem. These two properties will imply that all possible perturbation equals its conjugate.

In what follows, we will work in an abstract form. Let us consider $(z_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, be real-valued functions, and let (u_0, s_0, δ) be the functions obtained from Lemma 2.6.3 starting at (z_0, w_0), i.e., (u_0, s_0, δ) are such that

$$B_x + z_{0,x} - K_t - s_0 = \frac{1}{\beta + i\alpha + \delta} \sin\left(\frac{B + z_0 + K + u_0}{2}\right) + (\beta + i\alpha + \delta)\sin\left(\frac{B + z_0 - K - u_0}{2}\right), \tag{2.8.1}$$

$$B_t + w_0 - K_x - u_{0,x} = \frac{1}{\beta + i\alpha + \delta} \sin\left(\frac{B + z_0 + K + u_0}{2}\right) - (\beta + i\alpha + \delta)\sin\left(\frac{B + z_0 - K - u_0}{2}\right), \tag{2.8.2}$$

for some $\delta \in \mathbb{C}$ small. Considering $\eta_0 > 0$ small enough such that $C\eta < \nu_0$, we have the validity of the hypotheses in Lemma 2.6.4 for (u_0, s_0). With these in mind, we obtain $(y_0, v_0, \tilde{\delta}) \in H^1(\mathbb{R}; \mathbb{C}) \times L^2(\mathbb{R}; \mathbb{C}) \times \mathbb{C}$ satisfying (2.6.16), i.e.,

$$K_x + u_{0,x} - v_0 = \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{K + u_0 + y_0}{2}\right) + (\beta - i\alpha + \tilde{\delta})\sin\left(\frac{K + u_0 - y_0}{2}\right), \tag{2.8.3}$$

$$K_t + s_0 - y_{0,x} = \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{K + u_0 + y_0}{2}\right) - (\beta - i\alpha + \tilde{\delta})\sin\left(\frac{K + u_0 - y_0}{2}\right), \tag{2.8.4}$$

for some small $\tilde{\delta} \in \mathbb{C}$.

We want now to invert the order of the transformations. First, we apply Proposition 2.7.5, starting at (y_0, v_0), with fixed parameter $\beta + i\alpha + \delta$, and from Corollary 2.3.6 we obtain $(\tilde{u}_0, \tilde{s}_0) \in H^1(\mathbb{R}; \mathbb{C}) \times L^2(\mathbb{R}; \mathbb{C})$ satisfying (2.7.11) (using naturally condition (2.7.12) applied this time to $(\overline{K}, \overline{K}_t)$). Then, invoking Proposition 2.7.7 starting at $(\tilde{u}_0, \tilde{s}_0)$ with transformation parameter $\beta - i\alpha + \tilde{\delta}$, Corollary 2.4.5 ensures the existence of functions $(\tilde{z}_0, \tilde{w}_0) \in H^1(\mathbb{R}; \mathbb{C}) \times L^2(\mathbb{R}; \mathbb{C})$ such that

$$B_x + \widetilde{z}_{0,x} - \overline{K}_t - \widetilde{s}_0 = \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{B + \widetilde{z}_0 + \overline{K} + \widetilde{u}_0}{2}\right)$$

$$
+ (\beta - i\alpha + \tilde{\delta}) \sin\left(\frac{B + \widetilde{z}_0 - \overline{K} - \widetilde{u}_0}{2}\right), \tag{2.8.5}
$$

$$
\begin{aligned}
B_t + \widetilde{w}_0 - \overline{K}_x - \widetilde{u}_{0,x} \ &= \frac{1}{\beta - i\alpha + \tilde{\delta}} \sin\left(\frac{B + \widetilde{z}_0 + \overline{K} + \widetilde{u}_0}{2}\right) \\
&\quad - (\beta - i\alpha + \tilde{\delta}) \sin\left(\frac{B + \widetilde{z}_0 - \overline{K} - \widetilde{u}_0}{2}\right).
\end{aligned} \tag{2.8.6}
$$

2.8.2 Statement and proof

This being said, we are ready to announce and prove a permutability theorem.

Theorem 2.8.1 (Permutability Theorem)**.** *Let* (z_0, w_0) *and* $(\widetilde{z}_0, \widetilde{w}_0)$ *be the perturbacions defined by* (2.8.1)-(2.8.2) *and* (2.8.5)-(2.8.6) *respectively. Then, we have* $(z_0, w_0) \equiv (\widetilde{z}_0, \widetilde{w}_0)$*. In particular* $\widetilde{z}_0$ *and* $\widetilde{w}_0$ *are real-valued functions.*

Remark 2.8.1. The previous result can be represented by the diagram in Fig. 2.8.

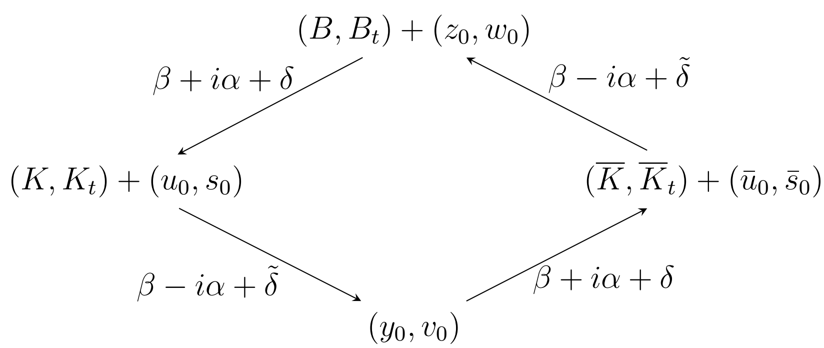

Figure 2.8: Theorem 2.8.1 about permutability, explained.

In order to prove this result, we will need the following auxiliary lemma.

Lemma 2.8.2. *Let* (B, B_t) *and* (K, K_t) *be breather and kink profiles with parameters* $\beta \in (-1, 1)$*,* $\beta \neq 0$*, and* $x_1, x_2 \in \mathbb{R}$*. Let also* $(\overline{K}, \overline{K}_t)$ *be the corresponding conjugate kink profile. Then, the following relations are satisfied:*

(i) Difference between K *and its conjugate:*

$$
K - \overline{K} = 4 \arctan\left(\frac{i\alpha \sin(\alpha x_1)}{\alpha \cosh(\beta(x + x_2))}\right). \tag{2.8.7}
$$

(ii) The following identities are satisfied:

$$\begin{aligned}\sec^2\left(\frac{B}{4}\right) &= 1+\left(\frac{\beta\sin(\alpha x_1)}{\alpha\cosh(\beta(x+x_2))}\right)^2,\\ \tan^2\left(\frac{B}{4}\right) &= \left(\frac{\beta\sin(\alpha x_1)}{\alpha\cosh(\beta(x+x_2))}\right)^2,\end{aligned} \tag{2.8.8}$$

and

$$\frac{B_t\sec^2\left(\frac{B}{4}\right)}{1+\ell^2\tan^2\left(\frac{B}{4}\right)} = \frac{4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\ell^2\beta^2\sin^2(\alpha x_1)}. \tag{2.8.9}$$

Proof. See Appendix E. □

Proof of Theorem 2.8.1. We divide the proof in several steps.

Step 1. Preliminaries. For the sake of notation we define

$$(\phi^{0,1},\phi^{0,2}) := (y_0,v_0), \qquad (\phi^{1,1},\phi^{1,2}) := (K+u_0, K_t+s_0),$$
$$(\phi^{2,1},\phi^{2,2}) := (\overline{K}+\tilde{u}_0, \overline{K}_t+\tilde{s}_0).$$

Also,

$$\begin{aligned}\varphi^1 &= (\varphi^{1,1},\varphi^{1,2}) := (B+z_0, B_t+w_0),\\ \varphi^2 &= (\varphi^{2,1},\varphi^{2,2}) := (B+\tilde{z}_0, B_t+\tilde{w}_0),\end{aligned}$$

and

$$a_1 := \beta+i\alpha+\delta, \quad a_2 := \beta-i\alpha+\tilde{\delta}.$$

Finally, let ℓ and $\tilde{\ell}$ denote

$$\ell := \frac{a_1-a_2}{a_1+a_2}, \quad \tilde{\ell} := \frac{a_1+a_2}{a_1-a_2}. \tag{2.8.10}$$

Note that both values ℓ and $\tilde{\ell}$ are well-defined, since $\delta,\tilde{\delta}$ are small. We want to prove $\varphi^1\equiv\varphi^2$. In order to prove this, let us define the auxiliary function $(\phi^{3,1},\phi^{3,2})$ via the identities

$$\phi^{3,1}-\phi^{1,1} = -4\arctan\left(\ell\tan\left(\frac{\varphi^{1,1}-\phi^{0,1}}{4}\right)\right), \tag{2.8.11}$$

and

$$\phi^{3,2}-\phi^{1,2} = \frac{-\ell(\varphi^{1,2}-\phi^{0,2})\sec^2\left(\frac{\varphi^{1,1}-\phi^{0,1}}{4}\right)}{1+\ell^2\tan^2\left(\frac{\varphi^{1,1}-\phi^{0,1}}{4}\right)}. \tag{2.8.12}$$

Step 2. First identities. Note that if

$$\begin{aligned}(\phi^{0,1},\phi^{0,2}) = (0,0), \quad (\phi^{1,1},\phi^{1,2}) = (K,K_t), \quad (\varphi^{1,1},\varphi^{1,2}) = (B,B_t),\\ a_1=\beta+i\alpha \quad \text{ y } \quad a_2=\beta-i\alpha,\end{aligned} \tag{2.8.13}$$

then from (2.8.7) we have

$$\phi^{3,1} = K - 4\arctan\left(\frac{2i\alpha}{2\beta}\frac{\beta\sin(\alpha x_1)}{\alpha\cosh(\beta(x+x_2))}\right) = \overline{K}.$$

Similarly, replacing (2.8.13) in (2.8.12), we obtain

$$\phi^{3,2} = K_t - \frac{\ell B_t \sec^2\left(\frac{B}{4}\right)}{1+\ell^2\tan^2\left(\frac{B}{4}\right)}. \tag{2.8.14}$$

Therefore, using (2.8.8) and (2.8.9), we obtain that (2.8.14) is reduced to simplifying the RHS of the identity

$$\phi^{3,2} = \frac{4i\alpha e^{\beta(x+x_2)+i\alpha x_1}}{1+e^{2\beta(x+x_2)+2i\alpha x_1}} - \frac{4\ell\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\ell^2\beta^2\sin^2(\alpha x_1)}.$$

Let us consider the notation

$$\theta_1 := \alpha x_1, \quad \theta_2 := \beta(x+x_2), \quad \theta := \beta(x+x_2)+i\alpha x_1. \tag{2.8.15}$$

We have,

$$\begin{aligned}
\phi^{3,2} &= \\
&= \frac{4i\alpha^3 e^{\theta}(\cosh^2(\theta_2)-\sin^2(\theta_1)) - 4i\alpha^3(1+e^{2\theta})\cos(\theta_1)\cosh(\theta_2)}{(1+e^{2\theta})(\alpha^2\cosh^2(\theta_2)-\alpha^2\sin^2(\theta_1))} \\
&= \frac{4i\alpha e^{\theta}(\cosh^2(\theta_2)-\sin^2(\theta_1)) - 4i\alpha(1+e^{2\theta})\cos(\theta_1)\cosh(\theta_2)}{(1+e^{2\theta})(\cosh^2(\theta_2)-\sin^2(\theta_1))} \\
&= \frac{i\alpha\big(e^{\theta+2\theta_2}+e^{\theta-2\theta_2}+e^{\theta+2i\theta_1}+e^{\theta-2i\theta_1}-(1+e^{2\theta})(e^{i\theta_1}+e^{-i\theta_1})(e^{\theta_2}+e^{-\theta_2})\big)}{\big(1+e^{2\theta}\big)\big(\cosh^2(\theta_2)-\sin^2(\theta_1)\big)} \\
&= \frac{-i\alpha\big(e^{3\theta}+e^{-\theta}+2e^{\theta}\big)}{\big(1+e^{2\theta}\big)\big(\cosh^2(\theta_2)-\sin^2(\theta_1)\big)} = \frac{-i\alpha e^{-\theta}\big(1+e^{2\theta}\big)^2}{\big(1+e^{2\theta}\big)\big(\cosh^2(\theta_2)-\sin^2(\theta_1)\big)} \\
&= \frac{-i\alpha e^{-\theta}\big(1+e^{2\theta}\big)}{\cosh^2(\theta_2)-\sin^2(\theta_1)} = \frac{-4i\alpha e^{-\theta}\big(1+e^{2\theta}\big)}{\big(1+e^{2\theta}\big)\big(e^{-2i\theta_1}+e^{-2\theta_2}\big)} \\
&= \frac{-4i\alpha e^{-\theta}}{e^{-2i\theta_1}+e^{-2\theta_2}} = \frac{-4i\alpha e^{\theta_2-i\theta_1}}{1+e^{2(\theta_2-i\theta_1)}} = \overline{K}_t.
\end{aligned}$$

Then, if (2.8.13) holds, necessarily

$$\phi^3 = (\phi^{3,1}, \phi^{3,2}) = (\overline{K}, \overline{K}_t). \tag{2.8.16}$$

Step 3. ODEs satisfied by ϕ^3. Let us consider now general values of ϕ^0, ϕ^1, φ^1 and a_1, a_2, as before. We shall prove that $\phi^3 = (\phi^{3,1}, \phi^{3,2})$ defined in (2.8.11)-(2.8.12) satisfy the identities

$$\phi^{3,1}_x - \phi^{0,2} = \frac{1}{a_1}\sin\left(\frac{\phi^{3,1}+\phi^{0,1}}{2}\right) + a_1\sin\left(\frac{\phi^{3,1}-\phi^{0,1}}{2}\right), \tag{2.8.17}$$

$$\phi^{3,2} - \phi_x^{0,1} = \frac{1}{a_1}\sin\left(\frac{\phi^{3,1}+\phi^{0,1}}{2}\right) - a_1\sin\left(\frac{\phi^{3,1}-\phi^{0,1}}{2}\right). \qquad (2.8.18)$$

Hence, from (2.8.16) we conclude that $(\phi^{3,1},\phi^{3,2}) \equiv (\phi^{2,1},\phi^{2,2})$. Similarly, denoting $\phi^4 := (\phi^{4,1},\phi^{4,2})$ the solution to

$$\phi^{2,1} - \phi^{4,1} = -4\arctan\left(\frac{a_1+a_2}{a_1-a_2}\tan\left(\frac{\varphi^{2,1}-\phi^{0,1}}{4}\right)\right),$$
$$\phi^{2,2} - \phi^{4,2} = -\frac{\tilde{\ell}\big(\varphi^{2,2}-\phi^{0,2}\big)\sec^2\left(\frac{\varphi^{2,1}-\phi^{0,1}}{4}\right)}{1+\tilde{\ell}^2\tan^2\left(\frac{\varphi^{2,1}-\phi^{0,1}}{4}\right)},$$

and proving that $(\phi^{4,1},\phi^{4,2})$ satisfy

$$\phi_x^4 - \phi_t^0 = \frac{1}{a_1}\sin\left(\frac{\phi^4+\phi^0}{2}\right) + a_1\sin\left(\frac{\phi^4-\phi^0}{2}\right),$$
$$\phi_t^4 - \phi_x^0 = \frac{1}{a_1}\sin\left(\frac{\phi^4+\phi^0}{2}\right) - a_1\sin\left(\frac{\phi^4-\phi^0}{2}\right),$$

then we have $(\phi^{4,1},\phi^{4,2}) \equiv (\phi^{1,1},\phi^{1,2})$. From here we conclude that $(\varphi^{1,1},\varphi^{1,2}) \equiv (\varphi^{2,1},\varphi^{2,2})$. Moreover,

$$\tan\left(\frac{\varphi^1-\phi^0}{4}\right) = -\frac{a_1+a_2}{a_1-a_2}\tan\left(\frac{\phi^2-\phi^1}{4}\right). \qquad (2.8.19)$$

This identity will be used a posteriori. Let us now show (2.8.17) and (2.8.18).

Step 4. Proof of (2.8.17)**.** In fact, from (2.8.11) we have

$$\varphi^{1,1} - \phi^{0,1} = -4\arctan\left(\ell^{-1}\tan\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right). \qquad (2.8.20)$$

Then, taking derivative wrt x,

$$\varphi_x^{1,1} - \phi_x^{0,1} = \frac{-\ell^{-1}(\phi_x^{3,1}-\phi_x^{1,1})\sec^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}{1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}, \qquad (2.8.21)$$

or

$$-\frac{1}{\ell}(\phi_x^{3,1}-\phi_x^{1,1})\sec^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right) = \left(1+\frac{1}{\ell^2}\tan\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right)\big(\varphi_x^{1,1}-\phi_x^{0,1}\big). \qquad (2.8.22)$$

On the other hand, from (2.8.20) it is not difficult to show that

$$\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right) = \frac{-2\ell^{-1}\tan\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}{1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)},$$
$$\cos\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right) = \frac{1-\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}{1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}. \qquad (2.8.23)$$

Since from Proposition 2.6.1 we have the connections

$$\mathbb{B}(\phi^{0,1},\phi^{0,2}) \xrightarrow{a_2} (\phi^{1,1},\phi^{1,2}), \qquad \mathbb{B}(\phi^{1,1},\phi^{1,2}) \xrightarrow{a_1} (\varphi^{1,1},\varphi^{1,2}),$$

which in particular imply

$$\begin{aligned}
\varphi_x^{1,1} - \phi^{1,2} &= \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}+\phi^{1,1}}{2}\right) + a_1\sin\left(\frac{\varphi^{1,1}-\phi^{1,1}}{2}\right)\\
\phi^{1,2} - \phi_x^{0,1} &= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right),
\end{aligned}$$

we can rewrite the LHS of (2.8.21) as follows:

$$\begin{aligned}
&\varphi_x^{1,1} - \phi_x^{0,1}\\
&\quad = \varphi_x^{1,1} - \phi^{1,2} + \phi^{1,2} - \phi_x^{0,1}\\
&\quad = \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}+\phi^{1,1}}{2}\right) + a_1\sin\left(\frac{\varphi^{1,1}-\phi^{1,1}}{2}\right)\\
&\qquad + \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
&\quad = \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}+\phi^{0,1}+\phi^{1,1}}{2}\right) + a_1\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}+\phi^{0,1}-\phi^{1,1}}{2}\right)\\
&\qquad + \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right).
\end{aligned}$$

Expanding terms,

$$\begin{aligned}
\varphi_x^{1,1} - \phi_x^{0,1} = {} & \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{0,1}+\phi^{1,1}}{2}\right)\\
& + \frac{1}{a_1}\cos\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\sin\left(\frac{\phi^{0,1}+\phi^{1,1}}{2}\right)\\
& + a_1\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
& - a_1\cos\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
& + \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right).
\end{aligned}$$

Replacing this last identity in the RHS of (2.8.22), and using the identities found in (2.8.23), we have

$$-\frac{1}{\ell}(\phi_x^{3,1} - \phi_x^{1,1})\sec^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)$$

$$
\begin{aligned}
&= \left(1 + \frac{1}{\ell^2}\tan^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\right)\left(\frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) - a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right) \\
&+ \left(1 - \frac{1}{\ell^2}\tan^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\right)\left(\frac{1}{a_1}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) - a_1 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right) \\
&- \frac{2}{\ell}\tan\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) + a_1 \cos\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right). \qquad (2.8.24)
\end{aligned}
$$

Then, using that the LHS of (2.8.21) can be rewritten as

$$\phi^{3,1}_x - \phi^{1,1}_x = \phi^{3,1}_x - \phi^{0,2} + \phi^{0,2} - \phi^{1,1}_x,$$

recalling that $\mathbb{B}(\phi^{0,1}, \phi^{0,2}) \xrightarrow{a_2} (\phi^{1,1}, \phi^{1,2})$, i.e.,

$$
\begin{aligned}
\phi^{1,1}_x - \phi^{0,2} &= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) + a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right) \\
\phi^{1,2} - \phi^{0,1}_x &= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) - a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right),
\end{aligned}
$$

we can replace (2.8.24) in (2.8.21) to get

$$
\begin{aligned}
\phi^{3,1}_x - \phi^{0,2} &= \phi^{3,1}_x - \phi^{1,1}_x + (\phi^{1,1}_x - \phi^{0,2}) \\
&= -\cos^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\left(\ell + \ell^{-1}\tan^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\right) \\
&\quad \left(\frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) - a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right) \\
&\quad - \cos^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\left(\ell - \ell^{-1}\tan^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\right) \\
&\quad \left(\frac{1}{a_1}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) - a_1 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right) \\
&\quad + 2\sin\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\cos\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right) \\
&\quad \left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) + a_1 \cos\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right)\right) \\
&\quad + \frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) + a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right).
\end{aligned}
$$

A further simplification gives

$$
\begin{aligned}
\phi^{3,1}_x - \phi^{0,2} = &\frac{1}{a_2}\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right) + a_2 \sin\left(\frac{\phi^{1,1} - \phi^{0,1}}{2}\right) \\
&- \cos^2\left(\frac{\phi^{3,1} - \phi^{1,1}}{4}\right)\sin\left(\frac{\phi^{1,1} + \phi^{0,1}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\left[\left(\frac{1}{a_2}+\frac{1}{a_1}\right)\ell+\ell^{-1}\left(\frac{1}{a_2}-\frac{1}{a_1}\right)\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&+\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
&\left[(a_1+a_2)\ell+(a_2-a_1)\ell^{-1}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&+\sin\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)+a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right).
\end{aligned}
$$

Thanks to (2.8.10), we have

$$
\begin{aligned}
&\phi_x^{3,1}-\phi^{0,2}\\
&=\frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)+a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
&\quad-\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)\\
&\quad\left[\left(\frac{1}{a_2}-\frac{1}{a_1}\right)\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)+\left(\frac{1}{a_2}+\frac{1}{a_1}\right)\sin^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&\quad+\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\\
&\quad\left[(a_1-a_2)\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)-(a_1+a_2)\sin^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&\quad+\sin\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)+a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right).
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
\phi_x^{3,1}-\phi^{0,2}&=-\frac{1}{a_1}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)\left[\sin^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)-\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&\quad+a_1\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\left[\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)-\sin^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right]\\
&\quad+\sin\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)+a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right)\\
&=\frac{1}{a_1}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\\
&\quad+a_1\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\\
&\quad+\sin\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)+a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right).
\end{aligned}
$$

Finally,

$$\phi_x^{3,1} - \phi^{0,2} = \frac{1}{a_1}\sin\left(\frac{\phi^{3,1}+\phi^{0,1}}{2}\right) + a_1\sin\left(\frac{\phi^{3,1}-\phi^{0,1}}{2}\right).$$

This ends the proof of the case (2.8.17).

Step 4. Proof of (2.8.18)**.** We proceed as before. First, we write the LHS of (2.8.12) as follows:

$$\phi^{3,2} - \phi^{1,2} = \phi^{3,2} - \phi_x^{0,1} + \phi_x^{0,1} - \phi^{1,2}.$$

Similarly, we have $\varphi^{1,2} - \phi^{0,2} = \varphi^{1,2} - \phi_x^{1,1} + \phi_x^{1,1} - \phi^{0,2}$. Thanks to (2.8.11), we have that (2.8.12) reads now

$$\begin{aligned}\phi^{3,2} - \phi_x^{0,1} &= \phi^{1,2} - \phi_x^{0,1} \\ &\quad - \ell\big(\varphi^{1,2} - \phi_x^{1,1} + \phi_x^{1,1} - \phi^{0,2}\big)\left(1 + \ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right) \\ &\quad \cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right).\end{aligned} \tag{2.8.25}$$

On the other hand, recall that

$$\phi_x^{1,1} - \phi^{0,2} = \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) + a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right). \tag{2.8.26}$$

Similarly, we have

$$\begin{aligned}&\varphi^{1,2} - \phi_x^{1,1} \\ &= \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}+\phi^{1,1}}{2}\right) - a_1\sin\left(\frac{\varphi^{1,1}-\phi^{1,1}}{2}\right) \\ &= \frac{1}{a_1}\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}+\phi^{0,1}+\phi^{1,1}}{2}\right) - a_1\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}+\phi^{0,1}-\phi^{1,1}}{2}\right) \\ &= \frac{1}{a_1}\left(\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) + \cos\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right)\right) \\ &\quad - a_1\left(\sin\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) - \cos\left(\frac{\varphi^{1,1}-\phi^{0,1}}{2}\right)\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right).\end{aligned}$$

Therefore, (2.8.23) implies

$$\begin{aligned}&\varphi^{1,2} - \phi_x^{1,1} \\ &= \frac{-2\ell^{-1}\tan\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}{1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\ &\quad + \frac{1-\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}{1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)}\left(\frac{1}{a_1}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_1\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right)\end{aligned} \tag{2.8.27}$$

Therefore, replacing (2.8.26) and (2.8.27) in (2.8.25) we get

$$
\begin{aligned}
&\phi^{3,2} - \phi^{0,1}_x \\
&= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) \\
&\quad + \sin\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\left(\frac{1}{a_1}\cos\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_1\cos\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\
&\quad - \left(\ell\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right) - \ell^{-1}\sin^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right) \\
&\qquad \left(\frac{1}{a_1}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_1\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\
&\quad - \ell\left(\frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) + a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\
&\qquad \left(1+\ell^{-2}\tan^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right)\right)\cos^2\left(\frac{\phi^{3,1}-\phi^{1,1}}{4}\right).
\end{aligned}
$$

Finally, gathering terms and using the value of ℓ we obtain

$$
\begin{aligned}
&\phi^{3,2} - \phi^{0,1}_x \\
&= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) \\
&\quad - \frac{1}{a_1^2-a_2^2}\left((a_1^2+a_2^2)\cos\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right) - 2a_1a_2\right) \\
&\qquad \left(\frac{1}{a_1}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) + a_1\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\
&\quad - \frac{1}{a_1^2-a_2^2}\left(a_1^2+a_2^2-2a_1a_2\cos\left(\frac{\phi^{3,1}-\phi^{1,1}}{2}\right)\right) \\
&\qquad \left(\frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) + a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right)\right) \\
&= \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}+\phi^{0,1}}{2}\right) - a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) + \frac{1}{a_1}\sin\left(\frac{\phi^{3,1}+\phi^{0,1}}{2}\right) \\
&\quad - a_1\sin\left(\frac{\phi^{3,1}-\phi^{0,1}}{2}\right) - \frac{1}{a_2}\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) + a_2\sin\left(\frac{\phi^{1,1}-\phi^{0,1}}{2}\right) \\
&= \frac{1}{a_1}\sin\left(\frac{\phi^{3,1}+\phi^{0,1}}{2}\right) - a_1\sin\left(\frac{\phi^{3,1}-\phi^{0,1}}{2}\right),
\end{aligned}
$$

which finally proves (2.8.18). □

Corollary 2.8.3. *Under the assumptions of Theorem 2.8.1 we have*

$$(u_0, s_0) = (\bar{\tilde{u}}_0, \bar{\bar{s}}_0), \quad \delta = \bar{\bar{\delta}}.$$

Proof. Theorem 2.8.1 implies $(z_0, w_0) \equiv (\tilde{z}_0, \tilde{w}_0)$. Then, after conjugation of (2.8.5) and (2.8.6) we have

$$
\begin{aligned}
B_x + z_{0,x} - K_t - \bar{\tilde{s}}_0 = {}& \frac{1}{\beta + i\alpha + \bar{\tilde{\delta}}} \sin\left(\frac{B + z_0 + K + \bar{\tilde{u}}_0}{2}\right) \\
& + (\beta + i\alpha + \bar{\tilde{\delta}}) \sin\left(\frac{B + z_0 - K - \bar{\tilde{u}}_0}{2}\right), \\
B_t + w_0 - K_x - \bar{\tilde{u}}_{0,x} = {}& \frac{1}{\beta + i\alpha + \bar{\tilde{\delta}}} \sin\left(\frac{B + z_0 + K + \bar{\tilde{u}}_0}{2}\right) \\
& - (\beta + i\alpha + \bar{\tilde{\delta}}) \sin\left(\frac{B + z_0 - K - \bar{\tilde{u}}_0}{2}\right).
\end{aligned}
$$

Therefore, thanks to the uniqueness of perturbations (via Implicit Function Theorem), and using (2.8.1) and (2.8.2), we conclude the result. □

The following result will be essential in the rest of the proof.

Corollary 2.8.4 (Real-valued character of the double BT)**.** *Let (z_0, w_0) be satisfying the hypotheses of Theorem 2.8.1. Then y_0, v_0 are real-valued.*
Remark 2.8.2. This last result finally proves Theorem 2.1.3.

Proof. Note that Corollary 2.8.3 implies $\delta = \bar{\tilde{\delta}}$. Then, from (2.8.19)

$$
\tan\left(\frac{B + z_0 - y_0}{4}\right) = \frac{2\beta + \delta + \tilde{\delta}}{2i\alpha + \delta - \tilde{\delta}} \tan\left(\frac{K + u_0 - \overline{K} - \bar{u}_0}{4}\right).
$$

Simplifying, we get

$$
\tan\left(\frac{B + z_0 - y_0}{4}\right) = \frac{\beta + \operatorname{Re}\delta}{\alpha + \operatorname{Im}\delta} \tanh\left(\frac{\operatorname{Im}(K + u_0)}{2}\right),
$$

so that $y_0(x)$ is real-valued. □

2.9 2-kinks and kink-antikink perturbations

In this section we will assume that $\mathbb{K} = \mathbb{R}$ in Definition 2.2.3. Consider $(D, D_t) = (R, R_t)$ or (A, A_t), 2-kink or kink-antikink profiles respectively, with shifts $x_1, x_2 \in \mathbb{R}$ and speed $\beta \in (-1, 1)$, $\beta \neq 0$. Also, we will consider (Q, Q_t) a real-valued kink profile with speed $-\beta$ and shift $x_1 + x_2$, see (2.4.18) for more details.

In what follows, we denote by d the parameter of the BT associated to (D, D_t): if $(D, D_t) = (R, R_t)$, then $d := a_3(\beta) = -a(\beta)$; and if $(D, D_t) = (A, A_t)$, then $d := a(\beta)$. See Fig. 2.7 for more details.

Proposition 2.9.1 (Connection to the zero solution)**.** *Let* (D, D_t) *be a kink-antikink or 2-kink profile, as in Definitions 2.4.2 and 2.4.3, with speed* $\beta \in (-1,1) \setminus \{0\}$ *and shifts* $x_1, x_2 \in \mathbb{R}$. *Let also* $(Q, Q_t)(\cdot; -\beta, x_1 + x_2)$ *be a real-valued kink profile associated to* (D, D_t)*, with BT parameter* d*. Then, there exist constants* $\eta_0 > 0$ *and* $C > 0$ *such that, for all* $0 < \eta < \eta_0$ *and for all* $(z_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *such that*

$$\|(z_0, w_0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} < \eta,$$

the following holds:

1. *There are unique* (u_0, s_0, b) *defined in an open subset of* $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}$ *such that*

$$\mathcal{F}(D + z_0, D_t + w_0, Q + u_0, Q_t + s_0, d + b) = (0,0), \tag{2.9.1}$$

and where

$$\|(u_0, s_0)\|_{H^1 \times L^2} + |b| < C\eta. \tag{2.9.2}$$

2. *Making* η_0 *smaller if necessary, there are unique* $(y_0, v_0, \tilde{b})$*, defined in an open subset of* $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}$*, and such that*

$$\mathcal{F}(Q + u_0, Q_t + s_0, y_0, v_0, a^{-1}(\beta) + \tilde{b}) = (0,0), \tag{2.9.3}$$

and moreover,

$$\|(y_0, v_0)\|_{H^1 \times L^2} + |\tilde{b}| < C\eta. \tag{2.9.4}$$

The proof of this result is very similar to the one of Proposition 2.6.1, so that we only indicate the main differences. First of all, we need the following integrant factor lemma. For the proofs, see Appendix D.

Lemma 2.9.2 (Integrant factor for the 2-kink)**.** *Let* (R, R_t) *and* (Q, Q_t) *be 2-kink and real-valued kink profiles as in Proposition 2.9.1. Let us consider*

$$\mu_R(x) := \frac{\cosh(\gamma(x + x_1 + x_2))}{\cosh^2(\gamma x_1) + \beta^2 \sinh^2(\gamma(x + x_2))} = \frac{1}{4\gamma} R_x - \frac{1}{4\beta\gamma} R_t.$$

Then, $\mu_R(x)$ *is smooth and solves the ODE:*

$$\mu_x - \frac{1}{2}\left(\frac{1}{d}\cos\left(\frac{R+Q}{2}\right) + d\cos\left(\frac{R-Q}{2}\right)\right)\mu = 0, \tag{2.9.5}$$

where $d = a_3 = -a(\beta)$*. Moreover, we have the nondegeneracy condition*

$$\int_{\mathbb{R}} \mu_R \cdot (R_x - Q_t) = \frac{4}{\beta} \neq 0.$$

Lemma 2.9.3 (Integrant factor for the kink-antikink)**.** *Let (A, A_t) and (Q, Q_t) be kink-antikink and real-valued kink profiles, respectively exactly as in Proposition 2.9.1. Let us consider*

$$\mu_A(x) := \frac{\cosh(\gamma(x + x_1 + x_2))}{\beta^2 \cosh^2(\gamma(x + x_2)) + \sinh^2(\gamma x_1)} = \frac{1}{4\beta^2\gamma} A_t - \frac{1}{4\beta\gamma} A_x.$$

Then, $\mu_A(x)$ is smooth and solves the ODE:

$$\mu_x - \frac{1}{2}\left(\frac{1}{d}\cos\left(\frac{A+Q}{2}\right) + d\cos\left(\frac{A-Q}{2}\right)\right)\mu = 0, \tag{2.9.6}$$

where $d = a = a(\beta)$. Moreover, we have

$$\int_{\mathbb{R}} \mu_A \cdot (A_x - Q_t) = -\frac{4}{\beta} \neq 0. \tag{2.9.7}$$

In order to show (2.9.1)-(2.9.2), first item in Proposition 2.9.1, we follow the proof in Lemma 2.6.3. After linearizing the BT, we must study whether or not the ODE

$$\begin{aligned} u_{0,x} + \left(\frac{1}{2d}\cos\left(\frac{D+Q}{2}\right) + \frac{d}{2}\cos\left(\frac{D-Q}{2}\right)\right) u_0 \\ = f + \frac{b}{d^2}\sin\left(\frac{D+Q}{2}\right) + b\sin\left(\frac{D-Q}{2}\right), \end{aligned}$$

has a unique solution (u_0, b) such that $u_0 \in H^1(\mathbb{R})$, for each $f \in H^1(\mathbb{R})$. Using μ as in Lemmas 2.9.2 or 2.9.3 depending on the cases $D = A, R$, we have

$$u_0 = \frac{1}{\mu}\int_{-\infty}^{x} \mu\left(f + \frac{b}{d}(D_x - Q_t)\right).$$

Additionally, Lemmas 2.9.2-2.9.3 imply that we can choose $b \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \mu\left(f + \frac{b}{d}(D_x - Q_t)\right) = 0.$$

The rest of the proof is similar to the one in Lemma 2.6.3.

Finally, (2.9.3) and (2.9.4), part of the second item in Proposition 2.9.1, are consequence of a new application of the Implicit Function Theorem. In fact, we must study whether or not the equation

$$-y_{0,x} + \frac{\tilde{b}}{a_2^2}\sin\left(\frac{Q}{2}\right) - \frac{y_0}{2a_2}\cos\left(\frac{Q}{2}\right) + \tilde{b}\sin\left(\frac{Q}{2}\right) - \frac{a_2 y_0}{2}\cos\left(\frac{Q}{2}\right) = f, \tag{2.9.8}$$

possesses a unique solution $(y_0, \tilde{b})$ such that $y_0 \in H^1(\mathbb{R})$, for each $f \in H^1(\mathbb{R})$. Simplifying (2.9.8) and recalling that $\gamma = (1-\beta^2)^{-1/2}$, we get

$$y_{0,x} + \gamma \cos\left(\frac{Q}{2}\right) y_0 = f + \frac{2\tilde{b}}{1-\beta}\sin\left(\frac{Q}{2}\right).$$

We define now the integrant factor $\mu_Q(x) := \operatorname{sech}(\gamma(x+x_0))$. Since μ_Q decays exponentially fast, we have

$$y_0 \ = \ \frac{1}{\mu_Q}\int_{-\infty}^{x} \mu_Q\left(f + \frac{2\tilde{b}}{1-\beta}\sin\left(\frac{Q}{2}\right)\right).$$

Note that $\int_{\mathbb{R}} \mu_Q \sin(\frac{Q}{2}) = \int_{\mathbb{R}} \operatorname{sech}^2(\gamma(x+x_0)) = \frac{2}{\gamma}$. Then, we can choose $\tilde{b} \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \mu_Q\left(f + \frac{2\tilde{b}}{1-\beta}\sin\left(\frac{Q}{2}\right)\right) = 0.$$

The rest of the proof is similar to the one in Lemma 2.6.4.

2.10 2-kink and kink-antikink perturbations: inverse dynamics

In this section we still assume $\mathbb{K} = \mathbb{R}$ in Definition 2.2.3. Our objective will be to show the following result, in the vein of Proposition 2.7.4.

Proposition 2.10.1 (Connection with 2-soliton solutions)**.** *Let (D, D_t) be a 2-kink or kink-antikink profile, as in Definitions 2.4.2-2.4.3, with speed $\beta \in (-1,1) \setminus \{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$. Let $(Q, Q_t) = (Q, Q_t)(\cdot; -\beta, x_1 + x_2)$ be the real-valued kink profile associated to (D, D_t). Then, there are constants $\eta_1 > 0$ and $C > 0$ such that, for all $0 < \eta < \eta_1$ and for all $(y, v, \tilde{b}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}$, if*

$$\|(y,v)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} + |\tilde{b}| < \eta,$$

then the following holds:

1. *There are unique (u, s) defined in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that*

$$\mathcal{F}(Q+u, Q_t+s, y, v, a(\beta)^{-1} + \tilde{b}) = (0,0),$$

and for some $\widetilde{D}_0$ in the Schwartz class and z given by the modulation (2.5.7),

$$\int_{\mathbb{R}} (u,s)\cdot(\widetilde{D}_0, D) = \mathcal{N}_D(\tilde{b}, u, z), \qquad \int \widetilde{D}_0 Q_x \neq 0, \tag{2.10.1}$$

and where $\mathcal{N}_D(\tilde{b}, u, z)$ is a nonlinear term in u, and where additionally

$$\|(u,s)\|_{H^1\times L^2} < C\eta.$$

2. *If $|b| < \eta$, and making η_1 smaller if necessary, there are unique (z, w), defined in a subset of $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, and such that*

$$\mathcal{F}(D + z, D_t + w, Q + u, Q_t + s, d + b) = (0, 0),$$

$$\int_{\mathbb{R}} (z, w) \cdot (D_1, (D_1)_t) = 0, \tag{2.10.2}$$

and finally, $\|(z, w)\|_{H^1 \times L^2} < C\eta$.

Since the proof of this result is similar to the proof of Proposition 2.7.4, we only sketch the main ideas. The first part of Proposition 2.10.1 requires to understand if the ODE

$$u_x - \gamma \cos\left(\frac{Q}{2}\right) u = f, \tag{2.10.3}$$

possesses a unique solution $u \in H^1(\mathbb{R})$ for all $f \in H^1(\mathbb{R})$. The associated integrating factor here is $\mu_Q(x) := \cosh(\gamma(x + x_0))$, and the solution u is given by

$$u = \frac{1}{\mu_Q} \mu_Q(0)\, u(0) + \frac{1}{\mu_Q} \int_0^x \mu_Q f.$$

Precisely, condition (2.10.1) allows us to choose u in a unique form. The value of $\widetilde{D}_0$, obtained in the same form as $\widetilde{B}_0$ was obtained in (2.7.2), is given by

$$\widetilde{D}_0 := D_{xxt} + \frac{1}{2d}(D - D_{t,x}) \cos\left(\frac{D + Q}{2}\right) - \frac{1}{2} d(D + D_{t,x}) \cos\left(\frac{D - Q}{2}\right).$$

The rest of the proof is the same as before. For the second part, we will need the following integrating factors:

$$\mu^A(x) =: \frac{1}{\mu_A}(x) = \frac{\beta^2 \cosh^2(\gamma(x + x_2)) + \sinh^2(\gamma x_1)}{\cosh(\gamma(x + x_1 + x_2))},$$

and

$$\mu^R(x) := \frac{1}{\mu_R}(x) = \frac{\beta^2 \sinh^2(\gamma(x + x_2)) + \cosh^2(\gamma x_1)}{\cosh(\gamma(x + x_1 + x_2))},$$

which are smooth and solve the ODE

$$\mu_x + \left(\frac{1}{2d} \cos\left(\frac{D + Q}{2}\right) + \frac{d}{2} \cos\left(\frac{D - Q}{2}\right)\right) \mu = 0,$$

with $D = A, R$, $d = a$ and $d = a_3 = -a$ respectively. Both integrant factors are exponentially increasing in space. With these functions on hand, we plan to conclude the proof. Indeed, the second part requires the study of the ODE

$$z_x - \left(\frac{1}{2d} \cos\left(\frac{D + Q}{2}\right) - \frac{d}{2} \cos\left(\frac{D - Q}{2}\right)\right) z = f.$$

Simplifying, and using the integrant factors before proposed, we have

$$z \;=\; \frac{1}{\mu}\mu(0)\, z(0) + \frac{1}{\mu}\int_0^x \mu f, \quad \mu = \mu^R,\ \mu^A. \tag{2.10.4}$$

Once again, the uniqueness is obtained by imposing (2.10.2). The rest of the proof is well-known.

2.11 Stability of 2-solitons. Proof of Theorem 2.1.1

In this section we prove Theorem 2.1.1. Let us consider (ϕ_0, ϕ_1) satisfying (2.1.8) for some $\eta < \eta_0$ small. Let also $(\phi(t), \phi_t(t))$ be the unique solution of (2.1.1) with initial condition $(\phi, \phi_t)(0) = (\phi_0, \phi_1)$. Note that $(\phi(t), \phi_t(t)) - (D, D_t)(t) \in H^1 \times L^2$.

Proof of Theorem 2.1.1. Let $\varepsilon_0 > 0$ be a fixed parameter. Let (D, D_t) be a profile defined as in 2.5.1. Consider the tubular neighborhood (2.5.4), for $t \leq T^* < +\infty$. Note that in order to recover the 2-soliton solutions of Remarks 2.4.2 and 2.4.3, it is enough to redefine

$$(D, D_t)(t, x; \beta, x_1, x_2) := (D, D_t)(x; \beta, x_1 + t, x_2).$$

At this point we split the proof into two cases: (i) breather, and (ii) 2-kink and kink-antikink.

Breather case

In what follow we split the proof in two cases: t is uniformly far from all t_k, and the case t close to some t_k.

1. Let us assume then that $(\phi, \phi_t)(t)$ satisfies (2.5.4) with T^* obeying

$$|T^* - t_k| \geq \varepsilon_0,$$

for all $k \in \mathbb{Z}$. We plan to show that (2.5.4) is satisfied with C^* replaced by $C^*/2$, proving Theorem 2.1.1 for all times t far from t_k. Indeed, taking $\eta_0 > 0$ small and $\eta \in (0, \eta_0)$, thanks to Corollary 2.5.3 we have unique functions $x_1(t), x_2(t) \in \mathbb{R}$, defined in $[0, T^*]$, and such that $(z, w)(t, x)$, defined in (2.5.6), satisfy the orthogonality conditions (2.5.7). Note also that we have (2.5.9). WLOG, we can assume (2.3.5) not satisfied and $x_1(0) = x_2(0) = 0$. We define $(z_0, w_0) := (z, w)(0)$. From Proposition 2.6.1 we obtain functions (y_0, v_0), (u_0, s_0) and parameters $\delta, \tilde{\delta}$. Moreover, Corollary 2.8.4, implies that $(y_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ are real-valued. Recall that the constants from Proposition 2.6.1 do not depend on C^*. Now, we evolve SG to a time $t > 0$, with initial data (y_0, v_0). Thanks to Theorem 2.2.6 we have

(2.2.14) for $(y(t), v(t))$, and Proposition 2.7.4 is valid for all $t \in \mathbb{R}$ far from t_k. On the other hand, from Corollary 2.5.3 we have

$$|x_1'(t)| + |x_2'(t)| \lesssim C^* \eta,$$

so that the set of times $\tilde{t}_k$ where (2.3.5) is satisfied is still a countable set of points with no accumulation points. Invoking Proposition 2.7.4, starting at $(y, v)(t)$, and considering for all time $t \in \mathbb{R}$ the 2-soliton and 1-soliton profiles

$$\begin{aligned}(B^*, B_t^*) &:= (B, B_t)(x; \beta, x_1(t), x_2(t)),\\ (\overline{K}^*, \overline{K}_t^*) &:= (\overline{K}, \overline{K}_t)(x; \beta, x_1(t), x_2(t)),\end{aligned}$$

and parameters $\beta - i\alpha + \tilde{\delta},\ \beta + i\alpha + \delta \in \mathbb{C}$, we obtain a function $(B^*, B_t^*)(t) + (z, w)(t)$. This form constructed coincides with the solution $(\phi, \phi_t)(t)$. Indeed, note that at time $t = 0$, both initial data coincide, so that, thanks to the uniqueness of the solutions associated to the Cauchy problem (2.1.1) (see also Theorems 2.2.6 and 2.2.8), we conclude that $(B^* + z, B_t^* + w)(t)$ obtained via BT is actually $(\phi, \phi_t)(t)$. Finally, we also have

$$\sup_{|t - t_k| \geq \varepsilon_0} \|(\phi, \phi_t)(t) - (B^*, B_t^*)(t)\|_{H^1 \times L^2} \leq C_0 \eta, \tag{2.11.1}$$

so that, considering C^* large such that $C_0 \leq \frac{1}{2} C^*$, we conclude that T^* must be infinite (see (2.5.5)). This idea is schematically represented in Fig. 2.9.

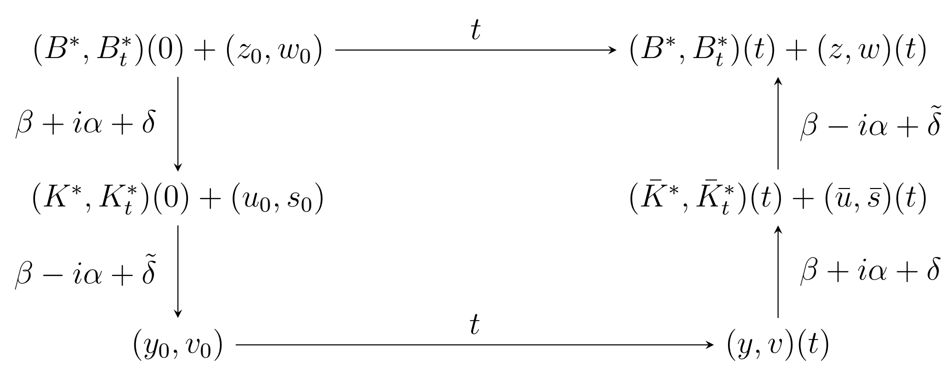

Figure 2.9: Diagram for the proof of Theorem 2.1.1 in the case where $x_1(t)$ does not follow (2.3.5).

2. Let us consider now the case $|T^* - t_k| < \varepsilon_0$ for some $k \in \mathbb{N}$ fixed. We shall prove that for ε_0 sufficiently small, but independent of k,

$$\sup_{|t - t_k| < \varepsilon_0} \|(\phi, \phi_t) - (B^*, B_t^*)\|_{H^1 \times L^2} \leq \frac{3}{4} C^* \eta. \tag{2.11.2}$$

Since C^* grows as ε_0 tends to zero, we must choose η_0 sufficiently small such that each step above holds properly. Let $I_k := (t_k - \varepsilon_0, t_k + \varepsilon_0]$. Let us consider

$$T_* := \sup \Big\{ T \in I_k : \ \forall\, t \in (t_k - \varepsilon_0, T], \ \|(z,w)(t)\|_{H^1\times L^2} \leq \frac{3}{4} C^* \eta \Big\}. \tag{2.11.3}$$

It is enough to show $T_* = t_k + \varepsilon_0$. Let us assume that $T_* < t_k + \varepsilon_0$. Note that, by the same argument as the previous step, using BT we have

$$\|(z,w)(t_k - \varepsilon_0)\|_{H^1\times L^2} = \frac{1}{2} C^* \eta.$$

Now, we use a bootstrap argument. Let $t \in [t_k - \varepsilon_0, T_*]$ and consider

$$\Delta := \frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} (z_x^2 + z^2 + w^2)(t,x)dx\right).$$

We claim that Δ is bounded by $C(C^*)^2\eta^2$, a contradiction to the definition of T_*. First, we will need (2.2.1) in terms of (z,w), using (2.5.6) with $D = B$. In fact,

$$\begin{cases} \partial_t B^* + z_t = B_t^* + w \\ \partial_t B_t^* + w_t = B_{xx}^* + z_{xx} - \sin(B^* + z). \end{cases}$$

Simplifying, we get

$$\begin{cases} z_t = w - x_1' B_t - x_2' B_x \\ w_t = z_{xx} - \sin(B^* + z) + \sin B^* - x_1' B_{tt}^* - x_2' B_{tx}^*. \end{cases}$$

Now, computing directly,

$$\begin{aligned} \Delta &= \int_{\mathbb{R}} (z z_t - z_{xx} z_t + w w_t) \\ &= \int_{\mathbb{R}} (z - z_{xx})(w - x_1' B_t^* - x_2' B_x^*) \\ &\quad + \int_{\mathbb{R}} w(z_{xx} - \sin(B^* + z) + \sin B^* - x_1' B_{tt}^* - x_2' B_{tx}^*) \\ &= \int_{\mathbb{R}} z(w - x_1' B_t^* - x_2' B_x^*) + \int_{\mathbb{R}} z_{xx}(x_1' B_t^* + x_2' B_x^*) \\ &\quad + \int_{\mathbb{R}} w\left(\sin B^*(\cos z - 1) + \cos(B^*)\sin z - x_1' B_{tt}^* - x_2' B_{tx}^*\right). \end{aligned}$$

Clearly if (z,w) are small,

$$|\Delta| \lesssim \int_{\mathbb{R}} (z_x^2 + z^2 + w^2) + |x_1'(t)|^2 + |x_2'(t)|^2.$$

Therefore, using (2.11.3) and (2.5.10) we obtain that for $t \in (t_k - \varepsilon_0, T_*]$ it holds

$$\left|\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}}(z^2 + z_x^2 + w^2)\right| = |\Delta| \leq C(C^*)^2\eta^2.$$

Consequently, integrating we have that for ε_0 sufficiently small (but fixed)

$$\begin{aligned}&\int_{\mathbb{R}} z^2(T^*) + z_x^2(T^*) + w^2(T^*)\\ &\qquad \leq \int_{\mathbb{R}} z^2(t_k - \varepsilon_0) + z_x^2(t_k - \varepsilon_0) + w^2(t_k - \varepsilon_0) + C\varepsilon_0(C^*)^2\eta^2 \leq \frac{3}{4}(C^*)^2\eta^2.\end{aligned}$$

Then, (2.11.3) has been improved, and $T_* = t_k + \varepsilon_0$. This estimate does not depend on $k \in \mathbb{Z}$, but only on the length of the interval $\sim \varepsilon_0$. Therefore, T^* in (2.5.4) is infinite for all C^* large enough. This proves (2.1.9) and the proof of Theorem 2.1.1 in the case of the breather solution.

2-kink or kink-antikink case

Here we can repeat the previous scheme but with no problem on the time t chosen. Since proofs are similar, we only sketch the main steps.

Let $(z, w)(t)$ be the functions defined in (2.5.6) and $x_1(t)$, $x_2(t)$ modulations from Corollary 2.5.3. Hence, applying Proposition 2.9.1 with perturbation $(z_0, w_0) = (z, w)(0)$ we obtain functions with real values (y_0, v_0). Then, we evolve SG with initial data $(y_0, v_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Finally, we consider functions $(Q, Q_t)(x; -\beta, x_1 + x_2)$ and a parameter of BT $d \in \mathbb{R}$ given as follows:

1. If $(D, D_t) = (A, A_t)$, then we have $d := a(\beta)$.

2. If $(D, D_t) = (R, R_t)$ then $d := -a(\beta)$.

Now we invoke Proposition 2.10.1 for each time t fixed, and with 2-soliton and 1-soliton profiles given by

$$\begin{aligned}(D^*, D_t^*) &:= (D, D_t)(x; \beta, x_1(t), x_2(t)),\\ (Q^*, Q_t^*) &:= (Q, Q_t)(x; -\beta, x_1(t) + x_2(t)).\end{aligned}$$

Thanks to the uniqueness of the solution to the Cauchy problem (2.1.1), we have coincidence between $(\phi, \phi_t)(t)$ and the functions returned via BT. Lastly, noticing that from Theorem 2.2.6 we have

$$\sup_{t\in\mathbb{R}} \|(y, v)(t)\|_{H^1\times L^2} \lesssim \|(y_0, v_0)\|_{H^1\times L^2},$$

we conclude from Proposition 2.10.1 that

$$\sup_{t\in\mathbb{R}} \|(\phi, \phi_t)(t) - (D^*, D_t^*)(t)\|_{H^1\times L^2} \leq C_0\eta.$$

The proof of Theorem 2.1.1 in these cases is complete. □

2.11.1 Proof of Corollary 2.1.4

We will show the breather case only, the other cases are very similar. Thanks to Lemma 2.2.4 and (2.4.8), it is enough to compute

$$\begin{aligned}
\ell_{\pm}^{+,1}(t) &= \lim_{x\to\pm\infty}\left(1-\cos\left(\frac{B+z+K+u}{2}\right)\right) \\
&= \lim_{x\to\pm\infty}\left(1-\cos\left(\frac{B+K}{2}\right)\right) = \begin{cases}2, & x\to+\infty\\ 0 & x\to-\infty\end{cases}, \\
\ell_{\pm}^{-,1}(t) &= \lim_{x\to\pm\infty}\left(1-\cos\left(\frac{B+z-(K+u)}{2}\right)\right) = \begin{cases}2, & x\to+\infty\\ 0 & x\to-\infty\end{cases}.
\end{aligned}$$

and

$$\begin{aligned}
\ell_{\pm}^{+,2}(t) &= \lim_{x\to\pm\infty}\left(1-\cos\left(\frac{K+u+y}{2}\right)\right) = \begin{cases}2, & x\to+\infty\\ 0 & x\to-\infty\end{cases}, \\
\ell_{\pm}^{-,2}(t) &= \lim_{x\to\pm\infty}\left(1-\cos\left(\frac{K+u-y}{2}\right)\right) = \begin{cases}2, & x\to+\infty\\ 0 & x\to-\infty\end{cases}.
\end{aligned}$$

Hence, using these values, and Proposition 2.6.1 and (2.2.8),

$$\begin{aligned}
E[B+z,B_t+w] &= E[K+u,K_t+s] + \frac{4}{\beta+i\alpha+\delta} + 4(\beta+i\alpha+\delta), \\
E[K+u,K_t+s] &= E[y,v] + \frac{4}{\beta-i\alpha+\tilde{\delta}} + 4(\beta-i\alpha+\tilde{\delta}).
\end{aligned}$$

Since $\tilde{\delta}=\overline{\delta}$ (see Corollary 2.8.3), we obtain

$$E[B+z,B_t+w] = E[y,v] + \frac{8(\beta+\operatorname{Re}\delta)}{(\beta+\operatorname{Re}\delta)^2+(\alpha+\operatorname{Im}\delta)^2} + 8(\beta+\operatorname{Re}\delta),$$

from which we obtain (2.1.12), since $\alpha^2+\beta^2=1$. For the momentum part, we proceed in the same fashion, obtaining (2.1.13).

Chapter 3

Dispersive blow-up and persistence properties for the Schrödinger-Korteweg de Vries system

In this chapter we are concerned with persistence properties of solutions of the initial value problem associated to the Schrödinger-Korteweg-de Vries system in well-chosen fractional weighted Sobolev spaces. This persistence result, in addition to be interesting by itself, is then used to prove the existence of finite time point singularities, usually described as *dispersive blow-up*. It is believed that this mathematical phenomenon is one of the conceivable explanations for oceanic and optical rogue waves. In our case, our main goal is to prove dispersive blow-up for initial data in $H^{2^-}(\mathbb{R}) \times H^{3/2^-}(\mathbb{R})$. This is, as far as we understand, the first dispersive blow up result known for this system of equations.

This chapter is part of the article

- F. Linares, and J. M. Palacios, *On the persistence properties of the Schrd̈inger-Korteweg-de Vries system, and applications to dispersive blow-up*, preprint 2018 [55].

3.1 Introduction and main results

3.1.1 The model

This chapter is concerned with the Initial Value Problem (IVP) associated to the Schrödinger-Korteweg-de Vries (NLS-KdV) system in $\mathbb{R}_t \times \mathbb{R}_x$,

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = \alpha uv, \qquad t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{3.1.1}$$

where $u = u(t,x)$ is a complex-valued function and $v(t,x)$ is a real-valued function. This system governs the interactions between short-waves $u = u(t,x)$ and longwaves $v = v(t,x)$ and has been studied in several fields of physics and fluid dynamics: an electron-plasma, ion-field interaction, a diatomic lattice system, and water waves theory. See [32], [39], [40] and [77] for these applications.

The Schrödinger-Korteweg-de Vries system (3.1.1) has been shown not to be a completely integrable system (see [13]). Therefore the solvability of (3.1.1) is dependent upon the method of evolution equations.

Definitions. We recall the definition of the Sobolev spaces $H^s(\mathbb{R})$ for index $s \geq 0$:

$$H^s(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \ \|f\|_{s,2} = \|J^s f\|_2 < \infty\},$$

where

$$\|J^s f\|_2 = \left(\int_{-\infty}^{\infty} (1+\xi^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2},$$

and $\widehat{f}$ denotes the Fourier transform of f. In particular, if $s > s' > 0$, then

$$H^s(\mathbb{R}) \subsetneq H^{s'}(\mathbb{R}) \subsetneq L^2(\mathbb{R}).$$

Several works related to the well-posedness for the single equations has been done. For the single nonlinear Schrödinger equation with cubic term ($|u|^2 u$) Y. Tsutsumi [83] established local and global well-posedness for data in $L^2(\mathbb{R})$. On the other hand, for the Korteweg-de Vries (KdV) equation Kenig, Ponce and Vega [44] have proved local well-posedness for data in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$. See also [19] and [45] for other local well-posedness results in Sobolev spaces with negative exponents.

In general, a coupled system like (3.1.1) is more difficult to handle in the same spaces as in the space the single equation is solved. In the case of the system (3.1.1) this is due to the antisymmetric nature of the characteristics of each linear part. In [12] Bekiranov, Ogawa and Ponce showed that the coupled system (3.1.1) is locally well-posed in $H^s(\mathbb{R})\times H^{s-\frac{1}{2}}(\mathbb{R})$ with $s\geq 0$. To obtain this result, the authors used the Fourier restriction norm method introduced by Bourgain in [19] to study the NLS and KdV equations. In [22] Corcho and Linares extended this result for weak initial data $(u_0,v_0)\in H^k(\mathbb{R})\times H^s(\mathbb{R})$ for various values of k and s, where the lowest admissible values are $k=0$ and $s=-\frac{3}{4}+\delta$ with $0<\delta\leq\frac{1}{4}$. In this case, the authors deduced some new bilinear estimates for the coupling terms in system (3.1.1). The exact statement of this result is the following.

Theorem 3.1.1 (see [22])**.** *Let $k\geq 0$ and $s>-\frac{3}{4}$. Then for any $(u_0,v_0)\in H^k(\mathbb{R})\times H^s(\mathbb{R})$ such that:*

1. $k-1\leq s\leq 2k-\frac{1}{2}$ *for* $k\in[0,\frac{1}{2}]$,
2. $k-1\leq s<k+\frac{1}{2}$ *for* $k\in(\frac{1}{2},\infty)$,

the following is satisfied. There exist a positive time $T=T(\|u_0\|_{H^k},\|v_0\|_{H^s})$ and a unique solution $(u(t),v(t))$ of the initial value problem (3.1.1)*, satisfying*

$$u\in C([0,T];H^k(\mathbb{R}))\quad \textit{and}\quad v\in C([0,T];H^s(\mathbb{R})).$$

Moreover, the map $(u_0,v_0)\mapsto(u(t),v(t))$ is locally Lipschitz from $H^k(\mathbb{R})\times H^s(\mathbb{R})$ into $C([0,T];H^k(\mathbb{R})\times H^s(\mathbb{R}))$.

The endpoint $(k,s)=(0,-\frac{3}{4})$ was proved by Z. Guo and Y. Wang in [35]. The key ingredient in their proof is the use of $\bar{F}^s$-type spaces (introduced by Guo in [36]) to deal with the KdV part of the system and the coupling terms. In order to overcome the difficulty caused by the lack of scaling invariance, they proved uniform estimates for the multiplier.

3.1.2 Main results: persistence

The aim of this work is to prove the following persistence property for local solutions to the IVP (3.1.1) associated to the NLS-KdV system in weighted spaces.

Theorem 3.1.2. *Let $s\in\mathbb{R}$, $r_1,r_2\geq 0$ be fixed parameters. Let*

$$(u_0,v_0)\in\left(H^{s+1/2}(\mathbb{R})\cap L^2(|x|^{r_1}dx)\right)\times\left(H^s(\mathbb{R})\cap L^2(|x|^{r_2}dx)\right),$$

with

$$s>\frac{3}{4},\quad s+\frac{1}{2}>r_1,\quad \textit{and}\quad s>2r_2.$$

Then there exist $T=T(\|u_0\|_{s+\frac{1}{2}}+\|v_0\|_s)>0$ and a unique solution $(u(t),v(t))$ of the IVP (1.6.4) *satisfying*

$$u\in C([0,T];H^{s+\frac{1}{2}}(\mathbb{R})\cap L^2(|x|^{r_1}dx)),\quad v\in C([0,T];H^s(\mathbb{R})\cap L^2(|x|^{r_2}dx)),$$

Furthermore,

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} + \|D_x^{s-\frac{1}{2}} \partial_x v\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty,$$
$$\|u\|_{L_x^2 L_T^\infty} + \|v\|_{L_x^2 \| L_T^\infty} < \infty,$$
$$\|\partial_x u\|_{L_T^4 L_x^\infty} + \|\partial_x v\|_{L_T^4 L_x^\infty} < \infty.$$

Moreover, given $T' \in (0, T)$, the map data solution is Lipschitz continuous.

The proof of Theorem 3.1.2 is based on the contraction map principle. As an application, we consider the question of existence of dispersive blow-up for the IVP (3.1.1) associated to the NLS-KdV system.

3.1.3 Dispersive blow up: definitions and properties

Dispersive blow up of (dispersive) equations is a focusing phenomenon of smooth initial disturbances with finite mass (or finite energy, depending on the physical context), that relies upon the dispersion relation guaranteeing that, in the linear regime, different wavelengths propagate at different speeds. This is especially the case for models wherein the linear dispersion is unbounded, so that energy can be moved around at arbitrarily high speeds, but even bounded dispersion can exhibit this type of singularity formation [17]. In Theorem 2.1 of [18], Bona and Saut shown that for any given point $(t_*, x_*) \in \mathbb{R}^n \times \mathbb{R}_+$ there exists initial data $u_0 \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that the solution $u(t, x)$ of the corresponding initial-value problem for the free Schrödinger equation:

$$i\partial_t u + \Delta u = 0, \qquad u(t = 0) = u_0(x), \tag{3.1.2}$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{(t_*, x_*)\}$, but

$$\lim_{(t,x)\to(t_*,x_*)} |u(t, x)| = +\infty.$$

This fact is referred to as (finite-time) dispersive blow up. The analogous phenomena also appears in other linear dispersive equations, such as the linear Korteweg-de Vries equation [14] and the free surface water waves system linearized around the rest state [18].

As Bona et al. explain in [17], at first sight, one would expect that nonlinear terms would destroy dispersive blow up. What is a little surprising is that even the inclusion of physically relevant nonlinearities in various models of wave propagation does not prevent dispersive blow up. Indeed, theory shows in some important cases that initial data leading to this focusing singularity under the linear evolution continues to blow up in exactly the same way when nonlinear terms are included. In [16], this was shown to be true for the Korteweg-de Vries equation, a model for shallow water waves and other simple wave phenomena.

Regarding the dispersive blow-up question, in [16] Bona and Saut studied the dispersive blow-up of the generalized Korteweg-de Vries equation

$$v_t + v_{xxx} + v^k v_x = 0, \qquad k \in \mathbb{N}.$$

In their proof the main tool was the use of weighted Sobolev spaces. Then, in [57] Linares and Scialom proved that the use of weighted Sobolev spaces was not necessary for the case $k \geq 2$. Later, Linares, Ponce and Smith improved the previous proof to include the case $k = 1$, i.e., for the KdV equation. On the other hand, Bona and Saut showed in [18] the existence of DBU for the one dimensional nonlinear Schrödinger with nonlinearity $|u|^{p-1}u$ with p in the range $1 \leq p \leq 3$. Later, Bona et al. showed in [17] that the same conclusions holds for the complete range of nonlinearities $p \geq \lfloor \frac{n}{2} \rfloor$ and any dimension $n \geq 1$.

In what follows, we will adopt the following notation: for $s \geq 0$

$$H^{s^+}(\mathbb{R}) = \bigcup_{s'>s} H^{s'}(\mathbb{R}), \qquad H^{s^-}(\mathbb{R}) = \bigcup_{0 \leq s' < s} H^{s'}(\mathbb{R}).$$

Let $1 \leq p, q \leq \infty$ and $f : \mathbb{R} \times [0,T] \to \mathbb{R}$. We also define the norms

$$\|f\|_{L^p_x L^q_T} := \left(\int_{\mathbb{R}} \left(\int_0^T |f|^q dt \right)^{p/q} dx \right)^{1/p}, \qquad \|f\|_{L^p_T L^q_x} := \left(\int_0^T \left(\int_{\mathbb{R}} |f|^q dx \right)^{p/q} dt \right)^{1/p}.$$

3.1.4 Existence of dispersive blow up for NLS-KdV

As a consequence of Theorem 3.1.2, we are able to generalize the results of DBU for the single equations mentioned above to the Schrödinger-Korteweg-de Vries system.

Theorem 3.1.3. *There exist initial data*

$$u_0 \in C^\infty(\mathbb{R}) \cap H^{2^-}(\mathbb{R}), \quad v_0 \in C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}),$$

such that the following holds: there exists $t^* \in [0,T]$ such that the corresponding solution $(u,v)(\cdot,\cdot)$ of the IVP 3.1.1:

$$u \in C([0,T] : H^{2^-}(\mathbb{R})), \qquad v \in C([0,T] : H^{3/2^-}(\mathbb{R})),$$

provided by Theorem 3.1.2 is such that

$$u(t^*,\cdot) \notin H^2(\mathbb{R}), \quad v(t^*,\cdot) \notin C^1(\mathbb{R}).$$

The ideas of the proof of Theorem 3.1.3 follow from the ideas of Bona and Saut in [16] and Linares, Ponce and Smith in [56]. Nevertheless, as we shall see, in our case the NLS-KdV system presents several new difficulties because its coupling terms. The key ingredient on the demonstration is the previous persistence property on weigthed spaces, which allow us to prove some nonlinear estimates on the solution.

It should be said that there is no dispersive blow-up for the solution of the nonlinear Schrödinger equation, since neither its solution nor any of its derivatives is loosing regularity in terms of the L^∞-norm (at least we were not able to prove that). Nevertheless, we shall prove a smoothing effect by a quarter of derivative associated to its nonlinear term. The main issue to stablish the dispersive blow-up for the NLS solution is to deal with the coupled term (KdV), which has both worse regularity and worse persistence in weighted Sobolev spaces, and prove that the solution regularizes. More explicitly, to show that the NLS solution has dispersive blow-up we must prove that the solution $u(t,x)$ is in $H^{\frac{5}{2}+\varepsilon}$ even when it has a term which is only in $H^{\frac{3}{2}^-}(\mathbb{R})$.

3.1.5 Organization of this chapter

This paper is organized as follows. In section 3.2 we state a series of results needed in the remainder of this chapter. The dispersive blow-up for each of the linear equation is established in section 3.3. In this section we show how to construct the initial data which shall develop dispersive blow-up. Section 3.4 is devoted to prove the main theorem 3.1.2. The dispersive blow-up for the coupled system (theorem 3.1.3) is proved in the last section 3.5.

3.2 Preliminaries

3.2.1 Smoothing properties for Korteweg and Schrödinger linear evolutions

In this subsection we review some standard results about smoothing properties of the free Schrödinger group $S(t) = e^{it\Delta}$ and the KdV group $V(t) = e^{-t\partial_x^3}$.

First, the following lemma provides the smoothing effect of Kato type for solutions of the linear KdV equation.

Lemma 3.2.1 ([43]).

$$\sup_x \|\partial_x V(t)v_0\|_{L^2_T} \leq c\|v_0\|_{L^2}, \tag{3.2.1}$$

and

$$\|\partial_x \int_0^t V(t-t')F(\cdot,t')\,dt'\|_{L^2_x} \leq c\|F\|_{L^1_x L^2_T}. \tag{3.2.2}$$

From now on, we will denote the so-called homogeneous derivatives of order $s > 0$ by

$$D^s f(x) := \mathcal{F}^{-1}\big(|\xi|^s \hat{f}(\xi)\big)(x).$$

Next lemma give us the smoothing effects of Kato type for solutions of the linear Schrödinger equation in dimension $n = 1$.

Lemma 3.2.2 ([43])**.**

$$\sup_x \|D_x^{1/2} e^{it\Delta} u_0\|_{L^2_T} \leq c\|u_0\|_{L^2}, \tag{3.2.3}$$

$$\|D_x^{1/2} \int_0^t e^{i(t-t')\Delta} F(\cdot, t')\, dt'\|_{L^2_x} \leq c\|F\|_{L^1_x L^2_T}, \tag{3.2.4}$$

and

$$\sup_x \|\partial_x \int_0^t e^{i(t-t')\Delta} F(\cdot, t')\, dt'\|_{L^2_T} \leq c\|F\|_{L^1_x L^2_T}. \tag{3.2.5}$$

This last Lemma is sharp, in the sense that there exists a class of initial data u_0 in which (3.2.3) becomes an identity. Next, we present Strichartz estimates.

Lemma 3.2.3 (Strichartz and smoothing estimates)**.** *Let $2 \leq p, q \leq \infty$ be admissible, i.e., such that $\frac{2}{q} = \frac{1}{2} - \frac{1}{p}$. Then the following holds:*

$$\|e^{it\Delta} f\|_{L^q_t L^p_x} \leq c\|f\|_{L^2_x}. \tag{3.2.6}$$

Moreover, for $(\alpha, \theta) = [0, 1/2] \times [0, 1]$ it holds that

$$\|D_x^{\alpha\theta/2} V(t) f\|_{L^q_t L^p_x} \leq c\|f\|_{L^2_x}, \tag{3.2.7}$$

where $(q, p) = (6/\theta(\alpha + 1), 2/(1 - \theta)))$.

For a proof of these estimates see for instace [54]. Finally, we complete the set of estimates introducing the next maximal function estimate norms for the linear solutions.

Lemma 3.2.4 ([43])**.** *For $s > 1/2$ and $\rho_1 > 1/4$, it holds that*

$$\|e^{it\Delta} f\|_{L^2_x L^\infty_T} \leq c(1 + T)^{\rho_1} \|f\|_{s,2}. \tag{3.2.8}$$

For $s > 3/4$ and $\rho_2 > 3/4$ it holds that

$$\|V(t) f\|_{L^2_x L^\infty_T} \leq c(1 + T)^{\rho_2} \|f\|_{s,2}. \tag{3.2.9}$$

Proof. For a proof of the first inequality see [84, 75]. For a proof of the second one see [85]. □

3.2.2 Weighted estimates

The following Lemma allow us to interpolate weighted Sobolev spaces by weighted L^2-based spaces and standard Sobolev spaces.

Lemma 3.2.5 ([70]). *Let $a, b > 0$. Assume that $J^a f = (1-\Delta)^{a/2} f \in L^2(\mathbb{R}^n)$ and $\langle x \rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R}^n)$. Then, for any $\theta \in (0,1)$*

$$\|\langle x \rangle^{\theta b} J^{(1-\theta)a} f\|_2 \leq c \|\langle x \rangle^b f\|_2^{\theta} \|J^a f\|_2^{1-\theta}. \tag{3.2.10}$$

3.2.3 Leibnitz rules

The following Lemma is the standard Leibnitz rule and commutator estimate for fractional derivatives in L^p-based spaces, which will be employed to deal with the nonlinear terms.

Theorem 3.2.6 ([43]).

1. *For $s > 0$ and $1 < p < \infty$, it holds*

$$\|D_x^s(fg) - f D_x^s g - g D_x^s f\|_p \leq c \|f\|_\infty \|D_x^s g\|_p. \tag{3.2.11}$$

2. *Let $0 < b < 1$. For $1 \leq p < \infty$, $b_1, b_2 \in [0, b]$ such that $b_1 + b_2 = b$ and $p_1, p_2 \in (1, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, it holds*

$$\|D^b(fg) - f D^b g - g D^b f\|_p \leq c \|D^{b_1} f\|_{p_1} \|D^{b_2} g\|_{p_2}. \tag{3.2.12}$$

3.3 Dispersive blow-up: construction of the initial data

In this Section, we construct the initial data for which there will be dispersive blow-up. Let us divide the analysis in two cases, the linear case for the Schrödinger equation and the linear case for the KdV equation.

3.3.1 Linear Schrödinger equation.

We will follow the argument employed in [17] and [18] with some modifications. Consider the IVP associated to the linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{3.3.1}$$

Our goal is to construct an initial data $u_0 \in H^{2^-}$ with enough decay and such that for any time $t \in \mathbb{R}$, the corresponding solution of the IVP (3.3.1) satisfies

$$u(t,\cdot) \notin H^2(\mathbb{R}).$$

Now, recall that for any $u_0 \in L^2(\mathbb{R})$, the unique solution u of (3.3.1) has the representation:

$$u(t,x) = \frac{1}{(4i\pi t)^{1/2}} \int_{\mathbb{R}} e^{\frac{i|x-y|^2}{4t}} u_0(y)dy,$$

where the integral is taken in the improper Riemann sense. Choose the initial data u_0 to be

$$u_0(x) := \frac{e^{-ix^2}}{1+|x|^{5/2}}. \tag{3.3.2}$$

The Sobolev regularity of the initial data u_0 is explained in the next result.

Lemma 3.3.1. *Let u_0 be as described in* (3.3.2)*. Then $u_0 \in H^s(\mathbb{R})$ for any $s \in [0,2)$, but $u_0 \notin H^2(\mathbb{R})$. Moreover,*

$$\langle x\rangle^{2^-} u_0(x) \in L^2(\mathbb{R}).$$

Proof. The fact that $\langle x\rangle^{2-\varepsilon} u_0(x) \in L^2(\mathbb{R})$ for $\varepsilon \in (0,2)$ we just notice that:

$$\|\langle x\rangle^{2-\varepsilon} u_0(x)\|_{L^2(\mathbb{R})} \le c \left\| \frac{1}{1+|x|^{1/2+\varepsilon}} \right\|_{L^2(\mathbb{R})} < \infty.$$

Now, let us prove that $u_0 \in H^s(\mathbb{R})$ for any $s \in (0,2)$. Consider first the case $0 < s < 1$. For s in this range, Propositions 1 and 2 in [70] provide the inequalities:

$$|\mathcal{D}^s e^{i|x|^2}| \le c_n(1+|x|^s), \quad \forall x \in \mathbb{R}^n, \quad s \in (0,1), \tag{3.3.3}$$

and

$$\|\mathcal{D}^s(fg)\|_{L^2(\mathbb{R}^n)} \le \|f\mathcal{D}^s g\|_{L^2(\mathbb{R}^n)} + \|g\mathcal{D}^s f\|_{L^2(\mathbb{R}^n)}, \qquad s \in (0,1), \tag{3.3.4}$$

where $\mathcal{D}^s$ is defined as

$$\mathcal{D}^s f := \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}} \right)^{1/2} dy.$$

On the other hand, a straightforward calculation reveals that

$$\|\mathcal{D}^s f\|_{L^2(\mathbb{R}^n)} = c_n \||\xi|^s \hat{f}\|_{L^2(\mathbb{R}^n)} \equiv \|D^s f\|_{L^2(\mathbb{R}^n)}. \tag{3.3.5}$$

Combining estimates (3.3.3) and (3.3.4) with identity (3.3.5) and using interpolation, one arrives at the inequality

$$\left\| \mathcal{D}^s \left(\frac{e^{-ix^2}}{1+|x|^{5/2}} \right) \right\|_{L^2(\mathbb{R})} \le \left\| \frac{1}{1+|x|^{5/2}} \mathcal{D}^s \big(e^{-ix^2}\big) \right\|_{L^2} + \left\| \mathcal{D}^s \left(\frac{1}{1+|x|^{5/2}} \right) \right\|_{L^2}$$

$$\leq c\left\|\frac{1}{1+|x|^{5/2}}\right\|_{L^2} + c\left\|\frac{|x|^s}{1+|x|^{5/2}}\right\|_{L^2}$$
$$+\left\|\frac{1}{1+|x|^{5/2}}\right\|_{L^2}^{1-s}\left\|D_x\left(\frac{1}{1+|x|^{5/2}}\right)\right\|_{L^2}^{s}.$$

The right-hand side of this inequality is finite for any $0 \leq s \leq 1$. Note that if instead $s = s' + 1$, where $0 < s' < 1$, then simply apply the above analysis to the derivative:

$$\partial_x u_0 = \frac{\big(-2ix(1+|x|^{5/2}) + 2x\big)e^{-ix^2}}{(1+|x|^{5/2})^2},$$

from which it follows that $\|\mathcal{D}^s(\partial_x u_0)\|_{L^2(\mathbb{R})} < \infty$ if and only if $0 \leq s < 1$. Thus $u_0 \in H^s(\mathbb{R})$ for any $s \in (0,2)$ but $u_0 \notin H^2(\mathbb{R})$. $\square$

The following result is a direct consequence of the previous analysis, and Lemma 2.1 in [17].

Corollary 3.3.2. *Let $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and*

$$u_0(x) = \frac{e^{-i\alpha(x-x_0)^2}}{1+|x|^{5/2}}.$$

Then,

$$u_0 \in C^\infty(\mathbb{R}) \cap H^{2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{2^-}dx) \cap L^\infty(\mathbb{R}),$$

and the associated global-in-time solution $u \in C(\mathbb{R}; H^{2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{2^-}dx))$ of (3.3.1), with initial datum u_0 satisfies that, for all $t \in \mathbb{R}$, the solution $u(t,x) \notin H^2(\mathbb{R})$.

Proof. The fact that

$$u \in C\big(\mathbb{R}; H^{2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{2^-}dx)\big),$$

is just a consequence of the well-posedness theory for the Cauchy problem associated to the free Schrödinger equation and the persistences properties of the solution (see [70]). Now, to prove that $u \notin H^2(\mathbb{R})$ it is enough to prove that $\partial_x^2 u \notin L^2(\mathbb{R})$. This is just a consequence of the fact that the free Schrödinger group is an isometry on $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. In fact, since the Schrödinger group commutes with derivatives, we have

$$\|\partial_x^2 u(t,x)\|_{L^2_x} = \|e^{it\Delta}\partial_x^2 u_0(x)\|_{L^2_x}. \tag{3.3.6}$$

The latter norm is equals to $+\infty$ due to the main term of $\partial_x^2 u_0(x)$, which is:

$$c\frac{x^2 e^{-ix^2}}{1+|x|^{5/2}},$$

where c is a non-zero constant of no consequence. In fact, since $e^{it\Delta}$ is an unitary group and all the other terms in (3.3.6) have L^2-norm finite, we have that the norm of the solution is bounded below by,

$$\|\partial_x^2 u(t,x)\|_{L^2_x} = \|\partial_x^2 u_0(x)\|_{L^2} \geq \left\| \frac{x^2 e^{-ix^2}}{1+|x|^{5/2}} \right\|_{L^2} = +\infty,$$

Thus, for any $t \in \mathbb{R}$ we have $u(t,\cdot) \notin H^2(\mathbb{R})$, which completes the demonstration. $\square$

This concludes the case of the free Schrödinger equation. Now the attention is turned to construct the initial data for the linear Korteweg-de Vries equation.

3.3.2 Linear Korteweg-de Vries equation.

This part was proved in Linares et al. (see Section 3 in [56]). For completeness reasons, we include it.

Consider the linear IVP associated to the linear Korteweg-de Vries equation:

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & x \in \mathbb{R},\ t > 0, \\ v(0,x) = v_0(x), \end{cases} \tag{3.3.7}$$

whose solution is given by

$$v(t,x) = V(t)v_0(x) = e^{-t\partial_x^3} v_0 = S_t * v_0(x), \tag{3.3.8}$$

where,

$$S_t(x) := \frac{1}{3\sqrt{3t}} A_i \left(\frac{x}{3\sqrt{3t}} \right),$$

and $A_i(\cdot)$ denotes the classical Airy function. Our goal is to construct an smooth initial data

$$v_0 \in C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

with some decay and such that, for some prescribed time t^*, the corresponding solution of the IVP (3.3.7) satisfies

$$v(t^*,\cdot) \notin C^1(\mathbb{R}).$$

In this case we will construct an initial data such that for any $t \in \mathbb{N}$ the solution satisfies $v(t,\cdot) \notin C^1(\mathbb{R})$. The following Lemma give us the detailed statement for the dispersive blow-up for the initial-value problem associated to the linear KdV equation (3.3.7).

Lemma 3.3.3 ([56]). *Consider the initial data*

$$v_0(x) := \sum_{j=1}^{\infty} \alpha_j V(-j)\phi(x), \quad \alpha_j > 0,$$

where $\alpha_j = ce^{-j^2}$ with $c > 0$ small enough and $\phi(x) := e^{-2|x|}$. Then,

$$v_0 \in C^{\infty}(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{3/4^-} dx) \cap L^{\infty}(\mathbb{R}),$$

and the associated global in-time solution $v \in C(\mathbb{R}; H^{3/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{3/4^-} dx))$ of (3.3.7) *is such that*

1. *For any $t > 0$ with $t \notin \mathbb{Z}$, we have $v(t, \cdot) \in C^1(\mathbb{R})$.*
2. *For any $t \in \mathbb{N}$ we have $v(t, \cdot) \notin C^1(\mathbb{R})$.*

Proof. Define the initial data

$$v_0(x) := \sum_{j=1}^{\infty} \alpha_j V(-j)\phi(x), \qquad \alpha_j > 0.$$

Note that if

$$\sum_{j=1}^{\infty} \alpha_j \ll 1,$$

we have that $v_0 \in H^1(\mathbb{R})$, in fact in $v_0 \in H^{3/2^-}(\mathbb{R})$. This in particular guarantees the global existence of solutions in $H^1(\mathbb{R})$ for the IVP (3.3.7). Now, we will divide the analysis en four steps.

Step 1: Estimate for $V(t)\phi$ for $t > 0$.

Assume that $\tilde{v}_0 \in L^2(\mathbb{R})$ and $e^x\tilde{v}_0 \in L^2(\mathbb{R})$. Now, consider $w(t,x) := e^x\tilde{v}(t,x)$. Following Kato [42], we set $\tilde{v}(t,x) = e^{-x}w(t,x)$ where w is solution of the IVP:

$$\begin{cases} \partial_t w + (\partial_x - 1)^3 w = 0, \\ w(0,x) = e^x \tilde{v}_0(x). \end{cases} \tag{3.3.9}$$

Since $(\partial_x - 1)^3 = \partial_x^3 - 3\partial_x^2 + 3\partial_x - 1$, one has that

$$\begin{aligned} w(x,t) &= V(t)e^{3t\partial_x^2}e^{-3t\partial_x}e^t\big(e^x\tilde{v}_0(x)\big) \\ &= V(t)e^{3t\partial_x^2}e^{-3t\partial_x}\big(e^{x+t}\tilde{v}_0(x)\big). \end{aligned}$$

Thus,

$$V(t)v_0 = \tilde{v}(t,x) = e^{-x}V(t)e^{3t\partial_x^2}\big(e^{x-2t}\tilde{v}_0(x-3t)\big).$$

We notice using the heat kernel properties that

$$\partial_x^m V(t)\tilde{v}_0 \sim e^{-x}V(t)\big(\partial_x^m e^{3t}\partial_x^2\big)\big(e^{x-2t}\tilde{v}_0(x-3t)\big).$$

It follows that

$$\|e^x\partial_x^m V(t)\tilde{v}_0\|_2 \sim \frac{c_m}{(3t)^{m/2}}\|e^{x-2t}\tilde{v}_0(x-3t)\|_2 \sim \frac{c_m}{(3t)^{m/2}}e^t,$$

since

$$\|e^{x-2t}\tilde{v}_0(x-3t)\|_2 = e^t\|e^{x-3t}\tilde{v}_0(x-3t)\|_2 = ce^t.$$

Similarly, if $t<0$, we have an IVP analogous to the one in (3.3.9) with the operator $-(\partial_x+1)^3$ instead of $(\partial_x-1)^3$. Thus,

$$\begin{aligned}V(t)\tilde{v}_0 &= e^xV(t)e^{-3t\partial_x^2}e^{-3t\partial_x}(e^{-t}e^{-x}\tilde{v}_0)\\ &= e^xV(t)e^{-3t\partial_x^2}e^{-3t\partial_x}\big(e^{-x-t}\tilde{v}_0(x)\big)\\ &= e^xV(t)e^{-3t\partial_x^2}\big(e^{-x-4t}\tilde{v}_0(x-3t)\big),\end{aligned}$$

and so we have,

$$\partial_x^m V(t)\tilde{v}_0 \sim e^xV(t)\big(\partial_x^m e^{-3t\partial_x^2}\big)\big(e^{x-4t}\tilde{v}_0(x+3t)\big)$$

and

$$\|e^{-x}\partial_x^m V(t)\tilde{v}_0\|_2 \sim \frac{c_m}{(3t)^{m/2}}e^{-t}. \tag{3.3.10}$$

Step 2: Next we prove that

$$v_0(x) = \sum_{j=1}^{\infty}\alpha_j V(-j)\phi \in C^\infty(\mathbb{R}),$$

or equivalently

$$\sum_{j=1}^{\infty}\alpha_j e^{-x}V(-j)\phi \in C^\infty(\mathbb{R}).$$

To do this, it suffices to show that

$$\sum_{j=1}^{\infty}\alpha_j e^{-x}\big(\partial_x^m V(-j)\phi\big) \in L^2(\mathbb{R}) \quad \text{for all } m,$$

or equivalently,

$$\sum_{j=1}^{\infty}\alpha_j\frac{c_m}{(3j)^{m/2}}e^j < \infty.$$

Step 3: For each $t > 0$, $t \notin \mathbb{Z}^+$, we claim that

$$V(t)v_0 = \sum_{j=1}^{\infty} \alpha_j V(t-j)\phi \in C^1(\mathbb{R}).$$

In fact, combining (3.3.10) and the assumption,

$$\sum_{j=1}^{\infty} \alpha_j \frac{1}{3|t-j|} e^{|t-j|} < \infty,$$

one has $V(t)v_0 \in H^2_{\text{loc}}(\mathbb{R}) \subseteq C^1(\mathbb{R})$ which proof the claim.

Step 4: For $t = n \in \mathbb{Z}^+$ we affirm that

$$V(n)v_0 = \alpha_n \phi + \sum_{\substack{j=0 \\ j \neq n}}^{\infty} \alpha_j V(n-j)\phi \equiv \alpha_n \phi + \Phi_n \tag{3.3.11}$$

with $\Phi_n \in C^1$. As before, using (3.3.10) and taking

$$\sum_{\substack{j=0 \\ j \neq n}}^{\infty} \alpha_j \frac{1}{3|n-j|} e^{|n-j|} < +\infty$$

it follows that $\Phi_n \in H^2_{loc}(\mathbb{R})$ which yields (3.3.11). □

3.4 Persistence property: Proof of Theorem 3.1.2

In this Section we prove Theorem 3.1.2. For this purpose, we perform a contraction principle argument for the IVP problem (3.1.1).

Proof of Theorem 3.1.2. We consider the system of integral equations equivalent to (3.1.1), that is,

$$\begin{aligned} \Phi(u) &= S(t)u_0 + \int_0^t S(t-t')(uv)(t')\,dt' + \int_0^t S(t-t')|u|^2 u(t')\,dt', \\ \Psi(v) &= V(t)v_0 - \int_0^t V(t-t')v\partial_x v(t')\,dt' + \int_0^t V(t-t')\partial_x(|u|^2)(t')\,dt', \end{aligned} \tag{3.4.1}$$

where $\{S(t)\}$ and $\{V(t)\}$ are the unitary groups associated to the linear Schrödinger and the Airy equation respectively.

By using the definition, group properties, Minkowski's inequality, and Sobolev spaces properties we have

$$\begin{aligned}\|D_x^{s+\frac{1}{2}}\Phi(u)\|_{L^2} &\leq c\|D_x^{s+\frac{1}{2}}u_0\|_{L^2} + \int_0^T \|D_x^{s+\frac{1}{2}}(uv)\|_{L^2}\,dt' + \int_0^T \|D_x^{s+\frac{1}{2}}(|u|^2u)\|_{L^2}\,dt' \\ &\leq c\,\|u_0\|_{s+1/2} + \int_0^T \|D_x^{s-\frac{1}{2}}\partial_x(uv)\|_{L^2}\,dt' + c\,T\sup_{[0,T]}\|u(t)\|_{s+1/2}^3.\end{aligned} \tag{3.4.2}$$

To complete the estimate we use the commutator estimate (3.2.11), Sobolev spaces properties, the Cauchy-Schwarz inquality, and Hölder's inequality in time to led to

$$\begin{aligned}&\int_0^T \|D_x^{s-\frac{1}{2}}\partial_x(uv)\|_{L^2}\,dt \\ &\quad\leq c\int_0^T \|D_x^{s-1/2}(u\partial_x v)\|_{L^2}\,dt + c\int_0^T \|D_x^{s-1/2}(\partial_x uv)\|_{L^2}\,dt \\ &\quad\leq c\int_0^T \|\partial_x v\|_{L_x^\infty}\|D_x^{s-1/2}u\|_{L^2}\,dt + c\int_0^T \|uD_x^{s-1/2}\partial_x v\|_{L^2} \\ &\qquad + c\int_0^T \|\partial_x u\|_{L_x^\infty}\|D_x^{s-1/2}v\|_{L^2}\,dt + c\int_0^T \|vD_x^{s-1/2}\partial_x u\|_{L^2} \\ &\quad\leq cT^{4/3}\,\|\partial_x v\|_{L_T^4L_x^\infty}\sup_{[0,T]}\|u(t)\|_{s+1/2} + cT^{1/2}\|u\|_{L_x^2L_T^\infty}\|D_x^{s-1/2}\partial_x v\|_{L_x^\infty L_T^2} \\ &\qquad + cT^{4/3}\,\|\partial_x u\|_{L_T^4L_x^\infty}\sup_{[0,T]}\|v(t)\|_{s-1/2} + cT^{1/2}\|v\|_{L_x^2L_T^\infty}\|D_x^{s-1/2}\partial_x u\|_{L_x^\infty L_T^2}.\end{aligned} \tag{3.4.3}$$

Next we estimate the H^s-norm of $\Psi(v)$. It is enough to estimate $\|D^s\Psi(v)\|_{L^2}$. To do so, we use group properties, Minkowskii's and Holder's inequalities to obtain

$$\begin{aligned}\|D_x^s\Psi(v)\|_{L^2} &\leq c\|D_x^s v_0\|_{L^2} + \int_0^T \|D_x^s(v\partial_x v)\|_{L^2}\,dt' + \int_0^T \|D_x^s\partial_x(u\bar{u})\|_{L^2}\,dt' \\ &\leq c\,\|v_0\|_s + \int_0^T \|D_x^s(v\partial_x v)\|_{L^2}\,dt' + \int_0^T \|D_x^s(\bar{u}\partial_x u)\|_{L^2}\,dt' \\ &\quad + \int_0^T \|D_x^s(u\partial_x\bar{u})\|_{L^2}\,dt'.\end{aligned} \tag{3.4.4}$$

The commutator estimates (3.2.11) and Hölder's inequality yield

$$\int_0^T \|D_x^s(v\partial_x v)\|_{L^2}\,dt \leq cT^{4/3}\,\|\partial_x v\|_{L_T^4L_x^\infty}\sup_{[0,T]}\|v(t)\|_s + cT^{1/2}\|v\|_{L_x^2L_T^\infty}\|D_x^s\partial_x v\|_{L_x^\infty L_T^2}. \tag{3.4.5}$$

Similarly, we get

$$\begin{aligned}&\int_0^T \|D_x^s(\bar{u}\partial_x u)\|_{L^2}\,dt' + \int_0^T \|D_x^s(u\partial_x\bar{u})\|_{L^2}\,dt' \\ &\qquad\qquad \le cT^{4/3}\,\|\partial_x u\|_{L_T^4L_x^\infty}\sup_{[0,T]}\|u(t)\|_s + cT^{1/2}\|u\|_{L_x^2L_T^\infty}\|D_x^s\partial_x u\|_{L_x^\infty L_T^2}.\end{aligned} \tag{3.4.6}$$

On the other hand, using Kato's smoothing effect (3.2.3) and the analysis in (3.4.3)

$$\begin{aligned}&\|D^s\partial_x\Phi(u)\|_{L_x^\infty L_T^2} \\ &\qquad \le c\|D_x^{s+1/2}u_0\|_{L^2} + \int_0^T \|D_x^{s+1/2}(uv)\|_{L^2}\,dt + \int_0^T \|D_x^{s+1/2}(|u|^2u)\|_{L^2}\,dt \\ &\qquad \le c\|u_0\|_{s+1/2} + \int_0^T \|D_x^{s-1/2}\partial_x(uv)\|_{L^2}\,dt + \int_0^T \|D_x^{s+1/2}(|u|^2u)\|_{L^2}\,dt.\end{aligned} \tag{3.4.7}$$

Same argument as above, now applying Kato's smoothing effect (3.2.1) and the analysis in (3.4.5), lead to

$$\begin{aligned}\|D^s\partial_x\Psi(v)\|_{L_x^\infty L_T^2} &\le c\|D_x^s v_0\|_{L^2} + \int_0^T \|D_x^s(v\partial_x v)\|_{L^2}\,dt + \int_0^T \|D_x^s\partial_x(u\bar{u})\|_{L^2}\,dt \\ &\le c\,\|v_0\|_s + \int_0^T \|D_x^s(v\partial_x v)\|_{L^2}\,dt' + \int_0^T \|D_x^s(\bar{u}\partial_x u)\|_{L^2}\,dt' \\ &\quad + \int_0^T \|D_x^s(u\partial_x\bar{u})\|_{L^2}\,dt'.\end{aligned} \tag{3.4.8}$$

From the maximal norm estimates (3.2.8) and (3.2.9) it follows that

$$\|\Phi(u)\|_{L_x^2L_T^\infty} \le c\,(1+T)^{\rho_1}\Big\{\|u_0\|_{s+1/2} + \int_0^T \|(uv)(t)\|_{s+1/2}\,dt + \int_0^T \|u(t)\|_{s+1/2}^3\,dt\Big\} \tag{3.4.9}$$

and

$$\|\Psi(v)\|_{L_x^2L_T^\infty} \le c\,(1+T)^{\rho_2}\Big\{\|v_0\|_s + \int_0^T \|v\partial_x v(t)\|_s\,dt + \int_0^T \|\partial_x(u\bar{u})(t)\|_s\,dt\Big\}. \tag{3.4.10}$$

The Strichartz estimates (3.2.6) and (3.2.7) imply

$$\|\partial_x\Phi(u)\|_{L_T^4L_x^\infty} \le c\,\|u_0\|_{s+1/2} + \int_0^T \|(uv)(t)\|_{s+1/2}\,dt + \int_0^T \|u(t)\|_{s+1/2}^3\,dt \tag{3.4.11}$$

and

$$\|\partial_x\Psi(v)\|_{L_T^4L_x^\infty} \le c\,\|v_0\|_s + \int_0^T \|v\partial_x v(t)\|_s\,dt + \int_0^T \|\partial_x(u\bar{u})(t)\|_s\,dt, \tag{3.4.12}$$

respectively.

Finally, we need to estimate $\sup\limits_{[0,T]}\||x|^{r_1}\Phi(u)(t)\|_{L^2}$ and $\sup\limits_{[0,T]}\||x|^{r_2}\Psi(v)(t)\|_{L^2}$. To simplify the proof we only consider the case $s = 3/4^{+}$. In other words, we estimate $\sup\limits_{[0,T]}\||x|^{\frac{5}{8}^+}\Phi(u)(t)\|_{L^2}$ and $\sup\limits_{[0,T]}\||x|^{\frac{3}{8}^+}\Psi(v)(t)\|_{L^2}$.

We can also deduce a formula for solutions of the linear Schrödinger equation:

$$|x|^r S(t)u_0(x) = S(t)(|x|^r u_0) + S(t)\big\{\Lambda_{t,r}(\widehat{u}_0)(\xi)\big\}^{\vee}, \tag{3.4.13}$$

where

$$\|\big\{\Lambda_{t,r}(\widehat{u}_0)(\xi)\big\}^{\vee}\|_{L^2_x} \le c(1+|t|)\,\big(\|u_0\|_{L^2_x} + \|D^r_x u_0\|_{L^2_x}\big), \tag{3.4.14}$$

which is just an analogue to the formula deduced by Fonseca, Linares and Ponce in [31] for the Airy group: for $\beta \in (0,1)$ and $t \in \mathbb{R}$,

$$|x|^\beta V(t)f = V(t)\big(|x|^\beta f\big) + V(t)\{\Phi_{t,\beta}(\hat{f})\}, \tag{3.4.15}$$

with

$$\|\Phi_{t,\beta}(\hat{f})\|_2 \le c(1+|t|)\|f\|_{2\beta,2}. \tag{3.4.16}$$

Thus, applying formula (3.4.13), we have

$$\begin{aligned}
\||x|^{\frac{5}{8}^+}\Phi(u)(t)\|_{L^2} &\le c\||x|^{\frac{5}{8}^+}S(t)u_0\|_{L^2} + \||x|^{\frac{5}{8}^+}S(t)\int_0^t S(t')(uv+|u|^2u)(t')\,dt'\|_{L^2}\\
&\le \||x|^{\frac{5}{8}^+}u_0\|_{L^2} + c(1+|t|)\,(\|u_0\|_{L^2} + \|D_x^{\frac{5}{8}^+}u_0\|_{L^2})\\
&\quad + c\int_0^T \||x|^{\frac{5}{8}^+}(uv+|u|^2u)(t)\|_{L^2}\,dt\\
&\quad + c\,(1+T)\int_0^T \|(uv+|u|^2u)(t)\|_{L^2}\,dt\\
&\quad + c\,(1+T)\int_0^T \|D_x^{\frac{5}{8}^+}(uv+|u|^2u)(t)\|_{L^2}\,dt\\
&\le \||x|^{\frac{5}{8}^+}u_0\|_{L^2} + c(1+T)\|u_0\|_{\frac{5}{4}^+} + A_1 + A_2 + A_3.
\end{aligned} \tag{3.4.17}$$

Next we estimate A_i, $i = 1,2,3$. Hölder's inequality and Sobolev lemma lead to

$$\begin{aligned}
A_1 &\le \int_0^T (\||x|^{\frac{5}{8}^+}uv(t)\|_{L^2} + \||x|^{\frac{5}{8}^+}u|u|^2(t)\|_{L^2})\,dt\\
&\le c\,T\Big(\sup_{[0,T]}\||x|^{\frac{5}{8}^+}u(t)\|_{L^2} \times \sup_{[0,T]}\|v(t)\|_{\frac{3}{4}^+} + \sup_{[0,T]}\||x|^{\frac{5}{8}^+}u(t)\|_{L^2} \times \sup_{[0,T]}\|u(t)\|^2_{\frac{5}{4}^+}\Big).
\end{aligned} \tag{3.4.18}$$

Hölder's inequality and Sobolev lemma yield

$$A_2 \leq c\,(1+T)\,T\left(\sup_{[0,T]}\|u(t)\|_{\frac{5}{4}^+} \times \sup_{[0,T]}\|v(t)\|_{\frac{3}{4}^+} + \sup_{[0,T]}\|u(t)\|^3_{\frac{5}{4}^+}\right). \tag{3.4.19}$$

Applying Sobolev spaces properties we obtain

$$\begin{aligned} A_3 &\leq c\,(1+T)\int_0^T \left(\|D_x^{\frac{5}{8}^+}(uv)(t)\|_{L^2} + \|D_x^{\frac{5}{8}^+}(u|u|^2)(t)\|_{L^2}\right) dt \\ &\leq c\,(1+T)\,T\big(\sup_{[0,T]}\|u(t)\|_{\frac{5}{4}^+}\sup_{[0,T]}\|v(t)\|_{\frac{3}{4}^+} + \sup_{[0,T]}\|u(t)\|^3_{\frac{5}{4}^+}\big), \end{aligned} \tag{3.4.20}$$

Now we estimate $\||x|^{\frac{3}{8}^+}\Psi(v)\|_{L^2}$. Applying formula (3.4.15) we get

$$\begin{aligned} \||x|^{\frac{3}{8}^+}\Psi(v)\|_{L^2} &\leq \||x|^{\frac{3}{8}^+}v_0\|_{L^2} + c(1+T)\left(\|v_0\|_{L^2} + \|D_x^{\frac{3}{4}^+}v_0\|_{L^2}\right) \\ &\quad + \|\int_0^t V(t-t')|x|^{\frac{3}{8}^+}v\partial_x v\,dt'\|_{L^2} + \|\int_0^t V(t-t')|x|^{\frac{3}{8}^+}\partial_x|u|^2\,dt'\|_{L^2} \\ &\quad + c\,(1+T)\int_0^T \left(\|v\partial_x v\|_{L^2} + \|\partial_x|u|^2\|_{L^2}\right) dt \\ &\quad + c\,(1+T)\int_0^T \|D_x^{\frac{3}{4}^+}(v\partial_x v)\|_{L^2}\,dt \\ &\quad + c\,(1+T)\int_0^T \|D_x^{\frac{3}{4}^+}\partial_x(|u|^2)\|_{L^2}\,dt. \end{aligned} \tag{3.4.21}$$

The last three terms above were previously estimated. To obtain the desired estimate we need to bound the third and fourth term on the right hand side of (3.4.21). To do so we follow an argument introduced in [31].

We take $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1,\ |x| < 1/2$ and $\varphi = 0,\ |x| \geq 1$ write

$$\begin{aligned} |x|^{\frac{3}{8}^+}v\partial_x v &= \varphi(x)|x|^{\frac{3}{8}^+}v\partial_x v + (1-\varphi(x))|x|^{\frac{3}{8}^+}v\partial_x v \\ &= \varphi|x|^{\frac{3}{8}^+}v\partial_x v + \partial_x((1-\varphi)|x|^{\frac{3}{8}^+}v^2/2) - \partial_x((1-\varphi)|x|^{\frac{3}{8}^+})v^2/2 \\ &\equiv N_{11} + N_{12} + N_{13}. \end{aligned} \tag{3.4.22}$$

Hence

$$\begin{aligned} &\|\int_0^t V(t-t')|x|^{\frac{3}{8}^+}v\partial_x v(t')dt'\|_{L^2} \\ &\qquad = \|\int_0^t V(t-t')(N_{11}+N_{12}+N_{13})\,dt'\|_{L^2} \\ &\qquad \leq \int_0^T \|V(t-t')(N_{11}+N_{13})\|_{L^2}dt' + \|\int_0^t V(t-t')N_{12}\,dt'\|_{L^2}. \end{aligned} \tag{3.4.23}$$

Thus,

$$\int_0^T \|V(t-t')N_{11}\|_{L^2} dt' \leq \|v\partial_x v\|_{L^1_T L^2_x} \\ \leq c\,T^{3/4} \sup_{[0,T]} \|v(t)\|_{L^2} \|\partial_x v\|_{L^4_T L^\infty_x} \tag{3.4.24}$$

Using the dual version of Kato's smoothing effect (3.2.2) we obtain

$$\| \int_0^t V(t-t')N_{12}\, dt'\|_{L^2_x} \leq c\| \,(1-\varphi)|x|^{\frac{3}{8}^+} v^2\|_{L^1_x L^2_T} \\ \leq \||x|^{\frac{3}{8}^+} v\|_{L^2_x L^2_T} \|v\|_{L^2_x L^\infty_T} \leq T^{1/2} \||x|^{\frac{3}{8}^+} v\|_{L^\infty_T L^2_x} \|v\|_{L^2_x L^\infty_T}. \tag{3.4.25}$$

Finally,

$$\int_0^{T_0} \|V(t-t')N_{13}\|_{L^2_x} dt' \leq c\,\| \,v^2\|_{L^1_T L^2_x} \leq c\,T\|v\|^2_{L^\infty_T H^{\frac{3}{4}^+}}. \tag{3.4.26}$$

As above taking $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1,\ |x| < 1/2$ and $\varphi = 0,\ |x| \geq 1$ we write

$$|x|^{\frac{3}{8}^+} \partial_x(|u|^2) = \varphi|x|^{\frac{3}{8}^+} \partial_x(|u|^2) + \partial_x((1-\varphi)|x|^{\frac{3}{8}^+}|u|^2) - \partial_x((1-\varphi)|x|^{\frac{3}{8}^+})|u|^2 \\ \equiv N_{21} + N_{22} + N_{23}. \tag{3.4.27}$$

Hence

$$\| \int_0^t V(t-t')|x|^{\frac{3}{8}^+} \partial_x(|u|^2)(t')dt'\|_{L^2_x} \\ \leq \| \int_0^t V(t-t')(N_{21}+N_{22}+N_{23})\, dt'\|_{L^2_x} \\ \leq \int_0^{T_0} \|V(t-t')(N_{21}+N_{23})\|_{L^2_x} dt' + \| \int_0^t V(t-t')N_{22}\, dt'\|_{L^2_x}. \tag{3.4.28}$$

It follows that

$$\int_0^{T_0} \|V(t-t')N_{21}\|_{L^2_x} dt' \leq c \int_0^T \left(\|\bar{u}\partial_x u\|_{L^2} + \|u\partial_x \bar{u}\|_{L^2} \right) dt \\ \leq c\,T^{3/4}\, \|u\|_{L^\infty_T L^2_x} \|\partial_x u\|_{L^4_T L^\infty_x}. \tag{3.4.29}$$

The smoothing effect (3.2.2) yields

$$\| \int_0^t V(t-t')N_{22}\, dt'\|_{L^2_x} = \|\partial_x \int_0^T V(t-t')\,(1-\varphi)|x|^{\frac{3}{8}^+} u\bar{u}\, dt'\|_{L^2} \\ \leq c\| \,(1-\varphi)|x|^{\frac{3}{8}^+} u\bar{u}\|_{L^1_x L^2_T} \leq c\,\| \,|x|^{\frac{3}{8}^+} u\|_{L^2_x L^2_T}\, \|u\|_{L^2_x L^\infty_T} \\ \leq cT^{1/2}\| \,|x|^{\frac{5}{8}^+} u\|_{L^\infty_T L^2_x}\, \|u\|_{L^2_x L^\infty_T}. \tag{3.4.30}$$

Finally,

$$\int_0^{T_0} \|V(t-t')N_{23}\|_{L^2_x} dt' \leq c_0 \| u\bar{u}\|_{L^1_T L^2_x} \leq cT \sup_{[0,T]} \big(\|u(t)\|_{L^2}\|u(t)\|_{L^\infty}\big)$$
$$\leq c\,T \sup_{[0,T]} \|u(t)\|^2_{5/4^+}. \tag{3.4.31}$$

The proof is complete. □

3.5 Dispersive blow-up: Proof of Theorem 3.1.3

The proof is built upon the linear analysis appearing in Section 3.3. Consider the IVP associated to the Schrödinger-Korteweg-de Vries system (3.1.1),

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = \alpha uv, \qquad t, x \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases} \tag{3.5.1}$$

Let us consider

$$u_0 \in H^{2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{2^-}dx) \cap L^\infty(\mathbb{R}),$$

and

$$v_0 \in C^\infty(\mathbb{R}) \cap H^{3/2^-}(\mathbb{R}) \cap L^2(\langle x\rangle^{3/4^-}dx) \cap L^\infty(\mathbb{R}),$$

be the initial data constructed in the Section 3.3, which leads to linear, independent dispersive blow-up at every time $t \in \mathbb{Z}$. Now, using the Schrödinger group, the Airy group and the Duhamel's formula, we obtain that the solution can be represented as

$$u(t,x) = S(t)u_0 + C_1 \int_0^t S(t-t')(uv)(t')dt' + C_2 \int_0^t S(t-t')(|u|^2u)(t')dt', \tag{3.5.2}$$

and

$$v(t,x) = V(t)v_0 + C_3 \int_0^t V(t-t')(\partial_x(v^2))(t')dt' + C_4 \int_0^t V(t-t')(\partial_x|u|^2)(t')dt', \tag{3.5.3}$$

where C_1, C_2, C_3 and C_4 are non-zero constant of no consequence. If the integral terms in (3.5.2) and (3.5.3) are $H^{2+\varepsilon}_x(\mathbb{R})$ and $C^1_x(\mathbb{R})$ functions for all $t \in [0,T]$ (respectively), then the desired result follows from what we already known about $S(t)u_0$ and $V(t)v_0$ from Section 3.3. To do this we divide the analysis in two cases, the inhomogeneous terms at Schrödinger equation level (3.5.2) and the other ones at KdV equation level (3.5.3).

The following two propositions are sufficient to complete the proof of Theorem 3.1.3.

Proposition 3.5.1. *Consider initial data* $(u_0, v_0) \in H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{r_1}dx) \times H^s(\mathbb{R}) \cap L^2(|x|^{r_2}dx)$ *with* $s > 3/4$, $r_1 = (s+\frac{1}{2})^-$ *and* $r_2 = \frac{s}{2}^-$. *Let* $(u(t), v(t))$ *the corresponding solution for the IVP* (3.1.1) *given by Theorem 3.1.2. From* (3.5.2), *define* I *as follows:*

$$\begin{aligned} u(t,x) &= S(t)u_0 + C_1 \int_0^t S(t-t')(uv)(t')dt' + C_2 \int_0^t S(t-t')(|u|^2u)(t')dt' \\ &=: S(t)u_0 + \mathrm{I}(t,x), \end{aligned}$$

then $\mathrm{I} \in C([0,T]; H^{s+\frac{3}{4}}(\mathbb{R}))$.

In other words, the integral term I is "smother" than the free propagator $e^{it\Delta}u_0$ by a quarter of derivative. In particular, this implies that for initial data as at the beginning of this section, the integral term $\mathrm{I} \in C([0,T] : H^{\frac{9}{4}^-}(\mathbb{R}))$.

Proposition 3.5.2. *Consider initial data* $(u_0, v_0) \in H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{r_1}dx) \times H^s(\mathbb{R}) \cap L^2(|x|^{r_2}dx)$ *with* $s > 7/6$, $r_1 = (s+\frac{1}{2})^-$ *and* $r_2 = \frac{s}{2}^-$. *Let* $(u(t), v(t))$ *the corresponding solution for the IVP* (3.1.1) *given by Theorem 3.1.2.*

$$\begin{aligned} v(t,x) &= V(t)v_0 + C_1 \int_0^t V(t-t')(\partial_x(v^2))(t')dt' + C_2 \int_0^t V(t-t')(\partial_x|u|^2)(t')dt' \\ &=: V(t)v_0 + \mathrm{II}(t,x), \end{aligned}$$

then $\mathrm{II} \in C([0,T]; H^{s+\frac{1}{6}}(\mathbb{R}))$.

In other words, the integral term II is "smoother" than the free propagator $V(t)v_0$ by a one sixth derivative. In particular, this implies that for initial data as at the beginning of this section, the integral term $\mathrm{II} \in C([0,T] : C^1(\mathbb{R}))$.

Proof of Proposition 3.5.1. First of all, recall that Theorem 3.1.2 guarantees the existence of the solution

$$u \in C([0,T] : H^{s+\frac{1}{2}}(\mathbb{R}) \cap L^2(|x|^{s+\frac{1}{2}-\varepsilon}dx)), \quad v \in C([0,T] : H^s(\mathbb{R}) \cap L^2(|x|^{\frac{s}{2}-\varepsilon}dx)). \tag{3.5.4}$$

Now, let us divide the analysis in two steps. First, define

$$u_1(t) := \int_0^t S(t-t')(uv)(t')dt'.$$

Thus, we shall show that $u_1(t) \in H^{s+\frac{3}{4}}(\mathbb{R})$ for all $t \in [0,T]$. In fact, by (3.2.4) we have

$$\begin{aligned} \|D_x^{s+\frac{3}{4}}u_1\|_{L^2_x} &\le \Big\|D_x^{1/2}\int_0^t S(t-t')D_x^{s+\frac{1}{4}}(uv)(t')dt'\Big\|_{L^2} \\ &\le \|D_x^{s+\frac{1}{4}}(uv)\|_{L^1_xL^2_T} \\ &\le \|vD_x^{s+\frac{1}{4}}u\|_{L^1_xL^2_T} + \|uD_x^{s+\frac{1}{4}}v\|_{L^1_xL^2_T} + E_1, \end{aligned} \tag{3.5.5}$$

where E_1 are easy to control by considering the commutator estimates (see [43]) and interpolated norms of the previous terms to be considered below, so we omit this proof. Now, using Hölder's inequality we can bound the first term of (3.5.5) by

$$\begin{aligned}\|vD_x^{s+\frac{1}{4}}u\|_{L_x^1L_T^2} &\leq c\|v\|_{L_x^2L_T^\infty}\|D_x^{s+\frac{1}{4}}u\|_{L_x^2L_T^2}\\ &\leq cT^{1/2}\|v\|_{L_x^2L_T^\infty}\|D_x^{s+\frac{1}{4}}u\|_{L_T^\infty L_x^2}<\infty.\end{aligned}$$

On the other hand, we can bound the second term of (3.5.5) by

$$\begin{aligned}\|uD_x^{s+\frac{1}{4}}v\|_{L_x^1L_T^2} &\leq c\|\langle x\rangle^{s-\varepsilon}uD_x^{s+\frac{1}{4}}v\|_{L_x^1L_T^2}\\ &\leq cT^{3/4}\sup_{[0,T]}\|\langle x\rangle^{s-\varepsilon}u\|_{L_x^2}\|D_x^{s+\frac{1}{4}}v\|_{L_T^4L_x^\infty}<\infty.\end{aligned}$$

Thus we have $u_1(t)\in H^{s+\frac{3}{4}}(\mathbb{R})$ for all $t\in[0,T]$, which concludes the demonstration of the first step.

Now, let us consider the second integral term of the solution $u(t,x)$:

$$u_2(t):=\int_0^t S(t-t')(|u|^2u)(t')dt'.$$

Thus, we shall show that $u_2(t)\in H^{s+\frac{3}{4}}(\mathbb{R})$ for all $t\in[0,T]$. In fact, by the dual version of the Kato's smoothing effect 3.2.2 we have

$$\begin{aligned}\|D_x^{s+\frac{3}{4}}u_2\|_{L_x^2} &\leq c\|D_x^{s+\frac{1}{4}}(|u|^2u)\|_{L_x^1L_T^2}\\ &\leq c\|u\|_{L_x^2L_T^\infty}\|D_x^{s+\frac{1}{4}}u\|_{L_x^\infty L_T^2}+E_2<\infty.\end{aligned}$$

where, again, the terms in E_2 are easy to control by considering the commutator estimates (see [43]) and the interpolated norms of the previous terms, so we omit the proof. Thus, the proof is complete. □

Proof of Proposition 3.5.2. From the previous proof, we have (3.5.4). Even more, we have

$$\int_0^T\|D_x^{\alpha\theta/2}J^kv(\cdot,t)\|_p^q dt<\infty\qquad\text{for}\qquad 0\leq k\leq s, \tag{3.5.6}$$

for

$$(q,p)=\left(\tfrac{6}{\theta}(\alpha+1),\,\tfrac{2}{1-\theta}\right),\qquad \theta\in(0,1),\qquad 0\leq\alpha\leq\tfrac{1}{2}.$$

Now, let us divide the analysis in two steps. First, define

$$v_1(t):=\int_0^t V(t-t')\big(\partial_x(v^2)\big)(t')dt'\;\in\;C^1(\mathbb{R}).$$

Thus, we shall show that $v_1(t) \in H^{s+\frac{1}{6}}(\mathbb{R})$ for all $t \in [0,T]$. In fact, using the smoothing Kato effect (3.2.2), we obtain

$$\sup_{0\le t\le T} \|D_x^{s+1/6} \int_0^t V(t-t')(v\partial_x v)(t')dt'\|_2 \le \|D_x^{s-5/6}(v\partial_x v)\|_{L_x^1 L_T^2}$$
$$\le \|v\|_{L_x^{6/5} L_T^3} \|D_x^{s+1/6} v\|_{L_x^6 L_T^6} + E_1,$$

where E_1 are easy to control by considering the commutator estimates (see [43]) and interpolated norms of the previous terms to be considered below, so we omit this proof. Now, from (3.5.6) with $p=q=6$, $\theta=\frac{2}{3}$ and $\alpha=\frac{1}{2}$ we obtain:

$$\|D_x^{s+1/6} v\|_{L_x^6 L_T^6} < \infty.$$

On the other hand, using (3.2.10) in Lemma 3.2.5 we deduce:

$$\begin{aligned}\|v\|_{L_x^{6/5} L_T^3} &\le c\|\langle x\rangle^{1/2^+} v\|_{L_T^3 L_x^3}\\ &\le cT^{1/3}\|\langle x\rangle^{1/2^+} v\|_{L_T^\infty L_x^3}\\ &\le cT^{1/3}\|J^{1/6}\big(\langle x\rangle^{1/2^+} v\big)\|_{L_T^\infty L_x^2}\\ &\le cT^{1/3}\|J^s v\|_{L_T^\infty L_x^2}^{1-\gamma} \|\langle x\rangle^{\frac{s}{2}^-} v\|_{L_T^\infty L_x^2}^{\gamma},\end{aligned}$$

with γ such that $\frac{s\gamma^-}{2} = \frac{1}{2}^+$, i.e. $\gamma > \frac{1}{s}$, and such that $(1-\gamma)s > 1/6$. Note that the last inequality imposes the restriction $s > \frac{7}{6}$. Thus we have $v_1(t) \in H^{s+\frac{1}{6}}(\mathbb{R})$ for all $t \in [0,T]$, which concludes the demonstration of the first step.

Now, let us consider the second integral term of the solution $v(t,x)$:

$$v_2(t) := \int_0^t V(t-t')\big(\partial_x |u|^2\big)(t')dt' \in C^1(\mathbb{R}).$$

We shall show that $v_2(t) \in H^{s+\frac{1}{6}}(\mathbb{R})$ for all $t \in [0,T]$. For this, we use the inhomogeneous smothing Kato effect (3.2.2), thus we obtain:

$$\sup_{0\le t\le T} \big\|D_x^{s+\frac{1}{6}} \int_0^t V(t-t')\big(\partial_x |u|^2\big)dt'\big\|_2 \le c\big\|D_x^{s+\frac{1}{6}}\big(|u|^2\big)\big\|_{L_x^1 L_T^2}$$
$$\le c\|u\|_{L_x^1 L_T^\infty} \|D_x^{s+\frac{1}{6}} u\|_{L_x^\infty L_T^2} + E_2,$$

where the terms in E_2 are easy to control by considering the commutator estimates and the interpolated norms of the previous terms to be considered below, so we omit the proof.

Now, to estimate $\|D_x^{s+\frac{1}{6}} u\|_{L_x^\infty L_T^2}$, we use the homogeneous smoothing Kato effect (3.2.3), Minkowskii's integral inequality and group properties to obtain

$$\|D_x^{s+\frac{1}{6}} u\|_{L_x^\infty L_T^2} \le c\|u_0\|_{H^{s-\frac{1}{3}}} + c\int_0^T \|D_x^{s-\frac{1}{3}}(uv)\|_{L_x^2} dt + c\int_0^T \|D_x^{s-\frac{1}{3}}(|u|^2 u)\|_{L_x^2}$$

$$\leq c\|u_0\|_{H_x^{s-\frac{1}{3}}} + cT\|u\|_{L_T^\infty H_x^{s-\frac{1}{3}}}\|v\|_{L_T^\infty H_x^{s-\frac{1}{3}}} + cT\|u\|^3_{L_T^\infty H_x^{s-\frac{1}{3}}}. \tag{3.5.7}$$

Then, by the local-well posedness theory, we conclude that:

$$\|D_x^{s+\frac{1}{6}}u\|_{L_x^\infty L_T^2} < \infty.$$

which concludes the estimates for the solution u. Thus, we conclude that we can reduce ourselves to consider the linear associated problem so the nonlinearity after Section 3.3 is not relevant for our purposes. □

Appendix A

Proof of Proposition 2.4.6

We start proving that (2.4.20) is satisfied. We follow the same scheme of Proposition 2.4.4. Taking derivative of A wrt x we get

$$\begin{aligned} A_x &= \frac{4\beta^2\cosh^2(\gamma(x+x_2))}{\beta^2\cosh^2(\gamma(x+x_2))+\sinh^2(\gamma x_1)}\cdot\frac{-\sinh(\gamma x_1)}{\beta\cosh^2(\gamma(x+x_2))}\cdot\gamma\sinh(\gamma(x+x_2)) \\ &= -\frac{4\beta\gamma\sinh(\gamma x_1)\sinh(\gamma(x+x_2))}{\sinh^2(\gamma x_1)+\beta^2\cosh^2(\gamma(x+x_2))}. \end{aligned} \tag{A.0.1}$$

For the sake of simplicity we define $\theta := \gamma(x-x_1+x_2)$. Using basic trigonometric identities we have

$$\sin\left(\frac{A\pm Q}{2}\right) = \frac{2\tan\left(\arctan\left(\frac{\sinh(\gamma x_1)}{\beta\cosh(\gamma(x+x_2))}\right)\pm\arctan\left(e^{\theta}\right)\right)}{1+\tan^2\left(\arctan\left(\frac{\sinh(\gamma x_1)}{\beta\cosh(\gamma(x+x_2))}\right)\pm\arctan\left(e^{\theta}\right)\right)}. \tag{A.0.2}$$

Since $\tan(a\pm b)=\frac{\tan a\pm\tan b}{1\mp\tan a\tan b}$, (A.0.2) reads now

$$\sin\left(\frac{A\pm Q}{2}\right) = \frac{2\left(\dfrac{\sinh(\gamma x_1)\pm\beta\cosh(\gamma(x+x_2))e^{\theta}}{\beta\cosh(\gamma(x+x_2))\mp\sinh(\gamma x_1)e^{\theta}}\right)}{1+\left(\dfrac{\sinh(\gamma x_1)\pm\beta\cosh(\gamma(x+x_2))e^{\theta}}{\beta\cosh(\gamma(x+x_2))\mp\sinh(\gamma x_1)e^{\theta}}\right)^2},$$

and simplifying,

$$\sin\left(\frac{A\pm Q}{2}\right) = \frac{2f_2(x)}{\left(1+e^{2\theta}\right)\left(\sinh^2(\gamma x_1)+\beta^2\cosh^2(\gamma(x+x_2))\right)}, \tag{A.0.3}$$

where $f_2(x)=f_2(x;\beta,x_1,x_2)$ is such that

$$f_2(x) := \beta\sinh(\gamma x_1)\cosh(\gamma(x+x_2)) \mp e^{\theta}\sinh^2(\gamma x_1)$$

$$\pm \beta^2 e^{\theta} \cosh^2(\gamma(x+x_2)) - \beta e^{2\theta} \sinh(\gamma x_1)\cosh(\gamma(x+x_2)). \qquad \text{(A.0.4)}$$

We are now ready to show that (2.4.20) is satisfied. Substracting (2.3.2) from (A.0.1) we obtain

$$A_x - Q_t = -\frac{4\beta\gamma \sinh(\gamma x_1)\sinh(\gamma(x+x_2))}{\sinh^2(\gamma x_1) + \beta^2 \cosh^2(\gamma(x+x_2))} - \frac{4\beta\gamma e^{\theta}}{1+e^{2\theta}} = \frac{F_2}{F_3},$$

where

$$F_3 := \big(1+e^{2\theta}\big)\big(\sinh^2(\gamma x_1) + \beta^2\cosh^2(\gamma(x+x_2))\big), \qquad \text{(A.0.5)}$$

and

$$\begin{aligned} F_2 :=& -4\beta\gamma\Big[e^{\theta}\big(\beta^2\cosh^2(\gamma(x+x_2)) + \sinh^2(\gamma x_1)\big) \\ & + (1+e^{2\theta})\sinh(\gamma(x+x_2))\sinh(\gamma x_1)\Big]. \end{aligned}$$

On the other hand, recalling that $a := a(\beta)$ and $\gamma = 1/\sqrt{1-\beta^2}$, from (A.0.4) we conclude

$$\frac{1}{a}\sin\left(\frac{A+Q}{2}\right) + a\sin\left(\frac{A-Q}{2}\right) = \frac{F_4}{F_3} \qquad \text{(A.0.6)}$$

where F_3 is given by (A.0.5) and

$$F_4 := 4\beta\gamma\big[(1-e^{2\theta})\sinh(\gamma x_1)\cosh(\gamma(x+x_2)) + e^{\theta}\sinh^2(\gamma x_1) - \beta^2 e^{\theta}\cosh^2(\gamma(x+x_2))\big].$$

Therefore, (2.4.20) is reduced to show that $F_2 - F_4 \equiv 0$. Indeed,

$$\begin{aligned} F_2 - F_4 &= -4\beta\gamma\big[2e^{\theta}\sinh^2(\gamma x_1) + (1+e^{2\theta})\sinh(\gamma(x+x_2))\sinh(\gamma x_1)\big] \\ & \quad - 4\beta\gamma(1-e^{2\theta})\sinh(\gamma x_1)\cosh(\gamma(x+x_2)) = 0. \end{aligned}$$

This proves (2.4.20). We only need to show (2.4.21) now. We follow the same scheme as before: from (2.4.7) and (2.3.1) we have

$$A_t - Q_x = \frac{4\beta^2\gamma\cosh(\gamma(x+x_2))\cosh(\gamma x_1)}{\beta^2\cosh^2(\gamma(x+x_2)) + \sinh^2(\gamma x_1)} - \frac{4\gamma e^{\theta}}{1+e^{2\theta}} = \frac{\widetilde{F}_2}{F_3},$$

where F_3 is given in (A.0.5) and

$$\begin{aligned} \widetilde{F}_2 :=& 4\gamma\big[\beta^2\cosh(\gamma(x+x_2))\cosh(\gamma x_1)(1+e^{2\theta}) \\ & - (\beta^2\cosh^2(\gamma(x+x_2)) + \sinh^2(\gamma x_1))e^{\theta}\big]. \end{aligned} \qquad \text{(A.0.7)}$$

On the other hand, since $a = a(\beta)$ and $\gamma = 1/\sqrt{1-\beta^2}$, and following the same ideas as in the proof of (A.0.6), we have

$$\frac{1}{a}\sin\left(\frac{A+Q}{2}\right) - a\sin\left(\frac{A-Q}{2}\right) = \frac{\widetilde{F}_4}{F_3},$$

where F_3 came from (A.0.5) and $\widehat{F}_4$ denotes the quantity

$$\widetilde{F}_4 := -4\gamma\big[\beta^2 \sinh(\gamma x_1)\cosh(\gamma(x+x_2))(1-e^{2\theta}) \\ - e^{\theta}(\beta^2\cosh^2(\gamma(x+x_2)) - \sinh^2(\gamma x_1))\big]. \tag{A.0.8}$$

Therefore, (2.4.21) has been reduced to show that $\widetilde{F}_2 - \widetilde{F}_4 \equiv 0$. Indeed, from (A.0.7) and (A.0.8)

$$\widetilde{F}_2 - \widetilde{F}_4 = 4\gamma\beta^2 \cosh(\gamma x_1)\cosh(\gamma(x+x_2))(1+e^{2\theta}) - 8\gamma\beta^2\cosh^2(\gamma(x+x_2))e^{\theta} \\ + 4\gamma\beta^2(1-e^{2\theta})\sinh(\gamma x_1)\cosh(\gamma(x+x_2)) = 0,$$

which ends the proof.

Appendix B

Description of derivatives and orthogonality

B.1 Orthogonality for breather type functions

We start with the following result.

Lemma B.1.1. *Let (B, B_t) be a SG breather profile with scaling parameter $\beta \in (-1,1)\backslash\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$. Let us suppose that $x_2 = 0$. Then, B_t and B_x are even and odd respectively.*

Proof. It is enough to see that from (2.4.1), (2.4.11) and (2.4.2),

$$\begin{aligned} B_t = B_1 &= \frac{4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha x_1)}, \\ B_x = B_2 &= \frac{-4\beta^2\alpha\sin(\alpha x_1)\sinh(\beta(x+x_2))}{\alpha^2\cosh^2(\beta(x+x_2))+\beta^2\sin^2(\alpha x_1)}, \end{aligned} \tag{B.1.1}$$

so that if $x_2 = 0$ we get

$$B_t = \frac{4\alpha^2\beta\cos(\alpha x_1)\cosh(\beta x)}{\alpha^2\cosh^2(\beta x)+\beta^2\sin^2(\alpha x_1)}, \qquad B_x = \frac{-4\beta^2\alpha\sin(\alpha x_1)\sinh(\beta x)}{\alpha^2\cosh^2(\beta x)+\beta^2\sin^2(\alpha x_1)},$$

which readily gives the respective parity properties. □

Corollary B.1.2. *Let (B, B_t) be a SG breather with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} B_t B_x dx = 0.$$

Proof. A consequence of the previous lemma and the invariance under translations of the integral on $\mathbb{R}$. □

Lemma B.1.3. *Let (B, B_t) be a SG breather profile with scaling parameter $\beta \in (-1,1)$, $\beta \neq 0$, and shifts $x_1, x_2 \in \mathbb{R}$. Consider $(B_i, B_{t,i})$ the derivatives of B and B_t wrt the variables x_i, $i = 1,2$. Let us additionally suppose that $x_2 = 0$. Then, $B_{t,1}$ and $B_{t,2}$ are functions in the Schwartz class, even and odd in x respectively.*

Proof. For the sake of brevity we define $\theta_1 := \gamma x_1$ and $\theta_2 := \gamma(x + x_2) = \gamma x$. Since B_t in (2.4.2) is smooth, we have after differentiation

$$B_{t,1} = -4\alpha^3\beta \frac{\big(\sin\theta_1\cosh\theta_2(\alpha^2\cosh^2\theta_2 + \beta^2\sin^2\theta_1) + \beta^2\sin(2\theta_1)\cos\theta_1\cosh\theta_2\big)}{\big(\alpha^2\cosh^2\theta_2 + \beta^2\sin^2\theta_1\big)^2},$$
$$B_{t,2} = 4\alpha^2\beta^2 \frac{\big(\cos\theta_1\sinh\theta_2(\alpha^2\cosh^2\theta_2 + \beta^2\sin^2\theta_1) - \alpha^2\sinh(2\theta_2)\cos\theta_1\cosh\theta_2\big)}{\big(\alpha^2\cosh^2\theta_2 + \beta^2\sin^2\theta_1\big)^2}.$$

The desired parity properties are then direct. □

Corollary B.1.4. *Let (B, B_t) be a SG breather profile with scaling parameter $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} B_{t,1}B_{t,2}\,dx = 0.$$

Proof. Direct from previous lemma. □

B.2 Orthogonality of 2-kink or kink-antikink type functions

In this section, we treat the case of 2-kink R and kink-antikink A. Since proofs are similar to the breather case, we only sketch the main ideas.

Lemma B.2.1. *Let (A, A_t) be a SG kink-antikink profile with speed $\beta \in (-1,1)\setminus\{0\}$ and shifts $x_1, x_2 \in \mathbb{R}$. Consider $(A_i, A_{t,i})$ the derivatives of A and A_t wrt the directions x_i, $i = 1,2$. Suppose again that $x_2 = 0$. Then, A_t and $A_{t,1}$ are even, and A_x and $A_{t,2}$ are odd. Each function above is in the Schwartz class.*

Proof. We define $\theta_1 := \gamma x_1$ y $\theta_2 := \gamma(x + x_2) = \gamma x$. Thanks to (A.0.1), (2.4.7) and direct computations, we have

$$A_t = \frac{4\beta^2\gamma\cosh\theta_1\cosh\theta_2}{\beta^2\cosh^2\theta_2 + \sinh^2\theta_1}, \qquad A_x = \frac{-4\beta\gamma\sinh\theta_1\sinh\theta_2}{\beta^2\cosh^2\theta_2 + \sinh^2\theta_1},$$

$$A_{t,1} = \frac{4\beta^2\gamma^2\big(\sinh\theta_1\cosh\theta_2(\beta^2\cosh^2\theta_2+\sinh^2\theta_1)-\sinh(2\theta_1)\cosh\theta_1\cosh\theta_2\big)}{\big(\beta^2\cosh^2\theta_2+\sinh^2\theta_1\big)^2},$$
$$A_{t,2} = \frac{4\beta^2\gamma^2\big(\cosh\theta_1\sinh\theta_2(\beta^2\cosh^2\theta_2+\sinh^2\theta_1)-\beta^2\sinh(2\theta_2)\cosh\theta_1\cosh\theta_2\big)}{\big(\beta^2\cosh^2\theta_2+\sinh^2\theta_1\big)^2}.$$

Here parities are concluded directly since $x_2 = 0$. □

Corollary B.2.2. *Let (A, A_t) be a kink-antikink profile with speed $\beta \in (-1,1)$ and shifts $x_1, x_2 \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} A_t A_x dx = 0, \qquad \int_{\mathbb{R}} A_{t,1}A_{t,2} = 0.$$

Proof. Direct form the previous lemma. □

Lemma B.2.3. *Let (R, R_t) be a SG 2-kink profile with speed $\beta \in (-1,1)$ and shifts $x_1, x_2 \in \mathbb{R}$. Let us consider $(R_i, R_{t,i})$ the derivatives of R y R_t in the directions x_i, $i = 1,2$. Let us assume additionally that $x_2 = 0$. Then, R_t and $R_{t,1}$ are odd, and R_x and $R_{t,2}$ are even. Each of the last four last functions is in the Schwartz class.*[1]

Proof. Using the same notation as in the proof of Lemma B.2.1, we have

$$R_t = \frac{-4\beta^2\gamma\sinh\theta_1\sinh\theta_2}{\cosh^2\theta_1+\beta^2\sinh^2\theta_2}, \qquad R_x = \frac{4\beta\gamma\cosh\theta_1\cosh\theta_2}{\cosh^2\theta_1+\beta^2\sinh^2\theta_2},$$

and

$$\frac{R_{t,1}}{4\beta^2\gamma^2} = -\frac{\cosh\theta_1\sinh\theta_2(\cosh^2\theta_1+\beta^2\sinh^2\theta_2)-\beta^2\sinh(2\theta_1)\sinh\theta_2\sinh\theta_1}{\big(\cosh^2\theta_1+\beta^2\sinh^2\theta_2\big)^2},$$
$$\frac{R_{t,2}}{4\beta^2\gamma^2} = -\frac{\sinh\theta_1\cosh\theta_2(\cosh^2\theta_1+\beta^2\sinh^2\theta_2)-\beta^2\sinh(2\theta_2)\sinh\theta_1\sinh\theta_2}{\big(\cosh^2\theta_1+\beta^2\sinh^2\theta_2\big)^2}.$$

□

Finally, the following result is direct:

Corollary B.2.4. *Let (R, R_t) be a 2-kink SG profile with speed $\beta \in (-1,1)$, $\beta \neq 0$, and shifts $x_1, x_2 \in \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} R_t R_x dx = 0, \qquad \int_{\mathbb{R}} R_{t,1}R_{t,2} = 0.$$

[1] Note that R is not in the Schwartz class.

Appendix C

Proof of Lemma 2.5.1

The proof of this result is standard, we only sketch the main ideas. Let us define H : $\mathbb{R}^2 \times \mathcal{U}(\eta) \to \mathbb{R}^2$, given by

$$\big(H(x_1, x_2, \phi, \phi_t)\big)_j := \Big(\langle \phi - D,\, D_j\rangle_{H^1}, \langle \phi_t - D_t, (D_t)_j\rangle_{L^2}\Big), \quad j = 1, 2,$$

where D, D_t, D_j y $D_{t,j}$ are evaluated at the point $(\cdot; \beta, x_1, x_2)$. Clearly we have $H(x_1, x_2, D, D_t) = (0, 0)$. Moreover, $H \in \mathcal{C}^1$ in a vicinity of (x_1, x_2, D, D_t). Differentiating, we get

$$\big(H_{x_i}(x_1, x_2, D, D_t)\big)_j = -\Big(\langle D_i, D_j\rangle, \big\langle D_{t,i}, D_{t,j}\big\rangle\Big), \quad i, j \in \{1, 2\}.$$

Let us show that $H'(x_1, x_2, D, D_t)$ is invertible. In what follows, we proceed by cases, depending on $D = A, B$ or R.

1. Case $D = B$. Thanks to Lemmas B.1.2 and B.1.4, we have H' diagonal and invertible.

2. Case $D = A, R$. Thanks to Lemmas B.2.2 and B.2.4, we have the same situation as before.

From the last statements we conclude that the matrix $H'(x_1, x_2, D, D_t)$ is always invertible. Hence, the Implicit Function Theorem says that, if ν_0 is sufficiently small, and $\nu \in (0, \nu_0)$, we will have unique functions $(\tilde{x}_1, \tilde{x}_2)$ in $\mathcal{C}^1$, depending on $(\phi, \phi_t) \in \mathcal{U}(\nu)$, and such that $H(\tilde{x}_1(\phi, \phi_t), \tilde{x}_2(\phi, \phi_t), (\phi, \phi_t)) = (0, 0)$.

Appendix D

Proof of Lemmas 2.6.2, 2.9.2 and 2.9.3

D.1 Proof of Lemma 2.6.2

First of all, note that from (2.3.15) we have that μ_K in (2.6.1) satisfies

$$\mu_K = \frac{\cosh(\beta(x+x_2))\cos(\alpha x_1) - i\sinh(\beta(x+x_2))\sin(\alpha x_1)}{\cosh^2(\beta(x+x_2))\cos^2(\alpha x_1) + \sinh^2(\beta(x+x_2))\sin^2(\alpha x_1)} = \frac{1}{2\beta}K_x.$$

Therefore, it is necessary that x_1 does not satisfy (2.3.5) in order to get μ_K well-defined for any x. In this case, μ_K is smooth and decays to zero exponentially in space.

Proving (2.6.5), notice that since x_1 does not satisfy (2.3.5), we can use (2.6.1) and (2.3.11):

$$\int_{\mathbb{R}} \mu_K \sin\left(\frac{K}{2}\right) = \int_{\mathbb{R}} \frac{dx}{\cosh^2(\beta(x+x_2)+i\alpha x_1)} = \frac{2}{\beta}.$$

Now we prove (2.6.6). It is enough to notice that

$$\partial_x\left(\frac{\beta^2\sin(2\alpha x_1) - i\alpha^2\sinh(2\beta(x+x_2))}{\alpha\beta(\alpha^2\cosh(\beta(x+x_2))^2 + \beta^2\sin(\alpha x_1)^2)}\right) = \Phi_1 - \Phi_2,$$

where

$$\Phi_1 = -\frac{8\alpha\beta^2\cosh(\beta(x+x_2)+i\alpha x_1)\sinh(\beta(x+x_2))\sin(\alpha x_1)}{\left(\alpha^2\cosh(\beta(x+x_2))^2 + \beta^2\sin(\alpha x_1)^2\right)^2} = \mu(x)B_x,$$

$$\Phi_2 = \frac{2i\alpha}{\alpha^2\cosh(\beta(x+x_2))^2 + \beta^2\sin(\alpha x_1)^2} = \mu(x)K_t.$$

Integrating on $\mathbb{R}$ we obtain $\int_{\mathbb{R}} \mu\cdot\left(B_x - K_t\right) = -\frac{4i}{\alpha\beta}$, i.e. (2.6.6).

Let us show (2.6.3). We have from (2.3.11) $\beta\cos\left(\frac{K}{2}\right) = -\beta\tanh(\beta(x+x_2)+i\alpha x_1)$, hence, from (2.6.1),

$$(\mu_K)_x = -\frac{\beta\sinh(\beta(x+x_2)+i\alpha x_1)}{\cosh^2(\beta(x+x_2)+i\alpha x_1)} = \beta\cos\left(\frac{K}{2}\right)\mu_K,$$

which proves (2.6.3).

In order to finish, we only need to prove (2.6.4). Recall the notation in (2.8.15). First we have

$$\begin{aligned}\mu_x(x) &= \frac{\beta\sinh(\theta)\big(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)\big)-\alpha^2\beta\sinh(2\theta_2)\cosh(\theta)}{\big(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)\big)^2}\\ &= \left(\frac{\beta\tanh(\theta)\big(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)\big)-\alpha^2\beta\sinh(2\theta_2)}{\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)}\right)\mu(x)\\ &= \left(\beta\tanh(\theta)-\frac{\alpha^2\beta\sinh(2\theta_2)}{\alpha^2\cosh^2(\theta_1)+\beta^2\sin^2(\theta_1)}\right)\mu(x).\end{aligned}$$

Consequently, our problem now is to show that

$$\begin{aligned}&\beta\tanh(\theta)-\frac{\alpha^2\beta\sinh(2\theta_2)}{\alpha^2\cosh^2(\theta_1)+\beta^2\sin^2(\theta_1)}\\ &\quad=\frac{(\beta-i\alpha)}{2}\cos\left(\frac{B+K}{2}\right)+\frac{(\beta+i\alpha)}{2}\cos\left(\frac{B-K}{2}\right).\end{aligned}\tag{D.1.1}$$

Let us compute explicitly the RHS of the last equation. Using basic trigonometric identities

$$\begin{aligned}\cos\left(\frac{B\pm K}{2}\right) &= \left(1-\tan^2\left(\frac{B\pm K}{4}\right)\right)\left(1+\tan^2\left(\frac{B\pm K}{4}\right)\right)^{-1}\\ &= \frac{(1-e^{2\theta})(\alpha^2\cosh^2(\theta_2)-\beta^2\sin^2(\theta_1))\mp 4\alpha\beta\cosh(\theta_2)\sin(\theta_1)e^{\theta}}{(1+e^{2\theta})(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1))}.\end{aligned}$$

Then, using this the RHS of (D.1.1) reads now

$$\begin{aligned}&\text{RHS(D.1.1)}\\ &= \frac{\beta(1-e^{2\theta})(\alpha^2\cosh^2(\theta_2)-\beta^2\sin^2(\theta_1))+4i\alpha^2\beta\cosh(\theta_2)\sin(\theta_1)e^{\theta}}{(1+e^{2\theta})(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1))}\\ &= \frac{\beta\tanh(\theta)(\beta^2\sin^2(\theta_1)-\alpha^2\cosh^2(\theta_2))}{\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)}+\frac{2i\alpha^2\beta\cosh(\theta_2)\sin(\theta_1)}{\cosh(\theta)(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1))}\\ &= \beta\tanh(\theta)-\frac{2\alpha^2\beta\tanh(\theta)\cosh^2(\theta_2)}{\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1)}+\frac{2i\alpha^2\beta\cosh(\theta_2)\sin(\theta_1)}{\cosh(\theta)(\alpha^2\cosh^2(\theta_2)+\beta^2\sin^2(\theta_1))}\end{aligned}$$

$$
\begin{aligned}
&= \beta \tanh(\theta) - \frac{2\alpha^2\beta\cosh(\theta_2)\left(\sinh(\theta)\cosh(\theta_2) - \sinh(i\theta_1)\right)}{\cosh(\theta)(\alpha^2\cosh^2(\theta_2) + \beta^2\sin^2(\theta_1))} \\
&= \beta \tanh(\theta) - \frac{2\alpha^2\beta\sinh(\theta_2)\cosh(\theta_2)}{\alpha^2\cosh^2(\theta_2) + \beta^2\sin^2(\theta_1)} \qquad \text{(D.1.2)} \\
&= \beta \tanh(\theta) - \frac{\alpha^2\beta\sinh(2\theta_2)}{\alpha^2\cosh^2(\theta_2) + \beta^2\sin^2(\theta_1)};
\end{aligned}
$$

where in (D.1.2) we used

$$
\begin{aligned}
&\sinh(\theta)\cosh(\theta_2) - \sinh(i\theta_1) = \sinh(\theta)\cosh(\theta_2) - \sinh(\theta - \theta_2) \\
&\quad = \sinh(\theta)\cosh(\theta_2) - \sinh(\theta)\cosh(\theta_2) + \cosh(\theta)\sinh(\theta_2) = \cosh(\theta)\sinh(\theta_2),
\end{aligned}
$$

which ends the proof.

D.2 Proof of Lemma 2.9.2

First we prove (2.6.6). Indeed, note that

$$
\partial_x\left(\frac{\beta^2\sinh^2(2\gamma(x+x_2)) - \sinh(2\gamma x_1)}{\beta\big(\beta^2\sinh^2(\gamma(x+x_2)) + \cosh^2(\gamma x_1)\big)}\right) = \Phi_1 - \Phi_2,
$$

where

$$
\begin{aligned}
\Phi_1 &= \frac{4\beta\gamma\cosh(\gamma(x+x_1+x_2))\cosh(\gamma x_1)\cosh(\gamma(x+x_2))}{\big(\beta^2\sinh^2(\gamma(x+x_2)) + \cosh^2(\gamma x_1)\big)^2} = \mu(x)R_x, \\
\Phi_2 &= \frac{2\beta\gamma}{\beta^2\sinh^2(\gamma(x+x_2)) + \cosh^2(\gamma x_1)} = \mu(x)Q_t.
\end{aligned}
$$

Integrating on $\mathbb{R}$ we obtain (2.6.6). We prove now (2.9.5). We will compute each term involved in the equation. For the sake of simplicity, we denote

$$
\theta_1 := \gamma x_1, \quad \theta_2 := \gamma(x+x_2), \quad \theta = \gamma(x+x_1+x_2).
$$

First we have

$$
\begin{aligned}
\mu_x(x) &= \frac{\gamma\sinh(\theta)\big(\cosh^2(\theta_1) + \beta^2\sinh^2(\theta_2)\big) - \beta^2\gamma\sinh(2\theta_2)\cosh(\theta)}{\big(\cosh^2(\theta_1) + \beta^2\sinh^2(\theta_2)\big)^2} \\
&= \left(\gamma\tanh(\theta) - \frac{\beta^2\gamma\sinh(2\theta_2)}{\cosh^2(\theta_1) + \beta^2\sinh^2(\theta_2)}\right)\mu(x).
\end{aligned}
$$

Consequently, our problem now is to show that

$$
\gamma\tanh(\theta) - \frac{\beta^2\gamma\sinh(2\theta_2)}{\cosh^2(\theta_1) + \beta^2\sinh^2(\theta_2)} = \frac{1}{2a_3}\cos\left(\frac{R+Q}{2}\right) + \frac{a_3}{2}\cos\left(\frac{R-Q}{2}\right).
$$

Let us compute the RHS of the last equation. For this, we use trigonometric identities:

$$
\begin{aligned}
\cos\left(\frac{R\pm Q}{2}\right) &= \left(1-\tan^2\left(\frac{R\pm Q}{4}\right)\right)\left(1+\tan^2\left(\frac{R\pm Q}{4}\right)\right)^{-1}\\
&= \frac{(1-e^{2\theta})(\cosh^2(\theta_1)-\beta^2\sinh^2(\theta_2)) \mp 4\beta\cosh(\theta_1)\sinh(\theta_2)e^{\theta}}{(1+e^{2\theta})(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}
\end{aligned}
$$

Hence, using this last identity, the RHS of (2.9.5) is reduced to

$$
\begin{aligned}
\text{RHS} &= \frac{-\gamma(1-e^{2\theta})(\cosh^2(\theta_1)-\beta^2\sinh^2(\theta_2)) - 4\beta^2\gamma\cosh(\theta_1)\sinh(\theta_2)e^{\theta}}{(1+e^{2\theta})(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}\\
&= \frac{\gamma\tanh(\theta)(\cosh^2(\theta_1)-\beta^2\sinh^2(\theta_2))}{(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))} - \frac{2\beta^2\gamma\cosh(\theta_1)\sinh(\theta_2)}{\cosh(\theta)(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}\\
&= \gamma\tanh(\theta) - \frac{2\beta^2\gamma\tanh(\theta)\sinh^2(\theta_2)}{\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2)} - \frac{2\beta^2\gamma\cosh(\theta_1)\sinh(\theta_2)}{\cosh(\theta)(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}\\
&= \gamma\tanh(\theta) - \frac{2\beta^2\gamma\sinh(\theta_2)\big(\sinh(\theta)\sinh(\theta_2)+\cosh(\theta_1)\big)}{\cosh(\theta)(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}\\
&= \gamma\tanh(\theta) - \frac{2\beta^2\gamma\sinh(\theta_2)\cosh(\theta)\cosh(\theta_2)}{\cosh(\theta)(\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2))}\\
&= \gamma\tanh(\theta) - \frac{\beta^2\gamma\sinh(2\theta_2)}{\cosh^2(\theta_1)+\beta^2\sinh^2(\theta_2)}.
\end{aligned}
$$

The proof is complete.

D.3 Proof of Lemma 2.9.3

Same as the proof of Lemma 2.9.2.

Appendix E

Proof of Lemma 2.8.2

Proof of (i). We use the same notation as in (2.8.15). We have

$$K - 4\arctan\left(\frac{2i\alpha}{2\beta}\frac{\beta\sin(\theta_1)}{\alpha\cosh(\theta_2)}\right) = 4\arctan\left(e^{\theta}\right) - 4\arctan\left(\frac{i\sin(\theta_1)}{\cosh(\theta_2)}\right)$$
$$= 4\arctan\left(e^{\theta}\right) - 4\arctan\left(\frac{e^{i\theta_1} - e^{-i\theta_1}}{e^{\theta_2} + e^{-\theta_2}}\right).$$

Therefore, using that $\arctan(u) - \arctan(v) = \arctan(\frac{u-v}{1+uv})$, we obtain

$$\phi^{3,1} = 4\arctan\left(e^{\theta}\right) - 4\arctan\left(\frac{e^{i\alpha x_1} - e^{-i\theta_1}}{e^{\theta_2} + e^{-\theta_2}}\right) = 4\arctan\left(\frac{e^{\theta} - \dfrac{e^{i\theta_1} - e^{-i\theta_1}}{e^{\theta_2} + e^{-\theta_2}}}{1 + e^{\theta}\dfrac{e^{i\theta_1} - e^{-i\theta_1}}{e^{\theta_2} + e^{-\theta_2}}}\right)$$
$$= 4\arctan\left(\frac{e^{\theta}\left(e^{\theta_2} + e^{-\theta_2}\right) - e^{i\theta_1} + e^{-i\theta_1}}{e^{\theta_2} + e^{-\theta_2} + e^{\theta}\left(e^{i\theta_1} - e^{-i\theta_1}\right)}\right)$$
$$= 4\arctan\left(\frac{e^{2\beta(x+x_2)+i\alpha x_1} + e^{-i\theta_1}}{e^{-\theta_2} + e^{\beta(x+x_2)+2i\alpha x_1}}\right)$$
$$= 4\arctan\left(e^{\bar{\theta}} \cdot \frac{e^{\beta(x+x_2)+2i\alpha x_1} + e^{-\theta_2}}{e^{-\theta_2} + e^{\beta(x+x_2)+2i\alpha x_1}}\right) = 4\arctan\left(e^{\bar{\theta}}\right) = \overline{K}.$$

Proof (ii). The identities in (2.8.8) are straightforward. In order to show (2.8.9), we have

$$\frac{B_t\sec^2\left(\frac{B}{4}\right)}{1 + \ell^2\tan^2\left(\frac{B}{4}\right)} = \frac{\left(\dfrac{4\alpha^2\beta\cos(\theta_1)\cosh(\theta_2)}{\alpha^2\cosh^2(\theta_2) + \beta^2\sin^2(\theta_1)}\right)\left(1 + \left(\dfrac{\beta\sin(\theta_1)}{\alpha\cosh(\theta_2)}\right)^2\right)}{1 + \ell^2\left(\dfrac{\beta\sin(\theta_1)}{\alpha\cosh(\theta_2)}\right)^2}$$

$$= \frac{4\alpha^2\beta\cos(\theta_1)\cosh(\theta_2)}{\alpha^2\cosh^2(\theta_2) + \ell^2\beta^2\sin^2(\theta_1)}.$$

Appendix F

Proof of (2.7.3)

By the analysis made in chapter 3, it is direct that $\mathrm{Re}\left(\widetilde{B}_0 K_x\right)$ is even and $\mathrm{Im}\left(\widetilde{B}_0 K_x\right)$ is odd. Then, from the fact that $\widetilde{B}_0 K_x$ belongs to the Schwartz class, whenever x_1 does not satisfy (2.3.5), we conclude

$$\int_{\mathbb{R}} \widetilde{B}_0 K_x \in \mathbb{R}.$$

Now, to show that the integral is different from zero, first we note that from the invariance under shifts of the integral we can assume $x_2 = 0$. Then, since we only measure the sign of the integral, using a proper scaling in x we can assume $\beta = 0.2$ and x_1 is its only remaining independent variable. A numeric computation performed in `Mathematica` obtains that the integral is never zero, as shows Fig. F.1.

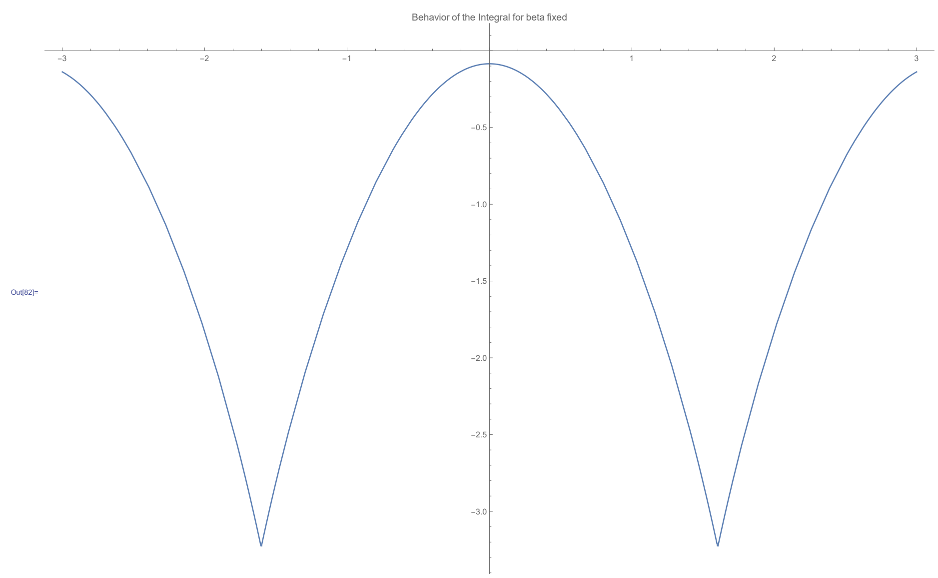

Figure F.1: Behavior of $I(x_1) = \int_{\mathbb{R}} \widetilde{B}_0 K_x$ in x_1 for $\beta = 0.2$.

Conclusions and Perspectives

Along this work, several interesting open questions appeared from the already proved results. Here we mention some of the most interesting ones, in our personal opinion:

1. **Asymptotic stability for odd data for SG 2-solitons**. This is an open question that we could not solve because of some bad preservation properties of the Bäcklund transformations in SG. We plan to attack this problem in the near future.

2. **Stability of Gardner breathers**. This question is currently under work by Alejo and myself. We plan to show that periodic and nonperiodic Gardner breathers are stable using the techniques involved in this work. Probably these ideas are useful to prove other low regularity stability results for nonlocal equations.

3. **Sharp dispersive blow-up for NLS-KdV**. This is an open question related to the non sharp character of the proof in Chapter 3. We believe that, improving some estimates to their best possible level, it is possible to show dispersive blow up at the $(C_x^2(\mathbb{R}), C_x^1(\mathbb{R}))$ level for each equation, respectively. This is part of the current research by Linares and myself, as well as extensions to other nonlocal dispersive systems.

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