UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE FÍSICA
DEPARTAMENTO DE INGENIERIA MATEMÁTICAS

SKYRMION EXPLOSIONS

TESIS PARA OPTAR AL GRADO DE
MAGÍSTER EN CIENCIAS, MENCIÓN FÍSICA
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMATICO

CARLOS ANDRÉS SAJI SANTANDER

PROFESOR GUÍA:
ÁLVARO NÚÑEZ VÁSQUEZ

MIEMBROS DE LA COMISIÓN:
JUAN DÁVILA BONCZOS
LUIS FOA TORRES
RODRIGO ARIAS FEDERICI

Este trabajo ha sido parcialmente financiado por Proyecto Fondecyt No. 1150072 and Center for the Development of Nanoscience and Nanotechnology CEDENNA FB0807

SANTIAGO DE CHILE
2018

RESUMEN DE LA MEMORIA PARA OPTAR
AL TÍTULO DE MAGÍSTER EN CIENCIAS, MENCIÓN FÍSICA
POR: CARLOS ANDRÉS SAJI SANTANDER
FECHA: 2018
PROF. GUÍA: ÁLVARO NÚÑEZ VÁSQUEZ

SKYRMION EXPLOSIONS

Los skyrmions son texturas magneticas que poseen propiedades que las hacen el objeto de estudio de diversas areas de la fisica teorica, matematica, nanotecnologia, etc. Una de ellas es su proteccion topologica que les otorga una estabilidad emergente de gran interes tecnológico. En este contexto es de mucha importancia el estudiar las pequenas fluctuaciones en torno a un skyrmion, las cuales se conocen como ondas de spin o campo de magnones.

En esta tesis, estudiaremos la dinamica conjunta del sistema skyrmion-magnones, en contraste con la literatura, donde tipicamente son consideradas como independientes. Especificamente, veremos como la dinamica propia del skyrmion genera ondas de spin y, como estas, a su vez, afectan al skyrmion en forma de reaccion de radiación eEn una coreografia análoga al modelo de Abraham-Lorentz del electrón. Estudiaremos además, basando nuestro análisis en dicha analogía, el origen de la masa de los skyrmion, entregando su dependencia en cantidades experimentalmente accequibles.

Por otra parte, actualmente los skyrmions son de mucho interes en el posible nuevo desarrollo de circuitos logicos, en los cuales los skirmions representan bits binarios. De esta manera el estudio de la aniquilacion de un skyrmions es de suma importancia. Estudiaremos el problema de la explosion de un skymion (blow up) y derivaremos la dinámica de la explosion con la consecuente emision de ondas de spin en forma de radiacion. El espectro de emisión correspondiente es completamente caracterizado, entregando una distribución de Fock-Breit-Wigner.

"Although the wind blow against
the powerful work continues,
You can make a stanza.
Never stop dreaming,
because in a dream, man is free (...)
(Walt Whitman)"

Agradecimientos

A mis profesores guía Álvaro Núñez y Juan Dávila por aceptarme en este largo proyecto, y por su siempre amable e incondicional disposición de ayudarme a lo largo de la realización de éste.

A mis amigos del $DIM \cup DFI$ por su compañia y amistad, por todos los buenos y maravillosos momentos compartidos. Especial agradecimientos a mis amigos: Calisto, Bob, JP, Ricardo, Mauro, Camilo, Panchito, Pedro, Seba, Vale, Ocho, Edgardo, Chaparron, Piero, Paul, Hasson, Hugo, Alvarito, Ambuli, Joselito. A todos, muchas gracias por su amistad, y compañia por tantos años.

A mis padres por brindarme siempre su apoyo incondicional en todo y motivarme a dar siempre lo mejor de mi.

Table of Contents

Introduction

We will study the dynamics of $2D-$magnetic textures within the theoretical framework established in the micro magnetic formalism. The latter is an effective theory of large wavelength excitations based on the phenomenology developed by Landau and Lifshitz. In the Landau-Lifshitz theory [18], magnetization is a continuous field in space-time at values on the unit sphere $\vec{s} : \mathbb{R}^{2+1} \to S^2$, whose dynamics is governed by the Landau-Lifshitz equation:

$$\frac{\partial \vec{s}(x,t)}{\partial t} = \vec{s}(x,t) \times \frac{\delta \mathcal{H}}{\delta \vec{s}(x,t)} \tag{1}$$

$$\vec{s}(x,0) = \vec{s}_0(x) \in S^2 \quad \text{for } x \in \Omega \tag{2}$$

where Ω is the spacedomain in the space of the system, and $\mathcal{H}$ is the energy that, in the continuous approach, can be written as [12]:

$$\mathscr{H}(\vec{s}) = \int_{\Omega} J \ |\nabla \vec{s}|^2 + D \ \vec{s} \cdot \nabla \times \vec{s} + \vec{B} \cdot \vec{s} \ \ \mathrm{d}^2 x \tag{3}$$

where the first term is associated with the energy of exchange, the second is the term of Dzyloshinskii-Moriya, and the third is the energy by an external field $\vec{B}$.

In addition, if we consider dissipation into the problem, the dynamics is governed the Landau-Lifshitz-Gilbert equation [4]:

$$\frac{\partial \vec{s}(x,t)}{\partial t} = \gamma \ \vec{s}(x,t) \times \frac{\delta \mathcal{H}}{\delta \vec{s}(x,t)} - \alpha \ \vec{s}(x,t) \times (\vec{s}(x,t) \times \frac{\delta \mathcal{H}}{\delta \vec{s}(x,t)}) \tag{4}$$

$$\vec{s}(x,0) = \vec{s}_0(x) \in S^2 \in \Omega \tag{5}$$

. In this context, skyrmions are stationary solutions of the Landau-Lifshitz-Gilbert equation in the plane 2D, distinguished meanly by his topological protection, which allows them to be considered as particles objects (solitons) described by a finite number of parameters. From a mathematical point of view, they play a fundamental role in a wide branch of physical theories: condensed matter, particle physics, string theory, etc.

In the limiting case where only exchange energy is relevant, we can find, by inspection, the skyrmion bubble, which has the form:

$$\vec{U}(r,\theta) = \mathcal{R}_\omega \begin{pmatrix} \sin(\Theta(r))\cos(\theta) \\ \sin(\Theta(r))\sin(\theta) \\ \cos(\Theta(\rho)) \end{pmatrix}$$

where:

$$\Theta(r) = 2\arctan\left(\frac{|x-\xi|}{\lambda}\right)$$

here ξ the position of the skyrmion, λ the radius of the skyrmion, and $\mathcal{R}_\omega$ a rotation in the phase sphere ω around axis z. Skyrmions are topological solitons with topological charge (see the appendix):

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} \vec{U}(x)\cdot\left(\frac{\partial \vec{U}(x)}{\partial x^1}\times\frac{\partial \vec{U}}{\partial x^2}\right)\, \mathrm{d}^2x = \deg(\vec{U}) = 1$$

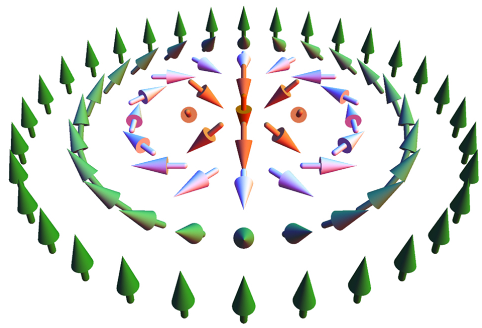


Figura 1: skyrmion of topological charge $Q = 1$, where the violet arrows demarcates the perimeter of the skyrmion and phase.

We explore the method of collective variables and the Lagrangian method, in order to study the linealizated Landau-Lifshitz-Gilbert equation in the proximity of a Skyrmion with time dependent parameters. We will see that such exitations, called spin waves [4], can be related to a Schrodinger equation with a vector and scalar potential determined by the evolution of the skyrmion parameters, in addition with a source term. In this way, we will explore the problem the emergent electrodynamics and the Aharonov-Bohm effects in this context [12], [30], [6]. Since the source term in the evolution equation of the spin waves, a wave generation is expected at distances far away from the skyrmion. We will study the radiation emitted by the skyrmion collapsing, as well as its spectrum, and we will examine the back reaction effect of this radiation field over the skyrmion dynamics. We derive a reaction force on the skyrmion by the magnons field, in a direct analogy with the Lorentz electron model. In particular, we find how appears a effective mass of the skyrmion due to this interaction phenomena.

Furthermore, nowadays skyrmions are very studied in the design of logic circuits in nanotechnology, where skyrmions do the role of binary bits, specifically, the creation or annihilation of a skyrmion represents the change of a bit. In chapter 3 we will study the dynamics of skyrmion near the blow up the, which by the way, is strongly related with the change of the topological charge toward the ferromagnetic ground state [32].

Capítulo 1

Dynamics of the Magnon-Skyrmion System

1.1. Preliminaries

Linear excitation of a dynamical system play a fundamental role, for instance, in the study of the stability of a particular configuration. From a conceptual point of view, this excitations can also emerges by fluctuating degrees of freedom. The mean goal in this chapter is to study this excitations, called the magnons fields in the literature [12], [4], and how the dynamics of the skyrmion affects it and vice-versa.

Specifically, in this chapter we will study the dynamics of the skyrmion-magnons system using different approaches:

- Collective Variables
- Effective Field Theory

Throughout this chapter, we will focus in the case $D = 0$ and $\vec{B} = 0$ in (3). Furthermore, in order to simplify the equations as much as possible, it will be convenient do the scaling $x \to D^{1/2}x$. In consequence, the Hamiltonian for a field u will be:

$$\mathscr{H} = \int_{\mathbb{R}^2} |\nabla u|^2 \ \mathrm{d}^2 x$$

It is well known that the variation of this energy functional is [19]:

$$\frac{\delta \mathscr{H}}{\delta u} = \Delta u + |\nabla u|^2 u$$

where the term $|\nabla u|^2$ in the right hand side, comes from the Lagrange multiplier due to the geometric restriction $|u|^2 = 1$.

Definición 1.1 *Let $\alpha > 0, \beta$ real fixed numbers, and $u : \Omega \to \mathbb{R}^3$. We denote J_u the matrix* *valued function in Ω defined by:*

$$v \to J_u v = \alpha\ v + \beta\ u \wedge v$$

where $\wedge$ denotes the cross product in $\mathbb{R}^3$.

Lema 1.2 *For $\alpha > 0, \beta$ rel numbers such that $\alpha^2 + \beta^2 = 1$, then $J_u(x)$ is a orthogonal matrix for all $x \in \Omega$*

With this notation, the LLG equation can be written in the form:

$$\partial_t u = J_u \frac{\delta \mathscr{H}}{\delta u} = J_u \left(\Delta u + |\nabla u|^2 u \right) \tag{1.1}$$

From this, the steady solutions u^0 of (1.1) are given by:

$$\Delta u^0 + \left|\nabla u^0\right|^2 u^0 = 0$$

solutions of this equation are known as harmonic maps onto the sphere in the literature [19]. In this context, skyrmions bubble are a particular case of harmonic map, given by the following definition:

Definición 1.3 *We call the skyrmion bubble of parameters $q = (\lambda, \omega, \xi)$, where ξ denotes the position, λ the radius, and $\mathcal{R}_\omega$ a rotation in the sphere in an angle ω around axis z; to the map:*

$$U_q(r, \theta) = \mathcal{R}_\omega \begin{pmatrix} \sin(\Theta(r)) \cos(\theta) \\ \sin(\Theta(r)) \sin(\theta) \\ \cos(\Theta(r)) \end{pmatrix} \tag{1.2}$$

where:

$$\Theta(r) = 2 \arctan \left(\frac{|x - \xi|}{\lambda} \right)$$

Alternatively, for sake of brevity, we will use sometimes the spinor representation, namely:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x + \mathrm{i}y \\ z \end{pmatrix}$$

In this way, in the spinor representation :

$$U_q = \begin{pmatrix} \sin(\Theta)\ \mathrm{e}^{\mathrm{i}\theta} \\ \cos(\Theta) \end{pmatrix}$$

Consider $q = (\lambda_0, \omega_0, \xi_0)$ a constant set of parameters, then from (A.1), U_q is an harmonic map, and hence a steady solution of (1.1).

We are interested in describe the linear excitations φ around U, so we need a convenient basis to work in this space.

Definición 1.4 *Let $U : \mathbb{R}^2 \to S^2$ be from (4.11), and consider $\varphi : \mathbb{R}^2 \to \mathbb{T}S^2$, i.e, such that $\varphi(x) \in \mathbb{T}_{U(x)}S^2$ for all x. We define:*

$$E_1 = \frac{\partial_r U}{|\partial_r U|} \ , \qquad E_2 = \frac{\partial_\theta U}{|\partial_\theta U|}$$

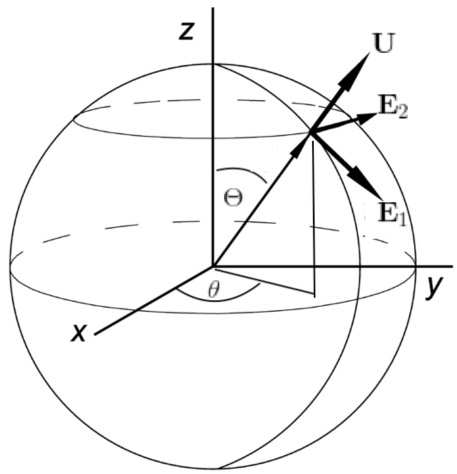


Figura 1.1: Graphics of the orthogonal frame $\{E_1, E_2, U\}$

A straightforward derivation using the spinor representation, gives the formulas:

$$E_1 = \begin{pmatrix} \cos(\Theta)e^{i\phi} \\ -\sin(\Theta) \end{pmatrix}, \qquad E_2 = \begin{pmatrix} ie^{i\phi} \\ 0 \end{pmatrix}$$

By definition $\{E_1(x), E_2(x)\}$ forms an orthonormal basis in the tangent space of the unit sphere at the point $U(x)$: $\mathbb{T}_{U(x)}S^2$. This basis allows us to compute the partial derivatives of U_q now seen as function of $q = (\lambda, \omega, \xi)$.

Lema 1.5 *[17] Consider $U_q(x)$ and $\{E_1(x), E_2(x)\}$. Then we have the formulas:*

$$\begin{aligned}
\partial_\lambda U_q &= \frac{r}{r^2+\lambda^2} E_1 \\
\partial_\omega U_q &= \frac{\lambda r}{r^2+\lambda^2} E_2 \\
\partial_{\xi_1} U_q &= \frac{\lambda}{r^2+\lambda^2} (\cos(\theta)\ E_1 + \sin(\theta)\ E_2) \\
\partial_{\xi_2} U_q &= \frac{\lambda}{r^2+\lambda^2} (\sin(\theta)\ E_1 - \cos(\theta)\ E_2)
\end{aligned}$$

A very important and useful result derived from differential geometry, that we will need is the following:

Teorema 1.6 *Let $U : \mathbb{R}^2 \to S^2$ be an harmonic map given by:*

$$U = \begin{bmatrix} \sin(\Theta)e^{i\theta} \\ \cos(\Theta) \end{bmatrix}$$

Then the second variation of $\mathscr{H}$ at U evaluated in the direction φ, is given by:

$$\frac{\delta^2 \mathscr{H}}{\delta^2 u}[U](\varphi, \varphi) = \int_{\mathbb{R}^2} \left|\nabla^E \varphi\right|^2 + V(x) \left|\varphi\right|^2 \; \mathrm{d}^2 x \tag{1.3}$$

for all ϕ, where ∇^E is the pullback of the Levi-Civita connection in TS^2 by U

$$\begin{aligned} \nabla^E_{E_1} E_1 = 0 \;, &\quad \nabla^E_{E_1} E_2 = 0 \\ \nabla^E_{E_2} E_1 = E_2 \;, &\quad \nabla^E_{E_1} E_2 = -E_1 \end{aligned}$$

and:

$$V(x) = -\frac{1}{|x|^2} \sin^2(\Theta)$$

is scalar function.

Proof can be find in the appendix.

Definición 1.7 *We define the operator $\mathcal{L}_U$ given by*

$$\varphi \mapsto \mathcal{L}_U \varphi = (\nabla^E)^2(\varphi) - V\varphi \tag{1.4}$$

This allows us to write the equation (1.3) in the form

$$\frac{\delta^2 \mathscr{H}}{\delta^2 u}[U](\varphi, \varphi) = \int_{\mathbb{R}^2} \langle \varphi, -\mathcal{L}_U \varphi \rangle \; \mathrm{d}^2 x \tag{1.5}$$

At this point, it is useful to do the representation:

Definición 1.8 *We define the representation of $\mathbb{T}_U S^2$ onto the complex numbers given by:*

$$\mathbb{T}_U S^2 \ni \varphi = \varphi_1 \; E_1 + \varphi_2 \; E_2 \;\; \longmapsto \;\; \varphi_1 + \mathrm{i} \; \varphi_2 \in \mathbb{C} \tag{1.6}$$

We see that this representation allow us to work the perturbation φ as scalar complex field on the plane (More rigorously speaking, it is a complex line bundle). Also, in this representation we have: $J_U \cdot () \longmapsto \gamma \cdot ()$ where $\gamma = \alpha + \mathrm{i}\beta$, since $U \wedge E_1 = E_2$ and $U \wedge E_2 = -E_1$.

Using (1.4), and the complex representation, $\mathcal{L}_U$ turns into the operator:

$$\mathcal{L}_U = (\nabla - \mathrm{i}\mathbf{A})^2 + V \qquad \text{where:} \quad \mathbf{A} = \frac{\cos(\Theta)}{r} \; \hat{\theta} \tag{1.7}$$

This operator corresponds precisely to the Hamiltonian of a charged particle in a electromagnetic field, in quantum theory. In fact, we will see in the next section that the equation of the magnon field corresponds to a Schrodinger equation (with a source term which depend on the dynamics of the skyrmion).

Finally, we can find the value of $\partial_t U$, using the formulas in (1.5) and the complex representation, it is:

$$\partial_t U = \frac{r}{r^2+\lambda^2}(\dot{\lambda}+\lambda\dot{\omega}) + \frac{1}{r^2+\lambda^2}(\dot{\xi}_1 - \mathrm{i}\dot{\xi}_2)\,\mathrm{e}^{\mathrm{i}\theta} \tag{1.8}$$

Let us denote $Z_{\mathrm{i}} = \dot{q}^{\mathrm{i}}\partial_{q_{\mathrm{i}}}U$, then using the complex representation (1.8) and formula (1.7), we can check out the equation:

$$\mathcal{L}_U Z_{\mathrm{i}} = 0 \;\Rightarrow\; Z_{\mathrm{i}} \in \ker\, \mathcal{L}_U$$

In conclusion, we obtain an element of the kernel (a zero modes of $\mathcal{L}_U$ in physical literature) for each parameter of the skyrmion. On the other hand, we have seen that an arbitrary variation of parameters of the skyrmion preserves the energy $\mathcal{E}$, In consequence Z_{i} corresponds to the generators of the group of symmetries of the energy. We call it the *conformal group*, and it plays a fundamental role in the dynamics of the skyrmion near the singularity, as discussed below.

1.2. Spin Waves around a Dynamical skyrmion

This section we will find the dynamics of the magnon field around a dynamical skyrmion. The analysis here differs from literature, for instance: [6], [10], [12], by the fact that we do not consider rigid skyrmion necessarily.

In order not to abuse notation, denote $U(x,t) = U_q(x,t)$, where $q(t) = (\lambda(t), \omega(t), \xi(t))$ will be a fixed function driven the dynamics of the skyrmion, which we do not worry about the moment. Consider the spin field $u(x,t) = U(x,t) + \psi(x,t)$ with $\psi \perp U$ small enough such that we retain only linear order terms in LLG equation. Thus the linear part of LLG will be

$$\begin{aligned}
\partial_t \varphi &= \frac{\delta}{\delta u}\left(J_U \frac{\delta \mathscr{H}}{\delta u}\right)\varphi - \partial_t U && (1.9)\\
&= \frac{\delta}{\delta u}(J_U)\,\frac{\delta \mathscr{H}}{\delta u}\varphi + J_U \frac{\delta}{\delta u}\left(\frac{\delta \mathscr{H}}{\delta u}\right)\varphi - \partial_t U && (1.10)\\
&= J_U \frac{\delta}{\delta u}\left(\frac{\delta \mathscr{H}}{\delta u}\right)\varphi - \partial_t U && (1.11)
\end{aligned}$$

where we have used:

$$\frac{\delta \mathscr{H}}{\delta u}(U) = 0$$

since U is a steady solution of LLG.

In consequence, using (1.7), (1.9) can be written in the form:

$$\partial_t \varphi = \gamma\left((\nabla - \mathrm{i}\mathbf{A})^2 + V\right)\varphi - \partial_t U \tag{1.12}$$

For $\gamma = \mathrm{i}$ (i,e without damping term: $\alpha = 0$), 1.12 corresponds to a Schrodinger equation of a charged particle coupled to the vectorial potential $\mathbf{A}$ and scalar potential V, in addition with a source term $\mathcal{J} = -\mathrm{i}\partial_t U$. Note that, since $\mathcal{L}_U$ is hermitian, so $\mathcal{H}$ is hermitian if and only if $\alpha = 0$ (i.e without damping term), and it has complex spectrum given by $\gamma\, \sigma(\mathcal{L}_U) = \gamma\lambda^{-2}\, \sigma(\mathcal{L}_{U_0})$ (where $\sigma(T)$ denotes the set of eigenvalues of T).

A fundamental point to remark in the equation (1.9) is that for $\alpha = 0$ and $\partial_t U = 0$, then (1.12) is invariant under gauge transformations defined by:

$$\tilde{\psi} = \mathrm{e}^{\mathrm{i}\chi}\psi \quad \tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi \quad \tilde{V} = V + \partial_t\chi \tag{1.13}$$

for an arbitrary function $\chi(x,t)$. According to the Noether theorem, this symmetry must generate a conserved quantity.

We define the number of magnons [4] by the L^2 norm of ϕ:

$$\mathcal{N}_{\mathrm{mag}} = \int_\Omega |\phi|^2 \, \mathrm{d}^2x$$

and the magnon current by:

$$\mathcal{J}_{\mathrm{mag}} = \Im\left(\phi^\dagger(\nabla - \mathrm{i}\mathbf{A})\phi\right)$$

Then we have the conservation law of the number of magnons:

$$\frac{\mathrm{d}\mathcal{N}_{\mathrm{mag}}}{\mathrm{d}t} = \int_{\partial\Omega} \mathcal{J}_{\mathrm{mag}} \cdot \mathrm{d}\mathbf{S}$$

Remark: Usually it is preferable to work with fields $\mathbf{A}$ and V with more decay in space. For this, note that if $\chi = \theta$ then $\tilde{\mathbf{A}} = \frac{\cos\Theta + 1}{r}\partial_\phi$, so we get that $\tilde{\mathbf{A}} = o(\frac{1}{r})$ and so the magnetic flux is zero at the infinity except by a Dirac delta at the origin $B(x) = 2\pi\delta(x)$.

1.2.1. Force and dynamics of a rigid skyrmion

A measurable effects of the magnon field on the skyrmion, is the effective force that it does on the skyrmion. This effect has been studied in many papers [27], [37], [31], in the context of scattering of magnons by a skyrmion. We will review the force of the spin waves on a skyrmion.

The force over the skyrmion can be found using the LLG equation (see [12]) (1.1):

$$F_{\mathrm{i}} = \partial_{\xi_{\mathrm{i}}}\mathcal{E} = \int_{\mathbf{R}^2} \left\langle \frac{\delta\mathscr{H}}{\delta u}, \partial_{\xi_{\mathrm{i}}} u \right\rangle \, \mathrm{d}^2x = \int_{\mathbf{R}^2} \left\langle J_U^\top \partial_t u, \partial_{\xi_{\mathrm{i}}} u \right\rangle \, \mathrm{d}^2x \tag{1.14}$$

$$= \left\langle J_U^\top \partial_{\xi_{\mathrm{i}}} u, \partial_t u \right\rangle_{L^2} \tag{1.15}$$

For a single skyrmion $u(x,t) = U(x - \xi(t))$ with $\dot{\lambda} = \dot{\omega} = 0$, and using the formula:

$$\int_0^\infty \frac{\lambda^2}{(r^2+\lambda^2)^2} \, \mathrm{d}^2x = 4\pi$$

Thus (1.14) leads to:

$$F_{\mathrm{i}} = \int_{\mathbf{R}^2} \left\langle J_U^\top \partial_t U, \partial_{\xi_\mathrm{i}} U \right\rangle \, \mathrm{d}^2x = \left[\int_{\mathbf{R}^2} \left\langle J_U^\top \partial_{\xi_j} U, \partial_{\xi_\mathrm{i}} U \right\rangle \, \mathrm{d}^2x \right] \dot{\xi}_j$$

so we obtain the expression of force in vector form:

$$\mathbf{F} = \boldsymbol{\mathcal{D}}\mathbf{v} + \boldsymbol{\mathcal{G}}\mathbf{v} = 4\pi\alpha\mathbf{v} + 4\pi\beta\hat{\mathbf{z}} \wedge \mathbf{v} \tag{1.16}$$

where $\mathbf{v} = (\dot{\xi}_1, \dot{\xi}_2)^\top$ and

$$\mathcal{D}_{\mathrm{i}j} = \alpha \, \langle \partial_\mathrm{i} U, \partial_j U \rangle_{L^2} = 4\pi\alpha\delta_{\mathrm{i}j}$$
$$\mathcal{G}_{\mathrm{i}j} = \beta \, \langle U \wedge \partial_\mathrm{i} U, \partial_j U \rangle_{L^2} = 4\pi\beta\varepsilon_{\mathrm{i}j}$$

Equation (1.16) is the well known Thiele equation [12], where $\mathbf{v}$ is the force acting on the skyrmion.

At this point we have considered a rigid skyrmion ansatz. Let see how this expression changes if we add a small perturbation field φ. For $u(x,t) = U(x - \xi(t)) + \varphi(x,t)$ with φ small enough such that we retain only linear order terms:

$$F_{\mathrm{i}} = \left\langle J_U^\top \partial_t u, \partial_{\xi_\mathrm{i}} u \right\rangle_{L^2} = \left\langle J_U^\top \partial_{\xi_\mathrm{i}} U, \partial_t u \right\rangle_{L^2} \tag{1.17}$$

Suppose that there is not external forces over the system, hence $\mathbf{F}^{\mathrm{Thiele}} + \mathbf{F}^{\mathrm{mag}} = 0$, where it is added correction to the equation of Thiele (1.17):

$$F_{\mathrm{i}}^{\mathrm{mag}} = -\left\langle J_U \partial_{\xi_\mathrm{i}} U, \partial_t \varphi \right\rangle_{L^2} \tag{1.18}$$

The right side of (1.18) represents the reaction force by the spin waves on the skyrmion.

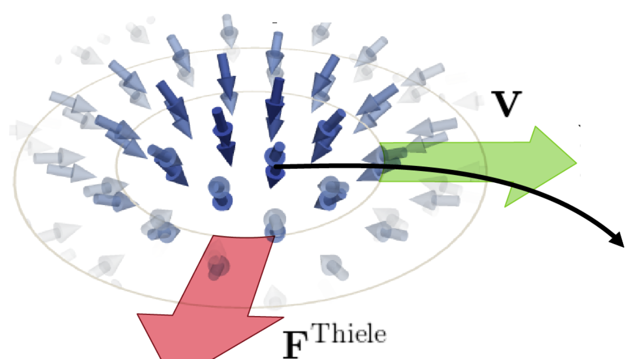


Figura 1.2: Direction of the Thiele's force $\mathbf{F}^{\mathrm{Thiele}} = 4\pi\hat{\mathbf{z}}\wedge\mathbf{v}$, and the trajectory of the skyrmion

1.3. Spectrum of Magnon field around a Skyrmion

It will be convenient to expand φ in angular Fourier Modes:

$$\varphi(r,\theta,t) = \sum_{l\in\mathbb{Z}} \varphi_l(r,t) \, \mathrm{e}^{\mathrm{i}l\theta} \tag{1.19}$$

Replacing (4.38) in (1.9) we have the set of evolution equations for $\varphi_l(r,t)$ for every mode $l \in \mathbb{Z}$:

$$\partial_t \varphi_l = \gamma \mathcal{L}_l \varphi_l - \partial_t U = \gamma \left(-\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \ V_l(r)\right) \varphi_l - \partial_t U \tag{1.20}$$

where we derive the effective potential for the angular modes:

$$V_l(r) = \frac{l^2 + 2l\ \cos(\Theta(r)) + \cos(2\Theta(r))}{r^2} \tag{1.21}$$

In this section we study the spectrum problem:

$$\mathcal{L}_l \phi_l = \varepsilon \phi_l \qquad l \in \mathbb{Z} \tag{1.22}$$

It will be convenient to do the change of the function: $\phi_l(r) = \frac{1}{\sqrt{r}} f_l(r)$, consequently we new operator will have to be:

Definición 1.9 *We introduce the operator:*

$$\mathcal{H}_l^f = -\partial_r^2 - \frac{1}{4r^2} + V_l$$

In this way, observe:

$$\mathcal{H}_l \phi_l = \frac{1}{\sqrt{r}}(\ -\partial_r^2 + [-\frac{1}{4r^2} + V_l]\ )\ f_l \equiv \frac{1}{\sqrt{r}}\ \mathcal{H}_l^f f_l$$

A fundamental result in this section is the following property:

Lema 1.10 [1]

$$\mathcal{H}_l^f = A_l^\dagger A_l \quad \textit{where} \quad A_l = \partial_r + F_l(r), \ \textit{and } F_l(r) = \frac{2r}{r^2+\lambda^2} + \frac{2l-3}{2r}$$

Definición 1.11 *We denote the zero modes of $\mathcal{H}_l$ by:*

$$z_l(r) = \sqrt{r} Z_l(r) = \sqrt{r}\frac{r^{1-l}}{r^2+\lambda^2}$$

the element of the kernel of $\mathcal{H}_l$, obtained by means of the equation $A_l z_l = 0$, and hence $\mathcal{H} Z_l = 0$

[1] *Proof:*

$$\begin{aligned}(-\partial_r + F_l(r))(\partial_r + F_l(r)) &= -\partial_r^2 + F_l^2(r) - F_l'(r)\\ &= -\partial_r^2 + \frac{4l}{r^2+\lambda^2} + \frac{(l-1)^2 - 1/4}{r^2} - \frac{8\lambda^2}{(r^2+\lambda^2)^2}\\ &= -\partial_r^2 - \frac{1}{4r^2} + V_l(r) = \mathcal{H}_l^f\end{aligned}$$

Consider now $(\mathcal{H}_l^f)^\dagger = A_l A_l^\dagger$. We can see that $(\mathcal{H}_l^f)$ and $(\mathcal{H}_l^f)^\dagger$ share the same spectrum, except by the zero mode. In effect: Let f be a proper vector of $\mathcal{H}_l^f$ with proper value $\omega \neq 0$ then:

$$A_l^\dagger A_l f = \omega f \implies A_l A_l^\dagger \tilde{f} = \omega \tilde{f}$$

where $\tilde{f} = A_l f$ is a proper vector of $(\mathcal{H}_l^f)^\dagger$. So evaluating $(\mathcal{H}_l^f)^\dagger = -\partial_r^2 + [F_l^2 + F_l']$ we have:

$$(\mathcal{H}_l^f)^\dagger = -\partial_r^2 \; + \; [\frac{4(l-1)}{r^2+\lambda^2} + \frac{(l-2)^2 - 1/4}{r^2} \;]$$

It is interesting to note that in the limit $\lambda \to 0$, the mode $\phi_l = J_{l+1}(kr)\mathrm{e}^{\mathrm{i}\theta}$, for all l. This results is expected as a consequence of Ahronov-Bohm effect due to the magnetic field

$$\mathbf{B}(x) = \frac{\lambda^2}{(r^2+\lambda^2)^2} \to \delta^2(x)$$

It is useful to have an explicit formula for $(\mathcal{H}_l^f)^{-1}$

Lema 1.12

$$(\mathcal{H}_l^f)^{-1} f(r) = A_l^{-1}(A_l^\dagger)^{-1} f(r) = z_l(r)\left(\int_0^r \frac{1}{z_l^2(r_1)}\left[\int_0^{r_1} z_l(r_2) f(r_2)\, \mathrm{d}r_2\right] \mathrm{d}r_1 + c\right)$$

where c is an arbitrary constant.

Proof: We can invert the operators A_l and $A_l^\dagger$ solving the equations $A_l f = g$ and $A_l^\dagger f = g$, respectively: $A_l f = g$:

$$A_l^{-1}(g)(r) = z_l\left(\int_0^r z_l^{-1}(r') f(r')\, \mathrm{d}r' + c\right) \tag{1.23}$$

$$(A_l^\dagger)^{-1}(g)(r) = z_l^{-1}\left(\int_0^r z_l(r') f(r')\, \mathrm{d}r' + c\right) \tag{1.24}$$

composing we have the result.

Observe that $\mathcal{H}_l^f = A_l^\dagger A_l$ is a positive operator then $E > 0$ for every eigenvalue, so we have continuum spectrum [2].

[2] For small energy $E = k^2$ values, we can do perturbation theory in order to find a serie expansion for ϕ_{lk}:

$$\phi(r) = \sum_{s\geq 0} \phi_s(r) k^{2s}$$

where we set $\phi_0(r) = Z_l(r)$, and where the $f_s(r)$ functions satisfies the following recurrence relation:

$$(-\nabla_r^2 + U_l(r))\; \phi_s(r) = \phi_{s-1}(r)$$

Thus we can solve the left hand side:

$$\phi_s(r) = \; \mathcal{H}_l^{-1}\phi_{s-1}(r) = Z_l(r)\int_r^\infty \frac{1}{r_1 Z_l^2(r_1)}\left[\int_0^{r_1} Z_l(r_2)\phi_{s-1}(r_2)\; r_2 \mathrm{d}r_2\right] \mathrm{d}r_1$$

This method allows us to know the spectrum of $\mathcal{H}_l$ in a analitic expansion serie in k.

Now let's study the asymptotic behaviour of the eigenfunctions. We can see directly that for $r \to 0$:

$$U(r) \approx \frac{(l-1)^2}{r^2}$$

Thus we have the asimptotics equation for $r \to 0$:

$$(-\nabla_r^2 + \frac{(l-1)^2}{r^2})\ \psi_{l,k}(r) \approx k^2\ \psi$$

whose solution give us the asymptotic behaviour of $\psi(r)$ for $r \to \infty$:

$$\psi_{l,k}(r) \approx J_{l-1}(kr)$$

were we have dropped the part from the Newman function $Y_{l-1}(r)$, since we are looking for bounded functions. On the other hand, if $r \to \infty$:

$$U(r) \approx \frac{(l+1)^2}{r^2}$$

Thus we have the asymptotic equation:

$$(-\nabla_r^2 + \frac{(l+1)^2}{r^2})\ \psi_{l,k}(r) \approx k^2\ \psi_{l,k}(r)$$

whose solution give us the asymptotic behavior of $\psi(r)$ for $r \to \infty$:

$$\psi_{l,k}(r) \approx A_l J_{l+1}(kr) + B_l Y_{l+1}(kr)$$

It is important to notice that when $l = 1$ then the spectrum have exact solution, since:

$$(\mathcal{H}^f_{l=1})^\dagger = -\partial_r^2\ + \frac{1-1/4}{r^2}$$

which have solution $\tilde{f}(r) = J_1(kr)$. Thus $f = A_1^{-1}\tilde{f}$. It is possible to prove that [6]:

$$\phi_{l=1,k}(r) = J_2(kr) + \frac{J_1(kr)}{kr}\frac{2}{1+(r/\lambda)^2} \tag{1.25}$$

Nevertheless, in general, it is not known yet an analytic expression for ϕ_{kl}.

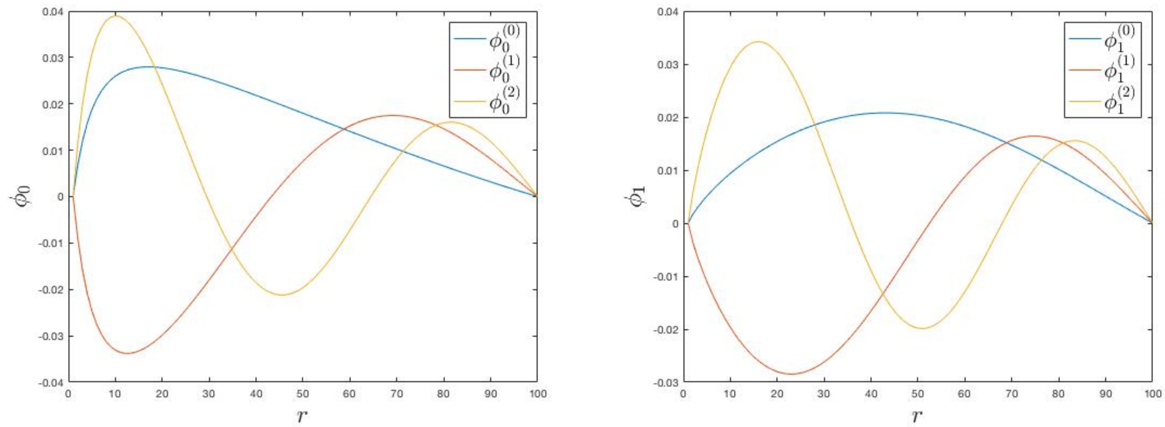
It is important remark that, since $\mathcal{H}_l$ is a hermitian operator, the set $\left\{\phi_{l,k}(r)\mathrm{e}^{\mathrm{i}l\theta}\right\}_{l,k}$ forms a complete orthogonal system in $L^2(\mathbb{R}^2)$ that is

$$\int_{\mathbb{R}^2} \phi_{l,k}(r)\phi_{l',k'}(r)\mathrm{e}^{\mathrm{i}(l-l')\theta}\ r\mathrm{d}r\ \mathrm{d}\theta = \frac{1}{k}\delta(k-k')\delta_{l,l'}$$

and the Green function is given by [1]:

$$G_\omega(r,r',\theta,\theta') = \lim_{\varepsilon\to 0}\sum_{l,k}\frac{\phi_{lk}(r)\phi_{lk}(r')}{\omega - k^2 + \mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}l(\theta-\theta')} \tag{1.26}$$

In chapter 2 we will find an approximated formula for the Green function (retarded).



(a) Graphics of $\phi_{l=0}(r)$ (b) Graphics of $\phi_{l=1}(r)$

Figura 1.3: Graphics of $\phi_l(r)$ with Dirichlet boundary condition

1.4. Collective variables approach

Let $u : \mathbb{R}^2 \times [0, T] \to S^2$ be a field which we looking for satisfying the Landau-Lifshitz equation, then it is well known we have a variational principle [33], [2] where the action is given by:

$$\mathcal{S}[u] = \int_0^T \int_{\mathbb{R}^2} \left(\mathcal{A}(u) \cdot \partial_t u - |\nabla u|^2 \right) \, \mathrm{d}^2 x \, \mathrm{d}t \tag{1.27}$$

where

$$\mathcal{A}(u) = \frac{(\cos(\Theta) - 1)}{\sin(\Theta)} \begin{bmatrix} \mathrm{i}e^{\mathrm{i}\theta} \\ 0 \end{bmatrix}, \qquad u = \begin{bmatrix} \sin(\Theta)e^{\mathrm{i}\theta} \\ \cos(\Theta) \end{bmatrix}$$

Evaluating at $u = U_q$ (4.11) with $q = q(t)$ time dependent, we have:

$$\begin{aligned} \mathcal{S}[U] &= \int_0^T \int_{\mathbb{R}^2} \left(\mathcal{A}(U) \cdot \partial_t U - |\nabla U|^2 \right) \, \mathrm{d}^2 x \, \mathrm{d}t & (1.28) \\ &= \int_0^T \left\{ \int_{\mathbb{R}^2} \left(\mathcal{A}(U) \cdot \left(\dot{\xi}^{\mathrm{i}} \partial_{\xi^{\mathrm{i}}} U + \dot{\omega} \partial_\omega U + \dot{\lambda} \partial_\lambda U \right) \; - \; |\nabla U|^2 \right) \, \mathrm{d}^2 x \right\} \, \mathrm{d}t & (1.29) \\ &= \int_0^T \left(\dot{\xi}^{\mathrm{i}} A_{\mathrm{i}}(\xi) + \dot{\omega} \mathcal{N} \right) \, \mathrm{d}t - 4\pi & (1.30) \end{aligned}$$

where:

$$A_{\xi_{\mathrm{i}}}(\xi) = \int_{\mathbb{R}^2} \mathcal{A}(U) \cdot \partial_{\xi_{\mathrm{i}}} U \, \mathrm{d}^2 x, \quad \mathcal{N}(\lambda) = \int_{\mathbb{R}^2} (\cos(\Theta) - 1) \, \mathrm{d}^2 x$$

We see that (1.28) is equivalent to the Lagrangian in electrodynamics of a charged particle coupled to the vector potential $A(\xi) = A_{\xi_{\mathrm{i}}}(\xi) \, \mathrm{d}\xi_{\mathrm{i}}$ and the scalar potential $-\mathcal{N}(\lambda)\dot{\omega}$, moving according to the path $t \mapsto \xi(t)$. From this we deduce the motion equation:

$$0 = \frac{\delta \mathcal{S}}{\delta \xi_{\mathrm{i}}} = F_{\mathrm{i}j}(\xi) \dot{\xi}_j \tag{1.31}$$

where:

$$F_{ij}(\xi) = \partial_{\xi_j} A_{\xi_i}(\xi) - \partial_{\xi_i} A_{\xi_j}(\xi) = \int_{\mathbb{R}^2} \left(\partial_{\xi_i} U \wedge \partial_{\xi_j} U\right) \cdot U \;\mathrm{d}^2 x$$

We obtain directly $F_{ij} = 4\pi\varepsilon_{ij}$ where $\varepsilon_{12} = -\varepsilon_{21} = 1$. This effect of dependence on ξ is due to the pole of the field $\mathcal{A}$ at the core of the skyrmion. (F is the pullback curvature of the sphere onto the plane by the map $\xi \mapsto U[\xi]$). Remark that the left side of (1.31) is, by definition, the components of the force on the skyrmion which is $\mathbf{F}^{magnus} = 4\pi\hat{z} \wedge \dot{\xi}$, the Magnus force. Motion equation of ω implies that $\mathcal{N}$ is constant, and hence $\dot{\lambda}(t) = 0$, since $\mathcal{N}$ depends only on λ. It means that the dynamics in ω is redundant, i.e, does not give information about the dynamics. Without loss of generality, we suppose that $\dot{\omega} = 0$. Roughly speaking, the dynamics of ω and λ have to be coupled. Accordingly the skyrmion preserves its shape through the motion. We will see actually that it is valid even for correction at first order.

Consider now $u = U + \psi$ where $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{T}S^2$ a small perturbation of the skyrmion bubble $U = U_q$ where $\lambda(t) = \lambda_0$ and $\omega(t) = \omega_0$ are assumed constant. The value of action of u, $\mathcal{S}[u]$, can be approximated up to second order in ψ:

$$\mathcal{S}[u] = \mathcal{S}[U] + \frac{\delta \mathcal{S}}{\delta u}[U](\psi) + \frac{1}{2}\frac{\delta^2 \mathcal{S}}{\delta^2 u}[U](\psi,\psi) \tag{1.32}$$

$$= \mathcal{S}[U] \; + \int_{\mathbb{R}^2} \langle (\partial_t U \wedge U), \psi \rangle + \frac{1}{2} \langle \psi, (U \wedge \partial_t \; - \mathcal{L}_U)\psi \rangle \;\mathrm{d}^2 x \;\mathrm{d}t \tag{1.33}$$

At this point it is convenient to use the complex representation of the tangent space $\mathbb{T}S^2$, so the Lagrangian of the magnon field around the skyrmion U can be written:

$$\mathscr{L}_{mag} = \int_{\mathbb{R}^2} \frac{1}{2}\psi^\dagger(\mathrm{i}\;\partial_t - \mathcal{L}_U)\psi - \mathcal{J}^\dagger \psi + h.c \;\; \mathrm{d}^2 x \quad \text{and} \quad \mathcal{S}_{mag} = \int_0^T \mathscr{L}_{mag} \;\mathrm{d}t \tag{1.34}$$

where $\mathcal{J} = \mathrm{i}\partial_t U = \mathrm{i}\dot{\xi} \cdot \nabla U + \mathrm{i}\dot{\omega}\hat{J}_z U$ is coupled to ψ like a source term. Therefore the dynamics of the skyrmion-magnon system is determined by the Euler equation of the Lagrangian:

$$0 = \frac{\delta \mathcal{S}}{\delta \psi^\dagger} = \frac{\partial \mathscr{L}_{mag}}{\partial \psi^\dagger} - \partial_\mu \left(\frac{\partial \mathscr{L}_{mag}}{\partial(\partial_\mu \psi^\dagger)} \right) \quad \Rightarrow \quad \mathrm{i}\;\partial_t \psi = \mathcal{L}_U \psi \; + \; \mathcal{J}$$

and taking variation respect to ξ:

$$0 = \frac{\delta \mathcal{S}}{\delta \xi_i} = F_{ij}\xi^i + \frac{\delta \mathcal{S}_{mag}}{\delta \xi^j} \tag{1.35}$$

where we have used the implicit dependence $U = U[\xi]$. Thus we note that appears an additional force over the skyrmion due to the magnon field around it, which we name henceforth the *reaction force* over the skyrmion. Later we will explore methods to compute this one. On the other hand the variation respect to ω:

$$0 = \frac{\delta \mathcal{S}}{\delta \omega} = -\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t} + \frac{\delta \mathcal{S}_{mag}}{\delta \omega}$$

As we will see below, this is precisely the conservation of angular moment.

Nevertheless, it is important to remark that $\mathcal{L}_U$ depends implicitly on ξ, ω through U. In order to separate correctly the degrees freedom we do the change of field $\psi \mapsto \phi$ defined by $\psi(x) = R_\omega \phi(x-\xi)$, in this way $\psi^\dagger \mathcal{L}_U \psi = \phi^\dagger \mathcal{L}_{U_0}\phi$. Therefore the magnon Lagrangian is:

$$\begin{aligned}
\mathscr{L}_{mag} &= \int_{\mathbb{R}^2} \mathrm{i}\dot{\xi}\cdot\nabla U_0^\dagger\ \phi + \mathrm{i}\dot{\omega}\hat{J}_z U_0^\dagger\ \phi + \frac{1}{2}\phi^\dagger(\mathrm{i}\ \partial_t - \mathcal{L}_{U_0})\phi + \frac{\mathrm{i}}{2}\ \phi^\dagger(\dot{\xi}\cdot\nabla + \dot{\omega}\hat{J}_z)\phi + h.c\ \ \mathrm{d}^2x \\
&= J_{\xi_\mathrm{i}}[\phi]\dot{\xi}^\mathrm{i} + J_\omega[\phi]\dot{\omega} + \frac{1}{2}\int_{\mathbb{R}^2}\phi^\dagger(\mathrm{i}\ \partial_t - \mathcal{L}_{U_0})\phi + h.c\ \ \mathrm{d}^2y
\end{aligned}$$

where we have defined:

$$J_{\xi_\mathrm{i}}[\phi] = \int_{\mathbb{R}^2}\mathrm{i}\left(\partial_{\xi_\mathrm{i}}U^\dagger\ \phi + \frac{1}{2}\phi^\dagger\partial_\mathrm{i}\phi\right) + h.c\ \ \mathrm{d}^2y\ ,\quad J_\omega[\phi] = \int_{\mathbb{R}^2}\mathrm{i}\left(\partial_\omega U^\dagger\ \phi + \frac{1}{2}\phi^\dagger\hat{J}_z\phi\right) + h.c\ \ \mathrm{d}^2y$$

Thus we obtain:

$$F^{\mathrm{self}} \equiv \frac{\delta\mathcal{S}_{mag}}{\delta\xi^\mathrm{i}} = -\frac{\mathrm{d}J_{\xi_\mathrm{i}}[\phi]}{\mathrm{d}t} \tag{1.36}$$

The above equation gives the force over the skyrmion combining (1.35) and (1.36) we have the equation of motion: $\mathbf{F}^{\mathrm{magnus}} + \mathbf{F}^{\mathrm{self}} = 0$. We note that it has the same form than the Lorentz's force of electrodynamics. In the same way, we have

$$\frac{\delta\mathcal{S}_{mag}}{\delta\omega} = -\frac{\mathrm{d}J_\omega[\phi]}{\mathrm{d}t}$$

Introducing $\mathcal{J}_0(x,t) = R_\omega^{-1}\mathcal{J}(x+\xi,t)$, the whole system of equation of motion leads to the non linear dynamical system:

$$\partial_t\phi = -\mathrm{i}\mathcal{L}_{U_0}\phi + \mathrm{i}\ \dot{\xi}\cdot\nabla_y\phi - \mathrm{i}\mathcal{J}_0[\phi,\xi] \tag{1.37}$$

$$\dot{\xi}_\mathrm{i} = \frac{1}{4\pi}\varepsilon_{\mathrm{i}j}\ \frac{\mathrm{d}J_{\xi_j}[\phi]}{\mathrm{d}t} \tag{1.38}$$

Teorema 1.13 *There exists a global solution of the dynamical system (1.37) and (1.38).*

Proof: Let us denote $X = (\phi,\xi)$ and consider the space: $\mathcal{X} = C^1([0,\bar{t}], W^{1,2}(\mathbb{R}^2)) \times C^1([0,\bar{t}])$, where $W^{1,2}$ denotes the Sobolev space, which is endowed with the norm $\|X\| = \sup_{t\in[0,\bar{t}]}\|\phi\|_{W^{1,2}(\mathbb{R}^2))} + \|\xi\|_{C^0([0,\bar{t}])} + |\dot{\xi}|_{C^0([0,\bar{t}])}$, and the energy $\mathcal{E}(X) = \left\langle\phi^\dagger, \mathcal{L}_{U_0}\phi\right\rangle_{L^2} + \langle\mathcal{J}_0^\dagger,\phi\rangle_{L^2} + h.c.$ We can see that $\frac{\mathrm{d}\mathcal{E}(X(t))}{\mathrm{d}t} = 0$ for all regular solution $\phi \in C^1([0,\bar{t}], C^1(\mathbb{R}^2))$ and $\xi \in C^1([0,\bar{t}],\mathbb{R}^2)$ of (1.37). In effect the Hamiltonian of the system is given by:

$$\frac{\partial\mathscr{L}}{\partial\dot{\xi}}\dot{\xi} + \frac{\partial\mathscr{L}}{\partial(\partial_t\phi)}\partial_t\phi + h.c - \mathscr{L} = \mathcal{E}$$

Since $\mathscr{L}$ does not depend explicitly on t, by the Noether theorem, $\mathcal{E}$ is a constant of motion. Let $G(x,y,t-s)$ be the Green function of $\mathrm{i}\partial_t - \mathcal{L}_0$:

$$(\mathrm{i}\partial_t - \mathcal{L}_0)\ G(x,y,t-s) = \delta(x-y,t-s)$$

We define the map $\mathcal{F} : \mathcal{X} \to \mathcal{X}$, which maps $\phi \mapsto \tilde{\phi} = \mathcal{F}_1[\phi, \xi](x,t)$ and $\xi \mapsto \tilde{\xi} = \mathcal{F}_2[\phi, \xi](x,t)$, given by:

$$\begin{aligned}
\tilde{\phi} &= \int G(x,y,t)\phi(x,0)\ \mathrm{d}^2 y - \int_0^t \int \dot{\xi}(s) \cdot \nabla_y G(x,y,t-s)\ \phi(y,s)\ \mathrm{d}^2 y\ \mathrm{d}s \\
&\qquad\qquad + \int_0^t \int G(x,y,t-s)\mathcal{J}_0(y,s)\ \mathrm{d}^2 y\ \mathrm{d}s \\
\tilde{\xi} &= \frac{1}{4\pi}\varepsilon_{\mathrm{i}j}\left(J_{\mathrm{i}}[\tilde{\phi}] - J_{\mathrm{i}}[\phi_0]\right) + \xi(0)
\end{aligned}$$

Then the system (1.37) turns into the abstract form:

$$X = \mathcal{F}(X) \tag{1.39}$$

Define $X_0 = (\int G(x,y,t)\phi(x,0)\ \mathrm{d}^2 y, 0)$, and let us consider the ball:

$$B_R = \{X \in \mathcal{X} : \|X - X_0\| \leq R\}$$

for some $R > 0$. Since G and J are smooth functions in their respective typologies, then we can take $\bar{t}$ such that $\mathcal{F}(X) \in B_\delta$ for any R. In addition, we can see that $\mathcal{F}$ is a contraction:

$$\|\mathcal{F}(X_1) - \mathcal{F}(X_2)\| \leq \delta(R, \bar{t})\, \|X_1 - X_2\|$$

In effect, we have the inequality:

$$\begin{aligned}
|\tilde{\phi}_1(x,t) - \tilde{\phi}_2(x,t)| \leq & \int_0^t \int |\nabla_y G(x,y,t-s)|\ \ |\dot{\xi}_1(s)\phi_1(y,s) - \dot{\xi}_2(s)\phi_2(y,s)|\ \mathrm{d}^2 y\ \mathrm{d}s \\
&\qquad + \int G(x,y,t-s)\, |\nabla U|\, |\dot{\xi}_1(s) - \dot{\xi}_2(s)|\ \mathrm{d}^2 y\ \mathrm{d}s \\
\leq & \ \varepsilon R\, \|X_1 - X_2\|
\end{aligned}$$

and similarly we have too the other ones. Finally, using the Banach fixed point theorem, we conclude that there exist a unique local solution X of the system. Furthermore, since $\mathcal{E}$ is constant, thus $\mathcal{E}(\bar{t}) = \mathcal{E}(0)$. Finally, it is possible to extend the solution up to $t = 2\bar{t}$ and, by induction, we obtain a global solution $X(t)$ for all $t > 0$ $\blacksquare$.

Now we are interested in how the magnon field produced by the skyrmion affects itself, so we do not consider the scattering part coming from $\phi(x,0)$. In addition notice that at low velocities $|v|$, we can neglect the term $v\nabla\phi$ in the integral equation for ϕ. Thus the approximate solution at first order in $|v|$ will be:

$$\phi^{\mathrm{self}}(x,t) = \int_0^t \int G(x,y,t-s)\mathcal{J}_0(y,s)\ \mathrm{d}^2 y\ \mathrm{d}s \tag{1.40}$$

The self force will be:

$$F_{\mathrm{i}}^{\mathrm{self}}(t) = -\frac{\mathrm{d}J_{\mathrm{i}}[\phi^{\mathrm{self}}]}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\int \mathrm{i}\left(\partial_{\mathrm{i}}U_0^{\dagger}\ \phi + \phi^{\dagger}\partial_{\mathrm{i}}\phi\right) + h.c\ \ \mathrm{d}^2x\right]$$
$$= -\int \mathrm{i}\left(\partial_{\mathrm{i}}U_0^{\dagger}\ \partial_t\phi + \partial_t(\phi^{\dagger}\partial_{\mathrm{i}}\phi)\right) + h.c\ \ \mathrm{d}^2x$$
$$= -\int_0^t \int \mathrm{i}\left(\partial_{\mathrm{i}}U_0^{\dagger}\partial_t G(x,y,t-s)\mathcal{J}_0(y,s)\right)\ \mathrm{d}^2y\ \mathrm{d}s\ \ \mathrm{d}^2x$$
$$-\int \mathrm{i}\left(\partial_{\mathrm{i}}U_0^{\dagger}(x,t)\ \mathcal{J}_0(x,t)\right)\ \mathrm{d}^2x\ +\ h.c$$

Since the force depend on the values of $\{\xi(s) : s < t\}$ there exists a memory effect in the skyrmion dynamics.

An alternative approach used in [2] is to consider an ansatz (phenomenological) $u[\xi,\dot{\xi}]$ where we treat the set $\left\{\xi,\dot{\xi}\right\}$ as collective variables. Then the equation of motion:

$$0 = \frac{\delta\mathcal{S}_{mag}}{\delta\xi^{\mathrm{i}}} = F_{\mathrm{i}j}^{(0)}(\xi,\dot{\xi})\ \dot{\xi}_j + F_{\mathrm{i}j}^{(1)}(\xi,\dot{\xi})\ \ddot{\xi}_j$$

where

$$F_{\mathrm{i}j}^{(0)} = \int_{\mathbb{R}^2} u\cdot\left(\frac{\partial u}{\partial\xi_{\mathrm{i}}}\wedge\frac{\partial u}{\partial\xi_j}\right)\ \mathrm{d}^2x \qquad F_{\mathrm{i}j}^{(1)} = \int_{\mathbb{R}^2} u\cdot\left(\frac{\partial u}{\partial\xi_{\mathrm{i}}}\wedge\frac{\partial u}{\partial\dot{\xi}_j}\right)\ \mathrm{d}^2x$$

The last equation is a particular case of the force derived from the LLG (1.17). Indeed, using the implicit differentiation formula

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial\xi_{\mathrm{i}}}\dot{\xi}_{\mathrm{i}} + \frac{\partial u}{\partial\dot{\xi}_{\mathrm{i}}}\ddot{\xi}_{\mathrm{i}}$$

we obtain:

$$F_{\mathrm{i}} = \langle J(U)\partial_{\xi_{\mathrm{i}}}U, \partial_t u\rangle_{L^2} = \int_{\mathbb{R}^2}\langle J(U)\partial_{\xi_{\mathrm{i}}}U, \partial_t u\rangle\ \mathrm{d}^2x = \int_{\mathbb{R}^2}\langle U, \partial_{\xi_{\mathrm{i}}}U\wedge\partial_t u\rangle\ \mathrm{d}^2x$$
$$= F_{\mathrm{i}j}^{(0)}(\xi,\dot{\xi})\ \dot{\xi}_j + F_{\mathrm{i}j}^{(1)}(\xi,\dot{\xi})\ \ddot{\xi}_j$$

However, there is not explained the functional dependence of $\dot{\xi}$ in the ansatz u. Our approach gives this dependence explicitly through the formula for $\phi^{self} = \phi^{self}[\xi,\dot{\xi}]$ in 1.40, in consequence $u[\xi,\dot{\xi}] = U[\xi] + \psi[\xi,\dot{\xi}]$. We can observe that the term $F^{(1)}(\dot{\xi}) \approx F^{(1)}(0)$ at first order in $|\dot{\xi}|$ (slow skrymion), in consequence it appears an effective mass given by:

$$M_{\mathrm{eff}} = F^{(1)}(0) \tag{1.41}$$

In chapter 2 we will compute $F^{(1)}(\dot{\xi})$ through a explicit formula for $\psi[\xi,\dot{\xi}]$.

In conclusion, the spin excitations around the skyrmion affects its dynamics and vice versa, likes the field produced by an electron in classical electrodynamics, which is suggested also by the emergent electrodynamics for the single skyrmion. Hence, likely the electromagnetic contribution to the mass of the electron, is expected analogously effect to the dynamical of the skymion. This is the affairs treated in the following section.

Consider now a general dynamical skyrmion with a time dependent radius $\lambda(t)$. We can remove the dependence on ξ, λ, ω from the second order term in the Lagrangian doing the variable change: $x \mapsto y = (x-\xi)/\lambda$ and changing the field $\psi \mapsto \phi$ with $\psi(x,t) = R_\omega \phi(y,t)$, hence:

$$\mathscr{L}_{mag} = \lambda^2 \left(J_{\xi_\mathrm{i}}[\phi]\dot{\xi}^\mathrm{i} + J_\omega[\phi]\dot{\omega} + J_\lambda[\phi]\dot{\lambda} \right) + \frac{1}{2}\int_{\mathbb{R}^2} \phi^\dagger (\ \mathrm{i}\lambda^2\ \partial_t - \mathcal{L}_{U_0})\phi + h.c\ \ \mathrm{d}^2 y$$

where:

$$J_\lambda[\phi] = \int_{\mathbb{R}^2} \mathrm{i}\left(\ \partial_\lambda U^\dagger\ \phi + \frac{1}{2}\phi^\dagger\ y \cdot \nabla_y \phi \right) + h.c\ \ \mathrm{d}^2 y$$

On the other hand, observe that doing the change of variable: $t \mapsto \tau$ and scaling the field $\phi \mapsto \eta$ defined by the following relations:

$$\tau = \int \frac{\mathrm{d}t}{\lambda^2(t)}\ , \qquad \eta(x,\tau) = \lambda(t)\ \phi(x,t)$$

then:

$$\int_{\mathbb{R}^2} \phi^\dagger (\mathrm{i}\lambda^2\ \partial_t - \mathcal{L}_{U_0})\phi\ \ + h.c\ \ \mathrm{d}^2 y\ \mathrm{d}t = \int_{\mathbb{R}^2} \eta^\dagger\left(\mathrm{i}\partial_\tau - \mathcal{L}_{U_0} - V(\tau)\right)\eta\ \ + h.c\ \ \mathrm{d}^2 y\ \mathrm{d}\tau$$

where we have introduced the time dependent function $V(\tau)$:

$$V(\tau) = \mathrm{i}\ \frac{\lambda_\tau}{\lambda}(\tau)$$

Therefore, the Lagrangian will be:

$$\mathscr{L}_{mag} = J_{q_\mathrm{i}}[\eta]\ \dot{q}^\mathrm{i} + \int_{\mathbb{R}^2} \frac{1}{2}\eta^\dagger\left(\mathrm{i}\partial_\tau - \mathcal{L}_{U_0} - V(\tau)\right)\eta\ \ + h.c\ \mathrm{d}^2 y$$

We observe that fixing the field ϕ, then the first term in $\mathscr{L}$ above is linear in $\dot{q}_\mathrm{i}$, hence it looks like a Berry Lagrangian in the parameter space $\{q_\mathrm{i}\}$. In the section we will point out this concept. On the other hand, the term $V(\tau)$ corresponds to a time dependent potential on the magnon field.

1.4.1. The Berry phase

We have seen in previous section that the evolution equation of the magnons field is a Schrodinger equation $\mathrm{i}\partial_t \psi = \mathcal{H}_U \psi$ where the Hamiltonian $\mathcal{H}_U = \mathcal{H}_U[q]$, $q = (\lambda, \omega, \xi)$ depends implicitly on the parameters of the skyrmion. Let $\phi_n[q]$ and $\varepsilon_n[q]$ the set of eigenfunctions and eigenvalues of $\mathcal{H}_{U_q}$, respectively. Note that the index n is determined by the angular and radial spectrum $n = (l,k)$. The mean assumption of the adiabatic hypothesis is that the unitary evolution of $\phi_n[q](t) = \phi_n[q(t)]$ for a slow variation of the parameters $t \mapsto q(t) = (\lambda(t), \omega(t), \xi(t))$. According to the Berry's theorem [36], the adiabatic evolution of the wave function is given by:

$$\phi(t) = \mathrm{e}^{\mathrm{i}\gamma_n(t)}\mathrm{e}^{-\mathrm{i}\int_0^t \varepsilon_n[q(t')]\ \mathrm{d}t'}\phi_n[q(t)], \quad \text{with} \quad \phi(t=0) = \phi_n(q_0) \tag{1.42}$$

where

$$\gamma_n(t) = \int_0^t \left\langle \phi_n[q(t)],\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_n[q(t)] \right\rangle = \int_{q(0)}^{q(t)} A_{\mathrm{i}}^{(n)}[q]\ \mathrm{d}q^{\mathrm{i}}$$

is the Berry phase, and

$$A_{\mathrm{i}}^{(n)}[q] = \left\langle \phi_n[q] \left| \frac{\partial}{\partial q_{\mathrm{i}}} \right| \phi_n[q] \right\rangle$$

is the Berry connection.

We know, from the scaling property of the operator $\mathcal{H}$, that:

$$\phi[q](x) = R_\omega \phi_n\left(\frac{x-\xi}{\lambda}\right), \qquad \varepsilon_n[q] = \frac{1}{\lambda^2}\varepsilon_n$$

where $\{\phi_n\}_{n=1,\dots}$ are the eigenvectors of $\mathcal{H}_0$. The modes

using this expressions, we can compute the components of the Berry connection:

$$A_\lambda^{(n)} = \left\langle \phi_n\left(\frac{x-\xi}{\lambda}\right),\ \frac{\partial}{\partial\lambda}\ \phi_n\left(\frac{x-\xi}{\lambda}\right) \right\rangle = -\lambda \int_{\mathbb{R}^2} \varphi_n(y)^\dagger\ y^{\mathrm{i}}\nabla_{y^{\mathrm{i}}}\phi_n(y)\ \mathrm{d}^2y$$

Similarly we have

$$A_\omega^{(n)} = \lambda^2 \int_{\mathbb{R}^2} \phi_n(y)^\dagger\ J_z\phi_n(y)\ \mathrm{d}^2y$$

$$A_{\xi^{\mathrm{i}}}^{(n)} = -\lambda \int_{\mathbb{R}^2} \phi_n(y)^\dagger\ \nabla_{y^{\mathrm{i}}}\phi_n(y)\ \mathrm{d}^2y$$

where have defined the variable change $y = (x-\xi)/\lambda$ in order to split the dependence on q. Let us focus in the Berry phase acquired throughout the a closed path $\mathcal{C} = \{q(t) : \omega(0) = 0, \omega(t) = 2\pi\}$:

$$\gamma_n = \int_{\mathcal{C}} A_\omega\dot{\omega}\ \mathrm{d}t = 2\pi\lambda^2 \int_{\mathbb{R}^2} \phi_n(y)^\dagger\ J_z\phi_n(y)\ \mathrm{d}^2y$$

then the adiabatic evolution operator can be written:

$$U(t) = \exp\left\{-\mathrm{i}\varepsilon_n\tau + \mathrm{i}\int_0^t A_{\mathrm{i}}[q(s)]\ \dot{q}^{\mathrm{i}}(s)\ \mathrm{d}s\right\} \tag{1.43}$$

where as before:

$$\tau = \int_0^t \frac{\mathrm{d}t'}{\lambda(t')^2}$$

It is interesting to see that since $\tau \to 0$ as the skyrmion collapses, so the contribution of $\varepsilon_n\tau$ dominates in the unitary evolution. Roughly speaking, at the collapse, the spectral part drives the dynamics of the internals modes.

We can add corrections to the above evolution operator $U(t)$, supposing that the variations $\left|\dot{\lambda(t)}\right|, \lambda(t)\left|\dot{\omega}(t)\right|, \left|\dot{\xi(t)}\right|$ are small enough compared with the amplitude of the oscillations. This hypothesis allows us to neglect the source term in the magnon's evolution equation.

$$\mathrm{i}\partial_t \phi(x,t) = \mathcal{H}_x \phi(x,t) \tag{1.44}$$

Now let define the semi group $\phi \mapsto \mathcal{P}_\tau \phi = \mathrm{e}^{-\mathrm{i}\omega(\tau)} \phi\left(\lambda(\tau) y + \xi(\tau)\right)$ which operate on the space of continuous functions, and define the time dependent infinitesimal generator $\mathcal{A}(\tau) = \frac{\mathrm{d}\mathcal{P}_\tau}{\mathrm{d}\tau} = \frac{\lambda_\tau}{\lambda} y \nabla_y + \frac{1}{\lambda}\xi_\tau \nabla_y - \mathrm{i}\omega_\tau$. Thus we can write the equation 4.3) in the form:

$$\mathrm{i}\partial_\tau \phi = \left(\mathcal{H}_y + \mathcal{A}(\tau)\right)\phi \tag{1.45}$$

The solution can be computed using the Baker-Hausdorff-Campbell formula:

$$\begin{aligned}
\exp\left\{\mathrm{i}\tau\mathcal{H} + \mathrm{i}\int_0^\tau \mathcal{A}(\tau)\ \mathrm{d}\tau\right\} &= \exp\left\{\mathrm{i}\int_0^\tau \mathcal{A}\ \mathrm{d}\tau\right\} \exp\left\{\mathrm{i}\tau\mathcal{H}\right\} \exp\left\{-\frac{1}{2}\tau\int_0^\tau \left[\mathcal{A}(\tau),\mathcal{H}\right]\ \mathrm{d}\tau\right\} \\
&\approx \exp\left\{\mathrm{i}\int_0^\tau \mathcal{A}\ \mathrm{d}\tau\right\} \exp\left\{\mathrm{i}\tau\mathcal{H}\right\} \left\{1 + \frac{1}{2}\tau\mathrm{i}\int_0^\tau \left[\mathcal{A}(\tau),\mathcal{H}\right]\ \mathrm{d}\tau\right\} + o(\tau^2) \\
&\approx \mathcal{P}_t \left\{\exp\left\{\mathrm{i}\tau\mathcal{H}\right\} + \frac{1}{2}\tau\int_0^\tau \left[\mathcal{A}(\tau),\mathcal{H}\right]\ \mathrm{d}\tau\right\} + o(\tau^2) \\
&\approx \mathcal{P}_t \left\{\exp\left\{\mathrm{i}\tau\mathcal{H}\right\} + \frac{1}{2}\tau \log\lambda(\tau)\ \left[y\nabla_y,\mathcal{H}\right]\right\} + o(\tau^2)
\end{aligned}$$

The first term, in the first line of the right side, corresponds to the Berry phase, the next is the unitary Hamiltonian evolution, whereas the rest are the radiative corrections. Therefore a formal solution of (1.44) is:

$$\phi(x,t) = \exp\left\{\mathrm{i}\tau\mathcal{H} + \mathrm{i}\int_0^\tau \mathcal{A}(\tau)\ \mathrm{d}\tau\right\}\phi_0 \tag{1.46}$$

In order to see the effect of the Berry phase, let us consider the adiabatic transition from $\lambda(\tau) \to \lambda_0$, as $t \to -\infty$, to $\lambda(\tau) \to \lambda_1$, as $\tau \to \infty$. Therefore, by (1.46), we observe that the Berry phase shifts the energy level in the magnon spectra in the quantity:

$$\Delta E = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \mathcal{A}(\tau)\ \mathrm{d}\tau$$

During the annihilation process $\lambda(\tau) \to 0$ as $\tau \to \infty$, the term $\mathcal{A} \sim \lambda(\tau)$, hence $\Delta E = 0$ which implies that there is not topological contribution by the Berry phase during the collapse process.

1.5. Classical effective field theory approach

In this section we study the effects of the spin waves on the dynamics of the skyrmion, as an effective theory at low energy. Therefore, we will consider the approximation $\phi = \phi^{\text{self}}$ given by (1.40). Using Fourier transform, the Lagrangian of the excitations, at first order, yields:

$$\begin{aligned}\mathcal{S}[\phi, \mathcal{J}] &= \int \frac{1}{2}\phi^\dagger(x,t)(\mathrm{i}\partial_t - \mathcal{H})\phi(x,t) - \mathcal{J}^\dagger(x,t)\phi(x,t) + h.c \;\; \mathrm{d}^2x \; \mathrm{d}t \\ &= \int \frac{1}{2}\tilde{\phi}^\dagger(x,-\omega)(\mathrm{i}\omega - \mathcal{H})\tilde{\phi}(x,t) - \tilde{\mathcal{J}}^\dagger(x,-\omega)\tilde{\phi}(x,\omega) + h.c \;\; \mathrm{d}^2x \; \mathrm{d}\omega\end{aligned}$$

Replacing (1.40), the classical effective action will be:

$$\mathcal{S}_{\text{eff}}[\mathcal{J}] = -\frac{1}{2}\int\int \tilde{\mathcal{J}}^\dagger(x,-\omega)G(x,y,\omega)\tilde{\mathcal{J}}(y,\omega) + h.c \;\; \mathrm{d}^2y \; \mathrm{d}^2x \; \mathrm{d}\omega$$

Now, using the formula 1.26 for $G(x,y,\omega)$ and:

$$\widetilde{\mathcal{J}}(r,\theta,\omega) = \mathrm{i}\widetilde{\partial_t U}(r,\theta,\omega) = \mathrm{i}\widetilde{\dot{\xi}_\mathrm{i}} \; \partial_{\xi_\mathrm{i}} U = \mathrm{i}\omega \; \lambda\tilde{\zeta}(\omega) \; \frac{\mathrm{e}^{\mathrm{i}\theta}}{r^2+\lambda^2}$$

where $\zeta = \xi_1 - \mathrm{i}\xi_2$ (we are using the complex representation), we obtain:

$$\begin{aligned}\mathcal{S}_{\text{eff}} &= \frac{1}{2}\int\sum_k \left[\int \frac{\phi_{k,1}(r)}{r^2+\lambda^2} \; r\mathrm{d}r\right]^2 \lambda^2\omega^2 \; |\tilde{\zeta}(\omega)|^2\mathcal{P}\frac{1}{\omega-k^2} \; \mathrm{d}\omega \\ &= \frac{1}{2}\sum_k \left[\int \frac{\phi_{k,1}(r)}{r^2+\lambda^2} \; r\mathrm{d}r\right]^2 \lambda^2\omega^2 \; |\tilde{\zeta}(k^2)|^2 \; \mathcal{P}\frac{1}{\omega-k^2} \; \mathrm{d}\omega \\ &= \frac{1}{2}\int\sum_k \lambda^2 F(k^2)^2 \; \omega^2 \; |\tilde{\zeta}(\omega)|^2\mathcal{P}\frac{1}{\omega-k^2} \; \mathrm{d}\omega\end{aligned}$$

where, using (1.25):

$$\begin{aligned}F(k^2) = \int \frac{\phi_{k,1}(r)}{r^2+\lambda^2} \; r\mathrm{d}r &= \int \frac{1}{c_k(\lambda)}\left(J_2(kr) + \frac{J_1(kr)}{kr}\frac{2}{1+r^2/\lambda^2}\right)\frac{r\mathrm{d}r}{r^2+\lambda^2} \\ &= F_0(\lambda) + F_1(\lambda)k^2 + F_2(\lambda)k^4 + \ldots\end{aligned}$$

where $c_\lambda = \|\phi_{k,1}\|_{L^2}$ is the normalization constant. Therefore, the effective action of the skyrmion at low energy limit will be:

$$\mathcal{S}_{\text{eff}}^{\omega\approx 0} = -\int \frac{1}{2}M_{\text{eff}} \; \omega^2|\tilde{\zeta}(\omega)|^2 \; \mathrm{d}\omega = \int \frac{1}{2}M_{\text{eff}} \; |\dot{\xi}(t)|^2 \; \mathrm{d}t \tag{1.47}$$

where we see the effective mass of the skyrmion.

$$M_{\text{eff}} = \lambda^2\sum_n \frac{F_0(\omega_n)^2}{\omega_n} \qquad \text{where:} \quad \omega_n = k^2 \tag{1.48}$$

Now, using the Weyl's law, we have the asymptotic behavior for the distribution of the eigenvalues of the $\mathcal{L}$.

$$\frac{N(\omega)}{\omega} \approx \frac{1}{4\pi} L^2$$

where $N(\omega)$ denotes the number of eigenvalues less or equal to ω, and L is the size of the system (with Dirichlet boundary conditions). Hence

$$M_{\text{eff}} \approx \frac{1}{4\pi} L^2 \lambda^2 \sum_n \frac{1}{c_{k_n} n}$$

Therefore, at first order (in the serie) and using $c_{k_1} \sim 2\pi L^{-2}$, we obtain the effective mass of the skyrmion due to the spin waves fluctuations around it: $M_{\text{eff}} \sim \lambda^2$. Mass of skyrmion has been studied recently by several authors [14], [12] using a different approach considering deviations from the skyrmion bubble by means of a spatially variation of its parameters:

$$U(r,\theta) = R_{\omega(t,\theta)} U\left(\frac{x - \xi(0)}{\lambda(t,\theta)}\right)$$

where $\lambda(r,\theta)$ and $\omega(t,\theta)$ are expanded in Fourier modes:

$$\lambda(t,\theta) = \sum_l \lambda_l(t) \mathrm{e}^{\mathrm{i}\theta} \ , \quad \omega(t,\theta) = \sum_l \omega_l(t) \mathrm{e}^{\mathrm{i}\theta}$$

The effective dynamics of is found integrating over all the the modes $l \neq 1, 0, -1$ in the action. This leads the effective actions $\mathcal{S}_{\text{eff}}[\lambda_0, \lambda_1, \omega_0, \omega_1]$. Finally one identify $\dot{\xi}_{\mathrm{i}}$ projecting on the mode $l = 1$.

1.5.1. Discussion about the mass of the skyrmion

At this point we need to comment the discrepancy between the different values (compare e.g with (1.48)), this fact is due to the ambiguity in the splitting of skyrmion-magnon freedom degrees. To survey this issue, let $\varrho = U \cdot (\partial_{x_1} U \wedge \partial_{x_2} U)$ be the topological density, and let u be an arbitrary configuration of spins with topological degree $\deg(u) = 1$. One could define the position of the skyrmion by the formula:

$$\xi_{cm} = \int_{\mathbb{R}^2} \varrho \ \mathbf{r} \ \mathrm{d}^2 x$$

on the other hand, one could also define:

$$\xi_{rad} = \int_{\mathbb{R}^2} (1 - \cos(\Theta)) \ \mathbf{r} \ \mathrm{d}^2 x$$

The first one has been proved in [33] having a vanish mass, whereas the second one has a mass depending on the spectrum of the magnon spectrum. In order to understand this dependence, following [10], [8] for instances, we have to look at the motion equation of Rigid skyrmion [1.17] adding a mass term:

$$\mathbf{F} = M\ddot{\xi} + 4\pi \mathbf{e}_3 \wedge \dot{\xi} = 0$$

Hence we obtain the characteristic frequencies: $\omega_0 = 0$ and $\omega_1 = 4\pi/M$ (do not confuse with the phase of the skyrmion!). The zero frequency is associated with the transnational symmetry, whereas the second one is due precisely to the interaction of the skyrmion and the "cloud" of magnons around it. In this way, the mass will be given by:

$$M = \frac{4\pi}{\omega_1}$$

However, the value of ω_1 depend on the radius of the skyrmion λ, as well as the size of the system L (for finite systems).

1.5.2. Conservation of the magnon number and spin current.

Using the Noether theorem, we can derive a conservation law from the rotational symmetry about the axis $\hat{z}$, the momenta angular onto the $\hat{z}$ axis, L_z, defined by:

$$L_z(t) = \int_\Omega u(x,t)\cdot\hat{z}\ \mathrm{d}^2x \tag{1.49}$$

Indeed, just multiplying the Landau-Lifshitz equation by $\hat{z}$ and integrating, we get

$$\frac{\mathrm{d}L_z(t)}{\mathrm{d}t} = \int \partial_t u\cdot\hat{z}\ \mathrm{d}^2x = \int_\Omega u\wedge\left(\Delta u + |\nabla u|^2 u\right)\cdot\hat{z}\ \mathrm{d}^2x = -\int_\Omega \nabla\cdot\mathcal{J}^z = -\int_{\partial\Omega}\mathcal{J}^z\cdot\mathrm{d}\mathbf{S} \tag{1.50}$$

where $\mathcal{J}^z_{\mathrm{i}} = (\nabla_{\mathrm{i}} u\wedge u)\cdot\hat{z}$ is called the spin current.

For a skyrmion U of radius λ and phase ω, we define the spin number of the skyrmion

$$\mathcal{N}_U = \int_\Omega U(x,t)\cdot\hat{z}\ \mathrm{d}^2x = \int_\Omega \cos(\Theta)\ \mathrm{d}^2x \tag{1.51}$$

We also can compute the contribution of a spin wave $\psi(x,t) = \begin{bmatrix}\psi_1(x,t)+\mathrm{i}\psi_2(x,t)\\ \psi_3(x,t)\end{bmatrix}$ to L_z, where $\psi_3 = \sqrt{1-\psi_1^2-\psi_2^2}$, since $|\psi|^2 = 1$, it is

$$L_z = \int_\Omega \psi_3\ \mathrm{d}^2x \approx |\Omega| - \frac{1}{2}\int_\Omega |\psi|^2\ \mathrm{d}^2x$$

This last expression gives us a direct relation between the angular momentum L_z and the magnon number

$$\mathcal{N}_{mag} = \int |\psi|^2\ \mathrm{d}^2x$$

In the same way, for a perturbation $u = \psi^1 E_1 + \psi^2 E_2 + \psi_3 U$ of the skyrmion U, and remembering that $E_1 = \begin{bmatrix}\cos(\Theta)\mathrm{e}^{\mathrm{i}\theta}\\ -\sin(\Theta)\end{bmatrix}$ and $E_2 = \begin{bmatrix}\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\\ 0\end{bmatrix}$, we have

$$\begin{aligned} L_z = \int_\Omega u\cdot\hat{z}\ \mathrm{d}^2x &= -\int_\Omega \sin(\Theta)\psi^1\ \mathrm{d}^2x + \int_\Omega \psi_3\cos(\Theta)\ \mathrm{d}^2x \\ &= \langle\partial_\lambda U,\psi\rangle_{L^2} + \int_\Omega \psi_3\cos(\Theta)\ \mathrm{d}^2x \\ &\approx \langle\partial_\lambda U,\psi\rangle_{L^2} + \mathcal{N}_U - \frac{1}{2}\int_\Omega |\psi|^2\cos(\Theta)\ \mathrm{d}^2x \end{aligned}$$

We remember that $\partial_\lambda U$ is the generator of the kernel of mode $l = 0$, so the orthogonality condition $\langle \partial_\lambda U, \psi \rangle_{L^2} = 0$ implies we can write the law of angular momentum in the form:

$$L_z = \mathcal{N}_U - \frac{1}{2} \int_\Omega |\psi|^2 \cos(\Theta) \, \mathrm{d}^2 x \tag{1.52}$$

For a skyrmion of small enough radius, we can approximate

$$\int_\Omega |\psi|^2 \cos(\Theta) \, \mathrm{d}^2 x \approx \int_\Omega |\psi|^2 \, \mathrm{d}^2 x \tag{1.53}$$

Finally combining (1.50), (1.52), (1.53), we obtain:

$$\frac{\mathrm{d}\mathcal{N}_U(t)}{\mathrm{d}t} \approx \frac{\mathrm{d}L_z(t)}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\psi|^2 \, \mathrm{d}^2 x = - \int_{\partial\Omega} (\mathcal{J}^z + \frac{1}{2}\mathcal{J}) \cdot \mathrm{d}\mathbf{S} \tag{1.54}$$

where we have used the continuity equation of the magnon current: $\partial_t(|\psi|^2) = -\nabla \cdot \mathcal{J}$. Consider a ball $\Omega = B_R$ of radius R, which we will take in the same order than the size of the skyrmion R (in order to control the orthogonality condition), and centered at the core of the skyrmion. Then we interpret the equation (1.54) as a balance between the dynamics of the skyrmion, through $\lambda(t)$, and the radiation field emitted, specifically the asymptotic behavior of ψ at the scale R).

More specifically, as the skyrmion collapses, it actually repels the excess radiation away from the origin. This evacuation process only causes the skyrmion to collapse at a faster rate. In conclusion, as λ decrease, the magnon number increase in same amount in radiation form. We will see in the followings chapter that, in fact, the total number of magnons emitted in the process can be estimated by the area of the skyrmion $2\pi\lambda^2$, which is, at the same time, an estimation of $\mathcal{N}_U$ for large values of λ (for domain wall skyrmion) [4]

References in this section: [21] studies how this effect is responsible of the instability in discrete lattice models, since the breaking of the conformal symmetry.

1.6. Singular Perturbation Approach

Our aim this section is to find dynamics's equations for λ in the singular perturbation approach, which is a very useful method in different branches of physics as well as mathematics, especially in vortex theory (superconductivity, hydrodynamics, etc...). Roughly speaking, this consists in a perturbation method in some parameter $\varepsilon > 0$ which in turn plays the role of regularize some model (e.g an evolution equation).

Definición 1.14 *Consider $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a map. The Ginzburg -Landau functional is defined by:*

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \ \mathrm{d}^2 x$$

Let u_ε be a solution of

$$\partial_t u_\varepsilon = J(u_\varepsilon) \frac{\delta \mathcal{E}}{\delta u}(u_\varepsilon) = J(u_\varepsilon) \left(\Delta u_\varepsilon + \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2) u_\varepsilon \right) \tag{1.55}$$

Teorema 1.15 *In the limit $\varepsilon \to 0$, u_ε converges in weak sense to u, where u is a weak solution of the LLG equation.*

We follow the scheme given [19] (There is consider only the harmonic map flow case). We will use the followings lemmas:

Lema 1.16 *Let u be a solution of (1.55) with initial data u_0. Then:*

$$\mathcal{E}_\varepsilon(u) + \alpha \int_0^t \int_{\mathbb{R}^2} |\partial_t u|^2 \ \mathrm{d}x \ \mathrm{d}t = \mathcal{E}(u) \tag{1.56}$$

Proof: Multiply by $\partial_t u$ the equation (1.55) and integrating, gives the result.

Lema 1.17 *Let u be a solution of (1.55) with initial data u_0 up to a time $T > 0$. Then:*

$$\|u\|_{L^\infty(\mathbb{R}^2 \times (0,T))} \leq 1 \tag{1.57}$$

Proof: Multiply by u the equation (1.55) gives:

$$\left(\alpha \frac{\partial}{\partial t} - \Delta \right) \left(\frac{|u|^2}{2} \right) + |\nabla u|^2 + \frac{1}{\varepsilon}(|u|^2 - 1) |u|^2 = 0$$

so:

$$\left(\alpha \frac{\partial}{\partial t} - \Delta + \frac{1}{\varepsilon} \right) \left(|u|^2 - 1 \right) \leq 0$$

and from the maximum principle, we obtain: $|u(x,t)|^2 - 1 \leq |u_0|^2 - 1 \leq 0$

Using the Galerking's method we can obtain a solution u_ε of (1.55). Moreover, by the previous lemmas, we have: $|u_\varepsilon| \leq 1$ and $\|1 - u_\varepsilon\|_{L^2} \leq 4\varepsilon \ \mathcal{E}(u_0)$. Thus taking $\varepsilon \to 0$, there exists $u : \mathbb{R}^2 \times (0,T) \to S^2$ such that, up to subsequence, $u_\varepsilon \to u$ and $\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^2 \times (0,T))$. Now we use the following lemma:

Lema 1.18 *A map $u:\mathbb{R}^{2+1}\to S^2$ is a weak solution of LLG iff*

$$(J(u)^\top\partial_t u)\wedge u-\nabla\cdot(\nabla u\wedge u)=0 \tag{1.58}$$

Proof: Suppose u is a weak solution of (1.58) then:

$$(J(u)^\top\partial_t u-\Delta u)\wedge u=0 \quad\Rightarrow\quad J(u)^\top\partial_t u-\Delta u=\kappa u$$

for some κ. Multiply by ηu where η is an arbitrary test function and integrating over $\mathbb{R}^{2+1}$, we see u is weak solution of LLG. For the reciprocal it is enough to note that:

$$\int_{\mathbb{R}^{2+1}}(J(u)^\top\partial_t u-\Delta u)\wedge\eta u=\int_{\mathbb{R}^{2+1}}(J(u)^\top\partial_t u)\wedge u\eta+(\nabla u\wedge u)\cdot\nabla\eta=0$$

Proof of (1.15): Operate (1.55) by $\wedge\eta u_\varepsilon$ we get:

$$\int_{\mathbb{R}^{2+1}}(J(u)^\top\partial_t u)\wedge u\eta+(\nabla u\wedge u)\cdot\nabla\eta=0$$

hence u solution of (1.58) and by lemma (1.18), a weak solution of *LLG*, completing the theorem.

Now, in order to find the dynamics of λ, our strategy is to solve (1.55) and then study the modulated dynamics of u_ε when $\varepsilon\to 0$. A very well reference of this method is [24] where is applied to liquid crystals theory and hydrodynamics.

Suppose there exist a domain wall of size $\varepsilon>0$ which separates the phase

$$u=\begin{cases}-\mathbf{e}_3 & \text{if } r<\lambda(t,\theta)\\ \mathbf{e}_3 & \text{if } r>\lambda(t,\theta)\end{cases}$$

Define the function:

$$u(x,t)=U(R\equiv\frac{r-\lambda(t,\theta)}{\varepsilon},t,\theta,\varepsilon) \tag{1.59}$$

Remark here U is not necessarily an harmonic map skyrmion (from the previous sections). Set $V(u)=(1-|u|^2)^2$. Now we go the perturbative expansion

$$U=U_0+U_1\varepsilon+U_2\varepsilon^2+\dots$$

. Replacing it in (1.55), and retain terms up to order $O(\varepsilon)$, we obtain:

$$\begin{aligned}-\varepsilon J(U_0)^\top\partial_R U_0\;\partial_t\lambda=\partial_R^2\left(U_0+\varepsilon U_1\right)+\varepsilon\frac{1}{r}\partial_R U_0+\frac{1}{r^2}\partial_R^2\left(U_0+\varepsilon U_1\right)(\partial_\theta\lambda)^2\\ +\varepsilon\frac{1}{r^2}\partial_R U_0\;\partial_\theta^2\lambda-V'(U_0)-\varepsilon V''(U_0)U_1\end{aligned}$$

where $r=\lambda+O(\varepsilon)$. Comparing terms of orders $O(\varepsilon)$ we get:

$$-\left(1+\frac{(\partial_\theta\lambda)^2}{\lambda^2}\right)\partial_R^2 U_0+V'(U_0)=0\quad\text{with}\quad U_0=\begin{cases}-\mathbf{e}_3 & \text{if } R\to-\infty\\ \mathbf{e}_3 & \text{if } R\to\infty\end{cases} \tag{1.60}$$

and $O(\varepsilon^2)$ we get:

$$-\left(1+\frac{(\partial_\theta\lambda)^2}{\lambda^2}\right)\partial_R^2 U_1 + V''(U_0)U_1 = \frac{1}{\lambda}\left(1+\frac{(\partial_\theta^2\lambda)}{\lambda}\right)\partial_R U_0 + \partial_t\lambda\ J(U_0)^\top \partial_R U_0 \tag{1.61}$$

$$\text{with } U_1 = \begin{cases} 0 & \text{if } R\to-\infty \\ 0 & \text{if } R\to\infty \end{cases} \tag{1.62}$$

Let $\mathscr{R}$ be an differential operator given by

$$U \mapsto \mathscr{R}(U) = -\left(1+\frac{(\partial_\theta\lambda)^2}{\lambda^2}\right)\partial_R^2 U + V'(U)$$

then (1.60) is just $\mathscr{R}(U_0)=0$. On the other hand, since $\mathscr{R}$ does not depend on R due to the radial symmetry of the $\mathscr{R}$, we can differentiate the equation $\mathscr{R}(U_0)=0$ to get $\mathcal{D}\mathscr{R}_{U_0}(\partial_R U)=0$ where

$$\mathcal{D}\mathscr{R}_{U_0} = -\left(1+\frac{(\partial_\theta\lambda)^2}{\lambda^2}\right)\partial_R^2 + V''(U_0)$$

In other words $\partial_R U$ belongs to the kernel of $\mathcal{D}\mathscr{R}_{U_0}$. In addition we can notice that $\mathcal{D}\mathscr{R}_{U_0}$is a elliptic operator, therefore it is valid the Fredholm alternative theorem, namely $\mathcal{D}\mathscr{R}_{U_0}(U_1) = F$ is solvable for some U_1 if and only if $f \in \ker\mathcal{D}\mathscr{R}_{U_0}^{\perp}$. In particular, a necessary condition is that $\langle \partial_R U_0, F\rangle_{L^2} = 0$. From here, (1.61) is solvable iff:

$$0 = \left\langle \partial_R U_0, \frac{1}{\lambda}\left(1+\frac{(\partial_\theta^2\lambda)}{\lambda}\right)\partial_R U_0 + \partial_t\lambda\ J(U_0)^\top\partial_R U_0 \right\rangle_{L^2} \tag{1.63}$$

$$= \frac{1}{\lambda}\left(1+\frac{(\partial_\theta^2\lambda)}{\lambda}\right)\langle \partial_R U_0, \partial_R U_0\rangle_{L^2} + \partial_t\lambda\ \langle \partial_R U_0, J(U_0)^\top \partial_R U_0\rangle_{L^2} \tag{1.64}$$

A simple solution of the equation (1.60) can be obtained supposing that does not depend on θ, so

$$\partial_R^2 U_0 = -V'(U_0) = (1-|U_0|^2)U_0 \quad \text{with} \quad U_0 = \begin{cases} -\mathbf{e}_3 & \text{if } R\to-\infty \\ \mathbf{e}_3 & \text{if } R\to\infty \end{cases}$$

Using the method of separation of variables, we found the solution:

$$U_0(R) = \tanh(R)\ \mathbf{e}_3$$

Thus by (1.63) we obtain the dynamical equation for $\lambda(t)$:

$$\lambda\ \partial_t\lambda = -\frac{\langle \partial_R U_0, \partial_R U_0\rangle_{L^2}}{\langle \partial_R U_0, J(U_0)^\top\partial_R U_0\rangle_{L^2}} \equiv -c$$

Integrating, it gives the solution:

$$\lambda(t) = a\sqrt{T-t} \tag{1.65}$$

where $T = \lambda(t=0)c^{-1}$ and $a = c^{-1/2}$ it is a self similar behaviour. Finally, we obtain at first order in ε, the solution:

$$U(x,t) = \tanh\left(\frac{x-\lambda(t)}{\varepsilon}\right) \tag{1.66}$$

The ansatz (1.66) has been tested numerical in simulations. The fitting agree successfully with a self similar behaviour.

The self-similar regimen is expected during the adiabatic evolution, however for a time near the blow-up time, as we will see in the next chapters, the emission of spin waves from the skyrmion produces that the skyrmion accelerates its collapse, resulting in a faster decaying for $\lambda(t)$.

Capítulo 2

Spin waves Radiation far away a skyrmion

In this chapter we will focus on the field of magnons emitted by a dynamical skyrmion, far away the core of the skyrmion. The origin of this radiation field is due to purely classic effects by the source part $\partial_t U$ in (1.9). We will derive the effect of this radiation on the dynamics of the skyrmion.

2.1. Radiation from a collapsing skyrmion

Consider a skyrmion U with time dependent parameters $q(t)$ varying slowly in time, such that we can neglect the second order terms: $\dot{\lambda} y \nabla \phi, \dot{\xi} \nabla \phi$ in the spin waves equation (1.9). In this section we will derive analytic expression for the radiation far away the core of a collapsing skyrmion under this approximation.

Denoting $p(t) = \lambda(t) \mathrm{e}^{\mathrm{i}\omega(t)}$, and using (1.8), we have that far away from the core of the skyrmion, such that $cos(\Theta) \approx 1$:

$$\partial_t U = \dot{\lambda} \partial_\lambda U + \dot{\omega} \partial_\omega U = -\dot{p}(t) \frac{r}{r^2 + \lambda^2} \approx -\dot{p}(t) \frac{1}{r}$$

Then the equation of the spin waves, in the mode $l = 0$, is

$$\gamma \ \partial_t \psi = \mathcal{L}_0 \psi - \gamma \partial_t U \approx \Delta_r \psi - \frac{1}{r^2} \psi + \frac{\gamma \dot{p}(t)}{r}$$

Henceforth we will suppose that $\dot{p}(t) = 0$ for all $t < 0$. Using the Green Method:

$$\psi(r,t) = \int_0^t \ \gamma \ddot{p}(s) \Psi(r, t-s) \ \mathrm{d}s + \gamma \dot{p}(0) \Psi(r,t)$$

where $\Psi(r,t)$ satisfy the equation:

$$\gamma \ \partial_t \Psi = \Delta_r \Psi - \frac{1}{r^2} \Psi + \frac{1}{r}$$

We look for a self-similar solution of the form: $\Psi(r,t) = \sqrt{t}\; f(\frac{r}{\sqrt{t}})$, replacing this in the last equation, we obtain:

$$\xi^2 f''(\xi) + (\xi + \frac{\gamma}{2}\xi^3)f'(\xi) - (1 + \frac{\gamma}{2}\xi^2)f(\xi) = -\xi \tag{2.1}$$

where $\xi = r/\sqrt{t}$.

We can check out directly that $f(\xi) = \xi$ is a solution of the homogeneous equation, and the Wronskian is given by:

$$W(\xi) = \frac{1}{\xi}\mathrm{e}^{-\frac{\gamma}{4}\xi^2}$$

so the other on solution of the homogeneous equation will be:

$$f_0(\xi) = -\xi \int \frac{1}{\xi^3}\mathrm{e}^{-\frac{\gamma}{4}\xi^2}\; \mathrm{d}\xi$$

The general solution of (2.1) is:

$$f(\xi) = a_1\xi \int \frac{1}{\xi^3}\mathrm{e}^{-\frac{\gamma}{4}\xi^2}\; \mathrm{d}\xi + a_2\xi + \frac{1}{\gamma\xi} \tag{2.2}$$

Thus the bounded solution is:

$$f(\xi) = \frac{2}{\gamma}\xi \int_\xi^\infty \frac{1}{s^3}\left(1 - \mathrm{e}^{-\frac{\gamma}{4}s^2}\right)\; \mathrm{d}s \tag{2.3}$$

In particular it is illustrative look at the important cases

$$f(\xi) \approx \begin{cases} -\frac{1}{2}\xi\log\xi & \text{if } \xi \approx 0 \\ \frac{1}{\gamma\xi} & \text{if } \xi \to \infty \end{cases}$$

In particular $\Psi(r,0) = 0$ and we extend this function $\Psi(r,t) = 0$ for all $t < 0$. Thus using integration by parts we can write:

$$\begin{aligned} \psi(r,t) &= \int_0^t \dot{p}(s)\partial_t\Psi(r,t-s)\; \mathrm{d}s & (2.4) \\ &= -\int_0^t \dot{p}(s)\left(\frac{1 - \mathrm{e}^{-\gamma r^2/4(t-s)}}{r}\right)\; \mathrm{d}s & (2.5) \\ &= \int_0^t \dot{p}(s)\mathcal{K}(r,t-s)\; \mathrm{d}s & (2.6) \end{aligned}$$

where:

$$\mathcal{K}(r,t) = \frac{1 - \mathrm{e}^{-\gamma r^2/4t}}{r} \approx \begin{cases} \frac{\gamma r}{4t} & \text{if } r^2 \ll 2t \\ \frac{1}{r} & \text{if } r^2 \gg 2t \end{cases}$$

It is possible to find a asymptotic formula for $\mathcal{K}(r,t)$ when $\alpha = 0$, using the Plemlj's identity:

$$\lim_{a\to\infty} \frac{1 - \mathrm{e}^{\mathrm{i}ax}}{x} = \mathcal{P}\left(\frac{1}{x}\right) - \mathrm{i}\pi\delta(x)$$

Therefore, far away the skyrmion:

$$\psi(r,t) \approx \left(\int_0^t \dot{p}(s)\ \mathrm{d}s\right)\frac{1}{r} = \frac{p(t)}{r} \approx \frac{\lambda(t)}{r}$$

Using this relation, the rate of magnons emitted in the process of a skyrmion until the collapse can be estimated by:

$$\frac{\mathrm{d}\mathcal{N}_{mag}}{\mathrm{d}t} = \int_{\mathbb{R}^2} 2\psi^*\partial_t\psi\ \mathrm{d}^2x \approx 2\int_\lambda^R \frac{\lambda\dot{\lambda}}{r}\ \mathrm{d}r = 8\pi\lambda\dot{\lambda}\log\left(\frac{R}{\lambda}\right)$$

Observe we take the cut off of short distances λ since the range of validity of ψ above. This result agrees with the change of spin number own of the skyrmion (compare with [21]):

$$\frac{\mathrm{d}\mathcal{N}_U(t)}{\mathrm{d}t} = \int_{\mathbb{R}^2} \dot{\lambda}\partial_\lambda U_z\ \mathrm{d}^2x = -8\pi\lambda\dot{\lambda}\log\left(\frac{R}{\lambda}\right)$$

as was expected by the conservation law of spin number seen in the previous chapter. On the other hand, the power lost by the radiation:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int_{\mathbb{R}^2} \partial_t\mathcal{H}\ \mathrm{d}^2x \approx \int_{\mathbb{R}^2} \mathcal{J}^\dagger\partial_t\psi\ +h.c\ \mathrm{d}^2x = \int_0^R 2\frac{\dot{\lambda}^2}{r}\ \mathrm{d}r = 8\pi\dot{\lambda}^2\log\left(\frac{R}{\lambda}\right) \tag{2.7}$$

It is interesting note that:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \hbar\omega\ \frac{\mathrm{d}\mathcal{N}_{mag}(t)}{\mathrm{d}t} \qquad \text{with}\ \ \omega = \frac{\dot{\lambda}}{\lambda}$$

The above equation corresponds to the Planck equation for the quatization of the magnon field, where ω the characteristic frequency of the radiation emitted. At the end of the chapter we will see that, in fact, the radiation can be seen as quantum phenomena.

2.2. Radiation for an accelerating skyrmion

In this part we study the radiation produced by a skyrmion moving with slow velocity $\dot{\xi}$, $U(x-\xi(t))$ where U is a rigid skyrmion of radius λ fixed.

To find the spin waves around U, we proceed analogously to the equation for the mode $l=0$, far away the core of the skyrmion, such that $cos(\Theta) \approx 1$ and using (1.8):

$$\partial_t U = \dot{\xi}_\mathrm{i}\ \partial_{\xi_\mathrm{i}} U = -w(t)\frac{\mathrm{e}^{\mathrm{i}\theta}}{r^2+\lambda^2} \approx -\gamma w(t)\frac{\mathrm{e}^{\mathrm{i}\theta}}{r^2}$$

where $w(t) = \lambda(\dot{\xi}_1 - \mathrm{i}\dot{\xi}_2)$, the spin waves equation is:

$$\gamma\ \partial_t\psi - \gamma\ \dot{\xi}\nabla\psi = \mathcal{L}_1\psi - \partial_t U \approx \Delta_r\psi - \frac{4}{r^2}\psi + \frac{\gamma w(t)}{r^2}\mathrm{e}^{\mathrm{i}\theta}$$

Our strategy is to solve this equation making perturbation theory in $\dot{\xi}$, so at first order we can neglect the term $\dot{\xi}\nabla\psi$ and consider:

$$\gamma\ \partial_t\psi = \Delta_r\psi - \frac{4}{r^2}\psi + \frac{\gamma w(t)}{r^2}\mathrm{e}^{\mathrm{i}\theta}$$

Using the Green Method:

$$\psi(r,t) = \left[\int_0^t \gamma\dot{w}(s)\Psi(r,t-s)\ \mathrm{d}s + \gamma w(0)\Psi(r,t)\right]\mathrm{e}^{\mathrm{i}\theta} \tag{2.8}$$

where $\Psi(r,t)$ satisfies the equation:

$$\gamma\ \partial_t\Psi = \Delta_r\Psi - \frac{4}{r^2}\Psi + \frac{1}{r^2}$$

We look for a self-similar solution of the form: $\Psi(r,t) = f(\frac{r}{\sqrt{t}})$, replacing this in the last equation, we obtain:

$$\xi^2 f''(\xi) + (\xi + \gamma\xi^3)f'(\xi) - 4f(\xi) = -1 \tag{2.9}$$

where $\xi = r/\sqrt{t}$.

The general solution of (2.9) is:

$$f(\xi) = a_1\frac{1}{\xi^2}\mathrm{e}^{-\frac{\gamma}{2}\xi^2} + a_2\frac{(\gamma\xi^2 - 2)}{\gamma^2\xi^2} + \frac{1}{4} \tag{2.10}$$

Thus the bounded solution we choose $a_2 = -\gamma/4$ and $a_1 = -1/(2\gamma)$

$$f(\xi) = \frac{1}{2\gamma}\frac{\left(1 - \mathrm{e}^{-\frac{\gamma}{2}\xi^2}\right)}{\xi^2} \tag{2.11}$$

Therefore, using integration by parts in equation (2.8), the mode 1 magnon field far away the core of the skyrmion is:

$$\psi(r,t) = \left[\int_0^t \mathcal{K}(r,t-s)w(s)\ \mathrm{d}s\right]\mathrm{e}^{\mathrm{i}\theta} \tag{2.12}$$

where:

$$\mathcal{K}(r,t) = -\frac{1 - \mathrm{e}^{-\gamma r^2/(2t)}}{r^2} + \gamma\frac{\mathrm{e}^{-\gamma r^2/(2t)}}{2t} \approx \begin{cases} 0 & \text{if } r^2 \ll 2t \\ -\frac{1}{r^2} & \text{if } r^2 \gg 2t \end{cases}$$

The form of the radiation field given by (2.12) corresponds to a superposition of wave packets propagating since the moving core of the skyrmion, with group velocity given by the relation of dispersion $\omega_k = k^2$. We can obtain the back reaction force on the skyrmion:

$$\begin{aligned} F_1^{\mathrm{rad}}(t) &= \langle \partial_t\psi(x,t), J(U)\partial_{\xi_1}U(x,t)\rangle_{L^2} \\ &= \int_{\mathbb{R}^2}\left[\int_0^t \mathcal{K}(r,t-s)\dot{w}(s)\ \mathrm{d}s\right]\frac{\gamma^*\lambda}{r^2+\lambda^2} +\ h.c\ \ \mathrm{d}^2x \\ &= \int_0^t\left[\int_0^\infty \gamma^*\lambda\mathcal{K}(r,t-s)\frac{r}{r^2+\lambda^2}\ \mathrm{d}r\right]\dot{w}(s)\ \mathrm{d}s\ +\ h.c \end{aligned}$$

and

$$
\begin{aligned}
F_2^{\mathrm{rad}}(t) &= \langle \partial_t \psi(x,t), J(U)\partial_{\xi_2} U(x,t)\rangle_{L^2} \\
&= \int_{\mathbb{R}^2} \left[\int_0^t \mathcal{K}(r,t-s)\dot{w}(s)\ \mathrm{d}s\right] \frac{-\mathrm{i}\gamma^*\lambda}{r^2+\lambda^2} +\ h.c\ \ \mathrm{d}^2x \\
&= -\mathrm{i}\int_0^t \left[\int_0^\infty \gamma^*\lambda\mathcal{K}(r,t-s)\frac{r}{r^2+\lambda^2}\ \mathrm{d}r\right]\ \dot{w}(s)\ \mathrm{d}s\ +\ h.c
\end{aligned}
$$

Considering both components in the the complex form $\boldsymbol{F}^{\mathrm{rad}} = F_1^{\mathrm{rad}} + \mathrm{i}F_2^{\mathrm{rad}}$ and $\boldsymbol{\xi} = \xi_1 + \mathrm{i}\xi_2$, we obtain the reaction of radiation on the skyrmion:

$$
\boldsymbol{F}^{\mathrm{rad}}(t) = \int_0^t \Xi(t-s)\ddot{\boldsymbol{\xi}}(s)\ \mathrm{d}s \tag{2.13}
$$

where:

$$
\Xi(t) = \int_0^\infty \gamma^*\lambda\mathcal{K}(r,t-s)\frac{r}{r^2+\lambda^2}\ \mathrm{d}r
$$

Therefore, we get the equation of motion for the skyrmion:

$$
\begin{aligned}
M_{\mathrm{eff}}\ \ddot{\boldsymbol{\xi}}(t) &= \boldsymbol{F}^{\mathrm{ext}}(t) + \mathbf{F}^{\mathrm{Thiele}}(t) + \mathbf{F}^{\mathrm{rad}}(t) \\
&= \boldsymbol{F}^{\mathrm{ext}} + \gamma\dot{\boldsymbol{\xi}}(t) + \int_0^t \Xi(t-s)\ddot{\boldsymbol{\xi}}(s)\ \mathrm{d}s
\end{aligned}
$$

We can solve this equation using the Fourier transform, hence:

$$
\tilde{\boldsymbol{\xi}}(\omega) = \left(-\omega^2 M_{\mathrm{eff}} + \mathrm{i}\gamma\omega + \omega^2\tilde{\Xi}(\omega)\right)^{-1}\tilde{\boldsymbol{F}}^{\mathrm{ext}}(\omega) \tag{2.14}
$$

We can observe the similitude with the Lorentz's electron model for the problem of the back reaction [15]

From (2.14) we can realize that the real part of $\tilde{\Xi}(\omega)$ can be seen as a correction, in the frequency space, to the mass of the skyrmion: $M_{\mathrm{eff}}(\omega) = M_{\mathrm{eff}} - \Re(\tilde{\Xi}(\omega))$.

It is possible to see that:

$$
\begin{aligned}
\Xi(t) = \int_0^\infty \gamma^*\lambda\mathcal{K}(r,t-s)\frac{r}{r^2+\lambda^2}\ \mathrm{d}r &\approx -\gamma^*\lambda^2\int_{\sqrt{2t}}^\infty \frac{1}{r(r^2+\lambda^2)}\ \mathrm{d}r \\
&= \gamma^*\log\left(1+\frac{\lambda^2}{2t}\right)
\end{aligned}
$$

Then $\Xi(t)$ becomes singular as $t \to 0$. Similarly to the electron model, we can repair this defining the renormalized mass (what one should observe in experiments) absorbing the singular part of Ξ by means of M_{eff}. Let Λ a cutoff of energy measured in experiments, and consider the splitting: $\tilde{\Xi}(\omega) = \tilde{\Xi}^{<\Lambda}(\omega) + \tilde{\Xi}^{>\Lambda}(\omega)$, then we define the re normalized mass at scale Λ by the relation:

$$
M_{\mathrm{eff}}^{\mathrm{re}} = \lim_{\omega\to 0}\Re(\tilde{\Xi}^{<\Lambda}(\omega))
$$

2.3. Spectral analysis of the radiation

In the previous section we solved the radiation field by a slow time dependent parameters skrymion at first order. In general case we can looking for a perturbation serie for the solution of the magnon equation:

$$\partial_t\phi - \dot{\xi}\cdot\nabla\phi = \mathcal{L}\phi - \partial_t U$$

in the form:

$$\phi = \sum_n \phi^{(n)}[\xi]$$

where $\phi^{(n)}[\xi]$ are functional of order n in ξ, that means: $\phi^{(n)}[\varepsilon\xi] = \varepsilon^n\phi_n[\xi]$ (for $\varepsilon < 1$). Hence, at first order the equation is:

$$\partial_t\phi^{(1)} = \mathcal{L}\phi^{(1)} - \partial_t U$$

which was solved before. We denote in abstract form $\phi^{(1)} = \mathcal{F}[\partial_t U]$ the solution of this one. Now let's see the equation for the next order:

$$\partial_t\phi^{(2)} - \dot{\xi}\cdot\nabla\phi^{(1)} = \mathcal{L}\phi^{(2)} \quad \Rightarrow \quad \phi^{(2)} = \mathcal{F}[\dot{\xi}\cdot\nabla\phi^{(1)}]$$

and so on:

$$\phi^{(n+1)} = \mathcal{F}[\dot{\xi}\cdot\nabla\phi^{(n)}]$$

for all $n > 1$. Therefore, the set radiative corrections $\phi^{(n)}$ can be founded just solving the following evolution equation:

$$\mathrm{i}\ \partial_t\phi(r,t) = \mathcal{L}\phi(r,t) + h(r,t) \tag{2.15}$$

Expanding the fields ϕ and h in angular modes, it reduces to solve:

$$\mathrm{i}\ \partial_t\phi_l(r,t) = \mathcal{L}_l\phi_l(r,t) + h_l(r,t) \tag{2.16}$$

The solution of equation (2.16) can be solved by means of the retarded Green function G of $\mathcal{H}_l$:

$$\phi_l(r,t) = \int_0^t\int_0^\infty G_l(r,r',t-s)h_l(r',s)\ r'\mathrm{d}r'\ \mathrm{d}s \tag{2.17}$$

or expressed by through the Fourier transform:

$$\tilde{\phi}_l(r,\omega) = \int_0^\infty G_l(r,r',\omega)\tilde{h}_l(r',\omega)\ r'\mathrm{d}r' \tag{2.18}$$

Our task this section is to find the retarded Green function $G(x,y,\omega)$ solving:

$$(\omega - \mathcal{L}_{l,r})\,G(r,r',\omega) = \frac{1}{r'}\delta(r-r') \tag{2.19}$$

where the subscripts r in $\mathcal{L}_{l,r}$ denotes derivations respects to r. In addition, we have to set up the boundary conditions through a physical hypothesis that radiation at spacial infinity $r\to\infty$ is purely out coming and bounded in the whole space.

We use the matching method:

$$G(r,r') = \begin{cases} \phi^{>}_{l,k}(kr) & \text{if } r > r' \\ \phi^{<}_{l,k}(kr) & \text{if } r < r' \end{cases}$$

Where $\phi_{l,k}(r)$ are eigenvalues of $\mathcal{H}_{l,k}\phi_l = k^2\phi_{l,k}$ Typically we are interested in sources h such that its support is concentrated near the core of the skyrmion, so we will suposse that $r' \lesssim \lambda$. Moreover we want to evaluate $\phi_l(r)$ at point very far away the core, so $\lambda \ll r$. From here

$$\left(k^2 - \partial_r^2 - \frac{(l+1)^2}{r}\partial_r\right)\phi^{>}_{l,k}(r) = 0 \quad \text{with} \quad \phi^{r>r'}_{l,k}(r) \to \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \quad \text{for} \quad r \to \infty$$

and

$$\left(k^2 - \partial_r^2 - \frac{(l-1)^2}{r}\partial_r\right)\phi^{<}_{l,k}(r) = 0 \quad \text{with} \quad \phi^{r<r'}_{l,k}(r) < \infty \quad \text{for} \quad r \to 0$$

The solutions of this equations are

$$\begin{aligned} \phi^{>}_{l,k}(r) &= c^{>}_{l,k}(r')\ H^{(1)}_{l+1}(kr) \\ \phi^{<}_{l,k}(r) &= c^{<}_{l,k}(r')\ J_{l-1}(kr) \end{aligned}$$

where $H^{(1)}_l(kr) = J_l(kr) + \mathrm{i}\ Y_l(kr)$ is the Hankel function of first kind. We can determine $c^{<}_l(r')$ and $c^{>}_l(r')$ using the matching conditions at $r = r'$:

$$\phi^{>}_{l,k}(r') = \phi^{<}_{l,k}(r') \quad \Rightarrow \quad c^{>}_{l,k}(r')\ H^{(1)}_{l+1}(kr') - c^{<}_{l,k}(r')\ J_{l-1}(kr') = 0$$

Besides since $\int_{r'-\varepsilon}^{r'+\varepsilon} \partial_r G(r,r')\ r\mathrm{d}r = 2\pi$ so:

$$r'\partial_r\phi^{>}_{l,k}(r') - r'\partial_r\phi^{<}_{l,k}(r') = 2\pi \quad \Rightarrow \quad c^{>}_{l,k}(r')kr'\ \partial_r H^{(1)}_{l+1}(kr') - c^{<}_{l,k}(r')kr'\ \partial_r J_{l-1}(kr') = 2\pi$$

Solving the system of equations we get:

$$\begin{aligned} c^{>}_{l,k}(r') &= \frac{2\pi J_{l-1}(kr')}{kr'\left(J_{l-1}(kr')\partial_r H^{(1)}_{l+1}(kr') - H^{(1)}_{l+1}(kr')\partial_r J_{l-1}(kr')\right)} \\ c^{<}_{l,k}(r') &= \frac{2\pi\ H^{(1)}_{l+1}(kr')}{kr'\left(\partial_r H^{(1)}_{l+1}(kr')J_{l-1}(kr') - H^{(1)}_{l+1}(kr')\partial_r J_{l-1}(kr')\right)} \end{aligned}$$

Consequently we have:

$$\tilde{\phi}_l(r,\omega) = \left[2\pi\int_0^\infty c^{>}_{l,k}(r')\tilde{h}_l(r',\omega)\ r'\mathrm{d}r'\right] H^{(1)}_{l+1}(kr)$$

Using the asymptotic formula:

$$H^{(1)}_l(kr) \approx \sqrt{\frac{2}{\mathrm{i}\pi kr}}\mathrm{e}^{\mathrm{i}kr - \mathrm{i}\frac{l}{2}\pi}$$

Thus far away the core of the skyrmion, we have the asymptotic formula:

$$\tilde{\phi}_l(r,\omega) \approx f_l(k)\ \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}} \quad \Rightarrow \quad \tilde{\phi}(r,\omega) \approx f(k,\theta)\ \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}$$

where the angular distribution of the radiation is:

$$f(k,\theta) = \sum_l f_l(k)\ \mathrm{e}^{\mathrm{i}l\theta} = \sum_l 2\pi\sqrt{\frac{2}{\mathrm{i}\pi}}\left[\int_0^\infty c^{>}_{l,k}(r')\tilde{h}_l(r',\omega)\ r'\mathrm{d}r'\right]\ \mathrm{e}^{\mathrm{i}l\theta+\mathrm{i}\frac{l}{2}\pi}$$

Hence the scattering cross section of the magnon radiation will be $\sigma_l(k) = |f_l(k)|^2$. Integrating:

$$\sigma(k) = \lim_{r\to\infty}\int_0^{2\pi} \tilde{\mathcal{J}}(r,\theta,\omega)\cdot\mathbf{r}\ \mathrm{d}\theta = \lim_{r\to\infty}\int_0^{2\pi}\left(\mathrm{i}\tilde{\phi}^\dagger(r,\theta,\omega)\nabla_{\mathbf{r}}\tilde{\phi}(r,\theta,\omega)\cdot\mathbf{r} + h.c\right)\ \mathrm{d}\theta = \sum_l \sigma_l(k)$$

Now, using the formulas (A.8) and (A.11) from the appendix, we find that the spectral flux of energy is:

$$\begin{aligned}\tilde{\mathcal{E}}(\omega) = \lim_{r\to\infty}\int_0^{2\pi}\tilde{T}_{\mathrm{i}0}(r,\theta,\omega)\ r^{\mathrm{i}}\ \mathrm{d}\theta &= -\int_0^{2\pi}\left(\partial_t\tilde{\phi}^\dagger(r,\theta,\omega)\ \nabla_{\mathrm{i}}\tilde{\phi}(r,\theta,\omega)\ r^{\mathrm{i}} + h.c\right)\ \mathrm{d}\theta \\ &= \sum_l \omega\ \sigma_l(k)\end{aligned}$$

Whereas, the force on the skyrmion is:

$$\begin{aligned}\tilde{F}_j(\omega) = \lim_{r\to\infty}\int_0^{2\pi}\tilde{T}_{\mathrm{i}j}(r,\theta,\omega)\ r^{\mathrm{i}}\ \mathrm{d}\theta &= -\int_0^{2\pi}\left(\partial_j\tilde{\phi}^\dagger(r,\theta,\omega)\ \nabla_{\mathrm{i}}\tilde{\phi}(r,\theta,\omega)\ r^{\mathrm{i}} + h.c\right)\ \mathrm{d}\theta \\ &= k\int_0^{2\pi}|f(\theta)|^2\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}\ \mathrm{d}\theta\end{aligned}$$

The retarded Green function will be given by:

$$G_l(r,r',t) = \int_{-\infty+\mathrm{i}\varepsilon}^{+\infty+\mathrm{i}\varepsilon} G_l(r,r',\omega)\mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}\omega$$

The contour of integration is taken over the upper half of the complex plane, in order to satisfy the property $G_l(r,r',t) = 0$ for $t < 0$.

We would like to extend $\omega \mapsto G(r,r',\omega)$ to the complex plane, but observe that the since $\omega = k^2$, we must to do a branch cut on the negative real axis $\mathbb{R}_{(-)}$. Nevertheless it is possible that there exists another branch points, in that case, will be necessary to find the adequate Riemann surface. Done that, let $\omega_n \in \mathbb{C}$ the poles of $G(r,r',\omega)$. Using the Cauchy theorem of residua, we have:

$$G_l(r,r',t) = \sum_n 2\pi\mathrm{i}\ R(\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n t}$$

the frequencies $\omega_n = \Re(\omega_n) + \mathrm{i}\Im(\omega_n)$ are known as quasi normal modes and in the context of resonances in quantum theory. The real part $\varepsilon_n = \Re(\omega_n)$ gives the energy whereas the imaginary part $\tau_n = \Im(\omega_n)^{-1}$ defines the life time of the resonant state, respectively. The reason of this fact is due to the tunneling of magnons of modes l through the barrier potential $V_l(r)$.

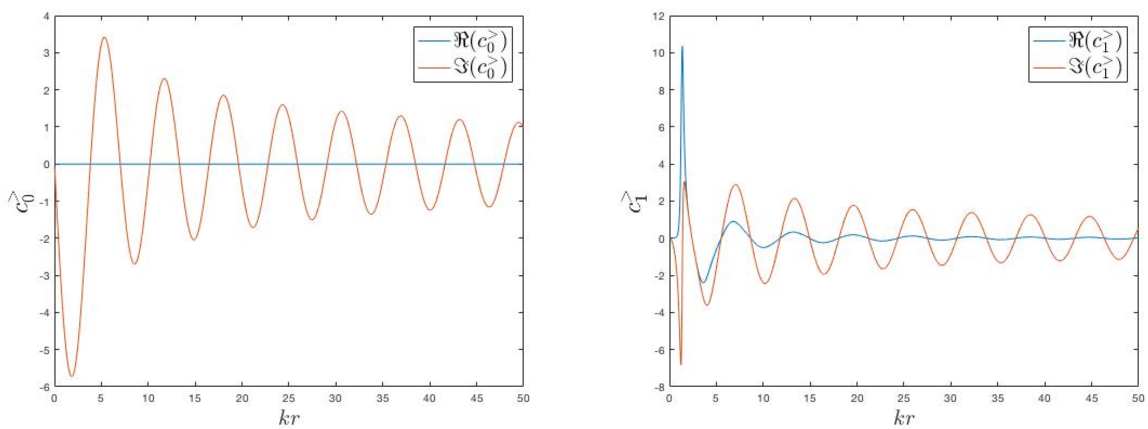
We can estimate $c_l^<$ in the limit of short wave length, using the asymptotic behaviour of the Bessel functions for small values of $kr \ll 1$:

$$\frac{\mathrm{d}H_{l+1}^{(1)}(x)}{\mathrm{d}x} J_{l-1}(x) - H_{l+1}^{(1)}(x) \frac{\mathrm{d}J_{l-1}(x)}{\mathrm{d}x} \approx \begin{cases} \mathrm{i}\frac{4^{-|l|}\Gamma(|l+1|)}{\pi\Gamma(|l-1|)} \frac{1}{x^3} & \text{if } l \neq -1 \\ \frac{1}{2\pi} \frac{1}{x \log(1/x)} & \text{if } l = -1 \end{cases}$$

therefore:

$$c_{l,k}^>(r) \approx \begin{cases} \mathrm{i}\frac{2^{-|l+1|}}{\Gamma(|l+1|)} (kr)^{|l+1|} & \text{if } l \neq -1 \\ \frac{kr}{\log(1/(kr))} & \text{if } l = -1 \end{cases}$$

We see the intensity of scattering is maximal for the mode $l = -1$.



(a) Graphics of $c_{l=0}(kr)$

(b) Graphics of $c_{l=1}(kr)$

Figura 2.1: Graphics of $c_l(kr)$

We can observe from the graphics that the real part of $c^>(kr)$ vanishes, in consequence, the mean contribution to the radiation in the zero angular mode comes from the breathing modes, since $h = \mathrm{i}\partial_t U = \mathrm{i}(\dot{\lambda} + \mathrm{i}\lambda\dot{\omega})Z_0(r) \approx \dot{\lambda} Z_0(r)$ for skyrmions of small radius.

2.3.1. Resonances and WKB semi-classical approach

In this section we will explore a classic method widely used in several papers [27], [37], [28] for scattering of magnons by a rigid skyrmion.

It will be useful reestablish the dimensional constant D into the Landau-Lifshitz equation. Define $\hbar = D^{-1}$, so the magnon equation take the form:

$$\mathrm{i}\hbar\ \partial_t \phi = \mathcal{H}\phi$$

In the the WKB approximation we look for an expansion $\psi(x,t) = (a_0(x) + a_1 t + \dots)\ \mathrm{e}^{\mathrm{i}S(x,t)}$

$$K(x,y,t) = \int_{\mathbf{R}^2} \mathrm{e}^{\mathrm{i}\hbar^1 (S(x,t,k) - \langle y,k\rangle)} a(x,y,t,k)\ \mathrm{d}^2 k$$

where $S(x,t,q)$ satisfies the Hamilton-Jacobi equation:

$$\begin{aligned} \partial_t S(x,t,k) &= |\nabla_x S(x,t,k) - \mathbf{A}(x)|^2 + V(x) \\ S(x,0,k) &= \langle x,k\rangle \end{aligned}$$

Let l fixed and consider the complex valued function $p(r) = \sqrt{k^2 - U_l(r)}$ (associated to the classical momentum), then we can solve approximately the stationary Schrodinger's equation:

$$(-\partial_r^2 + V_l(r))\ \psi = k^2\ \psi$$

$\Rightarrow$

$$\psi(r) \approx \begin{cases} \frac{2A}{\sqrt{p(r)}} \sin(\int_{r_0}^{r} p(r')\ \mathrm{d}r' + \frac{\pi}{4}) & \text{if } V_l(r) < k^2 \\ \frac{A}{\sqrt{|p(r)|}} \exp\left(-\int_{r_0}^{r} |p(r')|\ \mathrm{d}r'\right) & \text{if } V_l(r) > k^2 \end{cases}$$

where r_0 is any turning point, which are defined by the zeros of $p(r)$. Finally the phase shift of scattering will be:

$$\delta_l = \int_{r_0}^{\infty} \left(\sqrt{k^2 - V_l(r)} - k\right)\ \mathrm{d}r + \frac{\pi}{2}(l-1) - kr_0 \tag{2.20}$$

From this

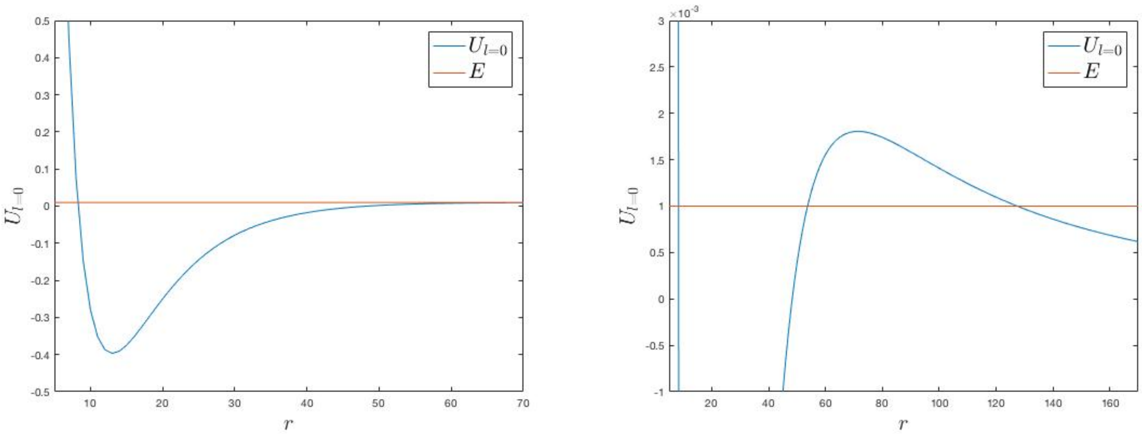
$$\psi(r,\theta) = f(\theta)\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \qquad \text{where } f(\theta) = \frac{\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}}{\sqrt{2\pi k}}(\mathrm{e}^{2\mathrm{i}\delta_l} - 1)$$

The scattering cross section will be given by the well known formula:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{4}{k}|f(\theta)|^2 = \sum_l \sin^2 \delta_l$$

The phase shift also gives information about the zero point energy:

For the mode 0 the effective potential $U_{l=0} = \frac{\cos(\Theta)}{r^2}$ has the shape of a potential well for low energies, so the quantum tunnel effect implies that the quantum of a local breathing mode



(a) Potential well form of U_0

(b) Near the turning points

Figura 2.2: Graphics of Potential U_0, $E = k^2$

excitation can escape to infinity (as magnon quasi-particle), even if it has not the enough energy to do it classically. The tunneling probability that a magnon of energy $\omega = k^2$ through the potential barrier V_l is given approximately :

$$P \approx \exp\left\{-2\int_{x_1}^{x_2}\left(\sqrt{V_l(r) - k^2}\right)\,\mathrm{d}r\right\}$$

In conclusion, this phenomena is analogous to the radioactive emission of particles by atomic nucleus.

In the next chapter we will that this phenomena is related to an instability in the vacuum of the magnon field due the presence of the skyrmion.

Capítulo 3

Quantum Magnon Radiation

3.1. Quantum origin of the radiation

We will determine the reaction force through an effective theory, following the approach given in [25], namely we would like get rid of the spin excitation freedom degrees 'integrating' this ones. This approach gives successfully a clever understanding about the Abraham-Lorentz's reaction force of the electron in electrodynamics. We are going to explain how it works. Let $\mathcal{S}$ be an action functional in ϕ with a source $\mathcal{J}$:

$$\mathcal{S}_{\text{eff}}[\mathcal{J}] = -\frac{1}{2}\int\int \tilde{\mathcal{J}}^{\dagger}(x,-\omega)G(x,y,\omega)\tilde{\mathcal{J}}(y,\omega) \ \ \mathrm{d}^2y\ \mathrm{d}^2x\ \mathrm{d}\omega$$

Where $G(x,y,\omega) = (\omega - \mathcal{H} + \mathrm{i}\varepsilon)^{-1}(x,y)$ is the propagator retarded propagator, given by:

$$G(x,y,\omega) = \lim_{\varepsilon\to 0}(\omega - \mathcal{H} + \mathrm{i}\varepsilon)^{-1}(x,y) = \sum_{k,l}\frac{\phi_{k,l}(r)\phi^{*}_{k,l}(r')}{\omega - k^2 + \mathrm{i}0^{+}}\mathrm{e}^{\mathrm{i}l(\theta-\theta')}$$

where:

$$\frac{1}{\omega - k^2 + \mathrm{i}0^{+}} = \mathcal{P}\frac{1}{\omega - k^2} - \mathrm{i}\pi\ \delta(\omega - k^2)$$

We know from quantum theory, that the real part contribute to the shift of the energy levels of the magnons modes due to the presence of the skyrmion. On the other hand, the imaginary part determines the spectrum of radiation emitted by the source. We know from quantum theory, that the real part contribute to the shift of the energy levels of the magnons modes due to the presence of the skyrmion. On the other hand, The imaginary part determines the spectrum of radiation emitted by the source. An sketch of this fact is presented, following [16]: The probability of permanence at time T of the vacuum is:

$$\mathbb{P}(T) = |\langle 0_{\text{out}}|0_{\text{in}}\rangle|^2 = |\ \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\text{eff}}}\ |^2 = \mathrm{e}^{-\frac{2}{\hbar}\Im(\mathcal{S}_{\text{eff}})}$$

The exponent $\frac{2}{\hbar}\Im(\mathcal{S})$ corresponds to the average number of particles emitted in the process (see [16]). In consequence, doing the spectral decomposition:

$$\frac{2}{\hbar}\Im(\mathcal{S}) = \sum_{k}\mathcal{N}_k$$

where $\mathcal{N}_k$ is the total number of particles with energy $\hbar\omega_k$ emitted in the whole process. Therefore the energy radiated will be:

$$\mathcal{E} = \sum_k \hbar\omega_k \mathcal{N}_k$$

Now, from (2.3):

$$\widetilde{\mathcal{J}}(r,\theta,\omega) = -\mathrm{i}\ \widetilde{\partial_t U}(r,\theta,\omega) = -\mathrm{i}\ \widetilde{\dot{\lambda}}(\omega)\ \partial_\lambda U \approx -\mathrm{i}\omega\ \tilde{\lambda}(\omega)\ \frac{1}{r}$$

Therefore:

$$\begin{aligned}
\mathcal{N}_k = \frac{1}{\hbar}\left[\int_0^\infty \phi_{k,0}(x)\mathcal{J}(x,\omega_k)\ \mathrm{d}^2x\right]^2 &= \frac{1}{\hbar}\left[\int_0^\infty \frac{\phi_{k,0}(r)}{r}\ r\ \mathrm{d}r\right]^2 \omega_k^2\ \tilde{\lambda}(\omega_k)^2 \\
&\approx \frac{1}{2\hbar}\left[\int_0^\infty \frac{r}{r^2+\lambda^2}\ \mathrm{d}r + O(k^2)\right]^2 \omega_k^2\ \tilde{\lambda}(\omega_k)^2 \\
&\approx \frac{1}{2\hbar}\log\left(\frac{R}{\lambda}\right)\omega_k^2\ \tilde{\lambda}(\omega_k)^2 + O(k^2)
\end{aligned}$$

In the next section we provide a quantum description of the radiation by means of creation-annihilation operators of magnons.

3.2. Canonical Quantization of the Radiation

We have seen in the chapter 1 that changing the field $\phi \mapsto \eta = \lambda\phi$ the magnon field action can be written as:

$$\mathscr{L}_{\mathrm{mag}} = J_{q_{\mathrm{i}}}[\eta]\ \dot{q}^{\mathrm{i}}\ +\ \frac{1}{2}\eta^\dagger\left(\mathrm{i}\partial_\tau - \mathcal{L}_{U_0} - V(\tau)\right)\eta\ + h.c$$

We observe that it is possible to neglect the Berry phase and the term $V(\tau)$ in the Lagrangian, as $\lambda \to 0$, leading to:

$$\mathscr{L}_{\mathrm{mag}} = \lambda\mathcal{J}^\dagger\eta\ +\ \frac{1}{2}\eta^\dagger\left(\mathrm{i}\partial_\tau - \mathcal{L}_{U_0}\right)\eta\ + h.c$$

in consequence the canonical momentum is given by:

$$\pi(y,\tau) = \frac{\partial\mathscr{L}}{\partial(\partial_\tau\eta(x,\tau))} = \mathrm{i}\eta^\dagger(y,\tau) \tag{3.1}$$

Canonical quantization proceeds in the standard way: namely $\eta \mapsto \hat{\eta}$ and $\pi \mapsto \hat{\pi}$ turns into operators which satisfy the following canonical commutation identity:

$$[\hat{\eta}(y,\tau), \hat{\pi}(y',\tau)] = \mathrm{i}\ \delta(y-y')$$

and:

$$[\hat{\eta}(y,\tau), \hat{\eta}(y',\tau)] = [\hat{\pi}(y,\tau), \hat{\pi}(y',\tau)] = 0$$

Now, for introduce the correspondent operators of creation-annihilation, we need to expand $\hat{\eta}$ in the basis of eigen-functions of $\mathcal{L}_{U_0}$:

$$\hat{\eta}(r,\theta,\tau)=\sum_{lk}\ \hat{a}_{lk}(\tau)\ \phi_{lk}(r)\mathrm{e}^{\mathrm{i}l\theta}$$

where $\left\{\phi_{lk}(r)\mathrm{e}^{\mathrm{i}l\theta}\right\}_{l,k}$ is the normalized set of eigenfunctions of $\mathcal{L}$. Additionally, we expand $\hat{\pi}$:

$$\hat{\pi}(r,\theta,\tau)=\sum_{lk}\ \mathrm{i}\ \hat{a}^{\dagger}_{lk}(\tau)\ \phi_{lk}(r)\mathrm{e}^{-\mathrm{i}l\theta}$$

Therefore, the set $\hat{a}_{kl}(\tau)$ satisfy:

$$\left[\hat{a}^{\dagger}_{kl}(\tau),\hat{a}_{k'l'}(\tau)\right]=\delta(k-k')\delta_{l,l'}$$

and

$$\left[\hat{a}_{kl}(\tau),\hat{a}_{k'l'}(\tau)\right]=\left[\hat{a}^{\dagger}_{kl}(\tau),\hat{a}^{\dagger}_{k'l'}(\tau)\right]=0$$

The correspondent Hamiltonian will be

$$\hat{\mathscr{H}}_{mag}(\tau)=\sum_{k,l}\ \lambda(\tau)\mathcal{J}^{\dagger}_{kl}(\tau)\ \hat{a}_{kl}(\tau)+\frac{1}{2}\omega_k\ \hat{a}^{\dagger}_{kl}(\tau)\hat{a}_{kl}(\tau)\ +h.c$$

where $\mathcal{J}_{kl}=\left\langle\mathcal{J},\phi_{kl}\mathrm{e}^{\mathrm{i}l\theta}\right\rangle_{L^2}$ Hence the motion equation for the modes is:

$$\frac{\mathrm{d}\hat{a}_{kl}(\tau)}{\mathrm{d}\tau}=\left[\hat{\mathscr{H}}_{mag}(\tau),\hat{a}_{kl}(\tau)\right]=-\mathrm{i}\omega_k\ \hat{a}_{kl}(\tau)+\lambda\mathcal{J}_{kl}(\tau) \tag{3.2}$$

thus:

$$\hat{a}_{kl}(\tau)=\mathrm{e}^{-\mathrm{i}\omega_k\tau}\hat{a}_{kl}(0)+\int_0^{\tau}\mathrm{e}^{\mathrm{i}\omega_k(\tau'-\tau)}\ \lambda(\tau')\mathcal{J}_{kl}(\tau')\ \mathrm{d}\tau'$$

Defining the quantum number: $\hat{\mathcal{N}}_{kl}(\tau)=\hat{a}^{\dagger}_{kl}(\tau)\hat{a}_{kl}(\tau)$, thus the average number of magnons in the state (k,l) produced at the time τ, is given by:

$$\begin{aligned}N_{kl}(\tau)=\left\langle 0\left|\hat{\mathcal{N}}_{kl}(\tau)\right|0\right\rangle&=\int_0^{\tau}\int_0^{\tau}\mathrm{e}^{\mathrm{i}\omega_k(\tau_2-\tau_1)}\ \lambda(\tau_1)\lambda(\tau_2)\ \ \mathcal{J}_{kl}(\tau_1)\mathcal{J}_{kl}(\tau_2)\ \mathrm{d}\tau_1\ \mathrm{d}\tau_2\\&=\left|\int_0^{\tau}\ \lambda(\tau')\mathcal{J}_{kl}(\tau')\mathrm{e}^{\mathrm{i}\omega_k\tau'}\ \mathrm{d}\tau'\right|^2\end{aligned}$$

On the other hand:

$$\mathcal{J}_{k0}(\tau)=\int_0^{\infty}\frac{\lambda(\tau)\dot{\lambda}(\tau)r}{r^2+1}\ \phi_{k,l=0}(r)\ r\mathrm{d}r=\lambda(\tau)\dot{\lambda}(\tau)c_k$$

where

$$c_k=\int_0^{\infty}\frac{r^2}{r^2+1}\ \phi_{k,l=0}(r)\ \mathrm{d}r$$

Hence

$$N_{k,l=0}(\tau) = \left| \int_0^\tau \ \lambda^2(\tau)\dot{\lambda}(\tau) c_k \ \mathrm{e}^{\mathrm{i}\omega_k \tau} \ \mathrm{d}\tau \right|^2$$

For small energy magnons, this last quantity can be estimated using that $\lambda(\infty) = 0$ and $\lambda^2 \ \mathrm{d}\tau = \mathrm{d}t$, hence:

$$N_{k,l=0}(\tau) \approx \left| \int_0^\infty \ \lambda^2(\tau)\dot{\lambda}(\tau) \ \mathrm{d}\tau \right|^2 c_k^2 = (\lambda_0 - \lambda(\tau))^2 c_k^2$$

This last quantity allows us to determine the average number of magnons emitted during the whole process of the collapse:

$$N_{k,l=0}(\tau = \infty) \approx \lambda_0^2 \ c_k^2$$

In consequence, the spectrum of energy:

$$\mathcal{E}_{k,l=0}(\tau = \infty) \approx \omega_k \lambda_0^2 \ c_k^2$$

The probability of create a n magnons in the state (k, l) is given by the Poisson distribution:

$$\mathcal{P}_{kl}(n) = \frac{N_{kl}^n}{n!} \mathrm{e}^{-N_{kl}}$$

We can study this result during the self-similar regimen: $\lambda(t) = \sqrt{T - t} \ \lambda_0$

$$\tau(t) = \int_0^t \frac{\mathrm{d}t'}{\lambda^2(t')} = \frac{1}{\lambda_0^2} \int_0^t \frac{\mathrm{d}t'}{T - t'} = \frac{1}{\lambda_0^2} \log(\frac{1}{T - t}) \to \infty$$

thus:

$$\lambda(\tau) = \lambda_0 \mathrm{e}^{-\frac{\lambda_0^2}{2}\tau}, \quad \lambda_\tau(\tau) = -\frac{\lambda_0^3}{2} \mathrm{e}^{-\frac{\lambda_0^2}{2}\tau}$$

and finally we obtain:

$$N_{k,l=0}(\infty) = \left| \int_0^\infty \ \lambda_\tau c_k \mathrm{e}^{\mathrm{i}\omega_k \tau} \ \mathrm{d}\tau \right|^2 = |c_k|^2 \left| \int_0^\infty \ \frac{\lambda_0^3}{2} \mathrm{e}^{-\frac{\lambda_0^2}{2}\tau} \mathrm{e}^{\mathrm{i}\omega_k \tau} \ \mathrm{d}\tau \right|^2 = \frac{\lambda_0^6}{\lambda_0^4 + \omega_k^2} \ |c_k|^2$$

The resulting spectrum is proportional to the Fock-Breit-Wigner distribution very known in nuclear physics (in the energy variable), where the energy of resonance is $\omega_{res} = 0$ and the spectral width is given by $\Gamma = \lambda^2$, which in turns determines the lifetime of the resonance: $\tau_{res} = \frac{1}{\lambda^2}$.

3.3. Quantum Path Integral and Effective Action

In this section we are interesting in deriving an effective action for the dynamics of the skyrmion-magnon system using the path integral method [7]. So we have to evaluate the formal path integral Using the formula (1.32) then:

$$\begin{aligned}\int \exp\left(\mathrm{i}\mathcal{S}[u]\right)\,\mathcal{D}u &= \int\int \exp\left(\mathrm{i}\mathcal{S}[U+\psi]\right)\,\mathcal{D}U\;\mathcal{D}\psi\mathcal{D}\psi^{\dagger}\\ &= \int\left\{\int \exp\left[\,\mathrm{i}\mathcal{S}[U]\;+\mathrm{i}\int \mathcal{J}^{\dagger}\psi\;-\psi^{\dagger}(\mathrm{i}\partial_t\;+\mathcal{L}_U)\psi + h.c\right]\mathcal{D}\psi\mathcal{D}\psi^{\dagger}\right\}\mathcal{D}U\\ &= \int \exp\left(\mathrm{i}\mathcal{S}_{\text{eff}}[U]\right)\,\mathcal{D}U\end{aligned}$$

Integrating the standard Gaussian integral in ψ, we obtain a non-local action effective action for the skyrmion:

$$\begin{aligned}\mathcal{S}_{\text{eff}}[U] = \int \mathcal{A}(U)\cdot\partial_t U\;\;\mathrm{d}^2x\;\mathrm{d}t\;+\int\int \mathcal{J}^{\dagger}(y,s)\;G_U(y-x,s-t)\mathcal{J}(x,t)\;\mathrm{d}^2y\;\mathrm{d}s\;\mathrm{d}^2x\;\mathrm{d}t\\ +\mathrm{i}\log(\det[\;\mathrm{i}\partial_t-\mathcal{L}_U\;])\end{aligned}$$

Hence defining $\mathcal{S}_{\text{eff}}[\xi]=\mathcal{S}_{\text{eff}}[U[\xi]]$ we obtain the effective action for the skyrmion dynamics:

$$\mathcal{S}_{\text{eff}}[q] = \int\; A(\xi(t))_{\mathrm{i}}\;\dot{q}^{\mathrm{i}}(t)\;\mathrm{d}t + \int\int\;\Sigma_U(q(s),q(t);s-t)_{\mathrm{ij}}\;\dot{\xi}^{j}(s)\;\dot{\xi}^{\mathrm{i}}(t)\;\mathrm{d}s\;\mathrm{d}t + \mathrm{i}\log(\det[\;\mathrm{i}\partial_t-\mathcal{L}_U\;]) \tag{3.3}$$

For instance, one obtain the motion equation by $\delta S[\xi]=0$:

$$\begin{aligned}F_{\mathrm{ij}}(\xi(t))\;\dot{\xi}^{j}(t) = \int_0^t C_{\mathrm{ij}}(\xi(s),\xi(t);t-s)\;\dot{\xi}^{j}(s)\;\mathrm{d}s + \left\{\int_0^t D_{\mathrm{ij}k}(\xi(s),\xi(t);t-s)\;\dot{\xi}^{k}(s)\;\mathrm{d}s\right\}\dot{\xi}^{j}(t)\\ +\mathrm{i}\delta_{\xi}\log(\det[\;\mathrm{i}\partial_t-\mathcal{L}_U\;])\end{aligned}$$

The left size corresponds to the classic's force over the skyrmion while the right size is the reaction force due to the quantum fluctuations.

For a static skyrmion U of radius λ, the last term in (3.3) gives the contribution from quantum fluctuations, thus corresponds to zero point energy of the skyrmion (see [5]):

$$\log(\det[\;\mathrm{i}\partial_t-\mathcal{L}_U\;]) = -\mathrm{i}\int \mathcal{E}^{vac}(\lambda)\;\mathrm{d}^2x\;\mathrm{d}t$$

where $\mathcal{E}^{vac}(\lambda)$ corresponds to the density of energy of the magnon's vacuum in presence of the skyrmion:

$$\mathcal{E}^{vac}(\lambda) = \sum_k \frac{1}{2}\;\omega_k(\lambda)$$

In order to regularize it, we use the Hawking's zeta function method by means of define:

$$\zeta_{\mathscr{A}}(s) = \sum_{\mu\in\sigma(\mathscr{A})}\frac{1}{\mu^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\;\mathrm{tr}\left(\mathrm{e}^{-t\mathscr{A}}\right)\;\mathrm{d}t$$

where $\sigma(\mathscr{A}) = \{\mu_k : k = 1, 2, \dots\}$ is the spectrum of $\mathscr{A}$. Thus:

$$\sum_k \omega_k(\lambda) \overset{\text{reg}}{=} \lim_{s\to -1} \sum_n \frac{1}{\omega_k^s} = \zeta_{\mathcal{L}_U}(-1) = \zeta_{\mathcal{L}_{U_0}}(-1)\frac{1}{\lambda^2}$$

It is possible to compute the value of $c_0 = -\zeta_{\mathcal{L}_{U_0}}(-1)$ using the asymptotic expansion of the heat Kernel of $\mathcal{L}_0$, but it is beyond the scope of this work, by the way, the zero point energy of skyrmion due to magnon field is

$$\mathcal{E}^{vac}(\lambda) = -c_0 \frac{1}{\lambda^2}$$

this effect is related with the the break of the conformal symmetry by quantum fields, a phenomena known as quantum conformal anomaly in context of sigma non linear models and string theory [5]. In [6] is proved using scattering theory that $\mathcal{E}(\lambda) \sim -\frac{a}{\lambda^2}$ with $a > 0$. This Break of conformal symetry in addition with loss of energy due to the magnon radiation, implies the collapse of skyrmion in a finite time. We can estimate the life time of the skyrmion using (2.7):

$$\frac{\mathrm{d}\mathcal{E}(\lambda)}{\mathrm{d}\lambda}\dot{\lambda} = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = 8\pi\dot{\lambda}^2 \log\left(\frac{R}{\lambda}\right)$$

Hence:

$$t_{\text{life}} = 8\pi \int_{\lambda_0}^{0} \log\left(\frac{R}{\lambda}\right) \frac{1}{\mathcal{E}'(\lambda)}\, \mathrm{d}\lambda \approx 8\pi \int_0^{\lambda_0} \log\left(\frac{R}{\lambda}\right) \lambda^3 \,\mathrm{d}\lambda \leq 2\pi \log\left(\frac{R}{\lambda_0}\right) \lambda_0^4$$

Now we focus in the particle production during the process of collapsing of the skyrmion:

$$\lambda(\tau) = \begin{cases} \lambda_0 & \text{if } \tau \to -\infty \\ 0 & \text{if } \tau \to \infty \end{cases}$$

We write the spin waves fields (quantized) for $\tau \to -\infty$ in the form:

$$\hat{\psi}_{\text{in}}(x,\tau) = \sum_l \int \hat{a}_{lk}^-\ u_{lk}(x)\mathrm{e}^{-\mathrm{i}\omega\tau} + \hat{b}_{lk}^+\ v_{lk}^*(x)\mathrm{e}^{\mathrm{i}\omega\tau}\ \mathrm{d}k$$

where $\hat{a}_k^-$ and $\hat{b}_k^+$ are the operators of annihilation of a magnon and creation of an anti-magnon, of momentum $\mathbf{k}$, respectively. Now replacing in the (1.44), we see that $u(x)$ and $v(x)$ are eigenfunctions of $\mathcal{L}_l$. In addition, in order to satisfies the commutation property $[a_{lk}, a_{lk}^\dagger] = [b_{lk}, b_{lk}^\dagger] = 1$, they must be normalizated.

On the other hand, for $\tau \to \infty$, we can expand the spin waves fields in the form:

$$\hat{\psi}_{\text{out}}(x,\tau) = \sum_l \int \hat{A}_k^-\ U_{lk}(x)\mathrm{e}^{-\mathrm{i}\omega\tau} + \hat{B}_k^+\ V_{lk}^*(x)\mathrm{e}^{\mathrm{i}\omega\tau}\ \mathrm{d}k$$

here $\hat{A}_{lk}^-$ and $\hat{B}_{lk}^+$ are the operators of annihilation of a magnon and creation of an anti-magnon, of momentum $\mathbf{k}$, but now $U_{lk}(x)$ and $V_{lk}(x)$ are eigenfunctions of Δ_{l+1}, since, as we have seen in chapter 1, if $\lambda \to 0$ the Ahronov-Bohm effect shifts the angular modes.

The standard Bogolyubov transformation [16] relates the fields $\hat{\psi}_{\text{out}}$ and $\hat{\psi}_{\text{out}}$ through the $\alpha_{kl,l'k'}$ and $\beta_{lk,l'k'}$ coefficients:

$$U_{lk}(x) = \sum_{l'} \int \alpha_{kl,l'k'}\ u_{k'}(x) + \beta_{k,l'k'}\ v^*_{lk'}(x)\ \mathrm{d}k'$$

Therefore we can calculate the average number of particles in the state (l, k) created in the annihilation process:

$$\langle \mathcal{N}_{lk} \rangle (\tau = \infty) = \left\langle 0_{\text{in}} \right| \hat{A}^-_k \hat{A}^+_k \left| 0_{\text{in}} \right\rangle = \sum_{l'} \int |\beta_{lk,l'k'}|^2\ \mathrm{d}k'$$

It is possible to understand this creation process in terms of the instability of the magnon vacuum due the presence of the skyrmion: $\Gamma = \frac{2}{\hbar}\ \Im(\mathcal{S}_{\text{eff}}) \neq 0$ (It should be interesting to find a non perturbative method to evaluate this quantity). Qualitatively this phenomena is analogous to the Schwinger effect [16] of the rate of creation of pairs of electron-positron under a electric field. In this case, it is manifested by the creation pairs of magnons of momentum $\mathbf{k}$ and $-\mathbf{k}$. The analogous electric field yields from the time dependent dynamics of the skyrmion (emergent electrodynamics in the chapter 1), and the Dirac field is replaced by the magnon field, where the more strong the electric field the more particle production. Since the collapse is a non adiabatic process, this implies the analogous electric field is increase at the rate $E \sim \frac{\dot{\lambda}}{\lambda}$, thus during the collapse, the rate of production increases considerably, in form of radiation of magnons, accelerating the process. Roughly speaking, it is similar to the collapse of black holes due to the Hawking radiation, pointed out in [6], and a similar approach in [3].

3.3.1. Integration over the zero modes and Radiative corrections

So far we have treated the fields U and ψ as independent fields, on the other hand, we note that there exists an ambiguity in the split $u = U[\xi] + \psi$ for the separation among this freedom degrees. In effect, it is due to the presence of Goldstone modes: $Z \in \text{Ker}(\mathcal{L}_U)$, thus we can write $u = (U[\xi] - \delta\xi^{\text{i}}\ \partial_{\xi^{\text{i}}} U) + (\psi - \delta\xi^{\text{i}}\ \partial_{\xi^{\text{i}}} U) = U[\xi + \delta\xi] + \tilde{\psi}$, and so one obtains a new split for u. Therefore, it would be necessary to introduce a *gauge* choosing for ψ, in order to not over counting u in the path integral. Consider the following action of the Lie algebra of the conformal group: $\phi \mapsto g(\phi) = \phi + \partial_{q_{\text{i}}} U \delta q$ and $U \mapsto g(U) = U - \partial_{q_{\text{i}}} U \delta q$ where $\delta q = (\delta\lambda, \delta\xi, \delta\omega)$. Let (G) be a fixing gauge functional, it is, such that $\mathcal{G}[g(\phi)] = 0$. Following the Faddev-Popov method, we define the functional $\Delta_{\mathcal{G}}[\phi]$ by means of the identity:

$$1 = \Delta_{\mathcal{G}}[\phi] \int \delta(\mathcal{G}[g(\phi)])\ \mathrm{d}g$$

where $\mathrm{d}g$ is the Haar measure of the conformal group. We can see from the definition that $\Delta_{\mathcal{G}}[g(\phi)] = \Delta_{\mathcal{G}}[\phi]$ for all $g \in G$. Let us denote $\mathcal{D}^2\psi = \mathcal{D}\psi\mathcal{D}^\dagger\psi$, then we have:

$$
\begin{aligned}
\exp(\mathrm{i}\mathcal{S}_{\text{eff}}) &= \int \exp(\mathrm{i}\mathcal{S}(\psi, U))\ \mathcal{D}^2\psi\mathcal{D}U \\
&= \int \left\{ \Delta_{\mathcal{G}}[\phi] \int \delta(\mathcal{G}[g(\phi)])\ \mathrm{d}g \right\} \exp(\mathrm{i}\mathcal{S}(\psi, U))\ \mathcal{D}^2\psi\mathcal{D}U \\
&= \int \int \Delta_{\mathcal{G}}[g(\phi')]\delta(\mathcal{G}[g(\phi')])\ \exp(\mathrm{i}\mathcal{S}(\psi', U'))\ \mathrm{d}g\ \mathcal{D}^2\psi'\mathcal{D}U' \\
&= \left[\int \mathrm{d}g \right] \int \Delta_{\mathcal{G}}[\phi]\delta(\mathcal{G}[\phi])\ \exp(\mathrm{i}\mathcal{S}(\psi, U))\ \mathcal{D}^2\psi\mathcal{D}U
\end{aligned}
$$

Hence, we obtain the correction to the effective action $\mathcal{S}_{\text{eff}}$, neglecting the term $\int \mathrm{d}g$ in the above equation. One natural gauge fixing is to take $\mathcal{G}^{\mathrm{i}}[\psi, U] = \langle \psi, \partial_{q_{\mathrm{i}}} U \rangle_{L^2} = 0$. Therefore denoting $c_{\mathrm{i}} = \|\partial_{q_{\mathrm{i}}} U\|_{L^2}$, we have $\Delta_{\mathcal{G}}[\phi] = \prod_{\mathrm{i}} c_{\mathrm{i}}$. On the othe hand, the delta function $\delta(\mathcal{G}[\psi])$ means that we must only integrate over the orthogonal space to the kernel $\psi \in Ker(\mathcal{L})^{\perp}$.

Now we can to evaluate perturbatively the effective action using the change $\phi(y, t) = \psi(x, t)$ with $y = (x - \xi)/\lambda$, seen previously, and we define the differential operator Σ_{i} by means of $J_{q_{\mathrm{i}}}[\phi] = \phi^\dagger \Sigma^{\mathrm{i}} \phi$. Thus:

$$
\begin{aligned}
\mathcal{S}_{\text{eff}}(U) &= \log \left[\int_{Ker(\mathcal{L}_0)^{\perp}} \exp \left\{ \int \phi^\dagger (\mathrm{i}\lambda^2 \partial_t - \mathcal{L}_0 + \dot{q}_{\mathrm{i}} \Sigma^{\mathrm{i}}) \phi\ \mathrm{d}^2 y\ \mathrm{d}t \right\} \mathcal{D}^2\phi \right] \\
&= \log \left[\det{}' \left(\mathrm{i}\lambda^2 \partial_t - \mathcal{L}_0 + \dot{q}_{\mathrm{i}} \Sigma^{\mathrm{i}} \right) \right] \\
&= \log[\det{}'(-\mathcal{L}_0)] - \operatorname{tr} \left(\mathrm{i}\lambda^2 \mathcal{L}_0^{-1} \partial_t + \dot{q}_{\mathrm{i}} \mathcal{L}_0^{-1} \Sigma^{\mathrm{i}} \right) + \frac{1}{2} \operatorname{tr} \left(\mathrm{i}\lambda^2 \mathcal{L}_0^{-1} \partial_t + \dot{q}_{\mathrm{i}} \mathcal{L}_0^{-1} \Sigma^{\mathrm{i}} \right)^2 + \dots
\end{aligned}
$$

where the $\det'(T) = \prod_{\mu \in \sigma(T) \setminus 0} \mu$. In particular, looking at the coefficient of $\dot{\xi}_{\mathrm{i}}^2$ we can see the radiative correction to the skyrmion mass:

$$
M_{\text{eff}} = \operatorname{tr} \left(\mathcal{L}_0^{-1} \Sigma^{\xi} \right)^2
$$

Remark: The expansion of $\mathcal{S}_{\text{eff}}$ above can be written diagrammatically using Feynman's diagrams, where successive terms give corrections to the effective theory, as well as, the scattering's section of magnon radiation.

Capítulo 4

Dynamics of a skyrmion near the blow up

In this section we will study the dynamics of the skyrmion near the annihilation. Specifically, we will study the asymptotic law of the evolution of the parameters: λ, ω and ξ. We will review the mean results in literature about the problem, and different approaches to study it, between them, the Dávila-Del Pino-Wei Approach [17]. In it scheme we will show that it is possible to construct solutions of the LLG equations which blows up exactly in an arbitrary set $(x_{\mathrm{i}}, t_{\mathrm{i}}) \in \Omega \times \mathbb{R}_+$ of events in space-time (fixed). That result could have applications, for instances, to logic circuits in nanotechnology, where skyrmion works like binary bits [34], and the annihilation of a skyrmion represents the change in a bit.

4.1. Dynamics and orthogonality condition

In this section we will explain the the origin of the orthogonality condition from a way to keep the stability of the skyrmion bubble. Additionally, we will use this condition to derive the dynamics of the parameters $\lambda(t)$ and $\omega(t)$ during the blow up process.

Consider ψ^1 the spin waves generated by the skyrmion by the formula (2.4) and let ϕ^0 be a free spin wave which satisfies the homogeneous equation:

$$\partial_t \phi^0 = \gamma \, \Delta \phi^0$$

The general solution solution of the spin waves far away the skyrmion will be the superposition of them: $\phi = \phi^0 + \phi^1$. Suppose that we set the orthogonality condition:

$$\int_{B(R)} \langle \phi(x,t), Z(x,t) \rangle \; \mathrm{d}^2 x = 0 \quad \forall \, t > 0 \tag{4.1}$$

where $Z \in Ker(\mathcal{L}_U)$ and R a cutoff such that $\lambda(t) \gtrsim R(t)$ and $R(t) \to 0$. We can determinate the dynamics of the skyrmion just as was mentioned in chapter 1. Indeed, evaluating (4.1) with $Z = \partial_\lambda U(t)$, we have:

$$
\begin{aligned}
0 &= \int_{B(R)} \left\langle \phi^1(x,t) + \phi^0(x,t), \partial_\lambda U(x,t) \right\rangle \mathrm{d}^2 x \\
&= -\int_0^t \int_0^R \Re\left[\dot{p}(s)\left(1 - \mathrm{e}^{-\gamma r^2/4(t-s)}\right)\right] \frac{r\cos\Theta(r/\lambda)}{r^2+\lambda^2}\ \mathrm{d}r\ \mathrm{d}s + \int_0^R \phi^0(\vec{r},t)\frac{r^2\cos\Theta(r/\lambda)}{r^2+\lambda^2}\ \mathrm{d}r \\
&= -\int_0^t \Re\left[\dot{p}(s) w(t-s)\right]\ \mathrm{d}s + R^2\ \mathrm{div}\ \phi^0(\xi,t) + o(R^2)
\end{aligned}
$$

where

$$
w(t) = \int_0^R \left(1 - \mathrm{e}^{-\gamma r^2/4t}\right) \frac{r\cos\Theta(r/\lambda)}{r^2+\lambda^2}\ \mathrm{d}r \approx \begin{cases} \gamma\ \frac{R^2}{t} & \text{if } t \gg \lambda^2 \\ o(R^2) & \text{if } \lambda^2 \ll t \end{cases}
$$

Analogously, evaluating (4.1) with $Z = \partial_\omega U(t)$, we have:

$$
0 = -\int_0^t \Im\left[\dot{p}(s) w(t-s)\right]\ \mathrm{d}s + R^2\ \mathrm{curl}\ \phi^0(\xi,t) + o(R^2)
$$

Therefore comparing the terms of high order $O(R^2)$, we get the following approximated integro-diferential equation for $p(t)$:

$$
\int_0^{t-\lambda^2} \frac{\dot{p}(s)}{t-s}\ \mathrm{d}s \approx \gamma^*\left(\mathrm{div}\ \phi^0(\xi(t)) + \mathrm{i}\ \mathrm{curl}\ \phi^0(\xi(t))\right)
$$

Define the complex function:

$$
Q(t) = \gamma^*\left(\mathrm{div}\ \phi^0(\xi(t)) + \mathrm{i}\ \mathrm{curl}\ \phi^0(\xi(t))\right)
$$

We can find a solution by setting the ansatz $\omega(t) = \omega_0$:

$$
\int_0^{t-\lambda^2} \frac{\dot{\lambda}(s)}{t-s}\ \mathrm{d}s \approx \Re(Q(T))
$$

Under the hypothesis that $\Re(Q(T)) < 0$, it is possible to solve this integral equation approximately [17] for $t \nearrow T$, giving the asymptotic behaviour:

$$
\lambda(t) \approx k_0\ \frac{T-t}{\log(T-t)^2} \tag{4.2}
$$

where κ_0 is a constant that depends on the u_0.

In the next section, we will derive again (4.2) by a different method.

4.2. Inner-Outer Matching Method

In this section we will study the behavior of the solution of LLG equation near the blow up point (x_0, T), from a formal asymptotic method, following the method used in [9]. It will be convenient to do a variable change which scales with the size of the skyrmion λ. One natural choose is define $y = \frac{x-\xi}{\lambda}$, $\rho = |y|$, and the modulated field $u(x,t) = R_\omega \tilde{u}(y,\tau)$, where we have introduced the new variable $\tau = \int_0^t \frac{\mathrm{d}t'}{\lambda^2(t')}$. Thus the evolution equation turns out:

$$\partial_\tau \tilde{u}(y,\tau) - \left(\frac{\lambda_\tau y + \xi_\tau}{\lambda}\right) \nabla_y \tilde{u}(y,\tau) + \omega_\tau \hat{J}\tilde{u}(y,\tau) = J_{\tilde{u}} \left(\Delta_y \tilde{u}(y,\tau) + |\nabla_y \tilde{u}(y,\tau)|^2 \tilde{u}(y,\tau)\right)$$

where subscripts denote derivation respect to τ. In order to describe fluctuations around the skyrmion, we will look for small perturbations of the steady solution $U_0(y)$ in the form: $\tilde{u}(y,\tau) = U_0(y) + \phi(y,\tau)$, with $U_0(y) \cdot \phi(y,\tau) = 0$ for all (y,τ). Therefore considering only linear order, the equation for ϕ is

$$\partial_\tau \phi = \mathcal{H}\phi + \frac{\lambda_\tau}{\lambda} y \nabla_y \phi - \omega_\tau \hat{J}\phi + \frac{\xi_\tau}{\lambda} \nabla_y \phi + \left[\frac{\lambda_\tau}{\lambda} y \nabla_y U_0 - \omega_\tau \hat{J} U_0 + \frac{\xi_\tau}{\lambda} \nabla_y U_0\right] \tag{4.3}$$

Since we are interested in the regimen $\lambda \to 0$ where $\lambda \dot{\xi}_t \approx 0$, we will omit the last term. Using the complex form $U_0 = \begin{bmatrix} \sin(\Theta)\mathrm{e}^{\mathrm{i}\theta} \\ \cos(\Theta) \end{bmatrix}$ where $\Theta(y) = 2\tan(\frac{1}{\rho})$, the inhomogeneous part of (4.3) is

$$\frac{\lambda_\tau}{\lambda} y \nabla_y U_0 - \omega_\tau \hat{J} U_0 + \frac{\xi_\tau}{\lambda} \nabla_y U_0 = -y\Theta'(y) \left(\frac{\lambda_\tau}{\lambda} E_1 + \omega_\tau E_2\right)$$

In addition, we have $J_z E_1 = \cos(\Theta) E_2$ and $J_z E_2 = -\cos(\Theta) E_1$, so $J_z \phi = \cos(\Theta) J_{U_0} \phi$. Henceforth we use the complex representation $\phi_1 E_1 + \phi_2 E_2 \mapsto \phi_1 + \mathrm{i}\phi_2$

$$\partial_\tau \phi = \mathcal{H}\phi + \frac{\lambda_\tau}{\lambda} y \nabla_y \phi - \mathrm{i}\omega_\tau \cos(\Theta)\ \phi - y\Theta'(y) \left(\frac{\lambda_\tau}{\lambda} + \mathrm{i}\omega_\tau\right) \tag{4.4}$$

We would like to do perturbation theory in the time dependent variables $\frac{\lambda_\tau}{\lambda} = \lambda\lambda_t \to 0, \omega_\tau = \lambda^2 \omega_t \to 0, \frac{\xi_\tau}{\lambda} = \lambda \xi_t \to 0$ as $t \to T$. This motivates to try the ansatz for the radiative corrections:

$$\phi(y,\tau) = \sum_{p,q,r,s} (\omega_\tau)^q \left(\frac{\lambda_\tau}{\lambda}\right)^p \left(\frac{\xi^1_\tau}{\lambda}\right)^r \left(\frac{\xi^2_\tau}{\lambda}\right)^s \ w_{p,q,r,s}(y) \tag{4.5}$$

At the first order: $\phi(y,\tau) \approx (\lambda_\tau/\lambda) w_{1,0}(y) + \omega_\tau w_{0,1}(y)$, then replacing this ansatz in the linearized equation (4.3), $w_{1,0}$ and $w_{0,1}$ must satisfy:

$$\gamma \mathcal{L}(\frac{\lambda_\tau}{\lambda} w_{1,0}(y) + \omega_\tau w_{0,1}) = \mathcal{H}\phi(y,\tau) = y\Theta'(y) \left(\frac{\lambda_\tau}{\lambda} + \mathrm{i}\omega_\tau\right) \tag{4.6}$$

Thus $\mathcal{L}_0 w_{1,0} = \gamma^* y\Theta'(y) = \gamma^* Z_0(y)$ and $\mathcal{L} w_{0,1} = \mathrm{i}\gamma^* y\Theta'(y) = \mathrm{i}\gamma^* Z_0(y)$

From these equations, we should expect $w_{1,0}$ and $w_{0,1}$ not be bounded (on the contrary, it does not satisfy the Fredholm alternative theorem).

$$w_{1,0}(y) = \gamma^*\ \mathcal{L}_0^{-1}(y\Theta'(y)) = \gamma^* Z_0(y) \int_0^y \frac{1}{y_1 Z_0^2(y_1)} \left[\int_0^{y_1} y_2 Z_0(y_2)^2\ \mathrm{d}y_2\right] \mathrm{d}y_1$$

From this last one we can obtain the asymptotic formula for $y \to \infty$

$$w_{1,0}(y) = -\mathrm{i} w_{0,1}(y) \approx -\frac{1}{2}\gamma^*(y\log(y) - y)$$

Therefore

$$\phi(y) = -\frac{1}{2}\gamma^*\left(\frac{\lambda_\tau}{\lambda} + \mathrm{i}\omega_\tau\right) w_{1,0}(y) \approx -\frac{1}{2}\gamma^*\left(\frac{\lambda_\tau}{\lambda} + \mathrm{i}\omega_\tau\right)(y\log(y) - y) \tag{4.7}$$

On the other hand, as he have seen previously, in the self similar scale we have the expression:

$$\phi(r,t) \approx -\frac{1}{2}\gamma^* \dot{p}(t)\ r\log(r) + q(t)r + \ldots \tag{4.8}$$

where $\dot{q}(t) = \frac{1}{4}\gamma^*\ddot{\lambda}\log(\lambda)$. The matching method consists in compare both expansion: (4.7) as $y \to \infty$, and (4.8) as $r \to 0$, term by term as a function in r.

$$-\frac{1}{2}\gamma^*\dot{p}(t)(r\log(r) - \log(\lambda)r - r) + o(r) = \phi(r,t) = -\frac{1}{2}\gamma^*\dot{p}(t)\ r\log(r) + q(t)r + o(r)$$

Therefore

$$\frac{1}{2}\gamma^*\dot{p}(t)(\log(\lambda) + 1) = q(t) \quad \Rightarrow \quad \frac{1}{2}\gamma^*\left(\frac{\dot{\lambda}^2}{\lambda} + \ddot{\lambda}\log(\lambda) + \ddot{\lambda}\right) = \dot{q}(t) = \frac{1}{4}\gamma^*\ddot{\lambda}\log(\lambda)$$

Since $\left|\ddot{\lambda}\right| \ll \left|\ddot{\lambda}\log(\lambda)\right|$ when $\lambda \to 0$, we can neglecting the term $\ddot{\lambda}$ in the last equation. Thus we obtain:

$$0 \approx \frac{\dot{\lambda}^2}{\lambda} + \frac{1}{2}\ddot{\lambda}\log(\lambda) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\lambda}\log(\lambda)^2\right)$$

Then $\dot{\lambda}\log(\lambda)^2 \approx \kappa_0$ is constant, so finally we get the asymptotic law of $\lambda(t) = \kappa_0\frac{T-t}{\log^2(T-t)}$

4.3. Davila-Del Pino Approach

In this section we will consider the Landau-Lifshitz-Gilbert equation with boundary conditions:

$$\partial_t u(x,t) = J_{u(x,t)}\ \left(\Delta u(x,t) + |\nabla u(x,t)|^2\, u(x,t)\right) \quad \text{for all } x \in \Omega \tag{4.9}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega \tag{4.10}$$

The problem to find solutions of the Landau-Lifshitz-Gilbert's equation that blows up in a finite time is widely study using several approaches, in the literature. In order to introduce the problem and shed light on the possible approaches, we give the mean results from the literature.

The following theorem is a generalization of the J.Qing's result [29] for the the LLG equation. This result means that the solution forms a *bubble tree* [13] around the singularity.

Teorema 4.1 (Paul Harpes [13]) *Let* $u_0 : \Omega \times [0,T) \to S^2$ *be an distributional solution of (4.9), and suppose that* u *does not admit any extension to* $B_R(x_0) \times [T-R^2, T]$ *for all* $R > 0$. *Then there exist sequences:* $x_{\mathrm{i}}^j \in \Omega$ *with* $x_j^{\mathrm{i}} \to x_0$, $t_j \nearrow T$, $\lambda_{\mathrm{i}}^j \to 0$ *as* $j \to \infty$, *and* $U_{\mathrm{i}} : \mathbb{R}^2 \to S^2$, *harmonic maps, for every* $\mathrm{i} = 1, \ldots, k$, *such that:*

$$u(x, t_j) - \sum_{\mathrm{i}=1}^{k} U_{\mathrm{i}} \left(\frac{x - x_{\mathrm{i}}^j}{\lambda_{\mathrm{i}}^j} \right) - U_{\mathrm{i}}(\infty) \to u(x, T)$$

in $W^{1,2}(\Omega, \mathbb{R}^3)$. *In addition:*

$$\lim_{t \nearrow T} \mathcal{E}(u(t)) = \mathcal{E}(u(T)) + \sum_{\mathrm{i}=1}^{k} \mathcal{E}(U_{\mathrm{i}})$$

If the initial condition is equivariant (see (A.1) in the appendix for a definition), the next theorem proves the existence of a class of equivariant solutions of the Landau-Lifshitz-Gilbert equation, that blows up in a finite time (which depend on the initial condition).

Teorema 4.2 (Merle-Raphaël-Rodniaski [11]) *There exist a set of maps* $u_0 : \Omega \to S^2$ *in the class of equivariant maps, arbitrarily closed to* U_0 *in* $\dot{H}^1$, *such that the problem:*

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= u(x,t) \wedge \Delta u(x,t) \quad \textit{for all } x \in \mathbb{R}^2 \\ u(x,0) &= u_0 \end{aligned}$$

has a solution that blows up at a finite time $T > 0$. *Furthermore it is of the form:*

$$u(x,t) = R_{\omega_0} U_0 \left(\frac{x}{\lambda(t)} \right) + \varphi(x,t)$$

with $\varphi \to 0$ *in* $\dot{H}^1$ *as* $t \to T$. *In addition, the rate of concentration is given by:*

$$\lambda(t) = \kappa_0 \frac{T-t}{\log(T-t)^2} (1 + o(1))$$

for some constant $\kappa_0 > 0$.

Nevertheless, the previous result does not give us more than only one singular point. The following theorem allows us to construct a solution which blows up in an arbitrary set of points for the harmonic map flow (case $\beta = 0$ in LLG's equation).

Teorema 4.3 (Dávila-Del Pino-Wei [17]) *Let* $u_0 : \Omega \to S^2$ *be an initial condition and consider the harmonic map flow equation:*

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + |\nabla u(x,t)|^2 u(x,t) \quad \textit{for all } x \in \Omega \\ u(x,t) &= 0 \quad \textit{for } \ x \in \partial\Omega \end{aligned}$$

Let $(x_j, t_j) \in \Omega \times \mathbb{R}_+$, $k = 1, 2 \ldots k$ *be arbitrary points. Then there exist* $u_0(x)$ *such that* $u(x,t)$ *blows up exactly at the points* $(x_{\mathrm{i}}, t_{\mathrm{i}})$, $\mathrm{i} = 1, \ldots k$.

Our aim in this chapter is to generalize the theorem (4.2) following a similar construction that is used for the proof of the theorem (4.3) in [17]. We propose the following generalization:

Teorema 4.4 *Let $u_0 : \Omega \to S^2$ be an initial condition and consider the LLG equation:*

$$\frac{\partial u(x,t)}{\partial t} = J_u \left(\Delta u(x,t) + |\nabla u(x,t)|^2 u(x,t)\right) \quad \text{for all } x \in \Omega$$
$$u(x,t) = 0 \quad \text{for } \; x \in \partial\Omega$$

Let $(x_j, t_j) \in \Omega \times \mathbb{R}_+$, $k = 1, 2 \ldots k$ be arbitrary points. Then there exist $u_0(x)$ such that $u(x,t)$ blows up exactly at the points $(\mathrm{x_i}, \mathrm{t_i})$, $\mathrm{i} = 1, \ldots k$. In addition, the rate of concentration satisfies the asymptotic rule:

$$\lambda(t) = \kappa_0 \frac{T-t}{\log(T-t)^2}(1 + o(1))$$

for some constant $\kappa_0 > 0$.

Proof : Let's define the bubble of parameters $q = (\lambda, \xi, \omega) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ as follows:

$$U_q = R_\omega \; U_0(\frac{x-\xi}{\lambda}) \quad \text{where:} \quad U_0(r,\theta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} \cos(\Theta(\rho)) \\ \sin(\Theta(\rho)) \end{pmatrix}, \quad \Theta(\rho) = \pi - 2\tan^{-1}(\rho) \quad (4.11)$$

where $R_\omega = \exp(\omega \cdot J_z)$ is the rotation matrix acting in the sphere S^2, namely

$$J_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We call to U_q in (4.11) the bubble of radius λ, center ξ and phase ω; and we call to the set of variables (λ, ξ, ω) the collective variables. It is easy to see that for a fixed q, U_q a stationary solution of the LLG.

We are looking forward solutions that blow up in a point ξ_∞, so we would like to consider $u(x,t) = U_{q(t)}(x)$ where $q(t) \to (0, \xi_\infty, \omega_\infty)$ when $t \to t^* < \infty$. Since $U_{q(t)}$ is not solution of LLG necessarily, we consider instead $u(x,t) = U_{q(t)}(x) + \varphi(x,t)$, where $\varphi(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}^3$ is a "small" perturbation (we will see what this means later). Let $t \to q(t)$ be fixed (later we will assume hypothesis on q). In order to simplify the notation, we write $U(t) \equiv U_{q(t)}$. Furthermore we can see that, in order to u satisfy LLG, it suffices that:

$$-\partial_t u + \mathcal{S}(u) = \chi \; U \tag{4.12}$$

for some scalar function χ. Indeed, let's suppose (4.12) then using that $|u| = 1$ we obtain:

$$\chi(U \cdot u) = -\partial_t u \cdot u + \mathcal{S}(u) \cdot u = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \frac{1}{2}\Delta(|u|^2) = 0$$

if $|\varphi| < \frac{1}{2}$, we find that $\partial u = \mathcal{S}(u)$ is just LLG.

Now evaluating the $u = U + \varphi$ ansatz in $\mathcal{S}$ and expanding (4.12) we obtain:

$$\partial_t \varphi = J_U \mathcal{L}_U(\varphi) - \partial_t U + \; \mathcal{N}(\varphi) + \chi \; U \tag{4.13}$$

where $\mathcal{L}_U = \mathcal{D}_U\mathcal{S}$ is the functional derivative of $\mathcal{S}$ in U, $\mathcal{N}(\varphi)$ are quadratic terms in φ, and χ is a scalar field such that the other terms in (4.13) are perpendicular to U. At this time we can note that if $\varphi \cdot U = 0$ then $\mathcal{L}_U(\varphi) \cdot U = 0$. Hence, without loss of generality, we will assume $\varphi \cdot U = 0$. In effect, the terms $\mathcal{L}_U(a(\varphi)U)$ can be absorbed in $\mathcal{N}(\varphi)$ and $\chi\, U$.

The philosophy now is resolve the equation (4.13) in the new variable φ which has the condition $\varphi \cdot U = 0$ or in other words $\varphi(\cdot) \in T_{U(\cdot)}S^2$ is a section in the pullback fiber bundle $U^{-1}TS^2$, hence (4.13) is the linearization of LLG, so we would like to use a sort of Lyapunov-Schmidt reduction.

It is convenient to split $\varphi = \varphi_{\text{in}} + \varphi_{out}$ in two parts, one internal φ_{in}, and the other one φ_{out} external. The internal part attaches to the bubble in his dynamics, and the external part matches to the spin waves very far away from the bubble, so that it does not "feel" the bubble. Thus we have to patch both parts in a dynamical region, but we will see later this last issue. Under the above, we assume the form of the functions:

$$\varphi_{in} = R_{\omega(t)}\phi\left(\frac{x-\xi(t)}{\lambda(t)}\right), \qquad \varphi_{out} = \Pi_{U^\perp}(\Psi^0 + \psi) \tag{4.14}$$

where Ψ^0 is the approximated solution of the inhomogeneous equation:

$$\partial_t\Psi = J_{u(\infty)}\Delta\Psi - \partial_t U \tag{4.15}$$

φ_{in} corresponds to the internal perturbation, whereas φ_{in} corresponds to the perturbation of the faraway from the bubble.

Therefore we can uncouple the equation (4.13) inserting $\varphi = \varphi_{\text{in}} + \varphi_{out}$ into this one, and writing the equation for ϕ in the following way with the variable change $z = (x-\xi)/\lambda$:

$$\partial_t\phi = \frac{1}{\lambda^2}J_{U_0}\mathcal{L}_{U_0}(\phi) + R_\omega^{-1}\,\left(J_U\tilde{\mathcal{L}}_U(\psi) + J_U\tilde{\mathcal{L}}_U(\Psi^0)\right) \tag{4.16}$$

$$+\,\Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U + (J_U - J_{U(\infty)})\Delta\psi) \tag{4.17}$$

where we have used the facts:

$$R_\omega^{-1}\mathcal{L}_U R_\omega = \mathcal{L}_{U_0}, \quad R_\omega^{-1}J_U R_\omega = J_{U_0}, \quad [J_U, \Pi_{U^\perp}] = 0, \quad R_\omega^{-1}\Pi_U R_\omega = \Pi_{U_0}$$

and we have defined $\tilde{\mathcal{L}}_U(w) \equiv \mathcal{L}_U(\Pi_{U^\perp}w) - \Pi_{U^\perp}(\Delta w)$ for all $w : \mathbb{R}^2 \to \mathbb{R}^3$

In addition we have the equation for ψ:

$$\partial_t\psi = J_{U(\infty)}\Delta\psi + R_\omega\left[\frac{1}{\lambda}(\dot\xi + \dot\lambda z)\nabla_z\phi - \dot\omega\ J_z\phi - R_\omega^{-1}\partial_t(\Pi_{U^\perp})(\Psi^0+\psi)\right] + \mathcal{N}(\phi,\psi) \tag{4.18}$$

where the Laplacian operator on the components of ψ , and the rest terms include all the terms producing an error in the approximation of the equation for ϕ and Ψ^0. Let's return to

(4.35). At this time, we will suppose hypothesis on the terms appearing in the second line of (4.35). We will suppose that all of them goes to zero when $\lambda \to 0$. Thus when $\lambda \to 0$ we have approximately:

$$0 = J_{U_0}\mathcal{L}_{U_0}(\phi) + \mathcal{E}[\psi, \Psi^0] \tag{4.19}$$

where:

$$\mathcal{E}[\psi, \Psi^0] = \lambda^2 \, R_\omega^{-1} \left(J_U\tilde{\mathcal{L}}_U(\psi) + J_U\tilde{\mathcal{L}}_U(\Psi^0) + \Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U + (J_U - J_{U(\infty)})\Delta\psi \right)$$

hence if $Z \in \ker(\mathcal{L}_{U_0})$ is any element in the kernel of $\mathcal{L}_{U_0}$, then:

$$\int_{\mathbb{R}^2} J_{U_0}Z(x) \cdot \mathcal{E}[\psi, \Psi^0] = 0 \tag{4.20}$$

On the other hand, we can see that $\Psi^0 = \Psi^0[q(\cdot)]$ as a result of solving (4.15), consequently the equation (4.20) can also be written:

$$\int_{\mathbb{R}^2} Z(x) \cdot J^\dagger_{U_0}\mathcal{E}[\psi, \Psi^0] = 0 \tag{4.21}$$

Since we have that $\dim(\ker(\mathcal{L}_{U_0})) = 4$ namely: $Z_{\mathrm{i}} = \frac{\partial}{\partial q_{\mathrm{i}}} U_q$. Therefore (4.21) is a set of integro-differential equations for the dynamics of $q(\cdot)$.

Using the properties of commutative:

$$\begin{aligned} J^\dagger_{U_0}\mathcal{E}[q] &= \lambda^2 \, R_\omega^{-1} \, (\tilde{\mathcal{L}}_U(\psi) + \tilde{\mathcal{L}}_U(\Psi^0) + J^\dagger_U\Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U + (J_U - J_{U(\infty)})\Delta\psi) \\ &= \lambda^2 \, R_\omega^{-1} \, (\tilde{\mathcal{L}}_U(\psi) + \mathcal{K}[q(\cdot)]) \end{aligned}$$

where $\mathcal{K}[q(\cdot)] = \tilde{\mathcal{L}}_U(\Psi^0) + J^\dagger_U\Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U + (J_U - J_{U(\infty)})\Delta\psi)$ depends on $q(\cdot)$ through $\Psi^0[q]$. We can rewrite (4.21) in the form:

$$\int_{\mathbb{R}^2} Z(x) \cdot R_\omega^{-1}\mathcal{K}[q] = \int_{\mathbb{R}^2} Z(x) \cdot R_\omega^{-1}\mathcal{L}_U(\psi) \tag{4.22}$$

We can notice that the right side term in (4.22) does not depend on α and β so we can use the same results than harmonic maps flows for this one. In particular for $Z = \partial_\lambda U$:

$$\int_{\mathbb{R}^2} \partial_\lambda U \cdot R_\omega^{-1} \, \tilde{\mathcal{L}}_U(\psi) \approx -\mathrm{e}^{-\mathrm{i}\omega(t)} \operatorname{div} \psi + o(\lambda) \tag{4.23}$$

and $Z = \partial_\omega U$:

$$\int_{\mathbb{R}^2} \partial_\omega U \cdot R_\omega^{-1} \, \tilde{\mathcal{L}}_U(\psi) \approx -\mathrm{e}^{-\mathrm{i}\omega(t)} \operatorname{curl} \psi + o(\lambda) \tag{4.24}$$

On the the other hand, in order to compute $\mathcal{K}$ we seek a solution of (4.15) of the form:

$$\Psi(r, \theta, t) = \begin{pmatrix} f(r,t)\mathrm{e}^{\mathrm{i}\theta} \\ 0 \end{pmatrix} \tag{4.25}$$

Then using corollary 2.2 in [17] that:

$$R_\omega^{-1}\ \tilde{\mathcal{L}}_U(\Psi) = \frac{2\Theta'(\rho)}{\lambda}(\Re(\mathrm{e}^{\mathrm{i}\omega}\partial_r f)\ E_1 + \frac{1}{r}\Im(\mathrm{e}^{-\mathrm{i}\omega} f)\ E_2) \tag{4.26}$$

We have additionally:

$$\partial_t U = \partial_t U^0 + \partial_t U^1$$

$$R_\omega^{-1}\ \partial_t U^0 = \rho\ \Theta'(\rho)[\ \frac{\dot\lambda}{\lambda}\ E_1 + \dot\omega\ E_2] \tag{4.27}$$

$$R_\omega^{-1}\ \partial_t U^1 = \frac{\Theta'(\rho)}{\lambda}[\ (\dot\xi_1\cos(\theta) + \dot\xi_2\sin(\theta))\ E_1 + (\dot\xi_1\sin(\theta) - \dot\xi_2\cos(\theta))\ E_2] \tag{4.28}$$

The reason to split it in this way is decompose in Fourier series in the θ variable later. Now we try the (4.25) ansatz for Ψ^* to solve (4.15):

Far away from bubble we can approximate:

$$\partial_t U = \partial_t U^0 \approx \frac{2r}{r^2+\lambda^2}\begin{pmatrix}\dot p(t)\ \mathrm{e}^{\mathrm{i}\theta}\\ 0\end{pmatrix}$$

where $p(t) = \lambda(t)\mathrm{e}^{\mathrm{i}\omega(t)}$. Moreover

$$J_{U(\infty)} = \begin{pmatrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & 0\end{pmatrix}$$

Note that for $\zeta \in \mathbb{C}$ we have $J_U\begin{pmatrix}\zeta\\ 0\end{pmatrix} = \begin{pmatrix}\gamma\cdot\zeta\\ 0\end{pmatrix}$ where $\gamma = \alpha + \mathrm{i}\beta$. Thus we can write the approximated equation (4.15) using the ansatz (4.25):

$$\partial_t f = \gamma\ \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) - \frac{1}{r^2}f\right) - \frac{2\dot p(t)}{r} \tag{4.29}$$

A bounded approximated solution of (4.29) is:

$$f(r,t) = \int_0^t\ k(r,t-s))\ \dot p(s)\ \mathrm{d}s \qquad \text{with } k(r,t) = \frac{1-\mathrm{e}^{-\gamma z^2/(4t)}}{z^2}\ r \tag{4.30}$$

and $z = \sqrt{r^2+\lambda^2}$.

Therefore replacing (4.30) in (4.26) we have:

$$\begin{aligned}R_\omega^{-1}\ \tilde{\mathcal{L}}_U(\Psi^0) = &\frac{2\Theta'(\rho)^2}{\lambda}\left[\int_0^t\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\partial_r k(r,t-s)\dot p(s))\ \mathrm{d}s\right]\ E_1\\ &+ \frac{2\Theta'(\rho)^2}{\lambda}\frac{1}{r}\left[\int_0^t\ \Im(\mathrm{e}^{-\mathrm{i}\omega}k(r,t-s)\dot p(s))\ \mathrm{d}s\right]\ E_2\end{aligned}$$

accordingly substituting this in (4.22) with $Z = z_1 E_1 + z_2 E_2 \in \ker(\mathcal{L}_{U_0})$:

$$\begin{aligned}
&\int_{\mathbb{R}^2} Z \cdot R_\omega^{-1}\ \tilde{\mathcal{L}}_U(\Psi^0) \\
&\qquad = \int_0^t \left\{ \int_{\mathbb{R}^2} \frac{2\Theta'(\rho)^2}{\lambda}\ [z_1\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\partial_r k(r,t-s)\dot{p}(s)) + \frac{1}{r} z_2\ \Im(\mathrm{e}^{-\mathrm{i}\omega} k(r,t-s)\dot{p}(s))\ ] \right\}\ \mathrm{d}s \\
&\qquad = \int_0^t \ \{\Re(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \Upsilon_1(\lambda(s),t-s)) + \Im(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \Upsilon_2(\lambda(s),t-s))\ ]\ \}\ \mathrm{d}s
\end{aligned}$$

where:

$$\Upsilon_1(\lambda,t-s) = \int_0^\infty \frac{2r\ \Theta'(\rho)^2}{\lambda}\, z_1\, \partial_r k(r,t-s)\, r\, \mathrm{d}r\ , \quad \Upsilon_2(\lambda,t-s) = \int_0^\infty \frac{2r\ \Theta'(\rho)^2}{\lambda}\frac{1}{r}\, z_2\, k(r,t-s)\, r\, \mathrm{d}r$$

On the other hand:

$$\int_{\mathbb{R}^2} Z\cdot R_\omega^{-1} J_U^\dagger \Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U)\ = \int_{\mathbb{R}^2} -Z\cdot J_{U_0}^\dagger R_\omega^{-1}\partial_t U + Z\cdot J_{U_0}^\dagger R_\omega^{-1}\Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0)$$

$$\begin{aligned}
\int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger R_\omega^{-1}\partial_t U &= \int_{\mathbb{R}^2} \rho\ \Theta'(\rho)\begin{pmatrix} z_1 & z_2\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\dot\lambda/\lambda\ +\ \dot\xi_1\cos(\theta)+\dot\xi_2\sin(\theta)\\ \dot\omega\ +\ \dot\xi_1\sin(\theta)-\dot\xi_2\cos(\theta)\end{pmatrix}\ r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \begin{cases} \Re(\gamma\ \mathrm{e}^{-\mathrm{i}\omega}\dot{p}) & \text{if } Z = \partial_\lambda U\\ \Im(\gamma\ \mathrm{e}^{-\mathrm{i}\omega}\dot{p}) & \text{if } Z = \partial_\omega U\\ \Re(\gamma^*\ \dot\xi) & \text{if } Z = \partial_{\xi_1} U\\ \Im(\gamma^*\ \dot\xi) & \text{if } Z = \partial_{\xi_2} U\end{cases}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger R_\omega^{-1}\Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0) &= -\int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger R_\omega^{-1}\Pi_{U^\perp}\ \partial_t\Psi^0\ +\ \int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger R_\omega^{-1}\Pi_{U^\perp} J_U\Delta\Psi^0 \\
&= -\int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger R_\omega^{-1}\Pi_{U^\perp}\ \partial_t\Psi^0\ +\ \int_{\mathbb{R}^2} Z\cdot R_\omega^{-1}\Pi_{U^\perp}\Delta\Psi^0 \\
&= -\int_{\mathbb{R}^2} Z\cdot J_{U_0}^\dagger \Pi_{U_0^\perp} R_\omega^{-1}\ \partial_t\Psi^0\ +\ \int_{\mathbb{R}^2} Z\cdot \Pi_{U_0^\perp} R_\omega^{-1}\Delta\Psi^0 \\
&= -\int_{\mathbb{R}^2}\begin{pmatrix} z_1 & z_2\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\cos(\Theta)\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\ \partial_t f)\\ \Im(\mathrm{e}^{-\mathrm{i}\omega}\partial_t f)\end{pmatrix} \\
&\qquad + \int_{\mathbb{R}^2}\begin{pmatrix} z_1 & z_2\end{pmatrix}\begin{pmatrix}\cos(\Theta)\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\ \Delta_r f)\\ \Im(\mathrm{e}^{-\mathrm{i}\omega}\Delta_r f)\end{pmatrix} \\
&= \int_0^t \{\ -\int_{\mathbb{R}^2}\begin{pmatrix} z_1 & z_2\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\cos(\Theta)\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \partial_t k)\\ \Im(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \partial_t k)\end{pmatrix}\ \mathrm{d}s \\
&\qquad + \int_{\mathbb{R}^2}\begin{pmatrix} z_1 & z_2\end{pmatrix}\begin{pmatrix}\cos(\Theta)\ \Re(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \Delta_r k)\\ \Im(\mathrm{e}^{-\mathrm{i}\omega}\dot{p}(s)\ \Delta_r k)\end{pmatrix}\ \}\ \mathrm{d}s
\end{aligned}$$

We can see that this does not vanish only for $Z = \partial_\lambda U$ and $Z = \partial_\omega U$.

Now we evaluate (4.22) with $Z = \partial_\lambda U$ then we obtain:

$$0 = \text{div } \psi(x_0, T) + \Re(\gamma e^{-i\omega(t)} \dot{p}(t)) + \int_0^t \Re[e^{-i\omega} \dot{p}(s) \Gamma_{1,1}(t-s, \lambda(s))] + \Im[e^{-i\omega} \dot{p}(s) \Gamma_{1,2}(t-s, \lambda(s))] \, ds \tag{4.31}$$

where:

$$\Gamma_{1,1}(t, \lambda) = \int_{\mathbb{R}^2} [\cos(\Theta)(-\alpha \partial_t k + \Delta_r k) + \frac{2r\ \Theta'(\rho)^2}{\lambda}\ \partial_r k]\ r^2 \Theta'\ dr\ , \quad \Gamma_{1,2}(t, \lambda) = \int_{\mathbb{R}^2} \beta\ \partial_t k\ r^2 \Theta'\ dr$$

Similarly, if we evaluate now (4.22) with $Z = \partial_\omega U$, we obtain:

$$0 = \text{curl } \psi(x_0, T) + \Im(\gamma e^{-i\omega(t)} \dot{p}(t)) + \int_0^t \Re[e^{-i\omega} \dot{p}(s) \Gamma_{2,1}(t-s, \lambda(s))] + \Im[e^{-i\omega} \dot{p}(s) \Gamma_{2,2}(t-s, \lambda(s))] \, ds \tag{4.32}$$

where:

$$\Gamma_{2,1}(t, \lambda) = \int_{\mathbb{R}^2} \beta \cos(\Theta) \partial_t k\ r^2 \Theta'\ dr\ ,\ \ \Gamma_{2,2}(t, \lambda) = \int_{\mathbb{R}^2} (-\alpha \partial_t k + \Delta_r k + \frac{2r\ \Theta'(\rho)^2}{\lambda} \frac{1}{r}\ k(r,t))\ r^2 \Theta'\ dr$$

Thus the dynamics of λ, ω consists in the coupled integro-differential equations (4.31) and (4.32). We see that we can join both equation in the form (4.31)+i (4.32) so we get:

$$\gamma\ \dot{p}(t) + \int_0^t \dot{p}(s)\ \mathcal{K}(t-s, \lambda(s))\ ds = (\text{div } \psi + i \text{ curl } \psi)(\xi(t), t) \tag{4.33}$$

The evaluation of $\mathcal{K}$ is similar to the case of the harmonic map flow in [17]. In particular we can obtain the approximated integral equation:

$$\int_0^{t-\lambda^2(t)} \frac{\dot{p}(s)}{t-s}\ ds = \gamma^* (\text{div } \psi + i \text{ curl } \psi)(\xi(t), t) \tag{4.34}$$

Therefore we deduce the same equation than the harmonic map flow case, hence the solution will be (approximately):

$$\lambda(t) \sim \frac{T-t}{\log^2(T-t)}$$

Now ϕ in general produce divergent integrals at the infinity, so it is necessary to introduce a smooth spacial cut off function $\eta \leq 1$ such that $\text{supp}\eta \subseteq B_2$, $\eta(x) = 1$ if $x \in B_1$, and a scale $R(t)$ such that

Final anzats the solution is given by:

$$u(x,t) = U + \eta\left(\frac{x-\xi}{R(t)\lambda(t)}\right) R_\omega \phi\left(\frac{x-\xi}{\lambda}\right) + \Pi_{U^\perp}(\Psi^0 + \psi) + a\left(R_\omega \phi(\frac{x-\xi}{\lambda}) + \Pi_{U^\perp}(\Psi^0 + \psi)\right) U$$

we write in the same way the new equation for ϕ (the inner problem):

$$\lambda^2 \partial_t \phi = J_{U_0} \mathcal{L}_{U_0}(\phi) + \lambda^2 h[\psi, \Psi^0] \tag{4.35}$$

where:

$$h[\psi, \Psi^0] = R_\omega^{-1} \ \Big(J_U \tilde{\mathcal{L}}_U(\psi) + J_U \tilde{\mathcal{L}}_U(\Psi^0) + \Pi_{U^\perp}(-\partial_t \Psi^0 + J_U \Delta \Psi^0 - \partial_t U + (J_U - J_{U(\infty)}) \Delta \psi \Big)$$

and the new equation for ψ (the outer problem):

$$\partial_t \psi = J_{U(\infty)} \Delta \psi + H[\phi, \psi, \Psi^0] \tag{4.36}$$

where

$$H[\phi, \psi, \Psi^0] = \eta R_\omega \left[\frac{1}{\lambda}(\dot{\xi} + \dot{\lambda} z) \nabla_z \phi - \dot{\omega} \ J_z \phi - R_\omega^{-1} \partial_t (\Pi_{U^\perp})(\Psi^0 + \psi) \right] + (1 - \eta) h[\psi, \Psi^0]$$
$$+ R_\omega J_U (\Delta \eta \phi + 2 \nabla \eta \cdot \nabla \phi - \partial_t \eta \phi) + \mathcal{N}(\phi, \psi)$$

4.4. Solving the inner problem

In this section we look for solve the problem

$$\lambda^2 \partial_t \phi = \gamma \ \mathcal{L}_0 \phi \ + h(\rho, t) \qquad \text{with} \quad \phi(y, \tau) = 0 \quad \forall (y, \tau) \in \mathcal{D}_{R(t)} \tag{4.37}$$

where $R(\tau) = \lambda(\tau)^{-\beta}$ whith $\beta \in (0, 1)$ In angular Fourier Modes :

$$\varphi(r, \theta, t) = \sum_{l \in \mathbb{Z}} \varphi_l(r, t) \ \mathrm{e}^{\mathrm{i} l \theta} \tag{4.38}$$

Doing the variable change: $\mathrm{d}\tau = \lambda(t)^{-2} \ \mathrm{d}t$; $\rho = r/\lambda(t)$ and $\varphi(r) = R_\omega \phi(\rho)$ allows write (4.37) turns into:

$$\partial_\tau \phi_l = \mathcal{H}_l^0 \phi_l \ + h[\phi, t] \tag{4.39}$$

where $\mathcal{H}_l^0 = \mathcal{H}_l[U = U_0]$ and $h[\phi, t]$ is an lower order operator (we will consider this one as an error term)

At this point is important to define the space where we are looking for solutions. Consider the norm:

$$\|h\|_{a,\nu} = \sup_{\mathbb{R}^2} \ (1 + |y|^a) \lambda_*(t)^{-\nu} h(y, \tau)$$

We will try the Heat Kernel Method. As the previous section, we will write hence on $\tau = t$

$$\phi(t) = \mathrm{e}^{\gamma t \mathcal{L}_l} \phi(0) + \int_0^t \mathrm{e}^{\gamma(t-s)\mathcal{L}_l} h(s) \ \mathrm{d}s$$

We henceforth suppose that $\phi \in \mathcal{C}^1(\mathbb{R}_+, L^2(B_R))$ by the semigroup property.

At this point, it will be useful solve the problem: $\mathcal{L}\phi = g$,

$$\phi(r) = Z_l(r) \int_r^R \frac{1}{Z_l^2(r_1)} \left[\int_0^{r_1} Z_l(r_2) g(r_2) \mathrm{d}r_2 \right] \mathrm{d}r_1$$

Let's suppose that we are looking for to solve the spectrum problem with Dirichlet boundary condition.

$$\mathcal{L}_l f(r) = \mu_l f(r) \quad \forall r \in \partial B_R, \qquad f(x) = 0 \quad \forall r \in \partial B_R$$

So using above formula we have:

$$f(r) = \mu_l Z_l(r) \int_r^R \frac{1}{Z_l^2(r_1)} \left[\int_0^{r_1} Z_l(r_2) f(r_2) \mathrm{d}r_2 \right] \mathrm{d}r_1$$

Now we take a normaliced f such that $\int_0^R f^2(r)\, \mathrm{d}r = 1$. Thus:

$$\begin{aligned}
|f(r)| &\leq \mu_l z_l(r) \int_r^R \frac{1}{Z_l^2(r_1)} \left| \int_0^{r_1} Z_l(r_2) f(r_2) \mathrm{d}r_2 \right| \mathrm{d}r_1 \\
&\leq \mu_l Z_l(r) \int_r^R \frac{1}{Z_l^2(r_1)} \left(\int_0^{r_1} Z_l^2(r_2) \mathrm{d}r_2 \right)^{1/2} \left(\int_0^{r_1} f^2(r_2) \mathrm{d}r_2 \right)^{1/2} \mathrm{d}r_1 \\
&\leq \mu_l Z_l(r) \int_r^R \frac{1}{Z_l^2(r_1)} \left(\int_0^{r_1} Z_l^2(r_2) \mathrm{d}r_2 \right)^{1/2} \mathrm{d}r_1 \\
&\leq \mu_l Z_l(r) \left(\int_0^R \frac{1}{Z_l^2(r_1)} \mathrm{d}r_1 \right) \left(\int_0^{r_1} Z_l^2(r_2) \mathrm{d}r_2 \right)^{1/2}
\end{aligned}$$

We can evaluate directly $Z_l(r) = \frac{r^{1-l}}{1+r^2}$ so for the case $l \neq 0, 1$ we have

$$|f(r)| \lesssim \mu_l R^{l+1} z_l(r) \implies \mu_l \gtrsim \frac{1}{R^2}$$

on the other hand, if $l = 0$ we get:

$$\mu_0 \gtrsim \frac{1}{R^2 \log R}$$

elseif $l = 1$ we get:

$$\mu_1 \gtrsim \frac{1}{R^2 \log R}$$

Since $\mathcal{L}_l$ is hermitian elliptic operator, we have by the variational principle:

$$0 < \mu_l = \text{mín} \frac{\langle \phi, \mathcal{L}_l \phi \rangle}{\langle \phi, \phi \rangle} = \text{mín} \frac{\langle \phi, A_l \phi \rangle}{\langle \phi, \phi \rangle}$$

Taking adjoint to the equation (4.37) we obtain:

$$\partial_\tau \bar{\phi}_l = \gamma^* \mathcal{L}_l \bar{\phi}_l + \bar{h}_l(\rho, t) \tag{4.40}$$

Thus combinig (4.37) and (4.40)

$$\frac{1}{2}\partial_\tau(|\phi_l|^2) = \Re(\gamma\ \bar{\phi}\mathcal{L}\phi) + \Re(\gamma\ \bar{\phi}h_l)$$

On the other hand by definition

$$\int_{B_R} \bar{\phi}\mathcal{L}_l\phi\ \mathrm{d}y \geq -\mu_l \int_{B_R} |\phi|^2\ \mathrm{d}y$$

Therefore integrating we get the inequality

$$\begin{aligned}\frac{1}{2}\partial_\tau(\|\phi_l(t)\|_2^2) &\leq -\alpha\ \mu_l\ \|\phi_l(t)\|_2^2 + \Re(\gamma\ \bar{\phi}(t)h_l(t)) \\ &\leq -\alpha\ \mu_l\ \|\phi_l(t)\|^2 + \|h_l(t)\|_2\ \|\phi_l(t)\|_2\end{aligned}$$

where we notice that the boundary condition of ϕ ensures that the integral on the boundary vanishes. Hence we have:

$$\partial_\tau(\|\phi_l\|_2^2) \leq -\alpha\ \mu_l(t)\ \|\phi_l(t)\| + \|h_l(t)\|_2\ \|\phi_l(t)\|_2$$

Thus ussing the Gronwall inequality and ussing $\|\phi_l(0)\|_2 = 0$:

$$\begin{aligned}\|\phi_l(t)\|_2 &\leq \int_0^t \mathrm{e}^{-(t-s)\alpha\mu}\ \|h_l(s)\|_2\ \mathrm{d}s \\ &\leq \frac{1}{\alpha\mu_l} \sup_{s\in[0,t]} \|h_l(s)\|_2 \\ &\leq \frac{1}{\alpha}R^2 \sup_{s\in[0,t]} \|h_l(s)\|_2\end{aligned}$$

Let's denote the heat kernel of $-\gamma\partial_t + \mathcal{L}$ on $L^2_\Omega(\mathbb{C})$ by $K_\gamma(x, y, t; \Omega)$, which is solution of the boundary problem: for $y \in \Omega$ a fixed point

$$\begin{aligned}(\gamma\partial_t + \mathcal{L}_x)K_\gamma(x, y, t; \Omega) &= \delta(x - y, t) \\ K_\gamma(x, y, t; \Omega) &= 0 \quad \forall x \notin \Omega\end{aligned}$$

where the subscript in $\mathcal{L}_x$ denotes derivative respect to x. Since $\mathcal{L}$ is elliptic operator then

$$K_\gamma(x, y, t; \Omega) = \sum_{\mu_k\in\sigma(\mathcal{L})} \mathrm{e}^{-\gamma\mu_k t}\phi_k(x)\phi_k(y) \tag{4.41}$$

This serie converge uniformily. Then we constuct the following solution solution:

$$\phi(x, t) = \int_0^t \left[\int_{\mathcal{D}_s} K_\gamma(x, y, t - s; \mathcal{D}_s)h(y, s)\ \mathrm{d}y\right]\ \mathrm{d}s \tag{4.42}$$

where here $\mathcal{D}_s = \mathcal{D} \cap \{t\} \times \mathbb{R}^2$. On the other hand

$$\begin{aligned}K_\gamma(x, y, t; \Omega) &= \sum_{\mu_k\in\sigma(\mathcal{L})} \mathrm{e}^{-\gamma\mu_k t}\phi_k(x)\phi_k(y) \\ &= \sum_{\mu_k\in\sigma(\mathcal{L})} (\cos(\beta t) + \mathrm{i}\sin(\beta t))\mathrm{e}^{-\alpha\mu_k t}\phi_k(x)\phi_k(y)\end{aligned}$$

In conclusion, we have

$$|\Re(K_\gamma(x,y,t;\Omega))| \leq |K_\alpha(x,y,t;\Omega)| \quad \text{and} \quad |\Im(K_\gamma(x,y,t;\Omega))| \leq |K_\alpha(x,y,t;\Omega)|$$

$$\Re(\phi(x,t)) \leq \int_0^t \left[\int_{\mathcal{D}_s} K_\alpha(x,y,t-s;\mathcal{D}_s)h(y,s)\ \mathrm{d}y\right]\ \mathrm{d}s \equiv \phi^0(x,t)$$

where ϕ^0 is solution of the parabolic problem

$$\partial_\tau \phi^0 = \alpha\mathcal{L}\phi^0\ + h(\rho,t) \qquad \text{with} \quad \phi^0(y,\tau)=0 \quad \forall(y,\tau)\in\mathcal{D}_{R(t)}$$

We use the estimation for $1<a<2$, $\nu>0$ for h such that $\|h\|_{a,\nu}<\infty$ there exist a solution ϕ of the equation $\partial_\tau\phi = \mathcal{L}_U\phi + h$, such that:

$$\begin{aligned}(1+|y|)\left|\nabla\phi^0(y,\tau)\right| + \left|\phi^0(y,\tau)\right| \lesssim \tau^{-\nu}R^{2-a}\left\|h^\perp\right\|_{a,\nu} &+ \frac{\tau^{-\nu}R^{2-a+\sigma}}{1+|y|^\sigma}\left\|h_1-\bar{h}_1\right\|_{a,\nu} \\ &+ \frac{\tau^{-\nu}R^4}{1+|y|^2}\left\|\bar{h}_1\right\|_{a,\nu} + \frac{\tau^{-\nu}R^{\frac{5-a}{2}}}{1+|y|}\operatorname{mín}\left\{1, R^{\frac{5-a}{2}}|y|^{-2}\right\}\left\|h_0-\bar{h}_0\right\|_{a,\nu} \\ &+ \frac{\tau^{-\nu}R^2}{1+|y|}\left\|\bar{h}_0\right\|_{a,\nu}\end{aligned}$$

where $\sigma\in(0,1)$

This motivates to introduce the norm:

$$\|\phi\|_{X(a,\nu)} = \sup_{(y,\tau)\in\mathcal{D}_{2R}} \lambda_0^{-\nu}\frac{R^{\frac{5-a}{2}}}{1+|y|}\operatorname{mín}\left\{1, R^{\frac{5-a}{2}}|y|^{-2}\right\}\left[(1+|y|)\left|\nabla\phi^0(y,\tau)\right| + \left|\phi^0(y,\tau)\right|\right]$$

4.5. Solving the outer problem

Considering $\psi:\mathbb{R}^{2+1}\to\mathbb{R}^3$, we set the outer problem as:

$$\partial_t\psi - J_{U(\infty)}\Delta\psi = H[\phi,\psi] \tag{4.43}$$
$$\psi = 0 \ \text{ on } \partial\Omega\times(0,T) \tag{4.44}$$
$$\psi(x_0,T) = 0 \tag{4.45}$$
$$\psi(x_0,0) = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \ \text{ in } \Omega \tag{4.46}$$

where $H[\phi,\psi]$ is given in the equation (4.36):

$$\begin{aligned}H[\phi,\psi,\Psi^0] = \eta R_\omega\left[\frac{1}{\lambda}(\dot{\xi}+\dot{\lambda}z)\nabla_z\phi - \dot{\omega}\ J_z\phi - R_\omega^{-1}\partial_t(\Pi_{U^\perp})(\Psi^0+\psi)\right] + (1-\eta)h[\psi,\Psi^0] \\ +R_\omega J_U(\Delta\eta\phi + 2\nabla\eta\cdot\nabla\phi - \partial_t\eta\phi) + \mathcal{N}(\phi,\psi)\end{aligned}$$

and the boundary condition is determined by $U+\psi+\Psi^0 = \mathbf{e}_3$ on $\partial\Omega$.

Consider the model Problem:

$$\partial_t \psi = J(U)\Delta\psi + f \quad \text{in } \Omega \times (0,T) \tag{4.47}$$
$$\psi(x,0) = 0 \ x \in \omega \tag{4.48}$$
$$\psi(x,0) = 0 \ x \in \partial\omega, \ t \in (0,T) \tag{4.49}$$

Using the parametrix method of Parabolic elliptic operators, and asuming the hyphotesis that U is a infinitely differentiable in a cilinder $[0,t] \times \Omega$, it is possible to find a Green function $G(x,t,x',t')$ [23], of the operator $J_{U(\infty)}\partial_t\psi - \Delta$, that is:

$$(J_{U(\infty)}\partial_t - \Delta_x)G(x,t,x',t') = \delta(x-x',t-t')$$

which satisfies the following inequality:

$$|G(x,t,x',t')| \le C\frac{1}{|t-t'|}\mathrm{e}^{-c|x-x'|^2/|t-t'|} \tag{4.50}$$

and

$$|\nabla_x G(x,t,x',t')| \le C\frac{1}{|t-t'|^{3/2}}\mathrm{e}^{-c|x-x'|^2/|t-t'|} \tag{4.51}$$

for some constants $C, c > 0$. In consequence we can solve (4.43):

$$\psi(x,t) = Z(x,t) + \int_{\mathbb{R}^{2+1}} G(x,t,x',t')H(x',t')\ \mathrm{d}^2x'\mathrm{d}t'$$

The first integral is the homogeneous solution: $J_{U(\infty)}\partial_t Z = \Delta Z$ This part has the role of fixing the boundary conditions such that ψ satisfies $U + \psi + \Psi^0 = \mathbf{e}_3$. Furthermore since we are looking for discribe the blow up behaviour at main order by U, we require $Z(x_0,T) = 0$ at the blow up point (x_0,T). These conditions can be set in the initial conditions $Z_0(x) = Z(x,0)$ as Lagrange multipliers.

From this we can establish the equivalent estimates of the heat equation for the Landau-Lifshitz Let define the the following weights:

$$\varrho_1 = \lambda^{\nu-2}R^{-a}\chi_{\{r<2R\lambda_0\}}$$
$$\varrho_2 = T^{-\sigma_0}(1-\eta)\frac{\lambda_0}{r^2+\lambda^2}$$
$$\varrho_3 = \frac{\lambda_0^{1/2+\sigma_2}}{r+\lambda_0}$$

and we define the followings L^∞-weight norms:

$$\|f\|_{**} \doteq \sum_{\Omega\times(0,T)} (2+\varrho_1+\varrho_2+\varrho_3)^{-1}|f(x,t)|$$

$$
\begin{aligned}
\|\psi\|_* \doteq\ & \lambda_0(0)^{-\nu} R(0)^{2-a} |\log T| \sup_{\Omega\dot{(}0,T)} |\psi(x,t)| \\
&+ \sup_{\Omega\dot{(}0,T)} \lambda_0(t)^{-\nu} R(t)^{a-2} |\log(T-t)| \, |\psi(x,t)-\psi(x,T)| \\
&+ \sup_{\Omega\dot{(}0,T)} \lambda_0(t)^{1-\nu} R(0)^{a-1} |\nabla\psi(x,t)| \\
&+ \sup_{\Omega\dot{(}0,T)} \lambda_0(t)^{1-\nu} R(t)^{a-1} |\nabla\psi(x,t)-\nabla\psi(x,T)| \\
&+ \sup_{\Omega\dot{(}0,T)} \lambda_0(t_2)^{1-\nu+2\gamma} R(t)^{a-1+2\gamma} \frac{|\nabla\psi(x,t)-\nabla\psi(x',t)|}{|x-x'|^{2\gamma}} \\
&+ \sup_{\Lambda^*} \lambda_0(t_2)^{1-\nu+2\gamma} R(t_2)^{a-1+2\gamma} \frac{|\nabla\psi(x,t_2)-\nabla\psi(x,t_1)|}{|t_2-t_1|^{\gamma}}
\end{aligned}
$$

where we have taken $\Lambda^* = \left\{ x\in\Omega, 0\le t_1 < t_2 \le T \ \text{ such that } \ t_2 - t_1 \le \frac{1}{10}(T-t_2)\right\}$

The main result is the estimation of the solution of (4.43):

Lema 4.5 *for $T, \varepsilon > 0$ there exist a lineal operator maping f such that $\|f\|_{**} < \infty$ to $(\psi[f], c_1.c_2, c_3)$ solution of (4.43) such that :*

$$
\|\psi\|_* \le C \, \|f\|_{**}
$$

for some constant $C > 0$.

Proof : It is identical to the proof of the Proposition 5.1 in [17], which is based in the estimations using the heat kernel, in this case the estimations (4.50) and (4.51), above.

4.6. Gluing inner-outer problem

In order to solve the full system (4.35), (4.36) we proceed as follows: Consider $(\phi, \psi, p, \xi) \in B_\varepsilon(\mathfrak{B})$ where $B_\varepsilon(\mathfrak{B})$ is the ball of radius $\varepsilon > 0$ which we choose sufficiently small, and set the system

$$
\begin{aligned}
\partial_\tau \tilde{\phi} - J_{U_0}\mathcal{L}_{U_0}\tilde{\phi} &= (h-\bar{h})[\phi,\psi,p,\xi] \\
\partial_\tau \tilde{\phi} - J_{U_0}\mathcal{L}_{U_0}\tilde{\phi} &= \bar{h}[\phi,\psi,p,\xi] \\
\partial_\tau \tilde{\phi} - J_U \Delta\tilde{\psi} &= H[\phi,\psi,p,\xi] \\
\mathcal{R}_{\mathrm{i}}[p,\xi] &= Q_{\mathrm{i}}[\psi] \qquad \text{for i} = 0,1
\end{aligned}
$$

where:

$$
\bar{h}[\phi,\psi,p,\xi] = \sum_{l,j} c_{l,j}(\tau)\frac{1}{1+|y|} Z_{l,j}(y)
$$

$$
\int_{\mathcal{D}_R(\tau)} (h-\bar{h}) \ Z_{l,j}(y) \ \mathrm{d}y = 0 \quad \tau\in(\tau_0,\infty)
$$

In this way we have that $\phi = \phi_1 + \phi_2$ is solution of (4.37).

$$\mathcal{R}[p,\xi]_0(t) = \int_0^{t-\lambda^2} \frac{\dot{p}(s)}{t-s}\,\mathrm{d}s, \qquad Q_0[\psi] = -\gamma^*(\text{div } \psi(\xi(t)) + \text{i curl } \psi(\xi(t)))$$

for $a' < a, \gamma' < \gamma$ we introduce the norm:

$$\begin{aligned}
\|\psi\|_{Y(a',\nu,\gamma')} &\doteq \lambda_0(0)^{-\nu} R(0)^{2-a'} \left|\log T\right| \sup_{\dot{\Omega}(0,T)} |\psi(x,t)| \\
&+ \sup_{\dot{\Omega}(0,T)} \lambda_0(t)^{-\nu} R(t)^{a'-2} \left|\log(T-t)\right| |\psi(x,t) - \psi(x,T)| \\
&+ \sup_{\dot{\Omega}(0,T)} \lambda_0(t)^{1-\nu} R(0)^{a'-1} \left|\nabla\psi(x,t)\right| \\
&+ \sup_{\dot{\Omega}(0,T)} \lambda_0(t)^{1-\nu} R(t)^{a'-1} \left|\nabla\psi(x,t) - \nabla\psi(x,T)\right| \\
&+ \sup_{\dot{\Omega}(0,T)} \lambda_0(t_2)^{1-\nu+2\gamma} R(t)^{a'-1+2\gamma} \frac{|\nabla\psi(x,t) - \nabla\psi(x',t)|}{|x-x'|^{2\gamma'}} \\
&+ \sup_{\Lambda^*} \lambda_0(t_2)^{1-\nu+2\gamma} R(t_2)^{a'-1+2\gamma} \frac{|\nabla\psi(x,t_2) - \nabla\psi(x,t_1)|}{|t_2-t_1|^{\gamma'}}
\end{aligned}$$

Proof theorem 4.4: Then we define the operator $\mathcal{F} : B_\varepsilon(\mathfrak{B}) \to \mathfrak{B}$ by:

$$\mathcal{F}(\phi,\psi,p,\xi) = (\tilde{\phi},\tilde{\psi},\tilde{p},\tilde{\xi})$$

We claim that for ε adequately small we have in fact $\mathcal{F} : B_\varepsilon(\mathfrak{B}) \to B_\varepsilon(\mathfrak{B})$

Let R_1 be small and fixed, then in order to apply the Schauder fixed point theorem, we proceed in the following way: take $\phi_1, \phi_2 \in \bar{B}_{R_1}(X(a,\nu))$ and $\psi \in \bar{B}_{R_1}(Y(a',\nu',\gamma'))$.

First we will prove the bound:

$$\|H[\phi,\psi,p,\xi]\|_{**} \leq C(T^{\sigma_0} + R_1)$$

Since $\psi(x_0,T) = 0$ and $\|\psi\|_{Y(a',\nu,\gamma')} \leq 1$, we have:

$$\begin{aligned}
|\psi(x,t)| &\leq |\psi(x,t) - \psi(x,T)| + |\psi(x,T) - \psi(x_0,T)| \\
&\leq (r + \lambda_0 R(t)^{2-a'} \left|\log(T-t)\right|) \|\psi\|_{Y(a',\nu,\gamma')}
\end{aligned}$$

The strategy is to estimate every term in:

$$h[\psi,\Psi^0] = R_\omega^{-1}\left(J_U\tilde{\mathcal{L}}_U(\psi) + J_U\tilde{\mathcal{L}}_U(\Psi^0) + \Pi_{U^\perp}(-\partial_t\Psi^0 + J_U\Delta\Psi^0 - \partial_t U + (J_U - J_{U(\infty)})\Delta\psi)\right)$$

$$\begin{aligned}
H[\phi,\psi,\Psi^0] = \eta R_\omega &\left[\frac{1}{\lambda}(\dot{\xi} + \dot{\lambda}z)\nabla_z\phi - \dot{\omega}\; J_z\phi - R_\omega^{-1}\partial_t(\Pi_{U^\perp})(\Psi^0 + \psi)\right] + (1-\eta)h[\psi,\Psi^0] \\
&+ R_\omega J_U(\Delta\eta\phi + 2\nabla\eta\cdot\nabla\phi - \partial_t\eta\phi) + \mathcal{N}(\phi,\psi)
\end{aligned}$$

separately. Since $|J_U\varphi| \leq |\varphi|$ for all $\varphi \cdot U$ then:

$$
\begin{aligned}
|(1-\eta)\,|J_U \nabla_x U|\,\psi| &\leq |(1-\eta)\,|\nabla_x U|\,\psi| \\
&\leq (1-\eta)\frac{\lambda_0^2}{(r+\lambda_0)^4}(|\psi(x,t)-\psi(x,T)| + |\psi(x,T)-\psi(x_0,T)|) \\
&\leq (1-\eta)\frac{\lambda_0^2}{(r+\lambda_0)^4}(r+\lambda_0 R(t)^{2-a'}\,|\log(T-t)|)\,\|\psi\|_{Y(a',\nu,\gamma')} \\
&\leq C(1-\eta)\frac{\lambda_0}{(r+\lambda_0)^2}\,\|\psi\|_{Y(a',\nu,\gamma')} \\
&\leq C\varrho_2\,\|\psi\|_{Y(a',\nu,\gamma')}
\end{aligned}
$$

thus $\left\|(1-\eta)\ J_U\,|\nabla U|^2\,\psi\right\|_{**} \leq C\,\|\psi\|_{Y(a',\nu,\gamma')}$

Similarly : $\|(1-\eta)\ J_U\,|\nabla U|\,|\nabla\psi|\|_{**} \leq C\,\|\psi\|_{Y(a',\nu,\gamma')}$

$$
\begin{aligned}
|(\psi\cdot U)\partial_t U| &\leq |\psi|\,|U_t| \\
&\leq C\left(r+\lambda_0 R^{2-a'}\,|\log(T-t)|\right)\left[\frac{\left|\dot\lambda_0\right|}{r+\lambda_0}+\frac{\left|\dot\lambda_0\right|}{r^2+\lambda_0^2}\right]\|\psi\|_{Y(a',\nu,\gamma')}
\end{aligned}
$$

and ussing that $\left|\dot\lambda + \lambda\,|\omega|\right| \leq C\left|\dot\lambda_0\right| \leq C$ and $\left|\dot\xi\right| \leq C$:

$$
\begin{aligned}
|(\psi\cdot U)\partial_t U| &\leq C\left[1+\frac{\lambda_0 R^{2-a'}}{r+\lambda}\right]\|\psi\|_{Y(a',\nu,\gamma')} \\
&\leq C(\varrho_1+\varrho_2)\,\|\psi\|_{Y(a',\nu,\gamma')}
\end{aligned}
$$

Where we take $a' \approx 2$. Thus $\|(\psi\cdot U)\partial_t U\| \leq C\,\|\psi\|_{Y(a',\nu,\gamma')}$

On the other hand, using the definition of the norm $\|\cdot\|_{X(a,\nu)}$, we have the estimation:

$$
|\psi(y,\tau)| + (1+|y|)\,|\nabla\psi(y,\tau)| \leq \|\phi\|_{X(a,\nu)}\,\lambda_0^\nu R^{2-a} \quad \text{for} \ \ R \leq |y| \leq 2R
$$

Hence:

$$
\|R_\omega \Delta_x \phi\|_{**} + \|\partial_t R_\omega \eta\phi\|_{**} + \leq C\,\|\phi\|_{X(a,\nu)}
$$

and:

$$
\|(1-\eta)\Pi_{U^\perp}\mathcal{E}^*\|_{**} \leq (1-\eta)\frac{\lambda_0}{r^2+\lambda_0^2} \leq CT^{\sigma_0}
$$

besides:

$$
\|\mathcal{N}[\phi,\psi]\|_{**} \leq C\,\|\phi\|_{X(a,\nu)} + C\,\|\psi\|_{Y(a',\nu',\gamma')}
$$

Now we estimate the term

$$\begin{aligned}\left|(1-\eta)(J_U - J_{U(\infty)})\Delta\psi\right| \leq (1-\eta)\left|U - U(\infty)\right|\left|\Delta\psi\right| &\leq (1-\eta)\sqrt{\sin^2(\Theta) + (\cos(\Theta)-1)^2}\left|\Delta\psi\right| \\ &\leq (1-\eta)\frac{2\lambda}{r}\left|\Delta\psi\right| \\ &\leq \varrho_3 \left\|\Delta\psi\right\|_{L^\infty}\end{aligned}$$

thus:

$$\left\|(J_U - J_{U_\infty})\Delta\psi\right\|_{**} \leq C\left\|\psi\right\|_{Y(a',\nu',\gamma')}$$

Finally using the estimations of the exterior problem, we get:

$$|\tilde{\psi}|_* \leq \left\|H[\phi,\psi]\right\|_{**} \leq C\left\|\phi\right\|_{X(a,\nu)} + C\left\|\psi\right\|_{Y(a',\nu',\gamma')} \leq CR_1$$

Now fix $a_1 \in (a,2)$ and $\nu_1 \in (\nu, 1)$. then using the interior estimation's lemma:

$$|\tilde{\phi}_1|_{X(a_1,\nu_1)} \leq C|(G-\tilde{G})[\lambda,\omega,\xi,\psi]|_{a_1,\nu_1} \leq CT^{1-\nu_1} + CT^{\nu-1+\beta(a'-1)}R_1$$

The estimation for $p(t)$ is identical to the given in [17] (since we have used the same ansatz).

Finally we see:

$$\left\|\phi_1\right\|_{X(a,\nu)} + \left\|\phi_2\right\|_{X(a,\nu)} + \left\|\psi\right\|_{Y(a',\nu',\gamma')} + \left\|p\right\|_{L^\infty} + \left\|\xi\right\|_{L^\infty} \leq R_1$$

as we wanted to prove.

Conclusion

In this thesis, we have studied the dynamics of the skyrmion-magnons system. Specifically, we studied:

- The dynamics of the magnon field is given by a Schrodinger equation with a vector potential and scalar determined by the skyrmion parameters with a source term. We explored the problem of the emergent electrodynamics and the Aharonov-Bohm effects.
- How the skyrmion's own dynamics generates spin waves, and how they, in turn, affect the skyrmion in form of back reaction radiation. We have studied the origin of the skyrmion mass.
- The explosion of a skyrmion (blow up) and the dynamics of its parameters during the explosion, with the consequent emission of spin waves in the form of radiation. We found the radiation of magnons emitted during the process and its spectrum.
- Finally, we explored miscellaneous topics associated: Quntum Fluctuation, SUSY aproach, acoustic exitations.

Apéndice A

Appendix

A.1. Magnetic Solitons

Topological solitons are magnetic textures with a topological features. Formally, we can say that a u is a magnetic soliton if his topological degree is non zero.

Let $u : \mathbb{R}^2 \to S^2$ be a continuous map. Define the topological degree of u by the quantity:

$$\deg() = \frac{1}{4\pi} \int_{\mathcal{M}} u(x) \cdot (\frac{\partial u(x)}{\partial x^1} \times \frac{\partial u(x)}{\partial x^2}) \, \mathrm{d}^2 x \tag{A.1}$$

This number represents how many times the image of u cover the sphere S^2. It is possible to prove that $\deg(u)$ is invariant under continuous deformations of u.

It is possible to prove the following characterization of magnetic solitons:

Lema A.1 *Let $u^0 : \mathbb{R}^2 \to S^2$ an harmonic map. Then u^0 is a holomorphic or anti-holomorphic function seen as function from the complex plane, In addition if $u^0(\infty)$ exists, then there exists $\lambda_{\mathrm{i}} \in \mathbb{R}_+, n_{\mathrm{i}} \in \mathbb{Z}, \xi_{\mathrm{i}} \in \mathbb{C})$ such that:*

$$w(z) = \prod_{\mathrm{i}} \left(\frac{z - \xi_{\mathrm{i}}}{\lambda_{\mathrm{i}}} \right)^{n_{\mathrm{i}}}$$

where $w(z = x + \mathrm{i}y)$ denotes the stereo-graphics projection of $u^0(x, y) \in S^2$ onto the complex plane. and the topological charge is given by:

$$Q = \sum_{\mathrm{i}} n_{\mathrm{i}}$$

Let $u : \mathbb{R}^2 \to S^2$ be a map parametriced by the polar coordinates $(\rho, \varphi) \in [0, \infty) \times [0, 2\pi) \mapsto (\Theta(\rho, \phi), \Phi(\rho, \phi))$.

We say that u is an equivariant map of degree m if $\Theta = \theta(\rho)$ y $\Phi = m\phi$

It is possible to prove that u is an equivariant harmonic map if and only if:

$$\frac{\partial \Theta(\rho,t)}{\partial t} = \frac{\partial^2}{\partial \rho^2}[\rho\ \theta(\rho,t)] - \frac{m^2}{\rho^2}\sin(\Theta(\rho,t))\cos(\Theta(\rho,t)) \tag{A.2}$$

and the steady solution with $\Theta(0) = -\pi$ and $\Theta(\infty) = 0$ is

$$\Theta(\rho) = 2\tan^{-1}\left(\frac{1}{\rho}\right)$$

A.2. Energy Functional

In general, we can define the energy function for maps between Riemannian manifolds:

Definición A.2 *Let $f : \mathcal{M} \to \mathcal{N}$ be a map between the Riemann manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$, with metric g and h. Suppose that $\mathcal{N} \hookrightarrow \mathbb{R}^{N+k}$ is an embedding isometric (It always exists buy the Nash's Theorem). We say $f \in \mathcal{C}^2(\mathcal{M},\mathcal{N})$ is an harmonic map if it is a critical point of the energy functional:*

$$\mathscr{E}[f] = \frac{1}{2}\int_{\mathcal{M}} h^{\alpha\beta}(f(x))g_{\mathrm{ij}}(x)\ \frac{\partial f^\alpha}{\partial x^{\mathrm{i}}}\frac{\partial f^\beta}{\partial x^j}\ \sqrt{g}\ \mathrm{d}^n x \tag{A.3}$$

In addition we call to $\tau(f)^\alpha = \Delta f^\alpha + A^\alpha(\nabla f, \nabla f) f$ the tension field of f, where $A : \mathbb{T}\mathcal{N} \times \mathbb{T}\mathcal{N} \to \mathbb{R}^{N+k}$ is the second fundamental form of $\mathcal{N}$.

Teorema A.3 *Let $f_t : \mathcal{M} \to \mathcal{N}$ be a family of maps, such that f_0 is an harmonic map. Denotes $X(x) = \partial_t f(x,0)$, then we have the formulas:*

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}(t=0) = \int_{\mathcal{M}} \langle \tau, X\rangle\ \sqrt{g}\ \mathrm{d}^n x \tag{A.4}$$

$$\frac{\mathrm{d}^2\mathscr{E}}{\mathrm{d}t^2}(t=0) = \int_{\mathcal{M}} \sum_{\mathrm{i}} \left\langle X, \nabla^E_{\mathrm{e_i}} X - R(X, \mathrm{d}f_*\mathrm{e_i})\ \mathrm{d}f_*\mathrm{e_i}\right\rangle\ \sqrt{g}\ \mathrm{d}^n x \tag{A.5}$$

where $\{\mathrm{e_i}\}_{\mathrm{i}=1,\dots M}$ is an orthonormal basis of $\mathbb{T}\mathcal{M}$.

Proof See [22].

Proof of 1.3: We apply the previous formula for $f = U$.

The first variation is a classic result (see [19]).

For the second variation, it is enough to use the formula of the Riemann curvature on the sphere:

$$R_{\mathrm{i}jkl} = g_{\mathrm{i}k}g_{jl} - g_{\mathrm{i}l}g_{jk}$$

thus:

$$\langle X, R(X, \mathrm{d}f_*\mathrm{e_i})\ \mathrm{d}f_*\mathrm{e_i}\rangle = \|X\|^2\,\|\mathrm{d}f_*\mathrm{e_i}\|^2 - \langle X, \mathrm{d}f_*\mathrm{e_i}\rangle^2$$

on the other hand, taking the polar basis $\mathrm{e}_1 = \hat{r}$ and $\mathrm{e}_2 = \hat{\theta}$ of the plane, we obtain:

$$\mathrm{d}f_*\mathrm{e}_1 = \frac{\partial}{\partial r}U = \frac{\sin(\Theta)}{r}E_1$$
$$\mathrm{d}f_*\mathrm{e}_2 = \frac{1}{r}\frac{\partial}{\partial\theta}U = \frac{\sin(\Theta)}{r}E_2$$

Replacing, we have the result.

A.3. Stress tensor and magnons radiation

In this section we calculate the stress tensor of the the skyrmion-magnon system.

Namely we define the stress tensor T of a Lagrangian by its Lie derivative:

$$\pounds_X\mathcal{S} \equiv \lim_{t\to 0}\frac{\mathcal{S}(u\circ \mathrm{e}^{tX}) - \mathcal{S}(u)}{t} = \int T\cdot\pounds_X g = \int T\cdot\nabla X \qquad \forall\ X \text{ vector field in } \mathbb{R}^2$$

where $A\cdot B = tr(AB)$ and ∇X is the Jacobian of X.

Let $T_{\mu\nu}$ be components stress tensor magnons ($\mu,\nu = 0,1,2$), then

$$T_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi)}\ \partial_\nu\psi\ + h.c - \mathscr{L}g_{\mu\nu} \tag{A.6}$$

using the formula (1.34) in the form:

$$\mathscr{L} = -\mathrm{i}\psi^\dagger\ \partial_t\psi\ - \nabla_{\mathbf{A}}\psi^\dagger\nabla_{\mathbf{A}}\psi - V\psi^\dagger\psi\ +\ J\psi \tag{A.7}$$

g is the metric tensor of $\mathbb{R}^2$

$$\Rightarrow\ \frac{\partial\mathscr{L}}{\partial(\partial_\mathrm{i}\psi)} = -(\nabla_{\mathbf{A}}\psi^\dagger)_\mathrm{i} \quad \Rightarrow\ T_{\mathrm{i}j} = -(\nabla_{\mathbf{A}}\psi^\dagger)_\mathrm{i}\partial_j\psi\ + h.c - \mathscr{L}g_{\mathrm{i}j}$$

The force that the magnons act on the skyrmion will be:

$$F_j = \int_{\partial\Omega} T_{\mathrm{i}j}\ \mathrm{d}n^\mathrm{i} \tag{A.8}$$
$$= \int_{\partial\Omega}\left\{-(\nabla_{\mathbf{A}}\psi^\dagger)_\mathrm{i}\partial_j\psi\ + h.c - \mathscr{L}g_{\mathrm{i}j}\right\}\ \mathrm{d}n^\mathrm{i} \tag{A.9}$$
$$= \int_{\partial B_R}\left\{-(\nabla_{\mathbf{A}}\psi^\dagger)_\mathrm{i}\partial_j\psi\ + h.c - \mathscr{L}g_{\mathrm{i}j}\right\}\ n^\mathrm{i}R\ \mathrm{d}\theta \tag{A.10}$$

Whereas, the rate of radiated energy by the skyrmion will be:

$$\partial_t\mathscr{E} = \int_{\partial\Omega} T_{\mathrm{i}0}\ \mathrm{d}n^\mathrm{i} \tag{A.11}$$
$$= \int_{\partial B_R}\left\{-(\nabla_{\mathbf{A}}\psi^\dagger)_\mathrm{i}\partial_t\psi\ + h.c\right\}\ n^\mathrm{i}R\ \mathrm{d}\theta \tag{A.12}$$
$$= \int_{\partial B_R}\left\{-\partial_r\psi^\dagger\partial_t\psi\ + h.c\right\} R\ \mathrm{d}\theta \tag{A.13}$$

A.4. Acoustic excitations around a skyrmion

As we have seen in previous chapters, infenitesimal spatial translations $x \mapsto x + \xi$ correspond to Goldstone modes in the magnon dynamics. In general we can consider local infeniotesimal translations $x \mapsto x + \xi(x)$ with continuous deformations $t \mapsto f_t \in \mathrm{Diff}(\Omega)$, where Ω is the space of diffeomorphisms of Ω defining $\xi(x) = \partial_t f_t(x)$. In this way an infenitesimal deformation generates a disturbance in the texture u given by $\delta u(x) = \xi \cdot \nabla u(x)$. Let's study the effective dynamic of these exitations. We expanding the action up to second order, and using the Lie derivative:

$$\mathcal{S}[u \circ f_t] = \mathcal{S}[u] + \int_0^T \int_\Omega \pounds_\xi \mathscr{L}(u)\ \mathrm{d}^2x\ \mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega \pounds_\xi(\pounds_\xi \mathscr{L}(u))\ \mathrm{d}^2x\ \mathrm{d}t$$

Let's consider $u = U_0$ an harmonic map skyrmion, so we will have

$$\int_\Omega \pounds_\xi \mathscr{L}(u)\ \mathrm{d}^2x\ \mathrm{d}t = \int_\Omega T_{\mathrm{i}j} \pounds_\xi\ g^{\mathrm{i}j}\ \mathrm{d}^2x\ \mathrm{d}t = 0$$

since $\nabla_\mathrm{i} T_{\mathrm{i}j} = 0$ by the Noether theorem. On the other hand:

$$\begin{aligned}\int_\Omega \pounds_\xi(\pounds_\xi \mathscr{L}(u))\ \mathrm{d}^2x\ \mathrm{d}t &= \int_\Omega U \cdot (\pounds_\xi U \wedge \pounds_\xi \partial_t U) - \pounds_\xi (T_{\mathrm{i}j} \pounds_\xi g^{\mathrm{i}j})\ \mathrm{d}^2x\ \mathrm{d}t \\ &= \int_\Omega \varrho \left(\varepsilon_{\mathrm{i}j} \xi_\mathrm{i} \partial_t \xi_\mathrm{i} - (\partial_\mathrm{i}\xi_j + \partial_j \xi_\mathrm{i})^2\right)\ \mathrm{d}^2x\ \mathrm{d}t\end{aligned}$$

where $\varrho = U \cdot (\partial_{x_1} U \wedge \partial_{x_2} U)$ is the topological density. Hence the action of the small spacial perturbation will be:

$$\mathcal{S}[\xi] = \int_0^T \int_\Omega \varrho \left(\varepsilon_{\mathrm{i}j} \xi_\mathrm{i} \partial_t \xi_\mathrm{i} - (\partial_\mathrm{i}\xi_j + \partial_j \xi_\mathrm{i})^2\right)\ \mathrm{d}^2x\ \mathrm{d}t \tag{A.14}$$

Remark that for the case $\xi_\mathrm{i}(x,t) = \xi_\mathrm{i}^0(t)$ the action (A.14) gives:

$$\mathcal{S}[\xi^0] = \int_0^T \int_\Omega \varrho\ \varepsilon_{\mathrm{i}j}\ \xi_\mathrm{i}^0\ \partial_t \xi_\mathrm{i}^0\ \mathrm{d}^2x\ \mathrm{d}t = 4\pi \int_0^T \varepsilon_{\mathrm{i}j}\ \xi_\mathrm{i}^0\ \dot{\xi}_\mathrm{i}^0\ \mathrm{d}t$$

which is the action for $U[\xi^0(t)]$ derived from collective variables approach (1.28).

As future work it would be interesting to study the spectral of this excitations.

A.5. Fluid analogies for magnons-skyrmion system

Recently, very interesting fluid analogy for the magnons around a skyrmion, is done in [35].

Multipliyng the magnon equation by $\psi^\dagger$ we get:

$$\mathrm{i}\psi^\dagger\partial_t\psi = -\psi^\dagger(\nabla - \mathrm{i}\mathbf{A})^2\psi + V\psi^\dagger\psi + \psi^\dagger\mathcal{J} \tag{A.15}$$

We will introduce the Madelung's transformation $\phi \mapsto (\varrho, \vartheta)$ defined by the expression $\psi(x,t) = \sqrt{\varrho}(x,t)\mathrm{e}^{\mathrm{i}\vartheta(x,t)}$, if we replace this into the equation (A.15) and we separate this into real and imaginary part, we obtain the system of equations: $\mathbf{v} = 2(\nabla\vartheta - \mathbf{A})$

$$\partial_t\varrho + \nabla\cdot(\varrho\mathbf{v}) = \Im(\psi^\dagger\mathcal{J}) \tag{A.16}$$

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{E} + \mathbf{v}\wedge\mathbf{B} - \nabla p \qquad \text{where} \quad p(x) = 2\left(-\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}} + \frac{1}{\varrho}\Re(\psi^\dagger\mathcal{J})\right) \tag{A.17}$$

Where $\mathbf{E} = -\partial_t\mathbf{A} - \nabla V$ and $\mathbf{B}(x) = \nabla\wedge\mathbf{A}(x)$ is the magnetic field produced by the connection $\mathbf{A}$. From (A.16) it is possible understand the magnon field as an ideal fluid identifying $\mathbf{v}$ with the velocity field, ϱ the density and p the pressure of the fluid, respectively. By the way, the term $\Delta\sqrt{\varrho}/\varrho$ is known as *quantum pressure*, in the literature.

As a future work, it would be very interesting to survey in the hydrodynamics analogies, for instances, the effects likes drag force or add mass.

A.6. Skyrmion in a magnetic field

Let's consider a skyrmion U in a magnetic field $B : \Omega \to \mathbb{R}^3$. In this case the LLG equation turns out:

$$\partial_t u = J_u\ (\Delta u + |\nabla u|^2\, u + \Pi_{u^\perp}B) \equiv \mathcal{S}_B(u) \tag{A.18}$$

Note that the variational derivative of $u \mapsto \Pi_{u^\perp}$ is the second fundamental form in the sphere, and proceeding analogously we can derive that the equation for the excitations φ is:

$$\partial_t\varphi = \gamma\ (-(\nabla_{\mathbf{A}})^2\varphi + V_B\varphi\ ) \equiv \mathcal{H}_B\varphi \qquad \text{where:}\ \ V_B = -\sin^2(\Theta)/r^2 +\ B\cdot U \tag{A.19}$$

and $\mathbf{A}$ is the same.

Roughly speaking we can expect that the term $B\cdot U$ shifts the spectrum approximately in B

Nevertheless we notice that $\mathcal{H}_B$ is not "homogeneous" this introduce a characteristic scale l and clearly a static solution of (A.18) will not be U considered in the previous section; so we will try ansatz approximated for Θ

For a fixed l we can write (A.19) the variational principle, for that let's consider the energy functional:

$$\mathcal{E}[\phi] = \int_0^\infty \left\{\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{(l^2 + 2l\cos(\Theta) + \cos(2\Theta))}{r^2} + B\cos(\Theta)\right)\phi^2\right\}\, r\mathrm{d}r \tag{A.20}$$

It is possible to study the stability of the equation (A.18) around a skyrmion U_λ. In virtue of the principle of stability linearized, it reduces to study $\mathcal{D}^2\mathcal{S}(\varphi,\varphi)$ for $\phi \in \mathbb{T}_U S^2$ arbitrary. Let e_j be the j canonical vector of $\mathbb{R}^3$, then

$$\sum_{\mathrm{i}=1}^{3} \langle \mathrm{e_i}, \mathcal{L}\mathrm{e_i}\rangle_{L^2} = 0$$

on the other hand:

$$\sum_{\mathrm{i}=1}^{3} \langle \phi_\mathrm{i}, B \cdot U\phi_\mathrm{i}\rangle_{L^2} = B\int_\Omega \cos(\Theta) \sum_{\mathrm{i}=1}^{3} |\phi_\mathrm{i}|^2 \ \mathrm{d}x = 2B\int_\Omega \cos(\Theta)\ \mathrm{d}x \equiv B_{\text{eff}}$$

Therefore the stability of (A.18) depends on the sign of B_{eff}. In particular if $B_{\text{eff}} > 0$ then (A.18) is stable around the skyrmion, otherwise it is unstable.

A.7. Chiral Skyrmions and Dzyaloshinskii-Moriya term

Consider the new energy functional:

$$\mathcal{E}(u) = |\nabla u|^2 + D\ u \times \nabla u \tag{A.21}$$

where D is known as the Dzyaloshinskii-Moriya constant [12].

We can write the equation for the spin waves around a skyrmion, in the form [12], [27]:

$$\mathrm{i}\partial_t\psi = \mathcal{H}\psi + \mathcal{W}\psi^* \tag{A.22}$$

and the adjoint equation:

$$-\mathrm{i}\partial_t\psi^* = \mathcal{H}\psi^* - \mathcal{W}\psi^* \tag{A.23}$$

Thus denoting the spinor field $\Psi = \begin{pmatrix}\psi \\ \psi^*\end{pmatrix}$ we have the new evolution equation:

$$\mathrm{i}\partial_t\Psi = (\sigma_z\mathcal{H} + \sigma_x\mathcal{W})\Psi \tag{A.24}$$

where $\mathcal{H} = -(\nabla - \mathrm{i}\mathbf{A})^2 + V$

$$\mathbf{A} = \left(\frac{\cos\Theta}{r} - D\sin\Theta\right)\hat{\theta}, \qquad V = 4\frac{(-1 + D\lambda - Dr^2)^2)\lambda^2}{(r^2+\lambda^2)^2}, \qquad \mathcal{W} = -4\frac{D\lambda r^2}{(r^2+\lambda^2)^2}$$

In polar coordinates

$$\mathcal{H} = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\left(\partial_\theta - \mathrm{i}\cos\Theta - \mathrm{i}Dr\sin\Theta\right)^2$$

by the mixing between ψ and ψ^* we decompose

$$\psi_l(r,t) = \psi_l^+(r,t)\mathrm{e}^{\mathrm{i}\theta} + \psi_l^-(r,t)\mathrm{e}^{-\mathrm{i}\theta}$$

and let be $\Psi_l(r,t) = \begin{pmatrix} \psi_l^+(r,t) \\ \psi_l^-(r,t) \end{pmatrix}$ then it satisfies

$$\mathrm{i}\partial_t \Psi_l = (\sigma_z \mathcal{H}_l + \sigma_x \mathcal{W})\Psi_l \tag{A.25}$$

where

$$\mathcal{H}_l = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(l\sigma_z - \cos\Theta - D\ r\sin\Theta\right)^2$$

Far away the skyrmion $\cos(\Theta) \approx 1$ and retaining only terms $O(R^{-2})$, we can approximate the Hamiltonian:

$$\mathcal{H}_l = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\left(l\sigma_z - 1\right)^2, \qquad \mathcal{W} = 0$$

Thus the equation for ψ_l^+ and ψ_l^- are decoupled. In fact, the Hamiltonian is the same than a skyrmion without Dzyaloshinskii-Moriya term. In conclusion, the previous results are valid in this case too.

A.8. Linear response theory

Consider a time dependent Hamiltonian $\mathcal{H}(t) = \mathcal{H}_0 + f(t)\mathcal{O}_1$ where we will assume $f : \mathbb{R} \to \mathbb{C}$ is such that $f(t) = 0$ for $t < t_0$, so $f(t)\mathcal{O}_1$ represents a source that is switched at time t_0. Denoting the average of an operator $\langle \mathcal{O}(t) \rangle = \langle \psi(t) \left| \mathcal{O}_2 \right| \psi(t) \rangle$ and $\langle \mathcal{O}(t) \rangle_0 = \langle \psi_0(t) \left| \mathcal{O}_2 \right| \psi_0(t) \rangle$ where $\psi(t) = \mathrm{e}^{-\mathrm{i}t\mathcal{H}}\psi_0$ and $\psi_0(t) = \mathrm{e}^{-\mathrm{i}t\mathcal{H}_0}\psi_0$

$$\delta \langle \mathcal{O}_2(t) \rangle \equiv \langle \mathcal{O}_2(t) \rangle - \langle \mathcal{O}_2(t) \rangle_0 = \mathrm{i}\int_{t_0}^{t} \langle [\mathcal{O}_2(t), \mathcal{O}_1(s)] \rangle_0\ f(s)\ \mathrm{d}s \tag{A.26}$$

$$= \int_{t_0}^{\infty} D_{\mathcal{O}_2,\mathcal{O}_1}(t-s)\ f(s)\ \mathrm{d}s \tag{A.27}$$

where $D_{\mathcal{O}_2,\mathcal{O}_1}(t-s) = \mathrm{i}\langle [\mathcal{O}_2(t), \mathcal{O}_1(s)] \rangle_0 H(t-s)$ only depends on the difference of time $t-s$ since the translational time invariance of the unperturbed dynamics, and here $H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ is the Heaviside's function. At this point it is more convenient working in the Fourier space in time.

$$\tilde{D}_{\mathcal{O}_2,\mathcal{O}_1}(\omega) = \int_0^{\infty} D_{\mathcal{O}_2,\mathcal{O}_1}(t)\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t$$

and $\delta \left\langle \tilde{\mathcal{O}}_2(\omega) \right\rangle = \int_{\mathbb{R}} \delta \langle \mathcal{O}_2(t) \rangle\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t$, and $\tilde{f}(\omega) = \int_{\mathbb{R}} f(t)\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t$. Thus, from (A.26) and the convolution formula of Fourier transforms, we deduce:

$$\delta \left\langle \tilde{\mathcal{O}}_2(\omega) \right\rangle = \tilde{D}_{\mathcal{O}_2,\mathcal{O}_1}(\omega)\ \tilde{f}(\omega) \tag{A.28}$$

Suppose that $f(t) = \Re[f_0 \mathrm{e}^{\mathrm{i}\omega_0}]$ so

The average energy change of the system in a cycle of period ω_0 will be given by:

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \int_0^{2\pi/\omega_0} \frac{\partial \langle \mathcal{H} \rangle}{\partial t}\ \mathrm{d}t = 2\omega_0\ \Im\left\{ \tilde{D}_{\mathcal{O}_1,\mathcal{O}_1}(\omega_0) \right\} |f_0|^2 \tag{A.29}$$

This measure the absorption of energy of the system, which in this case is the magnon field. This energy change is manifested in form of radiation emitted by the evolution of the skyrmion.

Now we define $\mathcal{O}_0 = y\nabla_y$, $\mathcal{O}_1 = \mathrm{i}$, $\mathcal{O}_2 = \nabla_{y^1}$, $\mathcal{O}_3 = \nabla_{y^2}$ so $\mathcal{H}(t) = \mathcal{H}_0 + q^{\mathrm{i}}(t)\mathcal{O}_{\mathrm{i}}$ and then

$$D_{\mathrm{i},j}(t) = \mathrm{i}\,\langle[\mathcal{O}_{\mathrm{i}}(t), \mathcal{O}_j(0)]\rangle_0\, H(t), \qquad \tilde{D}_{\mathrm{i},j}(\omega) = \mathrm{i}\int_0^{\infty} \langle[\mathcal{O}_{\mathrm{i}}(t), \mathcal{O}_j(0)]\rangle_0\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t$$

In general the average quantities in quantum mechanics are given by $\langle\mathcal{O}\rangle = \mathrm{tr}(\rho\mathcal{O})$, where ρ is the density matrix of the system. In particular for a thermal bath $\rho = \mathrm{e}^{-\beta\mathcal{H}}$, nevertheless the calculus is cumbersome. For references this section see [20].

A.9. Super Symmetric Theory

This method consists in shift the dynamics toward the dual field $\tilde{f}$, together with the evolution equation. This method has been used in [26] in the context of nonlinear Schrodinger equations, nevertheless our approach is a little bit different. In order to codify well the dynamics to the dual variable we will suppose that f is orthogonal to the kernel of $\mathcal{L}$ this ensure that we can recover $f = A^{-1}\tilde{f}$ if we solve the dual equation.

Let's fix l and, as before, $\tilde{f} = A_l(\lambda)f$ so

$$\begin{aligned}
\partial_t\tilde{f} &= \partial_t(A(\lambda)f) = \partial_t A(\lambda)f + A(\lambda)\partial_t f \\
&= \partial_t A(\lambda)f + A(\lambda)(\mathcal{H}f + h) \\
&= \partial_t A(\lambda)f + A(\lambda)A(\lambda)^{\dagger}A(\lambda)f \\
&= \partial_t A(\lambda)A(\lambda)^{-1}f + A(\lambda)A(\lambda)^{\dagger}A(\lambda)f \\
&= \partial_t A(\lambda)A(\lambda)^{-1}\tilde{f} + \tilde{\mathcal{H}}\tilde{f} \\
&= (\tilde{\mathcal{H}} + \mathcal{V})\tilde{f}
\end{aligned}$$

where $\mathcal{V} = \partial_t A(\lambda)A(\lambda)^{-1}$, and we have used that $A(\lambda)h = 0$ since h is in the kernel of $\mathcal{H}(\lambda)$. So the dual evolution equation is

$$\partial_t\tilde{f} = (\tilde{\mathcal{H}} + \mathcal{V})\tilde{f} \tag{A.30}$$

Then we can compute:

$$\begin{aligned}
\mathcal{V}\tilde{f} &= \partial_t A(\lambda)A(\lambda)^{-1}\tilde{f} \\
&= \dot{\lambda}\partial_\lambda A(\lambda)A(\lambda)^{-1}\tilde{f} \\
&= -\dot{\lambda}\lambda\ \frac{4r^2}{r^2+\lambda^2}A(\lambda)^{-1}\tilde{f}
\end{aligned}$$

Roughly speaking we can say that $\mathcal{V}$ is a perturbation potential where the parameter of perturbation is $\dot{\lambda}\lambda$. Thus we can solve approximately (A.30) using time perturbation theory and expanding in powers of $\dot{\lambda}\lambda$ we have:

$$\left\langle k|\tilde{f}(t)\right\rangle = -\int_{t_0}^{t}\int_{\mathbb{R}^2} \langle k\,|\mathcal{V}|\,k'\rangle\ \mathrm{e}^{(E(k)-E(k'))(s-t_0)}\left\langle k'|\tilde{f}(t)\right\rangle\ \mathrm{d}k'\ \mathrm{d}s$$

where $|k\rangle$ for $k \in \mathbb{R}^2$ is the set of eigenvectors of $\tilde{H}(0)$. For $l = 0$ is interesting to see that $\tilde{H}(0) = -\partial_r^2 - 1/(4r^2)$, hence $\langle r|k\rangle \sim J_0(kr)$ are the Bessel functions.

On the other hand, by hypothesis, $\langle Z_\lambda|f(t)\rangle = 0$ foal all t where $Z_\lambda = \partial_\lambda U$ is an element of the kernel of $\mathcal{H}$. Hence:

$$\begin{aligned}
\langle \partial_t Z_\lambda|f(t)\rangle &= -\langle Z_\lambda|\partial_t f(t)\rangle \quad\Rightarrow\\
\left\langle \dot{\lambda}\partial_\lambda Z_\lambda|f(t)\right\rangle &= -\langle Z_\lambda|\mathcal{H}f + h\rangle \quad\Rightarrow\\
\dot{\lambda}\langle \partial_\lambda Z_\lambda|f(t)\rangle &= -\langle Z_\lambda|(\mathcal{H}f - \partial_t U)\rangle \quad\Rightarrow\\
\dot{\lambda}\langle \partial_\lambda Z_\lambda|f(t)\rangle &= \left\langle Z_\lambda|\ \dot{\lambda}Z_\lambda\right\rangle = \dot{\lambda} \quad\Rightarrow\\
\langle \partial_\lambda Z_\lambda|f(t)\rangle &= 1
\end{aligned}$$

In order to compute $\langle \partial_\lambda Z_\lambda|f(t)\rangle$ we use the previous formulas for the dual $\tilde{f}$:

$$\begin{aligned}
\langle \partial_\lambda Z_\lambda|f(t)\rangle &= \left\langle \partial_\lambda Z_\lambda|A(\lambda)^{-1}\tilde{f}(t)\right\rangle\\
&= \left\langle (A(\lambda)^\dagger)^{-1}\partial_\lambda Z_\lambda|\tilde{f}(t)\right\rangle\\
&= \int_{\mathbb{R}^2} \left\langle (A(\lambda)^\dagger)^{-1}\partial_\lambda Z_\lambda|k\right\rangle \left\langle k|\tilde{f}(t)\right\rangle\ \mathrm{d}k\\
&= \int_{\mathbb{R}^2} \left\langle (A(\lambda)^\dagger)^{-1}\partial_\lambda Z_\lambda|k\right\rangle \left\{-\int_{t_0}^{t}\int_{\mathbb{R}^2} \langle k\,|\mathcal{V}|\,k'\rangle\ \mathrm{e}^{(E(k)-E(k'))(s-t_0)}\left\langle k'|\tilde{f}(t)\right\rangle\ \mathrm{d}k'\ \mathrm{d}s\right\}\ \mathrm{d}k\\
&= \int_{t_0}^{t} \Gamma(s-t_0,\lambda(s))\dot{\lambda}(s)\ \mathrm{d}s
\end{aligned}$$

where $\Gamma(s-t_0,\lambda(s))$ is the function after doing the integration in k and k'. Finally we can write the integro-differential equation for λ:

$$\int_{t_0}^{t} \Gamma(s-t_0,\lambda(s))\dot{\lambda}(s)\ \mathrm{d}s = 1 \tag{A.31}$$

We are interesting in the behavior near to the singularity, so we will take $t_0 \approx T$, So:

$$\Gamma(s-t_0,\lambda(s)) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left\langle (A(\lambda)^\dagger)^{-1}\partial_\lambda Z_\lambda|k\right\rangle \langle k\,|\mathcal{V}|\,k'\rangle\ \mathrm{e}^{(E(k)-E(k'))(s-t_0)}\ \mathrm{d}k'\ \mathrm{d}s\ \mathrm{d}k \tag{A.32}$$

Bibliografía

[1] *Quantum Field Theory of Many-body Systems, from the Origin of Sound to an Origin of Light and Electrons.*

[2] F. G. Mertens A. R. Bishop A. R. Völkel, G. M. Wysin and H. J. Schnitzer. *Collective-variable approach to the dynamics of nonlinear magnetic excitations with application to vortices.* Phys. Rev., 1 November 1994.

[3] Alvaro S. Nunez A. Roldán-Molina and R.A. Duine. *Magnonic black holes.* Phys. Rev. Lett, 8 February 2017.

[4] A.M.KosevichB.A.IvanovA.S.Kovalev. *Magnetic Solitons.* Physics Reports, 1990.

[5] A.Zee. *Quantum Field Theory in a Nutshell.* Princeton University Press, second edition edition, June 2015.

[6] Vasyl V. Kryvonos Boris A.Ivanov, Denis D. Sheka and Franz G. Mertens. *Quantum effects for the two-dimensional soliton in isotropic ferromagnets.* Phys. Rev., 2007.

[7] Jelena Klinovaja Christina Psaroudaki, Silas Hoffman and Daniel Loss. *Quantum Dynamics of Skyrmions in Chiral Magnets.* Phys. Rev., June 2017.

[8] Achim Rosch Christoph Schütte, Junichi Iwasaki and Naoto Nagaosa. *Inertia, diffusion, and dynamics of a driven skyrmion.* Phys. Rev., b 90, 174434 edition, 2014.

[9] Jan Bouwe Van den Berg and J.F.Williams. *(In-)stability of singular equivariant solutions to the Landau–Lifshitz–Gilbert equation.* European Journal of Applied Mathematics, 2009.

[10] Boris A. Ivanov Denis D. Sheka and Franz G. Mertens. *Internal modes and magnon scattering on topological solitons in two-dimensional easy-axis ferromagnets.* PHYSICAL REVIEW, 2001.

[11] PierreRaphaël Frank Merle and Igor Rodnianskic. *Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map.* Comptes Rendus Mathematique, Volume 349, 2011.

[12] Jung Hoon Han. *Skyrmions in Condensed Matter.* Springer, 2017.

[13] Paul Harpes. *Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow.* Calculus of Variations and Partial Differential Equations, June 2004.

[14] Benjamin Krüger Imam Makhfudz and Oleg Tchernyshyov. *Inertia and Chiral Edge Modes of a Skyrmion Magnetic Bubble.* Phys. Rev. Lett, 20 November 2012.

[15] John David Jackson. *Classical Electrodynamics.* Wiley, third edition edition, 2012.

[16] T. Padmanabhan Jayant v. NarIikar. *Gravity, Gauge Theories and Quantum Cosmology.* D.Reidel Publishing Company, 1986.

[17] Manuel del Pino Juan Dávila and Juncheng Wei. *Singularity formation for the two-dimensional harmonic map flow into S^2.* Banff International Research Station for Mathematical Innovation and Discovery, 2017.

[18] Landau and Lifshitz. *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies.* Phys. Z. Sowjetunion, 1935.

[19] Fanghua Lin and Changyou Wang. *The Analysis of Harmonic Maps and Their Heat Flows.* World Scientific Publishing, 2008.

[20] Shi-Zeng Lin. *Dynamics and inertia of a skyrmion in chiral magnets and interfaces: A linear response approach based on magnon excitations.* Phys. Rev., b 96, 014407 edition, 2017.

[21] Eugene M. Chudnovsky Liufei Cai and D. A. Garanin. *Collapse of skyrmions in two-dimensional ferromagnets and antiferromagnets.* Physical review., 24 July 2012.

[22] Shigeyuki Morita. *Geometry of Differential Forms.* American Mathematical Soc., 2001.

[23] Roger Moser. *Partial Regularity for Harmonic Maps and Related Problems.* World Scientific, 2005.

[24] John C. Neu. *Singular Perturbation in the Physical Sciences.* American Mathematical Society, volume 1, number 3 edition, 2015.

[25] Shahar Hadar Ofek Birnholtz and Barak Kol. *Radiation reaction at the level of the action.* International Journal of Modern Physics A, 2014.

[26] Igor Rodnianski and Jacob Sterbenz. *On the formation of singularities in the critical $O(3)$ σ-model.* Annals of Mathematics, July 2010.

[27] Sarah Schroeter and Markus Garst. *Scattering of high-energy magnons off a magnetic skyrmion.* Low Temperature Physics, July 22, 2018.

[28] Denis D. Sheka* and Gary M. Wysin Franz G. Mertens Ivan A. Yastremsky, Boris A. Ivanov. *Amplitudes for magnon scattering by vortices in two-dimensional weakly easy-plane ferromagnets.* PHYSICAL REVIEW, 2004.

[29] Michael Struwe. *Variational Methods. Applications to Nonlinear Partial Differential*

Equations and Hamiltonian Systems. Springer, 4nd edition, 2008.

[30] A. Bauer M. Halder M. Wagner C. Franz C. Pfleiderer K. Everschor M. Garst T. Schulz, R. Ritz and A. Rosch. *Emergent electrodynamics of skyrmions in a chiral magnet.* Nature Physics, 2012.

[31] Ki-Suk Lee Takayuki Shiino, Kab-Jin Kim and Byong-Guk Park. *Inertia-driven resonant excitation of a magnetic skyrmion.* Scientific Reportsvolume, 2nd edition, 2017.

[32] Alberto D. Verga. *Skyrmion to ferromagnetic state transition: A description of the topological change as a finite-time singularity in the skyrmion dynamics.* Phys. Rev., November 2014.

[33] Ulrich K. Rößler Jeroen van den Brink Volodymyr P. Kravchuk, Denis D. Sheka and Yuri Gaididei. *Spin eigen-excitations of the magnetic skyrmion and problem of the effective mass.* Phys. Rev., 7 February 2018.

[34] Motohiko Ezawa Yan Zhou Xichao Zhang. *Magnetic skyrmion logic gates: conversion, duplication and merging of skyrmions.* Nature, scientific reports volume 5, article number: 9400 edition, 2015.

[35] Jin-Hong Park Yun-Tak Oh, Hyunyong Lee and Jung Hoon Han. *Dynamics of magnon fluid in Dzyaloshinskii-Moriya magnet and its manifestation in magnon-Skyrmion scattering.* Phys. Rev., 31 March 2015.

[36] A. Bohm A. Mostafazadeh H. Koizumi Q. Niu 1. Zwanziger. *The Geometric Phase in Quantum Systems.* Springer, 2003.

[37] Christoph Schu tte and Markus Garst. *Magnon-skyrmion scattering in chiral magnets.* PHYSICAL REVIEW, May 2014.