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MATHEMATICAL MODELS FOR THE STUDY OF GRANULAR FLUIDS

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Abstract

This Ph.D. thesis aims to obtain and to develop some mathematical models to understand some aspects of the dynamics of heterogeneous granular fluids. More precisely, the expected result is to develop three models, one where the dynamics of the granular material is modeled using a mixture theory approach, and the other two, where we consider the granular fluid is modeled using a multiphase approach involving rigid structures and fluids. More precisely:

- In the first model, we obtained a set of equations based on the mixture theory using homogenization tools and a thermodynamic procedure. These equations reflect two essential properties of granular fluids: the viscous nature of the intersticial fluid and a Coulomb-type of behavior of the granular component. With our equations, we study the problem of a dense granular heterogeneous flow, composed by a Newtonian fluid and a solid component in the setting of the Couette flow between two infinite cylinders.

- In the second model, we consider the motion of a rigid body in a viscoplastic material. The 3D Bingham equations model this material, and the Newton laws govern the displacement of the rigid body. Our main result is the existence of a weak solution for the corresponding system. The weak formulation is an inequality (due to the plasticity of the fluid), and it involves a free boundary (due to the motion of the rigid body). The proof is achieved using an approximated problem and passing it to the limit. The approximated problems consider the regularization of the convex terms in the Bingham fluid and by using a penalty method to take into account the presence of the rigid body.

- In the third model, we consider the motion of a perfect heat conductor rigid body in a heat conducting Newtonian fluid. The 3D Fourier-Navier-Stokes equations model the fluid, and the Newton laws and the balance of internal energy model the rigid body. Our main result is the existence of a weak solution for the corresponding system. The weak formulation is composed by the balance of momentum and the balance of total energy equation which includes the pressure of the fluid, and it involves a free boundary (due to the motion of the rigid body). To obtain an integrable pressure, we consider a Navier slip boundary condition for the outer boundary and the mutual interface. As in the second problem, the proof is achieved using an approximated problem and passing it to the limit. The approximated problems consider a regularization of the convective term and a penalty method to take into account the presence of the rigid body.

Résumé

Cette thèse vise à obtenir et à développer des modèles mathématiques pour comprendre certains aspects de la dynamique des fluides granulaires hétérogènes. Plus précisément, le résultat attendu consiste à développer trois modèles. Nous supposons dans un premier temps que la dynamique du matériau granulaire est modélisée à l'aide d'une approche fondée sur la théorie du mélange. D'autre part pour le deux modèles restant, nous considérons que le fluide granulaire est modélisé à l'aide d'une approche multiphase associant des structures et des fluides rigides. Plus exactement:

- Dans le premier modèle, nous avons obtenu un ensemble d'équations basées sur la théorie du mélange en utilisant des outils d'homogénéisation et une procédure thermodynamique. Ces équations reflètent deux propriétés essentielles des fluides granulaires: la nature visqueuse du fluide interstitiel et un comportement de type Coulomb de la composante granulaire. Avec nos équations, nous étudions le problème de Couette entre deux cylindres infinis d'un écoulement hétérogène granulaire dense, composé d'un fluide newtonien et d'une composante solide.

- Dans le deuxième modèle, nous considérons le mouvement d'un corps rigide dans un matériau viscoplastique. Les équations 3D de Bingham modélisent ce matériau et les lois de Newton régissent le déplacement du corps rigide. Notre résultat principal est d'établir l'existence d'une solution faible pour le système correspondant. La formulation faible du problème est une inégalité en raison de la plasticité du fluide. Nous obtenons un problème à frontière libre dû au mouvement du corps rigide. Le résultat est obtenu en utilisant un problème approximé et en le passant à la limite. Les problèmes approchés considèrent la régularisation des termes convexes dans le fluide de Bingham. Nous utilisons une méthode de pénalisation pour prendre en compte la présence du corps rigide.

- Dans le troisième modèle, nous considérons le mouvement d'un corps rigide conducteur thermique parfait dans un fluide newtonien conducteur de la chaleur. Les équations 3D de Fourier-Navier-Stokes modélisent le fluide, tandis que les lois de Newton et l'équilibre de l'énergie interne modélisent le déplacement du corps rigide. Notre principal objectif dans cette partie est de prouver l'existence d'une solution faible pour le système correspondant. La formulation faible est composée de l'équilibre entre la quantité du mouvement et l'équation de l'énergie totale, qui inclut la pression du fluide, et implique une limite libre due au mouvement du corps rigide. Pour obtenir une pression intégrable, nous considérons une condition au limite de glissement de Navier pour la limite extérieure et l'interface mutuelle. Comme dans le deuxième problème, la preuve est obtenue en considérant un problème approché et en le passant à la limite. Les problèmes approchés considèrent une régularisation du terme de convection et une méthode de pénalisation pour prendre en compte la présence du corps rigide.

Resumen

Esta tesis de doctorado tiene como objetivo obtener y desarrollar algunos modelos matemáticos para comprender algunos aspectos de la dinámica de los fluidos granulares heterogéneos. Más precisamente, el resultado esperado es desarrollar tres modelos, uno donde se modela la dinámica del material granular utilizando un enfoque de teoría de mezclas, y los otros dos, donde consideramos que el fluido granular se modela utilizando un enfoque multifásico que involucra estructuras rígidas y fluidos. Más precisamente:

- En el primer modelo, desarrollamos un conjunto de ecuaciones basadas en la teoría de mezclas utilizando herramientas de homogeneización y un procedimiento termodinámico. Estas ecuaciones reflejan dos propiedades esenciales de los fluidos granulares: la naturaleza viscosa del fluido intersticial y el comportamiento tipo Coulomb de la componente granular. Con nuestras ecuaciones, estudiamos el problema de Couette para un un fluido granular heterogéneo denso, compuesto por un fluido newtoniano y una componente sólida, contenido entre dos cilindros infinitos.

- En el segundo modelo, consideramos el movimiento de un cuerpo rígido en un material viscoplástico. Las ecuaciones de Bingham en 3D modelan este material y las leyes de Newton rigen el desplazamiento del cuerpo rígido. Nuestro principal resultado es la existencia de una solución débil para el sistema correspondiente. La formulación débil es una desigualdad (debido a la plasticidad del fluido) e implica un problema de frontera libre(debido al movimiento del cuerpo rígido). La demostración se logra utilizando un problema aproximado y pasándolo al límite. Los problemas aproximados consideran la regularización del término plástico en el fluido de Bingham y el uso de un método de penalización para tener en cuenta la presencia del cuerpo rígido.

- En el tercer modelo, consideramos el movimiento de un cuerpo rígido, conductor del calor perfecto, en un fluido newtoniano conductor del calor. Las ecuaciones 3D de Fourier-Navier-Stokes modelan el fluido, y las leyes de Newton y el equilibrio de la energía interna modelan el cuerpo rígido. Nuestro principal resultado es la existencia de una solución débil para el sistema correspondiente. La formulación débil está compuesta por la ecuación de balance de momentum y la ecuación de balance de energía total que incluye la presión del fluido, e implica un prblema de frontera libre (debido al movimiento del cuerpo rígido). Para obtener una presión integrable, consideramos una condición de borde de tipo Navier para el borde exterior y la interfaz mutua. Como en el segundo problema, la demostración se logra utilizando un problema aproximado y pasándolo al límite. Los problemas aproximados consideran una regularización del término convectivo y un método de penalización para tener en cuenta la presencia del cuerpo rígido.

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Benjamín

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Chapter 1

Introduction

In this chapter we include an introduction of this thesis in English, French and Spanish.

1.1 Introduction

This thesis aims to develop some mathematical models in continuum fluid mechanics to understand some aspects of the dynamics of **granular heterogeneous flows**. The natural question that follows the above statement is: What is a granular flow?.

Definition 1.1 (Granular fluids)**.** *According to [**?**], a granular flow is a collection of solid particles immersed in a fluid that can be water or air. We consider that particles are bigger than 1* [μm] *to avoid the inclusion of Van der Waals forces.*

The importance of granular fluids lies in their existence in numerous industrial and environmental contexts. According to [**?**], measured in tons, the first material manipulated on earth is water; the second is granular matter. Several examples of granular materials can be found in the industry such as mine tailings, pharmaceutical tablets, and capsules; and in nature such as landslides, debris avalanches, pyroclastic flows, rice, and sand.

Figure 1.1: Landslides

Figure 1.2: Snow avalanches

Figure 1.3: Pyroclastic flows

Figure 1.4: Mine tailings

The understanding of the dynamics of granular flows involves different methods of the physical sciences, from experimental and theoretical studies, as well as field observations to numerical simulations and, a combination of all the above (for a comprehensive review of the development and state of the art, we refer the reader to [?, ?]). In particular, in this thesis, we contribute with a theoretical study and numerical simulations involving continuum mechanics, the mathematical theory of constitutive equations and applied mathematics.

In the framework of continuum mechanics there are two ways to represent the granular component in a granular fluid:

Discrete Every solid particle satisfies the kinematic equations of motion. Then, the fluid and the solid component are part of a fluid-solid interaction system.

Continuum The granular material is modeled like a subset of $\mathbb{R}^3$ that contains all the particles and moves in time (a continuum media).

The discrete approach is characterized by a well-defined set of equations where the only question left is to determine how the fluid and the solid parts interact. The fact that the liquid and the solid components occupy moving domains add a great difficulty in the mathematical analysis of the associated system. On the other hand, if we want to achieve successful numerical simulations, the number of solid particles is restricted since, the degrees of freedom of the problem increase with the number of particles.

There exist several theories in the framework of the continuum approach. Among them, the mixture theory of Truesdell [?] is the most remarkable one. It was developed for studying the dynamics of mixtures of gases and describes the mechanical and thermodynamic behavior of a mixture using a set of partial differential equations. These equations extend the principles of continuum mechanics for a single body to a body made of two or more different materials. The central hypothesis, called **C**ontinuum hypothesis, is that at each instant, every point of the space is occupied simultaneously by a particle of each component. At first sight, the continuum hypothesis seems more manageable than the discrete approach. However, this hypothesis adds three main difficulties. The first one is to understand how the materials interact since we no longer know where are the frontiers of the components. The second one is the lack of a proper constitutive equations for granular materials. The third one is that, even if we propose a constitutive equation for a granular material, we can't think that the constitutive properties of the components don't change from being isolated to being in the mixture, in other words, the constitutive properties of all the elements of the mixture are related to each other.

The discrete and continuum approaches serve different purposes. The discrete approach has been used to obtain information on the evolution of the internal microstructure of the granular flow[?, ?]. On the other hand, the mixture theory approach has been used to understand the dynamics of a granular flow where the mass of the interstitial fluid is comparable to that of the solid (see for example [?]), and to model the dynamics of the flow, when there exist 2 or more significant components in the mixture (see for instance [?]).

Regarding the difficulties of using the mixture theory approach, we can say the following:

- The first difficulty is overcome by Truesdell [?], by adding a source term in the momentum equation of all the constituents, which is defined in the whole space. This term represents the drag force between each constituent of the mixture. This approach is useful since it allows that the dynamics of the whole mixture satisfies the equations of continuum mechanics for a single component, which is one of the hypotheses of the mixture theory (Third metaphysical principle).

- Regarding the second difficulty, we emphasize that a comprehensive view of the mechanical and thermodynamical properties of materials is needed it to write constitutive equations. In particular, granular materials reveals various mechanical behaviors, similar to elastoplastic solids, in the case of a quasi-static regime, and to dense gazes, in the case of intense agitation. Then, the properties of granular materials are somewhere between those of a liquid and those of a real solid. Even at rest, granular material can sustain some shearing stress but only an amount proportional to the average stress. This yielding property is dominant in dense regimes and, several authors have proposed constitutive equations resembling a viscoplastic material. The most remarkable ones are the Drucker-Prager law [?], that is an extension of the Mohr-Coulomb yield criterion, and more recently the $\mu(I)$-rheology [?]. Both models are an extension of the Bingham constitutive equation. The Bingham constitutive equation is one of the simplest models for a viscoplastic fluid. It was proposed by Bingham [?] in 1916 and characterized by constant yield stress. In the Drucker-Prager model and the $\mu(I)$-rheology, the yield stress is no more constant but pressure dependent. However, these models face the lack of good mathematical properties and accurate numerical methods. For example, [?] proved that the Drucker-Prager constitutive equation is ill-posed in all two-dimensional contexts and all realistic three-dimensional contexts. However, the research of granular materials using the $\mu(I)$-rheology is promising. For example, [?] proved that the $\mu(I)$-rheology is well posed under certain conditions on a parameter called inertial number. In the numerical front, [?] obtained accurate results using an augmented Lagrangian method to simulate the collapse of a granular wall.

- Proposing constitutive equations is the root of the material theory. As we said, in the case of a mixture is not clear how we should take into consideration the effects of the other constituents in the constitutive equations. To overcome this difficulty, most of the authors use the entropy principle which establishes that the solutions of the equations of continuum mechanics must satisfy the second law of thermodynamics. Under this restriction, using a mathematical procedure called Müller-Liu [?], the authors find restrictions for the constitutive functions and they managed to propose suitable constitutive equations, see for example in the case of a mixture [?, ?] and [?, Chapter 5.8] or [?, Chapter 7] for the case of a single material. However, we emphasize that the results of applying the Müller-Liu method depend on the initial hypotheses of what are the independent variables of the problem and what balance laws we are considering. For example, if we want to study an isothermal dynamics, the temperature is not an independent variable. In the mixture theory, the volume fraction of a constituent is an independent variable that represents how much of a component is in every point of the space. Goodman and Cowin [?] introduced the approach of adding a new balance equation for the volume fraction. The main argument of the authors is that granular fluids exhibit microstructural effects on their macroscale, which is accounted for, in general, by adding a new balance equation. In their work, the authors add to the balance equations of mass, linear and angular momentum, and energy, another balance equation called the balance of equilibrated forces, which is a dynamical equation for the volume fraction of the solid component. Using the Müller-Liu method, they proposed a set of constitutive equations. According to [?] the theory of Goodman and Cowin is moderately successful since it predicts the Mohr-Coulomb criterion, and the solutions to the equations are similar to observed phenomena for such media, predicting the existence, for example, of a plug flow. Several authors have followed this approach. Some examples are [?, ?, ?], and more recently [?]. However, there exist other approaches, for example, Liu[?]. For a porous media, without considering the balance of equilibrated force equation and using the Müller-Liu method, Liu obtains different constitutive equations to the ones obtained by Goodman and Cowin.

Concerning the above, this thesis contains two main parts. The first part is about a continuum model for granular heterogeneous flows and the second is about the discrete approach.

In the first part thesis, contained in Chapter 3, the aim is to model the dynamics of a granular heterogeneous flow composed by a Newtonian fluid (interstitial fluid) and $N-1$ solid components, for $N \in \mathbb{N} \setminus \{0,1\}$. To achieved this, we developed a mixture theory made of a full set of balance equations, aiming that our equations satisfied 2 of the four principles of the mixture theory of Truesdell, the Continuum Hypothesis and the second metaphysical principle. We also calculate the numerical solution of the stationary Couette flow between two infinite concentric cylinders in the case where $N = 2$. This chapter is divided into four sections:

- In Section 3.1, the objective is to deduce a set of partial differential equations, resembling the mixture theory equations, using a mathematical approach that utilizes the tools of the homogenization theory. In this approach, we define a set of observable scales at which we can distinguish where are located the different components of the mixture and where they interact. Then, we write the balance equations for this system. Under specific technical hypothesis, we pass to the limit, and we obtain the balance equations at a homogenized scale where the mixture satisfies the continuum hypothesis. An essential difference between our model and the classical mixture theory approach relies on that, in our case, the interaction terms are not considered as source terms, and they are included as a part of the constitutive functions. Then, our model of the constitutive functions needs to consider these interactions. We emphasize that, in our deduction, we do not use the Goodman and Cowin [?] balance equation of equilibrated forces. Instead we consider a saturation constraint, which entails that the sum of all volume fraction is equal to one, or in other words, the mixture is saturated.

- In section 3.2, the objective is to use the Müller-Liu procedure based on the entropy principle, to obtain a set of restrictions for the constitutive functions. One of these restrictions is the residual inequality. This inequality allows us to write constitutive equations as a sub-differential inclusion. In this context, sub-differential inclusions are a useful tool, since allows us to characterize the yielding property of granular materials accurately. In Section 3.2.4, we clarify the above point by reviewing some classic examples like the Bingham fluid equation, Drucker-Prager law and the $\mu(I)$-rheology.

- In section 3.3, we propose constitutive equations for a granular heterogeneous mixture of N components, using subdifferential inclusions. The constitutive equations models two important behaviors. First, the viscous nature of the interstitial fluid and second, the plastic behavior of the granular component. However, our equations still depend on a function called the Helmholtz free energy. The Helmholtz free energy is the difference between the internal energy and the entropy times the temperature. Then, to close the system of equations, we propose a Helmholtz free energy function for each constituent that is the sum of two parts: the first is the Helmholtz free energy function of the constituent isolated from the rest of the components and the second part is an interaction term, resembling drag energy. With this final hypothesis, we obtain the full system of equations. To test our model, we study the stationary Couette flow between two infinite concentric cylinders in the case where we have the interstitial fluid and one granular component (N=2). The result is a DAE system (Differential Algebraical Equation) that is solved using MATLAB. We included the codes in Section B of the Appendix.

- In section 3.4 we discuss the results obtained with the numerical simulation and the future prospects of this research.

The second part of this thesis is divided into two Chapters, Chapter 4 and Chapter 5. The primary objective of this part is to study the existence of weak solutions to multiphase models involving rigid structures and fluids. In these models, we consider that the fluid and the rigid part occupy well-defined domains and the constitutive properties of the fluid are well determined by a set of simple constitutive equations. Regarding the first chapter, the main difference is that the continuum hypothesis is dismissed and the interactions between the solid and the liquid are only considered in the boundary conditions of the problem.

At first sight, this approach seems much more realistic. However, it adds new complications: the equations of motion are satisfied in moving domains, and we need to propose a model for how the materials interact in their mutual frontier. More precisely:

- Since the equations are satisfied in moving domains we can't use standard techniques of the existence of solutions of the equations of fluid mechanics (see for example [?]). To overcome this, most of the authors construct the solutions via a penalization method developed by Hoffmann and Starovoitov [?] and [?]. Regarding the penalization method, there exist a least two different approaches: a "L^2" penalization (see for instance [?]), and a "H^1" penalization (see for instance [?]). In existence proofs, the "L^2" penalization approach is useful when the problem lacks a bound in H^1 for the velocity (see for instance [?]). In numerical schemes, these two penalization methods are used to simulate the motion of rigid bodies in a fluid, but the drawback of the H^1 penalization method is that the solid can change its shape which is not adapted for rigid motions. We refer for instance to [?] for the analysis of a numerical scheme base on the L^2 penalization method and also [?], [?], [?], [?], etc. for some other works on the numerical study of fluid-rigid body systems.

- Regarding the boundary conditions, for a viscous fluid-rigid solid interaction system, it has been proved by [?] that if the solid touches the exterior boundary, then this contact is done with null relative velocity and null relative acceleration. This result suggests that if at the initial time two bodies are not touching, they will never be in contact at a finite time This no-collision paradox goes against Arquimides' principle and has been known since the 1960s [?, ?], in the context of Stokes equations and it has been rigorously proved in the 2-d incompressible Navier-Stokes case [?] and for 3-d incompressible Navier-Stokes equation when the rigid body is a ball[?]. Such a property indicates that this model should be modified to recover collisions between rigid bodies. For the case of an infinite cylinder surrounded by a perfect incompressible fluid, It is shown in [?], that collisions occur in finite time. If the fluid is viscous, according to [?], one possible explanation could the choice of a no-slip boundary condition (Dirichlet). In [?], the authors considered instead the Navier-slip boundary condition. In that case and considering a ball immersed in the fluid, in [?] they proved the existence of a solution that touches the boundary. Other types of boundary conditions more appropriate to an industrial framework has been considered by [?]. In their work, the authors consider a Newtonian fluid-rigid body system with the Coulomb's boundary law. This boundary condition resembles the Drucker-Prager constitutive equation and states that if the tangential component of stress tensor does not exceed a threshold, then the boundary condition remains the standard Dirichlet boundary condition, whereas if it equals to this threshold, then the boundary condition corresponds to a generalized Navier boundary condition. In their work, the authors prove the existence of weak solutions up to the first collision.

Considering the above, in Chapter 4 we study the problem of the existence of weak solutions up to the first collision of an incompressible and isothermal Bingham fluid-rigid body system with Dirichlet boundary conditions. On the modeling side, the advantage of studying this problem lies in the fact that the Bingham fluid model is the simplest constitutive equation that holds the yielding property which is dominant in dense regimes of granular fluids. Then, a Bingham fluid-rigid body system can be useful to understand and shed some light about granular materials. On the mathematical side, although an existence theorem up to the first collision is the first step, we contribute to the full range of models that can be useful to understand the fluid-rigid body interactions. This chapter is divided into five sections:

- In Section 4.1, we present the equations of the fluid-structure model, and we state the main result. The governing equations for the fluid flow are written by considering the Cauchy momentum equation of a Bingham incompressible fluid. Newton equations for linear and angular momentum govern the motion of the rigid body. Finally, the precedent equations are completed by boundary conditions, where we assume continuity of the velocity field both in the fluid-solid interface and in the external boundary (no-slip boundary conditions). At the end of this section we state the main existence result.

- In Section 4.2, we define a weak solution to this problem, and we introduce the energy inequality. We also introduce proper notation, and we give some important preliminary results.

- In Section 4.3, using a 1-stage Galerkin scheme, we state the approximated problem which is defined by three parameters: one that corresponds to the dimension in the

Galerkin method, one that corresponds to the approximation of the plastic term related to the Bingham fluid, and one that corresponds to the penalization term used to deal with the solid inside the fluid. Later in this section, we prove the existence of a solution to the approximated problem using Peano's Theorem.

- In Section 4.4, we pass to the limit the parameters related to the Galerkin discretization and the regularization of the plastic term at the same time. The main complication is that we can't prove in this part that the regularization of the plastic term converges to an identifiable limit. Then, to avoid dealing with this limit and to take advantage of the convexity of the plastic term, following the proof of Theorem 3.2 in [?, Chapter 6], we use a monotonicity argument that allows us to pass to the limit all the terms in the approximated problem. However, the result of this process is a variational inequality for the velocity that now depends only on the parameter related to the penalization.

- In Section 4.5, we pass to the limit the parameter related to the penalization term. To do this, we need the strong convergence of the velocity field. The standard approach to obtaining the strong convergence is to follow [?, Section 7]. This argument is based on the Aubin-Lions-Simon Lemma [?], where we need a bound for the acceleration of the approximated system. In our case, we cannot obtain the bound directly from the inequality. However, by using the first approximated problem, before passing to the limit the Galerkin dimension and the regularization term, we obtain a bound for the acceleration that depends on the limit of the regularization of the plastic term, which is not a problem since this term is bounded in a suitable space. Then, using standard techniques, we can pass to the limit the equations and prove the existence of a weak solution. Moreover, our solution satisfies an energy inequality that allows us to prove that is valid up to the first collision.

The last chapter of this work, Chapter 5, is dedicated to studying the existence of a weak solution of an incompressible heat conducting Newtonian fluid-rigid body system considering a Navier's slip boundary condition. There exist several motivations to investigate this problem, varying from physical modeling to mathematical challenges

Concerning the physical modeling of fluids, there exist fluid's properties that are temperature dependent. For example, as we see in Figure 1.5, the dynamic viscosity of the water can vary up to 70 % considering temperatures from 20 to 80 Celsius degrees, while the density remains nearly constant.

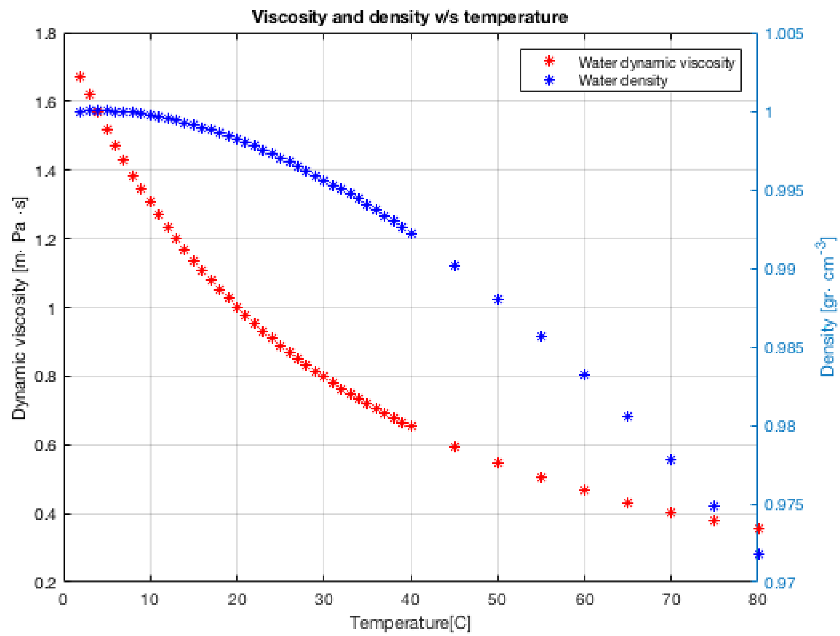

Figure 1.5: Viscosity v/s temperature and Density v/s temperature. Source: Release on the IAPWS Formulation 2008 for the Viscosity of Ordinary Water Substance. Berlin, Germany September 2008, available at http://www.iapws.org

Another example is the thermal conductivity, which is the property that indicates the ability of a substance to conduct heat. For instance, as we show in Figure 1.6, the thermal conductivity of copper and water vary for different values of temperature. This shows us that even in solid materials, some properties may exhibit a temperature dependence behavior.

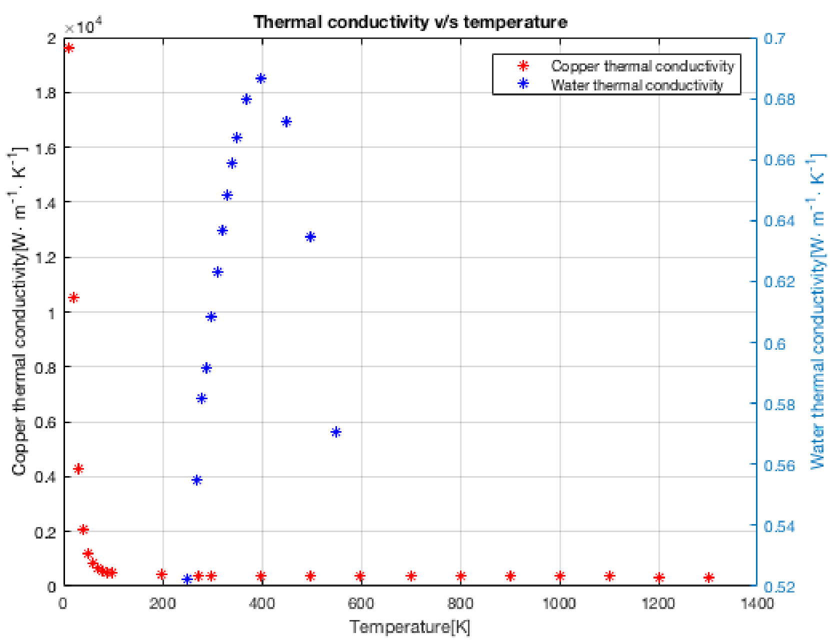

Figure 1.6: Thermal conductivity of copper and water v/s temperature [?]

Regarding the mathematical challenges, the study of a temperature dependent dynamics needs to consider the energy balance equation. However, in the context of continuum mechanics, there exist two formulations for this equation. One is referred to as the balance of total energy, which stands for the sum of the kinetic energy and the internal energy of the continuum media. The other one is the balance of internal energy. In the context of classical solutions, using the linear momentum balance equation, one may show that both formulations are equivalent. However, in the context of weak solutions, choosing one formulation over the other may lead to whole different problems. For instance, the problem of a 3d incompressible heat conducting Newtonian fluid with temperature dependent material coefficients, like in Figures 1.5 and 1.6, considering in the weak formulation the balance of internal energy, is still open [?]. The principal stumbling block is the presence of the dissipation term in the energy equation. The apriori estimate for the dissipation term is only L^1, then, in a Galerkin approximation scheme, is impossible to pass to the limit this term by just using this type of regularity.

This problem does not occur when the material coefficients are constant (we refer the reader to [?, Chapter 3.5]) or when there are better a priori estimates on the velocity gradient, for example, in the case of non-Newtonian fluids defined by power laws [?] or in two space dimensions [?, ?]. Another approach to this problem is to dismiss the dissipation term. This approximation received the name of Boussinesq and are useful to model atmospheric fronts, oceanic circulation, katabatic winds, dense gas dispersion, and fume cupboard. In this setting, the energy equation becomes a convection-diffusion equation, and the analysis of the coupled system is more straightforward (see for instance [?]). For example, the problem of the existence of a weak solution of an incompressible heat conducting Newtonian fluid-rigid body interaction with the Boussinesq approximation was solved in [?]. However, there are other methods to overcome the problem of the dissipation term.

In [?], the authors consider the balance of total energy instead of the balance of internal energy. This approach leads them to a weak formulation of the energy equation where all terms are properly bound, and the existence of a weak solution is proved. However, the corresponding study is more complicated since we need to obtain estimates of the pressure during the proof of existence. To get estimates for the pressure in [?], the authors used the Helmholtz-Weyl decomposition and proved that the pressure is an integrable function. However, this is only possible considering some slip boundary condition. In particular, in [?], the authors consider a Navier's slip boundary condition. In the case of a no-slip boundary condition, a method to obtain the pressure was developed by Wolf [?]. This pressure is called by the author "local" pressure and is the sum of an integrable pressure and of the time derivative of a harmonic function which is not integrable. Other settings where the pressure is an integrable function is in spatially periodic problems, see for instance [?].

The integrability of the pressure is not only relevant in the context of heat conducting fluids, but it also a crucial step in turbulence models [?] and incompressible non-newtonian fluids [?, ?, ?, ?]. Regarding the study of the pressure, In the context of fluid-structure interaction problems we refer the reader to [?] where the case of a non-Newtonian fluid-rigid body interaction is treated. For the boundary conditions, the authors consider a no-slip condition in the outer boundary and at the interface of the solid and the fluid. Then, part of this work is devoted to the study of the "local" pressure where the authors need the pressure estimates to pass to the limit in the nonlinearity associated with the stress tensor.

Considering the above, in Chapter 5 we study the problem of the existence of weak solutions up to the first collision of an incompressible heat conducting Newtonian fluid-rigid body system with temperature dependent material coefficients considering a Navier slip boundary condition. In our model, we assume that the rigid solid is a perfect heat conductor. This simplification hypothesis is accurate for many liquid-solid interaction problems, for example, the one describes in Figure 1.6, where the thermal capacity of the fluid is negligible regarding the thermal capacity of the solid. To proof the existence of a weak solution we follow the approach of [?], and we work with the total energy equation. With this framework, we are forced to impose a slip boundary condition to obtain estimates for the fluid's pressure. To deal with the Navier's slip boundary condition in the exterior domain and the fluid-solid interface, we follow the approach of [?].

Nevertheless, to obtain a suitable estimate for the fluid pressure, we need to make a further assumption: the densities of the fluid and the solid parts are equal. This hypothesis seems restrictive. However, up to our knowledge, this case has not been treated yet, and it can be applied in many natural and industrial contexts. For instance, the density of the fishes is equal to the density of the ocean in which they are swimming. There also exists solid materials that have nearly the same density of the water. For instance, there are many kinds of rubber which have a density close to $1[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$: common rubbers, polyisoprene, ethylene-propylene-diene rubber, polybutadiene and styrene-butadiene rubber have a density of around 0.92-0.98$[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$.

In the framework of granular fluids, our equations are a fully discrete model of a granular material with heat conducting properties. This is the main difference with the first model in Chapter 3, which is a fully continuous representation of an isothermal granular material, and concerning Chapter 4 which is a mix between continuous and a discrete model for an isothermal granular fluid. Although our model poses several restrictions, this is a first approach that can lead to several more robust models. This chapter is divided into six sections:

- In Section 5.1, we present the equations of the fluid-structure model and our main hypotheses. The governing equations for the fluid flow are the incompressible are the Fourier-Navier-Stokes equations with temperature dependent coefficients. Newton equations for linear and angular momentum govern the motion of the rigid body. For the thermic behavior, we consider the internal energy equation for the fluid and the solid parts. Finally, the precedent equations are completed by boundary conditions, where we assume continuity of the normal component of the velocity field both in the fluid-solid interface and in the external boundary and a Navier's slip condition for the tangential part of the velocity in the fluid-solid interface and the outer boundary. The hypothesis that the rigid body is a perfect heat conductor is translate in that the gradient of the temperature of the solid is 0. Then, since we need to force to the temperature to be constant on the solid, we need to use a scalar penalization method in the energy equation. Also in this section we define the weak solution of our problem, and we state the main existence theorem.

- In Section 5.2, we introduce proper notation, and we give some important preliminary results.

- In Section 5.3, we state the approximated problem. In this part, for the sake of clarity, the approximated problem is defined by only two parameters: one that corresponds to the penalization terms in the momentum equation and the energy equation, and the second one is a parameter that corresponds to a regularization of the convective term of our equations. In the momentum equation we use a L^2 penalization, and for the energy equation, we use a H^1 penalization method. The last parameter is introduced to be able to use the approximated velocity as a test function in our equations. Finally, in this section, we state an existence result for the approximated problem. We mention that in the approximated problem, the pressure is part of the momentum equation.

- In Section 5.4 we present the proof of the existence of a solution to the approximated problem. This proof is based on a 2-stage Galerkin scheme where the pressure is not introduced yet. One stage is related to momentum's equation and the other to the energy equation. As we did in Chapter 4, we prove the existence of a solution to the discrete problem using Peano's Theorem. Then, we pass to the limit the parameters related to the Galerkin discretization, and we introduced the pressure using the Helmholtz-Weyl decomposition.

- In Section5.5, we pass to the limit the parameter related to the penalization term and the regularization of the convective term at the same time. Then, as we did before in Chapter 4, to obtain the strong convergence of the velocity field and the temperature we follow [**?**, Section 7]. In this part, we face the main difficulty in dealing with the Navier's slip boundary condition which is the lack of H^1 estimates for the velocity in the solid. However, following the approach of [**?**] (see also [**?**]), this problem is overcome using a special kind of test functions. Finally, we pass to the limit the momentum equation. Then, following [**?**], we use the approximated momentum and energy equations to obtain the approximated total energy equation. In this equation, the dissipation term is no longer a problem and, after passing to the limit, we conclude the existence of a weak solution. Moreover, our solution satisfies an energy inequality that allows us to prove that is valid up to the first collision.

1.2 Introduction

Cette thèse vise à développer des modèles mathématiques en mécanique des fluides afin de comprendre certains aspects de la dynamique des fluides granulaires hétérogènes. La question naturelle qui convient à l'affirmation ci-dessus est la suivante: Qu'est-ce qu'un fluide granulaire?.

Definition 1.2 (Fluides granulaires)**.** *Selon [?], un fluide granulaire est un ensemble de particules solides immergées dans un fluide cela peut être de l'eau ou de l'air. Nous considérons que les particules sont plus grandes que 1* $[\mu m]$ *afin d'éviter l'inclusion des forces de Van der Waals.*

L'importance des fluides granulaires réside dans leur existence dans de nombreux contextes industriels et environnementaux. Selon [?], mesuré en tonnes, le premier matériau manipulé pour les serres humaines est l'eau; le seconde est la matière granulaire. On peut trouver plusieurs exemples de matériaux granulaires dans l'industrie, tels que les résidus miniers, les comprimés pharmaceutiques et les capsules ; et dans la nature tels que les glissements de terrain, les avalanches, les fluides pyroclastiques, le riz et le sable.

Figure 1.7: Glissements de terrain

Figure 1.9: Flux pyroclastiques

Figure 1.8: Avalanches de neige

Figure 1.10: Résidus miniers

La compréhension de la dynamique des fluides granulaires implique différentes méthodes des sciences physiques, allant des études expérimentales aux études théoriques, en passant par les observations de terrain, les simulations numériques et une combinaison de toutes les méthodes susmentionnées (pour un examen complet du développement et de l'état de l'art, nous renvoyons le lecteur à [?, ?]). En particulier, dans cette thèse, nous contribuons à une étude théorique et à des simulations numériques impliquant la mécanique des milieux continus, la théorie mathématique des équations constitutives et les mathématiques appliquées.

Dans le cadre de la mécanique, il existe deux manières de représenter le composant granulaire dans un fluide granulaire:

Discrète Une manière discrète où chaque particule solide satisfait les équations cinématiques du mouvement. Ensuite, le fluide et le composant solide font partie d'un système d'interaction fluide-solide.

Continu Une manière continu où le matériau granulaire est modélisé comme un sous-ensemble contenant toutes les particules et se déplaçant dans le temps (milieu continu).

L'approche discrète est caractérisée par un ensemble bien défini d'équations où la seule question qui reste est de déterminer la manière dont le fluide et les parties solides interagissent. Le fait que les composants liquides et solides occupent des domaines en mouvement ajoute une grande difficulté à l'analyse mathématique du système associé. D'autre part, si nous voulons réaliser des simulations numériques réussies, le nombre de particules solides est limité car, le degré de liberté du problème augmente avec le nombre de particules.

Il existe plusieurs théories dans le cadre de l'approche du milieu continu. Parmi eux, la théorie du mélange de Truesdell est la plus remarquable[**?**]. Elle a été développée pour étudier la dynamique des mélanges de gaz et décrit le comportement mécanique et thermodynamique d'un mélange à l'aide d'un ensemble d'équations aux dérivées partielles. Ces équations étendent les principes de la mécanique du continu pour un corps unique à un corps composé de deux matériaux différents ou plus. L'hypothèse centrale, appelée **hypothèse de continu**, est qu'à chaque instant, chaque point de l'espace est occupé simultanément par une particule de chaque composant. À première vue, l'hypothèse du continu semble plus facile à gérer que l'approche discrète. Cependant, cette hypothèse ajoute trois difficultés principales. La première consiste à comprendre comment les matériaux interagissent car nous ne savons plus où sont les frontières des composants. La seconde est l'absence d'équations constitutives appropriées pour les matériaux granulaires. La troisième est que, même si nous proposons une équation constitutive pour un matériau granulaire, nous ne pouvons pas penser que les propriétés constitutives des composants ne changent pas du fait qu'elles sont isolées au mélange, c'est-à-dire les propriétés de tous les éléments du mélange sont liés les uns aux autres.

Les approches discrète et continue servent différents objectifs. L'approche discrète a été utilisée pour obtenir des informations sur l'évolution de la microstructure interne du flux granulaire[**?**, **?**].. D'autre part, l'approche de la théorie des mélanges a été utilisée pour comprendre la dynamique d'un écoulement granulaire où la masse du fluide interstitiel est comparable à celle du solide (voir par exemple [**?**]) et pour modéliser la dynamique de l'écoulement, lorsqu'il existe 2 composants importants ou plus dans le mélange (voir par exemple [**?**]).

En ce qui concerne les difficultés d'utilisation de l'approche de la théorie du mélange, on peut dire ce qui suit:

- Truesdell surmonte la première difficulté en ajoutant un terme source dans l'équation de quantité de mouvement de tous les constituants, qui est définie dans tout l'espace. Ce terme représente la force de traînée entre chaque constituant du mélange. Cette approche est utile car elle permet que la dynamique de l'ensemble du mélange satisfasse les équations de la mécanique du continu pour un seul composant, ce qui est l'une des hypothèses de la théorie du mélange (Troisième principe métaphysique).

- En ce qui concerne la deuxième difficulté, nous soulignons qu'une vue complète des propriétés mécaniques et thermodynamiques des matériaux est nécessaire pour écrire des équations constitutives. En particulier, les matériaux granulaires révèlent divers comportements mécaniques, similaires aux solides élastoplastiques, dans le cas d'un régime quasi-statique, à des gaz trop denses, et lors d'une agitation intense. Ensuite, les propriétés des matériaux granulaires se situent quelque part entre celles d'un liquide et celles d'un solide réel. Même au repos, les matériaux granulaires peuvent supporter une certaine contrainte de cisaillement, mais seulement une quantité proportionnelle à la contrainte moyenne. Cette propriété est dominante dans les régimes denses et plusieurs auteurs ont proposé des équations constitutives ressemblant à un matériau viscoplastique. Les plus remarquables sont la loi Drucker-Prager [?], qui est une extension du critère de rendement de Mohr-Coulomb, et plus récemment le $mu(I)$-rhéologie [?]. Les deux modèles sont une extension de l'équation constitutive de Bingham. L'équation constitutive de Bingham est l'un des modèles les plus simples pour un fluide viscoplastique. Il a été proposé par Bingham en 1916[?] et caractérisé par une limite d'élasticité constante. Dans le modèle Drucker-Prager et la rhéologie $\mu(I)$, la limite d'élasticité n'est plus constante mais dépend de la pression. Cependant, ces modèles sont confrontés au manque de bonnes propriétés mathématiques et de méthodes numériques précises. Par exemple, [?] a prouvé que l'équation constitutive de Mohr-Coulomb est mal posée dans tous les contextes à deux dimensions et dans tous les contextes à trois dimensions réalistes. Cependant, la recherche de matériaux granulaires utilisant la rhéologie $\mu(I)$ est prometteuse. Par exemple, [?] a prouvé que la rhéologie $\mu(I)$ est bien posée dans certaines conditions sur un paramètre appelé nombre inertiel. Sur le front numérique, [?] a obtenu des résultats précis en utilisant une méthode lagrangienne augmentée pour simuler l'effondrement d'une paroi granulaire.

- Enfin, la proposition d'équations constitutives est la base de la théorie des matériaux. Comme nous l'avons dit précédemment, dans le cas d'un mélange, la manière dont nous devrions considérer les effets des autres constituants dans les équations constitutives n'est pas claire. Pour surmonter cette difficulté, la plupart des auteurs utilisent le principe de l'entropie qui établit que les solutions des équations de la mécanique du continu doivent satisfaire la deuxième loi de la thermodynamique. Sous cette restriction, en utilisant une procédure mathématique appelée Müller-Liu[?], les auteurs trouvent des restrictions pour les fonctions constitutives et parviennent à proposer des équations constitutives appropriées, voir par exemple dans le cas d'un mélange [?, ?] et [?, Chapitre 5.8] ou [?, Chapitre 7] pour le cas d'un seul matériau. Cependant, nous soulignons que les résultats de l'application de la méthode de Müller-Liu dépendent des hypothèses initiales de quelles sont les variables indépendantes du problème et des lois de balance que nous envisageons. Par exemple, si nous voulons étudier une dynamique isotherme , la température n'est pas une variable indépendante Dans la théorie des mélanges, la fraction volumique d'un constituant est une variable indépendante qui représente la quantité d'un composant en chaque point de l'espace. Goodman et Cowin [?] ont introduit l'approche consistant à ajouter une nouvelle équation de solde pour la fraction volumique. L'argument principal des auteurs est que les fluides granulaires présentent des effets microstructuraux sur leur échelle macroscopique, ce qui est généralement pris en compte par l'ajout d'une nouvelle équation d'équilibre. Dans leurs travaux, les auteurs ajoutent à la balance les équations de masse, de moment linéaire et angulaire et d'énergie, une autre équation de balance appelée balance des forces équilibrées, qui est une équation dynamique pour la fraction volumique de la composante solide. En utilisant la méthode de Müller-Liu, ils ont proposé un ensemble d'équations constitutives. Selon [?], la théorie de Goodman et Cowin est relativement fructueuse car elle prédit le critère de Mohr-Coulomb, et les solutions aux équations sont similaires aux phénomènes observés pour de tels milieux, prédisant l'existence, par exemple, d'un écoulement continu. Plusieurs auteurs ont suivi cette approche. Quelques exemples sont [?, ?, ?], et plus récemment [?]. Cependant, il existe d'autres approches, par exemple, Liu [?]. Pour un support poreux, sans prendre en compte l'équation d'équilibre des forces équilibrées et en utilisant la méthode de Müller-Liu, Liu obtient des équations constitutives différentes de celles obtenues par Goodman et Cowin.

En ce qui concerne ce qui précède, cette thèse contient deux parties principales. La première partie concerne un modèle de continu pour les flux hétérogènes granulaires et la seconde concerne l'approche discrète.

Dans la première partie de la thèse, contenue dans le Chapitre 3, l'objectif est de modéliser la dynamique d'un écoulement hétérogène granulaire composé d'un fluide newtonien (fluide interstitiel) et de $N-1$ composants solides, pour N tout nombre naturel supérieur à 1. Pour ce faire, nous avons développé de nouvelles équations basées sur la théorie du mélange et constituées d'un ensemble complet d'équations d'équilibres. Nous avons donc cherché à satisfaire deux des quatre principes de la théorie du mélange de Truesdell : l'hypothèse de continu et le second principe métaphysique. Nous calculons également la solution numérique du flux de Couette stationnaire entre deux cylindres concentriques infinis dans le cas où $N=2$. Ce chapitre est divisé en quatre sections.

- Dans la Section 3.1, l'objectif est de déduire un ensemble d'équations aux dérivées partielles, ressemblant aux équations de la théorie des mélanges, en utilisant une approche mathématique utilisant les outils de la théorie de l'homogénéisation. Dans cette approche, nous définissons un ensemble d'échelles observables auxquelles nous pouvons distinguer où se situent les différents composants du mélange et où ils interagissent. Ensuite, nous écrivons les équations de la balance pour ce système. Sous certaines hypothèses techniques, nous passons à la limite et nous obtenons les équations de la balance à une échelle homogénéisée où le mélange satisfait l'hypothèse du continu. Une différence essentielle entre notre modèle et l'approche classique de la théorie des mélanges repose sur le fait que, dans notre cas, les termes d'interaction ne sont pas considérés comme des termes sources et qu'ils font partie des fonctions constitutives. Ensuite, notre modèle des fonctions constitutives doit prendre en compte ces interactions.

- Dans la Section 3.2, l'objectif est d'utiliser la procédure de Müller-Liu basée sur le principe de l'entropie, afin d'obtenir un ensemble de restrictions pour les fonctions constitutives. L'une de ces restrictions est l'inégalité résiduelle. Cette inégalité nous permet d'écrire des équations constitutives comme une inclusion sous-différentielle. Dans ce contexte, les inclusions sous-différentielles sont un outil utile car elles permettent de caractériser avec précision la propriété de rendement des matériaux granulaires. Dans la section 3.2.4, nous clarifions le point ci-dessus en passant en revue quelques exemples classiques tels que l'équation du fluide de Bingham et la loi de Drucker-Prager et la $\mu(I)$-rhéologie.

- Dans la Section 3.3, nous proposons des équations constitutives pour un mélange granulaire hétérogène de N composants, utilisant des inclusions sub-différentielles. Les équations constitutives modélisent deux comportements critiques. Premièrement, la nature visqueuse du liquide interstitiel et, deuxièmement, le comportement plastique du composant granulaire. Cependant, nos équations dépendent toujours d'une fonction appelée énergie libre de Helmholtz. L'énergie libre de Helmholtz est la différence entre l'énergie interne et l'entropie multipliée par la température. Ensuite, pour fermer le système d'équations, nous proposons une fonction d'énergie libre de Helmholtz pour chaque constituant qui est la somme de deux parties : la première est la fonction d'énergie libre de Helmholtz du constituant isolé du reste des composants et la seconde partie est un terme d'interaction ressemblant à l'énergie de traînée. Avec cette dernière hypothèse, nous obtenons le système complet d'équations. Pour tester notre modèle, nous étudions l'écoulement stationnaire de Couette entre deux cylindres concentriques infinis dans le cas où nous avons le fluide interstitiel et un composant granulaire ($N = 2$). Le résultat est un système DAE (équation algébrique différentielle) résolu à l'aide de MATLAB. Nous avons inclus les codes dans la Section B de l'annexe.

- Dans la section 3.4, nous discutons des résultats obtenus avec la simulation numérique et des perspectives de cette recherche.

La deuxième partie de cette thèse est divisée en deux chapitres, les chapitres 4 et 5. L'objectif principal de cette partie est d'étudier l'existence de solutions faibles aux modèles multiphases impliquant des structures et des fluides rigides. Dans ces modèles, nous considérons que le fluide et la partie rigide occupent des domaines distincts et un ensemble d'équations constitutives simples détermine bien les propriétés constitutives du fluide. En ce qui concerne le premier chapitre, la principale différence est que l'hypothèse de continu est rejetée et que les interactions entre le solide et le liquide ne sont prises en compte que dans les conditions aux limites du problème.

À première vue, cette approche semble beaucoup plus réaliste. Cependant, cela ajoute de nouvelles complications : les équations du mouvement sont satisfaites dans les domaines en mouvement, et nous devons proposer un modèle pour l'interaction des matériaux dans leurs frontières mutuelles. Plus précisément:

- Puisque les équations sont satisfaites dans des domaines en mouvement, nous ne pouvons pas utiliser les techniques standard d'existence de solutions des équations de la mécanique des fluides (voir par exemple [?]). Pour remédier à cela, la plupart des auteurs construisent les solutions via une méthode de pénalisation développé par Hoffmann et Starovoitov [?] et [?]. En ce qui concerne la méthode de pénalisation, il existe au moins deux approches différentes: une pénalisation L^2 (voir par exemple [?]) et une pénalisation H^1 (voir pour plus de détails [?]). Dans les démonstration d'existence, l'approche de pénalisation L^2 est utile lorsque le problème manque d'une borne dans H^1 pour la vitesse (voir par exemple [?]). Dans les schémas numériques, ces deux méthodes de pénalisation sont utilisées pour simuler le mouvement de corps rigides dans un fluide, mais l'inconvénient de la méthode de pénalisation H^1 est que le solide peut changer de forme, ce qui n'est pas adapté aux mouvements rigides. Nous nous référons par exemple à [?] pour l'analyse d'un schéma basé sur la méthode de pénalisation L^2 ainsi que [?], [?], [?], [?], etc. pour d'autres travaux sur l'étude numérique des systèmes corporels semi-rigides.

- Dans les conditions aux limites, pour un système d'interaction fluide visqueux-solide rigide, il a été prouvé par [?] que si le solide touche la limite extérieure, ce contact est alors réalisé avec une vitesse relative nulle et une accélération relative nulle. Ce résultat suggère que si, au début, deux corps ne se touchent pas, ils ne seront jamais en contact à une heure finie. Ce paradoxe de non-collision va à l'encontre du principe d'Arquimides qui est connu depuis les années 1960[?, ?], dans le contexte des équations de Stokes et cela a été rigoureusement prouvé dans l'équations de Navier-Stokes incompressible à 2 dimensions [?] et dans l'équation de Navier-Stokes incompressible à 3 dimensions lorsque le corps rigide est une balle [?]. Une telle propriété indique que ce modèle devrait être modifié pour récupérer les collisions entre corps rigides. Pour le cas d'un cylindre infini entouré d'un fluide parfaitement incompressible, il est montré dans [?], que les collisions se produisent en temps fini. Si le fluide est visqueux, selon [?], une explication possible pourrait être le choix d'une condition aux limites sans glissement (Dirichlet). Dans [?], les auteurs ont plutôt considéré la condition aux limites de Navier-slip. Dans ce cas, et en considérant une boule immergée dans le fluide, ils ont prouvé, dans [?], l'existence d'une solution qui touche la limite. [?] a envisagé d'autres types de conditions aux limites plus appropriées à un cadre industriel. Dans leurs travaux, les auteurs considèrent un système de corps newtonien rigide-fluide avec la loi de limite de Coulomb. Cette condition aux limites ressemble à l'équation constitutive de Drucker-Prager et stipule que si la composante tangentielle du tenseur des contraintes ne dépasse pas un seuil, la condition aux limites reste la condition aux limites de Dirichlet standard, alors que si elle est égale à ce seuil, la condition aux limites correspond à une condition de limite de Navier généralisée. Dans leurs travaux, les auteurs prouvent l'existence de solutions faibles jusqu'à la première collision.

Considérant ce qui précède, au Chapitre 4, nous étudions le problème de l'existence de solutions faibles jusqu'à la première collision d'un système de corps rigide-fluide de Bingham incompressible et isotherme avec des conditions aux limites de Dirichlet. Du point de vue de la modélisation, l'avantage de l'étude de ce problème réside dans le fait que le modèle fluide de Bingham est l'équation constitutive la plus simple qui détient la propriété plastiqué qui prédomine dans les régimes denses de fluides granulaires. Ensuite, un système de corps Bingham fluide-rigide peut être utile pour comprendre et éclairer un peu les matériaux granulaires. Du point de vue mathématique, bien que la première étape soit un théorème de l'existence jusqu'à la première collision, nous contribuons à l'ensemble des modèles pouvant être utiles à la compréhension des interactions corps-fluide rigides. Ce chapitre est divisé en cinq sections:

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- Dans la Section 4.1, nous présentons les équations du modèle fluide-structure et et nous définissons le résultat principal. Les équations qui régissent l'écoulement du fluide sont écrites en considérant l'équation de Cauchy d'un fluide incompressible de Bingham. Les équations de Newton, pour le moment linéaire et angulaire, régissent le mouvement du corps rigide. Enfin, les équations précédentes sont complétées par des conditions aux limites, dans lesquelles nous supposons la continuité du champ de vitesse à la fois dans l'interface fluide-solide et dans la limite externe (conditions aux limites sans glissement). À la fin de cette section, nous énonçons le résultat principal de l'existence.

- Dans la Section 4.2, nous définissons une solution faible à ce problème et introduisons l'inégalité énergétique. Nous introduisons également la notation appropriée et donnons les résultats préliminaires nécessaires.

- Dans la section 4.3, en utilisant un schéma de Galerkin à un étage, nous énonçons le problème approché qui est défini par trois paramètres : le premier correspondant à la dimension de la méthode de Galerkin, le second correspondant à l'approximation du terme plastique lié au fluide de Bingham et le dernier correspondant au terme de pénalisation utilisé pour traiter le solide à l'intérieur du fluide. Plus loin dans cette section, nous prouvons l'existence d'une solution au problème approché utilisant le théorème de Peano.

- Dans la Section 4.4, nous passons à la limite des paramètres liés à la discrétisation de Galerkin et à la régularisation du terme plastique en même temps. La principale complication est que nous ne pouvons pas prouver dans cette partie que la régularisation du terme plastique converge vers une limite identifiable. Ensuite, pour éviter de traiter cette limite et tirer parti de la convexité du terme plastique, suite à la preuve du théorème 3.2 dans [?, Chapitre 6], nous utilisons un argument de monotonicité qui nous permet de passer à la limite de tous les termes du problème approché. Cependant, le résultat de ce processus est une inégalité variationnelle pour la vitesse qui ne dépend plus que du paramètre lié à la pénalisation.

- Dans la Section 4.5, nous passons à la limite du paramètre lié au terme de pénalisation. Pour ce faire, nous avons besoin de la forte convergence du champ de vitesse. L'approche standard pour obtenir la convergence forte est à suivre [?, Section 7]. Cet argument est basé sur la langue d'Aubin-Lions-Simon[?], où nous avons besoin d'une limite pour l'accélération du système approché. Dans notre cas, nous ne pouvons pas obtenir le lien directement à partir de l'inégalité. Cependant, en utilisant le premier problème approché, avant de passer à la limite la dimension de Galerkin et le terme de régularisation, nous obtenons une limite pour l'accélération qui dépend de la limite de régularisation du terme plastique, ce qui n'est pas un problème puisque ce terme est délimité dans un espace approprié. Ensuite, en utilisant des techniques standard, nous pouvons passer à la limite des équations et prouver l'existence d'une solution faible. De plus, notre solution répond à une inégalité énergétique nous permettant de prouver qu'elle est valable jusqu'à la première collision.

Le dernier chapitre de ce travail, le Chapitre 5, est consacré à l'étude de l'existence d'une solution faible d'un système de corps newtonien conductrice de la chaleur incompressible par la chaleur tenant compte de la condition limite de glissement de Navier. Il existe plusieurs motivations pour étudier ce problème, allant de la modélisation physique aux défis mathématiques.

En ce qui concerne la modélisation physique des fluides, il existe des propriétés de fluide dépendant de la température. Par exemple,comme on le voit à la figure 1.11, la viscosité dynamique de l'eau peut varier jusqu'à 70% en considérant des températures comprises entre 20 et 80 degrés Celsius alors que la densité reste presque constante.

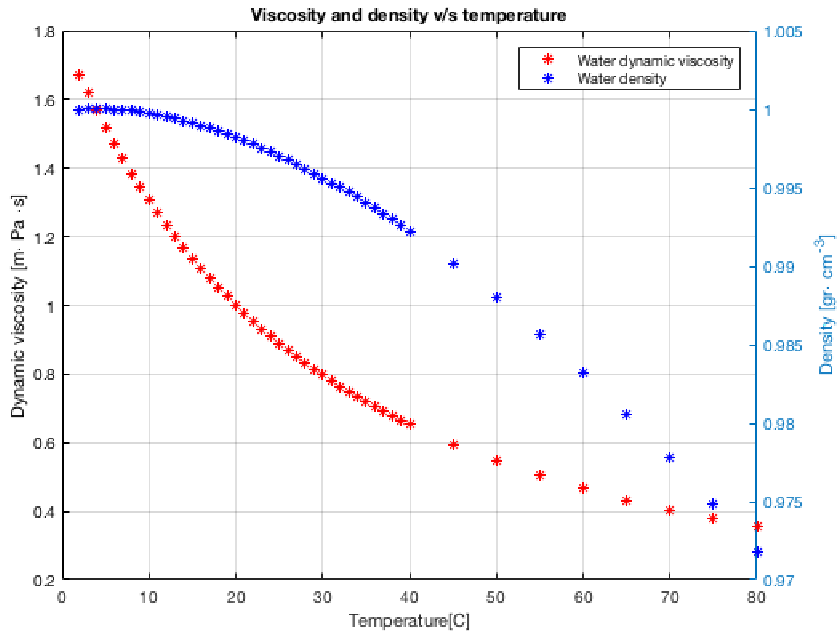

Figure 1.11: Viscosité v/s température et Densité v/s température. Source: Publication sur la formulation IAPWS de 2008 pour la viscosité d'une substance aqueuse ordinaire. Berlin, Allemagne, septembre 2008, disponible sur http://www.iapws.org

Un autre exemple est la conductivité thermique, propriété qui indique la capacité d'une substance à conduire de la chaleur. Par exemple, comme nous le montrons à la figure 1.12, la conductivité thermique du cuivre et de l'eau varie selon les valeurs de température. Cela nous montre que même dans les matériaux solides, certaines propriétés peuvent présenter un comportement dépendant de la température.

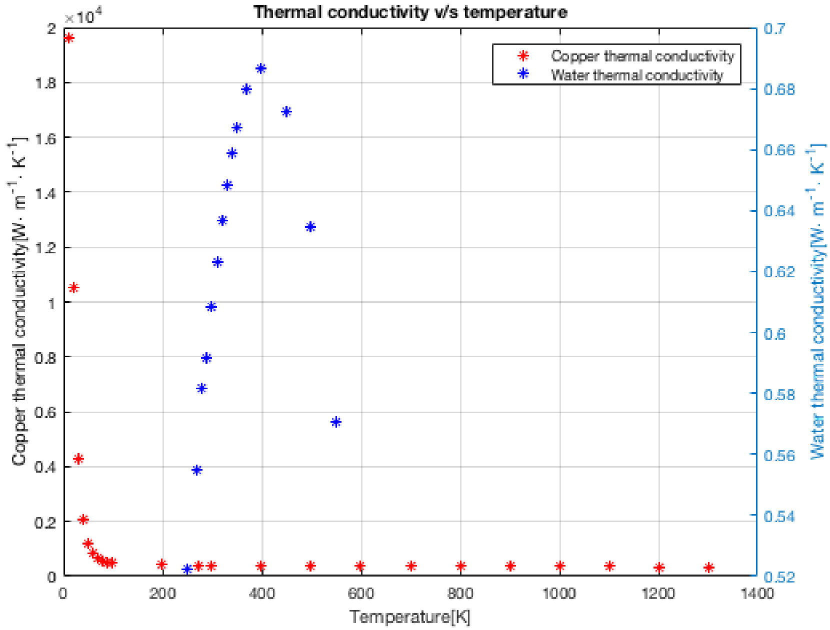

Figure 1.12: Conductivité thermique du cuivre et de l'eau en fonction de la température [?]

En ce qui concerne les défis mathématiques, l'étude d'une dynamique dépendante de la température doit prendre en compte l'équation du bilan énergétique. Cependant, dans le contexte de la mécanique du continu, il existe deux formulations pour cette équation. L'une est appelé le bilan de l'énergie totale et représente la somme de l'énergie cinétique et de l'énergie interne du continu. L'autre est l'équilibre de l'énergie interne .

Dans le contexte des solutions classiques, en utilisant l'équation d'équilibre de moment linéaire, on peut montrer que les deux formulations sont équivalentes.

Cependant, dans le contexte de solutions faibles, choisir une formulation plutôt qu'une autre peut entraîner des problèmes totalement différents.

Par exemple, le problème d'un fluide newtonien incompressible conducteur de chaleur avec des coefficients de matériau dépendants de la température considérant , comme dans les figures 1.11 et 1.12, considérant que la formulation faible avec l'équilibre en énergie interne est toujours ouverte [?]. La principale pierre d'achoppement est la présence du terme de dissipation dans l'équation d'énergie. L'estimation a priori du terme de dissipation n'est que de L^1 ; il est donc impossible, dans un schéma d'approximation de Galerkin, de passer à la limite de ce terme en utilisant simplement ce type de régularité.

Ce problème ne se pose pas lorsque les coefficients de matériau sont constants (nous renvoyons le lecteur à [?, chapitre 3.5]) ou lorsqu'il existe de meilleures estimations a priori du gradient de vitesse, par exemple dans le cas de fluides non newtoniens définis par des lois de puissance [?] ou dans deux dimensions d'espace [?, ?]. Une autre approche de ce problème consiste à écarter le terme de dissipation. Cette approximation a reçu le nom de Boussinesq et est utile pour modéliser les fronts atmosphériques, la circulation océanique, les vents catabatiques, la dispersion de gaz dense et les sorbonnes de fumée. Dans ce contexte, l'équation d'énergie devient une équation de convection-diffusion et l'analyse du système couplé est plus simple (voir par exemple [?]). Par exemple, le problème de l'existence d'une solution faible d'une probleme d'interaction fluide-structure avec un fluide Newtonien conducteur du chaleur avec l'approximation de Boussinesq a été solú dans [?]. Cependant, il existe d'autres méthodes pour résoudre le problème du terme de dissipation. Dans [?], les auteurs considèrent en particulier la condition de limite de glissement de Navier.

Dans [?], les auteurs considèrent le solde de l'énergie totale au lieu du solde de l'énergie interne. Cette approche conduit à une formulation faible de l'équation d'énergie où tous les termes sont correctement liés et où l'existence d'une solution faible est prouvée. Cependant, l'étude correspondante est plus compliquée car il faut obtenir des estimations de la pression lors de la preuve de l'existence. Pour obtenir des estimations de la pression dans [?], les auteurs utilisent la décomposition de Helmholtz-Weyl et prouvent que la pression est une fonction intégrable. Cependant, cela n'est possible qu'en considérant certaines conditions de limite de glissement. Dans le cas d'une condition limite sans glissement, une méthode permettant d'obtenir la pression a été développée par Wolf [?]. Cette pression est appelée par l'auteur "local" et est la somme d'une pression intégrable et de la dérivée temporelle de la fonction harmonique qui n'est pas intégrable. D'autres paramètres dans lesquels la pression est une fonction intégrable sont des problèmes spatialement périodiques, voir par exemple [?].

L'intégrabilité de la pression n'est pas seulement importante dans le contexte des fluides thermoconducteurs, elle constitue également une étape cruciale dans les modèles de turbulence [?] et dans les fluides incompressibles non newtoniens [?, ?, ?, ?]. En ce qui concerne l'étude de la pression, dans le contexte des problèmes d'interaction fluide-structure, nous renvoyons le lecteur à [?] où le cas d'une interaction corps-fluide non-newtonienne est traité. Pour les conditions aux limites, les auteurs considèrent une condition sans glissement dans la limite extérieure et à l'interface du solide et du fluide. Ensuite, une partie de ce travail est consacrée à l'étude de la pression "local" où les auteurs ont besoin que les estimations de pression passent à la limite de la non-linéarité associée au tenseur de contraintes.

Compte tenu de ce qui précède, au Chapitre 5, nous étudions le problème de l'existence de solutions faibles jusqu'à la première collision d'un système de corps fluide-rigide newtonien conducteur de la chaleur incompressible avec des coefficients de matériau dépendant de la température tenant compte d'une condition limite de glissement de Navier. Dans notre modèle, nous supposons que le solide rigide est un conducteur thermique parfait. Cette hypothèse de simplification est exacte pour de nombreux problèmes d'interaction liquide-solide, par exemple, celle décrite à la figure 1.12, où la capacité thermique du fluide est négligeable par rapport à la capacité thermique du solide.. Pour prouver l'existence d'une solution faible nous suivons l'approche de [?] et nous travaillons avec l'équation de l'énergie totale. Avec ce cadre, nous sommes obligés d'imposer une condition limite de glissement de Navier pour obtenir des estimations de la pression du fluide. Pour traiter de la condition limite de glissement de Navier dans le domaine extérieur et de l'interface fluide-solide, nous suivons l'approche de [?].

Néanmoins, pour obtenir une estimation appropriée de la pression du fluide, nous devons faire une hypothèse supplémentaire : les densités du fluide et des parties solides sont égales. Cette hypothèse semble restrictive. Cependant, à notre connaissance, ce cas n'a pas encore été traité et il peut être appliqué dans de nombreux contextes naturels et industriels. Par exemple, la densité des poissons est égale à la densité de l'océan dans lequel ils nagent. Il existe également des matériaux solides qui ont à peu près la même densité d'eau. Par exemple, de nombreux types de caoutchouc ont une densité proche de 1 $[g \cdot cm^3]$: les caoutchoucs ordinaires, le polyisoprène, le caoutchouc éthylène-propylène-diène, le polybutadiène et le caoutchouc styrène-butadiène ont une densité d'environ $0,92$-$0,98$ $[g \cdot cm^3]$.

Dans le cadre des fluides granulaires, nos équations sont un modèle totalement discret d'un matériau granulaire possédant des propriétés de conduction thermique. C'est la principale différence avec le premier modèle du Chapitre 3 3, qui est une représentation entièrement continue d'un matériau granulaire isotherme, et celui du Chapitre 4 qui est un mélange entre un modèle continu et un modèle discret pour un fluide granulaire isothermique. Bien que notre modèle pose plusieurs restrictions, il s'agit d'une première approche pouvant conduire à plusieurs modèles plus robustes.

Ce chapitre est divisé en cinq sections.

- Dans la section 5.1, nous présentons les équations du modèle fluide-structure et nos principales hypothèses. Les équations qui régissent l'écoulement des fluides sont les incompressibles sont les équations de Fourier-Navier-Stokes avec des coefficients dépendant de la température. Les équations de Newton pour le moment linéaire et angulaire régissent le mouvement du corps rigide. Pour le comportement thermique, nous considérons l'équation d'énergie interne pour le fluide et les parties solides. Enfin, les équations précédentes sont complétées par les conditions aux limites, où nous supposons la continuité de la composante normale du champ de vitesse, à la fois dans l'interface fluide-solide et dans la limite externe, et une condition de glissement de Navier pour la partie tangentielle de la vitesse dans le fluide-interface solide et la limite extérieure. L'hypothèse selon laquelle le corps rigide est un parfait conducteur de chaleur se traduit par le fait que le gradient de la température du solide est égal à 0. Ensuite, puisque nous devons forcer la température à rester constante sur le solide, nous devons utiliser une pénalisation scalaire. méthode dans l'équation d'énergie. Également dans cette section, nous définissons la solution faible de notre problème et nous énonçons le principal théorème de l'existence.

- Dans la section 5.2, nous introduisons la notation correcte et donnons les résultats préliminaires nécessaires.

- Dans la section 5.3, nous énonçons le problème approché. Dans cette partie, par souci de clarté, le problème approché n'est défini que par deux paramètres : l'un qui correspond aux termes de pénalisation dans l'équation de quantité de mouvement et l'équation d'énergie, et le second qui correspond à une régularisation du terme convectif de nos équations. Dans l'équation de quantité de mouvement, nous utilisons une pénalisation L^2, et pour l'équation d'énergie, nous utilisons une méthode de pénalisation H^1. Le dernier paramètre est introduit pour pouvoir utiliser la vitesse approximative comme fonction de test dans nos équations. Enfin, dans cette section, nous énonçons un résultat d'existence pour le problème approché. Nous mentionnons que dans le problème approché, la pression fait partie de l'équation du moment.

- Dans la section 5.4, nous présentons la preuve de l'existence d'une solution au problème approché. Cette preuve est basée sur un schéma de Galerkin en 2 étapes dans lequel la pression n'est pas encore introduite. Une étape est liée à l'équation de la quantité de mouvement et l'autre à l'équation de l'énergie. Comme nous l'avons fait au chapitre 4, nous prouvons l'existence d'une solution au problème discret utilisant le théorème de Peano. Ensuite, nous passons à la limite des paramètres liés à la discrétisation de Galerkin et nous avons introduit la pression en utilisant la décomposition de Helmholtz-Weyl.

- Dans la section 5.5, nous passons à la limite du paramètre relatif au terme de pénalisation et à la régularisation du terme de convection en même temps. Ensuite, comme nous l'avons déjà vu au Chapitre 4, pour obtenir la forte convergence du champ de vitesse, nous suivons [**?**, Section 7]. Dans cette partie, nous nous heurtons à la principale difficulté liée à la condition de limite de glissement de Navier, à savoir le manque d'estimations H^1 de la vitesse dans le solide. Cependant, suivant l'approche de [**?**] (voir aussi [**?**]), ce problème est résolu en utilisant un type spécial de fonctions de test. Enfin, nous passons à la limite de l'équation du moment. Ensuite, suivant l'approche de [**?**], nous utilisons les équations approximatives de quantité de mouvement et d'énergie pour obtenir l'équation approximative de l'énergie totale. Dans cette équation, le terme de dissipation n'est plus un problème et, une fois passée à la limite, nous concluons à l'existence d'une solution faible. De plus, notre solution répond à une inégalité énergétique nous permettant de prouver qu'elle est valable jusqu'à la première collision.

1.3 Introducción

Esta tesis tiene como objetivo desarrollar algunos modelos matemáticos en mecánica de fluidos para comprender algunos aspectos de la dinámica de **fluidos granulares heterogéneos**. Entonces , la pregunta natural es: ¿Qué es un fluido granular?

Definition 1.3.1 (Fluidos granulares)**.** *Según [?], un fluido granular es una colección de partículas sólidas sumergidas en un fluido que puede ser agua o aire. Consideramos que las partículas son más grandes que 1 μm para evitar la inclusión de las fuerzas de Van der Waals.*

La importancia de los fluidos granulares radica en su existencia en numerosos contextos industriales y ambientales. Según [?], medido en toneladas, el primer material manipulado en la tierra es el agua; El segundo es el material granular. Se pueden encontrar varios ejemplos de materiales granulares en la industria, tales como relaves, tabletas farmacéuticas y cápsulas; y en la naturaleza tales como deslizamientos de tierra, avalanchas, flujos piroclásticos, arroz y arena.

Figure 1.13: Desprendimientos de tierra

Figure 1.15: Flujos piroclásticos

Figure 1.14: Avalanchas

Figure 1.16: Relaves

La comprensión de la dinámica de los fluidos granulares implica diferentes métodos de las ciencias físicas, desde estudios experimentales y teóricos, así como observaciones de campo a simulaciones numéricas y, una combinación de todo lo anterior (para una revisión exhaustiva del desarrollo y el estado del arte, remitimos al lector a [?, ?]) . En particular, en esta tesis, contribuimos con un estudio teórico y simulaciones numéricas que involucran la mecánica de medios continuos, la teoría matemática de las ecuaciones constitutivas y las matemáticas aplicadas.

En el marco de la mecánica de medios continuos, hay dos formas de representar la componente granular en un fluido granular:

Discreta Cada partícula sólida satisface las ecuaciones cinemáticas del movimiento. Luego, el fluido y la componente sólido forman parte de un sistema de interacción fluido-sólido.

Continua El material granular se modela como un subconjunto de $\mathbb{R}^3$ que contiene todas las partículas y se mueve en el tiempo (un medio continuo).

El enfoque discreto se caracteriza por un conjunto de ecuaciones bien definidas donde la única pregunta que queda es determinar cómo interactúan el fluido y las partes sólidas. El hecho de que las componentes líquidas y sólidas ocupen dominios en movimiento agrega una gran dificultad en el análisis matemático del sistema. Por otro lado, si queremos lograr simulaciones numéricas, el número de partículas sólidas está restringido, ya que los grados de libertad del problema aumentan con el número de partículas.

Existen varias teorías en el marco del enfoque continuo. Entre ellas, la teoría de mezclas de Truesdell [?] es la más notable. Fue desarrollado para estudiar la dinámica de mezclas de gases y describe el comportamiento mecánico y termodinámico de una mezcla utilizando un conjunto de ecuaciones diferenciales parciales. Estas ecuaciones extienden los principios de la mecánica de medios continuos para un solo cuerpo a un cuerpo hecho de dos o más materiales diferentes. La hipótesis central, llamada **h**ipótesis del continuo, es que en cada instante, cada punto del espacio está ocupado simultáneamente por una partícula de cada componente. A primera vista, la hipótesis del continuo parece más manejable que el enfoque discreto. Sin embargo, esta hipótesis añade tres dificultades principales. La primera es entender cómo interactúan los materiales, ya que no sabemos dónde están las fronteras de cada una de las componentes. La segunda es la falta de ecuaciones constitutivas adecuadas para materiales granulares. La tercera es que, incluso si proponemos una ecuación constitutiva. para un material granular, no podemos pensar que las propiedades constitutivas de las componentes no cambian de estar aislados a estar en la mezcla, en otras palabras, las propiedades constitutivas de todos los elementos de la mezcla están relacionadas entre sí.

Los enfoques discretos y continuos sirven para diferentes propósitos. El enfoque discreto se ha utilizado para obtener información sobre la evolución de la microestructura interna del flujo granular [?, ?]. Por otro lado, el enfoque de la teoría de mezclas. para comprender la dinámica de un flujo granular donde la masa del fluido intersticial es comparable a la del sólido (ver, por ejemplo, [?]), y modelar la dinámica del flujo, cuando existen 2 o más componentes tienen un peso significativos en la mezcla (ver por ejemplo [?]).

Con respecto a las dificultades de utilizar el enfoque de la teoría de mezclas, podemos decir lo siguiente:

- La primera dificultad es superada por Truesdell [?], agregando un término fuente en la ecuación de momentum de todos los constituyentes, que se define en todo el espacio. Este término representa la fuerza de roce entre cada componente de la mezcla. Este enfoque es útil ya que permite que la dinámica de toda la mezcla satisfaga las ecuaciones de la mecánica de medios continuos para una sola componente, que es una de las hipótesis de la teoría de mezclas (tercer principio metafísico).

- Con respecto a la segunda dificultad, enfatizamos que se necesita una visión integral de las propiedades mecánicas y termodinámicas de los materiales para escribir ecuaciones constitutivas. En particular, los materiales granulares revelan diversos comportamientos mecánicos, similares a los sólidos elastoplásticos, en el caso de un régimen casi-estático, y a gases densos, en el caso de agitación intensa. Entonces, las propiedades de los materiales granulares están en algún lugar entre las de un líquido y las de un sólido real. Incluso en reposo, el material granular puede soportar algo de esfuerzo cortante, pero solo una cantidad proporcional al esfuerzo promedio. Esta propiedad de fluencia es dominante en regímenes densos y, varios autores han propuesto ecuaciones constitutivas que se asemejan a un material viscoplástico. Los más notables son la ley de Drucker-Prager [?], que es una extensión del criterio de rendimiento de Mohr-Coulomb, y más recientemente la reología $\mu(I)$[?]. Ambos modelos son una extensión de la ecuación constitutiva de Bingham. La ecuación constitutiva de Bingham es uno de los modelos más simples para un fluido viscoplástico. Fue propuesto por Bingham [?] en 1916 y se caracteriza por una tensión de fluencia constante. En el modelo de Drucker-Prager y la reología $\mu(I)$, la tensión de fluencia no es constante sino que depende de la presión. Sin embargo, estos modelos se enfrentan a la falta de buenas propiedades matemáticas y métodos numéricos precisos. Por ejemplo, [?] demostró que la ecuación constitutiva de Drucker-Prager está mal puesta a en todos los contextos bidimensionales y todos los contextos tridimensionales. Sin embargo, la investigación de materiales granulares utilizando la reología $\mu(I)$ es prometedora. Por ejemplo, [?] demostró que la reología $\mu(I)$ está bien puesta bajo ciertas condiciones sobre un parámetro llamado número de inercia. En el frente numérico, [?] obtuvo resultados precisos, utilizando un método Lagrangiano aumentado, para simular el colapso de una pared granular.

- Proponer ecuaciones constitutivas es la raíz de la teoría material. Como dijimos, en el caso de una mezcla no está claro cómo debemos tomar en consideración los efectos de los otros constituyentes en las ecuaciones constitutivas. Para superar esta dificultad, la mayoría de los autores utilizan el principio de entropía que establece que las soluciones de las ecuaciones de la mecánica de medios continuos deben satisfacer la segunda ley de la termodinámica. Bajo esta restricción, utilizando un procedimiento matemático llamado Müller-Liu [?], los autores encuentran restricciones para las funciones constitutivas y lograron proponer ecuaciones constitutivas adecuadas. Para el caso de una mezcla tenemos como ejemplo los trabajos de [?, ?] y para el caso de un solo material [?, Capítulo 5.8] o [?, Capítulo 7]. Sin embargo, enfatizamos que los resultados de aplicar el método Müller-Liu dependen de las hipótesis iniciales de cuáles son las variables independientes del problema y qué leyes de equilibrio estamos considerando. Por ejemplo, si queremos estudiar una dinámica isotérmica , la temperatura no es una variable independiente. En la teoría de mezclas, se introduce la concentración volumétrica de un constituyente que es una variable independiente que representa la cantidad de un componente en cada punto del espacio. Goodman y Cowin [?] introdujeron el enfoque de agregar una nueva ecuación de balance para la concentración volumétrica. El principal argumento de estos autores es que los fluidos granulares presentan efectos microestructurales en su macroescala, lo que se explica, en general, al agregar una nueva ecuación de balance. En su trabajo, los autores agregan a las ecuaciones de balance de masa, momento lineal y angular, y energía, otra ecuación de balance llamada balance de fuerzas equilibradas, que es una ecuación dinámica para concentración volumétrica de la componente sólida. Usando el método Müller-Liu, propusieron un conjunto de ecuaciones constitutivas. Según [?] la teoría de Goodman y Cowin es moderadamente exitosa ya que predice el criterio de Mohr-Coulomb (propiedad de fluencia), y las soluciones obtenidas describen exitosamente algunos fenómenos observados en fluidos granulares. Varios autores han seguido este enfoque. Algunos ejemplos son [?, ?, ?], y más recientemente [?]. Sin embargo, existen otros enfoques, por ejemplo, Liu [?]. Para un medio poroso, sin considerar el equilibrio de la ecuación de fuerza equilibrada y el uso del método Müller-Liu y obtine ecuaciones constitutivas diferentes.

En cuanto a lo anterior, esta tesis contiene dos partes principales. La primera parte trata sobre un modelo continuo para flujos heterogéneos granulares y la segunda trata sobre el enfoque discreto.

En la primera parte de la tesis, contenida en el Capítulo 3, el objetivo es modelar la dinámica de un flujo heterogéneo granular compuesto por un fluido Newtoniano (fluido intersticial) y $N-1$ componentes sólidas, con $N \in \mathbb{N} \setminus \{0, 1\}$. Para lograr esto, desarrollamos una teoría, parecida a la teoría de mezclas de Truesdell, compuesta de un conjunto de ecuaciones de balnce, con el objetivo de que nuestras ecuaciones cumplieran con 2 de los cuatro principios de la teoría de mezclas de Truesdell, la Hipótesis del Continuo y el segundo principio metafísico. También calculamos la solución numérica del flujo estacionario de Couette entre dos cilindros concéntricos infinitos en el caso de que $N = 2$. Este capítulo se divide en cuatro secciones:

- En la sección 3.1, el objetivo es deducir un conjunto de ecuaciones en derivadas parciales, que se asemejan a las ecuaciones de la teoría de mezclas, utilizando un enfoque matemático que utiliza las herramientas de la teoría de la homogeneización. En este enfoque, definimos un conjunto de escalas observables en las que podemos distinguir dónde se ubican las diferentes componentes de la mezcla y dónde interactúan. Luego, escribimos las ecuaciones de balance para este sistema. Bajo hipótesis técnicas, pasamos al límite y obtenemos las ecuaciones de equilibrio en una escala homogeneizada donde la mezcla satisface la hipótesis del continuo. Una diferencia esencial entre nuestro modelo y el enfoque de la teoría de mezclas clásica se basa en que, en nuestro caso, los términos de interacción no se consideran términos fuente y se incluyen como parte de las funciones constitutivas. Entonces, nuestro modelo de las funciones constitutivas necesita considerar estas interacciones. Enfatizamos que, en nuestra deducción, no usamos la ecuación de equilibrio de las fuerzas equilibradas de Goodman y Cowin [**?**]. En su lugar, consideramos una restricción de saturación, lo que implica que la suma de todas las fracciones de volumen es igual a uno, o en otras palabras, la mezcla está saturada.

- En la sección 3.2, el objetivo es usar el procedimiento M üller-Liu basado en el principio de entropía, para obtener un conjunto de restricciones para las funciones constitutivas. Una de estas restricciones es la desigualdad residual. Esta desigualdad nos permite escribir ecuaciones constitutivas como una inclusión sub-diferencial. En este contexto, las inclusiones sub-diferenciales son una herramienta útil, ya que nos permite caracterizar con precisión la propiedad de fluencia de los materiales granulares. En la Sección 3.2.4, claramos el punto anterior revisando algunos ejemplos clásicos como la ecuación de fluidos de Bingham, la ley de Drucker-Prager y la reología de $\mu(I)$.

- En la sección 3.3, proponemos ecuaciones constitutivas para una mezcla heterogénea de $N-1$ componentes granulares y 1 componente de fluido instersticial, utilizando inclusiones subdiferenciales. Las ecuaciones constitutivas modelan dos comportamientos importantes. Primero, la naturaleza viscosa del fluido intersticial y, segundo, el comportamiento plástico del componente granular. Sin embargo, nuestras ecuaciones todavía dependen de una función llamada energía libre de Helmholtz. La energía libre de Helmholtz es la diferencia entre la energía interna y la entropía multiplicada por la temperatura. Luego, para cerrar el sistema de ecuaciones, proponemos una función de energía libre de Helmholtz para cada constituyente que es la suma de dos partes: la primera es la función de energía libre de Helmholtz del constituyente aislado del resto de los componentes y la segunda parte es un término de interacción, que se asemeja a la energía de roce. Con esta hipótesis final, obtenemos el sistema completo de ecuaciones. Para contrastar nuestro modelo, estudiamos el flujo estacionario de Couette entre dos cilindros concéntricos infinitos en el caso donde tenemos el fluido intersticial y una componente granular ($N = 2$). El resultado es un sistema DAE (ecuación algebraica diferencial) que se resuelve utilizando MATLAB. Incluimos los códigos en la Sección B del Apéndice.

- En la Sección 3.4 discutimos los resultados obtenidos con la simulación numérica y las perspectivas futuras de esta investigación.

La segunda parte de esta tesis se divide en dos capítulos, Capítulo 4 y Capítulo 5. El objetivo principal de esta parte es estudiar la existencia de soluciones débiles para modelos multifásicos que involucran estructuras rígidas y fluidos. En estos modelos, consideramos que el fluido y la parte rígida ocupan dominios bien definidos y las propiedades constitutivas del fluido están bien determinadas por un conjunto de ecuaciones constitutivas simples. Con respecto al primer capítulo, la principal diferencia es que la hipótesis del continuo se descarta y las interacciones entre el sólido y el líquido solo se consideran en las condiciones de contorno del problema.

A primera vista, este enfoque parece mucho más realista. Sin embargo, agrega nuevas complicaciones: las ecuaciones de movimiento se satisfacen en los dominios en movimiento, y debemos proponer un modelo de cómo los materiales interactúan en la frontera mutua. Más precisamente:

- Dado que las ecuaciones se satisfacen en los dominios móviles, no podemos usar técnicas estándar de la existencia de soluciones de las ecuaciones de la mecánica de fluidos (ver, por ejemplo, [?]). Para superar esto, la mayoría de los autores construyen las soluciones a través de un método de penalización desarrollado por Hoffmann y Starovoitov [?] y San Martín et al.[?]. Con respecto al método de penalización, existen al menos dos enfoques diferentes: una penalización "L^2" (ver, por ejemplo, [?]), y una penalización "H^1" (ver por ejemplo [?]). En las pruebas de existencia, el enfoque de penalización "L^2" es útil cuando el problema carece de un límite en "H^1" para la velocidad (por ejemplo en [?]). En los esquemas numéricos, estos dos métodos de penalización se utilizan para simular el movimiento de cuerpos rígidos en un fluido, pero el inconveniente del método de penalización "H^1" es que el sólido puede cambiar su forma y no moverse como un sólido rígidos. Algunos trabajos importante en esta materia son, por ejemplo, [?] para el análisis de un esquema numérico basado en el método de penalización "L^2" y también a [?], [?], [?], [?], etc. para algunos otros trabajos sobre el estudio numérico de sistemas de cuerpos rígidos y fluidos.

- Con respecto a las condiciones de contorno, para un sistema de interacción fluido viscoso-sólido rígido, se demostró en [?] que si el sólido toca el límite exterior, este contacto se realiza con una velocidad y una aceleración relativa nula. Este resultado sugiere que si en el momento inicial dos cuerpos no se tocan, nunca estarán en contacto en un tiempo finito Esta paradoja de no colisión va en contra del principio de Arquimides y se conoce desde los años 1960 [?, ?], en el contexto de las ecuaciones de Stokes y se ha demostrado rigurosamente en el casode las ecuaciones de Navier-Stokes incompresible en 2d [?] y para la ecuación de Navier-Stokes incompresible en

3-d cuando el cuerpo rígido es una bola [?]. Dicha propiedad indica que este modelo debe modificarse para recuperar colisiones entre cuerpos rígidos. Para el caso de un cilindro infinito rodeado por un fluido incompresible perfecto, se muestra en [?], que las colisiones ocurren en tiempo finito. Si el fluido es viscoso, de acuerdo con [?], una posible explicación podría ser la elección de una condición de borde de no deslizamiento (Dirichlet). En [?], los autores consideraron la condición de borde de Navier. En ese caso, y considerando una bola inmersa en el fluido, en [?] se demostró la existencia de una solución que toca el borde. [?] ha considerado otros tipos de condiciones de borde más apropiadas para un marco industrial. En su trabajo, los autores consideran un sistema de interacció fluido newtoniano- sólido rígido con condición de borde tipo ley de Coulomb. Esta condición de borde se asemeja a la ecuación constitutiva de Drucker-Prager e indica que si la componente tangencial del tensor de tensión no excede una tensión umbral, entonces la condición de borde es una condición tipo Dirichlet, mientras que si es igual a este umbral, la condición de borde corresponde a una condición de tipo Navier generalizada. En su trabajo, los autores prueban la existencia de soluciones débiles hasta la primera colisión.

Teniendo en cuenta lo anterior, en el Capítulo 4 estudiamos el problema de la existencia de soluciones débiles hasta la primera colisión de un sistema de interaccion fluido-sólido rigido donde el fluido es del tipo Bingham incompresible e isotérmico con condiciones de borde del tipo Dirichlet en la frontera mutua y la frontera exterior. Desde el punto de vista del modelo, la ventaja de estudiar este problema radica en el hecho de que el modelo de un fluido Bingham es la ecuación constitutiva más simple que tiene la propiedad de fluencia que es dominante en regímenes densos de fluidos granulares. Luego, un sistema de interacción fluido Bingham-sólido rígido puede ser útil para comprender y arrojar algo de luz sobre los materiales granulares. Desde el punto de vista matemático, aunque nuestro teorema de existencia es valido hasta la primera colisión, es un primer paso y contribuimos a la gama completa de modelos que pueden ser útiles para comprender las interacciones entre fluidos y cuerpos rígidos. Este capítulo está dividido en cinco secciones:

- En la Sección 4.1, presentamos las ecuaciones del modelo de interacción fluido-estructura y establecemos el resultado principal. Las ecuaciones que modelan el movimiento del fluido son las ecuaciones de momentum de Cauchy para un fluido Bingham incompresible. Las ecuaciones de Newton del momento lineal y angular gobiernan el movimiento del sólido rígido. Finalmente, las ecuaciones anteriores se completan con condiciones de borde, donde asumimos la continuidad del campo de velocidades tanto en la interfaz fluido-sólido como en la frontera externa (condicion de Borde tipo Dirichlet). Al final de esta sección establecemos el resultado principal de este capitulo que es un teorema de existencia.

- En la Sección 4.2, definimos una solución débil para este problema e introducimos la desigualdad de energía. También introducimos notación adecuada, y damos algunos resultados preliminares importantes.

- En la Sección 4.3, utilizando un esquema de Galerkin, planteamos el problema aproximado que se define por tres parámetros: uno que corresponde a la dimensión en el método de Galerkin, otro que corresponde a la aproximación del término plástico asociado al fluido de Bingham, y finalmente uno que corresponde al término de penalización utilizado para tratar el sólido dentro del fluido. Más adelante en esta sección, comprobamos la existencia de una solución al problema aproximado utilizando el teorema de Peano.

- En la Sección 4.4, pasamos al límite los parámetros relacionados con la discretización de Galerkin y la regularización del término plástico al mismo tiempo. La principal complicación es que no podemos probar que la regularización del término plástico converge a un límite identificable. Entonces, para evitar tratar con este límite y aprovechar la convexidad del término plástico, siguiendo la prueba del Teorema 3.2 en [?, Capítulo 6], usamos un argumento de monotonía que nos permite pasar al límite todos los términos en el problema aproximado. Sin embargo, el resultado de este proceso es una desigualdad variacional para la velocidad que ahora depende solo del parámetro relacionado con la penalización.

- En la Sección 4.5, pasamos al límite el parámetro relacionado con el término de penalización. Para hacer esto, necesitamos la convergencia fuerte del campo de velocidad. El enfoque estándar para obtener la convergencia fuerte es seguir [?, Sección 7]. Este argumento se basa en el lema de Aubin-Lions-Simon [?], donde necesitamos una cota para la aceleración del sistema aproximado. En nuestro caso, no podemos obtener el límite directamente de la desigualdad. Sin embargo, al utilizar el primer problema aproximado, antes de pasar al límite en la dimensión de Galerkin y el término de regularización, obtenemos un límite para la aceleración que depende del límite de la regularización del término plástico. Esto no es un problema ya que este término está debidamente acotado en un espacio adecuado. Luego, utilizando técnicas estándar, podemos pasar al límite las ecuaciones y probar la existencia de una solución débil. Además, nuestra solución satisface una desigualdad de energía que nos permite probar que es válida hasta la primera colisión.

El último capítulo de este trabajo, Capítulo 5, se dedica a estudiar la existencia de una solución débil de un sistema de interacción fluido-sólido rígido donde el fluido es un fluido newtoniano incompresible conductor del calor y el sólido es un conductor perfecto de la temperatura. En este problema consideramos condiciones de borde del tipo Navier. Existen diversas motivaciones para investigar este problema, desde el modelamiento físico hasta los desafíos matemáticos.

Con respecto al modelamiento físico, muchas propiedades de los fluidos dependen de la temperatura. Por ejemplo, como vemos en la Figura 1.17, la viscosidad dinámica del agua puede variar hasta un 70 % considerando temperaturas de 20 a 80 grados Celsius, mientras que la densidad permanece casi constante.

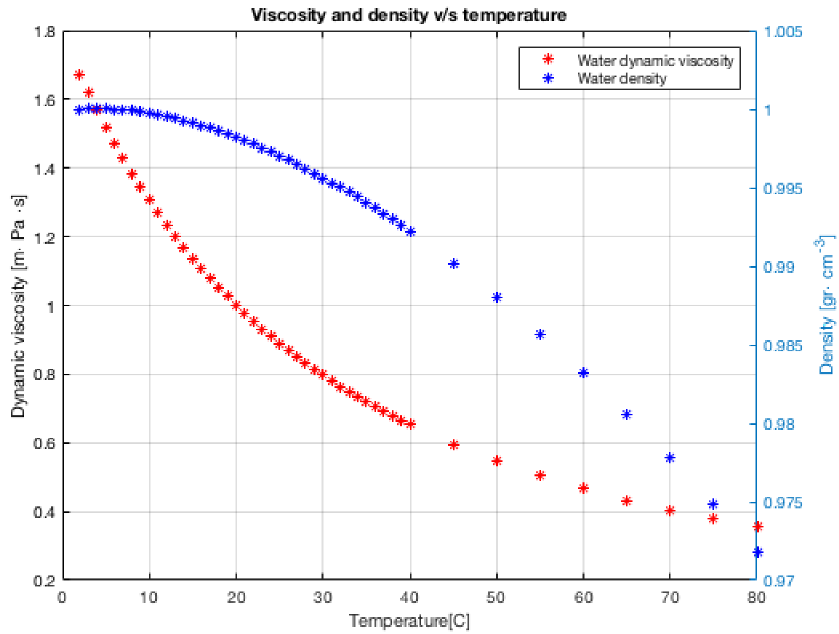

Figure 1.17: Viscosidad v/s temperatura y densidad v/s temperatura. Fuente: Publicación en la Formulación IAPWS 2008 para la viscosidad de la sustancia de agua ordinaria. Berlín, Alemania, septiembre de 2008, disponible en http://www.iapws.org

Otro ejemplo es la conductividad térmica, que es la propiedad que indica la capacidad de una sustancia para conducir el calor. Por ejemplo, como mostramos en la Figura 1.18, la conductividad térmica del cobre y el agua varían para diferentes valores de temperatura. Esto nos muestra que incluso en materiales sólidos, algunas propiedades pueden mostrar un comportamiento de dependencia de la temperatura.

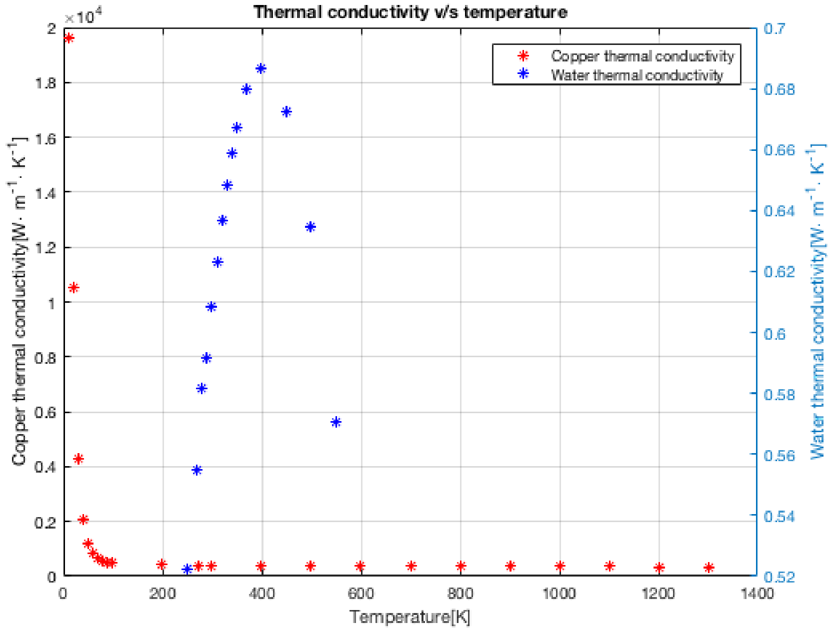

Figure 1.18: Conductividad térmica de cobre y agua v / s temperatura. [?]

Con respecto a los desafíos matemáticos, el estudio de una dinámica dependiente de la temperatura debe considerar la ecuación de balance de energía. Sin embargo, en el contexto de la mecánica de medios continuos, existen dos formulaciones para esta ecuación. Una se conoce como la ecuación de balance de la energía total, que representa la suma de la energía cinética y la energía interna de un medio continuo. La otra es el equilibrio de la energía interna. En el contexto de las soluciones clásicas, utilizando la ecuación momentum lineal, se puede mostrar que ambas formulaciones son equivalentes. Sin embargo, en el contexto de soluciones débiles, elegir una formulación sobre la otra puede llevar a problemas completamente diferentes. Por ejemplo, el problema de un fluido newtoniano conductor del calor e incompresible en 3d con coeficientes materiales dependientes de la temperatura, como en las Figuras 1.17 y 1.18, considerando en la formulación débil la ecuación de balance de la energía interna, sigue abierto [?]. El principal obstáculo es la presencia del término de disipación en la ecuación de energía. La estimación a priori para el término de disipación es solo L^1, por lo tanto, en un esquema de aproximación de Galerkin, es imposible pasar al límite de este término utilizando este tipo de regularidad.

Este problema no se produce cuando los coeficientes materiales no dependen de la temperatura (remitimos al lector a [?, Capítulo 3.5]) o cuando hay mejores estimaciones a priori sobre el gradiente de velocidad, por ejemplo, en el caso de fluidos no-Newtoniano definidos por una ley de potencia [?] o en dos dimensiones espaciales [?, ?]. Otro enfoque a este problema es descartar el término de disipación. Esta aproximación recibió el nombre de Boussinesq y es útil para modelar frentes atmosféricos, circulación oceánica, vientos catabáticos y dispersión de gases densos. En este contexto, la ecuación de energía se convierte en una ecuación de convección-difusión, y el análisis del sistema acoplado es más directo (ver, por ejemplo, [?]). Por ejemplo, el problema de la existencia de una solución débil para el problem de interacción fluido-estructura donde consideramos un fluido Newtoniano incompresible conductor del calor con coeficientes materiales dependiendo continuamente de la temperatura y donde para la ecuación de la energia consideramos la aproximación de Boussinesq se resolvió en [?].

Aunque la aproximación de Boussinesq resulta útil, existen otros métodos para superar el problema del término de disipación.

En [?], los autores consideran el balance de la energía total en lugar del balance de la energía interna. Este enfoque los lleva a una formulación débil de la ecuación de energía en la que todos los términos están correctamente acotados, y se demuestra la existencia de una solución débil. Sin embargo, El estudio correspondiente es más complicado ya que necesitamos obtener estimaciones de la presión durante la prueba de existencia. Para obtener estimaciones de la presión en [?], los autores utilizaron la descomposición de Helmholtz-Weyl y demostraron que la presión es una función integrable. Sin embargo, esto solo es posible considerando alguna condición de borde con deslizamiento. En particular, en [?], los autores consideran una condición de borde del tipo Navier. En el caso de una condición de borde del tipo Dirichlet, Wolf [?] desarrolló un método para obtener la presión. Esta presión es llamada por el autor presión"local" y es la suma de una presión integrable y de la derivada temporal de una función armónica que no es integrable. Otras configuraciones donde la presión es una función integrable se encuentran en problemas espacialmente periódicos, ver, por ejemplo, [?].

La integrabilidad de la presión no solo es relevante en el contexto de los fluidos conductores del calor, sino que también es un paso crucial en los modelos de turbulencia [?] y los fluidos no newtonianos incompresibles [?, ?, ?, ?]. Con respecto al estudio de la presión, en el contexto de los problemas de interacción fluido-estructura, remitimos al lector a [?] donde se trata el caso de una interacción de cuerpo sólido rígido y un fluido no-Newtoniano. Para las condiciones de borde, los autores consideran una condición de tipo Dirichlet en la frontera exterior y en la interfaz entre el sólido y el fluido. Luego, parte de este trabajo está dedicado al estudio de la presión "local" donde los autores obtienen estimaciones para la presión utilizando la no linealidad asociada tensor de esfuerzos de Cauchy.

Teniendo en cuenta lo anterior, en el Capítulo 5 estudiamos el problema de la existencia de soluciones débiles hasta la primera colisión de un sistema de interacción fluido-sólido rígido donde el consideramos que el fluido es Newtoniano incompressible, conductor del calor con coeficientes materiales dependientes continuamente de la temperatura con una condición de borde del tipo Navier. En nuestro modelo, asumimos que el sólido rígido es un conductor de calor perfecto. Esta hipótesis de simplificación es adecuada para muchos problemas de interacción fluido-sólido, por ejemplo, la que se describe en la Figura 1.18, Donde la capacidad térmica del fluido es insignificante con respecto a la capacidad térmica del sólido. Para probar la existencia de una solución débil, seguimos el enfoque de [?], y trabajamos con la ecuación de energía total. Con este marco, nos vemos obligados a imponer una condición de borde de tipo Navier para obtener estimaciones de la presión del fluido. Para hacer frente a la condición de borde de Navier en el dominio exterior y la interfaz fluido-sólido, seguimos el enfoque de [?]. Sin embargo, para obtener una estimación adecuada de la presión del fluido, necesitamos hacer una suposición adicional: las densidades del fluido y sólido son iguales. Esta hipótesis parece restrictiva. Sin embargo, hasta donde sabemos, este caso aún no se ha tratado y se puede aplicar en muchos contextos naturales e industriales. Por ejemplo, la densidad de los peces es igual a la densidad del océano en el que están nadando. También existen materiales sólidos que tienen casi la misma densidad del agua. Por ejemplo, hay muchos tipos de caucho que tienen una densidad cercana a $1[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$: cauchos comunes, poliisopreno, caucho de etileno-propileno-dieno, polibutadieno y el caucho de estireno-butadieno tiene una densidad de alrededor de 0.92 - $0.98[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$.

En el marco de los fluidos granulares, nuestras ecuaciones son un modelo basado en el enfoque discreto de un material granular con propiedades de conducción del calor. Esta es la principal diferencia con el primer modelo en el Capítulo 3, que es una representación continua de un material granular isotérmico, y en relación con el Capítulo 4 que es una mezcla entre un modelo continuo y un modelo discreto. Para un fluido granular isotérmico. Aunque nuestro modelo tiene varias restricciones, este es un primer paso que puede llevar a varios modelos más robustos. Este capítulo está dividido en seis secciones:

- En la Sección 5.1, presentamos las ecuaciones del modelo de estructura de fluidos y nuestras principales hipótesis. Las ecuaciones que gobiernan el fluido son las de Fourier-Navier-Stokes incompresible con coeficientes dependientes continuamente de la temperatura. Las ecuaciones de Newton para el momento lineal y angular gobiernan el movimiento del cuerpo rígido. Para el comportamiento térmico, consideramos la ecuación de energía interna para el fluido y las partes sólidas. Finalmente, las ecuaciones anteriores se completan con condiciones de borde, donde asumimos la continuidad del componente normal del campo de velocidades tanto en la interfaz fluido-sólido como en el límite externo y una condición de borde de Navier para la parte tangencial de la velocidad en el fluido en la interfaz fluido-sólida y el borde exterior. La hipótesis de que el cuerpo rígido es un conductor de calor perfecto se traduce en que el gradiente de la temperatura del sólido es 0. Entonces, dado que necesitamos forzar la temperatura para que sea constante en el sólido, necesitamos usar un método de penalización escalar en la ecuación de energía. También en esta sección definimos la solución débil de nuestro problema y establecemos el teorema principal de la existencia.

- En la Sección 5.2, introducimos notación adecuada y damos algunos resultados preliminares importantes.

- En la sección 5.3, introducimos el problema aproximado. En esta parte, para mayor claridad, el problema aproximado se define por solo dos parámetros: uno que corresponde a los términos de penalización en la ecuación de momentum y la ecuación de energía, y el segundo es un parámetro que corresponde a una regularización de la Término convectivo de nuestras ecuaciones. En la ecuación de momentum utilizamos una penalización L^2, y para la ecuación de energía, usamos un método de penalización H^1. El último parámetro se introduce para poder utilizar la velocidad aproximada como una función de test en nuestras ecuaciones. Finalmente, en esta sección, indicamos un resultado de existencia para el problema aproximado. En el problema aproximado, la presión es parte de la ecuación de momentum.

- En la Sección 5.4 presentamos la prueba de la existencia de una solución al problema aproximado. Esta prueba se basa en un esquema Galerkin de 2 etapas donde la presión aún no se ha introducido. Una etapa está relacionada con la ecuación de momentum y la otra con la ecuación de energía. Como hicimos en el Capítulo 4, demostramos la existencia de una solución al problema discreto usando el Teorema de Peano. Luego, pasamos al límite los parámetros relacionados con la discretización de Galerkin, e introducimos la presión utilizando la descomposición de Helmholtz-Weyl.

- En la Sección 5.5, pasamos al límite el parámetro relacionado con el término de penalización y la regularización del término convectivo al mismo tiempo. Luego, como hicimos antes en el Capítulo 4, para obtener la convergencia fuerte del campo de velocidades y la temperatura seguimos a [?, Sección 7]. En esta parte, nos enfrentamos a la dificultad de lidiar con la condición de borde de Navier, que se traduce en la falta de estimaciones H^1 para la velocidad en el sólido. Sin embargo, siguiendo el enfoque de [?] (ver también [?]), este problema se resuelve utilizando un tipo especial de funciones de test. Finalmente, pasamos al límite la ecuación de momento. Luego, siguiendo [?], usamos las ecuaciones de momentum y energía interna aproximadas para obtener la ecuación de energía total aproximada. En esta ecuación, el término de disipación ya no es un problema y, después de pasar al límite, concluimos la existencia de una solución débil. Además, nuestra solución satisface una desigualdad de energía que nos permite probar que es válida hasta la primera colisión.

Chapter 2

Preliminaries

In this chapter we establish some notation, we recall some essential definitions regarding function spaces, and we enounced some remarkable theorems of functional analysis that we use in the following chapters. We extracted most of the definitions and notations from [**?**, Chapter II].

2.1 Notation

- Let a and b two vectors in $\mathbb{R}^n$. We define the dot product as folllows

$$a \cdot b = \sum_{i=1}^{n} a_i b_i, \tag{2.1}$$

  or, using Einstein notation,

$$a \cdot b = a_i b_i.$$

- Let a and b two vectors in $\mathbb{R}^n$. We define the tensor product as folllows

$$(a \otimes b)_{i,j} = a_i b_j,$$

  for $i \in \{1, \ldots, n\}$.

- Let a and b two vectors in $\mathbb{R}^3$. We define the cross product as folllows

$$a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- We write $A = A_{ij}$ to mean A is an $m \times n$ matrix with $(i,j)^{th}$ entry $A_{i,j}$.

- $\mathbb{M}^{m \times n}$ is the space of real $m \times n$ matrices and $\mathbb{M}_s^{m \times n}$ to the space of real $m \times n$ symmetric matrices.

- We define the space

$$SO(n) = \{Q \in \mathbb{M}^{n \times n} \mid \; QQ^* = I_n, \text{ and } \det(Q) = 1\},$$

  where Q^* is the tranpose matrix of Q and I_n is the identity matrix in $\mathbb{M}^{n \times n}$.

- $\text{tr}(A)$=trace of the matrix A.
- If A and B are $n \times m$ matrices, then we define the trace product

$$A : B = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} B_{i,j}.$$

Using Einstein notation, we write:

$$A : B = A_{i,j} B_{i,j}.$$

- Given A in $\mathbb{M}^{m\times n}$, we define the Frobenious norm as follows

$$|A| = \sqrt{A : A}.$$

- Let $A \in \mathbb{M}^{n\times n}$ we define the traceless part of A as follows

$$\overline{A} = A - \frac{1}{n} \operatorname{tr}(A) I_n, \tag{2.2}$$

where I_n is the identity matrix in $\mathbb{M}^{n\times n}$.

2.2 Differential operators

- For a scalar function $v : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}$, a vectorial function $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^n$, for $n \in \mathbb{N} \setminus \{0, 1\}$ and a matrix function $\Sigma : \mathbb{R}^3 \mapsto \mathbb{M}^{3\times 3}$ we define the operators:

$$\nabla v = \begin{pmatrix} \frac{\partial v}{\partial x_1} \\ \frac{\partial v}{\partial x_2} \\ \frac{\partial v}{\partial x_3} \end{pmatrix},$$

$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} \end{pmatrix},$$

$$\operatorname{div}(v) = \sum_{i=1}^{3} \frac{\partial v}{\partial x_i},$$

$$\operatorname{div}(u) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}$$

and

$$\operatorname{div}(\Sigma) = \sum_{j=1}^{3} \frac{\partial \Sigma_{ij}}{\partial x_j} = \frac{\partial \Sigma_{ij}}{\partial x_j} \in \mathbb{R}^3.$$

We also define the symetric part of ∇u as

$$D(u) = \frac{1}{2} \left(\nabla u + \nabla u^* \right).$$

In the framework of fluid mechanics, if u is the velocity field, $D(u)$ is called the strain rate tensor. On the other hand we define the curl operator as the skew-symmetric matrix

$$\operatorname{curl}(u) = \nabla u - \nabla u^*.$$

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_i is a nonegative integer, is called a multiindex of order:
$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$
Let $u : \mathbb{R}^n \mapsto \mathbb{R}^m$ and a multindex α we define
$$\nabla^\alpha u = (\nabla^\alpha u_1, \dots, \nabla^\alpha u_m),$$
where
$$\nabla^\alpha u_i = \frac{\partial^{|\alpha| u_i}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

- Let $\Phi : \mathbb{M}^{n\times n} \mapsto \mathbb{R}$ a function and $D \in \mathbb{M}^{n\times n}$. We define the set of sub-derivatives of Φ at D as the set
$$\partial\Phi(D) = \{\Sigma \in \mathbb{M}^{n\times n} \mid \ \Sigma : (W - D) \leqslant \Phi(W) - \Phi(D) \quad \forall W \in \mathbb{M}^{n\times n}\}. \tag{2.3}$$
If $\partial\Phi(D) \neq \emptyset$, we say that Φ is subdifferentiable at D.

2.3 Function spaces

- We define some classical spaces of smooth functions. Given $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$, we denote $\mathcal{C}^k(\Omega)$ as the space k-times differentiable functions in Ω. To shorten notations, we set
$$\mathcal{C}^0(\Omega) := \mathcal{C}(\Omega).$$
We also set
$$\mathcal{C}^\infty(\Omega) := \bigcap_{k=0}^{\infty} \mathcal{C}^k(\Omega).$$
Moreover, by the symbols and $\mathcal{C}_0^k(\Omega)$ and $\mathcal{C}_0^\infty(\Omega)$ we indicate the sub- spaces of $\mathcal{C}^k(\Omega)$ and $\mathcal{C}^\infty(\Omega)$, respectively, of all those functions having compact support in Ω. Furthermore, $\mathcal{C}_0^k(\overline{\Omega})$, $0 \leqslant k \leqslant \infty$, denotes the class of restrictions to Ω of functions in $\mathcal{C}_0^k(\mathbb{R}^n)$. As before, we put
$$\mathcal{C}_0^0(\Omega) := \mathcal{C}_0(\Omega), \text{ and } \mathcal{C}_0^0(\Omega) := \mathcal{C}_0(\mathbb{R}^n).$$
We next define $\mathcal{C}^k(\overline{\Omega})$ ($\mathcal{C}(\overline{\Omega})$ for $k = 0$) as the space of all functions u for which $\nabla^\alpha u$ is bounded and uniformly continuous in Ω, for all multiindex $0 \leqslant |\alpha| \leqslant k$. We recall that for $k \leqslant \infty$, $C^k(\overline{\Omega})$ is a Banach space with respect to the norm
$$\|u\|_{\mathcal{C}^k(\Omega)} := \max_{0\leqslant|\alpha|\leqslant k} \sup_{\Omega} \|\nabla^\alpha u\|.$$
Finally, for $\lambda \in (0, 1]$ and $k \in \mathbb{N}$, by $\mathcal{C}^{k,\lambda}(\overline{\Omega})$ we denote the closed subspace of $\mathcal{C}^k(\overline{\Omega})$ consisting of all functions u whose derivatives up to the k-th order are Hölder continuous (Lipschitz continuous if $\lambda = 1$) in Ω, that is,
$$[u]_{k,\lambda} := \max_{0\leqslant|\alpha|\leqslant k} \sup_{x,y\in\Omega, x\neq y} \frac{\|\nabla^\alpha u(x) - \nabla^\alpha u(y)\|}{|x - y|^\lambda} \leqslant \infty.$$
$C^k, \lambda(\Omega)$ is a Banach space with respect to the norm
$$\|u\|_{\mathcal{C}^{k,\lambda}(\Omega)} = \|u\|_{\mathcal{C}^k(\Omega)} + [u]_{k,\lambda}.$$

- For $p \in [1,\infty)$, we define the space $L^p(\Omega)$ as the set of all real Lebesgue-measurable functions u defined in Ω, such that

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^q \, dx \right)^{1/p} < \infty,$$

  and for $p = \infty$

$$\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{\Omega} |u| < \infty.$$

- For $q \in [1,\infty]$ and $k \in \mathbb{N}$, we define the Sobolev space

$$W^{k,p}(\Omega) := \{ u \in L^p(\Omega) \mid \nabla^\alpha u \in L^p(\Omega),\ 0 \leqslant |\alpha| \leqslant m \},$$

  with the respective norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{0\leqslant|\alpha|\leqslant k} \|\nabla^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

  or

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{0\leqslant|\alpha|\leqslant k} \|\nabla^\alpha u\|_{L^\infty(\Omega)}.$$

  Finally, for the case $p = 2$ we denote $W^{k,2}(\Omega) = H^k(\Omega)$ and we denote by $W^{k,p}_0(\Omega)$, the completion of $C^\infty_0(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$

- Let X a Banach space and $\|\|_X$ its norm. For $p \in [1,\infty)$ we define the space $L^p(0,T;X)$ as the space of all mesurable functions $u : [0,T] \mapsto X$ such that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u\|_X \right)^{1/p} < \infty,$$

  and for $p = \infty$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{t\in[0,T]} \|u\|_X < \infty.$$

  We also introduce the space $\mathcal{C}([0,T];X)$ as the space of all continuous functions $u : [0,T] \mapsto X$ such that

$$\|u\|_{\mathcal{C}([0,T];X)} := \max_{t\in[0,T]} \|u\|_X < \infty.$$

  In general, $L^p(0,T;X)$ and $\mathcal{C}([0,T];X)$ are called Bochner spaces.

- For any set Ω with a bounded boundary, we say that Ω is of class $C^{k,\lambda}$ if the boundary $\partial\Omega$ can be parametrized with $C^{k,\lambda}$ functions (a precise definition of parametrized is given in [**?**, p.37]).

2.4 Important theorems of functional analysis

Theorem 2.4.1 (Hölder's inequality). *Let $\Omega \subset \mathbb{R}^n$ for $n \geqslant 1$. For any $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$*

$$\|uv\|_{L^r(\Omega)} \leqslant \|u\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)},$$

for $1/r = 1/q + 1/p$.

Corollary 2.4.2 (Interpolation inequality). *Let $\Omega \subset \mathbb{R}^n$ for $n \geqslant 1$. For any $v \in L^p(\Omega) \cap L^q(\Omega)$*

$$\|v\|_{L^r(\Omega)} \leqslant \|v\|^{\lambda}_{L^p(\Omega)} \|v\|^{1-\lambda}_{L^q(\Omega)},$$

for $1/r = \lambda/p + (1-\lambda)/q$. In particular, for $p = 2$ and $q = 6$, $\lambda = (6-r)/r$.

The proof of theorems 2.4.1 and 2.4.2 can be found in [**?**, Chapter 2].

Theorem 2.4.3 (Sobolev embeddings). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, meaning that the boundary can be parametrized by Lipschitz functions (a precise definition is given in [**?**, p.37]). Then,*

(i) Then, if $kp < N$ and $p \geqslant 1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any

$$1 \leqslant q \leqslant p^* = \frac{Np}{N-kp}.$$

Moreover, the embedding is compact if $k > 0$ and $q < p^*$.

(ii) If $kp = N$, the space $W^{k,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, \infty)$.

(iii) If $kp > N$, then $W^{k,p}(\Omega)$ is continuously embedded in $C^{k-[N/p]-1,\nu}(\Omega)$, where [ ] denotes the integer part and

$$\nu = \begin{cases} [N/p] + 1 - N/p & \text{if } N/p \in \mathbb{Z} \\ \text{arbitrary positive number in } (0,1) & \text{if } N/p \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < [N/p] + 1 - [N/p]$.

The proof of Theorem 2.4.3 can be found in [**?**, Theorem 2.5.1.]. A direct consequence of Theorem 2.4.3 is the following theorem:

Theorem 2.4.4 (Sobolev embeddings in dual spaces). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $k > 0$ and $q < \infty$ satisfy*

$$q > \frac{p^*}{p^*-1}, \text{ where } p^* = \frac{Np}{N-kp} \text{ if } kp < N,$$
$$q > 1 \text{ for } kp = N, \text{ or } q \geqslant 1 \text{ if } kp > N.$$

Then the space $L^q(\Omega)$ is compactly embedded into the space $W^{-k,p}(\Omega)$, $1/p + 1/p = 1$.

Theorem 2.4.5 (Trace theorem and Green's formula). *Let $\Omega \subset \mathbb{R}^N$, be a bounded Lipschitz domain and $p \geqslant 1$. Then there exists a linear operator $\gamma_0 : W^{1,p}(\Omega) \mapsto W^{1-1/p,p}(\partial\Omega)$ with the following properties:*

$$[\gamma_0(v)](x) = v(x) \text{ for } x \in \partial\Omega \text{ provided } v \in \mathcal{C}^\infty(\Omega),$$

$$\|\gamma_0(v)\|_{W^{1-1/p,p}(\partial\Omega)} \leqslant c\,\|v\|_{W^{1,p}(\Omega)} \text{ for all } v \in W^{1,p}(\Omega),$$
$$\ker[\gamma_0] = W_0^{1,p}(\Omega)$$

provided $1 < p < \infty$. *Conversely, there exists a continuous linear operator* $\ell : W^{1-1/p,p}(\partial\Omega) \mapsto W^{1,p}(\Omega)$ *such that*

$$\gamma_0(\ell(v)) = v \text{ for all } v \in W^{1-1/p,p}(\partial\Omega)$$

provided $1 < p < \infty$. *In addition, the following formula holds:*

$$\int_\Omega v\partial_{x_i}u\ dx + \int_\Omega u\partial_{x_i}v\ dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\widehat{n}_i\ dx,\ i = 1, \cdots, N.$$

for any $u \in W^{1,p}(\Omega)$, $v \in W^{1,p}(\Omega)$, *where* $\widehat{n}$ *is the outer normal vector to the boundary* $\partial\Omega$.

The proof of this theorem can be found in [**?**, Theorems 5.5, 5.7].

Theorem 2.4.6 (Aubin-Lions-Simon). *Let* $X \subset B \subset Y$ *three Banach spaces such that, the embbeding* $X \subset B$ *is compact and* $B \subset Y$ *is continuous:*

(i) Let F *be bounded in* $L^p(0,T;X)$, *where* $1 \leqslant p < \infty$, *and* $\frac{\partial F}{\partial t}$ *be bounded in* $L^1(0,T;Y)$. *Then* F *is relatively compact in* $L^p(0,T;B)$.

(ii) Let F *be bounded in* $L^\infty(0,T;X)$, *and* $\frac{\partial F}{\partial t}$ *be bounded in* $L^r(0,T;Y)$, *where* $r > 1$. *Then* F *is relatively compact in* $\mathcal{C}(0,T;B)$.

A proof of Theorem 2.4.6 can be found in [**?**, Corollary 4, Section 8].

Theorem 2.4.7 (Reynolds transport theorem). *Let* $T > 0$, $\Omega \subset \mathbb{R}^3$ *and* f *in* $\mathcal{C}^1([0,T], H^1(\Omega))$ *and* $u \in C([0,T]; H^1(\Omega))$. *Let* $\omega(t)$ *an open bounded sub-domain of* Ω *and such that* $\partial\omega(t)$ *moves with velocity* u. *Then:*

$$\begin{aligned}\frac{d}{dt}\int_{\omega(t)} f(t,x)\ dx &= \int_{\omega(t)} \frac{\partial}{\partial t}f(t,x) + \frac{\partial}{\partial x_i}\left(f(t,x)u_i(t,x)\right)\ dx \\ &= \int_{\omega_\alpha(t)} \frac{d}{dt}f(t,x) + f(t,x)\,\mathrm{div}(u(t,x))\ dx, \quad (2.4)\end{aligned}$$

where

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + u\cdot\nabla f,$$

is called the material derivative of f.

The proof of the Reynolds transport theorem can be found in [**?**, p.26] and is based on the dominated convergence theorem and the existence of the partial derivative in time of the function f.

Part I

Mixture Theory

Chapter 3

A mixture theory model for dense granular materials

The mixture theory of Truesdell [?] describes the mechanical and thermodynamic behavior of a mixture using a set of equations. These equations extend the principles of continuum mechanics for a single body to a body made of two or more different materials. The main hypotheses of this theory are the following:

Continuum hypothesis
: At each instant, every point of the space is occupied simultaneously by a particle of each component.

First metaphysical principle
: Every physical property of the mixture must have consequences of the components of the mixture.

Second metaphysical principle
: To describe the dynamics of a component, we have to imagine that the component is isolated from the mixture, but we have to consider the interactions with the components.

Third metaphysical principle
: The dynamics of the mixture is governed by the same equations of a continuous medium made of only one material.

Using these principles is possible to deduce a set of equations that represents the balance of mass, momentum, angular momentum, energy and entropy of each component of the mixture. The previous hypotheses seems a natural way to extend the equations of continuum mechanics to a mixture. However, the continuum hypothesis is physically impossible and only plays the role of a mathematical tool that allows describing the dynamics of the mixture more simply. Moreover, the continuum hypothesis adds three main difficulties. The first one is to understand how the materials interact since we do not know where the frontiers of the components are. The second one is the lack of a proper constitutive equations for granular materials. The third one is that, even if we propose a constitutive equation For a granular material, we can not think that the constitutive properties of the components do not change from being isolated to being in the mixture, in other words, the constitutive properties of all the elements of the mixture are related to each other. Regarding the difficulties of using the mixture theory approach, we can say the following:

- The first difficulty is overcome by Truesdell [?], by adding a source term in the momentum equation of all the constituents, which is defined in the whole space. This term represents the drag force between each component of the mixture. This approach is useful since it allows the dynamics of the entire mixture satisfied with the Third metaphysical principle.

- Regarding the second difficulty, we refer the reader to [?], where the author gives an extensive summary of the constitutive models avaliable for granular fluids. In particular, a viscoplastic behavior is dominant in dense regime of granular fluids. Therefore several authors have proposed constitutive equations resembling a viscoplastic material. The most remarkable ones are the Drucker-Prager [?], that is an extension of the Mohr-Coulomb yield criterion, and more recently the $\mu(I)$ - rheology [?]. Both models are an extension of the Bingham constitutive equation. The Bingham constitutive equation is one of the simplest models for a viscoplastic fluid. It was proposed by Bingham [?] in 1916 and characterized by constant yield stress.

- Regarding the third difficulty, to propose constitutive equations that consider the presence of other components, most of the authors use the entropy principle which establishes that the solutions of the equations of continuum mechanics must satisfy the second law of thermodynamics. Under this restriction, using a mathematical procedure called M üller-Liu [?], it is possible to find restrictions for the constitutive functions and to propose suitable constitutive equations, see for example in the case of a mixture [?, ?] and [?, Chapter 5.8] or [?, Chapter 7] for the case of a single material. The results of applying the Müller-Liu method depend on the initial hypotheses of what are the independent variables of the problem and what balance are we considering. represents how much of a component is in every point of the space. To deal with the volume fraction, Goodman and Cowin [?] introduced the balance law of equilibrated forces, which is a dynamic equation for the volume fraction of the solid component and adds up to the balance equations of mass, linear and angular momentum, and energy. Using the Müller-Liu method, they propose a set of constitutive equations According to [?] the theory of Goodman and Cowin is moderately successful since it predicts the Mohr-Coulomb criterion and the solutions to the equations are similar to observed phenomena for such media, predicting the existence, for example, of a plug flow. Several authors have followed this approach. Some examples are [?, ?, ?], and more recently [?]. However, there exist other approaches, for example, Liu[?]. For a porous media, without considering the balance of equilibrated force equation and using the Müller-Liu method, Liu obtains different constitutive equations to the ones of Goodman and Cowin.

Concerning the above, in this chapter, we developed a new set of balance equations using homogenization tools and we applied it to model the dynamics of a granular heterogeneous flow composed by a Newtonian fluid (interstitial fluid) and $N-1$ dense granular components that follows the Drucker-Prager constitutive equation. Also, we calculated a numerical solution for Couette flow of a two component granular fluid (N=2). The outline of this chapter is the following:

- In Section 3.1, we deduce a set of partial differential equations, resembling the mixture theory equations, using a mathematical approach that utilizes the tools of the homogenization theory. In this approach, we define a set of observable scales at which we can distinguish where are located the different components of the mixture and where they interact. Then, we write the balance equations for this system. Under specific technical hypothesis, we pass to the limit, and we obtain the balance equations at a homogenized scale where the mixture satisfies the continuum hypothesis. An essential difference between our model and the classical mixture theory approach relies on that, in our case, the interaction terms are not considered as source terms, and they are included as a part of the constitutive functions. Then, our model of the constitutive functions needs to consider these interactions. We emphasize that, in our deduction, we do not use the Goodman and Cowin [**?**] balance equation of equilibrated forces. Instead we consider a saturation constraint, which entails that the sum of all volume fraction is equal to one, or in other words, the mixture is saturated.

- In section 3.2, the objective is to use the Müller-Liu procedure based on the entropy principle, to obtain a set of restrictions for the constitutive functions. One of these restrictions is the residual inequality. This inequality allows us to write constitutive equations as a sub-differential inclusion. In this context, sub-differential inclusions are a useful tool, since allows us to characterize the yielding property of granular materials accurately. In Section 3.2.4, we clarify the above point by reviewing some classic examples like the Bingham fluid equation, Drucker-Prager law and the $\mu(I)$-rheology.

- In section 3.3, we propose constitutive equations for a granular heterogeneous mixture of N components, using sub-differential inclusions. The constitutive equations models two important behaviors. First, the viscous nature of the interstitial fluid and second, the plastic behavior of the granular component. However, our equations still depend on a function called the Helmholtz free energy. The Helmholtz free energy is the difference between the internal energy and the entropy times the temperature. Then, to close the system of equations, we propose a Helmholtz free energy function for each constituent that is the sum of two parts: the first is the Helmholtz free energy function of the constituent isolated from the rest of the components and the second part is an interaction term, resembling drag energy. With this final hypothesis, we obtain the full system of equations. To test our model, we study the stationary Couette flow between two infinite concentric cylinders in the case where we have the interstitial fluid and one granular component (N=2). The result is a DAE system (Differential Algebraical Equation) that is solved using MATLAB. We included the codes in Section B of the Appendix.

- In section 3.4 we discuss the results obtained with the numerical simulation and the future prospects of this research.

3.1 Balance Laws for an heterogeneous mixture

This section aims to deduce the balance equations for a heterogeneous mixture of N components using a homogenization process where, at the initial scale, every component has a

different domain, but in the homogenized scale the equations satisfy the Continuum hypothesis and the second metaphysical principle.
These equations are not the same equations of mixture theory. However, using the homogenization approach, we can understand how the components interact. The main difference is that the interactions terms are in the Cauchy strain tensor and the heat flux of each component. Let us describe the outline of this chapter. In Section 2.1.1, we define the problem in the real scale. In Section 2.1.2, we define the homogenization process that we use to derive the general balance laws equations for the mixture in Section 2.1.4. In Section 2.1.3 we give some examples of how this theoretical procedure works with real materials. Having these examples in mind, in Section 2.3 we formulate the main hypotheses need it to pass to the limit and obtain the general balance equations for the mixture.

3.1.1 General Framework

Our deduction of the mixture theory equations is based on the definition of an observable scale using a variable $\varepsilon > 0$. This variable indicates how the components are intertwined. At this scale, the system is composed of N components. In this work, we consider that every constituent is a group of particles that have the same mechanical and thermodynamic behavior. For example, we consider that:

- The component $\alpha = 1$ is a Newtonian fluid and
- the components $\alpha = 2, \dots, N$ are other particles or solid bodies immersed in the fluid.

Each component α has a volume V_ε^α, such that:

$$V_\varepsilon^\alpha \cap V_\varepsilon^\beta = \emptyset \qquad \forall \alpha \neq \beta,$$

and

$$\bigcup_{\alpha=1}^{N} V_\varepsilon^\alpha = V,$$

where V is the volume of the mixture.

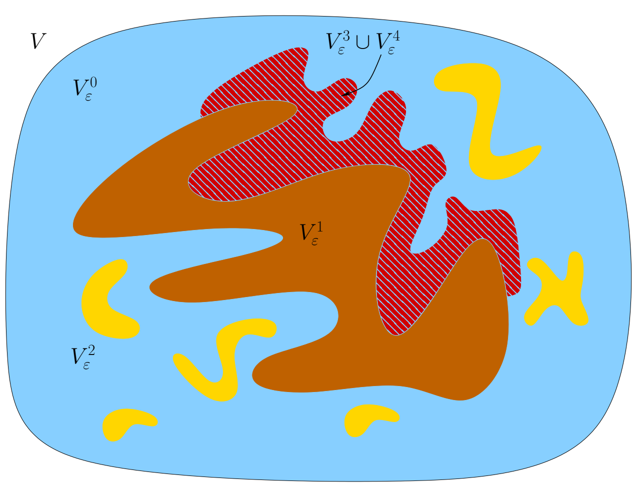

Figure 3.1: At scale ε, V is composed by 5 materials: 1 Newtonian fluid (V_0^ε) and 4 solid materials.

3.1.2 Homogenization process

The homogenization process is to take the limit when ε goes to 0. As a result of this technical process, we obtain a new material. We call this new material **homogenized continuum media**.

The question is: How to obtain the properties of the homogenized continuum media?

To answer this question, we make the first assumption:

Hypothesis 3.1. *The properties of the homogenized continuum media are the result of the properties of the heterogeneous media when ε goes to 0.*

A key ingredient, to use Hypothesis 3.1, is the characteristic function of V_ε^α defined as follows:

$$\chi_\alpha^\varepsilon(t,x) = \begin{cases} 1 & \text{if } (t,x) \in [0,T] \times V_\varepsilon^\alpha, \\ 0 & \text{if } (t,x) \in [0,T] \times (V_\varepsilon^\alpha)^c \end{cases}$$

For $\alpha \in \{1,\ldots,N\}$, the sequence $(\chi_\alpha^\varepsilon)_{\varepsilon>0}$ has the following properties:

- Since $(\chi_\alpha^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty((0,T)\times V)$, there exists a subsequence, still denoted $(\chi_\alpha^\varepsilon)_{\varepsilon>0}$, that converges weak star to a function $\theta_\alpha \in L^\infty((0,T)\times V)$. This function is such that
$$\theta_\alpha(t,x) \in [0,1] \quad \forall (t,x) \in [0,T]\times V.$$
- The function θ_α is the volume fraction of the constituent α in the homogenized media.

3.1.3 Example of a laminated material

In this section we will study the weak star convergence of the sequence $\{\chi_\alpha^\varepsilon\}_{\varepsilon\in\mathbb{R}}$ to θ_α,in the case of a laminated material composed by 2 different constituents, A and B.

Example 1:

For each $n \in \mathbb{N}\setminus\{0\}$, the material is made by n sheets of length $\frac{1}{n}$. Each of this sheets contains two sheets, one of the material A and one of the material B in the same proportion (Fig 3.2):

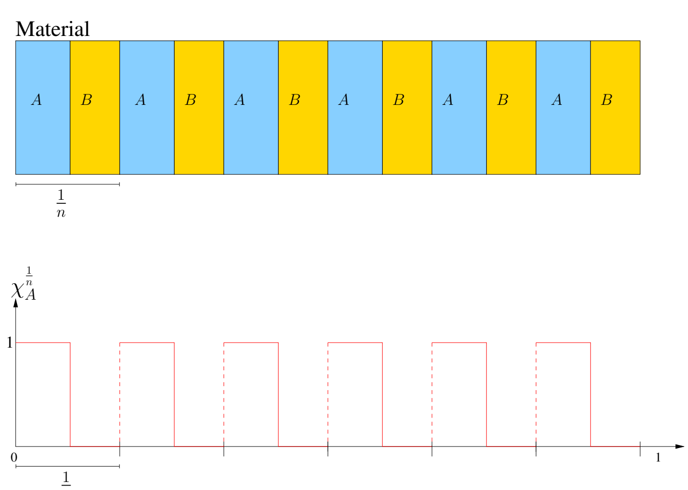

Figure 3.2: The mixture is composed by 2 different materials , A and B. At scale $\varepsilon = \frac{1}{n}$, the material is composed by n vertical sheets and each one of this are made of material A and B in the same proportion, with a length equals to $\frac{1}{2n}$.

To find the weak star limit of the above sequence we consider a continuous function f. Using Figure 3.2 we deduce that:

$$\int_0^1 \chi_A^{\frac{1}{n}}(x) f(x)\ dx = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{2n}} f(x)\ dx. \tag{3.1}$$

By the mean value theorem, there exists $\xi_k \in [\frac{k}{n}, \frac{2k+1}{n}]$ such that:

$$\int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{2n}} f(x)\ dx = f(\xi_k)\frac{1}{2n}.$$

Therefore:

$$\int_0^1 \chi_A^{\frac{1}{n}}(x) f(x)\ dx = \frac{1}{2}\sum_{k=0}^{n-1} f(\xi_k)\frac{1}{n}.$$

We take $n \to \infty$, in equation (3.1), and we obtain that:

$$\lim_{n\to\infty} \int_0^1 \chi_A^{\frac{1}{n}}(x) f(x)\ dx = \lim_{\Delta x_k \to 0} \frac{1}{2}\sum_{k=0}^{n-1} f(\xi_k)\frac{1}{n}$$

By the definition of the Riemann integral, we have that

$$\lim_{n\to\infty} \int_0^1 \chi_{\frac{1}{n}}^A(x) f(x)\ dx = \frac{1}{2}\int_0^1 f(x)\ dx.$$

Since the above relation holds for all continuous functions, we conclude that

$$\theta_A(x) = \frac{1}{2} \quad \forall x \in [0,1].$$

Example 2:
We consider a mixture where the distribution of the material A is given in the following figure:

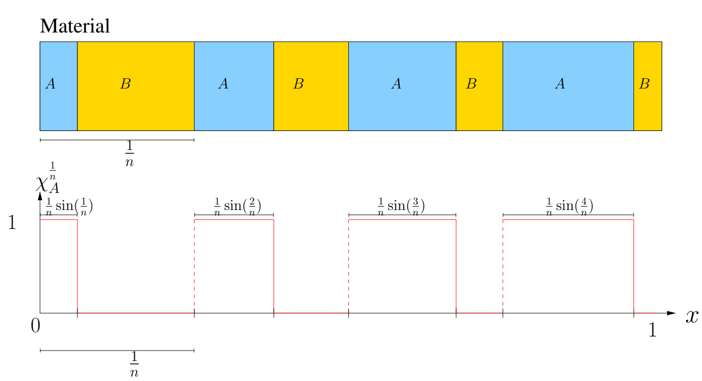

Figure 3.3: At $\varepsilon = \frac{1}{n}$ the material is made by n vertical sheets where, the sheet k of the material A has a length of $\frac{1}{n}\sin\left(\frac{k}{n}\right)$ and the sheet k of the material B has a length of $\frac{1}{n}\left(1-\sin\left(\frac{k}{n}\right)\right)$.

As in Example 1, $\chi_A^{\frac{1}{n}} \to \theta_A$ weak star in $L^\infty(0,1)$. However, θ_A is different. We take a continuous function f. Then, using Figure 3.3 we deduce that:

$$\int_0^1 \chi_A^{\frac{1}{n}}(x)f(x)\ dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k-1}{n}+\frac{1}{n}\sin\left(\frac{k}{n}\right)} f(x)\ dx. \tag{3.2}$$

By the mean value theorem, there exists $\xi_n \in \left[\frac{k-1}{n}, \frac{k-1}{n} + \frac{1}{n}\sin\left(\frac{k}{n}\right)\right]$ such that :

$$\int_{\frac{k-1}{n}}^{\frac{k-1}{n}+\frac{1}{n}\sin\left(\frac{k}{n}\right)} f(x)\ dx = f(\xi_k)\frac{1}{n}\sin\left(\frac{k}{n}\right)$$

As in Example 1, we take $n \to \infty$ in equation (3.2) and we obtain that:

$$\lim_{n\to\infty} \int_0^1 \chi_A^{\frac{1}{n}}(x)f(x)\ dx = \int_0^1 \sin(x)f(x)\ dx$$

Since the above equation holds for every continuous function f we conclude that:

$$\theta_A(x) = \sin(x) \quad x \in [0,1].$$

3.1.4 Hypotheses

Before we continue the deduction of the dynamics equations of the mixture we state a few important results, hypotheses, and notation.

Hypothesis on the weak convergence

To avoid physically meaningless sequences in the homogenization procedure, it is not enough that there exists a subsequence that converges to θ_α. An example of this is that if in the material of Example 1, we consider the sequence

$$\chi_A^{\frac{1}{n}}(x) = \begin{cases} 1 & \text{if } n \text{ is even and } x \in \bigcup\limits_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k}{n} + \frac{1}{2n}\right] \\ 0 & \text{if } n \text{ is even and } x \notin \bigcup\limits_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k}{n} + \frac{1}{2n}\right] \\ 1 & \quad\text{if } n \text{ is odd and } x \in \left[0, \frac{1}{2n}\right] \\ 0 & \quad\text{if } n \text{ is odd and } x \notin \left[0, \frac{1}{2n}\right] \end{cases},$$

the subsequence $\chi_A^{\frac{1}{2n}}(x)$ converges weakly star to $\frac{1}{2}$. However, the subsequence $\chi_A^{\frac{1}{2n+1}}(x)$ converge weakly star to 0. To avoid this, we make the following assumption:

Hypothesis 3.2. *The whole sequence χ_α^ε converges to θ_α weakly star in $L^\infty(0,T;L^\infty(V))$.*

Hypothesis on the regularity of θ_α

In the next sections we make the following assumption:

Hypothesis 3.3. *The derivatives of θ_α, $\frac{\partial\theta_\alpha}{\partial t}$ and $\frac{\partial\theta_\alpha}{\partial x_i}$, for all $i \in \{1,2,3\}$, exist and are continuous. More precisely,*

$$\theta_\alpha \in \mathcal{C}^1([0,T] \times \Omega)$$

Material derivative

In the mixture theory framework, as each homogenized component is a continuous media, there exists a function $u_\alpha : [0,T] \times V \to \mathbb{R}^3$ which is the velocity of the component α. Then, for f regular enough, we define the material derivative with respect to the component α, as follows:

$$\frac{d^\alpha}{dt} f = \frac{\partial f}{\partial t} + (u_\alpha \cdot \nabla) f. \tag{3.3}$$

3.1.5 Deduction of the dynamic equations

Saturation Constraint

At scale $\varepsilon > 0$, only one constituent occupies one point of the mixture. More precisely, for all $x \in V$, there exists only one material α such that $x \in V_\alpha^\varepsilon$:

$$\sum_{\alpha=1}^{N} \chi_\alpha^\varepsilon = 1.$$

Then for every integrable function f, we have the following:

$$\int_0^T \int_V \sum_{\alpha=1}^{N} \chi_\alpha^\varepsilon f \; dxdt = \int_0^T \int_V f \; dxdt. \tag{3.4}$$

By definition of weak star convergence, when $\varepsilon \to 0$ we have that

$$\int_0^T \int_V \sum_{\alpha=1}^{N} \chi_\alpha^\varepsilon f \; dxdt \to \int_0^T \int_V \sum_{\alpha=1}^{N} \theta_\alpha f \; dxdt.$$

Then, by equation (3.4), we have that:

$$\int_0^T \int_V \sum_{\alpha=1}^{N} \theta_\alpha f \; dxdt = \int_0^T \int_V f \; dxdt.$$

Since the above equation holds for every f we deduce the **saturation constrain**:

$$\sum_{\alpha=1}^{N} \theta_\alpha(t,x) = 1, \quad \forall (t,x) \in [0,T] \times V. \tag{3.5}$$

Mass balance

We denote by $\omega_\alpha(t) \subset \mathbb{R}^3$ a domain with velocity $u_\alpha(t)$ at instant t. Then **mass balance principle** states that the number of particles of the component α in ω_α remains constant all the time, namely that

$$\frac{d}{dt} M_{\omega_\alpha(t)} = 0,$$

where $M_{\omega_\alpha(t)}$ is the mass of the particles of the component α in $\omega_\alpha(t)$.

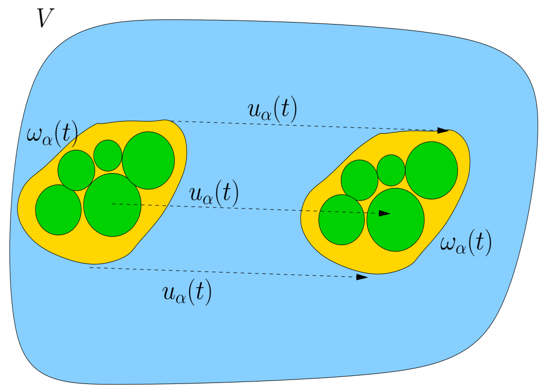

Figure 3.4: At instant t, the set $\omega_\alpha(t)$ moves along with the particles with velocity $u_\alpha(t)$.

By Hypothesis 3.1, we define $M_{\omega_\alpha(t)}$ at scale $\varepsilon > 0$ as follows:

$$M_{\omega_\alpha(t)} = \lim_{\varepsilon\to 0} M^\varepsilon_{\omega_\alpha(t)}.$$

Therefore, to compute the mass of the component α we have to compute the mass at scale $\varepsilon > 0$ and then, we take ε to 0. The mass at scale ε is:

$$M^\varepsilon_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)\cap V^\alpha_\varepsilon} \gamma_\alpha \; dx,$$

where γ_α is the real density of the material α.

Using the characteristic function of V^ε_α we can write the mass as follows:

$$M^\varepsilon_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \gamma_\alpha \chi^\varepsilon_\alpha \; dx.$$

Since $\chi^\varepsilon_\alpha \to \theta_\alpha$ weakly star in $L^\infty([0,T]\times V)$, we have the following:

$$\int_0^t M_{\omega_\alpha(t)} \; dt = \int_0^t \int_{\omega_\alpha(t)} \gamma_\alpha \theta_\alpha \; dxdt \quad \forall t \in (0,T).$$

Then we deduce

$$M_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \gamma_\alpha \theta_\alpha \; dx \quad \forall t \in (0,T).$$

Then, the mass balance equation is written as follows:

$$\frac{d}{dt}\int_{\omega_\alpha(t)} \gamma_\alpha \theta_\alpha \; dx = 0.$$

We define the **bulk density** of the component α as:

$$\rho_\alpha = \theta_\alpha \gamma_\alpha.$$

Then we write the mass balance equation as follows:

$$\frac{d}{dt}\int_{\omega_\alpha(t)} \rho_\alpha \; dx = 0.$$

Using the Reynolds transport theorem

$$\int_{\omega_\alpha(t)} \left(\frac{\partial}{\partial t}\rho_\alpha + \operatorname{div}(\rho_\alpha u_\alpha) \right) \, dx = 0.$$

As the above equation holds for every $\omega_\alpha(t)$, we conclude

$$\frac{\partial}{\partial t}\rho_\alpha + \operatorname{div}(\rho_\alpha u_\alpha) = 0.$$

We write the above equation using the material derivative of α and we conclude the local form of the mass balance equation::

$$\frac{d^\alpha}{dt}\rho_\alpha + \rho_\alpha \operatorname{div}(u_\alpha) = 0. \tag{3.6}$$

The above equation allows us to prove the following rule of calculus:

Proposition 3.1. *Let $f \in C([0,T]; L^2(V))$, ρ_α and u_α are regular solutions to the equations* (3.6)*. Then we have the following:*

$$\frac{d}{dt}\int_{\omega_\alpha(t)} \rho_\alpha f \, dx = \int_{\omega_\alpha(t)} \rho_\alpha \frac{d^\alpha}{dt} f \, dx. \tag{3.7}$$

Proof. Since

$$\frac{\partial}{\partial t}\rho_\alpha f + \operatorname{div}(\rho_\alpha u_\alpha f) = f\left(\frac{\partial \rho_\alpha}{\partial t} + \operatorname{div}(\rho_\alpha u_\alpha) \right) + \rho_\alpha \frac{\partial f}{\partial t} + \rho_\alpha u_\alpha \cdot \nabla f, \tag{3.8}$$

and considering that ρ_α and u_α are solutions of equation (3.6), we write equation (3.8) as follows:

$$\frac{\partial}{\partial t}\rho_\alpha f + \operatorname{div}(\rho_\alpha u_\alpha f) = \rho_\alpha \left(\frac{\partial f}{\partial t} + u_\alpha \cdot \nabla f \right).$$

By (3.3) we have that:

$$\frac{\partial}{\partial t}\rho_\alpha f + \operatorname{div}(\rho_\alpha u_\alpha f) = \rho_\alpha \frac{d^\alpha}{dt} f.$$

Using the above equation and the Reynolds transport theorem we deduce (3.7). □

Momentum Balance equation

The momentum balance principle states that:

$$\frac{d}{dt} P_{\omega_\alpha(t)} = F_{\omega_\alpha(t)}, \tag{3.9}$$

where $P_{\omega_\alpha(t)}$ and $F_{\omega_\alpha(t)}$ are the momentum and the external forces acting on the component α in $\omega_\alpha \alpha(t)$ respectively. By Hypothesis 3.1, to compute $P_{\omega_\alpha(t)}$ and $F_{\omega_\alpha(t)}$ we look the problem at scale $\varepsilon > 0$. Then,

$$P_{\omega_\alpha(t)} = \lim_{\varepsilon \to 0} P^\varepsilon_{\omega_\alpha(t)}$$

and

$$F_{\omega_\alpha(t)} = \lim_{\varepsilon \to 0} F^\varepsilon_{\omega_\alpha(t)}.$$

By definition, the momentum $P^{\varepsilon}_{\omega_\alpha(t)}$ is:

$$P^{\varepsilon}_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \chi^{\varepsilon}_{\alpha}\gamma_\alpha u_\alpha \, dx.$$

We take $\varepsilon \to 0$ in the above equation and we get that:

$$P_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \theta_\alpha\gamma_\alpha u_\alpha \, dx.$$

To define $F^{\varepsilon}_{\omega_\alpha(t)}$ we consider the action of 2 forces: the body forces acting on the homogenized component α, that we define using the density f_α, and the forces that are produce due to the interaction between the particles in $\omega_\alpha(t)$ and the particles outside of this domain.

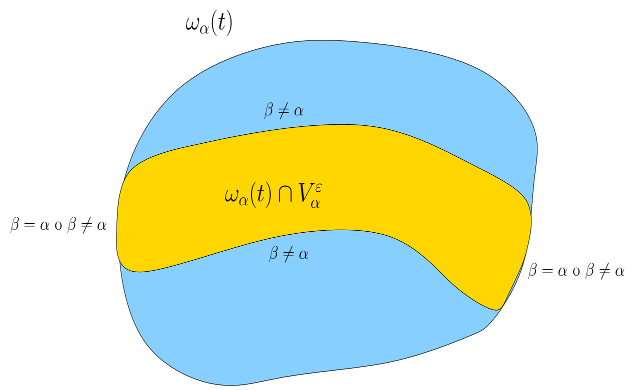

Figure 3.5: In the border of $\omega_\alpha(t) \cap V^{\varepsilon}_{\alpha}$, inside $\omega_\alpha(t)$, the component α only interacts with different components; however, in $\partial\omega_\alpha(t) \cap \partial\left(\omega_\alpha(t) \cap V^{\varepsilon}_{\alpha}\right)$ the component α interacts with other components and particles of the component α.

To define this interaction we have to consider that, at scale ε, the component α is also a continuous media and therefore, there exists a Cauchy stress tensor, see Theorem II.4 [**?**, p.41], that we call $\Sigma^{\varepsilon}_{\alpha}$. The Cauchy stress tensor represents the interaction between the component α in $\omega_\alpha(t) \cap V^{\alpha}_{\varepsilon}$ and the components outside this domain. Then $F^{\varepsilon}_{\omega_\alpha(t)}$ is written as follows:

$$F^{\varepsilon}_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)\cap V^{\alpha}_{\varepsilon}} \gamma_\alpha f_\alpha \, dx + \int_{\partial(\omega_\alpha(t)\cap V^{\alpha}_{\varepsilon})} \Sigma^{\varepsilon}_{\alpha} \cdot n ds,$$

where n is the normal vector. By the divergence theorem and using the characteristic function, we have that:

$$\int_{\partial(\omega_\alpha(t)\cap V^{\alpha}_{\varepsilon})} \Sigma^{\varepsilon}_{\alpha} \cdot n ds = \int_{\omega_\alpha(t)} \chi^{\alpha}_{\varepsilon} \operatorname{div}\left(\Sigma^{\varepsilon}_{\alpha}\right) \, dx.$$

To pass to the limit in the above equation we have to know the convergences properties of $\Sigma^{\varepsilon}_{\alpha}$. Therefore, we make the following hypothesis:

Hypothesis 3.4. *Let $\Sigma_\alpha : V \mapsto \mathbb{M}^{3\times 3}$. We assume that:*

$$\Sigma^{\varepsilon}_{\alpha} = \Sigma_\alpha.$$

Remark 3.1. *The Hypothesis 3.4 does not seem realistic in a homogenization procedure. However, in most cases, homogenization is used to calculate effective coefficients where we know the constitutive laws at a mesoscopic scale but, this type of problems has a different nature. In our case, we seek to deduce and understand the balance equations of the mixture theory through a simple homogenization procedure. More precisely, we need to deduce N systems of equations and understand how these systems interact through the volume fraction function without a micro-structure that relates the homogenized components.*

Using Hypothesis 3.4 we obtain:

$$F^{\varepsilon}_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \left(\gamma_\alpha f_\alpha + \operatorname{div}\left(\Sigma_\alpha\right)\right) \chi^{\alpha}_{\varepsilon}\ dx.$$

THen, taking $\varepsilon \to 0$ we deduce:

$$F_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \left(\gamma_\alpha f_\alpha\ dx + \operatorname{div}\left(\Sigma_\alpha\right)\right) \theta_\alpha\ dx.$$

Then, combining the above results with equation (3.9), we deduce:

$$\frac{d^\alpha}{dt}\int_{\omega_\alpha(t)} \rho_\alpha u_\alpha\ dx = \int_{\omega_\alpha(t)} \left(\gamma_\alpha f_\alpha + \operatorname{div}(\Sigma_\alpha)\right) \theta_\alpha\ dx.$$

Using the Reynolds transport theorem and the fact that the above equation holds for every $\omega_\alpha(t)$, we deduce the local form of the momentum balance equation:

$$\rho_\alpha \frac{d^\alpha}{dt} u_\alpha = \rho_\alpha f_\alpha + \theta_\alpha \operatorname{div}\left(\Sigma_\alpha\right). \tag{3.10}$$

Energy balance equation

The first **law of thermodynamics** states that:

$$\frac{d}{dt} E_{\omega_{\alpha(t)}} = W^f_{\omega_\alpha(t)} + Q_{\omega_\alpha(t)} + S_{\omega_\alpha(t)}, \tag{3.11}$$

where : E is the total energy of the system, given by the density of the internal energy, denoted by e, plus the kinetic energy; W^f is the work made by the forces applied to the system; Q is the energy dissipation produce by the heat flux, and S is an energy source. To compute the quantities in the above equation, we use Hypothesis 3.1:

$$E_{\omega_{\alpha(t)}} = \lim_{\varepsilon\to 0} E^{\varepsilon}_{\omega_{\alpha(t)}},$$

$$W^f_{\omega_\alpha(t)} = \lim_{\varepsilon\to 0} W^{f\varepsilon}_{\omega_\alpha(t)},$$

$$Q_{\omega_\alpha(t)} = \lim_{\varepsilon\to 0} Q^{\varepsilon}_{\omega_\alpha(t)}$$

and

$$S_{\omega_\alpha(t)} = \lim_{\varepsilon\to 0} S^{\varepsilon}_{\omega_\alpha(t)}.$$

For the total energy at scale ε we have that

$$E^{\varepsilon}_{\omega_{\alpha(t)}} = \int_{\omega_\alpha(t)\cap V^{\varepsilon}_{\alpha}} \gamma_\alpha \left(\frac{|u_\alpha|_2^2}{2} + e^\alpha\right)\ dx.$$

Then, taking $\varepsilon \to 0$ we have that:

$$E_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \theta_\alpha \gamma_\alpha \left(\frac{|u_\alpha|^2}{2} + e^\alpha \right) \, dx.$$

To compute $Q^\varepsilon_{\omega_\alpha(t)}$ we proceed as in the momentum equation. Since at level ε, the component α is a continuous media, we deduce the existence of a function Q^ε_α (Theorem II.4 [?, p.41]) such that:

$$Q^\varepsilon_{\omega_\alpha(t)} = - \int_{\partial(\omega_\alpha(t) \cap V^\varepsilon_\alpha)} Q^\varepsilon_\alpha \cdot n \, dx.$$

As for the Cauchy stress tensor we assume the following:

Hypothesis 3.5. *Let $Q_\alpha : V \mapsto \mathbb{R}^3$. We assume that:*

$$Q^\varepsilon_\alpha = Q_\alpha.$$

Using the divergence theorem and the characteristic function of V^ε_α we obtain:

$$\int_{\partial(\omega_\alpha(t) \cap V^\varepsilon_\alpha)} Q_\alpha \cdot n ds = \int_{\omega_\alpha(t)} \chi^\alpha_\varepsilon \operatorname{div}(Q_\alpha) \, dx.$$

Taking $\varepsilon \to 0$ in the above equation we deduce that

$$Q_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \theta_\alpha \operatorname{div}(Q_\alpha) \, dx.$$

To compute the energy produce by the mechanical forces, we have to consider the contribution of f_α and Σ_α. Then, at scale ε we have that:

$$W^{f\varepsilon}_{\omega_\alpha(t)} = \int_{\omega_\alpha(t) \cap V^\varepsilon_\alpha} \gamma_\alpha f_\alpha \cdot u_\alpha \, dx + \int_{\partial(\omega_\alpha(t) \cap V^\varepsilon_\alpha)} \Sigma_\alpha n \cdot u_\alpha ds.$$

Using the divergence theorem and the characteristic function of V^ε_α we have that:

$$\int_{\partial(\omega_\alpha(t) \cap V^\varepsilon_\alpha)} \Sigma_\alpha n \cdot u_\alpha ds = \int_{\partial\omega_\alpha(t)} \chi^\alpha_\varepsilon \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) \, ds.$$

Then,

$$W^{f\varepsilon}_{\omega_\alpha(t)} = \int_{\partial\omega_\alpha(t)} \chi^\alpha_\varepsilon \left(\gamma_\alpha f_\alpha + \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) \right) ds.$$

We take $\varepsilon \to 0$ in the above equation and we obtain

$$W^{f}_{\omega_\alpha(t)} = \int_{\partial\omega_\alpha(t)} \theta_\alpha \left(\gamma_\alpha f_\alpha + \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) \right) ds.$$

The last term is the work made by the external supply of energy that is defined by an energy density r_α as follows:

$$S^\varepsilon_{\omega_\alpha(t)} = \int_{\omega_\alpha(t) \cap V^\varepsilon_\alpha} \gamma_\alpha r_\alpha \, dx.$$

Using the characteristic function of V_ε^α and taking $\varepsilon \to 0$ we obtain:

$$S_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \theta_\alpha \gamma_\alpha r_\alpha \; dx$$

Then equation (3.11) can be written as:

$$\frac{d}{dt}\int_{\omega_\alpha(t)} \theta_\alpha \gamma_\alpha \left(\frac{|u_\alpha|^2}{2} + e^\alpha \right) \; dx = \int_{\omega_\alpha(t)} \left(\theta_\alpha \gamma_\alpha f_\alpha \cdot u_\alpha + \theta_\alpha \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) + \theta_\alpha \gamma_\alpha r_\alpha - \theta_\alpha \operatorname{div}(Q_\alpha) \right) \; dx.$$

Using the Reynolds transport theorem we write the above equation as follows:

$$\int_{\omega_\alpha(t)} \theta_\alpha \gamma_\alpha \frac{d^\alpha}{dt} \left(\frac{|u_\alpha|^2}{2} + e^\alpha \right) \; dx = \int_{\omega_\alpha(t)} \left(\theta_\alpha \gamma_\alpha f_\alpha \cdot u_\alpha + \theta_\alpha \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) + \theta_\alpha \gamma_\alpha r_\alpha - \theta_\alpha \operatorname{div}(Q_\alpha) \right) \; dx.$$

Since the above equation holds for every $\omega_\alpha(t)$ we have:

$$\theta_\alpha \gamma_\alpha \frac{d}{dt} \frac{|u_\alpha|^2}{2} + \theta_\alpha \gamma_\alpha \frac{d}{dt} e^\alpha = \theta_\alpha \gamma_\alpha f_\alpha \cdot u_\alpha + \theta_\alpha \operatorname{div}(\Sigma_\alpha \cdot u_\alpha) + \theta_\alpha \gamma_\alpha r_\alpha - \theta_\alpha \operatorname{div}(Q_\alpha)$$

We rewrite the above equation using the bulk density ρ_α as follows:

$$\rho_\alpha \frac{d}{dt} \frac{|u_\alpha|_2^2}{2} + \theta_\alpha \gamma_\alpha \frac{d}{dt} e^\alpha = \rho_\alpha f_\alpha \cdot u_\alpha + \theta_\alpha \operatorname{div}\left(\Sigma_\alpha \cdot u_\alpha\right) - \theta_\alpha \operatorname{div}\left(Q_\alpha\right) + \rho_\alpha r_\alpha. \tag{3.12}$$

Using

$$\frac{1}{2}\frac{d^\alpha}{dt} |u_\alpha|^2 = u_\alpha \cdot \frac{d^\alpha}{dt} u_\alpha,$$

$$\operatorname{div}(\Sigma_\alpha \cdot u_\alpha) = u_\alpha \operatorname{div}(\Sigma_\alpha) + \Sigma_\alpha : \nabla u_\alpha$$

and equation (3.12), we deduce

$$\rho_\alpha \frac{d^\alpha}{dt} e_\alpha + \theta_\alpha \operatorname{div}(Q_\alpha) - \Sigma_\alpha : \nabla u_\alpha - \rho_\alpha r_\alpha = \left(\rho_\alpha \frac{d^\alpha}{dt} u_\alpha - \theta_\alpha \operatorname{div}(\Sigma_\alpha) - \rho_\alpha f_\alpha \right) \cdot u_\alpha.$$

Finally, using the momentum balance equation (3.10) we deduce the local form of the balance equation of internal energy:

$$\rho_\alpha \frac{d^\alpha}{dt} e_\alpha = \theta_\alpha \Sigma_\alpha : \nabla u_\alpha - \theta_\alpha \operatorname{div}(Q_\alpha) + \rho_\alpha r_\alpha. \tag{3.13}$$

Second law of thermodynamics

The first law of thermodynamics establishes the equivalence between the heat and the work but do not specify how one is transformed into the other. For example, it is not forbidden that the heat flows from a colder body to a hotter one. The role of the second law of thermodynamics is to put limitations on how the heat flows and to reflect the direction of the process in nature [?]. Clausius in 1850 proposed the first equation of the second law of thermodynamics. This equation establishes heat cannot spontaneously flow from cold regions to hot regions without external work being performed on the system. The formulation of this principle is based on the Clausius-Planck inequality:

$$\triangle S \geqslant \int \frac{dQ}{T},$$

where $\triangle S$ is the variation of the entropy of a system between two equilibrium states, dQ is the heat transfer through the boundary of the body and T is the absolute temperature. For a continuum media of the density ρ under the action of an energy source defined by the density of energy r, which occupies a volume V, the Clausius-Planck equation is written as follows:

$$\frac{d}{dt}\int_V \rho s \; dx \leqslant -\int_{\partial V} \frac{Q}{T}\cdot n ds + \int_V \rho\frac{r}{T}\; dx, \tag{3.14}$$

where s is a density of entropy and Q is the heat flux. The quantity

$$\frac{d}{dt}\int_V \rho s \; dx + \int_{\partial V} \frac{Q}{T}\cdot n ds - \int_V \rho\frac{r}{T}\; dx$$

is the entropy production. Then, the inequality (3.14) states that the entropy production is always positive.
To deduce a similar inequality, in the mixture theory framework, there are two possible approaches:

(a) The first one is to consider that each component of the mixture satisfies an entropy inequality like (3.14)

(b) and the second one is to consider that the mixture as a whole satisfies an entropy inequality like (3.14).

According to Liu [?], the approach (a) is too restrictive and gives unrealistic results. On the other hand, the approach (b), allows that the entropy production of each constituent to be negative. However, the entropy production of the whole mixture has to be positive. In this work, we follow the second approach.

The entropy inequality for a sub-domain $\omega(t)$ is:

$$\frac{d}{dt}S_{\omega(t)} \leqslant \int_{\omega(t)} \frac{dQ}{T}\; dx, \tag{3.15}$$

where $S_{\omega(t)}$ is the entropy of the system in $\omega(t)$ and $\int_{\omega(t)} \frac{dQ}{T}$ is the heat transfer divided by the temperature. We consider that

$$\omega(t) = \bigcup_{\alpha=1}^{N} \omega_\alpha(t) \text{ for all } t \in [0, T],$$

where, as in the previous sections, $\omega_\alpha(t)$ is a domain with velocity u_α. Since $S_{\omega(t)}$ is an extensive variable, we define $S_{\omega(t)}$ as follows:

$$S_{\omega(t)} = \sum_{\alpha=1}^{N} S^\alpha_{\omega_\alpha(t)},$$

where $S^\alpha_{\omega_\alpha(t)}$ is the entropy of the α component in $\omega_\alpha(t)$. By Hypothesis 3.1, we define the entropy using the scale ε as follows:

$$S^\alpha_{\omega_\alpha(t)} = \lim_{\varepsilon\to 0} S^{\alpha\varepsilon}_{\omega_\alpha(t)},$$

where

$$S^{\alpha\varepsilon}_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)\cap V^\varepsilon_\alpha} \gamma_\alpha s_\alpha \; dx,$$

where s_α is the entropy density of the constituent α. Then,

$$S^\alpha_{\omega_\alpha(t)} = \int_{\omega_\alpha(t)} \theta_\alpha \gamma_\alpha s_\alpha \; dx$$

and the total entropy of the system is

$$S_{\omega(t)} = \sum_{\alpha=1}^{N} \int_{\omega_\alpha(t)} \rho_\alpha s_\alpha \; dx.$$

We define the heat transfer divided by the absolute temperature $\int_{\omega(t)} \frac{dQ}{T}$ as the sum of these quantities for each component, i.e.,

$$\int_{\omega(t)} \frac{dQ}{T} = \sum_{\alpha=1}^{N} \left(\int_{\omega(t)} \frac{dQ}{T} \right)_\alpha.$$

By Hypothesis (3.1), we define the heat transfer as follows:

$$\left(\int_{\omega(t)} \frac{dQ}{T} \right)_\alpha = \lim_{\varepsilon\to 0} \left(\int_{\omega(t)} \frac{dQ}{T} \right)_{\alpha,\varepsilon}.$$

For each component α the heat transfer divided by the temperature at scale ε is

$$\left(\int_{\omega(t)} \frac{dQ}{T} \right)_{\alpha,\varepsilon} = \int_{\omega_\alpha(t)\cap V^\varepsilon_\alpha} \frac{r_\alpha}{T_\alpha} \; dx - \int_{\partial(\omega_\alpha(t)\cap V^\varepsilon_\alpha)} \frac{Q_\alpha}{T_\alpha} \cdot n ds, \tag{3.16}$$

where Q_α is the heat flux of the component α, $T_\alpha$is the absolut temperature of the component α and r_α is an external energy source. Then, using the divergence theorem and the identity

$$\operatorname{div}\left(\frac{Q_\alpha}{T_\alpha}\right) = \frac{\operatorname{div}(Q_\alpha)}{T_\alpha} - \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2},$$

we write (3.16) as follows,

$$\int_{\partial(\omega_\alpha(t)\cap V^\varepsilon_\alpha)} \frac{Q_\alpha}{T} ds = \int_{\omega_\alpha(t)\cap V^\varepsilon_\alpha} \frac{\operatorname{div}(Q_\alpha)}{T_\alpha} - \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2} \; dx.$$

Then,

$$\left(\int \frac{dQ}{T}\right)_{\alpha\varepsilon} = \int_{\omega_\alpha(t)} \chi_\alpha^\varepsilon \left(\frac{r_\alpha}{T_\alpha} + \frac{\operatorname{div}(Q_\alpha)}{T_\alpha} - \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right)\, dx.$$

We take $\varepsilon \to 0$ and we obtain:

$$\int_{\omega(t)} \frac{dQ}{T} = \sum_{\alpha=1}^{N} \int_{\omega_\alpha(t)} \theta_\alpha \left(\frac{r_\alpha}{T_\alpha} + \frac{\operatorname{div}(Q_\alpha)}{T_\alpha} - \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right)\, dx.$$

Therefore, by (3.15),

$$\sum_{\alpha=1}^{N} \frac{d}{dt} \int_{\omega_\alpha(t)} \rho_\alpha s_\alpha \, dx \geqslant \sum_{\alpha=1}^{N} \int_{\omega_\alpha(t)} \frac{\rho_\alpha r_\alpha}{T_\alpha} - \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T_\alpha} - \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right)\, dx.$$

Using the Reynolds transport theorem we get the following inequality:

$$\sum_{\alpha=1}^{N} \int_{\omega_\alpha(t)} \rho_\alpha \frac{d^\alpha}{dt} s_\alpha - \frac{\rho_\alpha r_\alpha}{T_\alpha} + \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T_\alpha} + \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right)\, dx \geqslant 0.$$

Then we deduce

$$\int_{\omega(t)} \sum_{\alpha=1}^{N} \rho_\alpha \frac{d^\alpha}{dt} s_\alpha - \frac{\rho_\alpha r_\alpha}{T_\alpha} + \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T_\alpha} + \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right)\, dx \geqslant 0,$$

and, since the above inequality holds for every $\omega(t)$, we obtain,

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{d^\alpha}{dt} s_\alpha - \frac{\rho_\alpha r_\alpha}{T_\alpha} + \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T_\alpha} + \frac{Q_\alpha \cdot \nabla T_\alpha}{(T_\alpha)^2}\right) \geqslant 0. \tag{3.17}$$

The above inequality is equivalent to the one propose by Liu and Coll [**?**].

Remark 3.2. *The nature of the problem allows us to assume that the absolute temperature of every component is the same. More precisely,*

$$T_\alpha = T \quad \forall \alpha \in \{1, \dots, N\},$$

where T is the absolute temperature of the mixture. Then, we write the second law of thermodynamics as follows:

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{d^\alpha}{dt} s_\alpha - \frac{\rho_\alpha r_\alpha}{T} + \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T} + \frac{Q_\alpha \cdot \nabla T}{T^2}\right) \geqslant 0.$$

3.2 Constitutive equations

This chapter aims to obtain an equation for the Cauchy stress tensor in terms of a predefined set of dependent variables. These types of equations are called constitutive equations and are the root of the mathematical theory that studies the properties of materials. To obtain constitutive equations, that satisfies phenomenological properties of the materials, we use two primary tools: **the entropy principle** and **sub-differential equations**. In a broader sense, the entropy principle states that not every constitutive equation are admissible. On the other hand, we relate the Cauchy stress tensor to the sub-differential of a function that we called potential. The form of the potential express the mechanical and thermodynamical properties of the material. For example, in the framework of the mixture theory, with these tools, we can construct constitutive equations that consider how the existence of other constituents in the mixture affects the Cauchy stress tensor of one constituent and, how we express a yielding condition that is the main characteristic of the dense regime of a granular material[**?**]. Let us describe the outline of this part. In Section 3.1, we introduce the general framework and hypotheses that will we use to, in section 3.2, apply the entropy principle in the mixture theory framework. At the end of section 3.2, using the entropy principle, we derive a set of equations and an inequality known as the Liu's identities. In section 3.3, using these equations we formulated a sub-differential equation for the Cauchy stress tensors of all the constituents and, in section 3.4, we deduce some classical constitutive equations.

3.2.1 General settings

Under the theory developed in Chapter 3.1 we have that: given f_α and r_α, densities of body forces and energy supply, acting on the constituent α of a mixture of N components, occupying a bounded volume $V \subset \mathbb{R}^3$ in a homogenized sense, the balance equations that describes the dynamics of the constituent α are:

$$\frac{d^\alpha}{dt}\rho_\alpha + \rho_\alpha \operatorname{div}(u_\alpha) = 0, \tag{3.18}$$

$$\rho_\alpha \frac{d^\alpha}{dt} u_\alpha - \rho_\alpha f_\alpha - \theta_\alpha \operatorname{div}(\Sigma_\alpha) = 0 \tag{3.19}$$

and

$$\rho_\alpha \frac{d^\alpha}{dt} e_\alpha - \theta_\alpha \Sigma_\alpha : \nabla u_\alpha + \theta_\alpha \operatorname{div}(Q_\alpha) - \rho_\alpha r_\alpha = 0, \tag{3.20}$$

for all $\alpha \in \{1 \ldots, N\}$. The volume fraction functions of the constituents satisfy the saturation constraint:

$$\sum_{\beta=1}^{N} \theta_\beta = 1. \tag{3.21}$$

In what follows we will assume that γ_α is a constant. The above system is formed by $5N+1$ equations and $14N+1$ unknowns, namely θ_α, u_α, Σ_α, e_α, Q_α and T. To solve the set of equations (3.18)–(3.21), we need to formulate more equations to close the system. These equations are called constitutive equations and describe the material properties of the continuum media. The formulation of constitutive equations is the root of material theory, and the first step to developing them is to choose a set of independent variables.

In classical continuum mechanics, one may choose as the independent variables: the density, the velocity, and the absolute temperature or the internal energy. In the case of a mixture, we choose the volume fraction θ_α, the symmetric gradient of the velocity $D(u_\alpha)$ ($\alpha \in \{1, \ldots, N\}$) and the absolute temperature T. Then, to propose constitutive equations we need to find expressions for the dependent functions Σ_α, Q_α and e_α, in terms of the independent variables θ_α, $D(u_\alpha)$ and T. Therefore, we need to make the following hypothesis:

Hypothesis 3.6. *We denote*

$$\chi = \bigcup_\alpha \{\theta_\alpha, u_\alpha, D(u_\alpha)\} \bigcup \{T, \nabla T\}\,.$$

Then, there exist functions $\widehat{\Sigma}_\alpha$, $\widehat{e}_\alpha$ and $\widehat{Q}_\alpha$ such that

$$\Sigma_\alpha = \widehat{\Sigma}_\alpha(\chi), \tag{3.22}$$

$$e_\alpha = \widehat{e}_\alpha(\chi) \tag{3.23}$$

and

$$Q_\alpha = \widehat{Q}_\alpha(\chi). \tag{3.24}$$

The functions $\widehat{\Sigma}_\alpha$, $\widehat{Q}_\alpha$ and $\widehat{e}_\alpha$ are called response functions.

The choice of χ is the first constitutive assumption. The elements in this set represent the general material properties of the continuum media. For example, in the case of $N = 1$, we can represent a heat-conducting and viscous fluid by choosing $\chi = \{\rho, T, \nabla T, D(u)\}$, where ρ is the fluid density. By the same procedure that it will be used further in this chapter, we can deduce:

$$\widehat{\Sigma}\left(\rho, T, D\left(u\right)\right) = \left(-p + \lambda \operatorname{tr}(D(u))\right) I + 2\mu(D(u) - \frac{1}{3}\operatorname{tr}(D(u)), \tag{3.25}$$

and

$$\widehat{Q}\left(\nabla T\right) = -\kappa \nabla T, \tag{3.26}$$

p is the pressure function, λ is the compressibility constant, μ is the viscosity and κ is a heat conducting constant. The balance laws (3.18)–(3.5) with equation (3.25) and (3.26) are known as the Navier-Stokes-Fourier system. The first interpretation of equation (3.25) is that the material experiments internal friction due to the relative motion and an intrinsic internal compression. Equation 3.26 tell us that the heat flows from hot to cold.

3.2.2 Entropy Principle

The choice of χ is not enough to propose a well-defined set of constitutive equations since we could choose any type of dependence between the constitutive functions and the set of variables χ, regardless if they have physical meaning or not. Then, the first restriction that constitutive equations have to pass is the second law of thermodynamics, which is known as the **the entropy principle**.

The entropy principle states that every admissible set of independent variables satisfy the second law of thermodynamics. More precisely, a solution $(\theta_\alpha, u_\alpha, T)$ of the system of equations (3.18)–(3.21) satisfies the entropy inequality:

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{d^\alpha}{dt} s_\alpha + \sum_{\alpha=1}^{N} \theta_\alpha \left(\frac{\operatorname{div}(Q_\alpha)}{T} - \frac{\nabla T \cdot Q_\alpha}{T^2} \right) - \sum_{\alpha=1}^{N} \rho_\alpha \frac{r_\alpha}{T} \geqslant 0. \tag{3.27}$$

Since we introduce the entropy s_α, we assume the following:

Hypothesis 3.7. *There exists a response function of s_α that we call $\widehat{s}_\alpha$, such that*

$$s_\alpha = \widehat{s}_\alpha(\chi). \tag{3.28}$$

In what follows of this chapter we refer to $\widehat{\Sigma}_\alpha$, $\widehat{Q}_\alpha$, $\widehat{e}_\alpha$ and $\widehat{s}_\alpha$ as Σ_α, Q_α, e_α and s_α.

To use the entropy principle, we will reduce the number of equations using the Clausius–Duhem approach. This approach combines the energy balance equation (3.20) and the entropy inequality (3.27) to obtain a new inequality called **the Clausius–Duhem inequality**. To obtain the Clausius–Duhem inequality, we proceed as follows:

- First, we multiply the entropy inequality (3.27) by T and we obtain:

$$\sum_{\beta=1}^{N} \rho_\beta T \frac{d^\beta}{dt} s_\beta + (\theta_\beta \operatorname{div}(Q_\beta) - \rho_\beta r_\beta) - \theta_\beta \frac{\nabla T \cdot Q_\beta}{T} \geqslant 0. \tag{3.29}$$

- Adding up equations (3.20) we deduce:

$$\sum_{\beta=1}^{N} \theta_\beta \operatorname{div}(Q_\beta) - \rho_\beta r_\beta = \sum_{\beta=1}^{N} -\rho_\beta \frac{d^\beta}{dt} e_\beta + \theta_\beta \Sigma_\beta : \nabla u_\beta. \tag{3.30}$$

- We combine (3.29) and (3.30), and we obtain:

$$\sum_{\beta=1}^{N} \rho_\beta T \frac{d^\beta}{dt} s_\beta - \rho_\beta \frac{d^\beta}{dt} e_\beta + \theta_\beta \Sigma_\beta : \nabla u_\beta - \theta_\beta \frac{\nabla T \cdot Q_\beta}{T} \geqslant 0.$$

- Finally, we combine the material derivatives of s_α and e^α in the above equation and we obtain the Clausius-Duhem inequality for the mixture:

$$\sum_{\beta=1}^{N} \rho_\beta \frac{d^\beta}{dt} (T s_\beta - e_\beta) - \rho_\beta s_\beta \frac{d^\beta}{dt} T + \theta_\beta \Sigma_\beta : \nabla u_\beta - \theta_\beta \frac{\nabla T \cdot Q_\beta}{T} \geqslant 0. \tag{3.31}$$

  We rewrite the above inequality as follows:

$$\sum_{\beta=1}^{N} -\rho_\beta \frac{d^\beta}{dt} \psi_\beta - s_\beta \rho_\beta \frac{d^\beta}{dt} T + \theta_\beta \Sigma_\beta : \nabla u_\beta - \theta_\beta \frac{\nabla T \cdot Q_\beta}{T} \geqslant 0, \tag{3.32}$$

  where

$$\psi_\beta = e_\beta - T s_\beta$$

  is the Helmholtz free energy function of the constituent β.

With the Clausius-Duhem inequality, the problem of finding restrictions to the constitutive functions from the balance equations and the entropy inequality is reduced to obtaining restrictions from the saturation constraint, the mass balance equation, and the Clausius-Duhem inequality. We note that **in this approach, we do not need to consider the momentum balance equation since we already use it to obtain the expression of the energy equation in the deduction of the balance laws**.

The following step is to write (3.18), (3.21) and the Clausius-Duhem inequality (3.32) in terms of the independent variables χ. To do this we consider the following:

(a) We decompose the gradient of the velocity ∇u_α as the sum of two tensors:

$$\nabla u_\alpha = D(u_\alpha) + L(u_\alpha), \tag{3.33}$$

where $L(u_\alpha)$ is the skew symmetric part of the velocity gradient:

$$L(u_\alpha) = \frac{1}{2}\left(\nabla u_\alpha - (\nabla u_\alpha)^*\right).$$

This decomposition tells us that a part of ∇u_α is an independent variable. We recall that, since Σ_α is a symmetric matrix and $L(u_\alpha)$ is a skew-symmetric matrix, then:

$$\Sigma_\alpha : L(u_\alpha) = 0. \tag{3.34}$$

(b) By (3.3)

$$\frac{d^\alpha}{dt}\theta_\alpha = \frac{\partial}{\partial t}\theta_\alpha + u_\alpha \cdot \nabla\theta_\alpha, \tag{3.35}$$

$$\frac{d^\alpha}{dt}u_\alpha = \frac{\partial}{\partial t}u_\alpha + (u_\alpha \cdot \nabla)u_\alpha \tag{3.36}$$

and

$$\frac{d^\alpha}{dt}T = \frac{\partial}{\partial t}T + u_\alpha \cdot \nabla T. \tag{3.37}$$

We combine equation (3.33) and (3.36) and we obtain

$$\frac{d^\alpha}{dt}u_\alpha = \frac{\partial}{\partial t}u_\alpha + D(u_\alpha)\cdot u_\alpha + L(u_\alpha)\cdot u_\alpha. \tag{3.38}$$

(c) We write the material derivative of ψ_β as:

$$\frac{d^\beta}{dt}\psi_\beta = \sum_{\eta=1}^{N}\left(\frac{\partial\psi_\beta}{\partial\theta_\eta}\frac{d^\beta}{dt}\theta_\eta + \frac{\partial\psi_\beta}{\partial u_{\eta,k}}\frac{d^\beta}{dt}u_{\eta,k} + \frac{\partial\psi_\beta}{\partial D(u_\eta)_{kh}}\frac{d^\beta}{dt}D(u_\eta)_{kh}\right) + \frac{\partial\psi_\beta}{\partial T}\frac{d^\beta}{dt}T + \frac{\partial\psi_\beta}{\partial\frac{\partial T}{\partial x_k}}\frac{d^\beta}{dt}\frac{\partial T}{\partial x_k}.$$

Then, using (3.35), (3.36), (3.37) and (3.38) we obtain:

$$\begin{aligned}\frac{d^\beta}{dt}\psi_\beta = \sum_{\eta=1}^{N}\Bigg(&\frac{\partial\psi_\beta}{\partial\theta_\eta}\left(\frac{\partial}{\partial t}\theta_\eta + u_{\beta,j}\frac{\partial}{\partial x_j}\theta_\eta\right) + \frac{\partial\psi_\beta}{\partial u_{\eta,k}}\left(\frac{\partial}{\partial t}u_{\eta,k} + u_{\beta,j}\frac{\partial}{\partial x_j}u_{\eta,k}\right)\\ &+ \frac{\partial\psi_\beta}{\partial D(u_\eta)_{kh}}\left(\frac{\partial}{\partial t}D(u_\eta)_{kh} + u_{\beta,j}\frac{\partial}{\partial x_j}D(u_\eta)_{kh}\right)\Bigg) + \frac{\partial\psi_\beta}{\partial T}\left(\frac{\partial}{\partial t}T + u_{\beta,j}\frac{\partial}{\partial x_j}T\right)\\ &+ \frac{\partial\psi_\beta}{\partial\frac{\partial T}{\partial x_k}}\left(\frac{\partial}{\partial t}\frac{\partial T}{\partial x_k} + u_{\beta,j}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_k}\right).\end{aligned}$$

Using equation (3.33) we deduce the final form of the material derivative of ψ_β:

$$\frac{d^\beta}{dt}\psi_\beta = \frac{\partial \psi_\beta}{\partial T}\left(\frac{\partial}{\partial t}T + u^{\beta,j}\frac{\partial}{\partial x_j}T\right) + \frac{\partial \psi_\beta}{\partial \frac{\partial T}{\partial x_k}}\left(\frac{\partial}{\partial t}\frac{\partial T}{\partial x_k} + u_{\beta,j}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_k}\right)$$
$$+ \sum_{\eta=1}^{N}\left(\frac{\partial \psi_\beta}{\partial \theta_\eta}\left(\frac{\partial}{\partial t}\theta_\eta + u_{\beta,j}\frac{\partial}{\partial x_j}\theta_\eta\right)\right.$$
$$+ \frac{\partial \psi_\beta}{\partial u_{\eta,k}}\left(\frac{\partial}{\partial t}u_{\eta,k} + u_{\beta,j}D(u_\eta)_{kj} + u_{\beta,j}L(u_\eta)_{kj}\right)$$
$$\left. + \frac{\partial \psi_\beta}{\partial D(u_\eta)_{kh}}\left(\frac{\partial}{\partial t}D(u_\eta)_{kh} + u_{\beta,j}\frac{\partial}{\partial x_j}D(u_\eta)_{kh}\right)\right). \quad (3.39)$$

To write the mass balance equation (3.18), the saturation constraint (3.21) and the Clausius-Duhem inequality (3.32) in terms of the independent variables χ we proceed as follows:

- Since $\rho_\alpha = \gamma_\alpha\theta_\alpha$ and $\mathrm{div}(u_\alpha) = \mathrm{tr}(D(u_\alpha)) = I : D(u_\alpha)$, where I is the identity matrix, using equation (3.35) we write the (3.18) as

$$\gamma_\alpha\frac{\partial}{\partial t}\theta_\alpha + \gamma_\alpha u_\alpha\cdot\nabla\theta_\alpha + \gamma_\alpha\theta_\alpha I : D(u_\alpha) = 0. \quad (3.40)$$

- Using equations (3.37) and (3.39) we write the (3.32) as

$$\sum_{\beta=1}^{N}\sum_{\eta=1}^{N}\left(-\rho_\beta\frac{\partial \psi_\beta}{\partial \theta_\eta}\left(\frac{\partial}{\partial t}\theta_\eta + u_{\beta,j}\frac{\partial}{\partial x_j}\theta_\eta\right)\right.$$
$$-\rho_\beta\frac{\partial \psi_\beta}{\partial u_{\eta,k}}\left(\frac{\partial}{\partial t}u_{\eta,k} + u_{\beta,j}D(u_\eta)_{kj} + u_{\beta,j}L(u_\eta)_{kj}\right)$$
$$\left.-\rho_\beta\frac{\partial \psi_\beta}{\partial D(u_\eta)_{kh}}\left(\frac{\partial}{\partial t}D(u_\eta)_{kh} + u_{\beta,j}\frac{\partial}{\partial x_j}D(u_\eta)_{kh}\right)\right)$$
$$-\rho_\beta\frac{\partial \psi_\beta}{\partial T}\left(\frac{\partial}{\partial t}T + u_{\beta,j}\frac{\partial}{\partial x_j}T\right) - \rho_\beta\frac{\partial \psi_\beta}{\partial \frac{\partial T}{\partial x_k}}\left(\frac{\partial}{\partial t}\frac{\partial T}{\partial x_k} + u_{\beta,j}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_k}\right)$$
$$-\rho_\beta s_\beta\left(\frac{\partial}{\partial t}T + u_{\beta,j}\frac{\partial}{\partial x_j}T\right) + \theta_\beta\Sigma_\beta : D(u_\beta) - \theta_\beta\frac{\nabla T\cdot Q_\beta}{T} \geqslant 0. \quad (3.41)$$

- To deal with the **saturation constraint** we replace (3.21) by the following equations:

$$\sum_{\beta=1}^{N}\frac{\partial \theta_\beta}{\partial t} = 0 \quad (3.42)$$

and

$$\sum_{\beta=1}^{N}\nabla\theta_\beta = 0. \quad (3.43)$$

In what follows in this chapter we refer to the saturation constraint to the equations (3.42) and (3.43).

The following step is to identify which functions in (3.40), (3.41), (3.42) and (3.43) are not functions in χ. These functions are: $\frac{\partial\theta_\alpha}{\partial t}$, $\frac{\partial\theta_\alpha}{\partial x_i}$, $\frac{\partial u_\alpha}{\partial t}$, $L(u_\alpha)_{ij}$, $\frac{\partial}{\partial t}D\left(u_\alpha\right)_{ij}$, $\frac{\partial}{\partial x_k}D(u_\alpha)_{ij}$, $\frac{\partial^2}{\partial t\partial x_k}T$ and $\frac{\partial}{\partial t}T$ and $\frac{\partial^2 T}{\partial x_i\partial x_j}$, for every $\alpha\in\{1,\dots,N\}$. Then if we call the vector:

$$\mathbb{X}=\Big(\frac{\partial\theta_\alpha}{\partial t},\frac{\partial\theta_\alpha}{\partial x_i},\frac{\partial u_\alpha}{\partial t},L(u_\alpha)_{ij},\frac{\partial}{\partial t}D\left(u_\alpha\right)_{ij},\frac{\partial}{\partial x_k}D(u_\alpha)_{ij},\frac{\partial^2}{\partial t\partial x_k}T,\frac{\partial}{\partial t}T,\frac{\partial^2 T}{\partial x_i\partial x_j}\Big)^*,$$

we write equations (3.40), (3.42) and (3.43) as a system of equations:

$$A\left(\chi\right)\mathbb{X}+B\left(\chi\right)=0, \tag{3.44}$$

where $A\left(\chi\right)$ is a matrix and $B\left(\chi\right)$ is a vector and they only contain elements of χ. Similarly, we write inequality (3.41) as:

$$c\left(\chi\right)^T\mathbb{X}+d\left(\chi\right)\geqslant 0, \tag{3.45}$$

where $c\left(\chi\right)$ and $d\left(\chi\right)$ are vectors that only contain elements of χ.
Then, we have proved the following version of the **the entropy principle**:

Theorem 3.1 (Entropy Principle). *Let Σ_α, e_α, Q_α and s_α satisfying hypotheses 3.6 and 3.7, and $(\theta_1,\cdots,\theta_N,u_1,\cdots,u_N,T)$ be a solution of the system of equations* (3.18)–(3.21) *such that inequality* (3.27) *holds. Then*

$$\chi=\bigcup_\alpha\{\theta_\alpha,u_\alpha,D(u_\alpha)\}\bigcup\{T,\nabla T\}$$

satisfies the equation (3.44) *and the inequality* (3.45).

To exploit Theorem 3.1, we need to use the following algebraical theorem:

Theorem 3.2. *Let $A\in\mathbb{M}^{p\times N}$, $B\in\mathbb{R}^p$ such that there exists at least one solution of the equation.*

$$A\mathbb{X}+B=0.$$

Let $S\neq\emptyset$ a subset of $\mathbb{R}^N$ containing the solutions of (3.2). *Let $\alpha\in\mathbb{R}^N$ and $\beta\in\mathbb{R}$. Then, the following are equivalent:*

1. $$c^T\mathbb{X}+d\geqslant 0\quad\forall\mathbb{X}\in S.$$

2. *There exists $\Lambda\in\mathbb{R}^p(\neq 0)$ such that:*

$$c^T\mathbb{X}+d-\Lambda(A\mathbb{X}+B)\geqslant 0\quad\forall\mathbb{X}\in\mathbb{R}^N. \tag{3.46}$$

3. *There exists $\Lambda\in\mathbb{R}^p(\neq 0)$ such that:*

$$c^T-\Lambda A=0 \tag{3.47}$$

and

$$d-\Lambda\cdot B\geqslant 0. \tag{3.48}$$

The proof of the above theorem can be found in section 3 of [?]. The equation (3.47) is known as the Liu identity, the inequality (3.48) as the residual inequality and Λ as a Lagrange multiplier.
Theorem 3.2 is about an algebraic problem. However, Theorem 3.1 states that inequality (3.45) is satisfied for any solution of the system (3.44). This requirement is reduced to a purely algebraic problem if the following assumption can be justified:

Local solvability: *The system* (3.44) *is locally solvable at* (x^0, t^0) *if for any value of* $(\chi, \mathbb{X})$ *satisfying algebraically* (3.44) *there is a solution of the system of differential equations* (3.44), *in a neighborhood of* (x^0, t^0), *at which the solution is consistent with the given values.*

The key point to use the entropy principle and Theorem 3.2 is the local solvability assumption[?]. The proof of the local solvability assumption is made for a general case in the Proposition 1 from [?]. This proof uses the inverse function theorem and the Cauchy-Kowalevski theorem. In our the case, the local solvability assumption is simpler: Without lost of generality, we consider that $(x^0, t^0) = (0, 0)$ and let $(\chi, \mathbb{X})$ satisfies algebraically (3.44). Then we construct the following functions:

$$u_\alpha(t, x) = u_\alpha(0, 0) + t\frac{\partial}{\partial t}u_\alpha(0, 0) + x_k\frac{\partial}{\partial x_k}u_\alpha(0, 0) + x_k x_l\frac{\partial^2}{\partial x_l \partial x_k}u_\alpha(0, 0) + t x_k x_l\frac{\partial^3}{\partial t \partial x_l \partial x_k}u_\alpha(0, 0)$$

for $\alpha \in \{1, \ldots, N\}$ and

$$\theta_\alpha(t, x) = \theta_\alpha(0, 0) + x_k\frac{\partial}{\partial x_k}\theta_\alpha(0, 0) + t\frac{\partial}{\partial t}\theta_\alpha(0, 0)$$

for α between 2 and N, where the values $\frac{\partial u_\alpha(0,0)}{\partial x_k}$, $\frac{\partial u_\alpha(0,0)}{\partial t}$, $\frac{\partial^2 u_\alpha(0,0)}{\partial x_l \partial x_k}$, $\frac{\partial^3 u_\alpha(0,0)}{\partial t \partial x_l \partial x_k}$, $\theta_\alpha(0, 0)$ $\frac{\partial \theta_\alpha(0,0)}{\partial x_k}$ and $\frac{\partial \theta_\alpha(0,0)}{\partial t}$ are given by the algebraical solution $(\chi, \mathbb{X})$. We construct θ_1 as:

$$\theta_1 = 1 - \sum_{\beta=2}^{N} \theta_\beta.$$

Then we have construct a solution that solves locally the mass conservation equation and the saturation constraint and is consistent with the algebraical solution.
By Theorem 3.2, we use the Liu identities (3.47) and the residual inequality (3.48) to find constitutive functions. The advantage of this formulation is that we eliminate the elements in the vector $\mathbb{X}$, but we add the Lagrange multipliers Λ that are also functions of elements in χ. Another difficulty is to find the matrix A. However, using the implication (1)⇒(2) we can start to work with inequality (3.46) and, by standard calculations, from inequality (3.46) we obtain (3.47) and (3.48). In our scheme, there exist $N + 4$ Lagrange multipliers:

- 1 for the equation (3.42) that we call μ_0;
- 3 for the 3 equations (3.43) that we call μ_i with $i \in \{1, 2, 3\}$, and
- N for the N mass balance equations (3.40), that we call Λ^α.

Then, we write the inequality (3.46) as follows:

$$\sum_{\beta=1}^{N}\sum_{\eta=1}^{N}\Bigg(- \rho_\beta \frac{\partial \psi_\beta}{\partial \theta_\eta}\left(\frac{\partial}{\partial t}\theta_\eta + u_{\beta,j}\frac{\partial}{\partial x_j}\theta_\eta\right)$$
$$- \rho_\beta \frac{\partial \psi_\beta}{\partial u_{\eta,k}}\left(\frac{\partial}{\partial t}u_{\eta,k} + u_{\beta,j}D(u_\eta)_{kj} + u_{\beta,j}L(u_\eta)_{kj}\right)$$
$$- \rho_\beta \frac{\partial \psi_\beta}{\partial D(u_\eta)_{kh}}\left(\frac{\partial}{\partial t}D(u_\eta)_{kh} + u_{\beta,j}\frac{\partial}{\partial x_j}D(u_\eta)_{kh}\right)\Bigg) -$$
$$\rho_\beta \frac{\partial \psi_\beta}{\partial T}\left(\frac{\partial}{\partial t}T + u_{\beta,j}\frac{\partial}{\partial x_j}T\right) - \rho_\beta \frac{\partial \psi_\beta}{\partial \frac{\partial T}{\partial x_k}}\left(\frac{\partial}{\partial t}\frac{\partial T}{\partial x_k} + u_{\beta,j}\frac{\partial}{\partial x_j}\frac{\partial T}{\partial x_k}\right)$$
$$- \rho_\beta s_\beta\left(\frac{\partial}{\partial t}T + u_{\beta,j}\frac{\partial}{\partial x_j}T\right) + \theta_\beta \Sigma_\beta : D(u_\beta) - \theta_\beta \frac{\nabla T \cdot Q_\beta}{T} - \mu_0\left(\sum_{\beta=1}^{N}\frac{\partial}{\partial t}\theta_\beta\right)$$
$$- \sum_{i=1}^{3}\mu_i\left(\sum_{\beta=1}^{N}\frac{\partial}{\partial x_i}\theta_\beta\right) - \sum_{\alpha=1}^{N}\gamma_\alpha\Lambda_\alpha\Big(\frac{\partial}{\partial t}\theta_\alpha + u_\alpha\cdot\nabla\theta_\alpha + \theta_\alpha I : D(u_\alpha)\Big) \geqslant 0 \quad (3.49)$$

We rewrite the above inequality to deduce:

$$- \sum_{\beta=1}^{N}\frac{\partial \theta_\beta}{\partial t}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial \theta_\beta} + \gamma_\beta\Lambda_\beta + \mu_0\right)$$
$$- \sum_{\beta=1}^{N}\frac{\partial \theta_\beta}{\partial x_j}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial \theta_\beta}u_{\alpha,j} + \gamma_1\beta\Lambda_\beta u_{\beta,j} + \mu_j\right) - \sum_{\beta=1}^{N}\frac{\partial u_{\beta,k}}{\partial t}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial u_{\beta,k}}\right)$$
$$- \sum_{\beta=1}^{N}L\left(u_\beta\right)_{kj}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial u_{\beta,k}}u_{\alpha,j}\right) - \sum_{\beta=1}^{N}\frac{\partial}{\partial t}D\left(u_\beta\right)_{ij}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial D(u_\beta)_{ij}}\right)$$
$$- \sum_{\beta=1}^{N}\frac{\partial}{\partial x_j}D\left(u_\beta\right)_{kh}\left(\sum_{\alpha=1}^{N}\rho_\alpha u_{\alpha,j}\frac{\partial \psi_\alpha}{\partial D(u_\beta)_{kh}}\right)$$
$$- \frac{\partial^2 T}{\partial t\partial x_k}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial \frac{\partial T}{\partial x_k}}\right) - \frac{\partial^2 T}{\partial x_j\partial x_k}\left(\sum_{\alpha=1}^{N}\rho_\alpha u_{\alpha,j}\frac{\partial \psi_\alpha}{\partial \frac{\partial T}{\partial x_k}}\right)$$
$$- \frac{\partial T}{\partial t}\left(\sum_{\alpha=1}^{N}\rho_\alpha\frac{\partial \psi_\alpha}{\partial T} + \rho_\alpha s_\alpha\right)$$
$$+ \sum_{\alpha=1}^{N}D(u_\alpha)_{kj}\left(\theta_\alpha\Sigma_{\alpha,kj} - \sum_{\beta=1}^{N}\rho_\beta\frac{\partial \psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j} - \gamma_\alpha\Lambda_\alpha\theta_\alpha\delta_{kj}\right)$$
$$+ \frac{\partial T}{\partial x_j}\sum_{\alpha=1}^{N}\left[-\rho_\alpha u_{\alpha,j}\frac{\partial \psi_\alpha}{\partial T} - \rho_\alpha u_{\alpha,j}s_\alpha - \frac{\theta_\alpha Q_\alpha}{T}\right] \geqslant 0 \quad (3.50)$$

Since (3.50) holds for every $\frac{\partial \theta_\beta}{\partial t}$, $\frac{\partial \theta_\beta}{\partial x_j}$, $\frac{\partial u_{\beta,k}}{\partial t}$, $L\left(u_\beta\right)_{kj}$, $\frac{\partial}{\partial t}D\left(u_\beta\right)_{kh}$, $\frac{\partial}{\partial x_j}D\left(u_\beta\right)_{kh}$, $\frac{\partial^2 T}{\partial t\partial x_k}$, $\frac{\partial^2 T}{\partial x_j\partial x_k}$

and $\frac{\partial T}{\partial t}$ we arrive to the Liu identities (3.47):

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial \theta_\beta} + \gamma_\beta \Lambda_\beta + \mu_0 = 0, \tag{3.51}$$

for all $\beta \in \{1, \ldots, N\}$,

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial \theta_\beta} u_{\alpha,j} j + \gamma_\beta \Lambda_\beta u_{\beta,j} + \mu_j = 0, \tag{3.52}$$

for all $\beta \in \{1, \ldots, N\}$ and $j \in \{1, 2, 3\}$,

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial u_{\beta,k}} = 0, \tag{3.53}$$

for all $\beta \in \{1, \ldots, N\}$,

$$\left(\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial u_{\beta,k}} u_{\alpha,j} \right)_{\text{skew symmetric}} = 0, \tag{3.54}$$

for all $\beta \in \{1, \ldots, N\}$,

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial D(u_\beta)_{ij}} = 0, \tag{3.55}$$

for all $\beta \in \{1, \ldots, N\}$,

$$\sum_{\alpha=1}^{N} \rho_\alpha u_{\alpha,j} \frac{\partial \psi_\alpha}{\partial D(u_\beta)_{kh}} = 0, \tag{3.56}$$

for all $\beta \in \{1, \ldots, N\}$, $k \in \{1, 2, 3\}$ and $h \in \{1, 2, 3\}$,

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial \frac{\partial T}{\partial x_k}} = 0, \tag{3.57}$$

for all $k \in \{1, 2, 3\}$.

$$\sum_{\alpha=1}^{N} \rho_\alpha u_{\alpha,j} \frac{\partial \psi_\alpha}{\partial \frac{\partial T}{\partial x_k}} = 0, \tag{3.58}$$

for all k and j in $\{1, 2, 3\}$.

$$\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial \psi_\alpha}{\partial T} + \rho_\alpha s_\alpha = 0, \tag{3.59}$$

and the residual inequality (3.48),

$$\sum_{\alpha=1}^{N} D(u_\alpha)_{kj} \left[\theta_\alpha \Sigma_{\alpha,kj} - \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j} - \gamma_\alpha \Lambda_\alpha \rho_\alpha \delta_{kj} \right] + \frac{\partial T}{\partial x_j} \sum_{\alpha=1}^{N} \left[-\rho_\alpha u_{\alpha,j} \frac{\partial \psi_\alpha}{\partial T} - \rho_\alpha u_{\alpha,j} s_\alpha - \frac{\theta_\alpha Q_\alpha}{T} \right] \geqslant 0. \tag{3.60}$$

Remark 3.3. *Instead of using* (3.42) *and* (3.43) *as the saturation constraint, we can follow the approach of [**?**, Chapter 7], by considering the equation* (3.21) *and replacing one of the volume fractions in terms of the others* $N-1$*, for instance,*

$$\theta_1 = 1 - \sum_{\beta=2}^{N} \theta_\beta,$$

and then, we arrive to the same Liu identities using that

$$\frac{\partial}{\partial\theta_\alpha} f\left(\theta^1, \ldots, \theta^N\right) = \frac{\partial}{\partial\theta_\alpha} f\left(1 - \sum_{\beta=2}^{N} \theta_\beta, \ldots, \theta^N\right) = \frac{\partial}{\partial\theta_\alpha} f - \frac{\partial}{\partial\theta^1} f.$$

for all f *differentiable.*

The Liu equations (3.51)–(3.59) and the residual inequality (3.60) are useful tools to develop constitutive equations. However, we can still obtain better information from them:

- Since equation (3.51) holds for all β between 1 and N, we use the first equation to obtain

$$\mu_0 = -\sum_{\alpha=1}^{N} \rho_\alpha \frac{\partial\psi_\alpha}{\partial\theta_1} - \gamma_1\Lambda_1.$$

- Replacing μ_0 into equation (3.51) we get that:

$$\sum_{\alpha=1}^{N} \rho_\alpha \left(\frac{\partial\psi_\alpha}{\partial\theta_\beta} - \frac{\partial\psi_\alpha}{\partial\theta_1}\right) + \gamma_\beta\Lambda_\beta - \gamma_1\Lambda_1 = 0, \tag{3.61}$$

  for all $\beta \in \{2, \ldots, N\}$.

- Similarly, we write equation (3.52) as follows:

$$\sum_{\alpha=1}^{N} \rho_\alpha u_{\alpha,j} \left(\frac{\partial\psi_\alpha}{\partial\theta_\beta} - \frac{\partial\psi_\alpha}{\partial\theta_1}\right) + \gamma_\beta u_{\beta,j}\Lambda_\beta - \gamma_1 u_{1,j}\Lambda_1 = 0, \tag{3.62}$$

  for all $\beta \in \{2, \ldots, N\}$.

- We define the Helmholtz free energy function of the mixture as

$$\Psi_I = \sum_{\alpha=1}^{N} \rho_\alpha\psi_\alpha.$$

  Then, by (3.53), we deduce that:

$$\frac{\partial}{\partial u_{\beta,k}} \Psi_I = 0. \tag{3.63}$$

  for all $\beta \in \{1, \ldots, N\}$ and for all $k \in \{1, 2, 3\}$.

- By equation (3.55), we deduce that:

$$\frac{\partial}{\partial D\left(u_\beta\right)_{ij}}\Psi_I = 0. \tag{3.64}$$

- By equation (3.57), we deduce that:

$$\frac{\partial}{\partial \frac{\partial T}{\partial x_k}}\Psi_I = 0. \tag{3.65}$$

- By equations (3.63),(3.64) and (3.65) we can deduce that:

$$\Psi_I = \Psi_I(\theta_1, \ldots, \theta_N, T). \tag{3.66}$$

- We define the entropy of the mixture as follows:

$$S_I = \sum_{\alpha=1}^{N} \rho_\alpha s_\alpha.$$

Then, we can write equation (3.59) as follows:

$$\frac{\partial}{\partial T}\Psi_I + S_I = 0. \tag{3.67}$$

Using $\overline{D}(u_\alpha)$ and $\overline{\Sigma}_\alpha$, the traceless parts of $D(u_\alpha)$ and and Σ_α respectively, we rewrite the residual inequality (3.60) as follows:

$$\begin{aligned}
\sum_{\alpha=1}^{N}\left(\overline{D}(u_\alpha)_{kj} + \frac{1}{3}\operatorname{tr}\left(D(u_\alpha)\right)\delta_{kj}\right)&\left(\theta_\alpha \overline{\Sigma}_{\alpha,kj} + \theta_\alpha\frac{1}{3}\operatorname{tr}\left(\Sigma_\alpha\right)\delta_{kj}\right.\\
&- \left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j} - \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}\right)\delta_{kj}\right)\\
&\left. - \left(\frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}\right) + \Lambda_\alpha\rho_\alpha\right)\delta_{kj}\right)\\
&+ \frac{\partial T}{\partial x_j}\sum_{\alpha=1}^{N}\left(-\rho_\alpha u_{\alpha,j}\frac{\partial\psi_\alpha}{\partial T} - \rho_\alpha u_{\alpha,j}s_\alpha - \frac{\theta_\alpha Q_\alpha}{T}\right) \geqslant 0
\end{aligned}$$

Using that:

$$\operatorname{tr}(\overline{D}(u_\alpha)) = 0,$$

$$\operatorname{tr}\left(\overline{\Sigma}_\alpha\right) = 0,$$

and

$$\delta_{kj}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j} - \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}\right)\delta_{kj}\right) = 0,$$

and we deduce that:

$$\sum_{\alpha=1}^{N}\overline{D}(u_\alpha)_{kj}\left(\theta_\alpha\overline{\Sigma}_{\alpha,kj}-\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}-\frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}\right)\delta_{kj}\right)\right)$$
$$+\sum_{\alpha=1}^{N}\operatorname{tr}\left(D(u_\alpha)\right)\left(\frac{1}{3}\theta_\alpha\operatorname{tr}\left(\Sigma_\alpha\right)-\frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,k}\right)-\Lambda_\alpha\rho_\alpha\right)$$
$$+\frac{\partial T}{\partial x_j}\sum_{\alpha=1}^{N}\left[-\rho_\alpha u_{\alpha,j}\frac{\partial\psi_\alpha}{\partial T}-\rho_\alpha u_{\alpha,j}s_\alpha-\frac{\theta_\alpha Q_\alpha}{T}\right]\geqslant 0. \quad (3.68)$$

In order to exploit the residual inequality (3.68) we make use of the notion sub-differential inclusions.

3.2.3 Constitutive equation using sub-differential inclusions

This section aims to introduce the theory of constitutive equations using the sub-differential inclusions. Sub-differential inclusions are widely used to develop constitutive equations since they can easily extend one-dimensional phenomenological laws to three-dimensional constitutive equations (see for example [?]). In this framework, we use sub-differentials inclusions to relate the dependent variables Σ_α and Q_α and the sub-differential of a function Φ that we call potential. Let us introduce then, the following definition.

Definition 3.1. *A function $\Phi : \left(\mathbb{M}_s^{3\times3}\right)^N\times\mathbb{R}^N\times\mathbb{R}^3\to\mathbb{R}$ is called a* **potential** *if it is positive, convex, lower semi-continuous, proper and is such that*

$$\Phi\left(0\right)=0.$$

Within the framework of mixture theory, the central hypothesis that allows us the construction of constitutive equations is the following:

Hypothesis 3.8. *Let θ_α, u_α, Σ_α, e_α, Q_α, T and s_α satisfying the system* (3.18)–(3.21) *and the inequality* (3.32) *where $\psi_\alpha = e_\alpha - Ts_\alpha$. Then, there exist functions $\mathbb{P}_\alpha : \left(\mathbb{M}_s^{3\times3}\right)^N\times\mathbb{R}^N\times\mathbb{R}^3\to\mathbb{R}$, $\mathbb{S}_\alpha : \left(\mathbb{M}_s^{3\times3}\right)^N\times\mathbb{R}^N\times\mathbb{R}^3\to\mathbb{M}_s^{3\times3}$, $\mathbb{Q} : \left(\mathbb{M}_s^{3\times3}\right)^N\times\mathbb{R}^N\times\mathbb{R}^3\to\mathbb{R}^3$ and a potential $\Phi : \left(\mathbb{M}_s^{3\times3}\right)^N\times\mathbb{R}^N\times\mathbb{R}^3\to\mathbb{R}$, according to Definition 3.1, such that*

$$\theta_\alpha\Sigma_\alpha=\gamma_\alpha\theta_\alpha\Lambda_\alpha I_3+\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_\alpha}u_\beta+\mathbb{P}_\alpha(\overline{D}(u_1),\cdots,\overline{D}(u_N),\operatorname{tr}(D(u_1)),\cdots,\operatorname{tr}(D(u_N)),\nabla T)I$$
$$+\mathbb{S}_\alpha(\overline{D}(u_1),\cdots,\overline{D}(u_N),\operatorname{tr}(D(u_1)),\cdots,\operatorname{tr}(D(u_N)),\nabla T), \quad (3.69)$$

$$\sum_{\alpha=1}^{N}\theta_\alpha Q_\alpha=-T\left(\sum_{\alpha=1}^{N}\rho_\alpha u_{\alpha,j}\frac{\partial\psi_\alpha}{\partial T}+\rho_\alpha u_{\alpha,j}s_\alpha\right)$$
$$+\mathbb{Q}(\overline{D}(u_1),\cdots,\overline{D}(u_N),\operatorname{tr}(u_1),\cdots,\operatorname{tr}(u_N),\nabla T) \quad (3.70)$$

and

$$\begin{pmatrix} \mathbb{S}_1(D,\gamma,\Theta) \\ \vdots \\ \mathbb{S}_N(D,\gamma,\Theta) \\ \mathbb{P}_1(D,\gamma,\Theta) \\ \vdots \\ \mathbb{P}_N(D,\gamma,\Theta) \\ \mathbb{Q}(D,\gamma,\Theta) \end{pmatrix} \in \partial\Phi\left(D,\gamma,\Theta\right). \tag{3.71}$$

for all $(D,\gamma,\Theta) \in \left(\mathbb{M}_s^{3\times3}\right)^N \times \mathbb{R}^N \times \mathbb{R}^3$.

The above hypothesis is justified by the following proposition:

Lemma 3.1. *Let* Φ *a potential, according to Definition 3.1,* D *and* D' *in* $\left(\mathbb{M}_s^{3\times3}\right)^N$*,* Z *and* Z' *in* $\mathbb{R}^N$ *such that*

$$(D', Z') \in \partial\Phi(D, Z). \tag{3.72}$$

Then:

$$\sum_{j=1}^{N} D'_i : D_i + Z' \cdot Z \geqslant 0. \tag{3.73}$$

Proof. Let $D_i \in \mathbb{M}_s^{3\times3}(i \in \{1,\cdots,N\})$ and $Z \in \mathbb{R}^{N+3}$. We denote by $\partial\Phi(D,Z)$ the set of elements $D'_i \in \mathbb{M}_s^{3\times3}(i \in \{1,\cdots,N\})$ and $Z' \in \mathbb{R}^{N+3}$ such that the following inequality holds

$$\Phi(D_1, Z_1) - \Phi(D, Z) \geqslant \sum_{i=1}^{N} D'_i : (D_{1,i} - D_i) + Z' \cdot (Z_1 - Z),$$

for all $(D_1, Z_1) \in \left(\mathbb{M}_s^{3\times3}\right)^N \times \mathbb{R}^{N+1}$. In the above inequality we choose:

$$(D_1, Z_1) = \left(0_{\left(\mathbb{M}_s^{3\times3}\right)^N}, 0_{\mathbb{R}^{N+1}}, 0\right),$$

and we obtain

$$-\Phi(D, Z) \geqslant -\sum_{i=1}^{N} D'_i : (D_i) - Z' \cdot Z.$$

Then, by the fact that Φ is non negative we deduce (3.73). $\square$

Using the above lemma with

$$D = \left(\overline{D}\left(u_1\right), \ldots, \overline{D}\left(u_N\right)\right),$$

$$Z = \left(\operatorname{tr}\left(D\left(u_1\right)\right), \ldots, \operatorname{tr}\left(D\left(u_N\right)\right), \nabla T\right),$$

$$D'_{\alpha,kj} = \left(\theta_\alpha \overline{\Sigma}_{\alpha,kj} - \left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial\psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j} - \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial\psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j}\right)\delta_{kj}\right)\right),$$

$$Z'_\alpha = \left(\frac{1}{3}\theta_\alpha \operatorname{tr}\left(\Sigma\right) - \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial\psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j}\right) - \Lambda_\alpha \gamma_\alpha \theta_\alpha\right),$$

for $\alpha \in \{1, \dots, N\}$, and

$$Z'_{N+1} = \sum_{\alpha=1}^{N} \left(-\rho_\alpha u_{\alpha,j} \frac{\partial \psi_\alpha}{\partial T} - \rho_\alpha u_{\alpha,j} s_\alpha - \frac{\theta_\alpha Q_\alpha}{T} \right),$$

we obtain the residual inequality (3.68).

Using Hypothesis 3.8, when Φ is differentiable, we have the following proposition:

Proposition 3.2. *Let Φ a potential, according to Definition 3.1, such that is a differentiable function. Under Hypothesis 3.8, the Cauchy stress tensor of the constituent α is*

$$\theta_\alpha \Sigma_\alpha = \Lambda_\alpha \gamma_\alpha \theta_\alpha I_3 + \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta + \frac{\partial}{\partial \operatorname{tr}(D(u_\alpha))} \Phi + \frac{\partial}{\partial \overline{D}(u_\alpha)} \Phi, \tag{3.74}$$

and the heat flux is:

$$\sum_{\alpha=1}^{N} \frac{\theta_\alpha Q_\alpha}{T} = -\left(\sum_{\alpha=1}^{N} \rho_\alpha u_\alpha \frac{\partial \psi_\alpha}{\partial T} + \rho_\alpha u_\alpha s_\alpha \right) - \frac{\partial}{\partial \nabla T} \Phi. \tag{3.75}$$

Proof. Since Φ is differentiable then:

$$\partial \Phi = \{\nabla \Phi\}.$$

Then

$$\mathbb{P}_\alpha = \frac{\partial}{\partial \operatorname{tr}(D(u_\alpha))} \Phi,$$

$$\mathbb{S}_\alpha = \frac{\partial}{\partial \overline{D}(u_\alpha)} \Phi,$$

and

$$\mathbb{Q}_j = \frac{\partial}{\partial \nabla T} \Phi.$$

Then, we conclude by (3.69) and (3.70). □

The choice of Φ is a **constitutive assumption** and it is related to the material properties of the continuum media. In the next section we review some classical examples of constitutive equations.

3.2.4 Examples of constitutive equations

In this section, we review some of the classic examples of constitutive equations using the sub-differential inclusions. In the examples, we consider a body made of only one material and we set the set of independent variables as

$$\chi = \{\rho, D(u), T, \nabla T\}.$$

Considering that the saturation constraint (3.21) is not in the system of equations, using the entropy principle, Theorem 3.2 and the Liu's identities (3.51) and (3.59), as we did in the previous sections, we deduce the following residual inequality:

$$\overline{D}(u) : \overline{\Sigma} + \operatorname{tr}(D(u)) \left(\frac{1}{3} \operatorname{tr}(\Sigma) + \rho \frac{\partial \psi}{\partial \rho} \right) - \frac{\nabla T \cdot Q}{T} \geqslant 0, \tag{3.76}$$

where ψ is only a function of the density ρ and the temperature T. By Hypothesis 3.8, there exists a function $\Phi : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ such that:

$$\Sigma = \left(-\rho \frac{\partial \psi}{\partial \rho} + \mathbb{P}(\overline{D}(u), \operatorname{tr}(D(u)), \nabla T) \right) I_3 + \mathbb{S}(\overline{D}(u), \operatorname{tr}(D(u)), \nabla T), \tag{3.77}$$

and

$$Q = -\mathbb{Q}(\overline{D}(u), \operatorname{tr}(D(u)), \nabla T) \tag{3.78}$$

where

$$\begin{pmatrix} \mathbb{S}(D, \gamma, \Theta) \\ \mathbb{P}(D, \gamma, \Theta) \\ \mathbb{Q}(D, \gamma, \Theta) \end{pmatrix} \in \partial \Phi(D, \gamma, \Theta). \tag{3.79}$$

for all $(D, \gamma, \Theta) \in \mathbb{M}_s^{3\times 3} \times \mathbb{R} \times \mathbb{R}^3$.

Incompressible Navier-Stokes equations

The incompressible Navier-Stokes equations are the most known constitutive equation. These equations describe the motion of an isothermal incompressible Newtonian fluid. In this fluid, the spherical stresses are proportional to the rates of change of the fluid's velocity. More precisely, the Cauchy stress tensor is

$$\Sigma = -pI_3 + 2\mu D(u), \tag{3.80}$$

where μ is the proportionality constant, called molecular viscosity and $p = -1/3 \operatorname{tr}(\Sigma)$ is the pressure. The incompressibility means that

$$\operatorname{div}(u) = 0. \tag{3.81}$$

The above constitutive equation is obtained using the potential

$$\Phi(D, \gamma, \Theta) = 1_0(\gamma) + 1_0(\Theta) + \mu |D|^2,$$

where

$$1_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0. \end{cases} \tag{3.82}$$

To check this we note that, since $\Phi(\gamma, D) < \infty$ only if $\gamma \neq 0$, then the constitutive equation makes sens only if (3.81) holds. Since the inclusion (3.79) holds when $\operatorname{tr}(D(u)) = 0$, the term $\frac{1}{3} \operatorname{tr}(\Sigma) - \frac{\rho}{T} \frac{\partial \psi}{\partial \rho}$, suffers no restrictions from it and it can be any real number, say κ_1. Then, we write the trace of Σ as:

$$\frac{1}{3} \operatorname{tr}(\Sigma) = -\rho \frac{\partial \psi}{\partial \rho} + \kappa_1.$$

For $\gamma = 0$ and $\Theta = 0$, Φ is differentiable. Then, using Proposition (3.2) we deduce

$$\Sigma = \left(\kappa_1 - \rho \frac{\partial \psi}{\partial \rho}\right) I_3 + 2\mu D(u).$$

Finally, calling $p = -\kappa_1 + \rho \frac{\partial \psi}{\partial \rho}$ we deduce (3.81).

Remark 3.4. *Another way to impose the isothermal and incompressible constraints is to consider that the density and the temperature are constant and they are not in our selection of independent variables. In that case, the mass conservation equation implies equation* (3.81)*, and there is no need to add the function* (3.82) *into* Φ*.*

Compressible Navier-Stokes-Fourier equations

The compressible Navier-Stokes-Fourier equations model the motion of a heat conducting Newtonian compressible fluid, where the spherical part of the Cauchy stress tensor is proportional to the rate of change of the fluid's velocity, the heat flux is proportional to the rate of change of the temperature, and the fluid's density is variable. The constitutive equation for this type of fluid is obtained through the following potential:

$$\Phi(D, \gamma, \Theta) = \mu \left|D\right|^2 + \frac{(\lambda + 2\mu)}{2} \left|\gamma\right|^2 + \frac{\kappa}{2} \left|\Theta\right|^2. \tag{3.83}$$

Since Φ is differentiable for all (D, γ, Θ), using Proposition 3.2 we deduce

$$\Sigma = (-p + \lambda \operatorname{tr}(D(u)))I_3 + 2\mu D(u), \tag{3.84}$$

$$Q = -\kappa \nabla T, \tag{3.85}$$

where $p = \rho \frac{\partial \psi}{\partial \rho}$, μ is the molecular viscosity, and λ is a parameter called compressibility.

A further analysis allows us to obtain additional restrictions on the material coefficients. Combining the residual inequality (3.76) with (3.85) and (3.84) we obtain

$$2\mu \left|\overline{D}(u)\right|^2 + \frac{3\lambda + 2\mu}{3} (\operatorname{tr}(D(u)))^2 + \kappa \frac{\left|\nabla T\right|^2}{T} \geq 0.$$

From the above inequality, since tr(D) and $\overline{D}$ are mutually independent, we deduce that $\kappa \geq 0$, $3\lambda + 2\mu \geq 0$ and $\mu \geq 0$.

Bingham fluid equations

A Bingham fluid is a viscoplastic material, which means that it behaves as a liquid for high stresses and as a solid for low stresses. Viscoplastic materials can model various natural and industrial fluids, for instance, mudflows, snow avalanche, volcanic lava flows, toothpaste, mayonnaise, etc. The Bingham constitutive equation described below is the simplest models for a viscoplastic fluid. It was proposed by Bingham [?] in 1916. For simplicity, we consider that the fluid is incompressible and isothermal. Then $\operatorname{tr}(D(u)) = 0$, $\overline{D}(u) = 0$ and the Cauchy stress tensor (3.77) is

$$\Sigma = -pI_3 + \mathbb{S}(D(u), 0, 0)$$

where $p = \kappa_1 - \rho\frac{\partial \psi}{\partial \rho}$, $\mathbb{S}(D, 0, 0) \in \partial\Phi(D, 0, 0)$ for all $D \in \mathbb{M}^{3\times3}_s$, and

$$\Phi(D, \gamma, \Theta) = 1_0(\gamma) + 1_0(\Theta) + \mu\,|D|^2 + g\,|D|\,. \tag{3.86}$$

In the constitutive law given by Φ, the constant g is the yield stress, and μ is the molecular viscosity. Since Φ is not differentiable for $D = 0$, to express the Cauchy stress tensor we cannot rely only on Proposition (3.2). A comprehensive way of writing the Bingham law is given in the following lemma. To simplify the notation we write $\Phi(D)$ instead of $\Phi(D, \gamma, \Theta)$.

Lemma 3.2. *Let g and μ two positive constants. Let $\Phi : \mathbb{M}^{3\times3}_s \to \mathbb{R}$ a potential defined by equation* (3.86)*. Then, for $D \in \mathbb{M}^{3\times3}$ we have that the following are equivalent:*

(i)

$$\mathbb{S}(D) \in \partial\Phi(D), \tag{3.87}$$

and

(ii)

$$\begin{cases} |\mathbb{S}(D)| \leqslant g & \Leftrightarrow \quad D = 0, \\ |\mathbb{S}(D)| > g & \Leftrightarrow \quad \mathbb{S}(D) = 2\mu D + g\dfrac{D}{|D|}. \end{cases} \tag{3.88}$$

Proof. For simplicity we state the inclussion (3.87) in the following manner: since the function $D \in \mathbb{M}^{3\times3}_s \to |D|^2$ is differentiable, we write the inclusion (3.87) as follows:

$$\mathbb{S}(D) - 2\mu D \in \partial\Phi_1\,(D)\,, \tag{3.89}$$

where $\Phi_1 : \mathbb{M}^{3\times3}_s \to \mathbb{R}$ is defined by:

$$\Phi_1(D) = g\,|D|\,.$$

Then, by the definition of sub-differential, the inclusion (3.89) implies that:

$$(\mathbb{S}(D) - 2\mu D) : (D_1 - D) \leqslant \Phi_1(D_1) - \Phi_1(D) \tag{3.90}$$

holds for all $D_1 \in \mathbb{M}^{3\times3}_s$. Now, we prove the equivalences:

(i)⇒(ii) – We prove that if $D = 0$ then $|\mathbb{S}(0)| < g$. By inequality (3.90) we have that

$$\mathbb{S}(0) : D_1 \leqslant g\,|D_1|$$

for all $D_1 \in \mathbb{M}^{3\times3}_s$. Dividing by $|D_1|$ and taking supremun over D_1 we conclude $|\mathbb{S}(D)| \leqslant g$.

– We prove that if $|\mathbb{S}(D)| \leqslant g$ then $D = 0$. By (3.90), for $D_1 = 0$ we have that:

$$-\mathbb{S}(D) : D + 2\mu\,|D|^2 \leqslant -g\,|D|\,,$$

If we assume that $|D| > 0$, then we have that:

$$\frac{\mathbb{S}(D) : D}{|D|} > g,$$

which is a contradiction with $|\mathbb{S}(D)| \leqslant g$. Then $D = 0$.

- We prove that if $|\mathbb{S}(D)| \geqslant g$, then $\mathbb{S}(D) = 2\mu D + g\frac{D}{|D|}$. Since $|\mathbb{S}(D)| \geqslant g$, we have that $D \neq 0$. Then, since Φ_1 is differentiable and

$$\mathbb{S}(D) = 2\mu D + g\frac{D}{|D|}.$$

- The proof of $\mathbb{S}(D) = 2\mu D + g\frac{D}{|D|} \Rightarrow |\mathbb{S}(D)| \geqslant g$, is straigthforward from

$$|\mathbb{S}(D)| = (2\mu\,|D| + g)\,.$$

(ii)⇒(i)

- If $|\mathbb{S}(D)| > g$, we have that for every $D_1 \in \mathbb{M}_s^{3\times 3}$:

$$\mathbb{S}(D) : (D_1 - D) = 2\mu D_1 : D - 2\mu\,|D|^2 + g\frac{D : D_1}{|D|} - g\,|D|\,.$$

Then, using the Cauchy-Schwarz inequality and

$$2D_1 : D \leqslant |D|^2 + |D_1|^2\,,$$

we deduce

$$\mathbb{S}(D) : (D_1 - D) \leqslant \mu\,|D_1|^2 + g\,|D_1| - \mu\,|D|^2 - g\,|D|\,,$$

which implies that $\mathbb{S}(D) \in \partial\Phi(D)$.

- If $|\mathbb{S}(D)| \leqslant g$, we have:

$$\frac{\mathbb{S}(0) : D_1}{|D|} \leqslant g$$

for all $D_1 \in \mathbb{M}$. Then,

$$\mathbb{S}(D) : D_1 \leqslant \mu\,|D_1|_2 + g\,|D_1|_2\,,$$

and we conclude $\mathbb{S}(0) \in \partial\Phi(0)$.

□

Remark 3.5. *The above representation says that a Bingham fluid behaves like a viscous fluid if $|\mathbb{S}(D(u))| > g$, and as a rigid body otherwise. This propoerty is called yielding condition. We note that if $g = 0$, we recover the Navier-Stokes equations.*

Constitutive equations for granular materials: Mohr-Coulomb yield criterion

According to [?], a granular flow is a collection of solid particles immersed in a fluid that can be water or air. The modeling and understanding of granular materials represent a significant purpose of human activities since a broad range of materials can be considered as a granular media. According to [?], measured in tons, the first material manipulated on earth is water; the second is granular matter. Several examples of granular materials can be found in the industry: mine tailings; pharmaceutical tablets and capsules; and in nature: landslides, debris avalanches, pyroclastic flows, rice, and sand.

A comprehensive view of the mechanical and thermodynamical properties of materials is needed it to obtain constitutive equations. In particular, granular materials reveals various mechanical behaviors, similar to elastoplastic solids in the case of a quasi-static regime to dense gazes in the cases of strong agitation[?]. Then, the properties of a granular material are somewhere between those of a liquid and those of a real solid. Even at rest, granular material can sustain some shearing stress but only an amount proportional to the average stress. This yielding property, similar to a Bingham fluid 3.5, is dominant in dense regimes. For this reason and simplicity, we neglect other types of mechanical properties and regimes, and we consider only dense granular flows.

To obtain a constitutive equation that expresses the yielding property of a granular material we can use sub-differential inclusions. However, we first consider a more comprehensive way of deducing this constitutive equation based on the von-Misses yielding condition. The von-Misses yielding condition states that the material is at rest if

$$|\Sigma - \sigma I_3|^2 \leqslant k^2\sigma^2, \tag{3.91}$$

where k is a constant of the material and

$$\sigma = \frac{1}{3}\,\mathrm{tr}\,(\Sigma)\,.$$

Otherwise, if

$$|\Sigma - \sigma I_3|^2 = k^2\sigma^2, \tag{3.92}$$

the material experiences deformation. Equations (3.91) and (3.92) are deduced from the law of sliding friction applied to the individual particles in a granular material

To obtain a constitutive equation from the restrictions (3.91) and (3.92) we need to apply the flow rule. The flow rule is a quantitative formulation of the following idea: a granular material experiences deformation due to the application of different stresses in different directions. Intuitively, the response of the material to such different stresses should be to contract in the directions of higher stress and to expand in the directions of smaller stress. Explicitly, if we suppose that suppose that the material is rigid-perfectly plastic, which means that there is no viscosity and the fluid is incompressible, the flow rule states the existence of an scalar $q > 0$ such that:

$$D(u) = q(\Sigma - \sigma I). \tag{3.93}$$

Recovering our notation, calling $p = -\sigma$ and

$$\mathbb{S} = \Sigma + pI_3,$$

if the fluid is in motion, (3.91) implies

$$|\mathbb{S}| = kp$$

and, by the flow rule (3.93), we obtain

$$q = k\sigma\,|D(u)|^{-1}\,.$$

Then, if the fluid is in motion,

$$\Sigma = -pI_3 + kp\frac{D(u)}{|D(u)|} \quad \text{for } |D(u)| \neq 0. \tag{3.94}$$

If the fluid is not flowing, by (3.92) we have that

$$|\mathbb{S}|^2 \leqslant k^2 p^2 \quad \textbf{for} \quad |D(u)| = 0. \tag{3.95}$$

When the constant

$$k = \tan \phi,$$

where ϕ is the internal friction angle of the material. The material law (3.94) and (3.95) is known as **Mohr-Coulomb constitutive equation**. Later in this chapter, in Section 3.2.4, we review a constitutive equation where k is a function of $D(u)$.

To deduce the material (3.94) and (3.95) using sub-differential inclusions we use the potential

$$\Phi(D, \gamma, \Theta) = 1_0(\gamma) + 1_0(\Theta) + k\,|p|\,|D|\,, \tag{3.96}$$

where

$$p = -\frac{1}{3}\,\mathrm{tr}\,(\Sigma)\,,$$

and we consider that the fluid is isothermal and incompressible.

To obtain a material law from the potential (3.96) we proceed as follows

- Since the inclusion (3.76) must holds when $\mathrm{div}(u) = 0$, the term $\frac{1}{3}\,\mathrm{tr}\,(\Sigma) - \frac{\rho}{T}\frac{\partial \psi}{\partial \rho}$, suffers no restrictions from it and, it can be any real number, say κ_1. Then, we write the trace of Σ as:

  $$-p = \frac{1}{3}\,\mathrm{tr}\,(\Sigma) = -\rho\frac{\partial \psi}{\partial \rho} + \kappa_1.$$

  As usual, we also deduce that ∇T and T are constants. For simplicity, we call $\Phi(D) = \Phi(D, 0, 0)$. We notice that since T is constant, ψ depends only on ρ, we have that

  $$p = \hat{p}(\rho). \tag{3.97}$$

- If $|D| \neq 0$ then $\Phi(D)$ is a differentiable and, by Theorem 3.2 and (3.97), we have that:

  $$\mathbb{S}(D) = k\,|p|\,\frac{D}{|D|}. \tag{3.98}$$

- Otherwise, if $|D| = 0$, the only information that we have is

  $$\mathbb{S}(0) \in \partial\Phi(0).$$

  Similar to the example of a Bingham fluid, we deduce the threshold condition

  $$k\,|p| \geqslant |\mathbb{S}(0)|\,. \tag{3.99}$$

Gathering (3.98) and (3.99), we obtain the constitutive equation (3.94) and (3.95).

Constitutive equation for a granular material: $\mu(I)$ rheology

The equations of motion of a granular fluid considering the constitutive equations (3.94) and (3.95) are ill-posed in all two-dimensional contexts and all realistic three-dimensional contexts [?]. For this reason, many authors have proposed new constitutive equations that have better mathematical properties. Pouliquen et al.[?] give a constitutive equation for a dense granular material obtained through experiments in the one-dimensional case and then, generalized to the three-dimensional case. [?] proved that this model is well posed under certain conditions on the inertial number I, defined further in this section. We review the constitutive equation of [?] using sub-differential inclusions. For simplicity, we assume that the fluid is incompressible and isothermal. Then, the constitutive law in [?] is given by:

$$\Sigma = -pI_3 + \mathbb{S}(D(u)), \tag{3.100}$$

where p is the granular material pressure and $\mathbb{S}$ is given by

$$\mathbb{S}(D) = \frac{\sqrt{2}\mu(I(p,D))p}{|D|} D. \tag{3.101}$$

In the above constitutive equation: I is the inertial number and $\mu(I)$ is the friction coefficient. These quantities are given by:

$$I(p,D) = \frac{\sqrt{2}\,|D|\,d}{\sqrt{\frac{p}{\rho_s}}}$$

and

$$\mu(I) = \mu_s + (\mu_2 - \mu_s)\left(\frac{I_0}{I} + 1\right)^{-1},$$

where d is the particle diameter, ρ_s is the particle density and I_0 is a constant. The inertial number I is interpreted as the ratio between two time scales: a macroscopic deformation time scale $\frac{1}{\sqrt{2}|D(u)|_2}$ and an inertial time scale $\sqrt{\frac{d^2\rho_s}{P}}$. This material law is called the $\mu(I)$ rheology.

To express the $\mu(I)$ rheology (3.101) in the context of a sub-differential equation, the potential Φ must reflect the viscoplastic nature of the dense regime. Then, a suitable choice of Φ is the following:

$$\Phi(D,\gamma,\Theta) = 1_0(\gamma) + 1_0(\Theta) + A_1\,|D| + A_2 \ln\left(A_3\,|D| + 1\right) \tag{3.102}$$

where A_1, A_2 and A_3 do not depends on D and, in order to guarantee the convexity of Φ, we assume that $A_1 \geqslant 0$, $A_2 \leqslant 0$ and $A_3 \geqslant 0$.

By (3.102), we deduce that:

- Like in Section 3.2.4,

$$-p = \frac{1}{3}\,\mathrm{tr}\,(\Sigma) = -\rho\frac{\partial\psi}{\partial\rho} + \kappa_1.$$

  where κ_1 is any real number. We also deduce that $\nabla T = 0$, T is constant and we denote $\Phi(D) = \Phi(D,0,0)$. Finally, since T is constant, then, ψ and p depend only on ρ.

- If $|D| \neq 0$ then $\Phi(D)$ is a differentiable and by Theorem 3.2 we have that:

$$\mathbb{S}(D) = A_1\frac{D}{|D|} + A_2A_3\frac{D}{A_3\,|D|+1}.$$

- On the other hand, if $|D| = 0$, the only information that we have is that

$$\mathbb{S}(0) \in \partial\Phi(0).$$

By (3.79)

$$\Phi(D_1) \geqslant \mathbb{S}(0) : D_1,$$

for all $D_1 \in \mathbb{M}_s^{3\times 3}$. By (3.102), we write (3.2.4) as follows:

$$A_1 |D_1| + A_2 \ln(A_3 |D_1| + 1) \geqslant \mathbb{S}(0) : D_1.$$

Dividing the above inequality by $|D_1|$ we obtain:

$$A_1 + A_2 \frac{\ln(A_3 |D_1| + 1)}{|D_1|} \geqslant \frac{\mathbb{S}(0) : D_1}{|D_1|}. \tag{3.103}$$

Since (3.103) holds for every D_1 in $\mathbb{M}_s^{3\times 3}$, we take sup over D_1 and we obtain

$$\sup_{D_1 \in \mathbb{M}_s^{3\times 3}} \left(A_1 + A_2 \frac{\ln(A_3 |D_1| + 1)}{|D_1|} \right) \geqslant \sup_{D_1 \in \mathbb{M}_s^{3\times 3}} \frac{\mathbb{S}(0) : D_1}{|D_1|}. \tag{3.104}$$

The supremum on the left-hand side of (3.104) is equal to A_1 since $A_2 \leqslant 0$ and $A_3 \geqslant 0$. Then, ince

$$\sup_{D_1 \in \mathbb{M}_s^{3\times 3}} \frac{\langle \mathbb{S}(0), D_1 \rangle}{|D_1|} = |\mathbb{S}(0)|,$$

we deduce the threshold condition:

$$A_1 \geqslant |\mathbb{S}(0)|. \tag{3.105}$$

Due to the previous arguments, given a body force f acting on the granular material, the dynamics of the granular material can be model with the following set of equations:

$$\operatorname{div}(u) = 0$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \operatorname{div}(\Sigma) + f$$

$$\Sigma = \left(\rho \frac{\partial \psi}{\partial \rho} + \kappa \right) I_3 + \left(A_1 + \frac{A_2 A_3}{A_3 |D(u)| + 1} \right) \frac{D(u)}{|D(u)|} \quad \text{if } |D(u)| \neq 0 \tag{3.106}$$

$$A_1 \geqslant |\mathbb{S}(0)| \quad \text{if } |D(u)| = 0 \tag{3.107}$$

To obtain the constitutive equation (3.100) we combine (3.106) and (3.107) in the following way:

- We recall that, the pressure function p, defined as

$$p = -\rho \frac{\partial \psi}{\partial \rho} + \kappa_1.$$

is a function of ρ since ψ is a function of ρ. Then, we choose A_1, A_2 and A_3 as follow:

$$A_1 = \sqrt{2}\mu_2 p, \;\; A_2 = -\frac{(\mu_2 - \mu_s)\, p}{d} \sqrt{\frac{p}{\rho_s}} I_0 \text{ and } A_3 = \frac{\sqrt{2} d}{\sqrt{\frac{p}{\rho_s}} I_0}.$$

- Replacing A_1, A_2 and A_3 in (3.106) we obtain:

$$\mathbb{S}(D) = \sqrt{2}\mu_2 p \frac{D(u)}{|D(u)|} - \sqrt{2}(\mu_2 - \mu_s)p \frac{1}{|D(u)| \frac{\sqrt{2}d}{\sqrt{\frac{p}{\rho_s}}I_0} + 1} \frac{D(u)}{|D(u)|}.$$

Then, since

$$\sqrt{2}\mu_2 p \frac{D(u)}{|D(u)|} - \sqrt{2}(\mu_2 - \mu_s)p \frac{1}{|D(u)| \frac{\sqrt{2}d}{\sqrt{\frac{p}{\rho_s}}I_0} + 1} \frac{D(u)}{|D(u)|} = \left(\mu_2 - (\mu_2 - \mu_s) \frac{1}{|D(u)| \frac{\sqrt{2}d}{\sqrt{\frac{p}{\rho_s}}I_0} + 1} \right) \sqrt{2}p \frac{D(u)}{|D(u)|}, \quad (3.108)$$

and recognizing the inertial number $I = \sqrt{2}\,|D(u)|\, d\sqrt{\frac{\rho_s}{p}}$ in the above expression:

$$\left(\mu_2 - (\mu_2 - \mu_s) \frac{1}{|D(u)| \frac{\sqrt{2}d}{\sqrt{\frac{p}{\rho_s}}I_0} + 1} \right) \sqrt{2}p \frac{D(u)}{|D(u)|} = \left(\mu_2 - (\mu_2 - \mu_s) \frac{1}{\frac{I}{I_0} + 1} \right) \sqrt{2}p \frac{D(u)}{|D(u)|},$$

we obtain

$$\mathbb{S}(D) = \mu(I)\sqrt{2}p \frac{D}{|D|},$$

which is the material law propose in [?]. By inequality (3.107) we deduce that:

$$\mu_2 p \geqslant \frac{1}{\sqrt{2}} |\mathbb{S}(D)|.$$

The quantity $\frac{1}{\sqrt{2}} |\mathbb{S}(D)|$ is the second invariant of the tensor $\mathbb{S}(D)$ and is often written with the subindex II, i.e.,

$$\mathbb{S}(D)_{II} = \frac{1}{\sqrt{2}} |\mathbb{S}(D)|.$$

Then,

$$\mu_2 p \geqslant \mathbb{S}(D)_{II} \quad (3.109)$$

which is a threshold condition in the sense that the granular material behaves like a viscoplastic media with a yielding stress $\mu_2 p$ that depends on the pressure.

3.3 Dynamic equations for a dense granular heterogeneous flow

Based on the theory developed in Section 3.2, this chapter aims to construct the dynamic equations for a dense granular heterogeneous flow compose of $N-1$ granular flows and one Newtonian fluid. We consider that each granular material satisfy the Mohr-Coulomb constitutive equation (3.94) and (3.95) defined in 3.2.4 with different material parameters. For simplicity, we consider that the fluid and the granular components are isothermal and incompressible. The incorporation of other components adds several difficulties that do not arise in the one component case. The first problem that appears is that in the material laws for one component, that we review in Section 3.2.4, the Helmholtz free energy function ψ is a function of the temperature and the density. We arrived to this conclusion using the Liu identities (3.51)–(3.59). This condition simplifies the model, since the term

$$\sum \rho \frac{\partial \psi}{\partial u} u$$

is not include in the Cauchy stress tensor and there is no need to give a constitutive equation for ψ. The role of ψ is replaced by the pressure p. In the case of a mixture, we need to impose a constitutive equation for ψ_α for every $\alpha \in \{1, \cdots, N\}$, since we can't conclude that the functions ψ_α are independent of u_α using the Liu identities (3.51)–(3.59).

On the other hand, to propose a function Φ and ψ_α for $\alpha \in \{1, \ldots, N\}$, we need to consider the role of the interactions between the components.

The last and more technical problem is that the definition of the $\mathrm{tr}(\Sigma)$ is not necessarily independent of $D(u)$. This is a problem if we impose a yield condition depending on $\mathrm{tr}(\Sigma)$. We overcome the above challenges by proposing a suitable definition of ψ_α and the yield condition.

Let us describe the outline of this part. In Section 3.3.1, we propose an equation for the Cauchy stress tensor of a dense granular heterogeneous and isothermal mixture that depends on the Helmholtz free energy. In Section 3.3.2 we propose an expression for the Helmholtz free energy that contains the interaction terms. Then, in Section 3.3.3, we give a summary of all the equations that model the dynamics of the mixture. From Section 3.3.4 to Section 3.3.4 we develop the equations to study the case of a Couette flow between two concentric cylinders, and we provide a numerical solution.

3.3.1 Cauchy stress tensor for a dense granular mixture

According to Hypothesis 3.8 there exist functions $\mathbb{P}_\alpha : (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3 \mapsto \mathbb{R}$, $\mathbb{S}_\alpha : (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3 \mapsto \mathbb{M}_s^{3\times3}$, and a potential $\Phi : (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3 \mapsto \mathbb{R}$, that satisfies Definition 3.1, such that

$$\theta_\alpha \Sigma_\alpha = \gamma_\alpha \theta_\alpha \Lambda_\alpha I_3 + \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta + \mathbb{P}_\alpha(\overline{D}(u_1), \cdots, \overline{D}(u_N), \mathrm{tr}(D(u_1)), \cdots, \mathrm{tr}(D(u_N)), \nabla T) I$$
$$+ \mathbb{S}_\alpha(\overline{D}(u_1), \cdots, \overline{D}(u_N), \mathrm{tr}(D(u_1)), \cdots, \mathrm{tr}(D(u_N)), \nabla T), \quad (3.110)$$

$$\sum_{\alpha=1}^{N} \theta_\alpha Q_\alpha = -T\left(\sum_{\alpha=1}^{N} \rho_\alpha u_{\alpha,j} \frac{\partial \psi_\alpha}{\partial T} + \rho_\alpha u_{\alpha,j} s_\alpha\right) + \mathbb{Q}(\overline{D}(u_1), \cdots, \overline{D}(u_N), \mathrm{tr}(D(u_1)), \cdots, \mathrm{tr}(D(u_N)), \nabla T) \quad (3.111)$$

and

$$\begin{pmatrix} \mathbb{S}_1(D,\gamma,\Theta) \\ \vdots \\ \mathbb{S}_N(D,\gamma,\Theta) \\ \mathbb{P}_1(D,\gamma,\Theta) \\ \vdots \\ \mathbb{P}_N(D,\gamma,\Theta) \\ \mathbb{Q}(D,\gamma,\Theta) \end{pmatrix} \in \partial\Phi\left(D,\gamma,\Theta\right). \quad (3.112)$$

for all $(D,\gamma,\Theta) \in (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3$.

The choice of the potential Φ must impose that the temperature of the mixture is constant, as we are interested only in the mechanical behavior of the mixture. We also consider that the grannular components and the fluid are incompressible. Regarding to the mechanical behavior of the mixture, we consider that the fluid is newtonian and must satisfy an equation similar to (3.80), and we consider that the granular components satisfy a Mohr Coulomb constitutive equation similar to (3.94) and (3.95).

Finally, we assume that the form of Φ reflects only the mechanical behaviors of the fluid and the granular material without considering the interactions between the components. Then, we propose the following potential:

$$\Phi\left(D,\gamma,\Theta\right) = \mu_1\theta_1 \left|D\right|^2 + 1_0(\Theta) + \sum_{\beta=1}^{N} 1_0(\gamma_\beta) + \sum_{\beta=2}^{N} \theta_\beta A_\beta \left|D_\beta\right|, \quad (3.113)$$

for all $(D,\gamma,\Theta) \in (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3$, where μ_1 is the molecular viscosity of the fluid and A_β are the yielding stresses of the granular materials.

We recaall that the differential inclusion (3.112) means:

$$\Phi(D^*,\gamma^*,\Theta^*) - \Phi(D,\gamma,\Theta) \geqslant \sum_{\alpha=1}^{N} \mathbb{S}_\alpha : (D^* - D) + \sum_{\alpha=1}^{N} \mathbb{P}_\alpha\left(\gamma^* - \gamma\right) + \mathbb{Q}\cdot\left(\Theta^* - \Theta\right). \quad (3.114)$$

for all $(D^*,\gamma^*,\Theta^*) \in (\mathbb{M}_s^{3\times3})^N \times \mathbb{R}^N \times \mathbb{R}^3$.

From (3.113) and (3.114), we deduce the following:

- From inequality (3.114) we deduce that $\gamma^*_\alpha = 0$, for all $\alpha\{1,\ldots,N\}$. and $\Theta^* = 0$. Then,

$$\mathrm{div}(u_\alpha) = 0, \quad (3.115)$$

  and $\overline{D(u_\alpha)} = D(u_\alpha)$, for all $\alpha\{1,\ldots,N\}$, and T is constant. To simplify the notation we call $\Phi(D) = \Phi(D,0,0)$.

- By (3.115), $\mathbb{P}_\alpha$ can be any function, say $\theta_\alpha\kappa_\alpha$. To simplify the notation we rename Λ_α as $\Lambda_\alpha + \kappa_\alpha$. Then, we have that:

$$\frac{1}{3}\theta_\alpha \,\mathrm{tr}\left(\Sigma_\alpha\right) = \gamma_\alpha\theta_\alpha\Lambda_\alpha + \frac{1}{3}\,\mathrm{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial\psi_\beta}{\partial u_\alpha} u_\beta\right) \quad (3.116)$$

- Since the partial derivative of Φ with respect to D_1 exists, by Theorem 3.2, the Cauchy stress tensor of the newtonian fluid is:

$$\theta_1\Sigma_1 = \gamma_1\theta_1\Lambda_1 + \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_{1,k}} u_{\beta,j} + 2\mu\theta_1 D\left(u_1\right). \tag{3.117}$$

- For $\alpha \in \{2,\dots,N\}$, Φ is differentiable with respect to the variables D_α for $|D_\alpha| \neq 0$. Then, by Theorem 3.2,

$$\mathbb{S}_\alpha(D) = \theta_\alpha A_\alpha \frac{D_\alpha}{|D_\alpha|},$$

  for $D \in \left(\mathbb{M}_s^{3\times3}\right)^N$.

- To obtain a yielding condition for the granular component α, as we did in the case of Bingham fluid 3.2.4, the Mohr-Coulomb fluid 3.2.4 and the $\mu(I)$-rheology 3.2.4, we consider the case where the α-component of D is equals to 0, namely, $D_\alpha = 0$. Then we choose in (3.114)

$$D^* = \left(D_1,\dots,D_\alpha^*,\dots,D_N\right),$$

  where $D_\alpha^* \in \mathbb{M}_s^{3\times3}$. Then:

$$\Phi(D^*) - \Phi(D) = \theta_\alpha A_\alpha \left|D_\alpha^*\right|$$

  and

$$\sum_{\alpha=1}^{N} \mathbb{S}_\alpha : (D^* - D) = \mathbb{S}_\alpha.$$

  We combine the above expressions with inequality (3.114) and we obtain

$$\theta_\alpha A_\alpha \left|D_\alpha^*\right| \geqslant \mathbb{S}_\alpha : D_\alpha^*.$$

  We divide by D_α^* and we take the supremum over $D_\alpha^* \in \mathbb{M}_s^{3\times3}$, and we obtain the following threshold condition for the constituent α:

$$\theta_\alpha A_\alpha \geqslant \left|\mathbb{S}_\alpha\right|. \tag{3.118}$$

  Equation (3.118) together with (3.110)

$$\theta_\alpha A_\alpha \geqslant \left|\theta_\alpha\overline{\Sigma}_\alpha - \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta + \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta\right) I_3\right| \tag{3.119}$$

We still need to give a definition for the coeficcient A_α. By (3.116), the natural choice of A_α would be

$$A_\alpha = \mu_\alpha \left|p_\alpha\right|, \tag{3.120}$$

where

$$p_\alpha = \gamma_\alpha\theta_\alpha\Lambda_\alpha + \frac{1}{3}\operatorname{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta\right). \tag{3.121}$$

However, a desirable property is that

$$\frac{\partial p_\alpha}{\partial D\left(u_\gamma\right)_{ih}} = 0. \tag{3.122}$$

for all $\gamma \in \{1, \ldots, N\}$, otherwise the inclusion (3.112) would be confusing. In the case of a single component this problem do not arise since, by the Liu identities, the pressure is only a function of the density. In the case of a mixture we cannot prove that p_α, defined in (3.121), is independent of $D(u_\alpha)$. This is due to the inclusion of the saturation constraint that adds a multiplier μ_0 in the equation (3.51). To overcome this difficulty we define p_α as follows:

$$p_\alpha = -\frac{1}{3}\,\mathrm{tr}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j}\right) - \gamma_\alpha \theta_\alpha \Lambda_\alpha + \gamma_1 \theta_\alpha \Lambda_1.$$

To prove (3.122), first, using the Liu equation (3.61) for $\beta = \alpha$, we write the function p_α as follows:

$$p_\alpha = -\frac{1}{3}\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_{\alpha,i}} u_{\beta,i} + \theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial \theta_\alpha} - \theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial \theta_1}.$$

Then, the derivative of p_α with respect to $D\left(u_\eta\right)$ is:

$$\frac{\partial}{\partial D(u_\eta)_{ih}} p^\alpha = -\frac{1}{3}\frac{\partial}{\partial D(u_\eta)_{ih}}\left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_i^\alpha} u_i^\beta\right) + \frac{\partial}{\partial D(u_\eta)_{ih}}\left(\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi^\beta}{\partial \theta_\alpha}\right) - \frac{\partial}{\partial D(u_\eta)_{ih}}\left(\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi^\beta}{\partial \theta^1}\right).$$

Since θ_α, $D(u_\alpha)$ and u_α are independent variables and we assume that the function ψ is regular enough, we can change the order of the derivatives to obtain:

$$\frac{\partial}{\partial D(u_\eta)_{ih}} p_\alpha = -\frac{1}{3}\sum_{\beta=1}^{N} \rho_\beta u_{\beta,i} \frac{\partial}{\partial u_{\alpha,i}} \frac{\partial}{\partial D(u_\eta)_{ih}} \psi_\beta + \theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial \theta_\alpha} \frac{\partial}{\partial D(u_\eta)_{ih}} \psi_\beta - \theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial D(u_\eta)_{ih}} \psi_\beta.$$

We use the rule of the derivation of a product in both terms of the sum as follows:

$$\sum_{\beta=1}^{N} \rho_\beta u_{\beta,i} \frac{\partial}{\partial u_{\alpha,i}} \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta = \frac{\partial}{\partial u_{\alpha,i}}\left(\sum_{\beta=1}^{N} \rho_\beta u_{\beta,i} \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right) - \sum_{\beta=1}^{N} \rho_\beta \left(\frac{\partial}{\partial u_{\alpha,i}} u_{\beta,i}\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right),$$

$$\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial \theta_\alpha} \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta = \frac{\partial}{\partial \theta_\alpha}\left(\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right) - \theta_\alpha \sum_{\beta=1}^{N} \left(\frac{\partial}{\partial \theta_\alpha} \rho_\beta\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right)$$

and

$$\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta = \frac{\partial}{\partial \theta_1}\left(\theta_\alpha \sum_{\beta=1}^{N} \rho_\beta \frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right) - \theta_\alpha \sum_{\beta=1}^{N} \left(\frac{\partial}{\partial \theta_1} \rho_\beta\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}} \psi_\beta\right).$$

Using the Liu identities (3.55), (3.56) and:

$$\sum_{\beta=1}^{N}\rho_\beta\left(\frac{\partial}{\partial u_\alpha}u_{\beta,i}\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}}\psi_\beta\right)=3\rho_\alpha\frac{\partial}{\partial D(u_\gamma)_{ih}}\psi_\alpha,$$

$$\theta_\alpha\sum_{\beta=1}^{N}\left(\frac{\partial}{\partial\theta_\alpha}\rho_\beta\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}}\psi_\beta\right)=\theta_\alpha\gamma_\alpha\frac{\partial}{\partial D\left(u_\gamma\right)}\psi_\alpha$$

and

$$\theta_\alpha\sum_{\beta=1}^{N}\left(\frac{\partial}{\partial\theta_1}\rho_\beta\right)\left(\frac{\partial}{\partial D(u_\gamma)_{ih}}\psi_\beta\right)=\theta_\alpha\gamma_1\frac{\partial}{\partial D\left(u_\gamma\right)}\psi_1.$$

we obtain

$$\frac{\partial}{\partial D(u_\eta)_{ih}}p_\alpha=\theta_\alpha\gamma_1\frac{\partial}{\partial D\left(u_\gamma\right)}\psi_1. \tag{3.123}$$

To prove (3.122) we need to make assumptions on the function ψ_1.

3.3.2 Helmholtz free energy

To close the system of equations (3.145)–(3.145) we have to provide an expression for the Helmholtz free energy ψ_α. Consequently, we assume the following:

Hypothesis 3.9. *The Helmholtz free energy of a constituent α has the next form:*

$$\psi_\alpha=\psi_\rho^\alpha\left(\theta_\alpha\right)+\sum_{\eta=1}^{N}C_{\alpha\eta}\theta_\eta\left|u_\eta-u_\alpha\right|^2, \tag{3.124}$$

where ψ_ρ^α is a function that depends on the volume fraction of the constituent α and the second term is a contribution of the interaction between the constituent α and the other constituents in the mixture.

In the constitutive equation (3.124), we call $C_{\alpha\eta}$ the drag constant, since the second term of the right-hand side of (3.124) represents the role of a drag term in the momentum equation of the constituent α.

To clarify the above statement, we calculate the term $\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi^\beta}{\partial u_\alpha}u_\beta$ in the Cauchy stress tensor. First, we note that for every α,

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=\rho_\alpha\frac{\partial\psi_\alpha}{\partial u_{\alpha,k}}u_{\alpha,j}+\sum_{\beta\neq\alpha}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}.$$

Since the function ψ_ρ^α does not depend on the velocity we obtain:

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=\rho_\alpha u_{\alpha,j}\frac{\partial}{\partial u_{\alpha,k}}\left(\sum_\eta\theta_\eta C_{\alpha\eta}\left|u_\eta-u_\alpha\right|^2\right)+\sum_{\beta\neq\alpha}\rho_\beta u_{\beta,j}\frac{\partial}{\partial u_{\alpha,k}}\left(\sum_\eta\theta_\eta C_{\beta\eta}\left|u_\eta-u_\beta\right|^2\right).$$

Then,

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=-\sum_\eta\rho_\alpha\theta_\eta 2C_{\alpha\eta}\left(u_{\eta,k}-u_{\alpha,k}\right)u_{\alpha,j}+\sum_{\beta\neq\alpha}\rho_\beta\theta_\alpha 2C_{\beta\alpha}\left(u_{\alpha,k}-u_{\beta,k}\right)u_{\beta,j}$$

Gathering both summations in the previous equality, we have that:

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=2\rho_\alpha\sum_{\beta=1}^{N}\theta_\beta\left(C_{\beta\alpha}\left(u_{\alpha,k}-u_{\beta,k}\right)u_{\beta,j}-C_{\alpha\beta}\left(u_{\beta,k}-u_{\alpha,k}\right)u_{\alpha,j}\right).$$

We factorize by the term $u_{\alpha,k}-u_{\beta,k}$ in the summation and obtain:

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=2\rho_\alpha\sum_{\beta=1}^{N}\theta_\beta\left(u_{\alpha,k}-u_{\beta,k}\right)\left(C_{\beta\alpha}u_{\beta,j}+C_{\alpha\beta}u_{\alpha,j}\right)$$

The above equation implies that the drag force between the constituents vanishes if there is no interaction between the constituents. Under Hypothesis 3.9 and the Liu identities we prove that the drag force satisfies Newton's Third Law. To prove this, we notice that in order to satisfy the Liu identity:

$$\frac{\partial}{\partial u_{\alpha,k}}\psi_I=0,$$

since:

$$\frac{\partial}{\partial u_{\alpha,k}}\psi_I=2\rho_\beta\sum_{\eta}\theta_\eta(u_{\beta,k}-u_{\alpha,k})\left(C_{\alpha\beta}+C_{\beta\alpha}\right),$$

we need to choose $C_{\alpha\beta}$ and $C_{\beta\alpha}$ such that

$$C_{\beta\alpha}+C_{\alpha\beta}=0, \tag{3.125}$$

for every α and β in $\{1,\ldots,N\}$. Then, the drag term in the Cauchy stress tensor is:

$$\sum_{\beta=1}^{N}\rho_\beta\frac{\partial\psi_\beta}{\partial u_{\alpha,k}}u_{\beta,j}=2\rho_\alpha\sum_{\beta=1}^{N}\theta_\beta C_{\alpha\beta}\left(u_{\alpha,k}-u_{\beta,k}\right)\left(u_{\alpha,j}-u_{\beta,j}\right). \tag{3.126}$$

We notice that the above tensor is symmetric which is a restriction from the Liu identity (3.54). To complete the formula for the Helmholtz free energy, we assume the following:

Hypothesis 3.10. *For all $\alpha\in\{1,\ldots,N\}$, we define*

$$\psi_\rho^\alpha(\theta_\alpha)=\frac{c_\alpha}{\theta_\alpha\gamma_\alpha}\left(\theta_\alpha-\theta_{\alpha,c}\right)^2, \tag{3.127}$$

where c_α is a positive constant and $\theta_{\alpha,c}$ is called the critical volume fraction for solid particles, above which shearing of the material will cause dilatancy, below which it will cause contraction [?].

The above formula follows the approach of [?] (see also [?] and [?]) . Under the above assumptions on ψ_α, we have that:

$$\frac{\partial}{\partial\theta_\beta}\psi^\alpha=\begin{cases}\frac{c_\alpha}{\gamma_\alpha}\frac{(\theta_\alpha)^2-(\theta_{\alpha,c})^2}{(\theta_\alpha)^2} & \text{if } \beta=\alpha,\\ C_{\alpha\beta}\left|u_\beta-u_\alpha\right|^2 & \text{if } \beta\neq\alpha.\end{cases} \tag{3.128}$$

Finally, using Hypothesis 3.9 in (3.123), we deduce (3.122).

3.3.3 Dynamics equations of a dense granular flow

Regarding the discussion of Section (3.2) and sections 3.3.1 and 3.3.2, in this section we present a summary of the dynamics of a dense granular flow composed $N-1$ granular components, satisfying a Mohr-Coulomb constitutive equation, immenersed in an incompressible Newtonian fluid.

The dynamics of this system is given by the functions

$$(\theta_1, \dots, \theta_N, u_1, \dots, u_N, \Lambda_1, \dots, \Lambda_N)$$

and is governed by the following set of equations:

- The saturation constraint

$$\sum_{\alpha=1}^{N} \theta_\alpha = 1, \text{ in } (0,T) \times V. \tag{3.129}$$

- The mass conservation equations

$$\frac{\partial \theta_\alpha}{\partial t} + u_\alpha \cdot \nabla \theta_\alpha = 0, \text{ in } (0,T) \times V \quad \text{for } \alpha \in \{1, \dots, N\}. \tag{3.130}$$

- The momentum conservation equations

$$\rho_\alpha \left(\frac{\partial u_\alpha}{\partial t} + (u_\alpha \cdot \nabla) u_\alpha \right) = \theta_\alpha \operatorname{div}(\Sigma_\alpha) + \rho_\alpha f_\alpha,$$
$$\text{in } (0,T) \times V \text{ for } \alpha \in \{1, \dots, N\}. \tag{3.131}$$

Where the Cauchy stress tensor of the fluid is given by:

$$\theta_1 \Sigma_1 = \gamma_1 \theta \Lambda_1 I_3 + \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_1} u_\beta + 2\theta_1 \mu D(u_1), \tag{3.132}$$

and the Cauchy stress tensor of the solid components are given by:

$$\theta_\alpha \Sigma_\alpha = \gamma_\alpha \theta_\alpha \Lambda_\alpha I_3 + \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta + \mathbb{S}_\alpha(D(u_\alpha)) \quad \text{for } \alpha \in \{2, \dots, N\}, \tag{3.133}$$

where

$$\begin{pmatrix} \mathbb{S}_2(D) \\ \vdots \\ \mathbb{S}_\alpha(D) \end{pmatrix} \in \partial \Phi(D), \tag{3.134}$$

where $\Phi : (\mathbb{M}_s^{3\times 3})^{N-1} \mapsto \mathbb{R}$,

$$\Phi(D) = \sum_{\alpha=2}^{N} \mu_\alpha |p_\alpha| |D_\alpha|, \tag{3.135}$$

for $D = (D_2, \dots, D_N)$. In the definition of Φ

$$p_\alpha = -\frac{1}{3} \operatorname{tr} \left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_{\alpha,k}} u_{\beta,j} \right) - \gamma_\alpha \theta_\alpha \Lambda_\alpha - \gamma_1 \theta_\alpha \Lambda_1. \tag{3.136}$$

- The Helmholtz free energy function is given by:

$$\psi_\alpha = \frac{c_\alpha}{\gamma_\alpha \theta_\alpha} \left(\theta_\alpha - \theta_{\alpha,c}\right)^2 + \sum_{\eta=1}^{N} C_{\alpha\eta} \theta_\eta \left|u_\eta - u_\alpha\right|^2 \quad \text{for } \alpha \in \{1, \ldots, N\}, \tag{3.137}$$

  where $C_{\eta\alpha}$ is the drag constant satisfying the symmetry relation:

$$C_{\eta\alpha} = -C_{\alpha\eta}, \tag{3.138}$$

  and $\theta_{\alpha,c}$ is the critical volume fraction.

- To close the system of equations, we use the Liu identity:

$$\sum_{\alpha=1}^{N} \rho_\alpha \left(\frac{\partial \psi_\alpha}{\partial \theta_\beta} - \frac{\partial \psi_\alpha}{\partial \theta_1} \right) + \gamma_\beta \Lambda_\beta - \gamma_1 \Lambda_1 = 0. \tag{3.139}$$

3.3.4 Couette flow between two concentric cylinders

In this section, we study the problem of a stationary granular heterogeneous flow between two concentric cylinders using the equations developed in the previous parts (see Section 3.3.3 for a summary). This problem is called Couette flow, and its solution allows us to obtain a simple validation of our model.

Couette Flow

A Couette flow refers to the motion of a fluid between two parallel plates, one of which is moving relative to the other. The flow is driven by internal forces acting on the fluid and the applied pressure gradient parallel to the plates. The Couette flow is the most straightforward and meaningful case to illustrate fluid motion because we can provide analytical solutions.
A flow between two concentric cylinders of infinite length is a Couette flow in polar coordinates, in which the internal cylinder and the external cylinder are rotating with angular velocity Ω_1 and Ω_2 respectively.

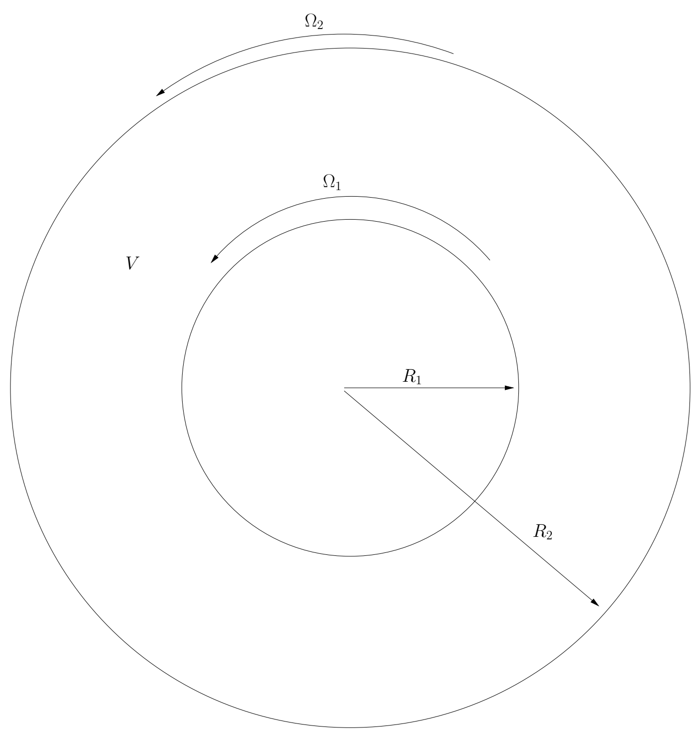

Figure 3.6: Scheme of two concentric infinite cylinders of radius R_1 and R_2 rotating with angular velocities Ω_1 and Ω_2, respectively.

In the framework of the mixture theory, we will study the stationary flow between two concentric cylinders of a mixture of a Newtonian fluid and a granular fluid satisfying the equations described in Section 3.3.3 for $N = 2$. The internal cylinder is rotating with angular velocity Ω_1, and the external cylinder is fixed ($\Omega_2 = 0$). The central hypothesis is that the velocity of the fluid and the granular component, still denoted by u_1 and u_2, the volume fractions θ_1 and θ_2, and the Lagrange multipliers Λ_1 and Λ_2 only depend on the radial variable and the velocities flow in the azimuthal direction indicated by e_θ. More precisely:

Hypothesis 3.11. *There exist two functions* $u_{1,\theta} : [R_1, R_2] \mapsto \mathbb{R}$ *and* $u_{2,\theta} : [R_1, R_2] \mapsto \mathbb{R}$ *such that:*

$$u_1 = u_{1,\theta}(r)e_\theta \text{ and } u_2 = u_{2,\theta}(r)e_\theta.$$

We also have that $\theta_1 : [R_1, R_2] \mapsto \mathbb{R}$, $\theta_2 : [R_1, R_2] \mapsto \mathbb{R}$, $\Lambda_1 : [R_1, R_2] \mapsto \mathbb{R}$ *and* $\Lambda_2 : [R_1, R_2] \mapsto \mathbb{R}$ *.*

To simplify the calculations we assume the following:

Hypothesis 3.12.

$$\theta_1 \in (0, 1) \text{ and } \theta_2 \in (0, 1).$$

The first step is to write the balance equations for two components. Under the Hypothesis 3.11, Hypothesis 3.12 and the incompressibility of the liquid and the granular component, the relevant equations are the momentum equations (3.131):

$$\operatorname{div}(\Sigma_\alpha) = 0, \tag{3.140}$$

for all $\alpha \in \{1, 2\}$, and the saturation constraint:

$$\theta_1 + \theta_2 = 1. \tag{3.141}$$

By Hypothesis 3.11 and the symmetry of the Cauchy stress tensor, we deduce

$$\Sigma_\alpha = \begin{pmatrix} 0 & \Sigma_{\alpha,r\theta} \\ \Sigma_{\alpha,r\theta} & 0 \end{pmatrix} \quad \alpha \in \{1, 2\}, \tag{3.142}$$

where we need to deduce $\Sigma_{\alpha,r\theta}$ from (3.132) and (3.133).

Dynamic equations for the motion of the fluid component

In the case of a two component mixture, the drag term in (3.132) is:

$$\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_1} u_\beta = 2\rho_1\theta_2 C_{12}(u_1 - u_2) \otimes (u_1 - u_2). \tag{3.143}$$

Then, from (3.132), we deduce

$$\begin{aligned} \operatorname{div}(\Sigma_1) = {} & \gamma_1 \nabla\Lambda_1 + 2\mu \operatorname{div}(D(u_1)) + 2\gamma_1 C_{12}(u_1 - u_2) \otimes (u_1 - u_2)\nabla(\theta_1\theta_2) \\ & + 2\rho_1 C_{12}\theta_2 \nabla(u_1 - u_2)(u_1 - u_2) + 2\rho_1 C_{12}\theta_2(u_1 - u_2) \operatorname{div}(u_1 - u_2). \end{aligned} \tag{3.144}$$

The next step is to write the momentum balance equation in the polar coordinates system (r, θ). Under Hypothesis 3.11 we have the following:

$$\nabla(u_1 - u_2) = \begin{pmatrix} 0 & -\frac{(u_{1,\theta} - u_{2,\theta})}{r} \\ \frac{\partial}{\partial r}(u_{1,\theta} - u_{2,\theta}) & 0 \end{pmatrix}, \tag{3.145}$$

$$\nabla\theta_2 = \begin{pmatrix} \frac{\partial \theta_2}{\partial r} \\ 0 \end{pmatrix}, \tag{3.146}$$

$$\nabla\Lambda_1 = \begin{pmatrix} \frac{\partial \Lambda_1}{\partial r} \\ 0 \end{pmatrix}, \tag{3.147}$$

$$(u_1 - u_2) \otimes (u_1 - u_2) = \begin{pmatrix} 0 & 0 \\ 0 & (u_{1,\theta} - u_{2,\theta})^2 \end{pmatrix}, \tag{3.148}$$

$$\nabla(u_1 - u_2) \cdot (u_1 - u_2) = -\begin{pmatrix} \frac{(u_{1,\theta} - u_{2,\theta})^2}{r} \\ 0 \end{pmatrix}, \tag{3.149}$$

and

$$\text{div}(u_1 - u_2) = 0. \tag{3.150}$$

Then, using (3.145)-(3.150), in (3.144), we write the momentum equation of the fluid (3.140):

$$0 = \frac{d\Lambda_1}{dr} - 2\theta_2 C_{12} \frac{(u_{1,\theta} - u_{1,\theta})^2}{r}, \tag{3.151}$$

$$0 = \frac{1}{r^2}\frac{d}{dr} r^2 \Sigma_{1,r\theta}, \tag{3.152}$$

where

$$\Sigma_{1,r\theta} = \mu r \frac{d}{dr}\frac{u_1}{r}. \tag{3.153}$$

Equation (3.152) implies that

$$\Sigma_{1,r\theta} = \frac{A_1}{r^2}, \tag{3.154}$$

where A_1 is constant. The constant A_1 represents a given torque applied to the inner cylinder.

Then, by (3.152) and (3.154), we deduce the following ODE for u_1:

$$\frac{d}{dr}\frac{u_1}{r} = \frac{A_1}{\mu r^3}.$$

The, above equation has the following solution:

$$u_{1,\theta}(r) = \frac{A_1}{2\mu}\left(\frac{1}{r} + rB\right),$$

where B is an integration constant. Considering the Dirichlet boundary condition in the outer cylinder:

$$u_{1,\theta}(R_2) = 0,$$

the velocity of the fluid is given by

$$u_{1,\theta}(r) = \frac{A_1}{2\mu}\left(\frac{1}{r} - \frac{r}{R_2^2}\right).$$

Then , the angular velocity of the fluid ω_1, which is defined as

$$\omega_1(r) = \frac{u_{1,\theta}}{r},$$

is given by the following expression:

$$\omega_1(r) = \frac{A_1}{2\mu}\left(\frac{1}{r^2} - \frac{1}{R_2^2}\right),$$

and (3.151) is written as:

$$\Lambda_1' = 2r\theta_2 C_{12}(\omega_1 - \omega_2)^2.$$

Dynamic equations for the granular component

In the solid component, the interaction term in the Cauchy stress tensor is

$$\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi^\beta}{\partial u_1} u_\beta = 2\rho_2\theta_1 C_{21}(u_2 - u_1) \otimes (u_2 - u_1).$$

Then, in the case when $|D(u_2)| \neq 0$, the force due to the Cauchy stress tensor in the momentum equation is:

$$\begin{aligned} \operatorname{div}(\Sigma_2) = \gamma_2 \nabla \Lambda_2 + \mu_2 p_2 \frac{D(u_2)}{|D(u_2)|} + 2\rho_2 C_{21}(u_2 - u_1) \otimes (u_2 - u_1) \nabla(\theta_1\theta_2) \\ + 2\rho_2 C_{21}\theta_1 \nabla(u_2 - u_1)(u_2 - u_1) + 2\rho_2 C_{21}\theta_1 (u_2 - u_1) \operatorname{div}(u_2 - u_1). \end{aligned} \tag{3.155}$$

By Hypothesis 3.11 and using (3.134), (3.145)–(3.150), (3.155) and the angular velocity of the solid

$$\omega_2(r) = \frac{u_{2,\theta}}{r},$$

the momentum balance equations of the solid component are:

$$\Lambda_2'(r) = 2r\theta_1 C_{21}(\omega_1 - \omega_2)^2, \tag{3.156}$$

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \Sigma_{2,r\theta}\right), \tag{3.157}$$

for all $r \in (0, r)$ such that $|\omega_2'| \neq 0$, where:

$$\Sigma_{2,r\theta} = \frac{\mu_2 \, |p_2|}{\sqrt{2}} \operatorname{sign}(\omega_2'). \tag{3.158}$$

where p_2 satisfies (3.136) which, using Hypothesis 3.11, is written as:

$$p_2(r) = -\frac{2}{3}\rho_2\theta_1 C_{21} r^2 (\omega_2 - \omega_1)^2 - \gamma_2\theta_2\Lambda_2 + \gamma_1\theta_2\Lambda_1, \tag{3.159}$$

Equation (3.157) implies that

$$\Sigma_{2,r\theta} = \frac{A_2}{r^2}, \tag{3.160}$$

where A_2 is a constant interpreted as the applied torque of the inner cylinder to the granular component. Assuming that A_2 is a positive constant, by equation (3.158) and since

$$\operatorname{sign}(\omega_2') = \operatorname{sign}(A_2),$$

we deduce

$$|p_2| = \frac{\sqrt{2}A_2}{\mu_2 r^2}, \tag{3.161}$$

where p_2 satisfies (3.136) which, using Hypothesis 3.11, is:

$$p_2 = -\frac{2}{3}\rho_2\theta_1 C_{21} r^2 \omega_{rel}^2 + \theta_2\left(\gamma_1\Lambda_1 - \gamma_2\Lambda_2\right). \tag{3.162}$$

The above equation holds when the granular component behaves like a fluid, i.e. for all $r \in (R_1, R_2)$ such that $|\omega_2'| \neq 0$. This is a free surface problem where the free surface is characterize by inequality (3.119). To express (3.119) in the context of the Couette problem, we start by the traceless part of $\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta$:

$$\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta - \frac{1}{2} \operatorname{tr} \left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_\alpha} u_\beta \right) I_2 = \rho_2 \theta_1 C_{21} r^2 (\omega_2 - \omega_1)^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\left| \theta_\alpha \overline{\Sigma}_2 - \sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_2} u_\beta + \frac{1}{2} \operatorname{tr} \left(\sum_{\beta=1}^{N} \rho_\beta \frac{\partial \psi_\beta}{\partial u_2} u_\beta \right) I_2 \right| = \sqrt{2} \theta_2 \sqrt{\overline{\Sigma}_{2,r\theta}^2 + \gamma_2^2 C_{21}^2 \theta_1^2 r^4 (\omega_1 - \omega_2)^4}.$$

Then, in the zone where the granular material is flowing, by inequality (3.119), we have that:

$$\sqrt{\overline{\Sigma}_{2,r\theta}^2 + \gamma_2^2 C_{21}^2 \theta_1^2 r^4 (\omega_1 - \omega_2)^4} \geqslant \frac{\mu_s p_2}{\sqrt{2}}.$$

Using the equation (3.160), The above inequality implies:

$$2\gamma_2^2 C_{21}^2 \theta_1^2 r^4 (\omega_1 - \omega_2)^4 \geqslant \mu_s^2 p_2^2 - \frac{2A_2^2}{r^4}. \tag{3.163}$$

Using equation (3.161), we deduce that inequality (3.163) does not contribute information to the problem. However, in the zone where the granular component is not flowing

$$\frac{2A_2^2}{r^4} \leqslant \mu_s^2 p_2^2 + 2\gamma_2^2 C_{21}^2 \theta_1^2 r^4 (\omega_1 - \omega_2)^4, \tag{3.164}$$

where p_2 is given by (3.162).

Equation for the Lagrange multipliers

To close the system of equations we consider the Liu identity (3.139). Using Hypothesis 3.11, we write (3.139) in terms of ω_1 and ω_2:

$$\gamma_1 \Lambda_1 - \gamma_2 \Lambda_2 = (\rho_1 + \rho_2) C_{12} r^2 (\omega_1 - \omega_2)^2 + c_2 \frac{\theta_2^2 - \theta_{2,c}^2}{\theta_2} - c_1 \frac{\theta_1^2 - \theta_{1,c}^2}{\theta_1}. \tag{3.165}$$

In the case of a two component mixture we consider the values of $\theta_{\alpha,c}$ given by [**?**, p.171]:

$$\theta_{2,c} = 0,52 \text{ and } \theta_{2,c} = \theta_{1,c} - 0,52. \tag{3.166}$$

Summary of the equations

We define the relative velocity

$$\omega_{rel} = \omega_1 - \omega_2, \tag{3.167}$$

where

$$\omega_1(r) = \frac{A_1}{2\mu} \left(\frac{1}{r^2} - \frac{1}{R_2^2} \right),$$

and the relative pressure

$$\Lambda_{rel} = \gamma_1 \Lambda_1 - \gamma_2 \Lambda_2. \tag{3.168}$$

Then, gathering the results of sections 3.3.4 and 3.3.4, the dynamics of the mixture is given by $(\omega_{rel}, \theta_2, \Lambda_{rel})$ solving the following system of equations:

$$\Lambda'_{rel} = -4rC_{21}(\gamma_1\theta_2 - \gamma_2\theta_1)\omega_{rel}^2,\ r \in \Omega_f, \tag{3.169}$$

$$\theta_1\theta_2\Lambda_{rel} = \theta_1\theta_2 C_{21}\left(\rho_1 + \rho_2\right) r^2\omega_{rel}^2 + c_2\theta_1\left(\theta_2^2 - \theta_c^2\right) - c_1\theta_2\left(\theta_1^2 - \theta_{1,c}^2\right),\ r \in \Omega_f, \tag{3.170}$$

$$\frac{\sqrt{2}A_2}{\mu_s r^2} = -\frac{2}{3}\rho_2\theta_1 C_{21} r^2\omega_{rel}^2 + \theta_2\Lambda_{rel},\ r \in \Omega_f, \tag{3.171}$$

$$\omega_{rel} = \omega_1,\ r \in \Omega_s, \tag{3.172}$$

where

$$\Omega_s = \left\{ r \in (R_1, R_2) \;\middle|\; \frac{1}{\mu_s}\sqrt{\frac{2A_2^2}{r^4} - 2\gamma_2^2 C_{21}^2\theta_1^2 r^4\omega_{rel}^4} + \frac{2}{3}\rho_2\theta_1 C_{21} r^2\omega_{rel}^2 \geqslant \theta_2\left|\Lambda_{rel}\right| \right\}, \tag{3.173}$$

$$\Omega_f = [R_1, R_2] \setminus \Omega_s, \tag{3.174}$$

c_1 and c_2 are a constants related to the Helmholtz free energy, A_1 and A_2 are the applied torque to the fluid and the granular component respectively and

$$\theta_1 = 1 - \theta_2, \quad r \in (R_1, R_2). \tag{3.175}$$

Numerical solution

If we consider that equation (3.171) is satisfied in $[R_1, R_2]$, then, we check that $\Omega_s = \emptyset$ and we deduce that $\Omega_f = [R_1, R_2]$. Considering the Dirichlet boundary conditions in the outer cylinder

$$\omega_{rel}(2) = 0,$$

we deduce from (3.170) and (3.171):

$$\theta_2(2)\Lambda_{rel}(2) = \frac{\sqrt{2}A_2}{\mu_s 4}, \tag{3.176}$$

$$\theta_1(2)\theta_2(2)\Lambda_{rel}(2) = c_2\theta_1\left(\theta_2(2)^2 - \theta_c(2)^2\right) - c_1\theta_2(2)\left(\theta_1(2)^2 - \theta_{1,c}(2)^2\right). \tag{3.177}$$

Then, the system of equations (3.169)–(3.171) is a differential algebraical system of equations (DAE). We solve system (3.169)–(3.171) using the MATLAB function ode15s (see Appendix B to see the MATLAB code). To obtain relevant and comparable results we consider the physical parameters taken in [?]. For the dimension of the inner and external cylinder we consider $R_1 = 1$[m] and $R_2 = 2$[m].

The figures 3.7–3.9 represent the solution of the system of equations (3.169)–(3.171) where we have considered the parameters: $A_1 + A_2 = 10^3[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]$, $\mu = 0.001[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}]$, $\mu_s = 723[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}]$, $\gamma_1 = 1000[\text{kg} \cdot \text{m}^{-3}]$, $\gamma_2 = 2200[\text{kg} \cdot \text{m}^{-3}]$, $c_1 = 1000[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]$, $c_2 = 2000[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]$ $C_{21} = 10^5[\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}]$, $\theta_{c,1} = 0.44$ and $\theta_{c,2} = 0.56$.

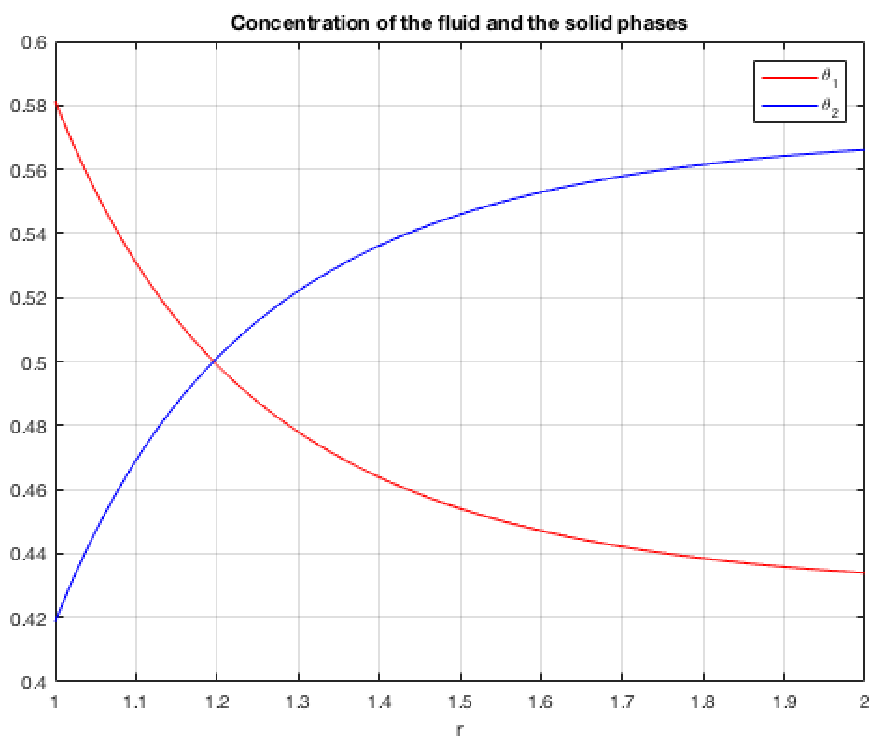

Figure 3.7: Volume fraction os the fluid and solid phases

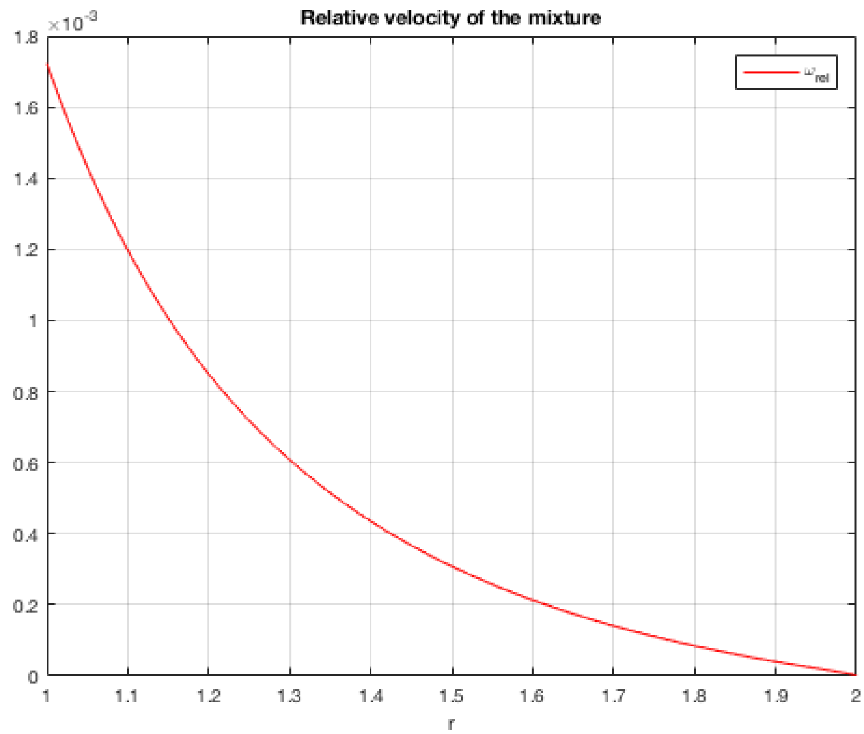

Figure 3.8: Relative velocity of the mixture.

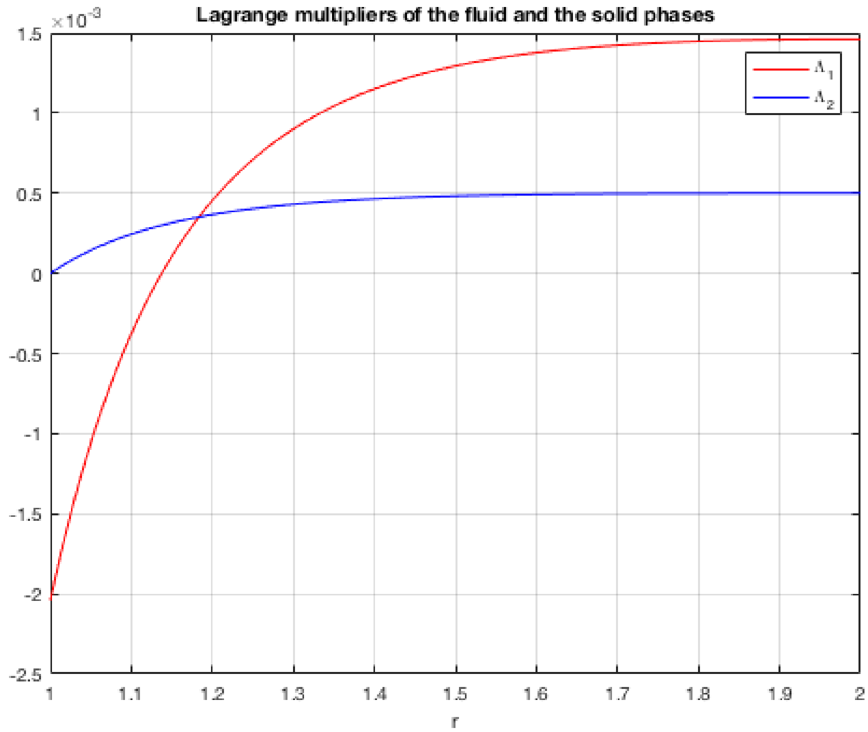

Figure 3.9: Lagrange multipliers of the fluid and the solid phases.

Figure 3.10 is the plot of $\omega_{rel}(1)$ (the relative angular velocity of the mixture in the inner cylinder) for different values of the applied total torque $A = A_1 + A_2$, where we have considered the parameters [kg $\cdot$ m^{-1} $\cdot$ s^{-2}], $\mu = 0.001$[kg $\cdot$ m $^{-1}$ $\cdot$ s^{-1}], $\mu_s = 723$[kg $\cdot$ m $^{-1}$ $\cdot$ s^{-1}], $\gamma_1 = 1000$[kg $\cdot$ m^{-3}], $\gamma_2 = 2200$[kg $\cdot$ m^{-3}], $c_1 = 10$[kg $\cdot$ m^{-1} $\cdot$ s^{-2}], $c_2 = 20$[kg $\cdot$ m^{-1} $\cdot$ s^{-2}] $C_{21} = 10^5$[kg $\cdot$ m^{-1} $\cdot$ s^{-2}], $\theta_{c,1} = 0.44$ and $\theta_{c,2} = 0.56$.

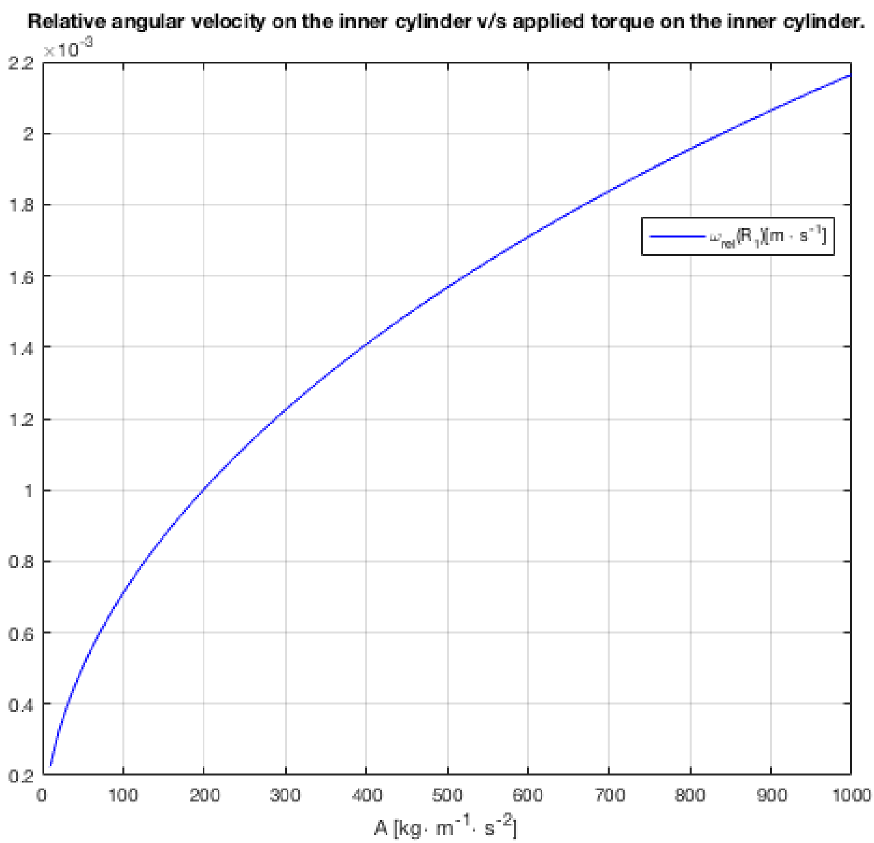

Figure 3.10: Relative angular velocity of the mixture at the inner cylinder $\omega(R_1)$ for different values of the applied torque A.

Figure 3.11 is the plot of $\omega_{rel}(r)$ for $r \in [1, 2]$ for different values of C_{21} where we have considered the parameters: $A = A_1 + A_2 = 1000[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $\mu = 0.001[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}]$, $\mu_s = 723[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}]$, $\gamma_1 = 1000[\mathrm{kg} \cdot \mathrm{m}^{-3}]$, $\gamma_2 = 2200[\mathrm{kg} \cdot \mathrm{m}^{-3}]$, $c_1 = 10[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $c_2 = 20[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$ $C_{21} = 10^5[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $\theta_{c,1} = 0.44$ and $\theta_{c,2} = 0.56$.

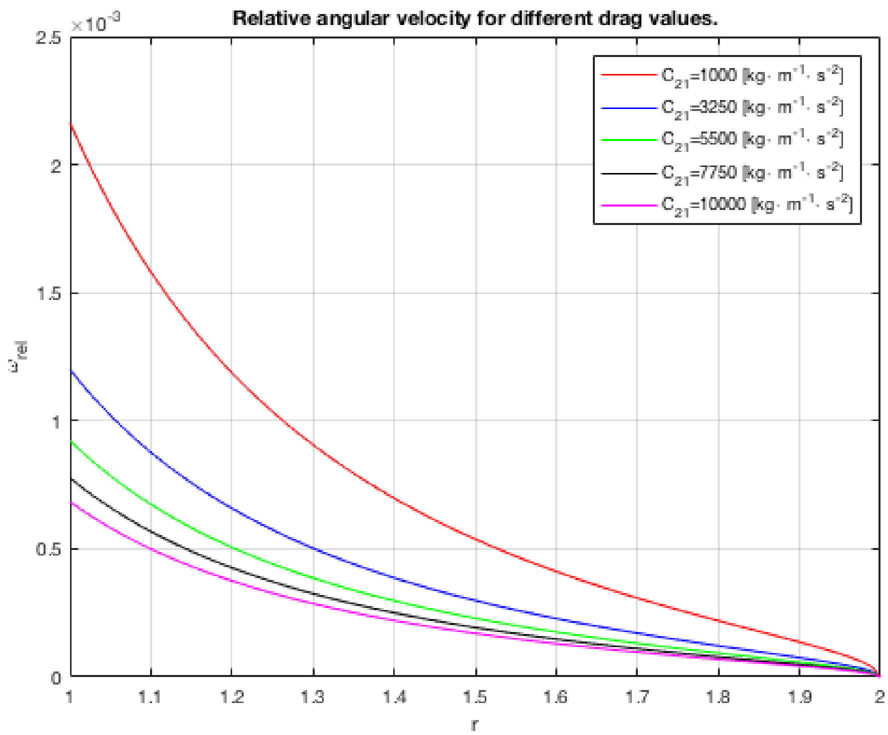

Figure 3.11: Relative angular velocity of the mixture for different values of the drag constant.

Figure 3.12 is the plot of $\theta_2(r)$ for $r \in [1, 2]$ for different values of c_1 and c_2 where we have considered the parameters: $A = A_1 + A_2 = 1000[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $\mu = 0.001[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}]$, $\mu_s = 723[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}]$, $\gamma_1 = 1000[\mathrm{kg} \cdot \mathrm{m}^{-3}]$, $\gamma_2 = 2200[\mathrm{kg} \cdot \mathrm{m}^{-3}]$, , $C_{21} = 5000[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $\theta_{c,1} = 0.44$ and $\theta_{c,2} = 0.56$.

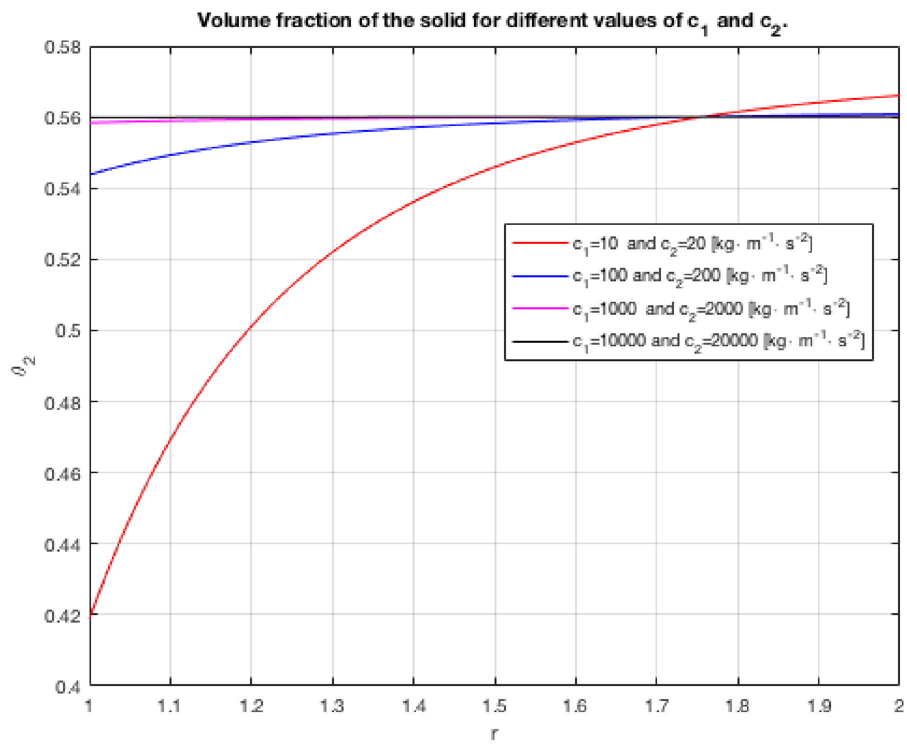

Figure 3.12: Volume fraction of the solid for different values of c_1 and c_2.

3.4 Discussion and conclusion

We recall that Wang and Hutter[**?**] propose a constitutive model for a multiphase mixture and study the Couette-flow of a saturated solid-fluid mixture. To derive their equations they use, besides the balance of mass, momentum and energy, the balance law for equilibrated forces, as proposed by Goodman and Cowin[**?**]. Also, their equations satisfy the four metaphysical principles from the mixture theory and are the result of using the Müller-Liu entropy principle imposing that the gradient of the volume fraction is an independent variable, which is a different assumption from ours. Other significant difference is the constitutive equation for the solid component. They use a viscous model with a volume fraction-dependant viscosity. At a first sight, from Figure 3.7 and Figure 3.8, our solution is similar to the one obtained by [**?**, Figure 2] (using the same physical parameters). In both cases, the volume fraction of the solid is an increasing function of r and is bounded in $[0.4, 0.6]$, and the relative velocity is nearly 0. Concerning Figure 3.9, we cannot compare our results directly with the ones of Wang and Hutter, since the Lagrange multipliers Λ do not appear directly in their equations.

An interesting and intuitive result is shown in Figure 3.10. The relative velocity at the inner cylinder increases when the applied torque increase, meaning that the liquid increase its velocity faster than the solid component. This could be understood as an early stage of a decantation phenomenon for the solid. However, the decantation makes sense only in a time-dependent dynamics.

Another interesting result is shown in Figure 3.11. The relative velocity decreases when the Drag constant increases. This results, also noticed by [**?**, Figure 2] and [**?**], is intuitive since the drag coefficient represents the attraction force between the different constituents.

Finally, in Figure 3.12 we show that the volume fraction of the solid approaches to a constant while the Helmholtz free energy constants c_1 and c_2 increase their value. Mathematically, this result is evident from equation (3.170), since the term related to Λ_{rel} and ω_{rel} loose importance when c_1 and c_2 are relatively high.

Concerning the above analysis, we conclude that our model satisfies features of a dense solid-fluid mixture qualitatively. The main advantages of our model concerning the model of Wang and Hutter are that our model is simpler since we only have one differential equation and two algebraical equations for the three main unknowns. On the other hand, Wang and Hutter have to deal with two differential equations for the volume fractions. This is complicated since an assumption of what is the volume fraction at walls needs to be made. In our case, this is not necessary, and we only need to impose that the volume fractions and the applied torques at the inner wall, A_1 and A_2, are consistent with the equations. Then, our approach seems promising. However, many important features of granular fluids and our model were left behind in this example. For instance:

- In this example, we were not able to find a solution where the granular component develops a yielding zone. This is because we were not able to adapt the main algorithms known for Bingham fluid's [**?**] to our equations.

- By the geometric characteristics of this problem, the only relevant part of the drag term was the quadratic one (see (3.144) and (3.155)). Then, for this problem, there is no difference between considering a quadratic drag term as a source term, rather than to impose a drag energy in the Helmholtz free energy (Hypothesis 3.9).

As we can see, there are many aspects of our model that needs development to fully conclude that our approach is useful to model a wide range of granular materials. A future investigation of this approach should consider the points mentioned above plus the study of the non-stationary dynamics.

Part II

Multiphase models involving rigid structures

Chapter 4

A Bingham fluid-rigid body system

In this chapter, we analyze a granular material using a multiphase fluid-structure interaction system where the fluid is an incompressible Bingham viscoplastic liquid, introduced in Section 3.2.4, and where the structure is a rigid body. This material is modeled by the 3D Bingham equations, and the Newton laws govern the displacement of the rigid body. The main result is the existence of a weak solution for the corresponding system. The weak formulation is an inequality (due to the plasticity of the fluid), and it involves a free boundary (due to the motion of the rigid body). We approximate it by regularizing the convex terms in the Bingham fluid and by using a penalty method to take into account the presence of the rigid body.

As we mentioned in Section 3.2.4, viscoplastic materials behave as a liquid for high stresses and as solid for low stresses an can model various natural and industrial fluids, for instance, mudflows, snow avalanche, volcanic lava flows, toothpaste, mayonnaise, etc. The Bingham constitutive equation is one of the simplest models for a viscoplastic fluid. It was proposed by Bingham [**?**] in 1916. The corresponding system of partial differential equations has been studied in many works, for instance, in Duvaut and Lions [**?**, Chapter VI], where the existence of weak solutions for the Bingham fluid (without structures) is proved.

Regarding Part I, in what follows we consider a different approach to modeling granular matter where we study a multiphase problem involving a rigid structure and a non-Newtonian fluid. The difference with this approach and the one used in Section 3.1 is that we no longer consider the mixture theory. More precisely, the Continuum hypothesis of the mixture theory is dismissed, and the equations of each component are satisfied in different domains.

At first sight, this approach seems much more realistic. However, we add new complications: the equations of motion are satisfied in moving domains, and we need to consider how the materials interact in their mutual frontier.

Nevertheless, in this type of multiphase modeling, we are allowed to use a more simple constitutive equation. In this line, the Bingham fluid model is the simplest constitutive equation that possesses the yielding property and, as we discuss in Section 3.2.4, the yielding property is dominant in dense regimes of granular materials. Then, a Bingham fluid-rigid body system can be useful to understand and shed some light about granular materials, which is high contrast with the Drucker-Prager law3.2.4 and the $\mu(I)$-rheology 3.2.4 characterized by complex constitutive equations

4.1 Introduction and main result

Let us describe our fluid-solid system: we consider $\Omega \subset \mathbb{R}^3$ an open, bounded and connected set containing a Bingham plastic fluid and a rigid body. We denote respectively by $\mathcal{S}(t)$ and by $\mathcal{F}(t)$ the domains of the structure and the fluid at instant t. We assume that the solid is a rigid body and its domain can be described from its initial configuration $\mathcal{S}_0$: for $a \in \mathbb{R}^3$ and $Q \in \mathcal{SO}(3)$ (the rotation group) we set

$$\widehat{\mathcal{S}}(a,Q) := a + Q\mathcal{S}_0 \quad \text{and} \quad \widehat{\mathcal{F}}(a,Q) := \Omega \setminus \overline{\widehat{\mathcal{S}}(a,Q)}. \tag{4.1}$$

Then,

$$\mathcal{S}(t) = \widehat{\mathcal{S}}(h(t), R(t)) \quad \text{and} \quad \mathcal{F}(t) = \widehat{\mathcal{F}}(h(t), R(t)).$$

We assume in what follows that the center of mass of $\mathcal{S}_0$ is located at the origin so that $h(t)$ is the position of the center of mass of the rigid body. We also suppose that $\mathcal{S}_0$ (and thus $\mathcal{S}(t)$) is open, bounded and connected and that $\mathcal{F}_0 := \Omega \setminus \overline{\mathcal{S}_0}$ (and thus $F(t)$ as long as the rigid body remains inside Ω) is connected.

The governing equations for the fluid flow is written by considering the Cauchy momentum equation of an incompressible fluid. We write the governing equations for the fluid flow by using the Cauchy momentum equation where the stress tensor is given by a sub-differential equation which represents the viscoplastic behavior of the Bingham fluid. The balance equations for linear and angular momentum govern the motion of the rigid body. The full system of equations modeling the motion of the fluid and the rigid body is:

$$\rho_f \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) - \operatorname{div} \Sigma(u,p) = 0, \qquad x \in \mathcal{F}(t), t \in (0,T), \tag{4.2}$$

$$\operatorname{div} u = 0, \qquad x \in \mathcal{F}(t), t \in (0,T), \tag{4.3}$$

$$u = 0, \qquad x \in \partial\Omega, t \in (0,T), \tag{4.4}$$

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)), \qquad x \in \partial\mathcal{S}(t), t \in (0,T), \tag{4.5}$$

$$m\ell' = -\int_{\partial\mathcal{S}} \Sigma(u,p)\widehat{n}\,dx, \qquad t \in (0,T), \tag{4.6}$$

$$(J\omega)' = -\int_{\partial\mathcal{S}} (x-h) \times \Sigma(u,p)\widehat{n}\,dx, \qquad t \in (0,T), \tag{4.7}$$

$$R' = \mathbb{A}(\omega)R, \qquad t \in (0,T), \tag{4.8}$$

$$h' = \ell, \qquad t \in (0,T), \tag{4.9}$$

$$u(0,\cdot) = u_0, \qquad x \in \mathcal{F}_0, \tag{4.10}$$

$$R(0) = I_3, \quad h(0) = 0, \tag{4.11}$$

$$\ell(0) = \ell_0, \quad \omega(0) = \omega_0. \tag{4.12}$$

In the above system the unknowns are $u(t,x)$ (velocity field of the fluid), $p(t,x)$ (pressure of the fluid), $h(t)$ and $\ell(t)$ (the position and the velocity of the center of mass of the rigid body), $R(t)$ and $\omega(t)$ (the orientation and the angular velocity of the rigid body). We have also denoted by $\widehat{n}$ the outward normal to $\mathcal{F}(t)$. For any $\omega \in \mathbb{R}^3$, $\mathbb{A}(\omega)$ is the skew-symmetric matrix:

$$\mathbb{A}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

We assume that the densities ρ_f and ρ_s of the fluid and the solid are positive constants. In that case, the mass of the solid m is given by

$$m = \rho_s \left|\mathcal{S}_0\right|,$$

where $|\mathcal{S}_0|$ is the volume of $\mathcal{S}_0$, and the moment of inertia J is given by:

$$J(t) = \widehat{J}(h(t), R(t)),$$

where

$$\widehat{J}(a, Q) = \rho_s \int_{\widehat{\mathcal{S}}(a,Q)} \left(|x-a|^2 I_3 - (x-a)\otimes(x-a)\right) dx.$$

We have denoted by $|a| = \sqrt{a \cdot a}$ the Euclidean norm in $\mathbb{R}^3$.

We can check that

$$\widehat{J}(h, Q) = QJ_0Q^*, \tag{4.13}$$

where we have denoted by M^* the transpose matrix of M and where

$$J_0 = \rho_s \int_{\mathcal{S}_0} \left(|x|^2 I_3 - x \otimes x\right) dx.$$

In particular, $J(t)$ is symmetric and positive definite.

The Cauchy stress tensor is given by the constitutive equation for a Bingham fluid. Similarly as we did in Section 3.2.4, to write this relation, first we decomposed the Cauchy stress tensor as follows:

$$\Sigma(u, p) = -pI_3 + \Sigma^d(D(u)),$$

where the function Σ^d is given by the following sub-differential inclusion:

$$\Sigma^d(D) \in \partial\Phi(D) \tag{4.14}$$

with $\Phi : \mathbb{M}^{3\times3} \to \mathbb{R}$ the convex function defined by:

$$\Phi(D) = \mu\left|D\right|^2 + g\left|D\right|. \tag{4.15}$$

We have denoted by $\mathbb{M}^{3\times3}$ the space of square matrices of order 3 and by $|D| = \sqrt{D : D}$ the corresponding Frobenius norm.

As in Chapter 3.2, we call Φ the energy potential and for a given $D \in \mathbb{M}^{3\times3}$ the set of sub-derivatives of $\Phi(D)$ is defined in Chapter 2.

In the constitutive law given by Φ, the constant $g > 0$ is the yield stress and the constant $\mu > 0$ is the molecular viscosity.

Just like we did in Lemma 3.2, we can prove that equation (4.14) is equivalent to:

$$\begin{cases} \left|\Sigma^d(D)\right| \leq g \iff D = 0, \\ \left|\Sigma^d(D)\right| > g \iff D \neq 0 \quad \text{and} \quad \Sigma^d(D) = 2\mu D + g\dfrac{D}{|D|}. \end{cases} \tag{4.16}$$

Indeed, if $D \neq 0$, then (4.14) is equivalent to $\Sigma^d(D) = 2\mu D + gD/\left|D\right|$. If we multiply $\Sigma^d(D)$ by D, we notice that $|\Sigma^d(D)|_2 > g$. If $D = 0$, (4.14) is equivalent to $|\Sigma^d(D)|_2 \leq g$. The above representation says that a Bingham fluid behaves like a viscous fluid if $\left|\Sigma^d(D(u))\right| > g$, and as a rigid body otherwise. We note that if $g = 0$, we recover the Navier-Stokes equations coupled with the equations of the rigid body.

To write the weak formulation associated with system (4.2)-(4.12), we first introduce some notation. We recall from Chapter 2, that L^q and H^q are the classical Lebesgue and Sobolev spaces. We also denote by $\mathcal{C}^q$ the space of q-times continuous differential functions. We write $\mathcal{C}_0^q$ the set of all functions in $\mathcal{C}^q$ with compact support.

We introduce the standard spaces in the study of the equations of fluid mechanics:

$$L^2_\sigma(\Omega) = \left\{ v \in L^2(\Omega) \; ; \; \operatorname{div}(v) = 0, \quad v \cdot \widehat{n} = 0 \quad \text{on } \partial\Omega \right\},$$
$$H^1_\sigma(\Omega) = L^2_\sigma(\Omega) \cap H^1_0(\Omega).$$

We define the space of rigid velocities:

$$\mathcal{R} = \{ x \mapsto \ell + \omega \times x \; ; \; \ell, \omega \in \mathbb{R}^3 \} \tag{4.17}$$

and we introduce the following spaces due to the presence of the rigid body:

$$L^2_{\mathcal{S}}(\Omega) = \left\{ v \in L^2_\sigma(\Omega) \; ; \; D(v) = 0 \quad \text{in } \mathcal{S} \right\},$$
$$H^1_{\mathcal{S}}(\Omega) = \left\{ v \in H^1_\sigma(\Omega) \; ; \; D(v) = 0 \quad \text{in } \mathcal{S} \right\}.$$

We recall (see, for instance, [**?**, Lemma 1.1, p.18]) that

$$D(v) = 0 \quad \text{in } \mathcal{S} \iff \exists v_R \in \mathcal{R} : v_{|\mathcal{S}} = v_{R|\mathcal{S}} \in \mathcal{R}.$$

We extend the fluid velocity u to the whole domain Ω by

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)) \quad x \in \mathcal{S}(t) \tag{4.18}$$

and similarly,

$$u_0(x) = \ell_0 + \omega_0 \times x \quad x \in \mathcal{S}_0. \tag{4.19}$$

In particular, $D(u) = 0$ in $\mathcal{S}(t)$ and $D(u_0) = 0$ in $\mathcal{S}_0$.

We also define a "global" density for the fluid-solid mixture as:

$$\rho(t,x) := \begin{cases} \rho_f & x \in \mathcal{F}(t), \\ \rho_s & x \in \mathcal{S}(t). \end{cases}$$

Then, we show in the next section the following result:

Proposition 4.1.1. *Assume that* $(u, p, \ell, \omega, h, R)$ *is a regular function satisfying* (4.2)–(4.12). *Then the following inequality holds:*

$$\int_0^T \int_\Omega \rho \left(\frac{\partial v}{\partial t} + (u \cdot \nabla) v \right) \cdot (v - u) dx dt + 2\mu \int_0^T \int_\Omega D(u) : D(v-u) dx dt$$
$$+ g \int_0^T \int_\Omega \left(|D(v)| - |D(u)| \right) dx dt \geqslant -\frac{1}{2} \int_\Omega \rho(0,x) \left| v(0,x) - u_0 \right|^2 dx, \tag{4.20}$$

for any $v \in \mathcal{C}^1\left([0,T]; H^1_{\mathcal{S}(t)}(\Omega)\right)$. *Moreover, the following energy inequality holds:*

$$\frac{1}{2} \int_\Omega \rho(t,x) \left| u(t,x) \right|^2 dx + 2\mu \int_0^T \int_\Omega |D(u)|^2 dx dt + g \int_0^T \int_\Omega |D(u)| dx dt \leqslant \frac{1}{2} \int_\Omega \rho(0,x) |u_0|^2 dx, \tag{4.21}$$

for all $t \in (0,T)$.

Remark 4.1. *Since the potential energy* Φ *(defined in* (4.15)*) is not differentiable, the Bingham constitutive equation* (4.14) *leads us to the variational inequality* (4.20)*. In this weak formulation, we also notice that the space of the test functions depends on the solution, which comes from the fact that we are working with a free boundary problem. We handle these two difficulties combining a monotonicity method and a penalization method.*

The above proposition allows us to introduce the notion of weak solution of the system (4.2)-(4.12):

Definition 4.1.2 (Weak Solution)**.** *A weak solution of the system of equations* (4.2)-(4.12) *is a triplet* (u, h, R) *with the following properties:*

- $(h, R) \in W^{1,\infty}(0, T; \mathbb{R}^3 \times SO(3))$ *and satisfy* (4.8), (4.9), (4.11).
- $u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_\sigma(\Omega))$ *and* $u(t, x) = \ell(t) + \omega(t) \times (x - h(t))$ *for* $x \in \mathcal{S}(t)$.
- *Inequality* (4.20) *holds for any* $v \in \mathcal{C}^1([0, T]; H^1_{\mathcal{S}(t)}(\Omega))$.
- *The energy inequality* (4.21) *holds true a.e. in* $(0, T)$.

Remark 4.2. *Following the arguments in [**?**, pp. 287–288], one can show that if* (u, h, R) *is a regular weak solution of the system* (4.2)-(4.12) *in the above sense and if* $|D(u)| \neq 0$ *in* $\mathcal{F}(t)$*, then* (u, h, R) *satisfies* (4.2)-(4.9)*. We prove this in Section 4.2, after the proof of Proposition 4.1.1.*

The main result of this chapter is the following result.

Theorem 4.1.3. *Assume* $\mathcal{S}_0 \Subset \Omega$, $\partial\Omega$ *and* $\partial\mathcal{S}_0$ *are of class* $\mathcal{C}^2$, $u_0 \in L^2_\sigma(\Omega)$, *with* $u_0(x) = \ell_0 + \omega_0 \times x$ *for* $x \in \mathcal{S}_0$*. Then, there exists a weak solution of the system* (4.2)-(4.12) *defined on a maximal time interval* $(0, T)$*, and one of the following alternatives holds true:*

1. $T = +\infty$;
2. $\lim_{t \to T} \operatorname{dist}(\mathcal{S}(t), \partial\Omega) = 0$.

Remark 4.3. *One can write a bi-dimensional version of system* (4.2)-(4.12) *and following the proof of the above theorem, it is possible to obtain the same existence result for the corresponding system. Let us mention that even in dimension 2 in space, the uniqueness of weak solutions can be a delicate question. For a Bingham fluid alone (without rigid bodies), it is done in [**?**, p.301]. However, for the system composed by a rigid body and a fluid governed by the Navier-Stokes system, this issue has been solved only recently (see [**?**], [**?**] and also [**?**] for a weak-strong uniqueness property).*

Remark 4.4. *As explained above our proof of Theorem 4.1.3 is based on a penalization method to deal with the free boundary due to the motion of the rigid body. Such a method is already used in several other articles on fluid-structure interaction systems. There exist a least two different penalization approaches: a "*L^2*" penalization (see for instance [**?**, **?**]), and a "*H^1*" penalization (see for instance [**?**]). We follow the first method (see (4.46)), but it could also be possible to consider a* H^1 *penalization. In that case, we would have to consider a variable viscosity in the approximation problems of Section 4.3, with a viscosity that goes to infinity in the solid domain. With such an approach, one would need to consider arguments* *from [?], whereas here we have used or adapted results both from [?] and [?]. These two penalization methods can be used in numerical schemes to simulate the motion of rigid bodies in a fluid, but the drawback of the H^1 penalization method is that the solid can change its shape which is not adapted for rigid motions. We refer for instance to [?] for the analysis of a numerical scheme base on the L^2 penalization method and also [?], [?], [?], [?], etc. for some other works on the numerical study of fluid-rigid body systems.*

Remark 4.5. *Let us note that the pressure of the fluid p does not appear in the weak formulation* (4.20) *due to the property of the test functions. One could also work with a mixed formulation where we keep the pressure and where the test functions do not satisfy the free divergence condition. The corresponding study is more complicated since we need to obtain estimates of the pressure during the proof of existence. Such an approach is made for the Bingham system without structures in [?] but the authors need to consider some slip boundary conditions to obtain their results. A method to obtain the pressure for a non-Newtonian fluid with Dirichlet boundary conditions is developed by Wolf [?]. This pressure is called by the author "local" pressure and is the sum of a regular pressure and of the time derivative of an harmonic function. We refer the reader to [?] where, the case of a non-Newtonian fluid with a power law and rigid body interaction is treated. Part of this work is devoted to the study of the "local" pressure where the authors manage to pass to the limit in the nonlinearity associated with the stress tensor taking advantage of the more regular structure of the stress tensor.*

Remark 4.6. *The interesting problem of obtaining some information on the set where the Bingham fluid behaves as a solid (where $D(u) = 0$), and also to know how this set interacts with the rigid body, is entirely open from the theoretical point of view, even without any rigid body. However, tackling these questions in a numerical study is possible. Lots of works have been done to solve numerically the Bingham fluid. We refer the reader to the book [?] and the review paper [?].*

The mathematical study of fluid-structure interaction systems has been the subject of an intensive research since around 2000. A large part of the articles devoted to this study concern the case of rigid bodies moving into a viscous incompressible fluid modeled by the Navier-Stokes system. We can quote for instance [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?], etc. Some works deal with different fluids [?, ?, ?] (incompressible perfect fluid), [?, ?, ?, ?] (viscous compressible fluid), [?] (viscous multipolar fluid), [?, ?] (incompressible non-Newtonian fluid). Let us also mention some results for the Navier-Stokes system but with other types of boundary conditions: [?, ?, ?]).

Up to our knowledge, the case of a Bingham fluid has not been treated yet. The first studies on Bingham fluid were done by Oldroyd [?] and Prager [?]. The works of Mosolov and Miasnikov in [?, ?] present a variational method and give some well-posedness results. We can also quote [?] where the authors consider the case of a stationary Bingham fluid around a rigid body. They consider a weak formulation and analyse the case where the motion of the rigid body is given. In [?], the authors provide a relation between the yield number and an eigenvalue problem.

Let us describe the outline of this chapter. In Section 4.2, we introduce some additional notation and we prove Proposition 4.1.1. We also give some technical results proved in [?] but that we state differently and that we prove for the sake of completeness. In Section 4.3, we introduce some approximations of the variational inequality (4.20). More precisely, we use a Galerkin method (of dimension M) where the plastic term is regularized (with a parameter ε) and where the free-boundary is replaced by a penalization term (with a parameter k). Section 4.4 is devoted to passing to the limit in M and ε. Finally, in Section 4.5, we prove the main result by passing to the limit in k.

4.2 Notation and preliminary results

Assume $(a, Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ and set

$$\mathcal{S} = \widehat{\mathcal{S}}(a, Q).$$

We denote by $P^{\mathcal{R}}_{\mathcal{S}}$ the orthogonal projection of $L^2(\mathcal{S})$ onto $\mathcal{R}$. By Lemma A.6, if

$$\ell + \omega \times (x - a) = P^{\mathcal{R}}_{\mathcal{S}} u,$$

then ℓ and ω are given by:

$$\ell = \frac{1}{m} \int_{\widehat{\mathcal{S}}(a,Q)} \rho_s u \, dx \tag{4.22}$$

and

$$\omega = \widehat{J}(a, Q)^{-1} \int_{\widehat{\mathcal{S}}(a,Q)} \rho_s (x - a) \times u \, dx. \tag{4.23}$$

We define the global density by

$$\widehat{\rho}_{a,Q} := \rho_f \mathbb{1}_{\widehat{\mathcal{F}}(a,Q)} + \rho_s \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}. \tag{4.24}$$

In what follows, we also need the following notation: for any set $\Omega_1 \subset \mathbb{R}^3$,

$$(\Omega_1)^{\delta} := \{x \in \mathbb{R}^3 \ ; \ \operatorname{dist}(x, \Omega_1) < \delta\} \tag{4.25}$$

and

$$(\Omega_1)_{\delta} := \{x \in \Omega_1 \ ; \ \operatorname{dist}(x, \partial\Omega_1) > \delta\}.$$

Given $(a, Q) \in \mathbb{R}^3 \times \mathcal{SO}(3)$ we define two operators of $L^2_{loc}(\mathbb{R}^3)$ as follows: assume $v \in L^2_{loc}(\mathbb{R}^3)$, then

$$\Phi_{a,Q}(v)(y) := Q^* v\left(a + Qy\right), \quad y \in \mathbb{R}^3 \tag{4.26}$$

and

$$\overline{\Phi}_{a,Q}(v)(x) := Qv\left(Q^*(x - a)\right), \quad x \in \mathbb{R}^3. \tag{4.27}$$

Let us notice the relation

$$\overline{\Phi}_{a,Q} \circ P^{\mathcal{R}}_{\mathcal{S}_0} \circ \Phi_{a,Q} = P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)}. \tag{4.28}$$

We will need the following result.

Lemma 4.1. *Assume $(h_n, R_n) \to (h, R)$ in $\mathbb{R}^3 \times \mathcal{SO}(3)$. Then,*

$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ in } L^p(\Omega) \quad \forall p \in [1, \infty). \tag{4.29}$$

Similarly, if $(h_n, R_n) \to (h, R)$ strongly in $\mathcal{C}([0, T]; \mathbb{R}^3 \times SO(3))$, then

$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ strongly in } \mathcal{C}([0, T]; L^p(\Omega)) \quad \forall p \in [1, \infty). \tag{4.30}$$

The proof of this lemma can be found in the appendix (Lemma A.4) and is based on the approximation of $\mathbb{1}_{\mathcal{S}_0}$ by a smooth function with compact support.

4.2.1 Weak form and energy inequality

In this section, we first prove Proposition 4.1.1:

Proof of Proposition 4.1.1. Let $v \in \mathcal{C}^1\left([0,T]; H^1_{\mathcal{S}(t)}(\Omega)\right)$. Using the results in Section 4.1, there exist two $\mathcal{C}^1$ functions, ℓ_v and ω_v, such that $v(t,x) = \ell_v(t) + \omega_v(t) \times (x - h(t))$ for $x \in \mathcal{S}(t)$.

We multiply equation (4.2) by $(v-u)$ and we integrate in $\mathcal{F}(t)$ and in $[0,T]$

$$\int_0^T \int_{\mathcal{F}(t)} \rho_f \left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot(v-u)\ dxdt = \int_0^T \int_{\mathcal{F}(t)} \operatorname{div}(\sigma(u,p))\cdot(v-u)\ dxdt. \quad (4.31)$$

By the divergence theorem,

$$\int_{\mathcal{F}(t)} \operatorname{div}(\sigma(u,p))\cdot(v-u)\ dx = -\int_{\mathcal{F}(t)} \sigma(u,p) : \nabla(v-u)\ dx + \int_{\partial\mathcal{F}(t)} \sigma(u,p)n\cdot(v-u)\ ds. \quad (4.32)$$

Using that $\operatorname{div}(v-u) = 0$, the boundary conditions of $v-u$ (see (4.4) and (4.5)) and the fact that $\Sigma^d(D(u))$ is a symmetric matrix, we deduce from (4.32)

$$\int_{\mathcal{F}(t)} \operatorname{div}\sigma(u,p)\cdot(v-u)dx = -\int_{\mathcal{F}(t)} \sigma^d(D(u)) : D(v-u)dx - m\ell'\cdot(\ell_v-\ell) - (J\omega)'\cdot(\omega_v-\omega). \quad (4.33)$$

Since $D \in \mathbb{M}^{3\times3} \mapsto |D|^2$ is differentiable, $\sigma^d(D(u)) \in \partial f(D(u))$ implies that

$$g\,|D(v)| - g\,|D(u)| \geqslant \left(\sigma^d(D(u)) - 2\mu D(u)\right) : (D(v)-D(u)).$$

Combining the above relation with (4.33) yields

$$\int_{\mathcal{F}(t)} \operatorname{div}\sigma(u,p)\cdot(v-u)\ dx + 2\mu\int_{\mathcal{F}(t)} D(u) : D(v-u)\ dx$$
$$+ g\int_{\mathcal{F}(t)} |D(v)| - |D(u)|\ \ dx + m\ell'\cdot(\ell_v-\ell) + (J\omega)'\cdot(\omega_v-\omega) \geqslant 0. \quad (4.34)$$

On the other hand, using the Reynolds transport theorem (see appendix Lemma A.5) and Lemma A.9 we deduce that:

$$\int_0^T \int_{\mathcal{F}(t)} \rho_f \left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot(v-u)\ dxdt + m\ell'\cdot(\ell_v-\ell) + (J\omega)'\cdot(\omega_v-\omega)$$
$$= \int_0^T \int_\Omega \rho\left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\cdot(v-u)\ dxdt$$
$$+ \frac{1}{2}\int_\Omega \rho(0,\cdot)\,|u(0,\cdot) - v(0,\cdot)|^2 - \rho(T,\cdot)\,|u(T,\cdot) - v(T,\cdot)|^2\,dx. \quad (4.35)$$

Then, using that $D(u) = D(v) = 0$ in $\mathcal{S}(t)$, and $|u(T,\cdot) - v(T,\cdot)|^2$ is non negative, gathering (4.31), (4.34) and (4.35) we arrive to (4.20).

To obtain the energy inequality, we replace equation (4.2) by u and we integrate in $F(s)$ and in $[0,t]$. Following the above calculations, this yields

$$\int_0^t \left(\int_{\mathcal{F}(s)} \rho_f\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot u\ dx + \int_{\mathcal{F}(s)} 2\mu|D(u)|_2^2 + g\,|D(u)|\ \ dx + m\ell'\cdot\ell + (J\omega)'\cdot\omega\right)\ ds \leq 0.$$

Using again the Reynolds transport theorem and standard calculation we deduce (4.21). □

We end this section by showing that if (u, h, R) is a regular weak solution of the system (4.2)-(4.12), it satisfies (4.2)-(4.12). We start with (4.20) and using that $D(u) \neq 0$ with the arguments in [**?**, pp. 287-288], we obtain

$$\int_0^T \int_\Omega \rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) \cdot v \; dxdt + 2\mu \int_0^T \int_\Omega D(u) : D(v) \; dxdt \\ + g \int_0^T \int_{\mathcal{F}(t)} \frac{D(u) : D(v)}{|D(u)|} \; dxdt = -\frac{1}{2} \int_\Omega \rho(0,x) \left| u(0,x) - u_0(x) \right|^2 dx, \quad (4.36)$$

for any $v \in \mathcal{C}^1([0,T]; H^1_{\mathcal{S}(t)}(\Omega))$ such that $v(T, \cdot) = v(0, \cdot) = 0$. Taking $v = 0$ we recover the initial conditions and taking v such that $\ell_v = 0$ and $\omega_v = 0$ we obtain

$$\int_0^T \int_{\mathcal{F}(t)} \left(\rho_f \frac{\partial u}{\partial t} + \rho_f (u \cdot \nabla) u - \operatorname{div} \Sigma^d(D(u)) \right) \cdot v \; dxdt = 0$$

Then, we recover the pressure p using Lemma III.1.1 in [**?**] and we obtain equation (4.2). Finally, combining (4.2) with (4.36), integrating by parts and using that $\operatorname{div} v = 0$, we obtain:

$$\int_0^T \left(m\ell' + \int_{\partial \mathcal{S}} \Sigma(u,p) n \; ds \right) \cdot \ell_v \; dt + \int_0^T \left((J\omega)' + \int_{\partial \mathcal{S}} (x - h) \times \sigma(u,p) n \; ds \right) \cdot \omega_v \; dt = 0.$$

Since the above equation holds for all ℓ_v and ω_v in $\mathcal{C}^1([0,T]; \mathbb{R}^3)$ with $\ell_v(T) = \omega_v(T) = 0$, we recover the equations (4.6) and (4.7).

4.3 Approximated Problems

To prove the existence of weak solutions of the system (4.2)-(4.12), we consider some approximations of (4.2)-(4.12). More precisely, we introduce 3 parameters:

- ε corresponds to the approximation of the plastic term,
- M corresponds to the dimension in the Galerkin method,
- k corresponds to the penalization term used to deal with the free boundary problem.

More precisely, we replace $j : \mathbb{M}^{3\times 3} \to \mathbb{R}, \; D \mapsto |D|$ by the $\mathcal{C}^1$ convex functions

$$j_\varepsilon : \mathbb{M}^{3\times 3} \to \mathbb{R}, \quad D \mapsto \frac{1}{1+\varepsilon} |D|^{1+\varepsilon}. \quad (4.37)$$

The gradient of j_ε is given by

$$\nabla j_\varepsilon(D) = |D|^{\varepsilon - 1} D \quad (4.38)$$

and satisfies

$$|\nabla j_\varepsilon(D)| = |D|^\varepsilon \leq 1 + |D| \quad (4.39)$$

if $\varepsilon \leq 1$.

Since $H^1_\sigma(\Omega)$ is a separable Hilbert space and $\mathcal{C}^\infty_0(\Omega) \cap H^1_\sigma(\Omega)$ is dense in $H^1_\sigma(\Omega)$, there exists an orthonormal basis $\{v_q\}_{q \in \mathbb{N}^*}$ of $H^1_\sigma(\Omega)$ such that $v_q \in \mathcal{C}^\infty_0(\Omega)$ for all $q \geq 1$. We define

$$V_M = \operatorname{span}\{v_1, \ldots, v_M\}$$

and we look for an approximated velocity in V_M.

This subspace does not impose that the velocity is rigid in the solid domain. That is why we add in the weak formulation a penalization term of the form

$$k \int_{\mathcal{S}} (u - P_{\mathcal{S}}^{\mathcal{R}}(u)) \cdot (v - P_{\mathcal{S}}^{\mathcal{R}}(v)) dx,$$

with $k \to \infty$.

Notation 4.3.1. *To simplify the notation, in this section we write*

$$n = (\varepsilon, k, M),$$

for instance u_n means $u_{\varepsilon,k,M}$.

Then, the approximated problem is defined as follows: to find

$$h_n \in \mathcal{C}^1([0,T]; \mathbb{R}^3), \quad R_n \in \mathcal{C}^1([0,T]; SO(3)) \quad \alpha_n \in \mathcal{C}^1([0,T]; \mathbb{R}^M) \tag{4.40}$$

satisfying the following properties:

$$\mathcal{S}_n(t) := \widehat{\mathcal{S}}(h_n(t), R_n(t)), \quad \mathcal{F}_n(t) := \widehat{\mathcal{F}}(h_n(t), R_n(t)), \tag{4.41}$$

where $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{F}}$ are defined in (4.1);

$$u_n := \sum_{j=1}^{M} \alpha_{n,j} v_j, \tag{4.42}$$

$$\ell_n + \omega_n \times (x - h_n) := P_{\mathcal{S}_n}^{\mathcal{R}}(u_n), \tag{4.43}$$

where $P_{\mathcal{S}_n}^{\mathcal{R}}$ is the projection defined in Section 4.2,

$$h_n'(t) = \ell_n(t), \quad h_n(0) = 0, \tag{4.44}$$

$$R_n'(t) = \mathbb{A}(\omega_n) R_n(t), \quad R_n(0) = I_3, , \tag{4.45}$$

$$\int_\Omega \rho_n \frac{\partial u_n}{\partial t} \cdot v_j dx + \int_\Omega \rho_n (Q_{\mathcal{S}_n}(u_n) \cdot \nabla) u_n \cdot v_j dx + 2\mu \int_\Omega D(u_n) : D(v_j) dx$$
$$+ g \int_\Omega \nabla j_\varepsilon (D(u_n)) : D(v_j) dx + k \int_{\mathcal{S}_n} (u_n - P_{\mathcal{S}_n}^{\mathcal{R}}(u_n)) \cdot (v_j - P_{\mathcal{S}_n}^{\mathcal{R}}(v_j)) dx = 0 \quad (j \in \{1, \cdots, M\}), \tag{4.46}$$

and

$$u_n(0, \cdot) = \mathbb{P}_{V_M}(u_0) := \sum_{j=1}^{M} \alpha_{n,0,j} v_j. \tag{4.47}$$

The operator $Q_{\mathcal{S}_n} := Q_{h^n, R^n}^{\delta, \frac{\delta}{k}}$ is given in Definition A.1.2, where δ is a positive constant. It satisfies in particular the following relation (see (A.15)):

$$Q_{\mathcal{S}_n}(u_n) = \begin{cases} u_n & \text{in } \Omega \setminus (\mathcal{S}_n)^\delta \\ P_{\mathcal{S}_n}^{\mathcal{R}} u_n & \text{in } \mathcal{S}_n \end{cases} \quad \text{and} \quad \operatorname{div} Q_{\mathcal{S}_n}(u_n) = 0. \tag{4.48}$$

The operator $\mathbb{P}_{V_M}$ is the L^2 orthogonal projection of $L^2(\Omega)$ onto V_M. The global density is defined by $\rho_n = \widehat{\rho}_{h_n,R_n}$ where $\widehat{\rho}$ is defined by (4.24).

Using the above properties of $Q_{\mathcal{S}_n}(u_n)$, we can show that (4.46) implies

$$-\int_0^T \int_\Omega \rho_n \left(\frac{\partial v}{\partial t} + (Q_{\mathcal{S}_n}(u_n) \cdot \nabla) v \right) \cdot u_n dx dt + \int_0^T \int_\Omega (2\mu D(u_n) + g \nabla j_\varepsilon(D(u_n))) : D(v) dx dt \\ + k \int_0^T \int_{\mathcal{S}_n} (u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)) \cdot (v - P^{\mathcal{R}}_{\mathcal{S}_n}(v)) dx dt \\ = \int_\Omega \left[\rho_n(0,\cdot) u_n(0,\cdot) \cdot v(0,\cdot) - \rho_n(T,\cdot) u_n(T,\cdot) \cdot v(T,\cdot) \right] \, dx, \quad (4.49)$$

for any $v \in \mathcal{C}^1([0,T]; V_M)$.

In the following proposition, we prove the existence of a solution of the approximated problems.

Proposition 4.3.2. *Assume $T > 0$, $M \in \mathbb{N}^*$, $k, \varepsilon > 0$. Then, there exists a solution (h_n, R_n, α_n) of the system (4.40)-(4.47). Moreover, we have the energy equality: for all $t \in [0,T]$,*

$$\frac{1}{2} \int_\Omega \rho_n(t,\cdot) \left| u_n(t,\cdot) \right|^2 \, dx + 2\mu \int_0^t \int_\Omega \left| D(u_n) \right|^2 \, dx dt + g \int_0^t \int_\Omega \nabla j_n(D(u_n)) : D(u_n) \, dx dt \\ + k \int_0^t \int_{\mathcal{S}_n} \left| u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n) \right|^2 \, dx dt = \frac{1}{2} \int_\Omega \rho_0 \left| \mathbb{P}_{V_M} u_0 \right|^2 \, dx. \quad (4.50)$$

Proof. We write (4.41)-(4.47) as a Cauchy problem

$$\frac{d}{dt} \begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix} = F\left(\begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix} \right), \quad \begin{pmatrix} h_n \\ R_n \\ \alpha_n \end{pmatrix}(0) = \begin{pmatrix} 0 \\ I_3 \\ \alpha_{n,0} \end{pmatrix} \quad (4.51)$$

where $F = (F_1, F_2, F_3)$ depends on n and can be expressed by using (4.41)-(4.47), (4.22)-(4.23) and (4.13):

$$F_1(a, Q, \beta) = \frac{\rho_s}{m} \sum_{i=1}^M \beta_i \int_\Omega \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} v_i \, dx,$$

$$F_2(a, Q, \beta) = \rho_s \sum_{i=1}^M \beta_i \mathbb{A} \left(Q J_0^{-1} Q^* \int_\Omega \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} (x - a) \times v_i(x) \, dx \right) Q,$$

and

$$F_3(a, Q, \beta) = C(a, Q)^{-1} G(a, Q, \beta),$$

where

$$C(a, Q)_{i,j} = \int_\Omega \widehat{\rho}_{a,Q} v_i \cdot v_j \, dx \quad (i, j \in \{1, \ldots, M\})$$

and

$$G(a,Q,\beta)_j = -2\mu \sum_{i=1}^{M} \beta_i \int_\Omega D(v_i) : D(v_j) dx$$
$$- \sum_{i=1}^{M} \beta_i \int_\Omega \widehat{\rho}_{a,Q} \left(Q_{a,Q}^{\delta,\delta/k} \left(\sum_{l=1}^{M} \beta_l v_l \right) \cdot \nabla \right) v_i \cdot v_j dx - g \int_\Omega \nabla j_\varepsilon \left(\sum_{i=1}^{M} \beta_i D(v_i) \right) : D(v_j) dx$$
$$- k \sum_{i=1}^{M} \beta_i \int_\Omega \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} (v_i - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)} v_i) \cdot (v_j - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)} v_j) dx \quad (4.52)$$

for $j \in \{1, \ldots, M\}$.

By Lemma 4.1 and (4.1) we have that

$$\mathbb{R}^3 \times \mathcal{SO}(3) \to L^1(\mathbb{R}^3), \quad (a,Q) \mapsto \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}$$

is continuous and thus F_1, F_2 and C are continuous functions. For the continuity of G, we gather the following arguments:

- Since j_ε is $\mathcal{C}^1$, then

  $$(a,Q,\beta) \mapsto \int_\Omega \nabla j_\varepsilon \left(\sum_{i=1}^{M} \beta_i D(v_i) \right) : D(v_j)\ dx$$

  is continuous.

- Using (4.22), (4.23) and (4.13), we have that

  $$(a,Q) \mapsto P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)} v_i \in \mathcal{R}$$

  is continuous.

- Using the definition (A.14) and the continuity of

  $$(a,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3) \mapsto \Phi_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)), \quad (a,Q) \in \mathbb{R}^3 \times \mathcal{SO}(3) \mapsto \overline{\Phi}_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)),$$

  we deduce

  $$(a,Q,\beta) \mapsto \int_\Omega \widehat{\rho}_{a,Q} \left(Q_{a,Q}^{\delta,\delta/k} \left(\sum_{l=1}^{M} \beta_l v_l \right) \cdot \nabla \right) v_i \cdot v_j\ dx$$

  is continuous.

Consequently, in (4.51), we have that F is continuous. Moreover, we note that relation (4.50) yields a bound of the form

$$|(h_n R_n \alpha_n)| \leq \kappa. \quad (4.53)$$

Let us consider a bound $\zeta > 0$ of $|F|$ on the closed ball $\overline{B(0,2\kappa)}$. Since

$$\overline{B((h_n,R_n,\alpha_n)(0),\kappa)} \subset \overline{B(0,2\kappa)},$$

the Peano theorem gives the existence of a solution for a time $\tau = \kappa/\zeta > 0$. This solution satisfies (4.50) and thus (4.53) in $[0,\tau]$. In particular,

$$\overline{B((h_n,R_n,\alpha_n)(\tau),\kappa)} \subset \overline{B(0,2\kappa)},$$

we can use again Peano theorem with initial condition $(h_n,R_n,\alpha_n)(\tau)$ and on the time interval $[\tau,2\tau]$. This solution satisfies (4.53) in $[\tau,2\tau]$ and we can use it to extend our solution on $[0,\tau]$ on the time interval $[0,2\tau]$. Then, by (4.51), we conclude that $(h_n,R_n,\alpha_n) \in \mathcal{C}^1([0,2\tau])$.

By induction, we deduce the existence of global solutions of the system (4.40)-(4.47). □

4.4 Passing to the limit $M \to \infty$ and $\varepsilon \to 0$

This section aims to pass to the limit for the parameters M and ε:

$$M \to \infty, \quad \varepsilon \to 0.$$

We take

$$\varepsilon = \frac{1}{M}$$

so that $n = (1/M, k, M)$. Again to simplify the notation, we write in this section the index (k, M) instead of $(1/M, k, M)$. For instance $u_{k,M}$ means $u_{1/M,k,M}$.

4.4.1 Weak convergences

Using (4.50) and that

$$\frac{1}{2}\int_\Omega \rho_0 \left|\mathbb{P}_{V_M} u_0\right|^2 \, dx \leq C\|u_0\|^2_{L^2(\Omega)}, \tag{4.54}$$

we deduce that

$$\{u_{k,M}\}_{k,M} \quad \text{is bounded in} \quad L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega)). \tag{4.55}$$

Therefore, there exists a subsequence of $\{u_{k,M}\}_{k,M}$ (still denoted $\{u_{k,M}\}_{k,M}$), and a function

$$u_k \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega))$$

such that:

$$u_{k,M} \overset{*}{\rightharpoonup} u_k \text{ weak star in } L^\infty(0,T; L^2_\sigma(\Omega)) \tag{4.56}$$

and

$$u_{k,M} \rightharpoonup u_k \text{ weakly in } L^2(0,T; H^1_\sigma(\Omega)). \tag{4.57}$$

We also deduce from (4.55):

$$(h_{k,M}, R_{k,M}) \overset{*}{\rightharpoonup} (h_k, R_k) \quad \text{weak star in } W^{1,\infty}(0,T; \mathbb{R}^3 \times \mathcal{SO}(3)), \tag{4.58}$$

and

$$(h_{k,M}, R_{k,M}) \to (h_k, R_k) \quad \text{strongly in } \mathcal{C}([0,T]; \mathbb{R}^3 \times \mathcal{SO}(3)). \tag{4.59}$$

We write

$$\mathcal{S}_k := \widehat{\mathcal{S}}(h_k, R_k)$$

and

$$J_{\mathcal{S}_k} := \widehat{J}(h_k, R_k).$$

By Lemma 4.1 we have that

$$\mathbb{1}_{\mathcal{S}_{k,M}} \to \mathbb{1}_{\mathcal{S}_k} \text{ strongly in } \mathcal{C}([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{4.60}$$

and thus

$$\rho_{k,M} \to \rho_k := \widehat{\rho}_{h_k,R_k} \text{ strongly in } \mathcal{C}([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{4.61}$$

Using (4.22) and (4.23), we deduce

$$P^{\mathcal{R}}_{\mathcal{S}_{k,M}} u_{k,M} \overset{*}{\rightharpoonup} P^{\mathcal{R}}_{\mathcal{S}_k} u_k \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{4.62}$$

4.4.2 Strong convergence of the velocity

As usual in the Navier-Stokes equations, we require the strong convergence of the velocity to pass to the limit the convective term. In the case of a Bingham fluid we also have to deal with the plastic term $\nabla j_{\frac{1}{k}}(D(u_{k,M}))$ which does not converge directly to $\nabla j_{\frac{1}{k}}(D(u_k))$ since the convergence of $\{D(u_{k,M})\}_{k,M}$ is only weak. We start by proving the strong convergence of $\{u_{k,M}\}_{k,M}$.

By (4.56) and (4.61) we have that

$$\rho_{k,M}u_{k,M} \to \rho_k u_k \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{4.63}$$

Let us fix $i \geq 1$ and take $M \geq i$. We recall that $\mathbb{P}_{V_i} : L^2(\Omega) \to V_i$ the orthogonal projection onto V_i. We can write (4.49) as follows:

$$\frac{\partial}{\partial t}\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M}) + \mathbb{P}_{V_i}A_{k,M} = 0,$$

in $(\mathcal{C}_0^\infty([0,T];H^1_\sigma(\Omega)))'$, where $A_{k,M}$ is defined by

$$\langle A_{k,M}, v\rangle := \int_0^T\int_\Omega \rho_{k,M}(Q_{\mathcal{S}_{k,M}}(u_{k,M})\cdot\nabla)v\cdot u_{k,M}dxdt - 2\mu\int_0^T\int_\Omega D(u_{k,M}):D(v)dxdt$$
$$-g\int_0^T\int_\Omega \nabla j_{\frac{1}{M}}(D(u_{k,M})):D(v)dxdt - k\int_0^T\int_{\mathcal{S}_{k,M}}(u_{k,M}-P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(u_{k,M}))\cdot(v-P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(v))dxdt \tag{4.64}$$

for all $v \in L^\infty(0,T;H^1_\sigma(\Omega))$. The next step is to prove that $\{A_{k,M}\}_M$ is bounded in $L^{4/3}(0,T;(H^1_\sigma(\Omega))')$. Using (4.39), we deduce

$$\left|\int_0^T\int_\Omega \nabla j_{\frac{1}{M}}(D(u_{k,M})):D(v)\ dxdt\right| \leqslant ((T\,|\Omega|)^{1/2} + \|u_{k,M}\|_{L^2(0,T;H^1_\sigma(\Omega))})\,\|v\|_{L^2(0,T;H^1_\sigma(\Omega))},$$

and, by using the property of the operator $Q^{\delta_1,\delta_2}_{a,Q}$ (see Definition A.1.2), we have that:

$$\|Q_{\mathcal{S}_{k,M}}(u_{k,M})\|_{L^2(0,T;H^1_\sigma(\Omega))} \leq C\,\|u_{k,M}\|_{L^2(0,T;H^1_\sigma(\Omega))}$$

with a constant $C = C(k) > 0$ and thus

$$\left|\int_0^T\int_\Omega \rho_{k,M}(Q_{\mathcal{S}_{k,M}}(u_{k,M})\cdot\nabla)v\cdot u_{k,M}dxdt\right|$$
$$\leqslant C\,\|u_{k,M}\|^{3/2}_{L^2(0,T;H^1_\sigma(\Omega))}\,\|u_{k,M}\|^{1/2}_{L^\infty(0,T;L^2_\sigma(\Omega))}\,\|v\|_{L^4(0,T;H^1_\sigma(\Omega))}.$$

The other terms in (4.64) can be estimated in a standard way and by using (4.55), this implies that $\frac{\partial}{\partial t}\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M})$ is bounded in $L^{4/3}(0,T;(H^1_\sigma(\Omega))')$. Using (4.63), we can apply the Aubin-Lions compactness result and we deduce that

$$\mathbb{P}_{V_i}(\rho_{k,M}u_{k,M}) \to \mathbb{P}_{V_i}(\rho_k u_k) \text{ strongly } L^2(0,T;(H^1_\sigma(\Omega))'). \tag{4.65}$$

Let us denote by $\mathbb{P} : L^2(\Omega) \to L^2_\sigma(\Omega)$ the orthogonal projection (the Leray projection). For any $z \in L^2(\Omega)$,

$$\|\mathbb{P}(z) - \mathbb{P}_{V_i}(z)\|_{(H^1_\sigma(\Omega))'} \leqslant \|z\|_{L^2(\Omega)}\sup_{\varphi\in H^1_\sigma(\Omega),\|\varphi\|_{H^1_\sigma(\Omega)}=1}\|\varphi - \mathbb{P}_{V_i}(\varphi)\|_{L^2(\Omega)}.$$

Using the compactness of the embedding $H^1_\sigma(\Omega) \subset L^2_\sigma(\Omega)$ and that $\{v_q\}$ is an orthonormal basis of $H^1_\sigma(\Omega)$,

$$\sup_{\varphi \in H^1_\sigma(\Omega), \|\varphi\|_{H^1_\sigma(\Omega)}=1} \|\varphi - \mathbb{P}_{V_i}(\varphi)\|_{L^2(\Omega)} \to 0 \text{ as } i \to \infty.$$

Combining this with (4.65) and with the fact that $\{\rho_{k,M} u_{k,M}\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ we deduce

$$\mathbb{P}(\rho_{k,M} u_{k,M}) \to \mathbb{P}(\rho_k u_k) \text{ strongly in } L^2(0,T;(H^1_\sigma(\Omega))'). \tag{4.66}$$

Now we follow an argument given in [**?**, p.47]: using (4.66) and (4.57), we first have

$$\begin{aligned}\int_0^T \int_\Omega \rho_{k,M} \left|u_{k,M}\right|^2 \, dxdt &= \int_0^T \langle \mathbb{P}(\rho_{k,M} u_{k,M}), u_{k,M} \rangle_{(H^1_\sigma(\Omega))', H^1_\sigma(\Omega)} \, dt \\ &\to \int_0^T \langle \mathbb{P}(\rho_k u_k), u_k \rangle_{(H^1_\sigma(\Omega))', H^1_\sigma(\Omega)} \, dt = \int_0^T \int_\Omega \rho_k \left|u_k\right|^2 \, dxdt. \end{aligned} \tag{4.67}$$

This yields

$$\sqrt{\rho_{k,M}} u_{k,M} \to \sqrt{\rho_k} u_k \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{4.68}$$

From (4.61) we have that

$$\frac{1}{\sqrt{\rho_{k,M}}} \to \frac{1}{\sqrt{\rho_k}} \text{ strongly in } \mathcal{C}([0,T];L^3(\Omega)).$$

The above convergence and (4.68) imply

$$u_{k,M} \to u_k \text{ strongly in } L^2(0,T;L^{\frac{6}{5}}(\Omega)). \tag{4.69}$$

From (4.55), we have that $\{u_{k,M}\}$ is bounded in $L^2(0,T;L^6(\Omega))$ and thus

$$u_{k,M} \to u_k \text{ strongly in } L^2(0,T;L^p(\Omega)) \quad (p < 6). \tag{4.70}$$

4.4.3 A monotonicity argument

In this section, we pass to the limit in the plastic term using a monotonicity argument. This type of technique is used to prove the existence of a weak solution of a Bingham fluid without the solid part, see [**?**, pp.296-297].

Let $\varphi \in \mathcal{C}^1([0,T];H^1_\sigma(\Omega))$. We denote by $\mathbb{P}^1_{V_M} : H^1_\sigma(\Omega) \to V_M$ the orthogonal projection and we define

$$\varphi_M := \mathbb{P}^1_{V_M} \varphi. \tag{4.71}$$

Then,

$$\varphi_M \to \varphi \text{ strongly in } \mathcal{C}^1([0,T];H^1_\sigma(\Omega)). \tag{4.72}$$

We set:

$$\begin{aligned} Z_M = \int_0^T \int_\Omega \rho_{k,M} \frac{\partial}{\partial t}(\varphi_M - u_{k,M}) \cdot (\varphi_M - u_{k,M}) dxdt + \frac{1}{2} \int_\Omega \rho(0,x) \left|u_{k,M}(0,x) - \varphi_M(0,x)\right|^2 dx. \\ + \int_0^T \int_\Omega \rho_{k,M} (Q_{\mathcal{S}_{k,M}}(u_{k,M}) \cdot \nabla)(\varphi_M - u_{k,M}) \cdot (\varphi_M - u_{k,M}) dxdt \\ + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) - j_{\frac{1}{M}}(D(u_{k,M})) - \nabla j_{\frac{1}{M}}(D(u_{k,M})) : D(\varphi_M - u_{k,M}) \, dxdt. \end{aligned} \tag{4.73}$$

By the Reynolds transport theorem and the convexity of $j_{\frac{1}{M}}$, we have that:

$$Z_M \geqslant 0.$$

Then, using equation (4.46) with the test function $\varphi_M - u_{k,M}$, Z_M can be written as follows:

$$\begin{aligned}
Z_M = &\int_0^T \int_\Omega \rho_{k,M} \frac{\partial \varphi_M}{\partial t} \cdot (\varphi_M - u_{k,M}) dxdt + \int_0^T \int_\Omega \rho_{k,M} (Q_{\mathcal{S}_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \cdot (\varphi_M - u_{k,M}) dxdt \\
&+ 2\mu \int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M - u_{k,M}) dxdt + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) - j_{\frac{1}{M}}(D(u_{k,M})) dxdt \\
&- k \int_0^T \int_{\mathcal{S}_{k,M}} \left| u_{k,M} - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(u_{k,M}) \right|^2 dxdt \\
&+ k \int_0^T \int_{\mathcal{S}_{k,M}} (u_{k,M} - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(u_{k,M})) \cdot (\varphi_M - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(\varphi_M))) dxdt + \frac{1}{2} \int_\Omega \rho(0,x) \left| u_{k,M}(0,x) - \varphi_M(0,x) \right|^2 dx.
\end{aligned}$$

Since $Z_M \geqslant 0$ and, by (4.50),

$$0 \leqslant \left\| u_{k,M} - P^{\mathcal{R}}_{\mathcal{S}_{k,M}} u_{k,M} \right\|_{L^2(0,T;L^2(\mathcal{S}_{k,M}))} \leqslant \frac{C}{\sqrt{k}},$$

we deduce the following inequality:

$$\begin{aligned}
&\int_0^T \int_\Omega \rho_{k,M} \left(\frac{\partial \varphi_M}{\partial t} + (Q_{\mathcal{S}_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \right) \cdot (\varphi_M - u_{k,M}) dxdt \\
&\quad + 2\mu \int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M) dxdt + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M)) dxdt \\
&\quad + C\sqrt{k} \left\| \varphi_M - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(\varphi) \right\|_{L^2(0,T;L^2(\mathcal{S}_{k,M}))} \geqslant -\frac{1}{2} \int_\Omega \rho(0,x) \left| u_{k,M}(0,x) - \varphi_M(0,x) \right|^2 dx \\
&\quad + g \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M})) dxdt + 2\mu \int_0^T \int_\Omega |D(u_{k,M})|_2^2 dxdt. \quad (4.74)
\end{aligned}$$

To conclude, we need to pass to the limit the terms in the above inequality as $M \to \infty$:

- Combining (4.61), (4.70) and (4.72), we deduce

$$\int_0^T \int_\Omega \rho_{k,M} \frac{\partial \varphi_M}{\partial t} \cdot (\varphi_M - u_{k,M}) \, dxdt \to \int_0^T \int_\Omega \rho_k \frac{\partial \varphi}{\partial t} \cdot (\varphi - u_k) \, dxdt. \quad (4.75)$$

- By Lemma A.1 and (4.70), we deduce that:

$$Q_{\mathcal{S}_{k,M}}(u_{k,M}) \to Q_{\mathcal{S}_k}(u_k) \text{ strongly in } L^2(0,T;L^5(\Omega)). \quad (4.76)$$

  Combining this with (4.61), (4.70) and (4.72) yields

$$\begin{aligned}
&\int_0^T \int_\Omega \rho_{k,M} (Q_{\mathcal{S}_{k,M}}(u_{k,M}) \cdot \nabla) \varphi_M \cdot (\varphi_M - u_{k,M}) \, dxdt \\
&\qquad \to \int_0^T \int_\Omega \rho_k (Q_{\mathcal{S}_k}(u_k) \cdot \nabla) \varphi \cdot (\varphi - u_k) \, dxdt. \quad (4.77)
\end{aligned}$$

- From (4.72) and (4.60), we obtain

$$\mathbb{1}_{\mathcal{S}_{k,M}} \varphi_{k,M} \to \mathbb{1}_{\mathcal{S}_k} \varphi \quad \text{in } L^2(0,T;L^2(\Omega))$$

and thus (with (4.22) and (4.23))

$$\mathbb{1}_{\mathcal{S}_{k,M}} P^{\mathcal{R}}_{\mathcal{S}_{k,M}} \varphi_{k,M} \to \mathbb{1}_{\mathcal{S}_k} P^{\mathcal{R}}_{\mathcal{S}_k} \varphi \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Consequently,

$$\left\| \varphi_M - P^{\mathcal{R}}_{\mathcal{S}_{k,M}} \varphi_M \right\|_{L^2(0,T;L^2(\mathcal{S}_{k,M}))} \to \left\| \varphi - P^{\mathcal{R}}_{\mathcal{S}_k} \varphi \right\|_{L^2(0,T;L^2(\mathcal{S}_k))}. \tag{4.78}$$

Similarly, since $u_{k,M} \to u_k$ strongly in $L^2(0,T;L^2(\Omega))$ and $\{u_{k,M}\}$ is bounded in $L^2(0,T;L^6(\Omega))$, we deduce that

$$\left\| u_{k,M} - P^{\mathcal{R}}_{\mathcal{S}_{k,M}} u_{k,M} \right\|_{L^2(0,T;L^2(\mathcal{S}_{k,M}))} \to \left\| u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k \right\|_{L^2(0,T;L^2(\mathcal{S}_k))}. \tag{4.79}$$

- From (4.72) and (4.57), we have that:

$$\int_0^T \int_\Omega D(u_{k,M}) : D(\varphi_M) dxdt \to \int_0^T \int_\Omega D(u_k) : D(\varphi)\ dxdt. \tag{4.80}$$

- From (4.57), we also have that

$$\liminf_{M\to\infty} \int_0^T \int_\Omega |D(u_{k,M})|_2^2\ dxdt \geqslant \int_0^T \int_\Omega |D(u_k)|_2^2\ dxdt. \tag{4.81}$$

- From the definition (4.37) of $j_{\frac{1}{M}}$,

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M))\ dxdt = \frac{M}{M+1} \int_0^T \int_\Omega |D(\varphi_M)|^{\frac{1}{M}+1}\ dxdt.$$

Using (4.72) and the dominated convergence theorem, we deduce

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(\varphi_M))\ dxdt \to \int_0^T \int_\Omega |D(\varphi)|\ dxdt. \tag{4.82}$$

- Following the argument of [**?**, p.298], we are going now to prove

$$\liminf_{M\to\infty} \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \int_0^T \int_\Omega |D(u_k)|\, dxdt. \tag{4.83}$$

First, by Hölder's inequality we have that:

$$\int_0^T \int_\Omega |D(u_{k,M})|\, dxdt \leqslant \left(\int_0^T \int_\Omega |D(u_{k,M})|^{1+\frac{1}{M}}\ dxdt \right)^{\frac{M}{1+M}} (T|\Omega|)^{\frac{1}{M+1}}$$

and thus

$$\int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \frac{M}{(1+M)(T|\Omega|)^{\frac{1}{M}}} \left(\int_0^T \int_\Omega |D(u_{k,M})|\, dxdt\right)^{1+\frac{1}{M}}.$$

Since $D(u_{k,M})$ is bounded in $L^1(0,T;L^1(\Omega))$, we have

$$\liminf_{M\to\infty} \int_0^T \int_\Omega j_{\frac{1}{M}}(D(u_{k,M}))dxdt \geqslant \liminf_{M\to\infty} \int_0^T \int_\Omega |D(u_{k,M})|\, dxdt$$

and since the application $v \mapsto \int_0^T \int_\Omega |D(v)|\, dxdt$ is continuous and convex on $L^2(0,T;H^1_\sigma(\Omega))$, it is lower semi-continuous for the weak topology. Using this with (4.57) yields (4.83).

- Using (4.47) and (4.72), we deduce that

$$\int_\Omega \rho_0 \left|u_{k,M}(0,\cdot) - \varphi_M(0,\cdot)\right|^2\, dx \to \int_\Omega \rho_0 \left|u_0 - \varphi(0,\cdot)\right|^2\, dx. \tag{4.84}$$

Gathering (4.74), (4.75), (4.77), (4.78), (4.80), (4.81), (4.82), (4.83), (4.84), we deduce the following inequality:

$$\int_0^T \int_\Omega \rho\left(\frac{\partial\varphi}{\partial t} + (Q_{\mathcal{S}_k}(u_k)\cdot\nabla)\varphi\right)\cdot(\varphi - u_k)dxdt + 2\mu \int_0^T \int_\Omega D(u_k) : D(\varphi)dxdt$$
$$+ g\int_0^T \int_\Omega |D(\varphi)|\, dxdt + C\sqrt{k}\left\|\varphi - P^{\mathcal{R}}_{\mathcal{S}_k}(\varphi)\right\|_{L^2(0,T;L^2(\mathcal{S}_k))} \geqslant -\frac{1}{2}\int_\Omega \rho(0,x)\left|u_0 - \varphi(0,x)\right|^2 dx$$
$$+ g\int_0^T \int_\Omega |D(u_k)|\, dxdt + 2\mu \int_0^T \int_\Omega |D(u_k)|_2^2\, dxdt \tag{4.85}$$

for any $\varphi \in \mathcal{C}^1([0,T];H^1_\sigma(\Omega))$. Using standard techniques, see for example [**?**, pp. 290-291], by (4.56), (4.79), (4.83) and (4.81) we deduce the following energy estimate for a.e. $t \in (0,T)$:

$$\frac{1}{2}\int_\Omega \rho_k(t,x)\left|u_k(t,x)\right|^2 dx + 2\mu\int_0^t \int_\Omega |D(u_k)|^2\, dxdt + g\int_0^t \int_\Omega |D(u_k)|\, dxdt$$
$$+ k\int_0^t \int_{\mathcal{S}_k} \left|u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right|^2 dxdt \leqslant \frac{1}{2}\int_\Omega \rho_0 \left|u_0\right|^2 dx. \tag{4.86}$$

In order to pass to the limit in k (next section), we need another relation. Let us take φ_M given by (4.71) with $\varphi \in \mathcal{C}^1_0((0,T);H^1_\sigma(\Omega))$ as a test function in (4.49).

$$-\int_0^T \int_\Omega \rho_{k,M}\left(\frac{\partial\varphi_M}{\partial t} + (Q_{\mathcal{S}_{k,M}}(u_{k,M})\cdot\nabla)\varphi_M\right)\cdot u_{k,M}\, dxdt$$
$$+ \int_0^T \int_\Omega (2\mu D(u_{k,M}) + g\nabla j_{\frac{1}{M}}(D(u_{k,M}))) : D(\varphi_M)\, dxdt$$
$$+ k\int_0^T \int_{\mathcal{S}_{k,M}} (u_{k,M} - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(u_{k,M}))\cdot(\varphi_M - P^{\mathcal{R}}_{\mathcal{S}_{k,M}}(\varphi_M))\, dxdt = 0. \tag{4.87}$$

Using (4.39), (4.50) and (4.54), we deduce that

$$\left\|\nabla j_{\frac{1}{M}}(D(u_{k,M}))\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C,$$

with a constant C independent of the solution and of k. Therefore, for any k, there exists an element $\chi_k \in L^2(0,T;L^2(\Omega))$ such that:

$$\nabla j_{\frac{1}{M}}(D(u_{k,M})) \rightharpoonup \chi_k \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{4.88}$$

and

$$\|\chi_k\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C. \tag{4.89}$$

Then taking $M \to \infty$ in equation (4.87) and using (4.75), (4.77), (4.78), (4.79), (4.80), (4.88), we obtain the following equation:

$$-\int_0^T \int_\Omega \rho_k \left(\frac{\partial \varphi}{\partial t} + (Q_{\mathcal{S}_k}(u_k)\cdot\nabla)\varphi\right)\cdot u_k dxdt + \int_0^T \int_\Omega (2\mu D(u_k) + g\chi_k) : D(\varphi) dxdt \\ + k\int_0^T \int_{\mathcal{S}_k} (u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k))\cdot(\varphi - P^{\mathcal{R}}_{\mathcal{S}_k}(\varphi)) dxdt = 0. \tag{4.90}$$

4.5 Passing to the limit $k \to \infty$

The aim of this section is to finish the proof of Theorem 4.1.3. From (4.86), we deduce that there exist

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_\sigma(\Omega)) \quad (h,R) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times \mathcal{S}O(3))$$

such that:

$$u_k \overset{*}{\rightharpoonup} u \text{ weak star in } L^\infty(0,T;L^2_\sigma(\Omega)), \tag{4.91}$$

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_\sigma(\Omega)), \tag{4.92}$$

$$(h_k,R_k) \overset{*}{\rightharpoonup} (h,R) \text{ weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times \mathcal{S}O(3)) \tag{4.93}$$

and

$$(h_k,R_k) \to (h,R) \text{ strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times \mathcal{S}O(3)). \tag{4.94}$$

We write

$$\mathcal{S} := \widehat{\mathcal{S}}(h,R)$$

and

$$J := \widehat{J}(h,R).$$

By Lemma 4.1 we have that

$$\mathbb{1}_{\mathcal{S}_k} \to \mathbb{1}_{\mathcal{S}} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{4.95}$$

and thus

$$\rho_k \to \rho := \widehat{\rho}_{h,R} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{4.96}$$

Using (4.22) and (4.23), we deduce

$$P^{\mathcal{R}}_{\mathcal{S}_k} u_k \overset{*}{\rightharpoonup} P^{\mathcal{R}}_{\mathcal{S}} u \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{4.97}$$

We write

$$P^{\mathcal{R}}_{\mathcal{S}} u =: \ell + \omega \times (x-h) \quad \text{ in } (0,T). \tag{4.98}$$

By the energy estimate (4.86) we deduce that:

$$\left\| u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k \right\|_{L^2(0,T;L^2(\mathcal{S}_k))} \leqslant \frac{C}{\sqrt{k}}. \tag{4.99}$$

Then, taking $k \to \infty$, we deduce that $u = P^{\mathcal{R}}_{\mathcal{S}} u$ in $\mathcal{S}$. Therefore, we conclude that $u(t,\cdot) \in H^1_{\mathcal{S}(t)}(\Omega)$ a.e. in $(0,T)$.

4.5.1 Strong Convergence of the velocity field

As in the limit in M, we require the strong convergence of the velocity to show the convergence of the convective term. To do this we follow the main steps of Section 7 of [?] (see also Section 5.5 of [?]).

We define the space

$$H^s_{\mathcal{S}}(\Omega) := \{v \in H^s_\sigma(\Omega);\ D(v) = 0 \text{ in } \mathcal{S}\}$$

and the projection

$$\mathcal{P}^s_{\mathcal{S}} : H^s_\sigma(\Omega) \mapsto H^s_{\mathcal{S}}(\Omega).$$

Using (4.94), we deduce that for all $d > 0$, there exists k_0 such that for all $k \geq k_0$,

$$\mathcal{S}_k(t) \subset (\mathcal{S}(t))^{\frac{d}{2}} \quad \forall t \in [0,T]. \tag{4.100}$$

Moreover, using the Heine theorem, there exists $N(d) > 0$ such that if

$$\tau := T/N \quad \text{and} \quad I_j := [j\tau, (j+1)\tau]$$

then

$$(\mathcal{S}(t))^{\frac{d}{2}} \subset (\mathcal{S}(j\tau))^d \subset (\mathcal{S}(t))^{2d} \quad (t \in I_j).$$

Then, we consider a test function $\varphi \in C^\infty_0((0,T), H^1_\sigma(\Omega))$ such that $D(\varphi(t,\cdot)) = 0$ in $(\mathcal{S}(j\tau))^d$ and $\varphi(t,\cdot) = 0$ if $t \notin I_j$. With such a test function in (4.90), the integral related to the penalization term vanishes, and we obtain the following estimate:

$$\left| \int_{I_j} \int_\Omega \rho_k u_k \cdot \frac{\partial \varphi}{\partial t} dxdt \right| \leqslant C\Big(\|Q_{\mathcal{S}_k}(u_k)\|_{L^2(0,T;L^4(\Omega))} \|u_k\|^{1/4}_{L^\infty(0,T;L^2_\sigma(\Omega))} \|u_k\|^{3/4}_{L^2(0,T;H^1_\sigma(\Omega))} + \|u_k\|_{L^2(0,T;H^1_\sigma(\Omega))} + \|\chi_k\|_{L^2(0,T;H^1_\sigma(\Omega))} \Big) \|\varphi\|_{L^8(I_j;H^1_\sigma(\Omega))}. \tag{4.101}$$

From (A.3) and (A.14), we have

$$\|Q_{\mathcal{S}_k}(u_k) - u_k\|_{L^p(\mathcal{F}_k)} \leqslant C\left(\left(\frac{1}{k}\right)^{\frac{1}{p}-\frac{1}{6}} \|u_k\|_{H^1(\mathcal{F}_k)} + \left\| \left(u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right) \cdot \widehat{n} \right\|_{L^p(\partial \mathcal{S}_k)} \right), \tag{4.102}$$

for $p \in [2,6]$. Moreover, using a Sobolev embedding, a trace theorem and an interpolation result, we can check that for $p \in [2,4]$,

$$\left\|\left(u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right) \cdot \widehat{n}\right\|_{L^p(\partial \mathcal{S}_k)} \leq C \left\|u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right\|^{2/p-1/2}_{L^2(\mathcal{S}_k)} \left\|u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right\|^{3/2-2/p}_{H^1(\mathcal{S}_k)}.$$

Combining this with (4.99), we deduce

$$\left\|\left(u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k\right) \cdot \widehat{n}\right\|_{L^p(\partial \mathcal{S}_k)} \leq C \left(\frac{1}{k}\right)^{1/p-1/4} \|u_k\|^{3/2-2/p}_{H^1(\mathcal{S}_k)}. \tag{4.103}$$

In particular,

$$\{Q_{\mathcal{S}_k}(u_k)\} \quad \text{is bounded in } L^2(0,T;L^4(\Omega)).$$

Using the above estimate, (4.86) and (4.89) in (4.101), we deduce that

$$\left\{\frac{\partial}{\partial t} \mathcal{P}^0_{(\mathcal{S}(j\tau))^d}(\rho_k u_k)\right\}_k \quad \text{is bounded in} \quad L^{8/7}(I_j; (H^1_{(\mathcal{S}(j\tau))^d}(\Omega))').$$

Using the Aubin-Lions lemma we deduce

$$\mathcal{P}^0_{(\mathcal{S}(j\tau))^d}(\rho_k u_k) \to \mathcal{P}^0_{(\mathcal{S}(j\tau))^d}(\rho u) \text{ strongly in } L^2(I_j; (H^s_{(\mathcal{S}(j\tau))^d}(\Omega))') \quad (s \in (0,1]). \tag{4.104}$$

Then using the relation

$$\mathcal{P}^0_{(\mathcal{S}(j\tau))^d} \mathcal{P}^s_{(\mathcal{S}(t))^{2d}} = \mathcal{P}^s_{(\mathcal{S}(t))^{2d}} \quad \forall t \in I_j,$$

we deduce for any $s \in (0,1]$,

$$\lim_{k\to\infty} \int_0^T \int_\Omega \rho_k u_k \cdot \mathcal{P}^s_{(\mathcal{S}(t))^{2d}}(u_k)\ dxdt = \int_0^T \int_\Omega \rho u \cdot \mathcal{P}^s_{(\mathcal{S}(t))^{2d}}(u)\ dxdt.$$

Then, using Corollary A.1.4 and (4.99), we have for $s \in (0,1/3)$

$$\int_0^T \left\|u_k(t,\cdot) - \mathcal{P}^s_{(\mathcal{S}_n(t))^d} u_k(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant C(d^{2(1/3-s)} + k^{-1/2}) \tag{4.105}$$

and

$$\int_0^T \left\|u(t,\cdot) - \mathcal{P}^s_{(\mathcal{S}(t))^d} u(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant C d^{2(1/3-s)}. \tag{4.106}$$

so that

$$\lim_{k\to\infty} \int_0^T \int_\Omega \rho_k |u_k|^2\ dxdt = \int_0^T \int_\Omega \rho |u|^2\ dxdt$$

and this allows us to deduce that

$$u_k \to u \text{ strongly in } L^2(0,T;L^p(\Omega)) \quad (p < 6). \tag{4.107}$$

4.5.2 Passing to the limit in the velocity inequality

Assume $v \in \mathcal{C}^1([0,T]; H^1_{\mathcal{S}(t)}(\Omega))$ with $\operatorname{supp} v \subset \Omega_\eta$, $\eta > 0$. We set

$$v_k := \overline{\Phi}_{h_k,R_k} \circ \Phi_{h,R}(v), \tag{4.108}$$

where Φ and $\overline{\Phi}$ are defined in (4.26) and (4.27). Then, for k large enough we have

$$v_k \in \mathcal{C}([0,T]; H^1_\sigma(\Omega)). \tag{4.109}$$

Moreover,

$$D(v_k) = 0 \quad \text{in } \mathcal{S}_k. \tag{4.110}$$

Using Lemma A.2 in [?] and (4.94), we deduce

$$v_k \to v \text{ strongly in } \mathcal{C}([0,T]; H^1_\sigma(\Omega)). \tag{4.111}$$

Similarly, deriving (4.108), we obtain

$$\frac{\partial v_k}{\partial t} + \left(P^{\mathcal{R}}_{\mathcal{S}_k} u_k \cdot \nabla\right) v_k - \omega_k \times v_k \to \frac{\partial v}{\partial t} + \left(P^{\mathcal{R}}_{\mathcal{S}} u \cdot \nabla\right) v - \omega \times v \text{ strongly in } L^\infty(0,T; L^2(\Omega)). \tag{4.112}$$

On the other hand, from (4.107) and (4.22), (4.23), we have

$$P^{\mathcal{R}}_{\mathcal{S}_k} u_k \to P^{\mathcal{R}}_{\mathcal{S}} u \text{ strongly in } L^2(0,T; L^2(\Omega)). \tag{4.113}$$

Finally, combining (4.112) with (4.111) and with (4.113), we conclude

$$\frac{\partial v_k}{\partial t} \to \frac{\partial v}{\partial t} \text{ strongly in } L^2(0,T; L^2(\Omega)). \tag{4.114}$$

Taking $\varphi = v_k$ in (4.85), we obtain

$$\begin{aligned}\int_0^T \int_\Omega \rho \left(\frac{\partial v_k}{\partial t} + (Q_{\mathcal{S}_k}(u_k) \cdot \nabla) v_k \right) \cdot (v_k - u_k) \; dxdt + 2\mu \int_0^T \int_\Omega D(u_k) : D(v_k) \; dxdt \\ + g \int_0^T \int_\Omega |D(v_k)| \; dxdt \geqslant -\frac{1}{2} \int_\Omega \rho(0,x) \left| u_0 - v_k(0,x) \right|^2 dx \\ + g \int_0^T \int_\Omega |D(u_k)| \, dxdt + 2\mu \int_0^T \int_\Omega |D(u_k)|^2 \, dxdt.\end{aligned} \tag{4.115}$$

We can pass to the limit as in Section 4.4, the only term that needs more details is the first term:

$$\begin{aligned}\int_0^T \int_\Omega \rho_k \left(\frac{\partial v_k}{\partial t} + (Q_{\mathcal{S}_k} u_k \cdot \nabla) v_k \right) \cdot (v_k - u_k) dxdt = \rho_f \int_0^T \int_\Omega \mathbb{1}_{\mathcal{F}_k} \frac{\partial v_k}{\partial t} \cdot (v_k - u_k) dxdt \\ + \rho_f \int_0^T \int_\Omega \mathbb{1}_{\mathcal{F}_k} (Q_{\mathcal{S}_k} u_k \cdot \nabla) v_k \cdot (v_k - u_k) dxdt \\ + \rho_s \int_0^T \int_\Omega \mathbb{1}_{\mathcal{S}_k} \left(\frac{\partial v_k}{\partial t} + \left(P^{\mathcal{R}}_{\mathcal{S}_k} u_k \cdot \nabla\right) v_k \right) \cdot (v_k - u_k) dxdt.\end{aligned} \tag{4.116}$$

Combining (4.95), (4.107), (4.111) and (4.114) we deduce that

$$\rho_f \int_0^T \int_\Omega \mathbb{1}_{\mathcal{F}_k} \frac{\partial v_k}{\partial t} \cdot (v_k - u_k)\, dxdt \to \rho_f \int_0^T \int_\Omega \mathbb{1}_{\mathcal{F}} \frac{\partial v}{\partial t} \cdot (v - u)\, dxdt.$$

Relation (4.103), (4.102) and (4.86) imply

$$\mathbb{1}_{\mathcal{F}_k}(Q_{\mathcal{S}_k}(u_k) - u_k) \to 0 \quad \text{strongly in } L^2(0,T;L^p(\Omega))$$

for $p < 4$. Gathering the above limit with (4.107) and (4.95), we deduce

$$\mathbb{1}_{\mathcal{F}_k} Q_{\mathcal{S}_k}(u_k) \to \mathbb{1}_{\mathcal{F}} u \text{ strongly in } L^2(0,T;L^p(\Omega))$$

for $p < 4$. Combining this with (4.107) and (4.111), we obtain

$$\rho_f \int_0^T \int_{\mathcal{F}_k} (Q_{\mathcal{S}_k}(u_k) \cdot \nabla) v_k \cdot (v_k - u_k)\, dxdt \to \rho_f \int_0^T \int_{\mathcal{F}} (u \cdot \nabla) v \cdot (v - u)\, dxdt.$$

Finally, from (4.95), (4.107), (4.111) and (4.112),

$$\int_0^T \int_\Omega \mathbb{1}_{\mathcal{S}_k} \left(\frac{\partial v_k}{\partial t} + \left(P^{\mathcal{R}}_{\mathcal{S}_k} u_k \cdot \nabla\right) v_k \right) \cdot (v_k - u_k)\, dxdt \to \int_0^T \int_\Omega \mathbb{1}_{\mathcal{S}} \left(\frac{\partial v}{\partial t} + \left(P^{\mathcal{R}}_{\mathcal{S}} u \cdot \nabla\right) v \right) \cdot (v - u)\, dxdt.$$

We thus conclude that u satisfies inequality (4.20). We can also pass to the limit in (4.86) we deduce (4.21). This finishes the proof of Theorem 4.1.3.

Chapter 5

A Navier-Stokes-Fourier-rigid body system

In this chapter, we study some mathematical aspects of a granular material with heat conducting properties using a multiphase fluid-structure interaction system where the fluid is a heat conducting incompressible Newtonian liquid, and where the structure is a perfect heat conductor rigid body. The 3D incompressible Fourier-Navier-Stokes equations model the fluid, and the Newton laws and balance of internal energy model the rigid body. For the boundary conditions, we consider the Navier slip in the exterior domain and in the fluid-solid interface. The main result is the existence of a weak solution for the corresponding system. The weak formulation involves the balance of momentum and total internal energy and a free boundary (due to the motion of the rigid body).

Regarding Part I, the model described in the above paragraph is a different approach to modeling granular matter. The difference lies in that the continuum hypothesis of the mixture theory is dismissed, and the equations of each component are satisfied in different domains. Concerning Chapter 4, the model presented in this chapter is considered a fully discrete model of a granular material with heat conducting properties. Meanwhile, the model shown in Chapter 4 is a mix between continuous and a discrete model for an isothermal granular fluid.

At first sight, this approach seems much more realistic than the one discuss in Part I and Chapter 4 since now the fluid and the solid have thermal properties. However, we also assume that the rigid solid is a perfect conductor and the densities of the fluid and the solid are equal. These two simplification hypotheses seem restrictive. Nevertheless, they can be applied in many natural and industrial contexts. In the case of the conductivity of the solid, this situation applies when the thermal capacity of the fluid is negligible regarding the thermal capacity of the solid. Regarding the densities, the densities of the fishes are similar to the density of the ocean in which they are swimming. There also exists solid materials that have nearly the same density of water. For instance, there are many kinds of rubber which have a density close to 1 $[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$: common rubbers, polyisoprene, ethylene-propylene-diene rubber, polybutadiene and styrene-butadiene rubber have a density of around 0.92-0.98 $[\mathrm{gr}\cdot\mathrm{cm}^{-3}]$.

Concerning the mathematical result, our work has two main pillars: to deal with the temperature we followed the ideas of [?] and to deal with the fluid-rigid body problem we followed the ideas of [?]. In [?], the authors established a framework to prove the existence of weak solutions to the incompressible Fourier-Navier-Stokes equations with heat-dependent coefficients. In their proof, the authors used the total energy equation in their weak formulation. This equation contains the pressure of the fluid. To get estimates for the pressure in [?], the authors used the Helmholtz-Weyl decomposition and proved that the pressure is an integrable function. However, this is only possible considering some slip boundary condition. In particular, they consider a Navier slip boundary condition. Up to our knowledge, a method to obtain an integrable pressure with no-slip boundary condition does not exist, instead, the pressure can be decomposed in 2 parts, one part is integrable, and the other is not [?]. Therefore, in our work, we consider Navier slip boundary conditions in the outer boundary and the mutual frontier between the solid and fluid. Then, to deal with fluid-rigid body interaction problem with Navier slip boundary condition, we follow the ideas of [?], where they treat the case of an incompressible Newtonian fluid-rigid body system with Navier slip boundary condition.

5.1 Introduction and main result

Let us describe our system: we consider $\Omega \subset \mathbb{R}^3$ an open, bounded and connected set containing $\{0\}$. The set Ω contains a Newtonian fluid and a rigid solid. We denote respectively by $\mathcal{S}(t)$ and by $\mathcal{F}(t)$ the domains of the solid and the fluid at instant t. We assume that the solid is a rigid body and its evolution domain can be described from its initial configuration $\mathcal{S}_0$ by the position of its center of mass $h(t)$ and the rotation with respect to the initial configuration, represented by the matrix $R(t)$. For the sake of simplicity, we suppose that $h(0) = 0$. That is

$$\mathcal{S}(t) = \widehat{\mathcal{S}}(h(t), R(t)),$$

where the function $\widehat{\mathcal{S}}$ is defined by

$$\widehat{\mathcal{S}}(a, Q) := a + Q\mathcal{S}_0, \tag{5.1}$$

for any $a \in \mathbb{R}^3$ and $Q \in SO(3)$ (the rotation group). We also denote

$$\widehat{\mathcal{F}}(a, Q) := \Omega \setminus \overline{\widehat{\mathcal{S}}(a, Q)} \quad \text{and} \quad \mathcal{F}(t) = \widehat{\mathcal{F}}(h(t), R(t)). \tag{5.2}$$

We assume in what follows that $\mathcal{S}_0$ (and thus $\mathcal{S}(t)$) is open, bounded and connected and that $\mathcal{F}_0 := \Omega \setminus \overline{\mathcal{S}_0}$ (and thus $\mathcal{F}(t)$ as long as the rigid body remains inside Ω) is connected. We denote by $\theta_f(t, x)$ and $\theta_s(t, x)$ the temperatures of the fluid and the solid respectively. We assume that the rigid body is a perfect rigid conductor. More precisely, for each t, the temperature of the solid is constant:

$$\nabla\theta_s(t, x) = 0 \quad x \in \mathcal{S}(t),\ t \in (0, T). \tag{5.3}$$

The governing equations for the fluid flow is written by considering the Cauchy momentum equation of an incompressible fluid. That is

$$\rho_f \left(\frac{\partial u_f}{\partial t} + (u_f \cdot \nabla)u_f \right) - \operatorname{div} \mathbf{\Sigma}_f = \rho_f b, \qquad \text{in } \mathcal{F}(t),\ t \in (0, T), \tag{5.4}$$

$$\operatorname{div} u_f = 0, \qquad \text{in } \mathcal{F}(t),\ t \in (0, T), \tag{5.5}$$

where u_f is the velocity of the fluid, $\mathbf{\Sigma}_f$ is the Cauchy stress tensor of the fluid, ρ_f is the density of the fluid which is a positive constant and, b is the body density force acting throughout the system. The Cauchy stress tensor $\mathbf{\Sigma}_f$ is given by the constitutive equation for a Newtonian fluid:

$$\mathbf{\Sigma}_f = -pI_3 + 2\mu(\theta_f)D(u_f), \tag{5.6}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ is the function that indicates how the mulecular viscosity of the fluid depends on the temperature.

The motion of the rigid body is governed by the Newton equations for linear and angular momentum:

$$m\ell' = -\int_{\partial\mathcal{S}} \Sigma_f \widehat{n}\, dx + \int_{\mathcal{S}} \rho_s b\, dx, \qquad \text{in } (0,T), \tag{5.7}$$

$$(J\omega)' = -\int_{\partial\mathcal{S}} (x-h) \times \mathbf{\Sigma}_f \widehat{n}\, dx + \int_{\mathcal{S}} (x-h) \times \rho_s b\, dx, \qquad \text{in } (0,T), \tag{5.8}$$

$$R' = \mathbb{A}(\omega)R, \qquad \text{in } (0,T), \tag{5.9}$$

$$h' = \ell, \qquad \text{in } (0,T), \tag{5.10}$$

where we denote by $\widehat{n}$ the outward normal of $\mathcal{F}$, ρ_s is the solid density,which is a positive constant ℓ is the velocity of the center of mass of the rigid body; ω is the angular velocity of the rigid body.

For any $\omega \in \mathbb{R}^3$, $\mathbb{A}(\omega)$ is the skew symmetric matrix:

$$\mathbb{A}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

The mass of the solid m is given by

$$m = \rho_s \left| S_0 \right|,$$

where $|S_0|$ is the volume of S_0, and the moment of inertia J is given by:

$$J(t) = \widehat{J}(h(t), R(t)),$$

where

$$\widehat{J}(a,Q) = \rho_s \int_{\widehat{\mathcal{S}}(a,Q)} \left(\left| x-a \right|^2 I_3 - (x-a) \otimes (x-a) \right) dx.$$

We can check that

$$\widehat{J}(a,Q) = QJ_0Q^*, \tag{5.11}$$

where Q^* is the transpose matrix of Q and where

$$J_0 = \rho_s \int_{\mathcal{S}_0} \left(\left| x \right|^2 I_3 - x \otimes x \right) dx.$$

In particular, $J(t)$ is symmetric and positive definite.

Moreover, for the thermic behavior, we consider the energy equation for the fluid and the solid parts:

$$\rho_f c_f \left(\frac{\partial \theta_f}{\partial t} + u_f \cdot \nabla \theta_f \right) = \Sigma_f : D(u_f) - \operatorname{div}(q_f) + \rho_f w, \qquad \text{in } \mathcal{F}(t),\ t \in (0,T), \tag{5.12}$$

$$mc_s\theta_s' = \int_{\partial\mathcal{S}} q_f \cdot \widehat{n}\, dx + \int_{\mathcal{S}} \rho_s w\, dx, \qquad \text{in } (0,T), \tag{5.13}$$

where q_f is the heat flux of the fluid and, c_f and c_s are the specific heats capacities of the fluid and the solid respectively, which are positive constants, and w is the energy density source acting throughout the system. The heat flux of the fluid satisfy the Fourier law given by:

$$q_f = -\kappa(\theta_f)\nabla\theta_f, \tag{5.14}$$

where $\kappa : \mathbb{R} \to \mathbb{R}$ is the function that indicates how the heat conductivity of the fluid depends on the temperature. The precedent equations are completed by boundary conditions, where we assume continuity of the normal component of the velocity field both in the fluid-solid interface and in the external boundary. For the tangential component of the velocity field, we consider a Navier boundary condition. Finally, for the temperature, we assume continuity in the fluid-solid interface, and we suppose the external boundary is adiabatic. That is:

$$u_f(t,x)\cdot\widehat{n}(t,x) = (\ell(t)+\omega(t)\times(x-h(t)))\cdot\widehat{n}(t,x), \quad x\in\partial\mathcal{S}(t),\ t\in(0,T), \tag{5.15}$$
$$\nu_{\mathcal{S}}(u_f(t,x)-(\ell(t)+\omega(t)\times(x-h(t))))_\tau + (\Sigma_f\widehat{n})_\tau = 0, \quad x\in\partial\mathcal{S}(t),\ t\in(0,T), \tag{5.16}$$
$$\theta_f(t,x) = \theta_s(t) \quad x\in\partial\mathcal{S}(t),\ t\in(0,T), \tag{5.17}$$
$$u_f\cdot\widehat{n} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{5.18}$$
$$\nu_\Omega(u_f)_\tau + (\boldsymbol{\Sigma}_f\widehat{n})_\tau = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{5.19}$$
$$\nabla\theta_f\cdot\widehat{n} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{5.20}$$

where ν_Ω and $\nu_{\mathcal{S}}$ are positive constants and, given a vector function $(t,x)\mapsto a(t,x)\in\mathbb{R}^3$, we denote the tangential component of a:

$$\begin{aligned} a_\tau(t,x) &= \widehat{n}(t,x)\times(a(t,x)\times\widehat{n}(t,x)) \\ &= a(t,x)-(a(t,x)\cdot\widehat{n}(t,x))\widehat{n}(t,x) \quad x\in\partial\mathcal{F}(t),\ t\in[0,T]. \end{aligned} \tag{5.21}$$

The above system is coupled with the initial conditions:

$$u_f(0,x) = u_{0,f} \text{ and } \theta_f(0,x) = \theta_{0,f}, \quad x\in\mathcal{F}(0), \tag{5.22}$$
$$\theta_s(0) = \theta_{s,0}, \quad x\in\mathcal{S}_0, \tag{5.23}$$
$$R(0) = I_3,\ h(0)=0,\ \ell(0)=\ell_0,\ \omega(0)=\omega_0. \quad \text{in } \mathbb{R}^3. \tag{5.24}$$

In what follows of this chapter, we consider the following: the densities ρ_f and ρ_s are equal, that is

$$\rho_f = \rho_s = \rho, \tag{5.25}$$

Assumption (5.25) is clearly restrictive but it is important for the forthcoming analysis (see Remark 5.5).

We also assume the following hypothesis on the continuity and boundness of the functions μ and κ:

Hypothesis 5.1. *The molecular viscosity $\mu : \mathbb{R}\to\mathbb{R}$ and the heat conductivity $\kappa : \mathbb{R}\mapsto\mathbb{R}$ are continuous functions such that there exist real numbers $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, $\overline{\kappa}$, such that:*

$$0 < \underline{\mu} \leqslant \mu(\vartheta) \leqslant \overline{\mu} \text{ for all } \vartheta\in\mathbb{R},$$

and

$$0 < \underline{\kappa} \leqslant \kappa(\vartheta) \leqslant \overline{\kappa} \text{ for all } \vartheta\in\mathbb{R}.$$

To write the weak formulation associated with system (5.4)-(5.24), we first recall some classical definitions and we introduce some notation.
For $q > 1$ and $k > 0$, we denote by L^q and $W^{k,q}$ the classical Lebesgue and Sobolev spaces. For the case $q = 2$, we write $W^{k,2} = H^k$ and we denote by W^{-k,q^*} the dual space of $W^{k,q}$, where $q^* = q/(q-1)$. We also denote by $\mathcal{C}^q$ the space of q-times continuous differential functions and by $\mathcal{C}^{1,1}$, the class of continuously differentiable functions such that the function and its derivative are Lipschitz functions . For any open set Ω , we write $\mathcal{C}_0^q(\Omega)$ the set of all functions in $\mathcal{C}^q(\Omega)$ with compact support included in Ω and, we denote by $W_0^{k,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$. We denote $\mathcal{C}^q(\overline{\Omega})$ to the class of function in $\mathcal{C}_0^q(\mathbb{R}^3)$ restricted to Ω. For Ω with a bounded boundary, we say that Ω is of class $\mathcal{C}^{1,1}$ if the boundary $\partial\Omega$ can be parametrized with $\mathcal{C}^{1,1}$ functions (a precise definition is given in [?, p.37]).

Now, we introduce the standard spaces in the study of the equations of fluid mechanics:

$$L^2_\sigma(\Omega) = \left\{ v \in L^2(\Omega) \; ; \; \operatorname{div}(v) = 0, \quad v \cdot n = 0 \quad \text{on } \partial\Omega \right\}, \tag{5.26}$$

and

$$H^1_\sigma(\Omega) = L^2_\sigma(\Omega) \cap H^1(\Omega). \tag{5.27}$$

We define the space of rigid velocities:

$$\mathcal{R} = \{ x \mapsto \ell + \omega \times x \; ; \; \ell, \omega \in \mathbb{R}^3 \}$$

and we introduce the following space of test functions due to the presence of the rigid body:

$$\begin{aligned} \mathcal{T}_T(\mathcal{S}(t)) := \{ & v \in \mathcal{C}([0,T); L^2_\sigma(\Omega))^3; \exists v_s \in \mathcal{C}^1([0,T]; \mathcal{R})^3, \; \exists v_f \in \mathcal{C}^1([0,T]; W^{1,5/2}(\Omega))^3 \\ & \text{such that } v_s \cdot \widehat{n} = v_f \cdot \widehat{n} \text{ on } \partial\mathcal{S}(t), \; v \cdot \widehat{n} = 0 \text{ on } \partial\Omega \text{ and } v = v_f \text{ in } \Omega \setminus \overline{\mathcal{S}(t)}, \\ & v = v_s \text{ in } \mathcal{S}(t) \text{ a.e. in } (0,T) \}. \end{aligned} \tag{5.28}$$

We recall (see, for instance, [?, Lemma 1.1, p.18]) that

$$D(v) = 0 \quad \text{in } \mathcal{S} \iff \exists v_R \in \mathcal{R} : v_{|\mathcal{S}} = v_{R|\mathcal{S}} \in \mathcal{R}.$$

In order to write a variational formulation of the problem, it is convenient to define a global velocity field by

$$u(t,x) = \begin{cases} u_f(t,x) & \text{if } x \in \mathcal{F}(t), \\ \ell(t) + \omega(t) \times (x - h(t)) & \text{if } x \in \mathcal{S}(t), \end{cases}$$

and similarly for the initial condition,

$$u_0(x) = \begin{cases} u_{0,f}(x) & \text{if } x \in \mathcal{F}_0, \\ \ell_0 + \omega_0 \times x & \text{if } x \in \mathcal{S}_0. \end{cases}$$

In particular, $D(u) = 0$ in $\mathcal{S}(t)$ and $D(u_0) = 0$ in $\mathcal{S}_0$.

We also define: the global temperature

$$\theta(t,x) := \begin{cases} \theta_f(t,x) & \text{if } x \in \mathcal{F}(t), \\ \theta_s(t) & \text{if } x \in \mathcal{S}(t); \end{cases}$$

the global initial temperature of the mixture

$$\theta_0 := \begin{cases} \theta_{0,f} & \text{if } x \in \mathcal{F}_0, \\ \theta_{0,s} & \text{if } x \in \mathcal{S}_0; \end{cases}$$

and the global specific heat capacity

$$c(t,x) := \begin{cases} c_f & \text{if } x \in \mathcal{F}(t), \\ c_s & \text{if } x \in \mathcal{S}(t). \end{cases}$$

To define the weak solution of our system we need the following proposition:

Proposition 5.1.1. *Assume that $(u, \theta, p, \ell, \omega, h, R)$ is a regular function satisfying* (5.4)-(5.24). *Then, the following identity holds:*

$$\begin{aligned} -\int_\Omega \rho u \cdot \left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\, dx - \int_{\mathcal{F}(t)} p \operatorname{div}(v)\, dx + \int_{\mathcal{F}(t)} 2\mu(\theta) D(u) : D(v)\, dx \\ + \nu_\Omega \int_{\partial\Omega} u\cdot v\, dx + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}(t)} ((u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}) \cdot ((v_{|\mathcal{F}(t)} - v_{|\mathcal{S}(t)}) \times \widehat{n})\, dx \\ = \int_\Omega \rho b \cdot v\, dx - \frac{d}{dt}\int_\Omega \rho u \cdot v\, dx, \end{aligned} \tag{5.29}$$

in $(0,T)$, for any $v \in \mathcal{T}_T(\mathcal{S}(t))$, and

$$\begin{aligned} -\int_\Omega \rho\left(\frac{|u|^2}{2} + c\theta\right)\left(\frac{\partial\varphi}{\partial t} + (u\cdot\nabla)\varphi\right)\, dx + \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\varphi dx \\ + \int_{\mathcal{F}(t)} (2\mu(\theta)D(u) - pI_3)\, u\cdot\nabla\varphi\, dx + \nu_\Omega \int_{\partial\Omega} |u|^2 \varphi\, dx \\ + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}(t)} \left|(u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}\right|^2 \varphi\, dx \\ = \int_\Omega \rho(b\cdot u + w)\varphi\, dx - \frac{d}{dt}\int_\Omega \rho\left(\frac{|u|^2}{2} + c\theta\right)\varphi\, dx \end{aligned} \tag{5.30}$$

in $(0,T)$, for any $\varphi \in \mathcal{C}^1([0,T]; H^1(\Omega))$ with $\nabla\varphi = 0$ in $\mathcal{S}(t)$ in $(0,T)$.

The proof of Proposition 5.1.1 will be done in Section 5.2.

Proposition 5.1.1 allows us to introduce the notion of weak solution of the system (5.4)-(5.24):

Definition 5.1.2 (Weak Solution)**.** *Given $T > 0$, a weak solution of the system of equations* (5.4)-(5.24) *in $(0,T)$ is a tuple $(u, \theta, \ell, \omega, p, h, R)$ with the following properties:*

- *$(h, R) \in W^{1,\infty}(0,T; \mathbb{R}^3 \times)$, $(\ell, \omega) \in L^\infty(0,T; \mathbb{R}^3 \times \mathbb{R}^3)$ and satisfy* (5.9), (5.10) *and* (5.24).
- *$u \in L^\infty(0,T; L^2_\sigma(\Omega))^3$, $u(t,x) = (\ell(t) + \omega(t) \times (x - h(t)))$ for $x \in \mathcal{S}(t)$ and $u \in L^2(0,T; H^1(\mathcal{F}(t)))$.*
- *$p \in L^{5/3}(0,T; L^{5/3}(\mathcal{F}(t)))$.*

- *$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^r((0,T)\times\Omega) \cap L^s(0,T;W^{1,s}(\Omega))$ for all $r \in [1,5/3)$, $s \in [1,5/4)$ and $\nabla\theta(t,x) = 0$ for $x \in \mathcal{S}(t)$ a.e. in $(0,T)$.*

- *For all $v \in \mathcal{T}_T(\mathcal{S}(t))$ such that $v(T,\cdot) = 0$, the following equation holds:*

$$-\int_0^T \int_\Omega \rho u \cdot \left(\frac{\partial v}{\partial t} + (u\cdot\nabla)v\right)\, dxdt - \int_0^T \int_{\mathcal{F}(t)} p \,\mathrm{div}\,(v)\, dxdt$$
$$+ \int_0^T \int_{\mathcal{F}(t)} 2\mu(\theta) D(u) : D(v)\, dxdt$$
$$+ \nu_\Omega \int_0^T \int_{\partial\Omega} u\cdot v\, dxdt + \nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}} (u_{|\mathcal{F}(t)} - u_{|\mathcal{S}(t)}) \times \widehat{n} \cdot (v_f - v_s)\times \widehat{n}\, dxdt$$
$$= \int_0^T \int_\Omega \rho b \cdot v\, dxdt + \int_\Omega \rho u_0 \cdot v(0,x)\, dx. \quad (5.31)$$

- *For all $\varphi \in \mathcal{C}^1([0,T];W^{1,\infty}(\Omega))$ such that $\nabla\varphi = 0$ in $\mathcal{S}(t)$ a.e. in $(0,T)$ and $\varphi(T,x) = 0$, the following equation holds:*

$$-\int_0^T \int_\Omega \rho \left(\frac{|u|^2}{2} + c\theta\right)\left(\frac{\partial \varphi}{\partial t} + u\cdot\nabla\varphi\right)\, dxdt + \int_0^T \int_\Omega \kappa(\theta)\nabla\theta\cdot\nabla\varphi\, dxdt$$
$$+ \int_0^T \int_{\mathcal{F}(t)} (pI_3 - 2\mu(\theta)D(u))\, u\cdot\nabla\varphi\, dxdt$$
$$+ \nu_\Omega \int_0^T \int_{\partial\Omega} |u|^2 \varphi\, dxdt + \nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}} \left|(u_{|\mathcal{F}(t)} - u_{|\mathcal{S}(t)}) \times \widehat{n}\right|^2 \varphi\, dxdt$$
$$= \int_0^T \int_\Omega \rho(b\cdot u + w)\varphi\, dxdt + \int_\Omega \rho\left(\frac{|u_0|^2}{2} + c\theta_0\right)\varphi\, dx. \quad (5.32)$$

Remark 5.1. *In the original model, equations* (5.12) *and* (5.13) *are the balance equations of internal energy. However, the weak solution in the sense of Definition 5.1.2 is defined using the total energy balance equation. The problem of finding a weak solution, without the rigid body, considering in the weak formulation the balance of internal energy, is still open [**?**]. The major difficulty lies in the presence of the dissipation term $\mu(\theta)|D(u)|^2$. Since the apriori estimate for the dissipation term is only L^1, then, in a Galerkin approximation scheme, is impossible to pass to the limit this term by just using this type of regularity. This problem does not occur when the material coefficients are constant (we refer the reader to [**?**, Chapter 3.5]) or when there are better a priori estimates on the velocity gradient, for example, in the case of non-Newtonian fluids defined by power laws [**?**] or in two space dimensions [**?**, **?**].*

Now, we state the main result of this chapter:

Theorem 5.1.3. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two $\mathcal{C}^{1,1}$ bounded domains of $\mathbb{R}^3$. Let $\mu,\ \kappa : \mathbb{R} \to \mathbb{R}$ two continuous functions such that Hypothesis 5.1 holds. Given*

$$b \in L^2((0,T)\times\Omega)^3 \text{ and } w \in L^2((0,T)\times\Omega) \text{ such that } w \geqslant 0, \quad (5.33)$$

and the initial conditions:

$$(\ell_0,\omega_0) \in \mathbb{R}^3\times\mathbb{R}^3,\ u_{0,f} \in L^2(\Omega),\ u_{0,s} \in \mathcal{R},\ \theta_{0,f} \in L^1(\Omega) \text{ and } \theta_{0,s} \in \mathbb{R}, \quad (5.34)$$

such that

$$\theta_{0,f} \geqslant \underline{\theta} \text{ and } \theta_{0,s} \geqslant \underline{\theta}, \text{ for } \underline{\theta} > 0, \tag{5.35}$$

there exists $T \in (0, \infty]$ *and a weak solution of the system of equations* (5.4)-(5.24) *in* $(0, T)$ *in the sense of Definition 5.1.2, such that the following estimate holds*

$$\frac{1}{2}\rho \int_{\Omega} |u(t,x)|^2 \, dx + \int_0^T \int_{\mathcal{F}(s)} 2\mu(\theta) |D(u)|^2 \, dxds \leqslant \frac{\rho}{2} \int_{\Omega} |u_0|^2 \, dx + \rho \int_0^T \int_{\Omega} u \cdot b \, dxds, \tag{5.36}$$

for a.e. $t \in (0, T)$, *and*

$$\theta(t,x) \geqslant \underline{\theta}, \text{ a.e. } (t,x) \in [0,T] \times \Omega. \tag{5.37}$$

Moreover, we can choose T *such that one of the following alternatives holds true*

1. $T = +\infty$;
2. $\lim_{t \to T} \operatorname{dist}(\mathcal{S}(t), \partial\Omega) = 0$.

Remark 5.2. *One can write a bidimensional version of system* (5.4)-(5.24) *using the balance of internal energy, and following the proof of the above theorem, it is possible to obtain the same existence result for the corresponding system. Let us mention that even in dimension 2 in space, the uniqueness of weak solutions can be a delicate question. For a non-heat conducting Newtonian fluid alone (without rigid bodies), it is done in [p. 198][?]. However, for the system composed by a rigid body and a fluid governed by the Navier-Stokes system, this issue has been solved only recently (see [?], [?] and also [?] for a weak-strong uniqueness property).*

Remark 5.3. *As explained above our proof of Theorem 4.1.3 is based on a penalization method to deal with the free boundary due to the motion of the rigid body. Such a method is already used in several other articles on fluid-structure interaction systems. There exist a least two different penalization approaches: a "L^2" penalization (see for instance [?, ?]), and a "H^1" penalization (see for instance [?]). In our case is imposible to follow the "H^1" penalization method due to the lack of "H^1" regularity in space of the test functions define in* (5.28). *Then we follow the first method (see 5.82).*

The mathematical study of fluid-structure interaction systems has been the subject of an intensive research since around 2000. A large part of the articles devoted to this study concern the case of rigid bodies moving into a viscous incompressible fluid modeled by the Navier-Stokes system. We can quote for instance [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?], etc. Some works deal with different fluids [?, ?, ?] (incompressible perfect fluid), [?, ?, ?, ?] (viscous compressible fluid), [?] (viscous multipolar fluid), [?, ?] (incompressible non-Newtonian fluid). Let us also mention some results for the Navier-Stokes system but with other types of boundary conditions: [?, ?, ?]).

On the other hand, the mathematical study of a 3D temperature dependent dynamics is strictly related to the study of the pressure (see Remark 5.1). We refer the reader to [?], where the authors prove the existence of a weak solution of a Navier-Stokes-Fourier incompressible fluid considering Navier slip conditions. In their work, they obtain estimates for the pressure using the Helmholtz-Weyl decomposition. The Helmholtz-Weyl decomposition is only compatible with slip boundary conditions. However, in the case of a no-slip boundary condition, a method to obtain the pressure was developed by Wolf [?]. This pressure is called by the author "local" pressure and is the sum of an integrable pressure and of the time derivative of a harmonic function which is not integrable. The question of the integrability of the pressure is not only relevant in the context of heat conducting fluids, but it also a crucial step in turbulence models [?] and incompressible non-newtonian fluids [?, ?, ?, ?]. Regarding the study of the pressure, In the context of fluid-structure interaction problems we refer the reader to [?] where the case of a non-Newtonian fluid with a power law and rigid body interaction is treated. For the boundary conditions, the authors consider a no-slip condition in the outer boundary and at the interface of the solid and the fluid. Then, part of this work is devoted to the study of the "local" pressure where the authors need the pressure estimates to pass to the limit in the nonlinearity associated with the stress tensor. Regarding heat conducting fluids and rigid bodies, up to our knowledge, the case of a heat conducting incompressible Newtonian-rigid body interaction with Navier slip boundary conditions has not been treated yet. However, we refer the reader to [?], where the global existence of weak solutions is proved for the problem of motion of one or several rigid bodies immersed in a non-Newtonian fluid of power-law type with heat conductivity considering the no-slip boundary condition and the Boussinesq approximation. The Boussinesq approximation implies the dismiss of the dissipation term in (5.12) and is useful to model atmospheric fronts, oceanic circulation, katabatic winds, dense gas dispersion, and fume cupboard. In this setting, the energy equation becomes a convection-diffusion equation, and the analysis of the coupled system is more straightforward (see for instance [?]).

Let us describe the outline of this chapter. In Section 5.2, we introduce some additional notation and we prove Proposition 5.1.1. In Section 5.3, we introduce the approximated problem of the weak solution 5.1.2. More precisely, the free-boundary is replaced by a penalization term (with a parameter k), and we replace the convection term $(u \cdot \nabla)$ by a regularization. Section 5.4 we prove the existence of a solution to the approximated problem using a 2-stage Galerkin discretization and introducing the pressure using the Helmholtz-Weyl decomposition. Finally, in Section 5.5, we prove the main result by passing to the limit in passing to the limit in the regularization and the penalization term at the same time.

5.2 Notation and preliminary results

Given $(a, Q) \in \mathbb{R}^3 \times SO(3)$, we set

$$\mathcal{S} = \widehat{\mathcal{S}}(a, Q),$$

and we make the following definitions:

- We denote by $P_{\mathcal{S}}^{\mathcal{R}}$ the orthogonal projection of $L^2(\mathcal{S})^3$ onto $\mathcal{R}$. By standard calculation, if

  $$P_{\mathcal{S}}^{\mathcal{R}} u = \ell + \omega \times (x - a),$$

  then ℓ and ω are given by:

  $$\ell = \frac{1}{|\mathcal{S}|} \int_{\mathcal{S}} u \, dx \tag{5.38}$$

  and

  $$\omega = \widehat{J}(a, Q)^{-1} \int_{\mathcal{S}} \rho (x - a) \times u \, dx. \tag{5.39}$$

- For scalar functions, we denote by $P^1_{\mathcal{S}}$ the orthogonal projection of $L^2(\mathcal{S})$ onto the space of constant functions in $\mathcal{S}$:

$$P^1_{\mathcal{S}}\varphi = \frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} \varphi \, dx. \tag{5.40}$$

- We define two operators of $L^2_{loc}(\mathbb{R}^3)$ as follows: assume $v \in L^2_{loc}(\mathbb{R}^3)$, then

$$\Phi_{a,Q}(v)(y) := Q^* v\,(a + Qy)\,, \quad y \in \mathbb{R}^3 \tag{5.41}$$

  and

$$\overline{\Phi}_{a,Q}(v)(x) := Qv\,(Q^*(x-a))\,, \quad x \in \mathbb{R}^3. \tag{5.42}$$

  Let us notice the relation

$$\overline{\Phi}_{a,Q} \circ P^{\mathcal{R}}_{\mathcal{S}_0} \circ \Phi_{a,Q} = P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)}. \tag{5.43}$$

- We define the global specific heat by

$$\widehat{c}_{a,Q} := c_f \mathbb{1}_{\widehat{\mathcal{F}}(a,Q)} + c_s \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}. \tag{5.44}$$

In what follows, we also need the following definitions:

- For any set $\Omega \subset \mathbb{R}^3$, we define

$$(\Omega)^\delta := \{x \in \mathbb{R}^3 \ ; \ \mathrm{dist}(x, \Omega) < \delta\} \tag{5.45}$$

  and

$$(\Omega)_\delta := \{x \in \Omega \ ; \ \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

- For $\varphi \in L^1(\Omega)$, we define the total energy

$$E(t,\varphi) = \int_\Omega \left(\rho c\theta(t,\cdot) + \frac{\rho}{2}\,|u(t,\cdot)|^2\right)\varphi \, dx,$$

  a.e. in t, and

$$E_0(\varphi) = \int_\Omega \left(\rho c_0\theta_0 + \frac{\rho}{2}\,|u_0|^2\right)\varphi \, dx.$$

Due to the presence of the rigid body inside we will need the following result:

Lemma 5.1. *Assume $(h_n, R_n) \to (h, R)$ in $\mathbb{R}^3 \times SO(3)$. Then,*

$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ in } L^p(\Omega) \quad \forall p \in [1,\infty).$$

Similarly, if $(h_n, R_n) \to (h, R)$ strongly in $\mathcal{C}([0,T]; \mathbb{R}^3 \times SO(3))$, then

$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ strongly in } \mathcal{C}([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty).$$

The proof of this lemma is in Lemma A.4 and is based on the approximation of $\mathbb{1}_{\mathcal{S}_0}$ by a smooth function with compact support.

By the Navier boundary condition (5.19), we will need the following version of Korn's Lemma:

Lemma 5.2. *Let $\Omega \in \mathcal{C}^{1,1}$, $d \geqslant 2$ and $q \in (1,+\infty)$. There exists a positive constant C depending only on Ω and q such that for all $v \in W^{1,q}(\Omega)^d$ with $v\,|_{\partial\Omega} \in L^2(\partial\Omega)^d$ there holds*

$$\|v\|_{W^{1,q}(\Omega)} \leqslant C(\|D(v)\|_{L^q(\Omega)} + \|v\|_{L^2(\partial\Omega)}). \tag{5.46}$$

A proof of this lemma can be found in [**?**, Lemma 1.11].

5.2.1 Proof of Proposition 5.1.1

Now, we proof Proposition 5.1.1

Proof of Proposition 5.1.1. Let $v \in \mathcal{T}_T(\mathcal{S}(t))$ such that $D(v) = 0$ in $\mathcal{S}(t)$. Using the results in Section 5.2, there exist $\mathcal{C}^1$ functions ℓ_v and ω_v such that $v(t,x) = \ell_v(t) + \omega_v(t) \times (x - h(t))$ for $x \in \mathcal{S}(t)$.

We multiply equation (5.4) by v and we integrate in $\mathcal{F}(t)$ for $t \in [0,T]$

$$\int_{\mathcal{F}} \rho_f \left(\frac{\partial u_f}{\partial t} + (u_f \cdot \nabla) u_f \right) \cdot v \, dx = \int_{\mathcal{F}} \operatorname{div}(\boldsymbol{\Sigma}_f) \cdot v \, dx + \int_{\mathcal{F}} b \cdot v \, dx. \tag{5.47}$$

Using integration by parts we obtain

$$\int_{\mathcal{F}} \operatorname{div}(\boldsymbol{\Sigma}_f) \cdot v \, dx = - \int_{\mathcal{F}} \boldsymbol{\Sigma}_f : \nabla v \, dx + \int_{\partial\Omega} \Sigma_f \widehat{n} \cdot v \, dx + \int_{\partial\mathcal{S}} \boldsymbol{\Sigma}_f \widehat{n} \cdot v \, dx. \tag{5.48}$$

Using that $\operatorname{div}(u_f) = 0$, the boundary conditions of v in $\partial\mathcal{S}$ (see (5.7) and (5.8)), (5.16), (5.19), (5.21) and the fact that Σ_f is a symmetric matrix, we deduce from (5.48)

$$\begin{aligned} \int_{\mathcal{F}} \operatorname{div}(\boldsymbol{\Sigma}_f) \cdot v \, dx = - \int_{\mathcal{F}} \boldsymbol{\Sigma}_f : D(v) \, dx - m\ell' \cdot \ell_v - (J\omega)' \cdot \omega_v . + \int_{\mathcal{S}} \rho b \cdot v \, dx \\ - \nu_\Omega \int_{\partial\Omega} u_f \cdot v \, dx - \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} ((u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}) \cdot ((v_{|\mathcal{F}(t)} - v_{|\mathcal{S}(t)}) \times \widehat{n}) \, dx. \end{aligned} \tag{5.49}$$

On the other hand, since

$$\int_{\mathcal{S}} \rho \left(\frac{\partial}{\partial t} u + (u \cdot \nabla) u \right) \cdot v \, dx = m\ell' \cdot \ell_v + (J\omega)' \cdot \omega_v,$$

and the Reynolds transport theorem, we deduce:

$$\begin{aligned} \int_{\mathcal{F}} \rho_f \left(\frac{\partial u_f}{\partial t} + (u_f \cdot \nabla) u_f \right) \cdot v \, dx + m\ell' \cdot \ell_v + (J\omega)' \cdot \omega_v \\ = \frac{d}{dt} \int_\Omega \rho v \cdot u \, dx - \int_\Omega \rho \left(\frac{\partial v}{\partial t} + (u \cdot \nabla) v \right) \cdot u \, dx. \end{aligned}$$

Then, combining the above relation with (5.47) and (5.49), we obtain:

$$\begin{aligned} \frac{d}{dt} \int_\Omega \rho u \cdot v \, dx - \int_\Omega \rho \left(\frac{\partial v}{\partial t} + (u \cdot \nabla) v \right) \cdot u \, dx + \int_{\mathcal{F}} \boldsymbol{\Sigma}_f : D(v) \, dx + \nu_\Omega \int_{\partial\Omega} u \cdot v \, dx \\ + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} (u_f - u_{|\mathcal{S}(t)}) \times \widehat{n} \cdot (v_{|\mathcal{F}(t)} - v_{|\mathcal{S}(t)}) \times \widehat{n} \, dx = \int_\Omega \rho b \cdot v \, dx. \end{aligned} \tag{5.50}$$

Then, using (5.6), we deduce (5.29).

Let $\varphi \in \mathcal{C}^1([0,T]; H^1(\Omega))$ with $\nabla\varphi = 0$ in $\mathcal{S}(t)$ a.e. in $(0,T)$. We denote $\varphi_s(t) := \varphi(t,x)$ in $\mathcal{S}(t)$. We multiply (5.12) by φ and we integrate in $\mathcal{F}(t)$ and in $[0,T]$:

$$\int_{\mathcal{F}} \rho_f c_f \left(\frac{\partial}{\partial t} \theta_f + u_f \cdot \nabla\theta_f \right) \varphi \, dx = \int_{\mathcal{F}} \boldsymbol{\Sigma}_f : D(u_f) \varphi \, dx - \int_{\mathcal{F}} \operatorname{div}(q_f) \varphi \, dx + \int_{\mathcal{F}} \rho_f w \varphi \, dx. \tag{5.51}$$

Using integration by parts we write the heat flux term as:

$$\int_{\mathcal{F}} \operatorname{div}(q_f)\varphi \, dx = \int_{\partial\mathcal{S}} \varphi q_f \cdot \widehat{n} \, dx - \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx.$$

Then, using equation (5.13) we have that:

$$\int_{\mathcal{F}} \operatorname{div}(q_f)\varphi \, dxdt = mc_s\theta_s'\varphi_s + \int_{\mathcal{S}} \varphi_s\rho w \, dx - \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx.$$

Using integration by parts in time we deduce:

$$\int_{\mathcal{F}} \operatorname{div}(q_f)\varphi \, dx = \frac{d}{dt} mc_s\theta_s\varphi_s - mc_s\theta_s\varphi_s' + \int_{\mathcal{S}} \varphi_s\rho w \, dx - \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx.$$

On the other hand, by the Reynolds transport theorem:

$$\int_{\mathcal{F}} \rho_f c_f \left(\frac{\partial}{\partial t}\theta_f + u_f \cdot \nabla\theta_f\right)\varphi \, dx = \frac{d}{dt}\int_{\mathcal{F}} c_f\rho_f\theta_f\varphi \, dx - \int_{\mathcal{F}} \rho_f c_f \left(\frac{\partial}{\partial t}\varphi + u_f \cdot \nabla\varphi\right)\theta_f \, dx.$$

Gathering the above in (5.51), we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{F}} c_f\rho_f\theta_f\varphi \, dx + \frac{d}{dt} mc_s\theta_s\varphi_s - \int_{\mathcal{F}} \rho_f c_f \left(\frac{\partial}{\partial t}\varphi + u_f \cdot \nabla\varphi\right)\theta_f \, dx - mc_s\theta_s\varphi_s' \\ = \int_{\mathcal{F}} \mathbf{\Sigma}_f : D(u_f)\varphi \, dx + \int_{\mathcal{F}} \rho_f w\varphi \, dx + \int_{\mathcal{S}} \rho w\varphi_s \, dx + \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx.\end{aligned}$$

Since $\nabla\varphi = 0$ in $\mathcal{S}$, we conclude:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} c\rho\theta\varphi \, dx - \int_{\Omega} \rho c \left(\frac{\partial}{\partial t}\varphi + u \cdot \nabla\varphi\right)\theta \, dx - \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx \\ = \int_{\mathcal{F}} \mathbf{\Sigma}_f : D(u_f)\varphi \, dx + \int_{\Omega} \rho w\varphi \, dx. \quad (5.52)\end{aligned}$$

To finish the proof of (5.30), we use $v = \varphi u$ in (5.50) and we obtain:

$$\begin{aligned}\int_{\mathcal{F}} \mathbf{\Sigma}_f : D(u)\varphi \, dxdt = -\nu_\Omega \int_{\partial\Omega} |u|^2 \varphi \, dx - \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} \left|(u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}\right|^2 \varphi \, dx \\ - \int_{\mathcal{F}} \mathbf{\Sigma}_f u \cdot \nabla\varphi \, dx + \int_{\Omega} \frac{\rho\varphi}{2}\left(\frac{\partial |u|^2}{\partial t} + u \cdot \nabla |u|^2\right) dx + \int_{\Omega} \rho |u|^2 \left(\frac{\partial\varphi}{\partial t} + u \cdot \nabla\varphi\right) dx \\ + \int_{\Omega} b \cdot u\varphi \, dx - \frac{d}{dt}\int_{\Omega} \rho |u|^2 \varphi \, dx. \quad (5.53)\end{aligned}$$

On the other hand, by the Reynolds transport theorem

$$\int_{\Omega} \frac{\rho\varphi}{2}\left(\frac{\partial |u|^2}{\partial t} + u \cdot \nabla |u|^2\right) dx = \frac{d}{dt}\int_{\Omega} \rho\frac{|u|^2}{2}\varphi \, dx - \int_{\Omega} \rho\frac{|u|^2}{2}\left(\frac{\partial\varphi}{\partial t} + u \cdot \nabla\varphi\right) dx.$$

Then, the dissipation term (5.53) is

$$\int_{\mathcal{F}} \mathbf{\Sigma}_f : D(u)\varphi \, dx = -\nu_\Omega \int_{\partial\Omega} |u|^2 \varphi \, dx - \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} \left|(u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}\right|^2 \varphi \, dx$$
$$- \int_{\mathcal{F}} \mathbf{\Sigma}_f u \cdot \nabla\varphi \, dx + \int_\Omega \rho \frac{|u|^2}{2} \left(\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) \, dx$$
$$+ \int_\Omega b \cdot u \varphi \, dx - \frac{1}{2} \frac{d}{dt} \int_\Omega \rho \, |u|^2 \varphi \, dx. \quad (5.54)$$

Combining (5.54) and (5.52) we deduce

$$\frac{d}{dt} \int_\Omega \left(\rho \frac{|u|^2}{2} + \rho c \theta \right) \varphi \, dx - \int_\Omega \left(\rho \frac{|u|^2}{2} + \rho c \theta \right) \frac{\partial}{\partial t} \varphi + \left(\rho \frac{|u|^2}{2} + \rho c \theta \right) u \cdot \nabla \varphi \, dx$$
$$- \int_{\mathcal{F}} q_f \cdot \nabla\varphi \, dx + \int_{\mathcal{F}} \mathbf{\Sigma}_f u \cdot \nabla\varphi \, dx + \nu_\Omega \int_{\partial\Omega} |u|^2 \varphi \, dx$$
$$+ \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} \left|(u_f - u_{|\mathcal{S}(t)}) \times \widehat{n}\right|^2 \varphi \, dx = \int_\Omega \rho (b \cdot u + w) \varphi \, dx. \quad (5.55)$$

Then, using the constitutive equations (5.14) and (5.6), and the fact that $\nabla\varphi = \nabla\theta = 0$ in $\mathcal{S}$, we deduce (5.30). □

5.3 Approximated Problems

To prove Theorem 5.1.3, we consider some approximations of it depending on 2 parameters:

- k corresponds to a penalization term used to deal with the free boundary problem and,
- δ corresponds to regularization in the convective term.

More precisely:

- Since the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ do not impose that the velocity and the temperature are rigid functions in the solid domain we add in the weak formulation two penalization terms:

  $$k \int_{\mathcal{S}} (u - P_{\mathcal{S}}^{\mathcal{R}}(u)) \cdot (v - P_{\mathcal{S}}^{\mathcal{R}}(v)) dx,$$

  in the momentum equation (5.31) and

  $$k \int_{\mathcal{S}} \nabla\theta \cdot \nabla\varphi \, dx,$$

  in the energy equation (5.32), where $k > 0$. In Section 5.5 we will take $k \to \infty$.

- We define the regularization of a function v in $L^2(\Omega)$:

  $$\overline{v}(x) = \varkappa_\delta * \eta_\delta v = \int_\Omega \eta_\delta(y) \varkappa_\delta(x - y) v(y) \, dy, \quad (5.56)$$

where:

$$\varkappa_\delta = \frac{1}{\delta^3}\varkappa(x/\delta), \tag{5.57}$$

and $\varkappa$ is a C^∞-function such tha thet supp($\varkappa$), is included in a ball of radius 1, $\varkappa(x) = \varkappa(-x) \geqslant 0$ and $\|\varkappa\|_{L^1(\Omega)} = 1$. The function η_δ is the cut-off function

$$\eta_\delta(x) = \begin{cases} 1 & \text{if } x \in \Omega_\delta, \\ 0 & \text{if } x \in \Omega \setminus \Omega_\delta. \end{cases} \tag{5.58}$$

Then, we use $\overline{u}$ in the convective terms in the balance of linear momentum (5.31) and the balance of energy (5.32).

Notation 5.3.1. *To simplify the notation, in this section we write*

$$n = (k, \delta),$$

for instance u_n means $u_{k,\delta}$.

Then the approximated problem is defined as follows: to find:

$$u_n \in L^\infty(0,T; L^2_\sigma(\Omega))^3 \cap L^2(0,T; H^1_\sigma(\Omega))^3, \tag{5.59}$$

$$\theta_n \in L^\infty(0,T;L^1(\Omega)) \cap L^s((0,T)\times\Omega) \cap L^r(0,T;W^{1,r}(\Omega)) \quad \text{for } s \in [1,5/3) \text{ and } r \in [1,5/4), \tag{5.60}$$

$$p_n \in L^2((0,T)\times\Omega), \tag{5.61}$$

$$(h_n, R_n) \in W^{1,\infty}(0,T; \mathbb{R}^3 \times SO(3)) \tag{5.62}$$

and

$$(\ell_n, \omega_n) \in W^{1,\infty}(0,T; \mathbb{R}^3 \times \mathbb{R}^3), \tag{5.63}$$

such that:

$$h_n'(t) = \ell_n(t), \quad h_n(0) = 0, \tag{5.64}$$

$$R_n'(t) = \mathbb{A}(\omega_n) R_n(t), \quad R_n(0) = I_3, \tag{5.65}$$

$$P^{\mathcal{R}}_{\mathcal{S}_n}(u_n) = \ell_n + \omega_n \times (x - h_n), \tag{5.66}$$

where $\mathcal{S}_n = \widehat{\mathcal{S}}(h_n, R_n)$,

$$\frac{\partial}{\partial t}u_n \in L^2(0,T;(H^1(\Omega))^*),$$

$$\rho\left\langle \frac{\partial u_n}{\partial t}, v\right\rangle_{(H^1)^*,H^1} - \rho\int_\Omega (Q_{S_n}(\overline{u_n})\cdot\nabla)v\cdot u_n\,dx + \int_\Omega 2\mu_k(\theta_n)D(u_n):D(v)\,dx$$
$$- \int_\Omega p_n \operatorname{div}(v)\,dx + \nu_\Omega\int_{\partial\Omega} u_n\cdot v\,dx + \nu_{\mathcal{S}}\int_{\partial\mathcal{S}}((u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n))\times\widehat{n})\cdot((v - P^{\mathcal{R}}_{\mathcal{S}_n}(v))\times\widehat{n})\,dx$$
$$+ k\int_{\mathcal{S}_n}(u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n))\cdot(v - P^{\mathcal{R}}_{\mathcal{S}_n}(v))\,dx = \rho\int_\Omega b\cdot v\,dx, \tag{5.67}$$

a.e. in $(0,T)$ for all $v \in H^1(\Omega)$ such that $v \cdot \widehat{n} = 0$ on $\partial\Omega$,

$$\lim_{t\to 0^+} \int_\Omega u(t,x)\cdot v(x)\, dx = \int_\Omega u_0 \cdot v\, dx, \tag{5.68}$$

for all $v \in H^1(\Omega)$,

$$\rho \left\langle \frac{\partial}{\partial t}(c_n\theta_n), \varphi \right\rangle_{W^{-1,p^*},W^{1,p}} - \rho \int_\Omega c_n\theta_n Q_{\mathcal{S}_n}(\overline{u_n})\cdot \nabla\varphi\, dx + \int_\Omega (\kappa(\theta_n) + k\mathbb{1}_{\mathcal{S}_n})\nabla\theta_n \cdot \nabla\varphi\, dx$$
$$- \int_\Omega \left(k\mathbb{1}_{\mathcal{S}_n} \left|u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n\right|^2 + 2\mu_k(\theta_n)\left|D(u_n)\right|^2 \right)\varphi\, dx = \rho \int_\Omega w\varphi\, dx \tag{5.69}$$

for all $p > 4$, where $1/p + 1/p^* = 1$, for all $\varphi \in W^{1,\infty}(\Omega)$, and $c_n = \widehat{c}_{(h_n,R_n)}$, where $\widehat{c}$ is defined in (5.75),

$$\lim_{t\to 0^+} \int_\Omega c(t,x)\theta(t,x)\psi(x)\, dx = \int_\Omega c_0\theta_0\psi\, dx, \tag{5.70}$$

for all $\psi \in \mathcal{C}(\Omega)$ and,

$$\theta_n(t,x) \geqslant \underline{\theta} > 0. \tag{5.71}$$

In the above system:

$$\mu_k(\theta_n) = k^{-2}\mathbb{1}_{\mathcal{S}_n} + \mu(\theta_n)\mathbb{1}_{\mathcal{F}_n}, \tag{5.72}$$

and for $\delta_1 \in (0, \operatorname{dist}(\mathcal{S}_0, \partial\Omega)/2)$ and $\mathcal{S} = \widehat{\mathcal{S}}(a,Q)$ for $a \in \mathbb{R}^3$ and $Q \in$, the operator $Q_{\mathcal{S}}$ is defined as follows:

$$Q_{\mathcal{S}} := Q^{\delta_1, \frac{\delta_1}{k}}_{a,Q},$$

where $Q^{\delta_1,\frac{\delta_1}{k}}_{a,Q}$ is given in Definition A.1.2. It satisfies, in particular, the following relation (see (A.15)):

$$Q_{\mathcal{S}}(u) = \begin{cases} u & \text{in } \Omega \setminus (\mathcal{S})^{\delta_1} \\ P^{\mathcal{R}}_{\mathcal{S}} u & \text{in } \mathcal{S} \end{cases} \quad \text{and} \quad \operatorname{div} Q_{\mathcal{S}}(u) = 0.$$

The next section aims to prove the following lemma:

Lemma 5.3. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two $\mathcal{C}^{1,1}$ bounded domains of $\mathbb{R}^3$. Let $\delta > 0$, $k \in \mathbb{N}$ and $n = (k,\delta)$. Assume that: μ and κ are continuous functions satisfying Hypothesis 5.1; u_0 and θ_0 satisfy (5.34); b and w satisfy (5.33). Then, there exists $(u_n, p_n, \theta_n, h_n, R_n)$ such that (5.59)-(5.71) hold. Moreover, we have that for all $t \in [0,T]$,*

$$\rho\frac{1}{2}\int_\Omega |u_n(t,x)|^2\, dx + \int_0^t \int_\Omega 2\mu_k(\theta_n)\left|D(u_n)\right|^2\, dxdt + \nu_\Omega \int_0^t \int_{\partial\Omega} |u_n|^2\, dxdt$$
$$+ \nu_{\mathcal{S}} \int_0^t \int_{\partial\mathcal{S}_n} \left|(u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)) \times \widehat{n}\right|^2\, dxdt + k\int_0^t \int_{\mathcal{S}_n} \left|u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n\right|^2\, dxdt$$
$$= \rho \int_0^t \int_\Omega b\cdot u_n\, dxdt + \frac{\rho}{2}\int_\Omega |u_0|^2\, dx, \tag{5.73}$$

and the following estimate holds

$$\|\theta_n\|_{L^\infty(0,T;L^1(\Omega))} + \|\theta_n\|_{L^r((0,T)\times\Omega)} + \|\theta_n\|_{L^s(0,T;W^{1,s}(\Omega))}$$
$$+ k\left\|\theta_n^\alpha - P^1_{\mathcal{S}_n}\theta_n^\alpha\right\|_{L^2(0,T;L^2(\mathcal{S}_n))} \leqslant C, \tag{5.74}$$

for $r \in [1, 5/3)$, $s \in [1, 5/4)$ and $\alpha \in (0, 1/2)$, where C is an independent constant.

5.4 Proof of Lemma 5.3

We decompose the proof of Lemma 5.3 in three parts. In the firsts two parts we use a two-stage Galerkin approximation where we introduce the parameters M and N corresponding to the dimension in the Galerkin method for the global velocity and the global temperature respectively. In the Galerkin approximation for the velocity we use the free divergence space $H^1_\sigma(\Omega)^3$, defined in (5.27), then the pressure will not appear in the momentum equation until the last part of the proof, where we introduce the pressure using the Helmholtz-Weyl decomposition and the Riesz representation theorem.

To introduce the Galerkin approximation we notice that, since $H^1_\sigma(\Omega)^3$ is a separable Hilbert space and $(\mathcal{C}^\infty(\Omega))^3 \cap H^1_\sigma(\Omega)^3$ is dense in $H^1_\sigma(\Omega)^3$, there exists an orthonormal basis $\{v_q\}_{q\in\mathbb{N}^*}$ of $H^1_\sigma(\Omega)^3$ such that $v_q \in \mathcal{C}^\infty(\Omega)^3$ for all $q \geq 1$. We define

$$V_M = \operatorname{span}\{v_1, \dots, v_M\}$$

and we look for an approximated velocity in V_M. For the energy equation, we consider a basis $\{w_q\}_{q\in\mathbb{N}^*}$ of $H^1(\Omega)$ such that $w_q \in \mathcal{C}^\infty(\Omega)$ for all $q \geq 1$ and we look for an approximated temperature in

$$W_N = \operatorname{span}\{w_1, \dots, w_N\}.$$

Considering the above, the Galerkin approximation is defined as follows: to find

$$h_{M,N} \in \mathcal{C}^1([0,T];\mathbb{R}^3),\ R_{M,N} \in \mathcal{C}^1([0,T];),\ \alpha_{M,N} \in \mathcal{C}^1([0,T];\mathbb{R}^M) \\ \text{and } \beta_{M,N} \in \mathcal{C}^1([0,T];\mathbb{R}^N), \quad (5.75)$$

satisfying the following properties:

$$\mathcal{S}_{M,N}(t) = \widehat{\mathcal{S}}(h_{M,N}(t), R_{M,N}(t)),\ \text{and } \mathcal{F}_{M,N}(t) = \widehat{\mathcal{F}}(h_{M,N}(t), R_{M,N}(t)), \quad (5.76)$$

where $\widehat{S}$ and $\widehat{\mathcal{F}}$ are defined in (5.1) and (5.2),

$$u_{M,N} := \sum_{j=1}^{M} \alpha_{M,N,j} v_j, \quad (5.77)$$

$$\theta_{M,N} := \sum_{j=1}^{N} \beta_{M,N,j} w_j, \quad (5.78)$$

$$P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N}) := \ell_{M,N} + \omega_{M,N} \times (x - h_{M,N}), \quad (5.79)$$

$$h'_{M,N}(t) = \ell_{M,N}(t), \quad h_{M,N}(0) = 0, \quad (5.80)$$

$$R'_{M,N}(t) = \mathbb{A}(\omega_n)\, R_{M,N}(t), \quad R_{M,N}(0) = I_3, \quad (5.81)$$

$$\begin{aligned}\rho \int_{\Omega} \frac{\partial u_{M,N}}{\partial t} \cdot v_i \, dx + \rho \int_{\Omega} \left(Q_{\mathcal{S}_{M,N}}(\overline{u_{M,N}}) \cdot \nabla\right) u_{M,N} \cdot v_i \, dx \\ + \int_{\Omega} 2\mu_k(\theta_{M,N}) D(u_{M,N}) : D(v_i) \, dx + \nu_{\Omega} \int_{\partial\Omega} u_{M,N} \cdot v_i \, dx \\ + \nu_{\mathcal{S}} \int_{\partial \mathcal{S}_{M,N}} ((u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})) \times \widehat{n}) \cdot ((v_i - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(v_i)) \times \widehat{n}) \, dx \\ + k \int_{\mathcal{S}_{M,N}} (u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})) \cdot (v_i - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(v_i)) \, dx = \rho \int_{\Omega} b \cdot v_i \, dx, \quad (5.82)\end{aligned}$$

for all $i \in \{1, \ldots, M\}$,

$$u_{M,N}(0, \cdot) = \mathbb{P}_{V_M}(u_0) := \sum_{i=1}^{M} \alpha_{M,N,0,i} v_i, \quad (5.83)$$

where the operator $\mathbb{P}_{V_M}$ is the L^2 orthogonal projection of $(L^2(\Omega))^3$ onto V_M,

$$\begin{aligned}\rho \int_{\Omega} c_{M.N} \frac{\partial \theta_{M,N}}{\partial t} w_i \, dx + \rho \int_{\Omega} c_{M,N} Q_{\mathcal{S}_{M,N}}(\overline{u_{M,N}}) \cdot \nabla \theta_{M,N} w_i \, dx \\ - \int_{\Omega} \left(k \mathbb{1}_{S_{M,N}} \left| u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N}) \right|^2 + 2\mu_k(\theta_{M,N}) \left| D(u_{M,N}) \right|^2 \right) w_i \, dx \\ + \int_{\Omega} \left(\kappa(\theta_{M,N}) + k \mathbb{1}_{\mathcal{S}_{M,N}} \right) \nabla \theta_{M,N} \cdot \nabla w_i \, dx = \rho \int_{\Omega} w w_i \, dx, \quad (5.84)\end{aligned}$$

for all $i \in \{1, \cdots, N\}$,

$$\theta_{M,N}(0, x) = \mathbb{P}_{W_N} \left(\theta_0 * \varkappa_{1/M} \right), \quad (5.85)$$

where: θ_0 is extended by $\underline{\theta}$ in $\mathbb{R}^3 \setminus \Omega$, the operator $\mathbb{P}_{W_M}$ is the L^2 orthogonal projection of $L^2(\Omega)$ onto V_M, and the global specific heat is defined by $c_{M,N} = \widehat{c}_{h_{M,N}, R_{M,N}}$ where $\widehat{c}$ is defined in (5.75).

In the following proposition, we prove the existence of a solution of the Galerkin approximation.

Proposition 5.4.1. *Assume $\delta_1 > 0$, $T > 0$ and $(M, N) \in (\mathbb{N} \setminus \{0\})^2$. Then, there exists a solution $(h_{M,N}, R_{M,N}, \alpha_{M,N}, \beta_{M,N})$ of the system (5.75)-(5.85). Moreover, for all $t \in [0, T]$,*

$$\begin{aligned}\frac{\rho}{2} \int_{\Omega} |u_{M,N}(t, x)|^2 \, dx + \int_0^t \int_{\Omega} 2\mu_k(\theta_{M,N}) \left| D(u_{M,N}) \right|^2 \, dxdt + \nu_{\Omega} \int_0^t \int_{\partial\Omega} |u_{M,N}|^2 \, dxdt \\ + \nu_{\mathcal{S}} \int_0^t \int_{\partial \mathcal{S}_{M,N}} \left| (u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})) \times \widehat{n} \right|^2 \, dxdt \\ + k \int_0^t \int_{\mathcal{S}_{M,N}} \left| u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N}) \right|^2 \, dxdt = \frac{\rho}{2} \int_{\Omega} |u_{M,N}(0, x)|^2 \, dx + \rho \int_0^t \int_{\Omega} b \cdot u_{M,N} \, dxdt, \\ (5.86)\end{aligned}$$

and

$$\frac{\rho}{2}\int_\Omega c_{M,N}(t,x)\left|\theta_{M,N}(t,x)\right|^2\,dx + \int_0^t\int_\Omega (\kappa(\theta_{M,N}) + k\mathbb{1}_{\mathcal{S}_{M,N}})\left|\nabla\theta_{M,N}\right|^2\,dxdt$$
$$-\int_0^t\int_\Omega\left(k\mathbb{1}_{S_{M,N}}\left|u_{M,N} - P^{\mathcal{R}}_{S_{M,N}}(u_{M,N})\right|^2 + 2\mu_k(\theta_{M,N})\left|D(u_{M,N})\right|^2\right)\theta_{M,N}\,dxdt$$
$$= \frac{\rho}{2}\int_\Omega c_0\left|\theta_{M,N}(0,x)\right|^2\,dx + \rho\int_0^T\int_\Omega w\theta_{M,N}\,dxdt. \quad (5.87)$$

Proof. We write (5.76)-(5.83) as a Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} h_{M,N} \\ R_{M,N} \\ \alpha_{M,N} \\ \beta_{M,N}\end{pmatrix} = F\left(\begin{pmatrix} h_{M,N} \\ R_{M,N} \\ \alpha_{M,N} \\ \beta_{M,N}\end{pmatrix}\right), \quad \begin{pmatrix} h_{M,N} \\ R_{M,N} \\ \alpha_{M,N} \\ \beta_{M,N}\end{pmatrix}(0) = \begin{pmatrix} 0 \\ I_3 \\ \alpha_{M,N,0} \\ \beta_{M,N,0}\end{pmatrix}, \quad (5.88)$$

where $F = (F_1, F_2, F_3, F_4)$ depends on n and can be expressed by using (5.76)-(5.83), (5.38)-(5.39) and (5.11):

$$F_1(a,Q,\beta) = \frac{\rho}{m}\sum_{i=1}^M \beta_i\int_\Omega \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}v_i\,dx,$$

$$F_2(a,Q,\beta) = \rho\sum_{i=1}^M\beta_i\mathbb{A}\left(QJ_0^{-1}Q^*\int_\Omega\mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}(x-a)\times v_i(x)\,dx\right)Q,$$

$$F_3(a,Q,\beta) = G(a,Q,\beta,\gamma),$$

and

$$F_4(a,Q,\beta) = H(a,Q)^{-1}K(a,Q,\beta,\gamma),$$

where

$$G(a,Q,\beta,\gamma)_j = -\sum_{i=1}^M\beta_i\int_\Omega 2\mu_k\left(\sum_{l=1}^N\gamma_l w_l\right)D(v_i):D(v_j)\,dx$$
$$-\rho\sum_{i=1}^M\beta_i\int_\Omega\left(Q^{\delta_1,\delta_1/k}_{a,Q}\left(\sum_{l=1}^M\beta_l\varkappa_\delta * v_l\right)\cdot\nabla\right)v_i\cdot v_j\,dx + \nu\sum_{i=1}^M\beta_i\int_{\partial\Omega}v_i\cdot v_j\,dx$$
$$-k\sum_{i=1}^M\beta_i\int_\Omega\mathbb{1}_{\widehat{\mathcal{S}}(a,Q)}\big(v_i - P_{\widehat{\mathcal{S}}(a,Q)}v_i\big)\cdot\big(v_j - P_{\widehat{\mathcal{S}}(a,Q)}v_j\big)\,dx + \rho\int_\Omega b\cdot v_j\,dx,$$

for all $j\in\{1,\dots,M\}$,

$$H(a,Q)_{i,j} = \int_\Omega\widehat{c}_{a,Q}w_iw_j\,dx,$$

for all $i, j \in \{1, \ldots, N\}$ and

$$K(a,Q,\beta,\gamma)_j = -\rho \sum_{l=1}^{N} \gamma_l \int_\Omega \widehat{c}_{a,Q} w_j Q_{a,Q}^{\delta_1,\delta_1/k} \left(\sum_{l=1}^{M} \beta_l \varkappa_\delta * v_l \right) \cdot \nabla w_l \, dx$$
$$+ k \int_\Omega \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} \left| \sum_{i=1}^{M} \beta_i (v_i - P_{\widehat{\mathcal{S}}(a,Q)} v_i) \right|^2 w_j \, dx + \int_\Omega 2\mu_k \left(\sum_{i=1}^{N} \gamma_i w_i \right) \left| \sum_{i=1}^{M} \beta_i D(v_i) \right|^2 w_j \, dx$$
$$- \sum_{i=1}^{N} \gamma_i \int_\Omega \left(\kappa \left(\sum_{l=1}^{N} \gamma_l w_l \right) + \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} \right) \nabla w_i \cdot \nabla w_j \, dx - \rho \int_\Omega w w_j \, dx,$$

for all $j \in \{1, \cdots, N\}$.

By Lemma 5.1 and (5.1) we have that

$$\mathbb{R}^3 \times SO(3) \to L^1(\mathbb{R}^3), \quad (a,Q) \mapsto \mathbb{1}_{\widehat{\mathcal{S}}(a,Q)} \tag{5.89}$$

is continuous and thus F_1, F_2 and H are continuous functions. For the continuity of G and K, we gather the following arguments:

- Using (5.11), (5.38) and (5.39) we have that

  $$(a,Q) \mapsto P_{\widehat{\mathcal{S}}(a,Q)} v_i \in \mathcal{R},$$

  is continuous.

- Using the definition (A.14) and the continuity of

  $$(a,Q) \in \mathbb{R}^3 \times SO(3) \mapsto \Phi_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)), \quad (a,Q) \in \mathbb{R}^3 \times SO(3) \mapsto \overline{\Phi}_{a,Q} \in \mathcal{L}(H^1(\mathbb{R}^3)),$$

  we deduce

  $$(a,Q,\beta) \mapsto \int_\Omega \left(Q_{a,Q}^{\delta_1,\delta_1/k} \left(\sum_{l=1}^{M} \beta_l \phi_\delta * v_l \right) \cdot \nabla \right) v_j \cdot v_i \, dx,$$

  and

  $$(a,Q,\beta) \mapsto \int_\Omega \widehat{c}_{a,Q} w_j Q_{a,Q}^{\delta_1,\delta_1/k} \left(\sum_{l=1}^{M} \beta_l \phi_\delta *, v_l \right) \cdot \nabla w_l \, dx$$

  are continuous.

- The rest of the terms are continuous by the continuity of μ, κ and (5.89).

Consequently, in (5.88), we have that F is continuous and we can apply the Peano result and deduce the existence of a local $\mathcal{C}^1$ solution. Then, we derive (5.86) in a standard way and deduce the existence of global solutions of the system (5.75)-(5.83). □

The aim of following sections is, to pass to the limit the parameters M and N. In the first step, we take $N \to \infty$ and then we take $M \to \infty$.

Remark 5.4. *We take $N \to \infty$ first since we are not able to deduce from* (5.87)*, convergence properties of $\theta_{M,N}$ without considering a higher regularity of $u_{M,N}$.*

5.4.1 Passing to the limit in N

Using Korn's Lemma (5.46), equation (5.86) and

$$\frac{1}{2}\int_\Omega \rho_0 \left|\mathbb{P}_{V_M} u_0\right|_2^2 \, dx \leq C\|u_0\|^2_{L^2(\Omega)}, \tag{5.90}$$

we deduce

$$\sup_{t\in(0,T)} \frac{\rho}{2}\int_\Omega |u_{M,N}(t,x)|^2\,dx + C_k\,\|u_{M,N}\|^2_{L^2(0,T;H^1(\Omega))} + k\left\|u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})\right\|^2_{L^2(0,T;L^2(\mathcal{S}_{M,N}))} \leqslant C. \tag{5.91}$$

Then, we conclude that

$$\{u_{M,N}\}_{M,N} \quad \text{is bounded in} \quad L^\infty(0,T;L^2_\sigma(\Omega))^3 \cap L^2(0,T;H^1_\sigma(\Omega))^3. \tag{5.92}$$

where $L^2_\sigma(\Omega)$ and $H^1_\sigma(\Omega)$ are defined in (5.26) and (5.27). Since we first pass to the limit in N, we deduce that exists a subsequence of $\{u_{M,N}\}_{M,N}$ (still denoted $\{u_{M,N}\}_{M,N}$), and a function

$$u_M \in L^\infty(0,T;L^2_\sigma(\Omega))^3 \cap L^2(0,T;H^1_\sigma(\Omega))^3$$

such that:

$$u_{M,N} \overset{*}{\rightharpoonup} u_M \text{ weak star in } L^\infty(0,T;L^2(\Omega))^3, \tag{5.93}$$

and

$$u_{M,N} \rightharpoonup u_M \text{ weakly in } L^2(0,T;H^1(\Omega))^3. \tag{5.94}$$

We also deduce from (5.80), (5.81) and (5.92):

$$(h_{M,N}, R_{M,N}) \overset{*}{\rightharpoonup} (h_M, R_M) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)) \tag{5.95}$$

and

$$(h_{M,N}, R_{M,N}) \to (h_M, R_M) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)). \tag{5.96}$$

We write

$$\mathcal{S}_M := \widehat{\mathcal{S}}(h_M, R_M)$$

and

$$J_{\mathcal{S}_M} := \widehat{J}(h_M, R_M).$$

By Lemma 5.1 we have that

$$\mathbb{1}_{\mathcal{S}_{M,N}} \to \mathbb{1}_{\mathcal{S}_M} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{5.97}$$

and thus:

$$c_{M,N} \to c_M := \widehat{c}_{h_M,R_M} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{5.98}$$

Using (5.38) and (5.39), we deduce that

$$P^{\mathcal{R}}_{\mathcal{S}_{M,N}} u_{M,N} \overset{*}{\rightharpoonup} P^{\mathcal{R}}_{\mathcal{S}_M} u_M \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{5.99}$$

As usual in the Navier Stokes equations, we need a result of the strong convergence of the velocity to deal with the convective term. Since

$$\alpha'_{M,N,i}(t) = -\rho\int_\Omega \left(Q_{\mathcal{S}_{N,M}}(\overline{u_{N,M}})\cdot\nabla\right)u_{N,M}\cdot v_i\;dx - \int_\Omega 2\mu_k(\theta_{M,N})D(u_{N,M}):D(v_i)\;dx$$
$$-\nu_\Omega\int_{\partial\Omega} u_{M,N}\cdot v_i\;dx - \nu_{\mathcal{S}}\int_{\partial\mathcal{S}}((u_{M,N}-P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N}))\times\widehat{n})\cdot((v_i-P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(v_i))\times\widehat{n})dx$$
$$-k\int_{\mathcal{S}_{N,M}}(u_{N,M}-P^{\mathcal{R}}_{\mathcal{S}_{N,M}}(u_{N,M}))\cdot(v_i-P^{\mathcal{R}}_{\mathcal{S}_{N,M}}(v_i))dx+\rho\int_\Omega b\cdot v_i\;dx \quad (5.100)$$

we deduce that:

$$\int_0^T \left|\alpha'_{N,M,i}(t)\right|^2 dt \leqslant c_{k,M}.$$

Then, since the injection $H^1(0,T)\subset\mathcal{C}([0,T])$ is compact,

$$u_{M,N}\to u_M \text{ strongly in } \mathcal{C}((0,T)\times\Omega)^3 \text{ and weakly in } H^1(0,T;H^1_\sigma(\Omega))^3. \quad (5.101)$$

For the temperature, by equation (5.87) we deduce

$$\frac{\rho}{2}\int_\Omega c_{M,N}(t,x)\left|\theta_{M,N}(t,x)\right|^2\;dx + C_k\int_0^t\int_\Omega\left|\nabla\theta_{M,N}\right|^2dxdt \leqslant \frac{1}{2}\int_\Omega \rho_0c_0\left|\theta_{M,N}(0,x)\right|^2\;dx$$
$$+C_k\left\|u_{M,N}\right\|_{L^2((0,T)\times\Omega)} + C_2\left(\int_0^t\left\|D(u_{M,N})\right\|^2_{L^4(\Omega)}ds\right)^{1/2}\int_0^t\int_\Omega\left|\theta_{M,N}\right|^2\;dxdt$$
$$+C\left\|w\right\|^2_{L^2((0,T)\times\Omega)}. \quad (5.102)$$

Using Grönwall's inequality and (5.91) we conclude:

$$\sup_{t\in(0,T)}\frac{1}{2}\int_\Omega\left|\theta_{M,N}\right|^2\;dx + \int_0^T\int_\Omega\left|\nabla\theta_{M,N}\right|^2\;dxdt \leqslant C_{k,M}. \quad (5.103)$$

Then, we deduce the existence of $\theta_M\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T,H^1(\Omega))$ such that

$$\theta_{M,N}\rightharpoonup\theta_M \text{ weakly in } L^2(0,T;H^1(\Omega)) \quad (5.104)$$

and

$$\theta_{M,N}\overset{*}{\rightharpoonup}\theta_M \text{ weakly star in } L^\infty(0,T;L^2(\Omega)). \quad (5.105)$$

By (5.98) and (5.105) we have that

$$c_{M,N}\theta_{M,N}\overset{*}{\rightharpoonup}c_M\theta_M \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \quad (5.106)$$

We notice that to pass to the limits the terms related to μ_k and κ, we need the strong convergence of the temperature. To obtain it, let us fix $i\geq 1$ and take $N\geq i$. We recall that $\mathbb{P}_{W_i}: L^2(\Omega)\to W_i$ is the orthogonal projection onto W_i and we write (5.84) as follows:

$$\rho\frac{\partial}{\partial t}\mathbb{P}_{W_i}(c_{M,N}\theta_{M,N})+\mathbb{P}_{W_i}A_{M,N}=0,$$

in $(\mathcal{C}_0^\infty((0,T);H^1(\Omega)))^*$, where $A_{M,N}$ is defined by

$$\langle A_{M,N},\varphi\rangle := -\rho\int_0^T\int_\Omega c_{M,N}\theta_{M,N}Q_{\mathcal{S}_{M,N}}(\overline{u_{M,N}})\cdot\nabla\varphi\,dxdt$$
$$+\int_0^T\int_\Omega(\kappa(\theta_{M,N})+k\mathbb{1}_{\mathcal{S}_{M,N}})\nabla\theta_{M,N}\cdot\nabla\varphi\,dxdt-k\int_0^T\int_\Omega\mathbb{1}_{\mathcal{S}_{M,N}}\left|u_{M,N}-P^{\mathcal{R}}_{\mathcal{S}_{M,N}(u_{M,N})}\right|^2\varphi\,dxdt$$
$$-\int_0^T\int_\Omega 2\mu_k(\theta_{M,N})\left|D(u_{M,N})\right|^2\varphi\,dxdt-\rho\int_0^T\int_\Omega w\varphi\,dxdt \quad (5.107)$$

for all $\varphi\in\mathcal{C}_0^\infty([0,T];H^1(\Omega))$. The next step is to prove that $\{A_{M,N}\}_M$ is bounded in a suitable space.

- By the fact that $P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})$ is the orthogonal projection and using Hölder's inequality we deduce

$$\int_0^T\int_\Omega\mathbb{1}_{\mathcal{S}_{M,N}}\left|u_{M,N}-P^{\mathcal{R}}_{\mathcal{S}_{M,N}(u_{M,N})}\right|^2\varphi\,dxdt\leqslant C\left\|u_{M,N}\right\|^2_{L^4(0,T;H^1(\Omega))}\left\|\varphi\right\|_{L^2((0,T)\times\Omega)}. \quad (5.108)$$

- Since ε is small we have that

$$\int_0^T\int_\Omega 2\mu_k(\theta_{M,N})\left|D(u_{M,N})\right|^2\varphi\,dxdt\leqslant C\int_0^T\int_\Omega\left|D(u_{M,N})\right|^2\varphi\,dxdt.$$

  Then, by Hölder's inequality, we deduce

$$\int_0^T\int_\Omega 2\mu_k(\theta_{M,N})\left|D(u_{M,N})\right|^2\varphi\,dxdt\leqslant C\left\|u_{M,N}\right\|^2_{L^2(0,T;W^{1,4}(\Omega))}\left\|\varphi\right\|_{L^2((0,T)\times\Omega)}. \quad (5.109)$$

- By (5.1), we deduce:

$$\int_0^T\int_\Omega\left(\kappa(\theta_{M,N})+k\mathbb{1}_{\mathcal{S}_{M,N}}\right)\nabla\theta_M\cdot\nabla\varphi\,dxdt\leqslant C_k\left\|\theta_{M,N}\right\|_{L^2(0,T;H^1(\Omega))}\left\|\varphi\right\|_{L^2(0,T;H^1(\Omega))}$$

- In the convective term, by Hölder's inequality, we have that:

$$\rho\int_0^T\int_\Omega c_{M,N}\theta_{M,N}Q_{\mathcal{S}_{M,N}}(u_{M,N})\cdot\nabla\varphi\,dxdt\leqslant C\int_0^T\left\|\theta_{M,N}\right\|_{L^2(\Omega)}\left\|Q_{\mathcal{S}_{M,N}}(\overline{u_{M,N}})\right\|_{L^\infty(\Omega)}\left\|\varphi\right\|_{H^1(\Omega)}\,dt.$$

  Using the properties of the operator $Q^{\delta_1,\delta_2}_{a,Q}$ (see Definition A.1.2), we deduce:

$$\left\|Q_{\mathcal{S}_{M,N}}(\overline{u_{M,N}})\right\|_{L^\infty((0,T)\times\Omega)}\le C\left(\left\|\overline{u_{M,N}}\right\|_{L^\infty((0,T)\times\Omega)}+\left\|P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(\overline{u_{M,N}})\right\|_{L^\infty(0,T;\mathcal{R})}\right) \quad (5.110)$$

  Since $u_{M,N}\in L^\infty(0,T;L^2(\Omega))$, we deduce

$$\int_0^T\int_\Omega c_{M,N}\theta_{M,N}Q_{\mathcal{S}_{M,N}}(u_{M,N})\cdot\nabla\varphi\,dxdt\leqslant C\left\|\theta_{M,N}\right\|_{L^\infty(0,T;L^2(\Omega))}\left\|\varphi\right\|_{L^2(0,T;H^1(\Omega))}.$$

The above implies:

$$\left\| \mathbb{P}_{W_i}\left(\frac{\partial}{\partial t} c_{M,N}\theta_{M,N}\right)\right\|_{L^2(0,T;(H^1(\Omega))^*)} \leqslant C_{M,k}.$$

Let us denote by $\mathbb{P} : L^2(\Omega) \to H^1(\Omega)$ the orthogonal projection. For any $z \in L^2(\Omega)$,

$$\|z - \mathbb{P}_{W_i}(z)\|_{(H^1(\Omega))^*} \leqslant \|z\|_{L^2(\Omega)} \sup_{\varphi\in H^1(\Omega), \|\varphi\|_{H^1(\Omega)}=1} \|\varphi - \mathbb{P}_{W_i}(\varphi)\|_{L^2(\Omega)} .$$

Using the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$ and that $\{W_q\}$ is an orthonormal basis of $H^1(\Omega)$,

$$\sup_{\varphi\in H^1(\Omega), \|\varphi\|_{H^1(\Omega)}=1} \|\varphi - \mathbb{P}_{W_i}(\varphi)\|_{L^2(\Omega)} \to 0 \text{ as } i \to \infty.$$

Then, we conclude

$$\left\| \frac{\partial}{\partial t} \left(c_{M,N}\theta_{M,N}\right)\right\|_{L^2(0,T;(H^1(\Omega))^*)} \leqslant C_{M,k},$$

and we deduce

$$\frac{\partial}{\partial t} c_{M,N}\theta_{M,N} \rightharpoonup \frac{\partial}{\partial t} c_M\theta_M \text{ weakly in } L^2(0,T;(H^1(\Omega))^*). \tag{5.111}$$

Using (5.106), we apply the Aubin-Lions compactness result [?, Section 8, Corollary 4] and we deduce that

$$c_{M,N}\theta_{M,N} \to c_M\theta_M \text{ strongly } L^2(0,T;(H^1(\Omega))^*). \tag{5.112}$$

Using the above convergence we have that:

$$\int_0^T \int_\Omega c_{M,N} \left|\theta_{M,N}\right|^2 \, dxdt = \int_0^T \left\langle c_{M,N}\theta_{M,N}, \theta_{M,N}\right\rangle_{(H^1)^*,H^1} \, dt$$
$$\to \int_0^T \left\langle c_M\theta_M, \theta_M\right\rangle_{(H^1)^*,H^1} \, dt = \int_0^T \int_\Omega c_M \left|\theta_M\right|^2 \, dxdt.$$

This yields

$$\sqrt{c_{M,N}}\theta_{M,N} \to \sqrt{c_M}\theta_M \text{ strongly in } L^2((0,T)\times\Omega). \tag{5.113}$$

From (5.98), we have

$$\frac{1}{\sqrt{c_{M,N}}} \to \frac{1}{\sqrt{c_M}} \text{ strongly in } \mathcal{C}([0,T];L^3(\Omega)).$$

The above convergence and (5.113) imply

$$\theta_{M,N} \to \theta_M \text{ strongly in } L^2(0,T;L^{\frac{6}{5}}(\Omega)). \tag{5.114}$$

From (5.104) we have that $\{\theta_{M,N}\}$ is bounded in $L^2(0,T;L^6(\Omega))$ and thus

$$\theta_{M,N} \to \theta_M \text{ strongly in } L^2(0,T;L^p(\Omega)) \quad (p<6) \tag{5.115}$$

or

$$\theta_{M,N} \to \theta_M \text{ strongly in } L^{10/3}((0,T)\times\Omega). \tag{5.116}$$

For the initial conditions we have that:

- By the strong convergence of $\alpha_{M,N}$ in $\mathcal{C}([0,T])$ we deduce that

$$u_{M,N}(0,\cdot) \to u_M(0,\cdot) = \mathbb{P}_{V_M}(u_0). \tag{5.117}$$

- By (5.85) the properties of the projection we have that

$$\theta_{M,N}(0,\cdot) \to \theta_{M,0} = \theta_0 * \varkappa_{1/M} \text{ strongly in } L^2(\Omega). \tag{5.118}$$

To pass to the limit the term in $\partial\mathcal{S}_{M,N}$ we notice that, using a change of variables, we obtain

$$\nu_{\mathcal{S}} \int_{\partial\mathcal{S}_{M,N}} ((u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})) \times \widehat{n}) \cdot ((v_i - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(v_i)) \times \widehat{n}) \, dx =$$
$$\sum_{j=1}^{M} \alpha_{M,N} \int_{\partial\mathcal{S}_0} (R_{M,N}(V_j^{M,N} - V_{j,s}^{M,N}) \times \widehat{n}) \cdot (R_{M,N}(V_i^{M,N} - V_{i,s}^{M,N}) \times \widehat{n}) \, dxdt,$$

where

$$V_j^{M,N} = \overline{\Phi}_{h_{M,N},R_{M,N}}(v_j) \text{ and } V_{j,s}^{M,N} = \overline{\Phi}_{h_{M,N},R_{M,N}}(P^{\mathcal{R}}_{\mathcal{S}_{M,N}} v_i).$$

for $i \in \{1,\ldots,M\}$. Since $P_{\mathcal{S}_{M,N}}(v_i) \to P_{\mathcal{S}_M}(v_i)$ strongly in $L^2((0,T)\times\Omega)$, using Lemma A.2 of [?], and the continuity of the trace operator we deduce that

$$V_j^{M,N} \to V_j^M = \overline{\Phi}_{h_M,R_M}(v_j) \text{ strongly in } L^2(0,T;L^2(\partial\mathcal{S}_0))$$

and

$$V_{j,s}^{M,N} \to V_{j,s}^M = \overline{\Phi}_{h_M,R_M}(P_{\mathcal{S}_M} v) \text{ strongly in } L^2(0,T;L^2(\partial\mathcal{S}_0)).$$

Gathering the above convergences, we conclude

$$\nu_{\mathcal{S}} \int_{\partial\mathcal{S}_{M,N}} ((u_{M,N} - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(u_{M,N})) \times \widehat{n}) \cdot ((v_i - P^{\mathcal{R}}_{\mathcal{S}_{M,N}}(v_i)) \times \widehat{n}) \, dx$$
$$\to \nu_{\mathcal{S}} \int_{\partial\mathcal{S}_M} ((u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \times \widehat{n}) \cdot ((v_i - P^{\mathcal{R}}_{\mathcal{S}_M}(v_i)) \times \widehat{n}) \, dx.$$

Then, by the continuity of μ and κ and the convergences (5.97), (5.101), (5.104), (5.111) and (5.116) we deduce:

$$\rho \int_{\Omega} \frac{\partial u_M}{\partial t} \cdot v_i \, dx + \rho \int_{\Omega} (Q_{\mathcal{S}_M}(\overline{u_M}) \cdot \nabla) \, u_M \cdot v_i \, dx + \int_{\Omega} 2\mu_k(\theta_M) D(u_M) : D(v_i) \, dx$$
$$+ \nu_{\Omega} \int_{\partial\Omega} u_M \cdot v_i \, dx + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}_M} ((u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \times \widehat{n}) \cdot ((v_i - P^{\mathcal{R}}_{\mathcal{S}_M}(v_i)) \times \widehat{n}) \, dx$$
$$+ k \int_{\mathcal{S}_M} (u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \cdot (v_i - P^{\mathcal{R}}_{\mathcal{S}_M}(v_i)) \, dx = \rho \int_{\Omega} b \cdot v_i \, dx, \tag{5.119}$$

a.e. in $(0,T)$ for $i \in \{1,\cdots,M\}$, and

$$\rho \left\langle \frac{\partial}{\partial t} c_M \theta_M, \varphi \right\rangle_{(H^1)^*,H^1} - \rho \int_{\Omega} c_M \theta_M Q_{\mathcal{S}_M}(u_M) \cdot \nabla\varphi \, dx + \int_{\Omega} (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M}) \nabla\theta_M \cdot \nabla\varphi \, dx$$
$$- \int_{\Omega} \left(k\mathbb{1}_{\mathcal{S}_M} \left| u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M) \right|^2 + 2\mu_k(\theta_M) \left| D(u_M) \right|^2) \right) \varphi \, dx = \rho \int_{\Omega} w\varphi \, dx, \tag{5.120}$$

for all $\varphi \in L^2(0,T;H^1(\Omega))$.

To prove that

$$\theta_M \geqslant \underline{\theta} \quad \text{a.e. in } (0,T)\times\Omega, \tag{5.121}$$

we use a minimum principle technique. We take

$$\varphi_M = \min\{0, \theta_M - \underline{\theta}\} \leqslant 0,$$

as a test function in temperature equation. Then, we deduce

$$\rho\left\langle \frac{\partial}{\partial t} c_M\theta_M, \varphi_M\right\rangle_{(H^1)^*,H^1} - \rho\int_\Omega c_M\theta_M Q_{\mathcal{S}_M}(u_M)\cdot\nabla\varphi_M\,dx \\ + \int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M})\nabla\theta_M\cdot\nabla\varphi_M\,dx \leqslant \rho\int_\Omega w\varphi_M\,dx, \tag{5.122}$$

a.e. in $(0,T)$. By the definition of φ_M:

$$\nabla\varphi_M = \mathbb{1}_{\theta_M\leqslant\underline{\theta}}\nabla\theta_M,$$

then,

$$\int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M})\nabla\theta_M\cdot\nabla\varphi_M\,dx \geqslant 0. \tag{5.123}$$

The rest of the proof is based on the following equation

$$\left\langle \frac{\partial}{\partial t} c_M\theta_M, \varphi_M\right\rangle_{(H^1)^*,H^1} - \int_\Omega c_M\theta_M Q_{\mathcal{S}_M}(u_M)\cdot\nabla\varphi_M\,dx = \frac{1}{2}\int_\Omega c_M\left|\varphi_M\right|^2\,dx. \tag{5.124}$$

Then, combining (5.124), (5.122) and (5.123) we deduce that

$$\varphi_M = 0,$$

and we conclude (5.121).

The proof (5.124) follows from a regularization procedure. Formally, if we assume that φ_M and θ_M are regular enough in time and using the definition of φ_M several times, we deduce

$$\begin{aligned}\int_\Omega c_M\varphi_M\left(\frac{\partial\theta_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\theta_M\right)\,dx &= \int_{\{\theta_M\leqslant\underline{\theta}\}} c_M\varphi_M\left(\frac{\partial\theta_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\theta_M\right)\,dx \\ &= \int_{\{\theta_M\leqslant\underline{\theta}\}} c_M\varphi_M\left(\frac{\partial\varphi_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\varphi_M\right)\,dx \\ &= \int_\Omega c_M\varphi_M\left(\frac{\partial\varphi_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\varphi_M\right)\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega c_M\left|\varphi_M\right|^2\,dx.\end{aligned}$$

On the other hand, by the Reynolds transport theorem for a fluid-solid system

$$\int_\Omega c_M\varphi_M\left(\frac{\partial\theta_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\theta_M\right)\,dx - \frac{d}{dt}\int_\Omega c_M\theta_M\varphi_M\,dx \\ = -\int_\Omega c_M\theta_M\left(\frac{\partial\varphi_M}{\partial t} + Q_{\mathcal{S}_M}(u_M)\cdot\nabla\varphi_M\right)\,dx.$$

If φ_M is regular enough, the duality bracket $(H^1)^*, H^1$ is

$$\left\langle \frac{\partial}{\partial t} c_M \theta_M, \varphi_M \right\rangle_{(H^1)^*,H^1} = -\int_\Omega c_M \theta_M \frac{\partial \varphi_M}{\partial t}\, dx + \frac{d}{dt}\int_\Omega c_M \theta_M \varphi_M\, dx.$$

Gathering the above, we conclude

$$\int_0^t \left\langle \frac{\partial}{\partial t} c_M \theta_M, \varphi_M \right\rangle_{(H^1)^*,H^1} dt - \int_0^t \int_\Omega c_M \theta_M Q_{\mathcal{S}_M}(u_M) \cdot \nabla \varphi_M\, dxdt$$
$$= \frac{1}{2}\int_\Omega c_M \left|\varphi_M\right|^2 dx, \quad (5.125)$$

for φ_M and θ_M regular enough. Then using a regularizing procedure we can prove in the same manner (5.125) for φ_M and θ_M. Then, gathering (5.122), (5.123) and (5.125), we deduce (5.121).

5.4.2 Uniform estimates in M

To pass to the limit in M, we need to deduce estimates independent of M. Using equation (5.119) and inequality (5.46), we deduce:

$$\sup_{t\in(0,T)} \frac{\rho}{2}\int_\Omega \left|u_M(t,x)\right|^2 dx + C_k \left\|u_M\right\|^2_{L^2(0,T;H^1(\Omega))}$$
$$+ k\left\|u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)\right\|^2_{L^2(0,T;L^2(\mathcal{S}_M))} \leqslant C. \quad (5.126)$$

As we did in Section 5.4.1, we deduce the existence of a subsequence of $\{(u_M, h_M, R_M)\}_M$ (still denoted $\{(u_M, h_M, R_M)\}_M$), and functions

$$u_n \in L^\infty(0,T;L^2_\sigma(\Omega))^3 \cap L^2(0,T;H^1_\sigma(\Omega))^3,$$

where $L^2_\sigma(\Omega)$ and $H^1_\sigma(\Omega)$ are defined in (5.26) and (5.27), and

$$(h_n, R_n) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)),$$

such that:

$$u_M \overset{*}{\rightharpoonup} u_n \text{ weak star in } L^\infty(0,T;L^2(\Omega))^3, \quad (5.127)$$
$$u_M \rightharpoonup u_n \text{ weakly in } L^2(0,T;H^1(\Omega))^3, \quad (5.128)$$
$$(h_M, R_M) \overset{*}{\rightharpoonup} (h_n, R_n) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)), \quad (5.129)$$
$$(h_M, R_M) \to (h_n, R_n) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)), \quad (5.130)$$
$$\mathbb{1}_{S_M} \to \mathbb{1}_{S_n} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty), \quad (5.131)$$
$$c_M \to c_n := \widehat{c}_{h_n,R_n} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty), \quad (5.132)$$

and

$$P^{\mathcal{R}}_{\mathcal{S}_M} u_M \overset{*}{\rightharpoonup} P^{\mathcal{R}}_{\mathcal{S}_n} u_n \text{ weakly star in } L^\infty(0,T,\mathcal{R}), \quad (5.133)$$

where $\mathcal{S}_n := \widehat{\mathcal{S}}(h_n, R_n)$ and $J_{\mathcal{S}_n} := \widehat{J}(h_n, R_n)$.

To obtain the strong convergence of the velocity we fix $i \geq 1$ and take $M \geq i$. We recall that $\mathbb{P}_{V_i} : L^2(\Omega)^3 \to V_i$ is the orthogonal projection onto V_i. We write (5.119) as

$$\rho \frac{\partial}{\partial t} \mathbb{P}_{V_i}(u_M) + \mathbb{P}_{V_i} \mathcal{A}_M = 0,$$

in $(\mathcal{C}_0^\infty([0,T]; H^1(\Omega)))^*$, where $\mathcal{A}_M$ is defined by

$$\begin{aligned} \langle \mathcal{A}_M, v \rangle &:= -\rho \int_0^T \int_\Omega (Q_{\mathcal{S}_M}(u_M) \cdot \nabla) v \cdot u_M \; dxdt + \int_0^T \int_\Omega 2\mu_k(\theta_M) D(u_M) : D(v) \; dxdt \\ &+ \nu_\Omega \int_0^t \int_{\partial\Omega} u_M \cdot v \; dxdt + \nu_{\mathcal{S}} \int_0^t \int_{\partial \mathcal{S}_M} ((u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \times \widehat{n}) \cdot ((v - P^{\mathcal{R}}_{\mathcal{S}_M}(v)) \times \widehat{n}) \; dxdt \\ &+ k \int_0^T \int_{\mathcal{S}_M} (u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \cdot (v - P^{\mathcal{R}}_{\mathcal{S}_M}(v)) \; dxdt - \rho \int_0^T \int_\Omega b \cdot v \; dxdt, \quad (5.134) \end{aligned}$$

for all $v \in L^\infty(0,T; H^1(\Omega))^3$. The next step is to prove that $\{\mathcal{A}_M\}_M$ is bounded in $L^2(0,T; (H^1(\Omega))^*)^3$. Using (5.110) we deduce

$$\left| \int_0^T \int_\Omega (Q_{\mathcal{S}_M}(\overline{u_M}) \cdot \nabla) v \cdot u_M \; dxdt \right| \leqslant C \left\| u_M \right\|_{L^\infty(0,T;L^2(\Omega))} \left\| v \right\|_{L^2(0,T;H^1(\Omega))} .$$

The other terms in (5.134) can be estimated in a standard way, this implies that $\frac{\partial}{\partial t} \mathbb{P}_{V_i}(u_M)$ is bounded in $L^2(0,T; (H^1(\Omega))^*)^3$. As we did for the termperature in the previous section, we can prove that $\frac{\partial}{\partial t} u_M$ is bounded in $L^2(0,T; (H^1(\Omega))^*)^3$ and

$$\frac{\partial}{\partial t} u_M \rightharpoonup \frac{\partial}{\partial t} u_n \text{ weakly in } L^2(0,T; (H^1(\Omega))^*)^3. \quad (5.135)$$

Then, using (5.128), the fact that the injection $H^1(\Omega) \subset L^2(\Omega)$ is compact, we apply the Aubin-Lions compactness result, and we deduce

$$u_M \to u_n \text{ strongly in } L^2((0,T) \times \Omega)^3. \quad (5.136)$$

Then, using (5.128) and Hölder's inequality, we conclude

$$u_M \to u_n \text{ strongly in } L^2(0,T; L^p(\Omega))^3 \quad (p < 6) \quad (5.137)$$

or

$$u_M \to u_n \text{ strongly in } L^{10/3}((0,T) \times \Omega)^3. \quad (5.138)$$

Using Lemma 3.2 in [?], we deduce

$$u_M \to u_n \text{ strongly in } L^{8/3}((0,T) \times \partial\Omega)). \quad (5.139)$$

On the other hand, by (5.38), (5.39) and (5.136) we deduce

$$P^{\mathcal{R}}_{\mathcal{S}_M} u_M \to P^{\mathcal{R}}_{\mathcal{S}_n} u_n \text{ strongly in } L^2((0,T) \times \Omega)^3. \quad (5.140)$$

To obtain boundedness properties for the temperature, first we take $\varphi = \mathbb{1}_{(0,t)}$ in (5.120) and we conclude that

$$\theta_M \text{ is bounded in } L^\infty(0,T; L^1(\Omega)). \quad (5.141)$$

On the other hand, taking $\varphi = \theta_M^\lambda$ in (5.120) for $\lambda \in (-1, 0)$, we have:

$$\int_0^T \int_\Omega \left(k\mathbb{1}_{\mathcal{S}_M} \left| u - P^{\mathcal{R}}_{\mathcal{S}_M} \right|^2 + 2\mu_k(\theta_M) \left| D(u_M) \right|^2 \right) \theta_M^\lambda \, dxdt + \rho \int_0^T \int_\Omega w \theta_M^\lambda \, dxdt$$
$$- \int_0^T \int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M}) \nabla\theta_M \cdot \nabla\theta_M^\lambda \, dxdt$$
$$= \rho \int_0^T \left\langle \frac{\partial c_M \theta_M}{\partial t}, \theta_M^\lambda \right\rangle_{(H^1)^*, H^1} dxdt - \rho \int_0^T \int_\Omega c_M \theta_M Q_{\mathcal{S}_M}(\overline{u_M}) \cdot \nabla\theta_M^\lambda \, dxdt. \quad (5.142)$$

We note that, by a regularizing procedure, the right hand side of the above equation is:

$$\rho \int_0^T \left\langle \frac{\partial c_M \theta_M}{\partial t}, \theta_M^\lambda \right\rangle_{(H^1)^*, H^1} dxdt - \rho \int_0^T \int_\Omega c_M \theta_M Q_{\mathcal{S}_M}(\overline{u_M}) \cdot \nabla\theta_M^\lambda \, dxdt$$
$$= \frac{\rho}{\lambda + 1} \left(\left\| c(T, \cdot) \theta(T, \cdot)^{\lambda+1} \right\|_{L^1(\Omega)} - \left\| c_0 \theta_0^{\lambda+1} \right\|_{L^1(\Omega)} \right).$$

Then, since

$$\int_0^T \int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M}) \nabla\theta_M \cdot \nabla\theta_M^\lambda \, dxdt = \lambda \int_0^T \int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M}) \theta_M^{\lambda-1} \left| \nabla\theta_M \right|_2^2 \, dxdt,$$

$\lambda \in (-1, 0)$, $\theta_M \geqslant \underline{\theta} > 0$ and Hypothesis 5.1, we deduce

$$\int_0^T \int_\Omega (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M}) \left| \nabla(\theta_M^{(\lambda+1)/2}) \right|^2 \, dxdt \leqslant C. \quad (5.143)$$

The above implies that $\theta_M^{(\lambda+1)/2} \in L^2(0, T; L^6(\Omega))$ for all $\lambda \in (-1, 0)$. Then, we deduce that

$$\|\theta_M\|_{L^r((0,T)\times\Omega)} \leqslant c, \text{ for } r \in [1, 5/3), \quad (5.144)$$

and we conclude (5.74). Moreover, since

$$\int_0^T \int_\Omega |\nabla\theta_M|^s \, dxdt = \int_0^T \int_\Omega |\nabla\theta_M|^s \theta_M^{(\lambda-1)s/2} \theta_M^{(1-\lambda)s/2} \, dxdt$$
$$\leqslant \left(\int_0^T \int_\Omega |\nabla\theta_M|^2 \theta_M^{\lambda-1} \, dxdt \right)^{s/2} \left(\int_0^T \int_\Omega \theta_M^{\frac{(1-\lambda)s}{2-s}} \, dxdt \right)^{(2-s)/2},$$

using (5.143) and (5.144), we deduce

$$\|\theta_M\|_{L^s(0,T;W^{1,s}(\Omega))} \leqslant C \text{ for } s \in [1, 5/4). \quad (5.145)$$

Gathering (5.144) and (5.145) we deduce the existence of a function $\theta_n \in L^r((0,T)) \cap L^s(0,T; W^{1,s}(\Omega))$ for $r \in [1, 5/3)$ and $s \in [1, 5/4)$ such that:

$$\theta_M \rightharpoonup \theta_n \text{ weakly in } L^r((0,T) \times \Omega) \text{ for } r \in [1, 5/3), \quad (5.146)$$

and

$$\theta_M \rightharpoonup \theta_n \text{ weakly in } L^s(0,T; W^{1,s}(\Omega)) \text{ for } s \in [1, 5/4). \quad (5.147)$$

As we did in Section 5.4.1, we need the strong convergence of the temperature to pass to the limit in $\mu_k(\theta_M)$ and $\kappa(\theta_M)$. To do this we apply the div-curl Lemma [**?**, p. 343] in the following manner:

- We define the vectors in $\mathbb{R}^4$:

$$a_M = (\rho c_M \theta_M, Q_1, Q_2, Q_3),$$

where

$$Q = \rho c_M \theta_M Q_{\mathcal{S}_M}(\overline{u_M}) + (\kappa(\theta_M) + k\mathbb{1}_{\mathcal{S}_M})\nabla\theta_M,$$

and

$$b_M = (\theta_M^\gamma, 0, 0, 0) \text{ for } \gamma > 0.$$

- Using (5.131), (5.146), (5.147) and (5.110), we deduce

$$a_M \rightharpoonup (\rho c_n \theta_n, \overline{Q_1}, \overline{Q_2}, \overline{Q_3}) \text{ weakly in } L^a((0,T)\times\Omega) \text{ for } a \in [1, 5/4),$$

and

$$b_M \rightharpoonup (\overline{\theta_n^\gamma}, 0, 0, 0) \text{ weakly in } L^b((0,T)\times\Omega) \text{ for } b \in [1, 5/3\gamma),$$

where $\overline{Q_1}$,$\overline{Q_2}$,$\overline{Q_3}$ and $\overline{\theta_M^\gamma}$ are weak limits.

- By the a Sobolev embedding

$$\int_0^T \int_\Omega \left(k\mathbb{1}_{\mathcal{S}_M} \left| u - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M) \right|^2 + 2\mu_k(\theta_M) \left| D(u_M) \right|^2 \right) \varphi \, dxdt$$
$$\leqslant C \left\| u_M \right\|_{L^2(0,T;H^1(\Omega))} \left\| \varphi \right\|_{W^{1,p}((0,T)\times\Omega)},$$

for $p > 4$. Then, using (5.120) we deduce that

$$\operatorname{div}_{t,x} a_M = \rho \frac{\partial}{\partial t}(c_M \theta_M) + \operatorname{div}_x(Q) \in L^1((0,T)\times\Omega).$$

Then, by the Sobolev embedding theorem we deduce that $\operatorname{div}_{t,x} a_M$ is precompact in $(W^{1,p}((0,T)\times\Omega))^*$ for $p^* \in (1, 4/3)$.

- By definition of the curl

$$\operatorname{curl} b_M = \nabla_{t,x} b_M - \nabla_{t,x} b_M^* = \gamma\theta_M^{\gamma-1} \begin{bmatrix} 0 & \frac{\partial\theta_M}{\partial x_1} & \frac{\partial\theta_M}{\partial x_2} & \frac{\partial\theta_M}{\partial x_3} \\ -\frac{\partial\theta_M}{\partial x_1} & 0 & 0 & 0 \\ -\frac{\partial\theta_M}{\partial x_2} & 0 & 0 & 0 \\ -\frac{\partial\theta_M}{\partial x_3} & 0 & 0 & 0 \end{bmatrix}.$$

Using (5.144) and (5.145), we deduce that $\operatorname{curl} b_M$ is bounded in $L^1((0,T)\times\Omega)$. Then by the Sobolev embedding theorem, $\operatorname{curl} b_M$ is precompact in $(W^{1,p}((0,T)\times\Omega))^*$ for $p^* \in (1, 4/3)$.

Gathering the above, and choosing a, b and γ such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{1+w} < 1$, for w small, we use the Div-Curl Lemma [**?**, p. 343] and we conclude that

$$c_M \theta_M \theta_M^\gamma \rightharpoonup c_n \theta_n \overline{\theta_n^\gamma} \text{ weakly in } L^{1+w}((0,T)\times\Omega). \tag{5.148}$$

To prove that $\overline{\theta_n^\gamma} = \theta_n^\gamma$, we use Minty's method. Since the function $x \in \mathbb{R}^+ \mapsto x^{\gamma+1}$ is increasing and continuous for $\gamma > 0$, we have:

$$\int_0^t \int_\Omega c_M(\theta_M^\gamma - h^\gamma)(\theta_M - h)dx \geqslant 0 \quad \forall h \in L^{1+w}((0,T)\times\Omega),\ h > 0.$$

Then using (5.132) and (5.148) in the above inequality we deduce

$$\int_0^t \int_\Omega c_n(\overline{\theta_n^\gamma} - h^\gamma)(\theta_n - h)\ dxdt \geqslant 0 \quad \forall h \in L^{1+w}((0,T)\times\Omega),\ h > 0.$$

Since $\theta_n > \underline{\theta}$, we can choose $h = \theta_n + \lambda\vartheta$ with $\vartheta \in L^{1+w}((0,T)\times\Omega),\ \vartheta > 0$ in the above inequality and taking $\lambda \to 0$ we deduce that:

$$\int_0^t \int_\Omega c_n(\overline{\theta_n^\gamma} - \theta_n^\gamma)\vartheta\ dxdt = 0.$$

Then, we conclude that $\overline{\theta_n^\gamma} = \theta_n^\gamma$ and we obtain

$$c_M\theta_M^{\gamma+1} \rightharpoonup c_n\theta_n^{\gamma+1} \text{ weakly in } L^{1+w}((0,T)\times\Omega). \tag{5.149}$$

Then,

$$c_M^{1/(\gamma+1)}\theta_M \rightharpoonup c_n^{1/(\gamma+1)}\theta_n \text{ weakly in } L^{1+\gamma}((0,T)\times\Omega),$$

and

$$\left\| c_M^{1/(\gamma+1)}\theta_M \right\|_{L^{1+\gamma}} \to \left\| c_n^{1/(1+\gamma)}\theta_n \right\|_{L^{1+\gamma}(\Omega)}.$$

Therefore, we deduce

$$c_M^{1/(\gamma+1)}\theta_M \to c_n^{1/(1+\gamma)}\theta_n \text{ strongly in } L^{1+\gamma}((0,T)\times\Omega).$$

Then, using (5.144) we deduce

$$\theta_M \to \theta_n \text{ strongly in } L^r((0,T)\times\Omega) \text{ for } r \in [1, 5/3). \tag{5.150}$$

5.4.3 Passing to the limit in M

To pass to the limit (5.119) we have to show the convergence of the boundary term in $\partial\mathcal{S}_M$. To do this, first we make the following change of variables:

$$\int_0^t \int_{\partial\mathcal{S}_M} ((u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)) \times \widehat{n}) \cdot ((v - P^{\mathcal{R}}_{\mathcal{S}_M}(v)) \times \widehat{n})\ dxdt =$$
$$\int_0^t \int_{\partial\mathcal{S}_0} (R_M(U^M - U_s^M) \times \widehat{n}) \cdot (R_M(V^M - V_s^M) \times \widehat{n})\ dxdt,$$

where

$$U^M = \overline{\Phi}_{h_M,R_M}(u_M),\ U_s^M = \overline{\Phi}_{h_M,R_M}(P^{\mathcal{R}}_{\mathcal{S}_M}u_M),\ V^M = \overline{\Phi}_{h_M,R_M}(v) \text{ and } V_s^M = \overline{\Phi}_{h_M,R_M}(P^{\mathcal{R}}_{\mathcal{S}_M}v).$$

By (5.137), Lemma A.2 of [?], an interpolation inequality and the continuity of the trace operator we deduce that

$$U^M \to U^n = \overline{\Phi}_{h_n,R_n}(u_n) \text{ strongly in } L^2(0,T;H^s(\partial\Omega)),$$

for all $s < 1/2$. In particular,

$$U^M \to U^n = \overline{\Phi}_{h_n,R_n}(u_n) \text{ strongly in } L^2((0,T)\times\partial\Omega).$$

Similarly, using (5.140) we deduce that

$$U_s^M \rightharpoonup U_s^n = \overline{\Phi}_{h_n,R_n}(P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)) \text{ strongly in } L^2((0,T)\times\partial\Omega).$$

Moreover,

$$V^M \to V = \overline{\Phi}_{h_n,R_n}(v) \text{ strongly in } L^2(0,T;H^{1/2}(\partial\mathcal{S}_0))$$

and

$$V^M_{\mathcal{S}} \to V_s = \overline{\Phi}_{h_n,R_n}(P_{\mathcal{S}_n}v) \text{ strongly in } L^2(0,T;H^{1/2}(\partial\mathcal{S}_0)).$$

Gathering the above convergences, we conclude

$$\int_0^T\int_{\partial\mathcal{S}_M}((u_M - P^{\mathcal{R}}_{\mathcal{S}_M}(u_M))\times n)\cdot((v-P^{\mathcal{R}}_{\mathcal{S}_M}(v))\times\widehat{n})\,dxdt \to \int_0^T\int_{\partial\mathcal{S}_n}((u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n))\times n)\cdot((v-P^{\mathcal{R}}_{\mathcal{S}_n}(v))\times\widehat{n})\,dxdt. \quad (5.151)$$

Using (5.128), (5.131), (5.132), (5.135),(5.137), (5.139), (5.150), (5.151) and (A.17), we apply integration by parts in (5.119) and we obtain:

$$\rho\int_0^t\left\langle\frac{\partial u_n}{\partial t},v\right\rangle_{(H^1)^*,H^1}dt-\rho\int_0^t\int_\Omega(Q_{\mathcal{S}_n}(\overline{u_n})\cdot\nabla)v\cdot u_n\,dxdt+\int_0^t\int_\Omega 2\mu_k(\theta_n)D(u_n):D(v)\,dxdt$$
$$+\nu_\Omega\int_0^t\int_{\partial\Omega}u_n\cdot v\,dxdt+\nu_{\mathcal{S}}\int_0^t\int_{\partial\mathcal{S}_n}((u_n-P^{\mathcal{R}}_{\mathcal{S}_n}(u_n))\times\widehat{n})\cdot((v-P^{\mathcal{R}}_{\mathcal{S}_n}(v))\times\widehat{n})\,dxdt$$
$$+k\int_0^t\int_{\mathcal{S}_n}(u_n-P^{\mathcal{R}}_{\mathcal{S}}u_n)\cdot(v-P^{\mathcal{R}}_{\mathcal{S}_n}(v))\,dxdt=\rho\int_0^t\int_\Omega b\cdot v\,dxdt, \quad (5.152)$$

for all $v\in L^2(0,T;H^1_\sigma(\Omega))$. Taking $v=u_n$ in the above equation we deduce (5.73), and taking $t\to 0^+$ in (5.152) we deduce (5.68).

To pass to the limit in equation (5.120), we need to prove

$$\int_0^t\int_\Omega k\mathbb{1}_{\mathcal{S}_M}\left|u_M-P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)\right|^2+2\mu_k(\theta_M)\left|D(u_M)\right|^2\,dxdt \to \int_0^t\int_\Omega k\mathbb{1}_{\mathcal{S}_n}\left|u_n-P^{\mathcal{R}}_{\mathcal{S}}(u_n)\right|^2+2\mu_k(\theta)\left|D(u_n)\right|^2\,dxdt. \quad (5.153)$$

We notice that by the strong convergences (5.137) and (5.131) we can prove that

$$\left\|u_M-P^{\mathcal{R}}_{\mathcal{S}_M}u_M\right\|_{L^2(0,T;L^2(\mathcal{S}_M))}\to\left\|u_n-P^{\mathcal{R}}_{\mathcal{S}_n}u_n\right\|_{L^2(0,T;L^2(\mathcal{S}_n))}.$$

By the weak convergence of $D(u)$ in $L^2((0,T)\times\Omega)$, we have:

$$\int_0^t\int_\Omega 2\mu_k(\theta_n)\left|D(u_n)\right|^2\,dxdt\leqslant\liminf_{M\to\infty}\int_0^t\int_\Omega 2\mu_k(\theta_M)\left|D(u_M)\right|^2\,dxdt.$$

Taking u_M as a test function in (5.119), we deduce:

$$\liminf_{M\to\infty}\int_0^t\int_\Omega 2\mu_k(\theta_M)\left|D(u_M)\right|^2\,dxdt$$
$$=\liminf_{M\to\infty}\Bigg(-\frac{\rho}{2}\int_\Omega |u_M(t,\cdot)|^2\,dx+\rho\int_0^t\int_\Omega b\cdot u_M\,dxdt+\frac{\rho}{2}\int_\Omega |u_M(0,\cdot)|^2\,dx$$
$$-\nu_\Omega\int_0^t\int_{\partial\Omega}|u_M|^2\,dxdt-\nu_{\mathcal{S}}\int_0^t\int_{\partial\mathcal{S}}\left|(u-P^{\mathcal{R}}_{\mathcal{S}_M}u_M)\times n\right|^2\,dxdt-k\int_0^t\int_{\mathcal{S}_M}\left|u_M-P^{\mathcal{R}}_{\mathcal{S}_M}(u_M)\right|^2\,dxdt\Bigg).$$

Then, since $\liminf\leqslant\limsup$ and using (5.152) with $v=u$ we deduce

$$\liminf_{M\to\infty}\int_0^t\int_\Omega 2\mu_k(\theta_M)\left|D(u_M)\right|^2\,dxdt\leqslant\int_0^t\int_\Omega 2\mu_k(\theta_n)\left|D(u_n)\right|^2\,dxdt,$$

and we conclude (5.153).

Then, taking $\varphi\in\mathcal{C}^1([0,T];W^{1,\infty}(\Omega))$ such that $\varphi(T,\cdot)=0$ we pass to the limit equation(5.120), and we obtain

$$-\rho\int_0^T\int_\Omega c_n\theta_n\left(\frac{\partial\varphi}{\partial t}+Q_{S_n}(\overline{u_n})\cdot\nabla\varphi\right)\,dxdt+\int_0^T\int_\Omega(\kappa(\theta_n)+k\mathbb{1}_{\mathcal{S}_n})\nabla\theta_n\cdot\nabla\varphi\,dxdt$$
$$-\int_0^T\int_\Omega\left(k\mathbb{1}_{\mathcal{S}_n}\left|u-P^{\mathcal{R}}_{\mathcal{S}_n}u_n\right|^2+2\mu_k(\theta_n)\left|D(u_n)\right|^2\right)\varphi\,dxdt$$
$$=\rho\int_\Omega c(0,x)\theta_0\varphi(0,x)\,dx+\rho\int_0^T\int_\Omega w\varphi\,dxdt.\quad(5.154)$$

Let $\varphi\in W^{1,p}(\Omega)$ for $p>4$. Using (5.154) we deduce

$$\int_0^T\left\langle\frac{\partial}{\partial t}c_n\theta_n,\varphi\right\rangle_{W^{-1,p^*},W^{1,p}}dt\leqslant C\Big(\|Q_{\mathcal{S}_n}(\overline{u_n})\|_{L^\infty((0,T)\times\Omega)}\|\theta_n\|_{L^r((0,T)\times\Omega)}+C_{\overline{\kappa},k}\|\theta_n\|_{L^s(0,T;W^{1,s}(\Omega)}$$
$$+k\left\|u_n-P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)\right\|^2_{L^2(0,T;L^2(\mathcal{S}_n))}+\left\|\sqrt{\mu_k(\theta_n)}D(u_n)\right\|^2_{L^2((0,T)\times\Omega)}+\|w\|_{L^2((0,T)\times\Omega)}+\|\theta_0\|_{L^1(\Omega)}\Big)\|\varphi\|_{W^{1,p}(\Omega)}$$

for all $r\in[1,5/3)$ and for all $s\in[1,5/4)$ where $1/p+1/p^*=1$. Then

$$\frac{\partial}{\partial t}c_n\theta_n\text{ is bounded in }L^1(0,T;(W^{1,p}(\Omega))^*),$$

and we deduce (5.69).

To prove θ is bounded in $L^\infty(0,T;L^1(\Omega))$, we integrate in $[0,\tau]$ and we take as a test function $\varphi=1$ in (5.69) and we deduce that

$$\rho\int_\Omega c_n(\tau,x)\theta_n(\tau,x)\,dx=\int_\Omega c_0\theta_0\,dx+\int_0^\tau\int_\Omega k\mathbb{1}_{S_n}\left|u_n-P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)\right|^2+2\mu_k(\theta_n)\left|D(u_n)\right|^2\,dxdt+\rho\int_0^\tau\int_\Omega w\,dx.$$

Since the right-hand side of the above equation is bounded, taking sup over $\tau\in(0,T)$ in the above equation, we conclude

$$\|\theta_n\|_{L^\infty(0,T;L^1(\Omega))}\leqslant C$$

with C independent of k. Combining the above inequality with (5.143),(5.144) and (5.145) we deduce (5.74). To prove (5.70) we take $\psi \in C^1(\Omega)$ as a test function in (5.69) and we integrate in $[0,\tau]$ and we obtain

$$\rho \int_\Omega c_n(\tau,x)\theta_n(\tau,x)\psi(x)\ dx$$
$$= \int_\Omega c_0\theta_0\psi(x)\ dx + \int_0^\tau \int_\Omega (k\mathbb{1}_{S_n}\left|u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)\right|^2 + 2\mu_k(\theta_n)\left|D(u_n)\right|^2)\psi(x)\ dxdt + \rho\int_0^\tau \int_\Omega w\psi(x)\ dx.$$

Then, since the right hand side is bounded, we take the limit when $\tau \to 0^+$ and we deduce (5.70) .

5.4.4 Introduction of the pressure

To introduce the pressure we use the following lemma

Lemma 5.4. *Let Ω a $\mathcal{C}^{1,1}$ bounded domain of $\mathbb{R}^3$, $(h,R) \in \mathbb{R}^3 \times SO(3)$, $\mathbb{Y} \in L^{s_1}(\Omega)^{3\times 3}$, $b \in L^{s_2}(\Omega)^3$, $c \in L^{s_3}(\partial\Omega)^3$ and $d \in L^{s_4}(\partial\Omega)^3$ with s_1, s_2 and s_3 in $[1,\infty)$. Then, there exists a function $p \in L^s(\Omega)$, where $s = \min(s_1,s_2,s_3,s_4)$, such that*

$$\int_\Omega p\Delta z\ dx = \int_\Omega \mathbb{Y} : \nabla^2 z\ dx + \int_\Omega b\cdot\nabla z\ dx + \int_{\partial\Omega} c\cdot\nabla z\ dx + \int_{\partial\widehat{\mathcal{S}}(h,R)} (d\times\widehat{n})\cdot((\nabla z - P_{\widehat{\mathcal{S}}(h,R)}(\nabla z))\times\widehat{n})\ dx, \quad (5.155)$$

for all $z \in W^{2,r}(\Omega)$ with $\nabla z\cdot n = 0$ on $\partial\Omega$, $\int_\Omega z\ dx = 0$ and $\frac{1}{r}+\frac{1}{s} = 1$.

Proof. Let $z \in W^{2,r}(\Omega)$ with $\nabla z\cdot n = 0$ in $\partial\Omega$, $\int_\Omega z\ dx = 0$. We define the functional $\mathcal{D} : L^r(\Omega) \to \mathbb{R}$ such that: for all $f \in L^r(\Omega)$ with $\int_\Omega f\ dx = 0$,

$$\mathcal{D}(f) = \int_\Omega \mathbb{Y} : \nabla^2\psi\ dx + \int_\Omega b\cdot\nabla\psi\ dx + \int_{\partial\Omega} c\cdot\nabla\psi\ dx + \int_{\partial\widehat{\mathcal{S}}(h,R)} (d\times\widehat{n})\cdot((\nabla\psi - P_{\widehat{\mathcal{S}}(h,R)}(\nabla\psi))\times\widehat{n})\ dx, \quad (5.156)$$

where ψ is the solution to the Neumann problem

$$\Delta\psi = f \text{ in } \Omega,\ \nabla\psi\cdot\widehat{n} = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega \psi\ dx = 0. \quad (5.157)$$

By the properties of the Neuman problem $\mathcal{D} \in (L^r(\Omega))^*$. Then, by the Riesz representation theorem, there exists an element $p \in L^s(\Omega)$, such that

$$\mathcal{D}(f) = \int_\Omega pf\ dx, \quad (5.158)$$

for all $f \in L^r(\Omega)$. Then, taking $f = \Delta z$ and combining (5.156), (5.157) and (5.158) we deduce (5.155). □

To construct the pressure, we consider a function $v \in H^1(\Omega)^3$ such that $v \cdot \widehat{n} = 0$ on $\partial\Omega$ and, by the Helmholtz-Weyl decomposition, there exists two functions, $v_0 \in H^1_\sigma(\Omega)^3$ and $z \in W^{2,2}(\Omega)$ such that

$$v = v_0 + \nabla z.$$

In the above decomposition the function z satisfies the Neumann problem

$$\Delta z = \operatorname{div}(v) \text{ in } \Omega,\ \nabla z \cdot \widehat{n} = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega z\ dx = 0.$$

Then using v_0 as a test function in (5.152) we obtain:

$$\begin{aligned}
\rho \left\langle \frac{\partial u_n}{\partial t}, v \right\rangle_{(H^1)^*,H^1} &- \rho \int_\Omega (Q_{\mathcal{S}_n}(\overline{u_n}) \cdot \nabla) v \cdot u_n\ dx + \int_\Omega 2\mu_n(\theta_k) D(u_n) : D(v)\ dx \\
&+ \nu_\Omega \int_{\partial\Omega} u_n \cdot v\ dx + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}} ((u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)) \times \widehat{n}) \cdot ((v - P^{\mathcal{R}}_{\mathcal{S}_n}(v)) \times \widehat{n})\ dx \\
&+ k \int_{\mathcal{S}_n} (u_n - P^{\mathcal{R}}_{\mathcal{S}_n}(u_n)) \cdot (v - P^{\mathcal{R}}_{\mathcal{S}_n}(v))\ dx \\
- \rho \left\langle \frac{\partial u_n}{\partial t}, \nabla z \right\rangle_{(H^1)^*,H^1} &- \int_\Omega 2\mu_n(\theta_k) D(u_n) : D(\nabla z)\ dx + \rho \int_\Omega (Q_{\mathcal{S}_n}(\overline{u_n}) \cdot \nabla) \nabla z \cdot u_n\ dx \\
&- \nu_\Omega \int_{\partial\Omega} u_n \cdot \nabla z\ dx - \nu_{\mathcal{S}} \int_{\partial\mathcal{S}_n} ((u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n) \times n) \cdot ((\nabla z - P^{\mathcal{R}}_{\mathcal{S}_n}(\nabla z)) \times n)\ dx \\
&k \int_{\mathcal{S}_n} (u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n) \cdot (\nabla z - P^{\mathcal{R}}_{\mathcal{S}_n}(\nabla z))\ dx = \rho \int_\Omega b \cdot v\ dx - \rho \int_\Omega b \nabla z\ dx.
\end{aligned}$$

Using a regularization procedure, $\operatorname{div}(u_n) = 0$ in Ω and $u_n \cdot \widehat{n} = 0$ in $\partial\Omega$ we can prove that

$$\left\langle \frac{\partial u_n}{\partial t}, \nabla z \right\rangle_{(H^1)^*,H^1} = 0. \tag{5.159}$$

Moreover, using that

$$\|Q_{\mathcal{S}_n}(\overline{u_n}) \otimes u_n\|_{L^2((0,T)\times\Omega)} \leqslant C_k$$

by Lemma 5.4 there exists a function $p_n \in L^2((0,T) \times \Omega)$ such that

$$\begin{aligned}
\int_\Omega p_n \Delta z\ dx = \int_\Omega 2\mu_k(\theta_n) D(u_n) : D(\nabla z)\ dx - \rho \int_\Omega (Q_{\mathcal{S}_n}(\overline{u_n}) \cdot \nabla) \nabla z \cdot u_n\ dx \\
+ \nu_\Omega \int_{\partial\Omega} u_n \cdot \nabla z\ dx + \nu_{\mathcal{S}} \int_{\partial\mathcal{S}_n} ((u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n) \times n) \cdot ((\nabla z - P^{\mathcal{R}}_{\mathcal{S}_n}(\nabla z)) \times n)\ dx \\
+ k \int_{\mathcal{S}_n} (u_n - P^{\mathcal{R}}_{\mathcal{S}_n} u_n) \cdot (\nabla z - P^{\mathcal{R}}_{\mathcal{S}_n}(\nabla z))\ dx - \rho \int_\Omega b \cdot \nabla z\ dx.
\end{aligned} \tag{5.160}$$

Then, using that $\operatorname{div}(v) = \Delta z$ we deduce (5.67).

Remark 5.5. *We notice that if we dismiss the assumption that the densities of the fluid and the solid are equals (see equation* (5.25)*), then, equation* (5.159) *does not holds. Instead, if we called* $\rho = \mathbb{1}_{\mathcal{F}_n} \rho_f + \mathbb{1}_{\mathcal{S}_n} \rho_s$*, then,*

$$\left\langle \frac{\partial \rho_n u_n}{\partial t}, \nabla z \right\rangle_{(H^1)^*,H^1} = -(\rho_f - \rho_s) \int_{\partial\mathcal{F}_n} u_n \cdot \widehat{n} \frac{\partial z}{\partial t}\ dx + (\rho_f - \rho_s) \frac{d}{dt} \int_{\partial\mathcal{F}_n} u_n \cdot \widehat{n} z\ dx.$$

Remark 5.6. *We notice that the sequence* $\{p_n\}_n$ *is not bounded in* $L^2((0,T) \times \Omega)$ *when* $k \to \infty$*. Suitable bounds will be obtained in Section 5.5.2.*

5.5 Passing to the limit $k \to \infty$

This section aims to pass to the limit in the penalization parameter k and δ. Then, we consider

$$\delta = 1/k$$

and we take $k \to \infty$. To simplify and clarify the notation, we call $u_n = u_k$ omitting δ.

First we notice that (5.73) does not imply that u_k is bounded in $L^2(0,T;H^1(\Omega))$. Therefore, by (5.73), we deduce that there exists

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \quad (h,R) \in W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)),$$

where $L^2_\sigma(\Omega)$ and $H^1_\sigma(\Omega)$ are defined in (5.26) and (5.27), such that:

$$u_k \xrightharpoonup{*} u \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{5.161}$$

$$(h_k, R_k) \xrightharpoonup{*} (h,R) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)), \tag{5.162}$$

$$(h_k, R_k) \to (h,R) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)), \tag{5.163}$$

$$\mathbb{1}_{S_k} \to \mathbb{1}_S \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{5.164}$$

and

$$c_k \to c := \widehat{c}_{h,R} \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{5.165}$$

We write

$$P_S u := \ell + \omega \times (x-h) \quad \text{ in } (0,T).$$

By (5.161) and (5.163), we conclude:

$$P^{\mathcal{R}}_{\mathcal{S}_k} u_k \xrightharpoonup{*} P^{\mathcal{R}}_{\mathcal{S}} u \text{ weakly star in } L^\infty(0,T,\mathcal{R}). \tag{5.166}$$

where $\mathcal{S} := \widehat{\mathcal{S}}(h,R)$ and $J_{\mathcal{S}} := \widehat{J}(h,R)$. By the equation (5.73) we deduce that:

$$\left\| u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k \right\|_{L^2(0,T;L^2(\mathcal{S}_k))} \leqslant \frac{C}{\sqrt{k}}. \tag{5.167}$$

Then, taking $k \to \infty$, we deduce that $u = P^{\mathcal{R}}_{\mathcal{S}} u$ in $\mathcal{S}$.

For any (h,R) such that $\operatorname{dist}(\widehat{\mathcal{S}}(h,R), \partial\Omega) \geqslant 2\delta_0$ we can construct an extension operator:

$$E_{\widehat{\mathcal{F}}(h,R)} : \{u \in H^1_\sigma(\widehat{\mathcal{F}}(h,R));\ u \cdot n = 0 \text{ on } \partial\Omega\} \to H^1_\sigma(\Omega)$$

such that

$$\|E_{\widehat{\mathcal{F}}(h,R)}(v)\|_{H^1(\Omega)} \leqslant C_\delta \|v\|_{H^1(\widehat{\mathcal{F}}(h,R))}. \tag{5.168}$$

Taking δ_0 such that:

$$0 < \delta_0 < \frac{1}{4} \operatorname{dist}(\mathcal{S}_0, \partial\Omega),$$

and following the proof of Proposition 4.6 of [?], we deduce from (5.64), (5.65) and (5.73), that we can choose $T > 0$, uniform in k such that

$$\operatorname{dist}(\mathcal{S}_k(t), \partial\Omega) \geqslant 2\delta_0, \quad (t \in [0,T]).$$

Then, we define

$$u_{f,k} = E_{\widehat{\mathcal{F}}(h_k,R_k)}(u_k). \tag{5.169}$$

Combining (5.168) and (5.73), we deduce the existence of a function $v_f \in L^2(0,T;H^1_\sigma(\Omega))$ such that

$$u_{f,k} \rightharpoonup u_f \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{5.170}$$

Finally, since

$$u_k = u_k \mathbb{1}_{\mathcal{F}_k} + (u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k))\mathbb{1}_{\mathcal{S}_k} + P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)\mathbb{1}_{\mathcal{S}_k},$$

since (5.161), (5.164), (5.166), (5.167) and (5.170) we deduce that $u = u_f \mathbb{1}_{\mathcal{F}} + P^{\mathcal{R}}_{\mathcal{S}}(u)\mathbb{1}_{\mathcal{S}}$.

For the temperature, since (5.74) is independet of k, we deduce the existence of a function θ and a subsequence of $\{\theta_k\}_k$ (still denoted $\{\theta_k\}_k$) such that:

$$\theta_k \rightharpoonup \theta \text{ weakly in } L^r((0,T)\times\Omega) \text{ for all } r \in [1,5/3), \tag{5.171}$$

$$\theta_k \rightharpoonup \theta \text{ weakly in } L^s(0,T;W^{1,s}(\Omega)) \text{ for all } s \in [1,5/4), \tag{5.172}$$

and, using Poincaré inequality,

$$\left\|\theta_k^\alpha - P^1_{\mathcal{S}_k}(\theta_k^\alpha)\right\|^2_{L^2(0,T;L^2(\mathcal{S}_k))} \leqslant \frac{c}{k}. \tag{5.173}$$

Then, taking $k \to \infty$, we deduce that $\theta^\alpha = P^1_{\mathcal{S}}\theta^\alpha$ in $\mathcal{S}$. Therefore, since $\nabla\theta^\alpha = \alpha\theta^{\alpha-1}\nabla\theta$, we conclude that $\nabla\theta(t,\cdot) = 0$ in $\mathcal{S}(t)$ a.e. in $(0,T)$.

5.5.1 Strong Convergence of the velocity field

To pass to the limit the convective term, we need the strong convergence of the velocity. To do this we follow the main steps of Section 7 of [?] (see also Section 5.5 of [?]).

We define the space

$$H^s_{\mathcal{S}}(\Omega)^3 := \{v \in H^s_\sigma(\Omega)^3 \ ; \ D(v) = 0 \text{ in } \mathcal{S}\}$$

and we denote by

$$\mathcal{P}^s_{\mathcal{S}} : H^s_\sigma(\Omega)^3 \to H^s_{\mathcal{S}}(\Omega)^3 \tag{5.174}$$

the orthogonal projection.

Using (5.163), we deduce that for all $d > 0$, there exists k_0 such that for all $k \geq k_0$,

$$\mathcal{S}_k(t) \subset (\mathcal{S}(t))^{\frac{d}{2}} \quad \forall t \in [0,T]. \tag{5.175}$$

Moreover, using the Heine theorem, there exists $N(d) > 0$ such that if

$$\tau := T/N \quad \text{and} \quad I_j := [j\tau,(j+1)\tau]$$

then

$$(\mathcal{S}(t))^{\frac{d}{2}} \subset (\mathcal{S}(j\tau))^d \subset (\mathcal{S}(t))^{2d} \quad (t \in I_j).$$

Then, we consider a test function $v \in C_0^\infty((0,T), H^1_\sigma(\Omega))^3$ such that $D(v(t,\cdot)) = 0$ in $(S(j\tau))^d$ and $v(t,\cdot) = 0$ if $t \notin I_j$. With such a test function in (5.67), the integrals related to the penalization term and the pressure vanishes, and we obtain the following estimate:

$$\left| \int_{I_j} \int_\Omega \rho u_k \cdot \frac{\partial v}{\partial t}\, dxdt \right| \leqslant C\Big(\|Q_{\mathcal{S}_k}(\overline{u_k})\|_{L^2(0,T;L^4(\Omega))} \|u_k\|^{1/4}_{L^\infty(0,T;L^2(\Omega))} \|u_{f,k}\|^{3/4}_{L^2(0,T;H^1(\Omega))} + \left\|P^{\mathcal{R}}_{\mathcal{S}_k}(\overline{u_k})\right\|_{L^\infty(0,T;\mathcal{R})} \|u_k\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\sqrt{\mu_k(\theta_k)}D(u_k)\right\|_{L^2((0,T)\times\Omega)} + \|b\|_{L^2((0,T)\times\Omega)} \Big) \|v\|_{L^8(I_j;H^1(\Omega))} \quad (5.176)$$

From (A.3) and (A.14), we have

$$\|Q_{\mathcal{S}_k}(\overline{u_k}) - \overline{u_k}\|_{L^2(0,T;L^p(\mathcal{F}_k))} \leqslant C\left(\left(\frac{1}{k}\right)^{\frac{1}{p}-\frac{1}{6}} \|\overline{u_k}\|_{L^2(0,T;H^1(\mathcal{F}_k))} + \|(\overline{u_k} - P_{S_k}\overline{u_k})\cdot n\|_{L^2(0,T;L^p(\partial S_k))} \right), \quad (5.177)$$

for $p \in [2,6]$. Moreover, using a Sobolev embedding and an interpolation result, we can check that for $p \in [2,4]$,

$$\left\|\left(\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k}\right)\cdot\widehat{n}\right\|_{L^p(\partial\mathcal{S}_k)} \leqslant C \left\|(\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k})\cdot\widehat{n}\right\|^{3/2-2/p}_{H^{1/2}(\partial\mathcal{S}_k)} \left\|(\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k})\cdot\widehat{n}\right\|^{2/p-1/2}_{H^{-1/2}(\partial\mathcal{S}_k)}.$$

Using that

$$\left\|\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k}\right\|_{L^2(S_k)} \leqslant \|u_k - \overline{u_k}\|_{L^2(\Omega)} + \left\|u_k - P^{\mathcal{R}}_{\mathcal{S}}u_k\right\|_{L^2(S_k)},$$

a trace theorem and (5.167), we deduce

$$\int_0^T \left\|(\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k})\cdot\widehat{n}\right\|_{H^{-1/2}(\partial\mathcal{S}_k)} dt \leqslant \int_0^T \left\|\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k}\right\|_{L^2(\mathcal{S}_k)} dt \leqslant \|u_k - \overline{u_k}\|_{L^2((0,T)\times\Omega)} + \frac{c}{\sqrt{k}}.$$

On the other hand,

$$\left\|(\overline{u_k} - P^{\mathcal{R}}_{\mathcal{S}_k}\overline{u_k})\cdot\widehat{n}\right\|_{H^{1/2}(\partial\mathcal{S}_k)} \leqslant \|u_{f,k}\|_{H^1(\mathcal{F}_k)} + \|u_k\|_{L^2(\mathcal{S}_k)}.$$

Then, we deduce

$$\|(\overline{u_k} - P_{S_k}\overline{u_k})\cdot\widehat{n}\|_{L^2(0,T;L^p(\partial S_k))} \leqslant C\left(\|\overline{u_k} - u_k\|^{2/p-1/2}_{L^2((0,T)\times\Omega)} + \left(\frac{1}{k}\right)^{1/p-1/4} \right)\left(\|u_{f,k}\|^{3/2-2/p}_{L^2(0,T;H^1(\mathcal{F}_k))} + \|u_k\|^{3/2-2/p}_{L^2(0,T;L^2(\mathcal{S}_k))} \right). \quad (5.178)$$

In particular,

$$\{Q_{S_k}(\overline{u_k})\} \quad \text{is bounded in } L^2(0,T;L^4(\Omega))^3. \quad (5.179)$$

Combining the above estimate with (5.73) in (5.176), we deduce

$$\left\{\frac{\partial}{\partial t}\mathcal{P}^0_{(S(j\tau))^d}(u_k)\right\}_k \quad \text{is bounded in} \quad L^{8/7}(I_j;(H^1_{(S(j\tau))^d}(\Omega))^*)^3.$$

Using the Aubin-Lions lemma we deduce

$$\mathcal{P}^0_{(S(j\tau))^d}(u_k) \to \mathcal{P}^0_{(S(j\tau))^d}(u) \text{ strongly in } L^2(I_j;(H^s_{(S(j\tau))^d}(\Omega))^*)^3 \quad (s \in (0,1]).$$

Then using the relation

$$\mathcal{P}^0_{(S(j\tau))^d}\mathcal{P}^s_{(S(t))^{2d}} = \mathcal{P}^s_{(S(t))^{2d}} \quad \forall t \in I_j,$$

we deduce for any $s \in (0,1]$,

$$\lim_{k\to\infty} \int_0^T \int_\Omega u_k \cdot \mathcal{P}^s_{(S(t))^{2d}}(u_k)\ dxdt = \int_0^T \int_\Omega u \cdot \mathcal{P}^s_{(S(t))^{2d}}(u)\ dxdt.$$

Then, using Corollary A.1.4, (5.167) and (5.73), we have for $s \in (0,1/3)$

$$\int_0^T \left\| u_k(t,\cdot) - \mathcal{P}^s_{(\mathcal{S}_k(t))^d} u_k(t,\cdot) \right\|^2_{H^s(\Omega)} dt \leqslant C(d^{2(1/3-s)} + k^{-1/2} + k^{s-1})$$

and

$$\int_0^T \left\| u(t,\cdot) - \mathcal{P}^s_{(S(t))^d} u(t,\cdot) \right\|^2_{H^s(\Omega)} dt \leqslant C d^{2(1/3-s)},$$

so that

$$\lim_{k\to\infty} \int_0^T \int_\Omega |u_k|^2\ dxdt = \int_0^T \int_\Omega |u|^2\ dxdt$$

and this allows us to deduce that

$$u_k \to u \text{ strongly in } L^2((0,T)\times\Omega)^3. \tag{5.180}$$

Using (5.164) and (5.170), we deduce

$$\mathbb{1}_{\mathcal{F}_k} u_k \to \mathbb{1}_{\mathcal{F}} u \text{ strongly in } L^2(0,T;L^p(\Omega))^3 \quad (p<6). \tag{5.181}$$

On the other hand, from (5.180) and (5.38), (5.39), we have

$$P^{\mathcal{R}}_{\mathcal{S}_k} u_k \to P^{\mathcal{R}}_{\mathcal{S}} u \text{ strongly in } L^2((0,T)\times\Omega)^3. \tag{5.182}$$

Using (5.180) and the continuity of the trace operator we deduce that

$$u_{f,k} \to u_f \quad \text{ strongly in } L^1((0,T)\times\partial\Omega)^3. \tag{5.183}$$

However, for $p \leqslant 8/3$ we have

$$\|u_k\|^p_{L^p(\partial\Omega)} \leqslant \|u_k\|^p_{H^l(\partial\Omega)} \leqslant \|u_k\|^{(1/2-l)p}_{L^2(\mathcal{F}_k)} \|u_k\|^{p(l+1/2)}_{H^1(\mathcal{F}_k)},$$

for any $l \geqslant 1-2/p$. Combining the above identity with (5.183) we deduce that

$$u_k \to u \text{ strongly in } L^{8/3}((0,T)\times\partial\Omega)^3. \tag{5.184}$$

5.5.2 Decomposition of the pressure

As we discuss in Remark 5.6, we can't prove that the pressure p_k is bounded in Ω independently of k . Then, this section aims to decompose the pressure in 2 parts: a first part in $L^{5/3}(0,T;L^{5/3}(\mathcal{F}_k))$, like in the Navier-Stokes-Fourier model without a rigid body inside and, a second part in $L^{4/3}(0,T;L^{4/3}(\mathcal{F}_k))$.

We start by defining the partial pressure $p_{2,k} \in L^{4/3}(\Omega)$ as the solution to the equation

$$\int_\Omega p_{2,k} \cdot \Delta z \; dx = \rho \int_{(\mathcal{S}_k)^{\delta_1}\setminus \mathcal{S}_k} (Q_{\mathcal{S}_k}(\overline{u_k}) \cdot \nabla)\nabla z \cdot u_k \; dx, \tag{5.185}$$

for all $z \in H^2(\Omega)$ with $\int_\Omega p_{2,k} \; dx = 0$ and $\nabla z \cdot \widehat{n} = 0$ in $\partial\Omega$. The existence of $p_{2,k}$ is secured by Lemma 5.4.

Then, we define the pressure $p_{1,k}$ as:

$$p_{1,k} = p_k - p_{2,k} \tag{5.186}$$

and we claim that $p_{1,k}$ satisfies

$$\|p_{1,k}\|_{L^{5/3}(0,T;L^{5/3}(\mathcal{F}_k))} \leqslant C. \tag{5.187}$$

and, consequently, using $\mathbb{1}_{\mathcal{F}_k} \to \mathbb{1}_{\mathcal{F}}$ strongly in $\mathcal{C}([0,T];L^p(\Omega))$ for $p \in [1,\infty)$ there exists a function $p_1 \in L^{5/3}(0,T;L^{5/3}(\mathcal{F}))$ such that

$$\mathbb{1}_{\mathcal{F}_k} p_{1,k} \rightharpoonup \mathbb{1}_{\mathcal{F}} p_1 \text{ weakly in } L^{5/3}((0,T)\times\Omega). \tag{5.188}$$

To prove (5.187), let $z \in H^2(\Omega)$ the solution to the problem

$$\Delta z = \mathbb{1}_{\mathcal{F}_k}\Delta z_1 \quad \text{in } \Omega$$
$$\nabla z \cdot n = 0 \quad \text{on } \partial\Omega \text{ and } \int_\Omega z \; dx = 0,$$

where $z_1 \in H^2(\mathcal{F}_k)$ is the solution to the problem

$$\Delta z_1 = |p_{1,k}|^{\beta-2} p_{1,k} + \frac{1}{|\Omega\setminus\mathcal{S}_0|}\int_{\mathcal{F}_k} |p_{1,k}|^{\beta-2} p_{1,k} \; dx \quad \text{in } \mathcal{F}_k$$
$$\nabla z_1 \cdot \hat{n} = 0 \quad \text{on } \partial\mathcal{F}_k \cup \Omega \text{ and } \int_{\mathcal{F}_k} z_1 \; dx = 0,$$

ad, where $\beta \in (1,2]$. We recall that by standard theory of the Neuman problem

$$\|\nabla z\|^{\beta^*}_{W^{1,\beta^*}(\Omega)} \leqslant C\,\|p_{1,k}\|^{\beta}_{L^\beta(\mathcal{F}_k)},$$

where $1/\beta + 1/\beta^* = 1$. Since

$$\int_\Omega (\nabla z - \mathbb{1}_{\mathcal{F}_k}\nabla z_1)\cdot\nabla\varphi \; dx = 0 \quad \text{for all } \varphi \in \mathcal{C}^\infty(\Omega),$$

and

$$\int_{\partial\mathcal{F}_k} (\nabla z_1 - \nabla z)\cdot n \; dx = 0,$$

we deduce that

$$\nabla z = \mathbb{1}_{\mathcal{F}_k} \nabla z_1. \tag{5.189}$$

Combining (5.160), (5.160) and (5.189) we deduce that $p_{1,k}$ satisfies:

$$\begin{aligned}\int_\Omega p_{1,k} \Delta z \, dx = \int_\Omega 2\mu_k(\theta_k) D(u_k) : \nabla^2 z \, dx - \rho \int_{\Omega \setminus \mathcal{S}_k^{\delta_1}} \overline{u_k} \otimes u_k : \nabla^2 z \, dx \\ - \rho \int_{\mathcal{S}_k} P^{\mathcal{R}}_{\mathcal{S}_k}(\overline{u_k}) \otimes u_k : \nabla^2 z \, dx + \nu_\Omega \int_{\partial\Omega} u_k \cdot \nabla z \, dx \\ + \nu_{\mathcal{S}} \int_{\partial \mathcal{S}_k} ((u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k) \times \widehat{n}) \cdot ((\nabla z - P^{\mathcal{R}}_{\mathcal{S}_k}(\nabla z)) \times \widehat{n}) \, dx - \rho \int_\Omega b \nabla z \, dx.\end{aligned} \tag{5.190}$$

Then, we deduce

$$\begin{aligned}\int_0^t \|p_{1,k}\|^\beta_{L^\beta(\mathcal{F}_k)} \, dxdt = \int_0^t \int_\Omega 2\mu_k(\theta_k) D(u_k) : \nabla^2 z \, dxdt \\ - \rho \int_0^t \int_{\Omega \setminus \mathcal{S}_k^{\delta_1}} \overline{u_{f,k}} \otimes u_{f,k} : \nabla^2 z \, dxdt - \rho \int_0^t \int_{\mathcal{S}_k} P^{\mathcal{R}}_{\mathcal{S}_k}(\overline{u_k}) \otimes u_k : \nabla^2 z \, dxdt \\ + \nu_\Omega \int_0^t \int_{\partial\Omega} u_k \cdot \nabla z \, dxdt + \nu_{\mathcal{S}} \int_0^t \int_{\partial \mathcal{S}_k} ((u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k) \times \widehat{n}) \cdot ((\nabla z - P^{\mathcal{R}}_{\mathcal{S}_k}(\nabla z)) \times \widehat{n}) \, dxdt \\ - \rho \int_0^t \int_\Omega b \cdot \nabla z \, dxdt.\end{aligned} \tag{5.191}$$

Then, using Hölder's and Young's inequalities we have:

$$\int_0^t \int_\Omega 2\mu_k(\theta_k) D(u_k) : \nabla^2 z \, dxdt \leqslant c_1 \left\| \sqrt{\mu_k(\theta_k)} D(u_k) \right\|^\beta_{L^2((0,T)\times\Omega)} + \frac{1}{8} \int_0^T \|p_k\|^\beta_{L^\beta(\mathcal{F}_k)} \, dt,$$

$$\begin{aligned}\int_0^t \int_\Omega b \cdot \nabla z \, dxdt &\leqslant c_2 \left(\|b\|^2_{L^2((0,T)\times\Omega)} + \|\nabla z\|^2_{L^2((0,T)\times\Omega)} \right) \\ &\leqslant c_2 \left(\|b\|^2_{L^2((0,T)\times\Omega)} + c_3 \|\nabla z\|^2_{L^{\beta^*}((0,T)\times\Omega)} \right) \\ &\leqslant c_2 \|b\|^2_{L^2((0,T)\times\Omega)} + c_4 + \frac{1}{8} \int_0^T \|p_k\|^\beta_{L^\beta(\mathcal{F}_k)} \, dt,\end{aligned}$$

$$\int_0^t \int_{\Omega \setminus \mathcal{S}_k^{\delta_1}} \overline{u_k} \otimes u_k : \nabla^2 z \, dxdt \leqslant c_5 \int_0^T \int_\Omega |\overline{u_{f,k}} \otimes u_{f,k}|^\beta \, dxdt + \frac{1}{8} \int_0^T \|p_k\|^\beta_{L^\beta(\mathcal{F}_k)} \, dxdt,$$

$$\int_0^t \int_{\mathcal{S}_k} P^{\mathcal{R}}_{\mathcal{S}_k}(\overline{u_k}) \otimes u_k : \nabla^2 h \, dxdt \leqslant c_6 \left\| P^{\mathcal{R}}_{\mathcal{S}_k}(\overline{u_k}) \right\|^\beta_{L^\infty(0,T;\mathcal{R})} \|u_k\|^\beta_{L^2(0,T;L^2(\mathcal{S}_k))} + \frac{1}{8} \int_0^T \|p_k\|^\beta_{L^\beta(\mathcal{F}_k)} \, dxdt,$$

$$\nu_\Omega \int_0^t \int_{\partial\Omega} u_k \cdot \nabla z \, dx \leqslant \frac{\nu_\Omega}{2} \|u_k\|^2_{L^2((0,T)\times\partial\Omega)} + c_7 + \frac{1}{8} \int_0^T \|p_k\|^\beta_{L^\beta(\mathcal{F}_k)} \, dx,$$

and

$$\nu_{\mathcal{S}} \int_0^t \int_{\partial \mathcal{S}_k} ((u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k) \times \widehat{n}) \cdot ((\nabla z - P^{\mathcal{R}}_{\mathcal{S}_k}(\nabla z)) \times \widehat{n}) \, dxdt$$
$$\leqslant c_8 \left(\|u_k\|^2_{L^2(0,T;H^1(\mathcal{F}_k))} + \left\|P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)\right\|^2_{L^2(0,T;L^2(\mathcal{S}_k))} \right) + c_9 + \frac{1}{8} \int_0^T \|p_k\|^{\beta}_{L^{\beta}(\mathcal{F}_k)} \, dx.$$

Combining the above estimates with (5.191), we deduce

$$\int_0^T \|p_k\|^{\beta}_{L^{\beta}(\mathcal{F}_k)} \, dt \leqslant C + \int_0^T \int_{\Omega} |\overline{u_{f,k}} \otimes u_{f,k}|^{\beta} \, dxdt.$$

Finally choosing $\beta = 5/3$ we obtain

$$\int_0^T \|p_k\|^{5/3}_{L^{5/3}(\mathcal{F}_k)} \, dt \leqslant C + \|\overline{u_{f,k}}\|^{5/3}_{L^{10/3}(0,T;L^{10/3}(\Omega))} \|u_{f,k}\|^{5/3}_{L^{10/3}(0,T;L^{10/3}(\Omega))},$$

and since u_k is bounded in $L^{10/3}((0,T) \times \Omega)$ we deduce (5.187).

We notice that, by the same procedure that we proved (5.187), we can prove that

$$\int_0^T \|p_{2,k}\|^{4/3}_{L^{4/3}(\mathcal{F}_k)} \, dt \leqslant c.$$

However, we don't use this bound in the following sections.

5.5.3 Strong Convergence of the temperature

To pass to the limit the viscosity, the yield stress and the heat conductivity, the weak convergences (5.171) and (5.172) are not enough and we need the strong convergence of the temperature. To obtain the strong convergence, as we did before, we follow the main steps of Section 7 of [?] (see also Section 5.5 of [?]) with some minor changes due to the fact that the temperature is an scalar function and the a priori estimates are not the same.

We define the spaces

$$H^s_{\mathcal{S}}(\Omega) := \{\varphi \in H^s_{\sigma}(\Omega) \ ; \ \nabla \varphi = 0 \text{ in } \mathcal{S}\}$$

and

$$W^{k,p}_{\mathcal{S}}(\Omega) := \{\varphi \in W^{k,p}(\Omega) \ ; \ \nabla \varphi = 0 \text{ in } \mathcal{S}\},$$

and we denote by

$$\mathbf{P}^s_S : H^s(\Omega) \to H^s_{\mathcal{S}}(\Omega)$$

the orthogonal projection.

Then, we consider a function $\varphi \in W^{1,p}(\Omega)$, for $p > 20$, such that $\nabla \varphi(\cdot) = 0$ in $(S(j\tau))^d$ and we take $\mathbb{1}_{I_j}(t)\varphi$ as a test function in (5.69). Then, the integral related to the penalization term vanishes and we deduce:

$$\rho \left| \int_{I_j} \left\langle \frac{\partial}{\partial t}(c_k \theta_k), \varphi \right\rangle \, dt \right| \leqslant C \Big(\|\theta_k\|_{L^r(0,T;L^{10/7}(\Omega))} \|Q_{S_k}(\overline{u_k})\|_{L^2(0,T;L^4(\Omega))} + \|\nabla \theta_k\|_{L^s((0,T)\times\Omega)}$$
$$+ \left\|\sqrt{\mu_k(\theta_k)} D(u_k)\right\|^2_{L^2((0,T)\times\Omega)} + k \left\|u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)\right\|^2_{L^2(0,T;L^2(\mathcal{S}_k))} + \|w\|_{L^2((0,T)\times\Omega)} \Big) \|\varphi\|_{W^{1,p}(\Omega)}.$$

Combining the above estimate with (5.73) and (5.179) we deduce

$$\left\{\frac{\partial}{\partial t}\mathbf{P}^0_{(S(j\tau))^d}((c_k\theta_k)\right\}_k \quad \text{is bounded in} \quad L^1(I_j; W^{-1,20/19}_{(\mathcal{S}(j\tau))^d}(\Omega)).$$

Moreover, since θ_k is bounded in $L^\beta(0,T;L^2(\Omega))$ for $\beta \in [1,4/3)$ we have that

$$\mathbf{P}^0_{(\mathcal{S}(j\tau))^d}(c_k\theta_k) \quad \text{is bounded in } L^\beta(0,T;L^2(\Omega)) \quad \text{for } \beta \in [1,4/3).$$

Since the inclusion $L^2 \subset H^{-s}_{(\mathcal{S}(j\tau))^d}$ for $s \in (0,1)$ is compact. From Aubin-Lions Lemma (see Corollary 4 in Section 8 of [?]) we deduce

$$\mathbf{P}^0_{(\mathcal{S}(j\tau))^d}(c_k\theta_k) \to \mathbf{P}^0_{(\mathcal{S}(j\tau))^d}(c\theta) \text{ strongly in } L^\beta(I_j;(H^s_{(\mathcal{S}(j\tau))^d}(\Omega))') \quad (s \in (0,1]).$$

for $\beta \in [1,4/3)$. On the other hand, since

$$\left\|\mathbf{P}^s_{(\mathcal{S}(t))^{2d}}(\theta_k^\alpha)\right\|^{\beta^*}_{H^s(\Omega)} \leqslant \|\theta_k^\alpha\|^{(1-s)\beta^*}_{L^2(\Omega)} \|\theta_k^\alpha\|^{\beta^* s}_{H^1(\Omega)},$$

for all $\alpha \in (0,1/2)$, we deduce

$$\mathbf{P}^s_{(\mathcal{S}(t))^{2d}}(\theta_k^\alpha) \text{ is bounded in } L^{\beta^*}(0,T;H^s(\Omega)) \text{ for } \beta^* > 4 \text{ and } s \in (0,2/\beta^*). \tag{5.192}$$

Then, using the relation

$$\mathbf{P}^0_{(\mathcal{S}(j\tau))^d}\mathbf{P}^s_{(\mathcal{S}(t))^{2d}} = \mathbf{P}^s_{(\mathcal{S}(t))^{2d}} \quad \forall t \in I_j$$

and (5.192) we deduce

$$\lim_{k\to\infty}\int_0^T\int_\Omega c_k\theta_k\cdot\mathbf{P}^s_{(\mathcal{S}(t))^{2d}}(\theta_k^\alpha)\,dxdt = \int_0^T\int_\Omega c\theta\cdot\mathbf{P}^s_{(\mathcal{S}(t))^{2d}}(\theta^\alpha)\,dxdt.$$

for $\alpha \in (0,1/2)$. The following step is to prove that

$$\lim_{k\to\infty}\int_0^T\int_\Omega c_k\theta_k^{\alpha+1}\,dxdt = \int_0^T\int_\Omega c\theta^{\alpha+1}\,dxdt, \tag{5.193}$$

for $\alpha \in (0,1/2)$. From Corollary A.2 and (5.173), we have for $s \in (0,1/3)$

$$\int_0^T \left\|\theta_k^\alpha(t,\cdot) - \mathbf{P}^s_{(S_k(t))^d}\theta_k^\alpha(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant C\left(d^{2(1/3-s)} + \frac{1}{k}\right). \tag{5.194}$$

and

$$\int_0^T \left\|\theta_k^\alpha(t,\cdot) - \mathbf{P}^s_{(S(t))^d}\theta_k^\alpha(t,\cdot)\right\|^2_{H^s(\Omega)} dt \leqslant Cd^{2(1/3-s)}. \tag{5.195}$$

Then, using (5.194) and (5.195) in

$$\begin{aligned}\int_0^T\int_\Omega c_k\theta_k^{\alpha+1}\,dxdt - \int_0^T\int_\Omega c\theta^{\alpha+1}\,dxdt &= \int_0^T\int_\Omega c_k\theta_k\mathbf{P}^s_{(\mathcal{S})^d}(\theta_k^\alpha)\,dxdt - \int_0^T\int_\Omega c\theta\mathbf{P}^s_{(\mathcal{S})^d}(\theta^\alpha)\,dxdt \\ &+ \int_0^T\int_\Omega c_k\theta_k(\theta_k^\alpha - \mathbf{P}^s_{(\mathcal{S})^d}(\theta_k^\alpha))\,dxdt - \int_0^T\int_\Omega c\theta(\theta^\alpha - \mathbf{P}^s_{(\mathcal{S})^d}(\theta^\alpha))\,dxdt,\end{aligned}$$

we deduce (5.193). Since, $c_k \to c$ strongly in $C^\infty(0,T;L^p(\Omega))$ for $p \in [1,+\infty)$ we deduce

$$\lim_{k\to\infty} \int_0^T \int_\Omega \theta_k^{\alpha+1}\ dxdt = \int_0^T \int_\Omega \theta^{\alpha+1}\ dxdt$$

for $\alpha \in (0,1/2)$. Finally, by the weak convergence (5.171) we conclude

$$\theta_k \to \theta \text{ strongly in } L^r((0,T)\times\Omega) \text{ for all } r \in [1,5/3). \tag{5.196}$$

Using (5.196), we deduce that

$$\theta_k \to \theta \text{ almost everywhere in } [0,T]\times\Omega. \tag{5.197}$$

Then, using (5.74), (5.197) and Fatou's Lemma we deduce

$$\theta \in L^\infty(0,T;L^1(\Omega)).$$

5.5.4 Passing to the limit in the equations

Let us fix $\alpha \in (0,2)$. In this section we will pass to the limit the weak formulation of the momentum equation (5.67) first and after that, the weak formulation of the energy equation (5.69). The weak formulation of the momentum equation (5.31) involves discontinuous test functions. However, equation (5.67) involves H^1 test functions. Therefore, following the approach of [**?**](see also [**?**]), we work with a special sequence of test functions defined in the following lemma:

Lemma 5.5. *Let Ω a $\mathcal{C}^{1,1}$ bounded domain and, $(h_k,R_k)_{k\in\mathbb{N}}$ a sequence such that*

$$(h_k,R_k) \overset{*}{\rightharpoonup} (h,R) \quad \text{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3\times SO(3)),$$
$$(h_k,R_k) \to (h,R) \quad \text{strongly in } \mathcal{C}([0,T];\mathbb{R}^3\times SO(3)).$$

We define $\mathcal{S}_k = \widehat{\mathcal{S}}(h_k,R_k)$ and $\mathcal{S} := \widehat{\mathcal{S}}(h,R)$, where $\mathcal{S}$ is defined in (5.1). Let us fix $\alpha > 0$ and assume $v \in \mathcal{T}_T(\mathcal{S})$ (see (5.28)). Then, there exists a sequence

$$v_k \in W^{1,\infty}(0,T;L^2(\Omega))^3 \cap L^\infty(0,T;H^1(\Omega))^3 \tag{5.198}$$

with $v_k \cdot \widehat{n} = 0$ on $\partial\Omega$ and such that:

$$v_k = \mathbb{1}_{\mathcal{F}_k} v_f + \mathbb{1}_{\mathcal{S}_k} v_{s,k} \tag{5.199}$$

where

$$v_{s,k} \in L^2(0,T;H^1_\sigma(\mathcal{S}_k))^3, \tag{5.200}$$

$$\|\mathbb{1}_{\mathcal{S}_k}(v_{s,k} - v_{\mathcal{S}})\|_{\mathcal{C}[0,T];L^p(\Omega)} = O(k^{-3/4\alpha(1/p-1/6)}), \tag{5.201}$$

for all $p \in [2,6]$,

$$v_k \to v \text{ strongly in } \mathcal{C}([0,T];L^p(\Omega))^3, \tag{5.202}$$

for all $p \in [1,4)$,

$$\|v_k\|_{\mathcal{C}([0,T];H^1(\Omega))} = O(k^{\alpha/2}), \tag{5.203}$$

and

$$\mathbb{1}_{\mathcal{S}_k}\left(\frac{\partial}{\partial t} + P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)\cdot\nabla\right) v_n \overset{*}{\rightharpoonup} \mathbb{1}_{\mathcal{S}}\left(\frac{\partial}{\partial t} + P^{\mathcal{R}}_{\mathcal{S}}(u)\cdot\nabla\right) v \text{ weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{5.204}$$

Proof. We define

$$v_k = \overline{\Phi}_{h_k,R_k}\left(\widetilde{\Lambda}^{\delta_1,1/k^{3\alpha/4}}(\Phi_{h,R}(v_f),\Phi_{h,R}(v_s))\right), \tag{5.205}$$

where $\widetilde{\Lambda}^{\delta_1,\delta_2}$ is defined in (A.1.5), and Φ and $\overline{\Phi}$ are defined in (5.41) and (5.42). Then, it follows that (5.199), (5.200), (5.201) and (5.203) hold. Using (5.201) and (5.178) we deduce (5.202). Deriving (5.205), we deduce

$$\frac{\partial v_{s,k}}{\partial t}+(P_{S_k}u_k\cdot\nabla)\,v_{s,k}-\omega_k\times v_{s,k}\overset{*}{\rightharpoonup}\frac{\partial v_s}{\partial t}+(P_S u\cdot\nabla)\,v_s-\omega\times v_s \text{ weakly star } L^\infty(0,T;L^2(\Omega)).$$

Then, using (5.201) and that ω_k is bounded in $L^\infty(0,T;\mathbb{R}^3)$, we deduce (5.204). □

We assume that $v\in\mathcal{T}_T(\mathcal{S})$ (according to definition (5.28)) such that $v_f\in\mathcal{C}^1([0,T];L^q(\Omega))$ for $q\geqslant 4$ and $v(T,\cdot)=0$. By Lemma A.8 there exists a sequence of functions v_k satisfying (5.198)-(5.204). Taking $\varphi=v_k$ in (5.67) and integrating in $(0,T)$, we obtain

$$\begin{aligned}
-\rho\int_0^T\int_\Omega\left(\frac{\partial v_k}{\partial t}+(Q_{S_k}(\overline{u_k})\cdot\nabla)v_k\right)\cdot u_k\,dxdt+\int_0^T\int_\Omega 2\mu_k(\theta_k)D(u_k):D(v_k)\,dxdt\\
-\int_0^T\int_\Omega(p_{1,k}+p_{2,k})\operatorname{div}(v_k)\,dxdt+\nu_\Omega\int_0^T\int_{\partial\Omega}u_k\cdot v_k\,dxdt\\
+\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}_k}((u_k-P^{\mathcal{R}}_{\mathcal{S}_k}(u_k))\times\widehat{n})\cdot((v_k-P^{\mathcal{R}}_{\mathcal{S}_k}(v_k))\times\widehat{n})\,dxdt\\
+k\int_0^T\int_{\mathcal{S}_k}(u_k-P^{\mathcal{R}}_{\mathcal{S}_k}(u_k))\cdot(v_k-P^{\mathcal{R}}_{\mathcal{S}_k}(v_k))\,dxdt=\rho\int_\Omega u_0\cdot v_k(0,x)\,dx\\
+\rho\int_0^T\int_\Omega b\cdot v_k\,dxdt.
\end{aligned}\tag{5.206}$$

To pass to limit all the terms of the above equation we proceed as follows:

- To pass to the limit the first term of (5.206) we notice that

$$\begin{aligned}
\rho\int_0^T\int_\Omega\left(\frac{\partial v_k}{\partial t}+(Q_{S_k}(\overline{u_k})\cdot\nabla)\,v_k\right)u_k\,dxdt=\rho\int_0^T\int_\Omega\mathbb{1}_{F_k}\frac{\partial v_f}{\partial t}\cdot u_k\,dxdt\\
+\rho\int_0^T\int_\Omega\mathbb{1}_{F_k}(Q_{S_k}(\overline{u_k})\cdot\nabla)\,v_f\cdot u_k\,dxdt\\
+\rho\int_0^T\int_\Omega\mathbb{1}_{S_k}\left(\frac{\partial v_{s,k}}{\partial t}+(P_{S_k}(\overline{u_k})\cdot\nabla)\,v_{s,k}\right)\cdot u_k\,dxdt.
\end{aligned}$$

  Combining (5.164) and (5.181) we deduce

$$\rho\int_0^T\int_\Omega\mathbb{1}_{\mathcal{F}_k}\frac{\partial v_f}{\partial t}\cdot u_k\,dxdt\to\rho\int_0^T\int_\Omega\mathbb{1}_{\mathcal{F}}\frac{\partial v_f}{\partial t}\cdot u\,dxdt.$$

  Relation (5.56), (5.178), (5.177) and (5.73) imply

$$\mathbb{1}_{\mathcal{F}_k}(Q_{S_k}(\overline{u_k})-u_k)\to 0\quad\text{strongly in } L^2(0,T;L^p(\Omega))$$

for all $p < 4$. Gathering the above limit with (5.181) and (5.164), we deduce

$$\mathbb{1}_{\mathcal{F}_k} Q_{S_k}(\overline{u_k}) \to \mathbb{1}_{\mathcal{F}} u \text{ strongly in } L^2(0,T;L^p(\Omega)) \tag{5.207}$$

for all $p < 4$. Combining this with (5.181) and (5.199), we obtain

$$\rho \int_0^T \int_{\mathcal{F}_k} (Q_{S_k}(\overline{u_k}) \cdot \nabla) v_k \cdot u_k \, dxdt \to \rho \int_0^T \int_{\mathcal{F}} (u \cdot \nabla) v_f \cdot u \, dxdt.$$

Finally, from (5.56), (5.164) and (5.204), we deduce

$$\int_0^T \int_\Omega \mathbb{1}_{S_k} \left(\frac{\partial v_k}{\partial t} + (P_{S_k}(\overline{u_k}) \cdot \nabla) \, v_k \right) \cdot u_k \, dxdt \to \int_0^T \int_\Omega \mathbb{1}_S \left(\frac{\partial v}{\partial t} + (P_{\mathcal{S}} u \cdot \nabla) \, v \right) \cdot u \, dxdt.$$

- Since $v_n \in L^2(0,T;H^1(\Omega))$ we have that

$$\operatorname{div}(v_k) = \mathbb{1}_{\mathcal{F}_k} \operatorname{div}(v_f) \tag{5.208}$$

and

$$D(v_k) = \mathbb{1}_{\mathcal{F}_k} D(v_f) + \mathbb{1}_{\mathcal{S}_k} D(v_{s,k}). \tag{5.209}$$

Using (5.209) we have that

$$\int_0^T \int_\Omega 2\mu_k(\theta_k) D(u_k) : D(v_k) \, dxdt = \int_0^T \int_{\mathcal{F}_k} 2\mu_k(\theta_k) D(u_k) : D(v_f) \, dxdt$$
$$+ k^{-2} \int_0^T \int_{\mathcal{S}_k} 2\mu_k(\theta_k) D(u_k) : D(v_{s,k}) \, dxdt.$$

Using (5.73), Hölder's inequality and (5.203) we deduce

$$k^{-2} \int_0^T \int_{\mathcal{S}_k} 2\mu(\theta_k) \left| D(u_k) : D(v_{s,k}) \right| \, dxdt \leqslant C k^{\alpha/2 - 1}.$$

Then, choosing $\alpha < 2$ and using Hypothesis (5.1), (5.170) and (5.196) we conclude

$$\int_0^T \int_\Omega 2\mu_k(\theta_k) D(u_k) : D(v_k) \, dxdt \to \int_0^T \int_{\mathcal{F}} 2\mu(\theta) D(u) : D(v_f) \, dxdt.$$

- To deal with the boundary term in $\partial\Omega$ we use the weak convergence (5.184) and $v_k = v_f$ in $\partial\Omega$ and we deduce:

$$\nu_\Omega \int_0^T \int_{\partial\Omega} u_k \cdot v_k \, dxdt = \nu_\Omega \int_0^T \int_{\partial\Omega} u_k \cdot v_f \, dxdt \to \nu_\Omega \int_0^T \int_{\partial\Omega} u \cdot v_f \, dxdt.$$

- To deal with the boundary term in $\partial\mathcal{S}_k$, as usual, we make the change of variables:

$$\nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}_k} ((u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)) \times \widehat{n}) \cdot ((v_k - P^{\mathcal{R}}_{\mathcal{S}}(v_k)) \times \widehat{n}) \, dxdt =$$
$$\nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}_0} (R_k(U^k_f - U^k_s) \times \widehat{n}) \cdot (R_k(V^k_f - V^k_s) \times \widehat{n}) \, dxdt,$$

where

$$U^k = \overline{\Phi}_{h_k,R_k}(u_k), \tag{5.210}$$
$$U^k_s = \overline{\Phi}_{h_k,R_k}(P^{\mathcal{R}}_{\mathcal{S}_k} u_k), \tag{5.211}$$
$$V^k_f = \overline{\Phi}_{h_k,R_k}(v_f)$$

and

$$V^k_s = \overline{\Phi}_{h_k,R_k}(P^{\mathcal{R}}_{\mathcal{S}_k} v_k).$$

By (5.181), Lemma A.2 of [?], an interpolation inequality and the continuity of the trace operator we deduce that

$$U^k \to U = \overline{\Phi}_{h,R}(u) \text{ strongly in } L^2(0,T;H^s(\partial\mathcal{S}_0)),$$

for all $s<1$. In particular,

$$U^k \to U = \overline{\Phi}_{h,R}(u) \text{ strongly in } L^2((0,T)\times\partial\mathcal{S}_0)), \tag{5.212}$$

Similarly, using (5.182) we deduce that

$$U^k_s \rightharpoonup U_s = \overline{\Phi}_{h,R}(P^{\mathcal{R}}_{\mathcal{S}}(u)) \text{ strongly in } L^2((0,T)\times\partial\mathcal{S}_0). \tag{5.213}$$

Moreover, by (5.202),

$$V^k_f \to V_f = \overline{\Phi}_{h,R}(v_f) \text{ strongly in } L^2(0,T;H^{1/2}(\partial\mathcal{S}_0))$$

and

$$V^k_s \to V_s = \overline{\Phi}_{h,R}(P^{\mathcal{R}}_{\mathcal{S}} v) \text{ strongly in } L^2(0,T;H^{1/2}(\partial\mathcal{S}_0)).$$

Gathering the above convergences, we conclude

$$\nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}_k} ((u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k))\times\widehat{n})\cdot((v_k - P^{\mathcal{R}}_{\mathcal{S}_k}(v_k))\times\widehat{n})\ dxdt \to$$
$$\nu_{\mathcal{S}} \int_0^T \int_{\partial\mathcal{S}} ((u - P^{\mathcal{R}}_{\mathcal{S}}(u))\times\widehat{n})\cdot((v - P^{\mathcal{R}}_{\mathcal{S}}(v))\times\widehat{n})\ dxdt.$$

- For the pressure term, using that $\operatorname{div}(v_n) = \mathbb{1}_{\mathcal{F}_k}\operatorname{div}(v_f)$ we deduce

$$\int_0^T \int_\Omega (p_{1,k}+p_{2,k})\operatorname{div}(v_k)\ dxdt = \int_0^T \int_{\mathcal{F}_k} p_{1,k}\operatorname{div}(v_f)\ dxdt + \int_0^T \int_\Omega p_2 \operatorname{div}(v_k).$$

Since $\operatorname{div}(v_f)\in\mathcal{C}([0,T];L^q(\Omega))$ and since $\mathbb{1}_{\mathcal{F}_k}p_{1,k} \rightharpoonup \mathbb{1}_{\mathcal{F}}p_1$ in $L^{5/3}((0,T)\times\Omega)$ we conclude

$$\int_0^T \int_{\mathcal{F}_k} p_{1,k}\operatorname{div}(v_f)\ dxdt \to \int_0^T \int_{\mathcal{F}} p_1 \operatorname{div}(v_f)\ dxdt.$$

To deal with the convergence of $p_{2,k}$, first we apply the Helmholtz decomposition to the test function v_k:

$$v_k = v_{0,k} + \nabla z_k,$$

where $z_k \in W^{2,2}(\Omega)$ is the solution to the problem:

$$\begin{aligned} \Delta z_k &= \operatorname{div}(v_k) \quad \text{in } \Omega, \\ \nabla z_k \cdot \widehat{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then, using (5.185) we have

$$\int_0^T \int_\Omega p_{2,k} \operatorname{div}(v_k) = \int_0^T \int_{(\mathcal{S}_k)^{\delta_1} \setminus \mathcal{S}_k} (Q_{\mathcal{S}_k}(\overline{u_k}) \cdot \nabla) \nabla z_k \cdot u_k \; dxdt. \tag{5.214}$$

To pass to the limit the right hand side of (5.214), we notice that

$$\|\nabla z_k\|_{W^{1,q}(\Omega)} = \|\nabla z_k\|_{L^q(\Omega)} + \left\|\nabla^2 z_k\right\|_{L^q(\Omega)} \leqslant \|\operatorname{div}(v_k)\|_{L^q(\Omega)} = \|\operatorname{div}(v_f)\|_{L^q(\mathcal{F}_k)}.$$

Then since $\operatorname{div}(v_f) \in \mathcal{C}([0,T]; L^q(\Omega))$ we conclude that

$$\nabla^2 z_k \overset{*}{\rightharpoonup} \nabla^2 z \text{ weakly star in } L^\infty(0,T; L^q(\Omega)), \tag{5.215}$$

where z_f is the solution to

$$\begin{aligned} \Delta z &= \operatorname{div}(v_f) \quad \text{in } \Omega, \\ \nabla z \cdot \widehat{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using that

$$\mathbb{1}_{\mathcal{F}_k} Q_{S_k}(\overline{u_k}) \to \mathbb{1}_{\mathcal{F}} u \text{ strongly in } L^2(0,T; L^p(\Omega)) \tag{5.216}$$

for all $p < 4$ and (5.215) we deduce:

$$\int_0^T \int_{(\mathcal{S}_k)^{\delta_1} \setminus \mathcal{S}_k} (Q_{\mathcal{S}_k}(\overline{u_k}) \cdot \nabla) \nabla z \cdot u_k \; dxdt \to \int_0^T \int_{(\mathcal{S})^{\delta_1} \setminus \mathcal{S}} (u \cdot \nabla) \nabla z \cdot u \; dxdt.$$

We call

$$p = p_1 + p_2, \tag{5.217}$$

where p_2 is such that nd $\int_\Omega p_2 \; dx = 0$ and is the solution of the equation

$$\int_\Omega p_2 \Delta z \; dxdt = \int_{(\mathcal{S})^{\delta_1} \setminus \mathcal{S}} (u \cdot \nabla) \nabla z \cdot u \; dxdt, \tag{5.218}$$

for all $z \in H^2(\Omega)$ and $\nabla z \cdot \widehat{n} = 0$. Then, as we did for (5.187), we can prove that

$$\int_0^T \|p_2\|_{L^{5/3}(\mathcal{F}(t))}^{5/3} \; dt \leqslant C.$$

Then we deduce that $p \in L^{5/3}(0, T : L^{5/3}(\mathcal{F}))$ and we conclude

$$\int_0^T \int_\Omega (p_{1,k} + p_{2,k}) \operatorname{div}(v_k) \; dxdt \to \int_0^T \int_{\mathcal{F}} p \operatorname{div}(v_f) \; dxdt.$$

- To deal with the penalization term, we use (5.167) and (5.201) to deduce that

$$k \int_0^T \int_{\mathcal{S}_k} (u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)) \cdot (v_k - P^{\mathcal{R}}_{\mathcal{S}_k}(v_k)) \, dxdt \leqslant ck^{1/2-\alpha/2},$$

  and we conclude

$$k \int_0^T \int_{\mathcal{S}_k} (u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k)) \cdot (v_k - P^{\mathcal{R}}_{\mathcal{S}_k}(v_k)) \, dxdt \to 0.$$

- By (5.202), we deduce

$$\rho \int_\Omega u_0 \cdot v_k(0,x) \, dx + \rho \int_0^T \int_\Omega b \cdot v_k \, dxdt \to \rho \int_\Omega u_0 \cdot v(0,x) \, dx + \rho \int_0^T \int_\Omega b \cdot v \, dxdt.$$

We thus conclude that u satisfies (5.31) for all $v \in \mathcal{T}_T(\mathcal{S})$ with $v_f \in \mathcal{C}^1([0,T]; W^{1,q}(\Omega))^3$ for $q \geqslant 4$. However, since p_2 satisfies (5.218), as we did for (5.187), we conclude that

$$\int_0^T \|p_2\|^{5/3}_{L^{5/3}(\mathcal{F}(t))} \, dt \leqslant C,$$

and then, we deduce that u satisfies (5.31) for all $v \in \mathcal{T}_T(\mathcal{S})$ with $v_f \in \mathcal{C}^1([0,T]; W^{1,5/2}(\Omega))^3$.

Using standard techniques, see for example [?, pp. 290-291], by (5.161), (5.170) deduce that (5.36).

To pass to limit (5.69) we face less dificulties regarding the test functions since, in the weak formulation (5.32), the test functions are scalar scalar functions in $\mathcal{C}^1([0,T]; W^{1,\infty}(\Omega))$ such that $\nabla\varphi = 0$ in $\mathcal{S}(t)$. However, we still need to find a sequence approximating the test functions to cancel the penalization term.

Assume $\varphi \in \mathcal{C}^1([0,T]; W^{1,\infty}(\Omega))$ with $\nabla\varphi = 0$ in $\mathcal{S}(t)$ and $\operatorname{supp}\varphi \subset \Omega_\eta$, $\eta > 0$, $\varphi(T,x) = 0$. We set

$$\varphi_k(t,x) := \varphi\left(t, h(t) + R(t)R_k^*(t)(x - h_k(t))\right). \tag{5.219}$$

A quick calculation reveals that

$$\nabla\varphi_k(t,x) = R_k(t)R^*(t) \cdot \nabla\varphi\left(t, h(t) + R(t)R_k^*(t)(x - h_k(t))\right). \tag{5.220}$$

By (5.219), for k large enough we have

$$\varphi_k \in \mathcal{C}([0,T]; W^{1,\infty}(\Omega)).$$

Moreover, by (5.220):

$$\nabla\varphi_k = 0 \quad \text{in } \mathcal{S}_k. \tag{5.221}$$

Using (5.162), we deduce

$$\varphi_k \to \varphi \text{ strongly in } \mathcal{C}([0,T]; W^{1,p}(\Omega)), \text{ for } p \in [1,\infty). \tag{5.222}$$

Applying $\frac{\partial}{\partial t}$ to the identity

$$\varphi_k(t, h_k + R_k y) = \varphi\left(t, h(t) + R(t)y\right),$$

we deduce

$$\frac{\partial \varphi_k}{\partial t} + P^{\mathcal{R}}_{\mathcal{S}_k} u_k \cdot \nabla \varphi_k \to \frac{\partial \varphi}{\partial t} + P^{\mathcal{R}}_{\mathcal{S}} u \cdot \nabla \varphi \text{ strongly in } L^\infty((0,T)\times\Omega).$$

Finally, combining (5.222) with (5.166), we conclude

$$\frac{\partial \varphi_k}{\partial t} \overset{*}{\rightharpoonup} \frac{\partial \varphi}{\partial t} \text{ weakly star in } L^\infty((0,T)\times\Omega). \tag{5.223}$$

In (5.69) is not possible to pass to the limit the term $\int_0^T \int_\Omega \mu_k(\theta_k) \left| D(u_k) \right|^2 \varphi_k \; dxdt$. However, using (5.67) with $v = u_k \varphi_k$ and identity (5.221), we obtain

$$\begin{aligned}
-\int_0^T \int_\Omega \left(k \mathbb{1}_{S_k} \left| u_k - P^{\mathcal{R}}_{\mathcal{S}_k}(u_k) \right|^2 + 2\mu_k(\theta_k) \left| D(u_k) \right|^2 \right) \varphi_k \; dxdt = -\frac{\rho}{2} \int_\Omega |u_0|^2 \varphi_k(0,x) \; dx \\
-\frac{\rho}{2} \int_0^T \int_\Omega |u_k|^2 \left(\frac{\partial \varphi_k}{\partial t} + Q_{\mathcal{S}_k}(u_k) \cdot \nabla \varphi_k \right) \; dxdt + \int_0^T \int_\Omega 2\mu_k(\theta_k) D(u_k) : u_k \otimes \nabla \varphi_k \; dxdt \\
+ \nu_\Omega \int_0^T \int_{\partial\Omega} |u_k|^2 \varphi_k \; dxdt + \nu_{\mathcal{S}} \int_0^T \int_{\partial \mathcal{S}_k} \left| (u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k) \times \widehat{n} \right|^2 \varphi_k \; dxdt \\
- \int_0^T \int_\Omega (p_{1,k} + p_{2,k}) u_k \cdot \nabla \varphi_k \; dxdt - \rho \int_0^T \int_\Omega b \cdot u_k \varphi_k \; dxdt.
\end{aligned}$$

We combine the above equation with (5.69) and we deduce the total energy equation:

$$\begin{aligned}
-\rho \int_0^T \int_\Omega \left(c_k \theta_k + \frac{|u_k|^2}{2} \right) \left(\frac{\partial \varphi_k}{\partial t} + Q_{S_k}(\overline{u_k}) \cdot \nabla \varphi_k \right) \; dxdt + \int_0^T \int_\Omega \kappa(\theta_k) \nabla \theta_k \cdot \nabla \varphi_k \; dx \\
+ \nu_\Omega \int_0^T \int_{\partial\Omega} |u_k|^2 \varphi_k \; dxdt + \nu_{\mathcal{S}} \int_0^T \int_{\partial \mathcal{S}_k} \left| (u_k - P^{\mathcal{R}}_{\mathcal{S}_k} u_k) \times n \right|^2 \varphi_k \; dxdt \\
+ \int_0^T \int_\Omega 2\mu(\theta_k) D(u_k) : u_k \otimes \nabla \varphi_k \; dxdt - \int_0^T \int_\Omega (p_{1,k} + p_{2,k}) u_k \cdot \nabla \varphi_k \; dxdt \\
= E_0(\varphi_k(0,x)) + \rho \int_0^T \int_\Omega (w + b \cdot u_k) \varphi_k \; dxdt.
\end{aligned} \tag{5.224}$$

To conclude, we need to pass to the limit the terms in the above equation as $k \to \infty$:

- Using (5.181)and (5.222), we deduce:

$$E_0(\varphi_k(0,x)) + \rho \int_0^T \int_\Omega (w + b \cdot u_k) \varphi_k \; dxdt \to E_0(\varphi(0,x)) + \rho \int_0^T \int_\Omega (w + b \cdot u) \varphi \; dxdt.$$

- Using (5.170), (5.181), (5.196), (5.222) and Hypothesis 5.1, we deduce:

$$\int_0^T \int_\Omega 2\mu_k(\theta_k) D(u_k) : u_k \otimes \nabla \varphi_k \; dxdt \to \int_0^T \int_\Omega 2\mu(\theta) D(u) : u \otimes \nabla \varphi \; dxdt.$$

- Using (5.172), (5.222) and Hypothesis 5.1 we deduce:

$$\int_0^T \int_\Omega \kappa(\theta_k) \nabla \theta_k \cdot \nabla \varphi_k \; dx \to \int_0^T \int_\Omega \kappa(\theta) \nabla \theta \cdot \nabla \varphi \; dx.$$

- Using (5.221), we decompose the first term of (5.224) as follows

$$
\begin{aligned}
&\rho\int_0^T\int_\Omega\left(c_k\theta_k+\frac{|u_k|^2}{2}\right)\left(\frac{\partial\varphi_k}{\partial t}+Q_{S_k}(\overline{u_k})\cdot\nabla\varphi_k\right)\,dxdt\\
&=\rho\int_0^T\int_\Omega\left(c_k\theta_k+\frac{|u_k|^2}{2}\right)\frac{\partial\varphi_k}{\partial t}dxdt+\rho\int_0^T\int_{\mathcal{F}_k}\left(c_k\theta_k+\frac{|u_k|^2}{2}\right)Q_{S_k}(\overline{u_k})\cdot\nabla\varphi_k\,dxdt.
\end{aligned}
$$

Using (5.165), (5.180), (5.196) and (5.223) we deduce

$$
\int_0^T\int_\Omega\left(c_k\theta_k+\frac{|u_k|^2}{2}\right)\frac{\partial\varphi_k}{\partial t}dxdt\to\int_0^T\int_\Omega\left(c\theta+\frac{|u|^2}{2}\right)\frac{\partial\varphi}{\partial t}dxdt.
$$

To pass to the limit the second term we notice that, by (5.74), (5.161), (5.165) (5.181), (5.196) we have

$$
c_k\theta_k+\frac{|u_{f,k}|^2}{2}\to c\theta+\frac{|u_f|^2}{2}\text{ strongly in }L^2(0,T;L^q(\Omega))\text{ for all }q\in(4/3,3/2).
$$

Then, using (5.216) and (5.222) we deduce

$$
\rho\int_0^T\int_{\mathcal{F}_k}\left(c_k\theta_k+\frac{|u_k|^2}{2}\right)Q_{S_k}(\overline{u_k})\cdot\nabla\varphi_k\,dxdt\to\rho\int_0^T\int_{\mathcal{F}}\left(c\theta+\frac{|u|^2}{2}\right)u\cdot\nabla\varphi\,dxdt.
$$

- Using (5.184) and (5.222) we deduce that

$$
\nu_\Omega\int_0^T\int_{\partial\Omega}|u_k|^2\varphi_k\,dxdt\to\nu_\Omega\int_0^T\int_{\partial\Omega}|u|^2\varphi\,dxdt.
$$

- To deal with the boundary term in $\partial\mathcal{S}_k$, we study the integral in $\partial\mathcal{S}_0$:

$$
\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}_k}\left|(u_k-P^{\mathcal{R}}_{\mathcal{S}_k}u_k)\times\widehat{n}\right|^2\varphi_k\,dxdt=\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}_0}\left|R_k(U^k-U^k_s)\times\widehat{n}\right|^2\varphi(t,h+Ry)\,dydt,
$$

where U^k and U^k_s are defined in (5.210) and (5.211).

As in (5.212) and (5.213), we deduce that

$$
U^k\rightharpoonup U=\overline{\Phi}_{h,R}(u)\text{ strongly in }L^2((0,T)\times\partial\mathcal{S}_0)),
$$

and

$$
U^k_s\rightharpoonup U_s=\overline{\Phi}_{h,R}(P^{\mathcal{R}}_{\mathcal{S}}(u))\text{ strongly in }L^2((0,T)\times\partial\mathcal{S}_0).
$$

Then, using (5.212) and (5.213) we conclude

$$
\begin{aligned}
&\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}_0}\left|R_k(U^k-U^k_{\mathcal{S}})\times\widehat{n}\right|^2\varphi\,dxdt\\
&\quad\to\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}_0}|R(U-U_{\mathcal{S}})\times\widehat{n}|^2\varphi\,dxdt=\nu_{\mathcal{S}}\int_0^T\int_{\partial\mathcal{S}}\left|(u-P^{\mathcal{R}}_{\mathcal{S}}u)\times\widehat{n}\right|^2\varphi_k\,dxdt.
\end{aligned}
$$

- We want to pass to the limit the term

$$\int_0^T \int_\Omega p_k \operatorname{div}(\varphi_k u_k)\ dxdt.$$

Since $\operatorname{div}(u_k) = 0$ and $p_k = p_{1,k} + p_{2,k}$

$$\int_0^T \int_\Omega p_k \operatorname{div}(\varphi_k u_k)\ dx = \int_0^T \int_\Omega p_{1,k} \nabla\varphi_k \cdot u_k\ dxdt + \int_0^T \int_\Omega p_{2,k} \operatorname{div}(\varphi_k u_k)\ dxdt$$

For the first term in the right hand side of the above equation, using (5.164),(5.181), (5.188) and (5.222) we deduce:

$$\int_0^T \int_\Omega p_{1,k} u_k \cdot \nabla\varphi_k\ dxdt = \int_0^T \int_{\mathcal{F}_k} p_{1,k} u_k \cdot \nabla\varphi_k\ dxdt \to \int_0^T \int_{\mathcal{F}} p_1 u \cdot \nabla\varphi\ dxdt. \quad (5.225)$$

To deal with the convergence of the term related to $p_{2,k}$ we use the Helmholtz decomposition of $\varphi_k u_k$:

$$\varphi_k u_k = (\varphi_k u_k)_0 + \nabla z_k,$$

where $z_k \in W^{2,2}(\Omega)$, such that

$$\begin{aligned} \Delta z_k &= u_k \nabla\varphi_k \quad && \text{in } \Omega, \\ \nabla z_k \cdot \widehat{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (5.226)$$

Then,

$$\int_0^T \int_\Omega p_{2,k} u_k \cdot \nabla\varphi_k\ dxdt = \rho \int_0^T \int_{\mathcal{S}_k^{\delta_1} \setminus \mathcal{S}_k} Q_{\mathcal{S}_k}(u_k) \otimes u_k : \nabla^2 z_k\ dxdt.$$

By (5.226) we have

$$\|\nabla z\|_{W^{1,q}} \leqslant C \|u_k \nabla\varphi_k\|_{L^q} \text{ and } \|z\|_{W^{1,q}(\Omega)} \leqslant C \|u_k \varphi_k\|_{L^q(\Omega)}.$$

Combining the above relations with Hölder's inequality for $q > 12/7$ we deduce that

$$\|z_k\|_{L^q(\Omega)} + \|\nabla z_k\|_{L^q(\Omega)} + \|\nabla^2 z_k\|_{L^q(\Omega)} \leqslant \|u_k\|_{L^2(\Omega)}(\|\nabla\varphi_k\|_{L^m(\Omega)} + \|\varphi_k\|_{L^m(\Omega)}),$$

for $m > 12$. Then, using $u_k \overset{*}{\rightharpoonup} u$ weakly star in $L^\infty(0,T; L^2(\Omega))$ and (5.222), we conclude that

$$\nabla^2 z_k \overset{*}{\rightharpoonup} \nabla^2 z \text{ weakly star in } L^\infty(0,T; L^q(\Omega)) \text{ for } q > 12/7.$$

where z is the solution to the problem

$$\begin{aligned} \Delta z &= u \nabla\varphi \quad && \text{in } \Omega, \\ \nabla z \cdot \widehat{n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then, using that $u_{f,k} \to u_f$ strongly in $L^{10/3}((0,T) \times \Omega)$ and (5.216)

$$\int_0^T \int_{\mathcal{S}_k^{\delta_1} \setminus \mathcal{S}_k} Q_{\mathcal{S}_k}(u_k) \otimes u_k : \nabla^2 z_k\ dxdt \to \int_0^T \int_{\mathcal{S}^{\delta_1} \setminus \mathcal{S}} u \otimes u : \nabla^2 z\ dxdt.$$

Gathering the above convergence with (5.218), (5.217) and (5.225) we conclude that

$$\int_0^T \int_\Omega (p_{1,k} + p_{2,k}) u_k \cdot \nabla\varphi_k\ dxdt \to \int_0^T \int_{\mathcal{F}} pu \cdot \nabla\varphi\ dxdt.$$

This finishes the proof of Theorem 5.1.3.

Appendix A

Lemmas

A.1 Junction of solenoidal fields

Here we state some technical results obtained and proved in [?]. The statements used in this article are slightly different and we thus recall the main steps of the proofs.

Theorem A.1.1. *Assume that $\delta_1 > \delta_2 > 0$. Then, there exists a family of bounded operators*

$$\Lambda^{\delta_1,\delta_2} : H^1_\sigma\left(\mathbb{R}^3\right)^3 \times \mathcal{R} \to H^1_\sigma(\mathbb{R}^3)^3,$$

such that, for all $(u_1, u_2) \in H^1_\sigma(\mathbb{R}^3) \times \mathcal{R}$ we have that:

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_2 \qquad \text{in } \mathcal{S}_0, \tag{A.1}$$

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_1 \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}_0^{\delta_1}, \tag{A.2}$$

where $\mathcal{S}_0^{\delta_1}$ is defined by (4.25), and the following inequality holds:

$$\left\|\Lambda^{\delta_1,\delta_2}(u_1, u_2) - u_1\right\|_{L^p(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0)} \leqslant C_{\delta_1,\mathcal{S}_0}\Big(\delta_2^{\frac{1}{p}-\frac{1}{6}}\left\|u_1 - u_2\right\|_{H^1(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0)} + \left\|(u_1 - u_2)\cdot\widehat{n}\right\|_{L^p(\partial\mathcal{S}_0)}\Big), \tag{A.3}$$

for $p \in [2, 6]$.

Proof. We consider an orthogonal curvilinear coordinate system (s_1, s_2, z) defined around $\partial\mathcal{S}_0$ such that $\partial\mathcal{S}_0 = \{z = 0\}$. For δ_1 small enough we have $\partial\mathcal{S}_0^{\delta_1} = \{z = \delta_1\}$. We consider $\varphi \in \mathcal{C}_0^\infty([0, 1); [0, 1])$ such that $\varphi(0) = 1$ and we define the function $\varphi_\delta(z) := \varphi(\frac{z}{\delta})$. Notice that:

$$\left\|\varphi_\delta\right\|_{L^a(\mathbb{R})} = \delta^{\frac{1}{a}}\left\|\varphi\right\|_{L^a(\mathbb{R})}. \tag{A.4}$$

We define $\Lambda^{\delta_1,\delta_2}(u_1, u_2)$ in $\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0$ as follows:

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) := V_1 + V_2 + V_3$$

where

$$V_1 = (1 - \varphi_{\delta_2})u_1 + \varphi_{\delta_2}\left(u_2 - ((u_2 - u_1)\cdot e_z)e_z\right),$$

$$V_2 = \{(u_2 - u_1)\cdot e_z\}\Big|_{z=0}\varphi_{\delta_1}e_z,$$

and V_3 solution of the system

$$\begin{cases} \operatorname{div} V_3 = -\operatorname{div}(V_1 + V_2) & \text{in } \mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0, \\ V_3 = 0. & \text{on } \partial\mathcal{S}_0^{\delta_1} \cup \partial\mathcal{S}_0. \end{cases} \tag{A.5}$$

From [?, Theorem III.3.1, p.171], the above system admits a solution since the compatibility condition holds. We can also check that

$$\Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_2 \quad \text{on } \partial\mathcal{S}_0, \quad \Lambda^{\delta_1,\delta_2}(u_1, u_2) = u_1 \quad \text{on } \partial\mathcal{S}_0^{\delta_1}.$$

Moreover we have the following properties:

$$|V_1 - u_1| \leqslant \varphi_{\delta_2} |u_2 - u_1|, \quad |V_2| \leqslant \left| (u_2 - u_1) \cdot \widehat{n} \big|_{z=0} \right|, \tag{A.6}$$

$$\operatorname{div} V_1 = -\varphi_{\delta_2} \operatorname{div}([(u_1 - u_2) \cdot e_z] e_z), \tag{A.7}$$

$$\operatorname{div} V_2 = ((u_1 - u_2) \cdot e_z)\big|_{z=0} (\varphi_{\delta_1} \operatorname{div} e_z + e_z \cdot \nabla \varphi_{\delta_1}). \tag{A.8}$$

Combining (A.6), (A.4) and a Sobolev embedding we deduce for any $p \in [1, 6]$

$$\|V_1 - u_1\|_{L^p(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \leqslant C \delta_2^{\frac{1}{p} - \frac{1}{6}} \|u_2 - u_1\|_{H^1(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \tag{A.9}$$

and

$$\|V_2\|_{L^p(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \leqslant C_{\delta_1, \mathcal{S}_0} \|(u_2 - u_1) \cdot \widehat{n}\|_{L^p(\partial\mathcal{S}_0)}. \tag{A.10}$$

Assume $q \leq 2$. From (A.7) and (A.8) we deduce that:

$$\|\operatorname{div} V_1\|_{L^q(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \leqslant C_{\delta_1, \mathcal{S}_0} \delta_2^{\frac{1}{q} - \frac{1}{2}} \|u_1 - u_2\|_{H^1\left(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0\right)}, \tag{A.11}$$

$$\|\operatorname{div} V_2\|_{L^q(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \leqslant C_{\delta_1, \mathcal{S}_0} \|(u_2 - u_1) \cdot \widehat{n}\|_{L^q(\partial\mathcal{S}_0)}. \tag{A.12}$$

Using [?, Theorem III.3.1, p.171] and a Sobolev embedding, we conclude that if $\frac{1}{p} = \frac{1}{q} - \frac{1}{3}$,

$$\begin{aligned} \|V_3\|_{L^p(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} &\leqslant C_{\delta_1, \mathcal{S}_0} \|V_3\|_{W^{1,q}(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0)} \\ &\leqslant C_{\delta_1, \mathcal{S}_0} \Big(\delta_2^{\frac{1}{p} - \frac{1}{6}} \|u_1 - u_2\|_{H^1\left(\mathcal{S}^{\delta_1} \setminus \mathcal{S}_0\right)} + \|(u_1 - u_2) \cdot \widehat{n}\|_{L^q(\partial\mathcal{S}_0)} \Big). \end{aligned} \tag{A.13}$$

Gathering (A.11), (A.10) and (A.13) yields (A.3). □

Definition A.1.2. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be two $\mathcal{C}^{1,1}$ bounded domains of $\mathbb{R}^3$. Assume $(a, Q) \in \mathbb{R}^3 \times SO(3)$ and $\delta_1 > \delta_2 > 0$ and we recall*

$$\widehat{\mathcal{S}}(a, Q) := a + Q\mathcal{S}_0 \quad \textit{and} \quad \widehat{\mathcal{F}}(a, Q) := \Omega \setminus \overline{\widehat{\mathcal{S}}(a, Q)}.$$

We define the operator $Q_{a,Q}^{\delta_1,\delta_2} \in \mathcal{L}(H^1_\sigma(\Omega)^3)$ as follows:

$$Q_{a,Q}^{\delta_1,\delta_2}(u) := \overline{\Phi}_{a,Q} \left(\Lambda^{\delta_1,\delta_2} \left(\Phi_{a,Q}(u), P^{\mathcal{R}}_{\mathcal{S}_0} \Phi_{a,Q}(u) \right) \right) \quad (u \in H^1_\sigma(\Omega)), \tag{A.14}$$

where we denote by $P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)}$ *the orthogonal projection of* $L^2(\widehat{\mathcal{S}}(a,Q))$ *onto* $\mathcal{R}$ *and, given* $(a,Q) \in \mathbb{R}^3 \times SO(3)$*, we define two operators of* $L^2_{loc}(\mathbb{R}^3)$ *as follows: assume* $v \in L^2_{loc}(\mathbb{R}^3)^3$*, then*

$$\Phi_{a,Q}(v)(y) := Q^* v\,(a + Qy)\,, \quad y \in \mathbb{R}^3$$

and

$$\overline{\Phi}_{a,Q}(v)(x) := Qv\,(Q^*(x-a))\,, \quad x \in \mathbb{R}^3.$$

Using (4.28), we can check that

$$Q^{\delta_1,\delta_2}_{a,Q}(u) = \begin{cases} u & \text{in } \Omega \setminus \widehat{\mathcal{S}}(a,Q)^{\delta_1}, \\ P^{\mathcal{R}}_{\widehat{\mathcal{S}}(a,Q)} u & \text{in } \widehat{\mathcal{S}}(a,Q). \end{cases} \tag{A.15}$$

Moreover, if $(h,R) \in L^\infty(0,T;\mathbb{R}^3 \times SO(3))$, then we deduce from (A.14) that $Q^{\delta_1,\delta_2}_{h,R}$ is a linear bounded operator in $L^2(0,T;H^1_\sigma(\Omega))^3$ into itself.

Lemma A.1. *Assume* $\delta_1 > \delta_2 > 0$ *and*

$$(h_M, R_M) \overset{*}{\rightharpoonup} (h,R) \quad \textit{weak star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)),$$
$$(h_M, R_M) \to (h,R) \quad \textit{strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)).$$

We define $\mathcal{S}_M := \widehat{\mathcal{S}}(h_M, R_M)$ *and* $\mathcal{S} := \widehat{\mathcal{S}}(h,R)$*. We also assume*

$$u_M \rightharpoonup u \quad \textit{weakly in } L^2(0,T;H^1_\sigma(\Omega))^3,$$
$$u_M \to u \quad \textit{strongly in } L^2(0,T;L^2(\Omega))^3.$$

Then we have that

$$Q^{\delta_1,\delta_2}_{\mathcal{S}_M}(u_M) \rightharpoonup Q^{\delta_1,\delta_2}_{\mathcal{S}}(u) \textit{ weakly in } L^2(0,T;H^1_\sigma(\Omega))^3 \tag{A.16}$$

and

$$Q^{\delta_1,\delta_2}_{\mathcal{S}_M}(u_M) \to Q^{\delta_1,\delta_2}_{\mathcal{S}}(u) \textit{ strongly in } L^2(0,T;L^2(\Omega))^3. \tag{A.17}$$

Proof. The proof of (A.16) and (A.17) are similar, so we only proof (A.16). We set

$$U_M := \Phi_{h_M,R_M}(u_M) \text{ and } U := \Phi_{h,R}(u).$$

Using Lemma A.2 of [?] we deduce that

$$U_M \rightharpoonup U \quad \text{weakly in } L^2(0,T;H^1_\sigma(\mathbb{R}^3))^3,$$

and thus

$$\Lambda^{\delta_1,\delta_2}(U_M, P^{\mathcal{R}}_{\mathcal{S}_0} U_M) \rightharpoonup \Lambda^{\delta_1,\delta_2}(U, P^{\mathcal{R}}_{\mathcal{S}_0} U) \text{ weakly in } L^2(0,T;H^1_\sigma(\mathbb{R}^3))^3.$$

Then, using again Lemma A.2 of [?] we conclude (A.16). □

The second type of junction we consider here is given by the following result. It corresponds to Lemma 5.3 of [?].

Theorem A.1.3. *Assume $\delta_1 > 2\delta_2 > 0$ and $s < \frac{1}{3}$. Then, there exists a family of bounded operators*

$$\widehat{\Lambda}^{\delta_1,\delta_2} : H^1_\sigma\left(\mathbb{R}^3\right)^3 \times \mathcal{R} \to H^s_\sigma(\mathbb{R}^3)^3$$

such that, for all $(u_1, u_2) \in H^1_\sigma(\mathbb{R}^3)^3 \times \mathcal{R}$ we have that:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) = u_2 \qquad \text{in } \mathcal{S}_0^{\delta_2} \tag{A.18}$$

$$\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) = u_1 \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}_0^{\delta_1}, \tag{A.19}$$

where $\mathcal{S}_0^{\delta_1}$ and $\mathcal{S}_0^{\delta_2}$ are defined in (4.25), and, for $s < \frac{1}{3}$, the following inequality holds

$$\begin{aligned}\left\|\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) - u_1\right\|_{H^s(\Omega\setminus\mathcal{S}_0^{\delta_2})} \\ \leqslant C_{\delta_1,\mathcal{S}_0}\left(\delta_2^{\frac{1}{3}-s}(\|u_1\|_{H^1(\mathcal{F}_0)} + \|u_2\|_{H^1(\mathcal{F}_0)}) + \|(u_1-u_2)\cdot\widehat{n}\|_{L^2(\partial\mathcal{S}_0)}\right).\end{aligned} \tag{A.20}$$

Proof. The proof of this theorem is similar to the proof of Theorem A.1.1. We use the same notation for the orthogonal curvilinear coordinate system (s_1, s_2, z) and for the functions φ_δ. We define $\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2)$ in $\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}$ as follows:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(u_1, u_2) = V_1 + V_2 + V_3$$

where

$$V_1 = (1 - \varphi_{\delta_2}(z - \delta_2))u_1 + \varphi_{\delta_2}(z - \delta_2)\left(u_2 - ((u_2 - u_1)\cdot e_z)e_z\right),$$

V_2 is solution of the equation

$$\operatorname{div} V_2 = -\operatorname{div} V_1 \text{ in } \mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}, \tag{A.21}$$

$$V_2 = 0 \text{ in } \partial\mathcal{S}_0^{\delta_1} \cup \partial\mathcal{S}_0^{\delta_2}, \tag{A.22}$$

and $V_3 = \nabla Y_3$ where

$$\Delta Y_3 = 0 \text{ in } \mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}, \tag{A.23}$$

$$\frac{\partial Y_3}{\partial n} = 0 \text{ in } \partial\mathcal{S}_0^{\delta_1}, \tag{A.24}$$

$$\frac{\partial Y_3}{\partial n} = (u_2 - u_1)\cdot e_z \text{ in } \partial\mathcal{S}_0^{\delta_2}. \tag{A.25}$$

One can check that the compatibility conditions are satisfied so that (A.21)-(A.22) and (A.23)-(A.25) are well-posed with the estimates

$$\|V_2\|_{H^1(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0^{\delta_2})} \le C_{\delta_1,\mathcal{S}_0}(\|u_1\|_{H^1(\mathcal{F}_0)} + \|u_2\|_{H^1(\mathcal{F}_0)}, \tag{A.26}$$

$$\|V_2\|_{L^2(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0^{\delta_2})} \le C_{\delta_1,\mathcal{S}_0}\|V_2\|_{W^{1,6/5}(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0^{\delta_2})} \le C_{\delta_1,\mathcal{S}_0}\delta_2^{1/3}(\|u_1\|_{H^1(\mathcal{F}_0)} + \|u_2\|_{H^1(\mathcal{F}_0)}), \tag{A.27}$$

and

$$\|V_3\|_{H^{1/2}(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0^{\delta_2})} \le C_{\delta_1,\mathcal{S}_0}\,\|(u_2-u_1)\cdot\widehat{n}\|_{L^2(\partial\mathcal{S}_0^{\delta_2})}.$$

Using Lemma 5.10 of [**?**], the above estimate yields

$$\|V_3\|_{H^{1/2}(\mathcal{S}_0^{\delta_1}\setminus\mathcal{S}_0^{\delta_2})} \le C_{\delta_1,\mathcal{S}_0}\left(\delta_2^{1/2}\,\|u_2-u_1\|_{H^1(\mathcal{F}_0)} + \|(u_2-u_1)\cdot\widehat{n}\|_{L^2(\partial\mathcal{S}_0)}\right). \tag{A.28}$$

We also remark that

$$(V_1 + V_2 + V_3) \cdot \widehat{n} = u_2 \cdot \widehat{n} \quad \text{on} \quad \partial \mathcal{S}_0^{\delta_2},$$
$$(V_1 + V_2 + V_3) \cdot \widehat{n} = u_1 \cdot \widehat{n} \quad \text{on} \quad \partial \mathcal{S}_0^{\delta_1}.$$

Using the definition of V_1 and (A.4) we deduce that:

$$\|V_1 - u_1\|_{L^2(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2})} \leqslant C_{\delta_1, \mathcal{S}_0} \delta_2^{\frac{1}{3}} \|u_1 - u_2\|_{H^1(\mathcal{F}_0)},$$
$$\|\nabla(V_1 - u_1)\|_{L^2(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}))} \leqslant C_{\delta_1, \mathcal{S}_0} \delta_2^{-\frac{2}{3}} \|u_1 - u_2\|_{H^1(\mathcal{F}_0)},$$

so that

$$\|V_1 - u_1\|_{H^s(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2})} \leqslant C_{\delta_1, \mathcal{S}_0} \delta_2^{\frac{1}{3} - s} \|u_1 - u_2\|_{H^1(\mathcal{F}_0)}. \tag{A.29}$$

Then combining (A.26), (A.27), (A.28) and the above estimate, we deduce (A.20). □

For the following result we define the space

$$H^s_{\mathcal{S}}(\Omega) := \{v \in H^s_\sigma(\Omega);\ D(v) = 0 \text{ in } \mathcal{S}\}$$

and the othogonal projection

$$\mathcal{P}^s_{\mathcal{S}} : H^s_\sigma(\Omega) \mapsto H^s_{\mathcal{S}}(\Omega).$$

Then, as a consequence of the previous lemma, we obtain the following result on the orthogonal projection:

Corollary A.1.4. *Let $u \in H^1(\Omega)^3$ and $(h, R) \in \mathbb{R}^3 \times SO(3)$ such that $\operatorname{dist}(\widehat{\mathcal{S}}(h, R), \partial\Omega) \geqslant \delta_1 > 0$. Then for all $d < \frac{\delta_1}{2}$ and $s \in (0, 1/3)$, we have that*

$$\left\| u - \mathcal{P}^s_{(\widehat{\mathcal{S}}(h,R))^d} u \right\|_{H^s(\Omega)} \leqslant C_{\delta_1, \mathcal{S}_0} \left(d^{\frac{1}{3} - s} \|u\|_{H^1(\widehat{\mathcal{F}}(h,R))} + \|u\|^{1/2}_{H^1(\Omega)} \left\| u - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(h,R)} u \right\|^{1/2}_{L^2(\widehat{\mathcal{S}}(h,R))} \right.$$
$$\left. + \|u\|^s_{H^1(\Omega)} \left\| u - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(h,R)} u \right\|^{1-s}_{L^2(\widehat{\mathcal{S}}(h,R))} \right). \tag{A.30}$$

Proof. We set

$$v := \overline{\Phi}_{h,R} \left(\widehat{\Lambda}^{\delta_1, d} \left(\Phi_{h,R}(u), P^{\mathcal{R}}_{\mathcal{S}_0} \Phi_{h,R}(u) \right) \right)$$

where Φ and $\overline{\Phi}$ are defined in (4.26) and (4.27). Then, by Theorem A.1.3, we have $v = P_{\widehat{\mathcal{S}}(h,R)} u$ in $\widehat{\mathcal{S}}(h, R)^d$ $v = u$ in $\Omega \setminus \widehat{\mathcal{S}}(h, R)^{\delta_1}$, and

$$\|v - u\|_{H^s(\Omega \setminus \widehat{\mathcal{S}}(h,R)^d)} \leqslant C \left(d^{\frac{1}{3} - s} \|u\|_{H^1(\widehat{\mathcal{F}}(h,R))} + \left\| (u - P_{\widehat{\mathcal{S}}(h,R)} u) \cdot \widehat{n} \right\|_{L^2(\partial \widehat{\mathcal{S}}(h,R))} \right). \tag{A.31}$$

We deduce that

$$\|v - u\|_{H^s(\Omega)} \leqslant C \left(d^{\frac{1}{3} - s} \|u\|_{H^1(\widehat{\mathcal{F}}(h,R))} + \left\| (u - P_{\widehat{\mathcal{S}}(h,R)} u) \cdot \widehat{n} \right\|_{L^2(\partial \widehat{\mathcal{S}}(h,R))} \right.$$
$$\left. + \left\| u - P_{\widehat{\mathcal{S}}(h,R)} u \right\|_{H^s(\widehat{\mathcal{S}}(h,R)^d \setminus \widehat{\mathcal{S}}(h,R))} + \left\| u - P_{\widehat{\mathcal{S}}(h,R)} u \right\|_{H^s(\widehat{\mathcal{S}}(h,R))} \right). \tag{A.32}$$

Now we have the following relations

$$\left\|(u-P_{\widehat{\mathcal{S}}(h,R)}u)\cdot\widehat{n}\right\|_{L^2(\partial\widehat{\mathcal{S}}(h,R))}\leq C\left(\|u\|^{1/2}_{H^1(\widehat{\mathcal{F}}(h,R))}+\|u\|^{1/2}_{L^2(\mathcal{S})}\right)\left\|u-P_{\widehat{\mathcal{S}}(h,R)}u\right\|^{1/2}_{L^2(\widehat{\mathcal{S}}(h,R))},$$

$$\left\|u-P_{\widehat{\mathcal{S}}(h,R)}u\right\|_{H^s(\widehat{\mathcal{S}}(h,R)^d\setminus\widehat{\mathcal{S}}(h,R))}\leq Cd^{1/3(1-s)}\|u\|_{H^1(\widehat{\mathcal{F}}(h,R))}$$

and

$$\left\|u-P_{\widehat{\mathcal{S}}(h,R)}u\right\|_{H^s(\widehat{\mathcal{S}}(h,R))}\leq C\|u\|^s_{H^1(\Omega)}\left\|u-P_{\widehat{\mathcal{S}}(h,R)}u\right\|^{1-s}_{L^2(\widehat{\mathcal{S}}(h,R))}.$$

Combining these relations with (A.32), we deduce the result. □

Now we introduce an interior type of junction:

Theorem A.1.5. *Assume that $\delta_1>\delta_2>0$. Then, there exists a family of bounded operators*

$$\widetilde{\Lambda}^{\delta_1,\delta_2}:H^1\left(\mathbb{R}^3\setminus\mathcal{S}_0\right)^3\times H^1_\sigma(\mathcal{S}_0)^3\to H^1(\mathbb{R}^3)$$

such that, for all $(u_1,u_2)\in W^{1,q}\left(\mathbb{R}^3\setminus\mathcal{S}_0\right)^3\times\mathcal{R}$ we have that:

$$\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)=u_2\ \text{ in }(\mathcal{S}_0)_{\delta_1}$$
$$\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)=u_1\ \text{ in }\mathbb{R}^3\setminus\mathcal{S}_0$$

and

$$\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)\in H^1_\sigma(\mathcal{S}_0),\tag{A.33}$$

where $(\mathcal{S}_0)_{\delta_1}$ is defined by (4.25), and the following inequality holds:

$$\left\|\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)-u_1\right\|_{L^p(\mathcal{S}_0)}\leqslant C_{\delta_1,\mathcal{S}_0}\Big(\delta_2^{1/p-1/6}(\|u_1\|_{H^1(\mathbb{R}^3\setminus\mathcal{S}_0)}+\|u_2\|_{H^1(\mathcal{S}_0)})+\|(u_1-u_2)\cdot\widehat{n}\|_{L^p(\partial\mathcal{S}_0)}\Big),\tag{A.34}$$

for $p\in[2,6]$.

$$\|\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)\|_{H^1(\mathcal{S}_0)}\leqslant C_{\delta_1}\left(\|(u_1-u_2)\cdot\widehat{n}\|_{L^2(\partial\mathcal{S}_0)}+\delta_2^{-2/3}(\|u_1\|_{H^1(\mathcal{S}_0)}+\|u_2\|_{H^1(\mathcal{S}_0)})\right)\tag{A.35}$$

Proof. We consider an orthogonal curvilinear coordinate system (s_1,s_2,z) defined around $\partial(\mathcal{S}_0)_{\delta_1}$ such that $\partial(\mathcal{S}_0)_{\delta_1}=\{z=0\}$. For δ_1 small enough we have $\partial\mathcal{S}_0=\{z=\delta_1\}$. As in Theorem A.1.1, we consider $\varphi\in\mathcal{C}^\infty_0([0,1);[0,1])$ such that $\varphi(0)=1$ and we define the function $\varphi_\delta(z):=\varphi(\frac{z}{\delta})$.

We define $\widetilde{\Lambda}^{\delta_1,\delta_2}(u_1,u_2)$ in $(\mathcal{S}_0)_{\delta_1}\setminus\mathcal{S}_0$ as follows:

$$\Lambda^{\delta_1,\delta_2}(u_1,u_2):=V_1+V_2+V_3$$

where

$$V_1=(1-\varphi_{\delta_2})u_1+\varphi_{\delta_2}\left(u_2-((u_2-u_1)\cdot e_z)e_z\right),$$

$$V_2=\{(u_2-u_1)\cdot e_z\}\Big|_{z=\delta_1}\varphi_{\delta_1}e_z,$$

and V_3 solution of the system

$$\begin{cases} \operatorname{div} V_3 = -\operatorname{div}(V_1 + V_2) & \text{in } \mathcal{S}_0 \setminus (\mathcal{S}_0)_{\delta_1} \\ V_3 = 0. & \text{on } \partial(\mathcal{S}_0)_{\delta_1} \cup \partial\mathcal{S}_0. \end{cases}$$

From [?, Theorem III.3.1, p.171], the above system admits a solution since the compatibility condition holds. Then, (A.33) holds. We can also check that

$$\Lambda^{\delta_1,\delta_2}(u_1,u_2) = u_1 \quad \text{on } \partial\mathcal{S}_0, \quad \Lambda^{\delta_1,\delta_2}(u_1,u_2) = u_2 \quad \text{on } \partial(\mathcal{S}_0)_{\delta_1}.$$

Moreover we have the following properties:

$$|V_1 - u_1| \leqslant \varphi_{\delta_2} |u_2 - u_1|, \quad |V_2| \leqslant \left| (u_2 - u_1)\cdot \widehat{n}\big|_{z=\delta_1} \right|, \tag{A.36}$$

$$\operatorname{div} V_1 = \operatorname{div}(u_1)(1-\varphi_{\delta_2}) - \varphi_{\delta_2} \operatorname{div}([(u_1-u_2)\cdot e_z]e_z), \tag{A.37}$$

$$\operatorname{div} V_2 = ((u_1-u_2)\cdot e_z)\big|_{z=\delta_1} (\varphi_{\delta_1} \operatorname{div} e_z + e_z \cdot \nabla\varphi_{\delta_1}). \tag{A.38}$$

Combining (A.36), (A.4) and a Sobolev embedding we deduce for any $p \in [1,6]$

$$\|V_1 - u_1\|_{L^p(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C\delta_2^{\frac{1}{p}-\frac{1}{6}} \|u_2 - u_1\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \tag{A.39}$$

and

$$\|V_2\|_{L^p(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C_{\delta_1,\mathcal{S}_0} \|(u_2-u_1)\cdot\widehat{n}\|_{L^p(\partial\mathcal{S}_0)}. \tag{A.40}$$

Assume $q \leq 2$. From (A.39) and (A.40) we deduce that:

$$\|\operatorname{div} V_1\|_{L^q(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C_{\delta_1,\mathcal{S}_0}\delta_2^{\frac{1}{q}-\frac{1}{2}} \|u_1-u_2\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})}, \tag{A.41}$$

and

$$\|\operatorname{div} V_2\|_{L^q(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C_{\delta_1,\mathcal{S}_0} \|(u_2-u_1)\cdot\widehat{n}\|_{L^q(\partial\mathcal{S}_0)}. \tag{A.42}$$

Using [?, Theorem III.3.1, p.171] and a Sobolev embedding, we conclude that if $\frac{1}{p} = \frac{1}{q} - \frac{1}{3}$,

$$\begin{aligned} \|V_3\|_{L^p(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} &\leqslant C_{\delta_1,\mathcal{S}_0} \|V_3\|_{W^{1,q}(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \\ &\leqslant C_{\delta_1,\mathcal{S}_0}\Big(\delta_2^{\frac{1}{p}-\frac{1}{6}} \|u_1-u_2\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} + \|(u_1-u_2)\cdot\widehat{n}\|_{L^q(\partial\mathcal{S}_0)}\Big). \end{aligned} \tag{A.43}$$

Gathering (A.41), (A.42) and (A.43) yields (A.34). To proof (A.35), we notice that by (A.37) and (A.38) we have

$$\|V_1\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C_{\mathcal{S}_0}\left(\|u_1\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} + \|u_2\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})}\right)$$

and

$$\|V_2\|_{H^1(\mathcal{S}_0\setminus(\mathcal{S}_0)_{\delta_1})} \leqslant C_{\mathcal{S}_0}(1+\delta_2^{-1/2})\|(u_2-u_1)\cdot\widehat{n}\|_{L^2(\partial\mathcal{S}_0)}$$

Combining the above estimations with (A.43) we deduce (A.35). □

A.2 Junction of scalar functions

In this section we adapt some lemmas of Section A.1 for scalar functions.

Theorem A.2.1. *Assume $\delta_1 > 2\delta_2 > 0$ and $s < \frac{1}{3}$. Then, there exists a family of bounded operators*

$$\widehat{\Lambda}^{\delta_1,\delta_2} : H^1\left(\mathbb{R}^3\right) \times \mathbb{R} \to H^s(\mathbb{R}^3)$$

such that, for all $(w_1, w_2) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ we have that:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) = w_2 \qquad \text{in } \mathcal{S}_0^{\delta_2} \tag{A.44}$$

$$\Lambda^{\delta_1,\delta_2}(w_1, w_2) = w_1 \quad \text{in } \mathbb{R}^3 \setminus \mathcal{S}_0^{\delta_1}, \tag{A.45}$$

where $\mathcal{S}_0^{\delta_1}$ and $\mathcal{S}_0^{\delta_2}$ are defined in (4.25)*, and the following inequality holds:*

$$\left\| \widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) - w_1 \right\|_{H^s(\Omega \setminus \mathcal{S}_0^{\delta_2})} \leqslant C_{\delta_1,\mathcal{S}_0} \delta_2^{\frac{1}{3}-s} (\|w_1\|_{H^1(\Omega)} + |w_2|). \tag{A.46}$$

Proof. The proof of this theorem is similar to the proof of Theorem A.1.1 and Theorem A.1.3. We use the same notation for the orthogonal curvilinear coordinate system (s_1, s_2, z) and for the functions φ_δ. We define $\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2)$ in $\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}$ as follows:

$$\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) = (1 - \varphi_{\delta_2}(z - \delta_2))w_1 + \varphi_{\delta_2}(z - \delta_2)w_2,$$

We also remark that

$$\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) = w_2 \quad \text{on} \quad \mathcal{S}_0^{\delta_2},$$

$$\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) = w_1 \quad \text{on} \quad \Omega \setminus \mathcal{S}_0^{\delta_1}.$$

Using (A.4) we deduce that:

$$\left\| \widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) - w_1 \right\|_{L^2(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2})} \leqslant C_{\delta_1,\mathcal{S}_0} \delta_2^{\frac{1}{3}} \left\| w_1 - w_2 \right\|_{H^1(\Omega)},$$

$$\left\| \nabla(\widehat{\Lambda}^{\delta_1,\delta_2}(w_1, w_2) - w_1) \right\|_{L^2(\mathcal{S}_0^{\delta_1} \setminus \mathcal{S}_0^{\delta_2}))} \leqslant C_{\delta_1,\mathcal{S}_0} \delta_2^{-\frac{2}{3}} \left\| w_1 - w_2 \right\|_{H^1(\Omega)},$$

and we conclude (A.46). ☐

As consequence of Theorem A.2.1, we obtain the following result on the orthogonal projection onto the space of rigid velocities:

Lemma A.2. *Let $w \in H^1(\Omega)$ and $(h, R) \in \mathbb{R}^3 \times SO(3)$ such that $\operatorname{dist}(\widehat{\mathcal{S}}(h, R), \partial\Omega) \geqslant \delta_1 > 0$. We set $\mathcal{S} = \widehat{\mathcal{S}}(h, R)$ and we define the space*

$$H^s_{\mathcal{S}}(\Omega) := \{\varphi \in H^s(\Omega);\ \nabla\varphi = 0 \text{ in } \mathcal{S}\}$$

and the projection

$$\mathbf{P}^s_{\mathcal{S}} : H^s(\Omega) \mapsto H^s_{\mathcal{S}}(\Omega).$$

We denote by $P^1_{\widehat{\mathcal{S}}(h,R)}$ the orthogonal projection of $L^2(\mathcal{S})$ onto the space of constant functions. Then, for all $d < \frac{\delta_1}{2}$ and $s \in (0, 1/3)$, we have that

$$\left\| w - \mathbf{P}^s_{(\widehat{\mathcal{S}}(h,R))^d} w \right\|_{H^s(\Omega)}$$
$$\leqslant C_{\delta_1,\mathcal{S}_0} d^{\frac{1}{3}-s} \left\| w \right\|_{H^1(\Omega)} + C \left\| w \right\|^{1/2}_{H^1(\Omega)} \left\| w - P^1_{\widehat{\mathcal{S}}(h,R)} w \right\|^{1/2}_{L^2(\widehat{\mathcal{S}}(h,R))}. \tag{A.47}$$

Proof. We set
$$v(x) = \left(\widehat{\Lambda}^{\delta_1,d}\left(w(h+Ry), P^1_{\mathcal{S}_0} w(h+Ry)\right)\right)|_{y=R^*(x-h)} \,.$$

for all $x \in \Omega$. Then, by Theorem A.2.1, we have $v = P^{\mathcal{R}}_{\widehat{\mathcal{S}}(h,R)} u$ in $\widehat{\mathcal{S}}(h,R)^d$ $v = w$ in $\Omega \setminus \widehat{\mathcal{S}}(h,R)^{\delta_1}$, and
$$\|v - w\|_{H^s(\Omega \setminus \widehat{\mathcal{S}}(h,R)^d)} \leqslant C d^{\frac{1}{3}-s} \|w\|_{H^1(\Omega)} \,.$$
We deduce that
$$\|v-w\|_{H^s(\Omega)} \leqslant C\left(d^{\frac{1}{3}-s}\|w\|_{H^1(\Omega)} + \left\|w - P^1_{\widehat{\mathcal{S}}(h,R)} w\right\|_{H^s(\widehat{\mathcal{S}}(h,R)^d \setminus \widehat{\mathcal{S}}(h,R))} + \left\|u - P^1_{\widehat{\mathcal{S}}(h,R)} w\right\|_{H^s(\widehat{\mathcal{S}}(h,R))}\right). \tag{A.48}$$

Now we have the following relations
$$\left\|w - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(h,R)} w\right\|_{H^s(\widehat{\mathcal{S}}(h,R)^d \setminus \widehat{\mathcal{S}}(h,R))} \leq C d^{1/3(1-s)} \|w\|_{H^1(\Omega)}$$
and
$$\left\|w - P^{\mathcal{R}}_{\widehat{\mathcal{S}}(h,R)} w\right\|_{H^s(\widehat{\mathcal{S}}(h,R))} \leq C \|w\|^s_{H^1(\Omega)} \left\|w - P^1_{\widehat{\mathcal{S}}(h,R)} w\right\|^{1-s}_{L^2(\widehat{\mathcal{S}}(h,R))} .$$
Combining these relations with (A.48), we deduce the result. □

A.3 Basis in a Hilbert space

Lemma A.3. *Assume X is a separable Hilbert space and assume that A is a dense subset of X. Then there exists an orthonormal basis $\{v_q\}_{q\geq 1}$ of X such that $v_q \in A$.*

Proof. First, we will prove that, since X is a separable space and A is dense in X, there exists a numerable subset of A dense in X. Since X is separable, there exists $\{\tilde{v}_r\}_{r\geqslant 1} \subset X$ a dense subset of X. Since A is dense, for any $r \geqslant 1$, there exists $V_{r,q} \in A$ such that
$$\|\tilde{v}_r - V_{r,q}\| \leqslant \frac{1}{q} \quad \forall q \geqslant 1.$$
We take $\{v_q\} = \{V_{r,q}, r \geqslant 1, q \geqslant 1\}$. Let $v \in X$ and $\epsilon > 0$, then we have that there exists $\tilde{v}_r \in X$ such that:
$$\|\tilde{v}_r - v\| \leqslant \frac{\varepsilon}{2}.$$
We choose q such that $\frac{1}{q} < \frac{\epsilon}{2}$, therefore we conclude that $\|v - V_{q,r}\| < \epsilon$.
Then, we can construct an orthonormal basis composed by elements of A. □

A.4 Convergence of the characteristics

Lemma A.4.

1. *Let $(h_n, R_n) \to (h, R)$ in $\mathbb{R}^3 \times SO(3)$. Then,*
$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ in } L^p(\Omega) \quad \forall p \in [1,\infty). \tag{A.49}$$

2. *Let* $(h_n, R_n) \to (h, R)$ *strongly in* $\mathcal{C}([0,T], \mathbb{R}^3 \times SO(3))$. *Then,*

$$\mathbb{1}_{\mathcal{S}(h_n,R_n)} \to \mathbb{1}_{\mathcal{S}(h,R)} \text{ strongly in } \mathcal{C}([0,T]; L^p(\Omega)) \quad \forall p \in [1,\infty). \tag{A.50}$$

Proof. The proofs of (A.49) and (A.50) are based on a similar argument. Then, we will prove just (A.49). By the definition of the cahracteristic function we have that

$$\mathbb{1}_{\mathcal{S}(h,R)} = \mathbb{1}_{\mathcal{S}_0}(R^*(x-h)).$$

Then, we deduce

$$\left\|\mathbb{1}_{\mathcal{S}(h_n,R_n)} - \mathbb{1}_{\mathcal{S}(h,R)}\right\|_{L^p(\Omega)} = \left\|\mathbb{1}_{\mathcal{S}_0}((R_n)^*(\cdot - h_n)) - \mathbb{1}_{\mathcal{S}_0}(R^*(\cdot - h))\right\|_{L^p(\mathbb{R}^3)}. \tag{A.51}$$

Let $\epsilon > 0$. Snce $\mathbb{1}_{\mathcal{S}_0} \in L^p(\mathbb{R}^3)$ for all p in $[1,\infty)$, there exists a continuous function with compact support g, such that:

$$\left\|\mathbb{1}_{\mathcal{S}_0} - g\right\|_{L^p(\mathbb{R}^3)} \leqslant \epsilon.$$

Then, we have the following:

- By the theorem of change of variables we have that:

$$\left\|\mathbb{1}_{\mathcal{S}_0}((R_n)^*(\cdot - h_n)) - g((R_n)^*(\cdot - h_n))\right\|_{L^p(\mathbb{R}^3)} = \left\|\mathbb{1}_{\mathcal{S}_0} - g\right\|_{L^p(\mathbb{R}^3)}$$

  and

$$\left\|g(R^*(\cdot - h)) - \mathbb{1}_{\mathcal{S}_0}(R^*(\cdot - h))\right\|_{L^p(\mathbb{R}^3)} = \left\|\mathbb{1}_{\mathcal{S}_0} - g\right\|_{L^p(\mathbb{R}^3)}.$$

  Then:

$$\left\|\mathbb{1}_{\mathcal{S}_0}((R_n)^*(\cdot - h_n)) - g((R_n)^*(\cdot - h_n))\right\|_{L^p(\mathbb{R}^3)} + \left\|g(R_n^*(\cdot - h_n)) - \mathbb{1}_{\mathcal{S}_0}(R^*(\cdot - h))\right\|_{L^p(\mathbb{R}^3)} \leqslant \frac{2\epsilon}{3}. \tag{A.52}$$

- Since g is continuous and it has compact support, it is uniformly continuous. Then, since $(h_n, R_n) \to (h, R)\mathbb{R}^3 \times SO(3)$, we deduce that

$$\left\|g((R_n)^*(\cdot - h_n)) - g(R^*(\cdot - h))\right\|_{L^p(\mathbb{R}^3)} \leqslant \frac{\epsilon}{3}, \tag{A.53}$$

  where ϵ does not depends on n.

Then, we combine (A.51), (A.52) and (A.53) and obtain

$$\left\|\mathbb{1}_{\mathcal{S}(h_n,R_n)} - \mathbb{1}_{\mathcal{S}(h,R)}\right\|_{L^p(\Omega)} \leqslant \epsilon.$$

Gathering the above we conclude (A.49). □

Remark A.1. *Lemma 4.1 part (2) can be proved using the compactness theory of Di Perna-Lions [**?**], see for example [**?**]. In our case we choose to do it straightforward from the definition due to the simplicity of the argument.*

A.5 Material derivative of a fluid-solid mixture

Lemma A.5. *Let $\rho_s > 0$, $(h, R) \in W^{1,\infty}(\mathbb{R}^3 \times SO(3))$, $u \in H^1(0, T; H^1_\sigma(\Omega))$ and $f \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. We define $\mathcal{S} = \widehat{\mathcal{S}}(h, R)$ and the density of the fluid-solid mixture $\rho := (1 - \mathbb{1}_{\mathcal{S}}) + \rho_s \mathbb{1}_{\mathcal{S}}$. Let $Q_{\mathcal{S}} : L^2(0, T; H^1_\sigma(\Omega)) \mapsto L^2(0, T; H^1_\sigma(\mathbb{R}^3))$ an operator such that: $Q_{\mathcal{S}}(u)(t, x) = h'(t) + \omega(t) \times (x - h(t))$ in $\mathcal{S}$. Then,*

$$\frac{d}{dt} \int_\Omega \rho f \; dx = \int_\Omega \rho \left(\frac{\partial f}{\partial t} + Q_{\mathcal{S}}(u) \cdot \nabla f \right) \; dx. \tag{A.54}$$

Proof. Since $\rho = 1 + (\rho_s - 1)\mathbb{1}_{\mathcal{S}}$, we have

$$\frac{d}{dt} \int_\Omega \rho f dx = \rho_s \frac{d}{dt} \int_{\mathcal{S}} f dx + \frac{d}{dt} \int_\Omega f \; dx. \tag{A.55}$$

By a change of variables, the first term of the right hand side of (A.55) is written as

$$\frac{d}{dt} \int_{\mathcal{S}(t)} f \; dx = \frac{d}{dt} \int_{\mathcal{S}_0} f(t, h(t) + R(t)y) \; dy.$$

Since $f \in H^1(0, T; L^2(\Omega))$, we pass the time derivative inside of the integral in the above equation and we obtain

$$\frac{d}{dt} \int_{\mathcal{S}(t)} f(t, x) \; dx = \int_{\mathcal{S}_0} \frac{\partial}{\partial t} f(t, h(t) + R(t)y) + \nabla f \cdot (h' + \omega(t) \times Ry) \; dy.$$

Then, by a change of variables and the fact that $Q_{\mathcal{S}}(u)(t, x) = h'(t) + \omega(t) \times (x - h(t))$ in $\mathcal{S}$, we deduce

$$\frac{d}{dt} \int_{\mathcal{S}} f \; dx = \int_{\mathcal{S}} \frac{\partial}{\partial t} f + \nabla f \cdot Q_{\mathcal{S}}(u) \; dx. \tag{A.56}$$

For the second term in the right hand side of (A.55) we obtain

$$\frac{d}{dt} \int_\Omega f \; dx = \int_\Omega \frac{\partial f}{\partial t} \; dx. \tag{A.57}$$

On the other hand, integrating by parts we have that:

$$\frac{d}{dt} \int_\Omega Q_{\mathcal{S}}(u) \cdot \nabla f \; dx = \int_{\partial\Omega} f Q_{\mathcal{S}}(u) \cdot n \; dx - \int_\Omega \operatorname{div}(Q_{\mathcal{S}}(u)) f \; dx.$$

Then, since $\operatorname{div}(Q_{\mathcal{S}}(u)) = 0$ and $Q_{\mathcal{S}}(u) = 0$ in $\partial\Omega$, gathering the above equation with (A.57) we deduce that:

$$\frac{d}{dt} \int_\Omega f \; dx = \int_\Omega \frac{\partial}{\partial t} f + \nabla f \cdot Q_{\mathcal{S}}(u) \; dx. \tag{A.58}$$

We combine (A.55), (A.56) and (A.58) and we obtain (A.54). □

A.6 Orthogonal projection over the set of rigid velocities

Lemma A.6. *Let $(h, R) \in \mathbb{R}^3 \times SO(3)$ and $\mathcal{R}$ the space of rigid velocities. We denote by $P_{\mathcal{S}}^{\mathcal{R}}$ the orthogonal projection of $L^2(\mathcal{S})$ onto $\mathcal{R}$, where $\mathcal{S} = \widehat{\mathcal{S}}(h, R)$. Then,*

$$P_{\mathcal{S}}^{\mathcal{R}} u = \ell_u + \omega_u \times (x - h)\,, \tag{A.59}$$

where

$$\ell_u = \frac{1}{m} \int_{\mathcal{S}} \rho_s u \; dx \tag{A.60}$$

and

$$\omega_u = J^{-1} \int_{\mathcal{S}} \rho_s (x - h) \times u \; dx. \tag{A.61}$$

Proof. Since $P_{\mathcal{S}}^{\mathcal{R}} u$ is a rigid velocity, there exists ℓ_u and ω_u such that (A.59) holds. Then, we have to prove equations (A.60) and (A.61). Since $P_{\mathcal{S}} u$ is the projection then we have that

$$\int_{\mathcal{S}} (P_{\mathcal{S}} u - u) \cdot v dx = 0 \tag{A.62}$$

for all $v \in \mathcal{R}$. Then if we choose $v = \rho_s b$ in (A.62) with $b \in \mathbb{R}^3$ and using the fact that

$$\int_{\mathcal{S}} (x - h) dx = 0$$

we have that:

$$\int_{\mathcal{S}} \rho_s \left(\ell_u - u\right) \cdot b = 0 \tag{A.63}$$

If we choose $v = \rho_s b \times (x - h)$ we arrive to the following equation:

$$\int_{\mathcal{S}} \rho_s \omega_u \times (x - h) \cdot b \times (x - h) dx = \int_{\mathcal{S}} \rho_s b \cdot (x - h) \times u dx \tag{A.64}$$

Using the identity:

$$\mathbb{A}_{ij}(\omega_u) \mathbb{A}_{ik}(b) = \omega_u \cdot b \delta_{jk} - \omega_{u,j} b_k,$$

we can write equation (A.64) as follows:

$$J \omega_u \cdot b = \int_{\mathcal{S}} \rho_s b \cdot (x - h) \times u dx. \tag{A.65}$$

Then if we choose b equals to canonical vectors in equations (A.63) and (A.65) we arrive to the equations (A.60) and (A.61). □

A.7 Convergence properties of the projection

Lemma A.7. *Let $v \in L^2([0, T] \times \Omega))$, and $(h_n, R_n) \to (h, R)$ strongly in $\mathcal{C}([0, T]; \mathbb{R}^3 \times SO(3))$. For all $t \in [0, T]$, we define $\mathcal{S}_n(t) := \widehat{\mathcal{S}}(h_k(t), R_k(t))$ and $\mathcal{S}(t) := \widehat{\mathcal{S}}(h(t), R(t))$. Then we have that:*

(a)

$$P^{\mathcal{R}}_{\mathcal{S}_n}(v) \to P^{\mathcal{R}}_{\mathcal{S}}(v) \text{ in } L^2(0,T;L^2(\Omega)) \tag{A.66}$$

and

(b)

$$\int_0^T \int_{\mathcal{S}_n} \left|v - P^{\mathcal{R}}_{\mathcal{S}_n}(v)\right|^2 dxdt \to \int_0^T \int_{\mathcal{S}} \left|v - P^{\mathcal{R}}_{\mathcal{S}}(v)\right|^2 \; dxdt. \tag{A.67}$$

Proof. (a) We recall that:

$$P^{\mathcal{R}}_{\mathcal{S}_n}(v)(t,x) = \ell_{n,v}(t) + \omega_{n,v}(t) \times (x - h_n(t)), \tag{A.68}$$

where

$$\ell n, v = \frac{1}{m} \int_{\mathcal{S}_n} \rho_s v \; dx,$$

where ρ_s is a positive constant that represents the density of the solid,

$$\omega^n_v = J^{-1}_{\mathcal{S}_n} \int_{\mathcal{S}_n} \rho_s (x - h_n(t)) \times v \; dx,$$

$$P^{\mathcal{R}}_{\mathcal{S}}(v) = \ell_v(t) + \omega_v(t) \times (x - h(t)),$$

where

$$\ell_v = \frac{1}{m} \int_{\mathcal{S}} \rho_s v \; dx,$$

and

$$\omega_v = J^{-1}_{\mathcal{S}} \int_{\mathcal{S}} \rho_s (x - h_n(t)) \times v \; dx.$$

Then, we check that

$$\begin{aligned}\int_0^T \int_\Omega \left|P^{\mathcal{R}}_{\mathcal{S}_n} v - P^{\mathcal{R}}_{\mathcal{S}} v\right|_2^2 dxdt \leqslant \int_0^T |\ell^n_v - \ell_v|_2^2 \, dt \, |\Omega| \\ + 4 \int_0^T |\omega_v|_\infty^2 \, |h_n - h|_2^2 \, dt \, |\Omega| \\ + 4 \int_0^T \int_\Omega |x - h|_\infty^2 \; dx \, |\omega^n_v - \omega_v|_2^2 \; dt.\end{aligned}$$

By the definition of $\ell_{v.n}$ and ℓ_v we have that

$$\int_0^T |\ell^n_v - \ell_v|_2^2 \, dt \leqslant \frac{\|v\|_{L^2([0,T]\times\Omega)}}{m} \left\| \mathbb{1}_{\mathcal{S}_n} - \mathbb{1}_{\mathcal{S}} \right\|_{L^2([0,T]\times\Omega)} .$$

Also, we write the difference of ω^n_v and ω as follows:

$$\begin{aligned}\omega_{n,v} - \omega_v = \left(J^{-1}_{\mathcal{S}_n} - J^{-1}_{\mathcal{S}}\right) \int_\Omega \rho_s \mathbb{1}_{\mathcal{S}_n} (x - h_n) \times v \; dx \\ + J^{-1}_{\mathcal{S}} \int_\Omega \rho_s \left(\mathbb{1}_{\mathcal{S}_n} - \mathbb{1}_{\mathcal{S}}\right)(x - h_n) \times v \; dx + J^{-1}_{\mathcal{S}} \int_\Omega \rho_s \mathbb{1}_{\mathcal{S}} (h - h_n) \times v \; dx\end{aligned}$$

Then, since $J^{-1}_{\mathcal{S}_n} \to J^{-1}_{\mathcal{S}}$ strongly in $\mathcal{C}([0,T];\mathbb{M}^{3\times3})$, by Lemma 4.1 we have that $\mathbb{1}_{\mathcal{S}_n} \to \mathbb{1}_{\mathcal{S}}$ strongly in $\mathcal{C}([0,T];L^p(\Omega))$ and since $h_n \to h$ strongly in $\mathcal{C}([0,T];\mathbb{R}^3)$ we deduce (A.66).

(b) Since $\mathbb{1}_{\mathcal{S}_n} \to \mathbb{1}_{\mathcal{S}}$ strongly in $\mathcal{C}([0,T];L^1(\Omega))$:

$$\int_0^T \int_{\mathcal{S}_n} |v|_2^2 \ dxdt \to \int_0^T \int_{\mathcal{S}} |v|_2^2 \ dxdt$$

Then, (A.66) we have that:

$$P^{\mathcal{R}}_{\mathcal{S}_n}(v) \to P^{\mathcal{R}}_{\mathcal{S}}(v) \text{ in } L^2(0,T;L^2(\Omega)).$$

Then, since $\mathbb{1}_{\mathcal{S}_n} \to \mathbb{1}_{\mathcal{S}}$ strongly in $\mathcal{C}([0,T];L^2(\Omega))$ we have that

$$\int_0^T \int_{\mathcal{S}_n} \left|P^{\mathcal{R}}_{\mathcal{S}_n}(v)\right|^2 \ dxdt \to \int_0^T \int_{\mathcal{S}} \left|P^{\mathcal{R}}_{\mathcal{S}}(v)\right|^2 \ dxdt$$

and we conclude (A.67).

□

A.8 Convergence of test functions

Lemma A.8. *Let $\eta > 0$ and*

$$(h_k, R_k) \to (h, R) \textit{ weakly star in } W^{1,\infty}(0,T;\mathbb{R}^3 \times SO(3)) \\ \textit{and strongly in } \mathcal{C}([0,T];\mathbb{R}^3 \times SO(3)). \quad \text{(A.69)}$$

For all $t \in [0,T]$, we define $S_k(t) := \widehat{S}(h_k(t), R_k(t))$ and $S(t) := \widehat{S}(h(t), R(t))$. Let $v \in \mathcal{C}^1([0,T], H_{S(t)}(\Omega))$ such that $\operatorname{supp} v \subset \Omega_\eta$. We write

$$v_k := \overline{\Phi}_{h_k,R_k} \circ \Phi_{h,R}(v),$$

where, given $(a,Q) \in \mathbb{R}^3 \times SO(3)$ we define two operators of $L^2_{loc}(\mathbb{R}^3)$ as follows: assume $v \in L^2_{loc}(\mathbb{R}^3)$, then

$$\Phi_{a,Q}(v)(y) := Q^* v\left(a + Qy\right), \quad y \in \mathbb{R}^3$$

and

$$\overline{\Phi}_{a,Q}(v)(x) := Qv\left(Q^*(x-a)\right), \quad x \in \mathbb{R}^3.$$

Then, for k large enough we have the following:

$$v_k \in \mathcal{C}([0,T];H^1_\sigma(\Omega))$$

with $\operatorname{supp} v_k \subset \Omega_{\frac{\eta}{2}}$;

$$D(v_k) = 0 \quad \textit{in } S_k, \tag{A.70}$$

$$v_k \to v \textit{ strongly in } \mathcal{C}([0,T];H^1_\sigma(\Omega)); \tag{A.71}$$

$$\frac{\partial v_k}{\partial t} + (w_k \cdot \nabla)v_k \to \frac{\partial v}{\partial t} + (w \cdot \nabla)v \textit{ strongly } L^\infty(0,T;L^2(\Omega)), \tag{A.72}$$

where ω_k and ω are defined by

$$w_k(t,x) = h_k'(t) + \omega_k(t) \times (x - h_k(t))$$

and

$$w(t,x) = h'(t) + \omega(t) \times (x - h(t)),$$

where $\mathbb{A}(\omega_k) = (R_k')^* R_k$ *and* $\mathbb{A}(\omega) = (R')^* R$*; and*

$$\frac{\partial v_k}{\partial t} \to \frac{\partial v}{\partial t} \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \tag{A.73}$$

Proof. First, by (4.26) and (4.27) we have that

$$v_k(t,x) = R_k(t)R^*(t)v(t, h(t) + R(t)R_k^*(t)(x - h_k(t))). \tag{A.74}$$

Then, by (A.69), if k is large enough we deduce that $\operatorname{supp} v_k \subset \Omega_{\frac{\eta}{2}}$. Then, we check that $v_k \in \mathcal{C}([0,T]; H^1_\sigma(\Omega))$.

Also from equation (A.74), we deduce (A.70), since if $x \in S_k(t)$ then

$$h(t) + R(t)R_k^*(t)(x - h_k(t)) \in S(t).$$

By (A.74) we deduce that:

$$\nabla v_k(t,x) = R_k(t)R^*(t)\nabla v(t, h(t) + R(t)R_k^*(t)(x - h_k(t)))R_k(t)R^*(t). \tag{A.75}$$

Then, the convergence (A.71) is straightforward from (A.74), (A.75) and the fact that the translations are continuous in $L^2(\Omega)$ (the same proof as Lemma A.4).

By the properties of Φ and $\overline{\Phi}$ we have the following identity :

$$\Phi_{h_k,R_k} \circ v_k = \Phi_{h,R} \circ v.$$

By derivation with respect to t and we deduce that

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_{h_k,R_k} \circ v_k &= \omega_k \times v_k(t, h_k(t) + R_k(t)y) \\ &\quad + R_k^*(t)\left(\frac{\partial}{\partial t} v_k(t, h_k(t) + R_k(t)y) + (w_k \cdot \nabla) v_k(t, h_k(t) + R_k(t)y)\right),\end{aligned}$$

and

$$\frac{\partial}{\partial t}\Phi_{h,R} \circ v = \omega(t) \times v(t,x) + R^*(t)\left(\frac{\partial}{\partial t} v(t,x) + (w \cdot \nabla) v(t,x)\right).$$

Then using (A.69) and (A.71), we deduce the convergence (A.72) .

By (A.72), (A.71) and the fact that $w_k \to w$ strongly in $L^2(0,T;\mathcal{R})$, where $\mathcal{R}$ is the space of rigid velocities, we deduce (A.73). □

A.9 Material derivative in the solid phase

Lemma A.9. *Let* $(h,R) \in W^{1,\infty}(0,T;\mathbb{R}^3)$ *for* $T > 0$*. We define* $\mathcal{S}(t) = \widehat{\mathcal{S}}(h(t), R(t))$ *a.e. in* $t \in [0,T]$ *in* $\mathcal{S}(t)$*. Let* u *and* v *two rigid velocities* S *such that*

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)) \tag{A.76}$$

and

$$v(t,x) = \ell_v(t) + \omega_v(t) \times (x - h(t)). \tag{A.77}$$

Then,

$$\int_{S(t)} \rho_s \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) \cdot (v - u) dx = m\ell' \cdot (\ell_v - \ell) + (J\omega)' \cdot (\omega_v - \omega), \tag{A.78}$$

a.e in $[0, T]$*, where* ρ_s *is the density of the solid, which is a positive constant,* m *is the mass of the solid given by*

$$m = \int_{\mathcal{S}} \rho_s \, dx,$$

and the moment of inertia J *is given by:*

$$J(t) = \widehat{J}(h(t), R(t)),$$

where

$$\widehat{J}(a, Q) = \rho_s \int_{\widehat{\mathcal{S}}(a,Q)} \left(|x - a|^2 I_3 - (x - a) \otimes (x - a) \right) dx,$$

for all $(a, Q) \in \mathbb{R}^3 \times \mathbb{M}^{3\times 3}$*.*

Proof. The strategy of this proof is to develop the term of the time derivative and the convective term and to prove that the sum of both is equal to the right-hand side of (A.78). Using (A.76) and (A.77), we have

$$\int_{S(t)} \rho_s (v - u) \cdot \frac{\partial}{\partial t} u \, dx = m\ell' \cdot (\ell_v - \ell) - m(\omega \times \ell) \cdot (\ell_v - \ell) + \int_{S(t)} \rho_s (\omega' \times (x - h)) \cdot ((\omega_v - \omega) \times (x - h)) \, dx. \tag{A.79}$$

Using that

$$(\omega_v - \omega) \times (x - h(t)) = -A(\omega_v - \omega)(x - h(t)),$$
$$\omega' \times (x - h(t)) = -A(\omega_v')(x - h(t)),$$

and

$$A(\omega_v - \omega)_{ij} A(\omega')_{ik} = (\omega_v - \omega) \cdot \omega_v' \delta_{jk} - (\omega_v - \omega)_j \omega_{v,k}',$$

we deduce that the last term of equation A.79 is be written as

$$\int_{S(t)} \rho_s (\omega' \times (x - h)) \cdot ((\omega_v - \omega) \times (x - h)) dx = J\omega' \cdot (\omega_v - \omega). \tag{A.80}$$

This alllows us to conclude that:

$$\int_{S(t)} \rho_s (v - u) \cdot \frac{\partial}{\partial t} u dx = m\ell' \cdot (\ell_v - \ell) - m(\omega \times \ell) \cdot (\ell_v - \ell) + J\omega' \cdot (\omega_v - \omega). \tag{A.81}$$

For the convective term we have that

$$(u \cdot \nabla) u = \omega \times u.$$

Then, we deduce

$$\int_{S(t)} (u \cdot \nabla)u \cdot \rho_s(v-u)dx = m(\omega \times \ell) \cdot (\ell_v - \ell) \\ + \int_{S(t)} \rho_s \omega \times (\omega \times (x-h)) \cdot (\omega_v \times (x-h))\, dx.$$

Since

$$J(t) = R(t)J(0))R^*(t),$$

and

$$R'(t)y = \omega \times R(t)y$$

for all $y \in \mathbb{R}^3$, we deduce that

$$J'y = \mathbb{A}(\omega)J + J\mathbb{A}(\omega),$$

where

$$\mathbb{A}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Using properties of the cross and dot product we arrive to the following equation for J':

$$J'\omega \cdot (\omega_v - \omega) = J\omega \cdot (\omega_v \times \omega).$$

Then, as we did before in (A.80), we have that:

$$J'\omega \cdot (\omega_v - \omega) = \int_{S(t)} \rho_s(\omega \times (x-h)) \cdot ((\omega_v \times \omega) \times (x-h))dx. \tag{A.82}$$

Using the following identity of the croos product:

$$(\omega_v \times \omega) \times (x-h) = -((x-h) \times \omega_v) \times \omega - (\omega \times (x-h)) \times \omega_v,$$

and orthogonality properties, we have that:

$$J'\omega \cdot (\omega_v - \omega) = \int_{S(t)} \rho_s \omega_v \times (x-h) \cdot (\omega \times (\omega \times (x-h)))dx.$$

Then we arrive to the following expression for the convective term:

$$\int_{S(t)} \rho_s(u \cdot \nabla)u \cdot (v-u)dx = m(\omega \times \ell) \cdot (\ell_v - \ell) + J'\omega \cdot (\omega_v - \omega). \tag{A.83}$$

We conclude by gathering (A.81) and (A.83). □

A.10 Rotation matrix

Lemma A.10. *Let R in $C^1(0,T;\mathbb{R}^{n\times n})$, satisfying the differential equation*

$$R'(t) = A(t)R(t), \tag{A.84}$$

with $R(0) = I_n$ and $A(t)$ an antisymmetric matrix. Then, R is a rotation matrix.

Proof. If we transpose equation (A.84) we get that

$$(R^*(t))' = -R^*(t)A(t).$$

Then, we multiply by $R(t)$ and we use again (A.84) and we obtain that

$$\frac{d}{dt}R(t)R^*(t) = 0.$$

Using the initial condition we can conclude that

$$R(t)R^*(t) = I_n.$$

Then, $R^*(t)$ is the inverse of $R(t)$. Since R is C^1, we use the formula

$$\det(R)' = tr\left(R'R^{-1}\right)\det(R),$$

and by equation (A.84), the fact that $tr(A(t)) = 0$ and $R(0) = I_n$ we can conclude that $\det R(t) = 1$ for all t in $[0, t^*)$.

□

A.11 The norm of the symmetric gradient

Lemma A.11. *Let* $u \in \left(H_0^1(\Omega)\right)^3$ *with* $\operatorname{div} u = 0$. *Then,*

$$\|u\|_{H_0^1(\Omega)} = 2\int_\Omega D(u) : D(u)dx. \tag{A.85}$$

Proof. Let u be a $\mathcal{C}_0^2(\Omega)$. The proof of this lemma is based on the following identity:

$$\operatorname{div}(\nabla u)\, u - (\operatorname{div} u)u = \nabla u^* : \nabla u - (\operatorname{div} u)^2.$$

Since

$$\nabla u : \nabla u^* = 2D(u) : D(u) - \nabla u : \nabla u,$$

we have that:

$$\operatorname{div}(\nabla u)\, u - (\operatorname{div} u)u = 2D(u) : D(u) - \nabla u : \nabla u - (\operatorname{div} u)^2.$$

We integrate the above equation over Ω and using that $\operatorname{div} u = 0$, we have that:

$$\int_\Omega \operatorname{div}(\nabla u)\, u\; dx = 2\int_\Omega D(u) : D(u)\; dx - \int_\Omega \nabla u : \nabla u\; dx.$$

Since u is in $\mathcal{C}_0^2(\Omega)$, using the divergence theorem, we have that:

$$\int_\Omega \operatorname{div}(\nabla u)\, u\; dx = \int_{\partial\Omega} (\nabla u)\, u\; dx = 0.$$

Then by density arguments we arrive to the formula (A.85). □

Appendix B

MATLAB codes for Section 3.3

Listing B.1: MATLAB code for figures 3.8, 3.7 and 3.9

```
% This functions solves the DAE system using the function ode15s
%\Lambda_{rel}'=-4rC_{21}(\gamma_1\theta_2-\gamma_2\theta_1)\omega_{rel}^2, &\ r\in\[1,2]\\
%\theta_1\theta_2\Lambda_{rel}=\theta_1\theta_2C_{21}\left(\rho_1+\rho_2\right)r^2\omega_{rel}^2
%+c_2\theta_1\left(\theta_2^2-\theta_c^2\right)
%-c_1\theta_2\left(\theta_1^2-\theta_{1,c}^2\right),&\ r\in [1,2]\\
%\frac{\sqrt{2}A_2}{\mu_s r^2}=-\frac{2}{3}\rho_2\theta_1 C_{21}r^2\omega_{rel}^2+\theta_{2}\Lambda_{rel},&\
%r\in \[1,2],\\
%\omega_{rel}=\omega_1,&\ r\in[1,2]
%

parametrosmixture2;
clf;
Rf=2;
A=1e3;
cbtheta1=0.2;
A1=cbtheta1*A;
A2=A-A1;
cbthetaaux=0;

set(1,'Userdata',[R2 A2]);
k=0;
eps1=1e-5;
while(k<100)

cbvel=A1/2/muf.*(1./Rf.^2-1./R2.^2);

Presion=sqrt(2)*A2./mus./Rf.^2;
theta02=findtheta20(Presion,cbvel,Rf);

```

```
if(isnan(theta02))
disp('theta_{02} is not a number');
else
    fprintf('theta02 %f\n',theta02);
end
%pause;

lambda0=Presion/theta02+gamma2*2/3*(1-theta02)*Cdrag*cbvel^2*Rf^2;

eps=1e-10;

M = [1 0; 0 0];

options = odeset('Mass',M,'RelTol',eps,'AbsTol',[eps eps]);
tic
[s,Y] = ode15s(@eqdiferencial,[0 Rf-R1],[lambda0 theta02],options);
toc
lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
cbthetaaux=theta1(end);
    if(abs(cbthetaaux-cbtheta1)<eps1)
        break;
    else
    cbtheta1=cbthetaaux;
    end
k=k+1;

A1=cbtheta1*A;
A2=A-A1;
set(1,'Userdata',[R2 A2]);

end

lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
r=Rf-s;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2.*lambda).*(theta2.*gamma2.*theta1*Cdrag.*r.^2)
   .^-1);
omega1=A1/2/muf.*(1./r.^2-1./R2.^2);
omega2=omega1-velr;

figure(1)
plot(r,theta1,'r',r,theta2,'b')
grid on;
```

```
legend('\theta_1','\theta_2')
title('Concentration of the fluid and the solid phases')
xlabel('r')
%
figure(2)
plot(r,omega1,'r*',r,omega2,'b')
grid on;
legend('\omega_1','\omega_2')
title('Angular velocities of the fluid and the solid phases')
xlabel('r')
figure(3)
plot(r,velr,'r')
grid on;
legend('\omega_{rel}')
title('Relative velocity of the mixture')
xlabel('r')

figure(4)
Lambda2=integra(s,0,2.*Cdrag.*theta1.*velr.^2.*r);
Lambda1=(lambda+gamma2*Lambda2')/gamma1;
 plot(r,Lambda1,'r',r,Lambda2,'b')
 legend('\Lambda_1','\Lambda_2')
title('Lagrange multipliers of the fluid and the solid phases')
xlabel('r')
grid on;


function dy = eqdiferencial(s,y)
parametrosmixture2;
a=get(1,'Userdata');
rfinal=a(1);
r=rfinal-s;
lambda=y(1);
theta2=y(2);
A2=a(2);
theta1=1-theta2;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2*lambda)*(theta2*gamma2*theta1*Cdrag*r^2)^-1);
dy=zeros(2,1);
dy(1)=4*r*Cdrag*(theta2*gamma1-theta1*gamma2)*velr^2;
dy(2)=-lambda*theta2*theta1+(theta2*gamma2+gamma1*theta1)*Cdrag*r^2*velr^2*
    theta2*theta1...
      +c2*(theta2^2-thetac2^2)*theta1-c1*(theta1^2-thetac1^2)*theta2;

end

function [theta02,Fval]=findtheta20(Presion,cbvel,Rf)
```

```
parametrosmixture2;
x=zeros(1,10);
ci=linspace(0,1,10);
tol=10e-4;
Fval=zeros(1,10);
for k=1:10
    [x(k), Fval(k)] = fsolve(@(x) FindBoundaryConditions(x,Presion,thetac2,
       thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf),ci(k));
end
i= x<1 & x>0 & abs(Fval)<tol;
theta02=mode(x(i));
end

function sys=FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,
   gamma2,Cdrag,cbvel,Rf)
sys=-(Presion+2/3*gamma2*x*(1-x)*Cdrag*Rf^2*cbvel^2)*(1-x)...
    +Rf^2*cbvel^2*Cdrag*(gamma1+(gamma2-gamma1)*x)*x*(1-x)...
    +c2*(x^2-thetac2^2)*(1-x)-c1*x*((1-x)^2-thetac1^2);
end
```

Listing B.2: MATLAB code for Figure 3.10

```
% This functions solves the DAE system using the function ode15s
%\Lambda_{rel}'=-4rC_{21}(\gamma_1\theta_2-\gamma_2\theta_1)\omega_{rel}^2,
   &\ r\in\[1,2]\\
%\theta_1\theta_2\Lambda_{rel}=\theta_1\theta_2C_{21}\left(\rho_1+\rho_2\
   right)r^2\omega_{rel}^2
%+c_2\theta_1\left(\theta_2^2-\theta_c^2\right)
%-c_1\theta_2\left(\theta_1^2-\theta_{1,c}^2\right),&\ r\in [1,2]\\
%\frac{\sqrt{2}A_2}{\mu_s r^2}=-\frac{2}{3}\rho_2\theta_1 C_{21}r^2\omega_{
   rel}^2+\theta_{2}\Lambda_{rel},&\
%r\in \[1,2],\\
%\omega_{rel}=\omega_1,&\ r\in[1,2]
%
% Plot relative velocity(1) v/s different values of applied torque to the
   inner cylinder

parametrosmixture2;
clf;
Rf=2;
A=linspace(10,1e3,100)';
cbtheta1=0.2;
cbthetaaux=0;



eps1=1e-3;
omega=zeros(length(A),3);
```

```
for j=1:1:length(A)

A1=cbtheta1*A(j);
A2=A(j)-A1;
k=1;
while(k<100)
cbvel=A1/2/muf.*(1./Rf.^2-1./R2.^2);

Presion=sqrt(2)*A2./mus./Rf.^2;
theta02=findtheta20(Presion,cbvel,Rf);

if(isnan(theta02))
disp('theta_{02} is not a number');
else
    fprintf('theta02 %f\n',theta02);
end

lambda0=Presion/theta02+gamma2*2/3*(1-theta02)*Cdrag*cbvel^2*Rf^2;

eps=1e-10;

M = [1 0; 0 0];
set(1,'Userdata',[R2 A2]);
options = odeset('Mass',M,'RelTol',eps,'AbsTol',[eps eps]);
tic
[s,Y] = ode15s(@eqdiferencial,[0 Rf-R1],[lambda0 theta02],options);
toc
lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
cbthetaaux=theta1(end);
    if(abs(cbthetaaux-cbtheta1)<eps1)
        break;
    else
    cbtheta1=cbthetaaux;
    end
k=k+1;

A1=cbtheta1*A(j);
A2=A(j)-A1;
set(1,'Userdata',[R2 A2]);

end
r=Rf-s;
Presion=sqrt(2)*A2./mus./r.^2;
```

```
velr=sqrt(3/2*(Presion-theta2.*lambda).*(theta2.*gamma2.*theta1*Cdrag.*r.^2).^-1);
omega11=A1/2/muf.*(1./r.^2-1./R2.^2);
omega22=omega11-velr;
omega(j,1)=omega11(end);
omega(j,2)=omega22(end);
omega(j,3)=velr(end);
end
figure(1)
plot(A,omega(:,1),'r*',A,omega(:,2),'b*');
grid on;
legend('\omega_1(R_1)','\omega_2(R_1)')
title('Angular velocities v/s applied torque in the inner cylinder.')
xlabel('A')
figure(2)
plot(A,omega(:,3),'b');
grid on;
legend('\omega_{rel}(R_1)[m \cdot s^{-1}]')
title('Relative angular velocity on the inner cylinder v/s applied torque on the inner cylinder.')
xlabel('A [kg\cdot m^{-1}\cdot s^{-2}]')

function dy = eqdiferencial(s,y)
parametrosmixture2;
a=get(1,'Userdata');
rfinal=a(1);
r=rfinal-s;
lambda=y(1);
theta2=y(2);
A2=a(2);
theta1=1-theta2;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2*lambda)*(theta2*gamma2*theta1*Cdrag*r^2)^-1);
dy=zeros(2,1);
dy(1)=4*r*Cdrag*(theta2*gamma1-theta1*gamma2)*velr^2;
dy(2)=-lambda*theta2*theta1+(theta2*gamma2+gamma1*theta1)*Cdrag*r^2*velr^2*theta2*theta1...
     +c2*(theta2^2-thetac2^2)*theta1-c1*(theta1^2-thetac1^2)*theta2;

end

function [theta02,Fval]=findtheta20(Presion,cbvel,Rf)
parametrosmixture2;
x=zeros(1,10);
ci=linspace(0,1,10);
tol=10e-4;
Fval=zeros(1,10);
```

```
for k=1:10
    [x(k), Fval(k)] = fsolve(@(x) FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf),ci(k));
end
i= x<1 & x>0 & abs(Fval)<tol;
theta02=mode(x(i));
end

function sys=FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf)
sys=-(Presion+2/3*gamma2*x*(1-x)*Cdrag*Rf^2*cbvel^2)*(1-x)...
    +Rf^2*cbvel^2*Cdrag*(gamma1+(gamma2-gamma1)*x)*x*(1-x)...
    +c2*(x^2-thetac2^2)*(1-x)-c1*x*((1-x)^2-thetac1^2);
end
```

Listing B.3: MATLAB code for Figure 3.11

```
% This functions solves the DAE system using the function ode15s
%\Lambda_{rel}'=-4rC_{21}(\gamma_1\theta_2-\gamma_2\theta_1)\omega_{rel}^2,&\ r\in\[1,2]\\
%\theta_1\theta_2\Lambda_{rel}=\theta_1\theta_2C_{21}\left(\rho_1+\rho_2\right)r^2\omega_{rel}^2
%+c_2\theta_1\left(\theta_2^2-\theta_c^2\right)
%-c_1\theta_2\left(\theta_1^2-\theta_{1,c}^2\right),&\ r\in [1,2]\\
%\frac{\sqrt{2}A_2}{\mu_s r^2}=-\frac{2}{3}\rho_2\theta_1 C_{21}r^2\omega_{rel}^2+\theta_{2}\Lambda_{rel},&\
%r\in \[1,2],\\
%\omega_{rel}=\omega_1,&\ r\in[1,2]
%
% Plot relative velocity v/s r for different values of C_{21}

parametrosmixture3;
clf;
Rf=2;
A=1e3;
Cdrag=linspace(1000,10000,5)';
cbtheta1=0.5;
cbthetaaux=0;



eps1=1e-3;
omega=zeros(300,length(Cdrag));
theta=zeros(100,length(Cdrag));
Lambda=zeros(100,length(Cdrag));
for j=1:1:length(Cdrag)
A1=cbtheta1*A;
A2=A-A1;
```

```matlab

k=1;
while(k<100)
cbvel=A1/2/muf.*(1./Rf.^2-1./R2.^2);

Presion=sqrt(2)*A2./mus./Rf.^2;
theta02=findtheta20(Presion,cbvel,Rf,Cdrag(j))

if(isnan(theta02))
disp('theta_{02} is not a number');
else
    fprintf('theta02 %f\n',theta02);
end

lambda0=Presion/theta02+gamma2*2/3*(1-theta02)*Cdrag(j)*cbvel^2*Rf^2;

eps=1e-10;

M = [1 0; 0 0];
set(1,'Userdata',[R2 A2 Cdrag(j)]);
options = odeset('Mass',M,'RelTol',eps,'AbsTol',[eps eps]);
tic
[s,Y] = ode15s(@eqdiferencial,[0 Rf-R1],[lambda0 theta02],options);
toc
lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
cbthetaaux=theta1(end);
    if(abs(cbthetaaux-cbtheta1)<eps1)
        break;
    else
    cbtheta1=cbthetaaux;
    end
k=k+1;

A1=cbtheta1*A;
A2=A-A1;
set(1,'Userdata',[R2 A2 Cdrag(j)]);

end

lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
r=Rf-s;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2.*lambda).*(theta2.*gamma2.*theta1*Cdrag(j).*r
```

```
    .^2).^-1);
omega11=A1/2/muf.*(1./r.^2-1./R2.^2);
omega22=omega11-velr;

omega111 = interp1(s,omega11,linspace(0,1,100)');
omega222 = interp1(s,omega22,linspace(0,1,100)');
velr333  = interp1(s,velr,linspace(0,1,100)');

theta222 = interp1(s,theta2,linspace(0,1,100)');
lambda333= interp1(s,lambda,linspace(0,1,100)');

omega(1:100,j)=omega111;
omega(101:200,j)=omega222;
omega(201:300,j)=velr333;
theta(1:100,j)=theta222;
Lambda(1:100,j)=lambda333;
end
figure(1)
r=linspace(2,1,100);
plot(r,omega(201:300,1),'r',r,omega(201:300,2),'b',r,omega(201:300,3),'g',r,
    omega(201:300,4),'k',r,omega(201:300,5),'m');
grid on;
legend('C_{21}=1000 [kg\cdot m^{-1}\cdot s^{-2}] ','C_{21}=3250 [kg\cdot m
    ^{-1}\cdot s^{-2}]','C_{21}=5500 [kg\cdot m^{-1}\cdot s^{-2}] '...
     ,'C_{21}=7750 [kg\cdot m^{-1}\cdot s^{-2}]','C_{21}=10000 [kg\cdot m
        ^{-1}\cdot s^{-2}]')
title('Relative angular velocity for different drag values.')
xlabel('r')
ylabel('\omega_{rel}')

figure(2)
plot(r,theta(1:100,1),'r',r,theta(1:100,2),'b',r,theta(1:100,3),'g',r,theta
    (1:100,4),'k',r,theta(1:100,5),'m');
grid on;
legend('C_{21}=1000 [kg\cdot m^{-1}\cdot s^{-2}] ','C_{21}=3250 [kg\cdot m
    ^{-1}\cdot s^{-2}]','C_{21}=5500 [kg\cdot m^{-1}\cdot s^{-2}] '...
     ,'C_{21}=7750 [kg\cdot m^{-1}\cdot s^{-2}]','C_{21}=10000 [kg\cdot m
        ^{-1}\cdot s^{-2}]')
title('Volume fraction for different drag values.')
xlabel('r')
ylabel('\theta')


figure(3)
plot(r,Lambda(1:100,1),'r',r,Lambda(1:100,2),'b',r,Lambda(1:100,3),'g',r,
    Lambda(1:100,4),'k',r,Lambda(1:100,5),'m');
grid on;
```

```
legend('C_{21}=1000 [kg\cdot m^{-1}\cdot s^{-2}] ','C_{21}=3250 [kg\cdot m^{-1}\cdot s^{-2}]','C_{21}=5500 [kg\cdot m^{-1}\cdot s^{-2}] '...
    ,'C_{21}=7750 [kg\cdot m^{-1}\cdot s^{-2}]','C_{21}=10000 [kg\cdot m^{-1}\cdot s^{-2}]')
title('Volume fraction for different drag values.')
xlabel('r')
ylabel('\Lambda')
function dy = eqdiferencial(s,y)
parametrosmixture3;
a=get(1,'Userdata');
rfinal=a(1);
r=rfinal-s;
lambda=y(1);
theta2=y(2);
A2=a(2);
Cdrag=a(3);
theta1=1-theta2;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2*lambda)*(theta2*gamma2*theta1*Cdrag*r^2)^-1);
dy=zeros(2,1);
dy(1)=4*r*Cdrag*(theta2*gamma1-theta1*gamma2)*velr^2;
dy(2)=-lambda*theta2*theta1+(theta2*gamma2+gamma1*theta1)*Cdrag*r^2*velr^2*theta2*theta1...
      +c2*(theta2^2-thetac2^2)*theta1-c1*(theta1^2-thetac1^2)*theta2;

end

function [theta02,Fval]=findtheta20(Presion,cbvel,Rf,Cdrag)
parametrosmixture3;
x=zeros(1,10);
ci=linspace(0,1,10);
tol=10e-4;
Fval=zeros(1,10);
for k=1:10
    [x(k), Fval(k)] = fsolve(@(x) FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf),ci(k));
end
i= x<1 & x>0 & abs(Fval)<tol;
theta02=mode(x(i));
end

function sys=FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf)
sys=-(Presion+2/3*gamma2*x*(1-x)*Cdrag*Rf^2*cbvel^2)*(1-x)...
    +Rf^2*cbvel^2*Cdrag*(gamma1+(gamma2-gamma1)*x)*x*(1-x)...
    +c2*(x^2-thetac2^2)*(1-x)-c1*x*((1-x)^2-thetac1^2);
end
```

Listing B.4: MATLAB code for Figure 3.12

```
% This functions solves the DAE system using the function ode15s
%\Lambda_{rel}'=-4rC_{21}(\gamma_1\theta_2-\gamma_2\theta_1)\omega_{rel}^2,
    &\ r\in\[1,2]\\
%\theta_1\theta_2\Lambda_{rel}=\theta_1\theta_2C_{21}\left(\rho_1+\rho_2\
    right)r^2\omega_{rel}^2
%+c_2\theta_1\left(\theta_2^2-\theta_c^2\right)
%-c_1\theta_2\left(\theta_1^2-\theta_{1,c}^2\right),&\ r\in [1,2]\\
%\frac{\sqrt{2}A_2}{\mu_s r^2}=-\frac{2}{3}\rho_2\theta_1 C_{21}r^2\omega_{
    rel}^2+\theta_{2}\Lambda_{rel},&\
%r\in \[1,2],\\
%\omega_{rel}=\omega_1,&\ r\in[1,2]
%
% Plot \theta_2 v/s r for diferent values of c_1 and c_2

parametrosmixture4;
clf;
Rf=2;
A=1e3;
Cdrag=5000;
cbtheta1=0.5;

c11=[10;100;1000;10000];
c22=[20;200;2000;20000];

eps1=1e-3;
omega=zeros(300,length(Cdrag));
theta=zeros(100,length(Cdrag));
Lambda=zeros(100,length(Cdrag));
for j=1:1:length(c11)
A1=cbtheta1*A;
A2=A-A1;
c1=c11(j);c2=c22(j);
k=1;
while(k<100)
cbvel=A1/2/muf.*(1./Rf.^2-1./R2.^2);

Presion=sqrt(2)*A2./mus./Rf.^2;
theta02=findtheta20(Presion,cbvel,Rf,Cdrag,c1,c2);

if(isnan(theta02))
disp('theta_{02} is not a number');
else
    fprintf('theta02 %f\n',theta02);
end
```

```
lambda0=Presion/theta02+gamma2*2/3*(1-theta02)*Cdrag*cbvel^2*Rf^2;

eps=1e-10;

M = [1 0; 0 0];
set(1,'Userdata',[R2 A2 Cdrag c1 c2]);
options = odeset('Mass',M,'RelTol',eps,'AbsTol',[eps eps]);
tic
[s,Y] = ode15s(@eqdiferencial,[0 Rf-R1],[lambda0 theta02],options);
toc
lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
cbthetaaux=theta1(end);
    if(abs(cbthetaaux-cbtheta1)<eps1)
        break;
    else
    cbtheta1=cbthetaaux;
    end
k=k+1;

A1=cbtheta1*A;
A2=A-A1;
set(1,'Userdata',[R2 A2 Cdrag c1 c2]);

end

lambda=Y(:,1);
theta2=Y(:,2);
theta1=1-theta2;
r=Rf-s;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2.*lambda).*(theta2.*gamma2.*theta1*Cdrag.*r.^2)
   .^-1);
omega11=A1/2/muf.*(1./r.^2-1./R2.^2);
omega22=omega11-velr;

omega111 = interp1(s,omega11,linspace(0,1,100)');
omega222 = interp1(s,omega22,linspace(0,1,100)');
velr333  = interp1(s,velr,linspace(0,1,100)');

theta222 = interp1(s,theta2,linspace(0,1,100)');
lambda333= interp1(s,lambda,linspace(0,1,100)');

omega(1:100,j)=omega111;
omega(101:200,j)=omega222;
```

```
omega(201:300,j)=velr333;
theta(1:100,j)=theta222;
Lambda(1:100,j)=lambda333;
end
figure(1)
r=linspace(2,1,100);
plot(r,omega(201:300,1),'r',r,omega(201:300,2),'b',r,omega(201:300,3),'m',r,
    omega(201:300,4),'k');
grid on;
legend('c_{1}=10  and c_{2}=20 [kg\cdot m^{-1}\cdot s^{-2}]',....
       'c_{1}=100 and c_{2}=200 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=1000  and c_{2}=2000 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=10000 and c_{2}=20000 [kg\cdot m^{-1}\cdot s^{-2}]')
title('Relative angular velocity for different drag values.')
xlabel('r')
ylabel('\omega_{rel}')

figure(2)
plot(r,theta(1:100,1),'r',r,theta(1:100,2),'b',r,theta(1:100,3),'m',r,theta
    (1:100,4),'k');
grid on;
legend('c_{1}=10  and c_{2}=20 [kg\cdot m^{-1}\cdot s^{-2}]',....
       'c_{1}=100 and c_{2}=200 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=1000  and c_{2}=2000 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=10000 and c_{2}=20000 [kg\cdot m^{-1}\cdot s^{-2}]')
title('Volume fraction of the solid for different values of c_1 and c_2.')
xlabel('r')
ylabel('\theta_2')


figure(3)
plot(r,Lambda(1:100,1),'r',r,Lambda(1:100,2),'b',r,Lambda(1:100,3),'m',r,
    Lambda(1:100,4),'k');
grid on;
legend('c_{1}=10  and c_{2}=20 [kg\cdot m^{-1}\cdot s^{-2}]',....
       'c_{1}=100 and c_{2}=200 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=1000  and c_{2}=2000 [kg\cdot m^{-1}\cdot s^{-2}]',...
       'c_{1}=10000 and c_{2}=20000 [kg\cdot m^{-1}\cdot s^{-2}]')
title('Relative pressure of the mixture for different drag values.')
xlabel('r')
ylabel('\Lambda')
function dy = eqdiferencial(s,y)
parametrosmixture4;
a=get(1,'Userdata');
rfinal=a(1);
r=rfinal-s;
lambda=y(1);
```

```
theta2=y(2);
A2=a(2);
Cdrag=a(3);
c1=a(4);
c2=a(5);
theta1=1-theta2;
Presion=sqrt(2)*A2./mus./r.^2;
velr=sqrt(3/2*(Presion-theta2*lambda)*(theta2*gamma2*theta1*Cdrag*r^2)^-1);
dy=zeros(2,1);
dy(1)=4*r*Cdrag*(theta2*gamma1-theta1*gamma2)*velr^2;
dy(2)=-lambda*theta2*theta1+(theta2*gamma2+gamma1*theta1)*Cdrag*r^2*velr^2*
    theta2*theta1...
      +c2*(theta2^2-thetac2^2)*theta1-c1*(theta1^2-thetac1^2)*theta2;

end

function [theta02,Fval]=findtheta20(Presion,cbvel,Rf,Cdrag,c1,c2)
parametrosmixture4;
x=zeros(1,10);
ci=linspace(0,1,10);
tol=10e-4;
Fval=zeros(1,10);
for k=1:10
    [x(k), Fval(k)] = fsolve(@(x) FindBoundaryConditions(x,Presion,thetac2,
        thetac1,c2,c1,gamma1,gamma2,Cdrag,cbvel,Rf),ci(k));
end
i= x<1 & x>0 & abs(Fval)<tol;
theta02=mode(x(i));
end

function sys=FindBoundaryConditions(x,Presion,thetac2,thetac1,c2,c1,gamma1,
    gamma2,Cdrag,cbvel,Rf)
sys=-(Presion+2/3*gamma2*x*(1-x)*Cdrag*Rf^2*cbvel^2)*(1-x)...
    +Rf^2*cbvel^2*Cdrag*(gamma1+(gamma2-gamma1)*x)*x*(1-x)...
    +c2*(x^2-thetac2^2)*(1-x)-c1*x*((1-x)^2-thetac1^2);
end
```

Listing B.5: MATLAB code of the auxiliary function integra.m

```

function f1=integra(y,ci,f)
for j=1:1:length(y)
    s=linspace(y(j),y(end),100);
    nu1=interp1(y,f,s);
    f1(j)=ci+trapz(s,nu1);
end
end
```

Listing B.6: MATLAB code of the auxiliary code parametrosmixture2.m

```
muf=0.001;
gamma1=1000;

mus=723;
gamma2=2200;
dchica=1e-4;

c2=20;
c1=10;

thetac2=0.56;
thetac1=1-thetac2;

R1=1;
R2=2;

Cdrag=1000;
```

Listing B.7: MATLAB code of the auxiliary code parametrosmixture3.m

```
muf=0.001;
gamma1=1000;

mus=723;
gamma2=2200;
dchica=1e-4;

c2=20;
c1=10;

thetac2=0.56;
thetac1=1-thetac2;



R1=1;
R2=2;
```

Listing B.8: MATLAB code of the auxiliary code parametrosmixture4.m

```
muf=0.001;
gamma1=1000;

mus=723;
gamma2=2200;
dchica=1e-4;

```

```
thetac2=0.56;
thetac1=1-thetac2;



R1=1;
R2=2;
```

Bibliography