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Implicit equations for thermoelastic bodies

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ABSTRACT

In this paper we generalize the recent implicit models that have been put into place to describe the elastic response of bodies when thermal effects come into play. The implicit constitutive relations for thermoelastic response presented here provide a very natural way to overcome a serious problem associated with the celebrated model due to Fourier, namely infinite speed of the propagation of temperature. We also study some boundary value problems within the context of the implicit equations that we have developed. We carry out a linearization based on the classical assumption that the displacement gradient is small and obtain constitutive relations that allow the linearized strain to be a non-linear function of the stress and temperature.

1. Introduction

The celebrated eponymous equation governing the conduction of heat, that is given the status of a 'law', namely Fourier's law, is merely an approximation which in fact predicts erroneously that temperature propagates with infinite velocity. In view of the fact that the propagation has finite speed, there has been considerable interest in developing a more meaningful equation for the conduction of heat. A pioneering study in this direction is that by Cattaneo [1]. Later, Lord and Shulman [2] studied the thermoelastic response of solids wherein they sought to ensure finite wave speeds for the propagation of temperature. These early works have been followed by papers too numerous to detail, provide minor improvements or generalization to the response of viscoelastic bodies and bodies described by higher gradient theories. The thermoelastic response studied by Lord and Shulman [2] as well as the others such as Ezzat [3], consider the response of Cauchy elastic bodies (or the sub-class of Green elastic bodies) with thermal effects being taken into consideration. In this paper, we study the response of a new class of elastic bodies that are not necessarily Cauchy elastic bodies, being described by implicit constitutive relationship between the stress and the deformation gradient, when thermal effects are included. At the outset, we would like to make a case for why the new class of implicit constitutive relations to describe the response of elastic bodies is worth studying in detail. As discussed in details by Rajagopal [4–9], there are several reasons for employing a theory wherein one has an implicit relationship between the deformation gradient and the Cauchy stress. From a philosophical standpoint the theory is in keeping with the demands of causality as the deformation is a consequence of the applied traction and the resulting stress field. Such an approach also allows for the material moduli to depend on for instance the mean value of the stress, namely the mechanical pressure, a feature exhibited by many polymeric solids (see Rajagopal and Saccomandi [10]). Furthermore, it allows the strain to have a nonlinear relationship with regard to the stress even in what would be considered the 'small strain' regime, a response characteristic of many intermetallic alloys (see Rajagopal [9], Devendiran et al. [11]). Also, a Cauchy elastic body cannot describe an elastic body which exhibits limiting strains, while a fully implicit constitutive relation or a constitutive expression wherein the Cauchy-Green strain as a function of the stress models can describe such constrained response (see [4]). Moreover, while using the linearized version of such implicit theories one does not necessarily have to face glaring inconsistencies such as those encountered while studying the state of strain at a crack tip within the context of the linearized theory of elasticity. As Cauchy elastic bodies are a very special sub-class of the class of bodies characterized by the implicit constitutive relation between the stress and the deformation gradient, the classical results of thermoelasticity are recovered when attention is directed to the sub-class of Cauchy elastic bodies.

In addition to the issue of ensuring finite speed for the propagation of temperature, we also consider the counterpart to the celebrated Oberbeck-Boussinesq equations (see Oberbeck [12,13] and Boussinesq [14]) that has been developed to describe the response of fluids that can only undergo isochoric motions in isothermal processes, but can undergo compression or expansion in non-isothermal processes. The Oberbeck-Boussinesq approximation is one of the most useful approximations in fluid mechanics, and is employed to study problems in

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astrophysical and geophysical fluid dynamics as well as several other technological applications. It is important to bear in mind that the Oberbeck-Boussinesq approximation does not stem from retaining terms in a proper perturbation expansion but in fact includes terms of different orders in the same equation. A detailed discussion of the status of the Oberbeck-Boussinesq approximation within the context of the full Navier-Stokes-Fourier theory can be found in the paper by Rajagopal et al. [15]. The Oberbeck-Boussinesq approximation has been extended for various other constitutive equations governing the response of fluids (see [16,17] and the references cited therein). The basic approach to the problem is the assumption that the deformation gradient meets the restriction that the motion is isochoric in isothermal processes.

The counterpart of the above problem within the context of classical nonlinear thermoelasticity is however not straightforward. It is well known that the above constraint leads to physically unrealistic situations such as that of instability of wave propagation (see Chadwick and Scott [18], Scott [19,20], Leslie and Scott [21], Scott [22]). Since Cauchy elastic bodies are a sub-class of the general class of implicit elastic bodies, and also overlap with bodies wherein the Cauchy-Green strain is an explicit function of the stress when the relationship is invertible, for such models the physically unrealistic situation will persist. For models wherein the relationship between the Cauchy-Green strain and the Cauchy stress is not invertible we do not know if this problem will recur. This is the object of an ongoing investigation. Here, we look at the problem wherein the constitutive relation is a non-linear relationship between the linearized strain and the Cauchy stress in which case we do not have the possibility of inverting the nonlinear expression for the linearized strain in terms of the stress. It is possible that even this class of models might exhibit the physically unacceptable behaviour observed by Scott and co-workers in the case of Cauchy elastic bodies. This is also being looked into in the ongoing investigation mentioned above.

The organization of the paper is as follows. In the next section we provide a brief introduction of the kinematics and the basic equations (see Section 2) and this is followed in Section 3 where the implicit constitutive relation between the Cauchy stress, the Cauchy-Green tensor, heat-flux vector and temperature is proposed to describe the response of a thermoelastic body. The first, and a specific form of the second, laws of thermodynamics are introduced, and a generalization of the Fourier model for heat transfer by conduction is proposed. In Section 4 the special case of isotropic relations is considered, and some subclasses of constitutive relations and equations are derived from that, assuming for some cases that some of the variables are small enough; of particular interest is the case of assuming small gradient of the displacement field. In Section 5 several simple boundary value problems are analyzed, for the special case of the constitutive equation obtained assuming that the gradient of the displacement field is small. In Section 6 the constraints of incompressibility and inextensibility are studied for two of the subclasses of constitutive equations proposed in Section 4. Finally, in Section 7 concluding remarks are made.

2. Basic equations

A point in a body $\mathcal{B}$ is denoted X and in the reference configuration the point occupies the position $\mathbf{X} = \boldsymbol{\kappa}_{\rm r}(X)$. The reference configuration is denoted $\kappa_{\rm r}(\mathcal{B})$. In the current configuration the position of the point is denoted $\mathbf{x}$, and it is assumed that there exists a one-to-one mapping $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The current configuration is denoted $\kappa_{\rm t}(\mathcal{B})$.

The displacement field, the deformation gradient, the right Cauchy-Green stretch tensor, the Lagrange strain and the linearized strain tensors are defined, respectively, as:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \nabla_{\rm r}\mathbf{x}, \quad \mathbf{C} = \mathbf{F}^{\rm T}\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}),$$
$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\rm T}), \tag{1}$$

where we assume $J = \det\mathbf{F} > 0$, $\nabla_{\rm r}$ and ∇ are the gradient operators in the reference and the current configuration, respectively.

The Cauchy stress tensor is denoted $\mathbf{T}$ and it satisfies the equation of motion

$$\rho\ddot{\mathbf{x}} = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \tag{2}$$

where ρ is the density of the body and $\mathbf{b}$ represents the body forces in the current configuration, and where we use the notation $(\dot{\ })$ for the time derivative.

The second Piola-Kirchhoff stress tensor $\mathbf{S}$ is defined as

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\rm T}. \tag{3}$$

More details about kinematics and the above definitions can be found, for example, in [23].

3. Implicit relations for thermoelastic bodies

We will be interested in studying some subclasses of the general implicit relation for a thermoelastic body (see [4,5] for the original formulation for elastic bodies)

$$\mathfrak{F}(\mathbf{S}, \mathbf{E}, \theta) = \mathbf{0}, \tag{4}$$

where θ is the absolute temperature and $\mathfrak{F}$ is a second order tensor relation. Relation (4) would be a generalization of the classical explicit model $\mathbf{S} = \mathfrak{K}(\mathbf{E}, \theta)$, where now in (4) $\mathbf{S}$ cannot be obtained in general explicitly in terms of $\mathbf{E}$. Additionally, we have added θ as one of the fundamental variables for the heat transfer problem.

The first law of thermodynamics in the reference configuration is (see, for example, [24])

$$\rho_{\rm r}\dot{\epsilon} = w + \mathrm{Div}\mathbf{h}_{\rm r} + \rho_{\rm r}\mathrm{r}, \tag{5}$$

where $\mathbf{h}_{\rm r}$ is the heat flux in the reference configuration, ϵ is the internal energy, $\rho_{\rm r}$ is the density in the reference configuration, $w = \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}})$ is the rate of work and r the rate of heat generated internally by the body.

The dissipation d is defined as (see, for example, [25])

$$d = \theta\dot{\eta} - \dot{\epsilon} + \frac{w}{\rho_{\rm r}}, \tag{6}$$

where this dissipation must satisfy the inequality

$$d \geq 0. \tag{7}$$

The heat flux must satisfy the inequality

$$-\mathbf{h}_{\rm r}\cdot\boldsymbol{\gamma} \geq 0, \quad \text{where} \quad \boldsymbol{\gamma} = \theta\nabla_{\rm r}\left(\frac{1}{\theta}\right). \tag{8}$$

Adding these two inequalities (7), (8) we obtain

$$\rho_{\rm r}\left(\theta\dot{\eta} - \dot{\epsilon} + \frac{1}{\rho_{\rm r}}w\right) - \mathbf{h}_{\rm r}\cdot\boldsymbol{\gamma} \geq 0. \tag{9}$$

Let us introduce the Helmotz potential ψ, which we assume is of the form

$$\psi = \psi(\mathbf{S}, \mathbf{E}, \theta). \tag{10}$$

The relation between the Helmholtz potential and the internal energy is

$$\psi = \epsilon - \theta\eta. \tag{11}$$

From (11) we have $\dot{\epsilon} = \dot{\psi} + \dot{\theta}\eta + \theta\dot{\eta}$ and replacing in (9) we obtain

$$-\rho_{\rm r}(\dot{\psi} + \dot{\theta}\eta) + w - \mathbf{h}_{\rm r}\cdot\boldsymbol{\gamma} \geq 0. \tag{12}$$

For $\dot{\psi}$ we have (in index notation and Cartesian co-ordinates)

$$\dot{\psi} = \frac{\partial \psi}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} + \frac{\partial \psi}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta} + \frac{\partial \psi}{\partial \theta}\dot{\theta}, \tag{13}$$

where the repetition of the index here and elsewhere means summation from 1 to 3 unless stated otherwise. Using (13) and (11) in (12) considering the above calculations we obtain

$$-\frac{\partial \psi}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} - \frac{\partial \psi}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta} - \frac{\partial \psi}{\partial \theta}\dot{\theta} - \dot{\theta}\eta + \frac{1}{\rho_{\mathrm{r}}}S_{\alpha\beta}\dot{E}_{\alpha\beta} - h_{\mathrm{r}_\alpha}\gamma_\alpha \geq 0, \tag{14}$$

which is satisfied if

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{15}$$

and

$$-\frac{\partial \psi}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} - \frac{\partial \psi}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta} + \frac{1}{\rho_{\mathrm{r}}}S_{\alpha\beta}\dot{E}_{\alpha\beta} - h_{\mathrm{r}_\alpha}\gamma_\alpha \geq 0. \tag{16}$$

If we take the derivative of $\mathfrak{F}$ in time we have

$$\frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial S_{\alpha\beta}}\dot{S}_{\alpha\beta} + \frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial E_{\alpha\beta}}\dot{E}_{\alpha\beta} + \frac{\partial \mathfrak{F}_{\gamma\zeta}}{\partial \theta}\dot{\theta} = 0. \tag{17}$$

The relation $\mathfrak{F}$ and the function ψ must satisfy the above two conditions for any $\dot{\mathbf{S}}$, $\dot{\mathbf{E}}$ and $\dot{\theta}$.

Using (11) and (13) in (5) (considering (15)) we have (in index notation and Cartesian co-ordinates)

$$\rho_{\mathrm{r}}\left[\left(\frac{\partial \psi}{\partial S_{\alpha\beta}} - \theta\frac{\partial^2 \psi}{\partial \theta \partial S_{\alpha\beta}}\right)\dot{S}_{\alpha\beta} + \left(\frac{\partial \psi}{\partial E_{\alpha\beta}} - \theta\frac{\partial^2 \psi}{\partial \theta \partial E_{\alpha\beta}}\right)\dot{E}_{\alpha\beta} - \theta\frac{\partial^2 \psi}{\partial \theta^2}\dot{\theta}\right]$$
$$-S_{\alpha\beta}\dot{E}_{\alpha\beta} = \frac{\partial h_{\mathrm{r}_\alpha}}{\partial X_\alpha} + \rho_{\mathrm{r}} r. \tag{18}$$

In this work we assume that $\mathbf{h}_{\mathrm{r}}$ can be found from the following vector implicit relation

$$\mathfrak{g}(\mathbf{S}, \mathbf{E}, \mathbf{h}_{\mathrm{r}}, \dot{\mathbf{h}}_{\mathrm{r}}, \nabla_{\mathrm{r}}\theta, \theta) = \mathbf{0}, \tag{19}$$

As it will be shown in Section 4.1.1 this implicit vector constitutive relation can be seen as a generalization of some models presented in the literature [1,2,26], which are of the form $\mathbf{h}_{\mathrm{r}} + c_o\dot{\mathbf{h}}_{\mathrm{r}} = -k\nabla_{\mathrm{r}}\theta$, and which have been proposed as replacements of the classical Fourier model $\mathbf{h}_{\mathrm{r}} = -k\nabla_{\mathrm{r}}\theta$ for heat transfer, in order to address some problems that such classical model has with the propagation of a heat wave, see, for example, the Introduction in [26].

In the following sections we study the case when $\mathfrak{F}$, $\mathfrak{g}$ and ψ are isotropic functions and relations, and we consider some special cases, where the strains, the stresses, the temperature, etc. are 'small' in comparison with some reference values. In order to do so, let us assume that there exist characteristic or reference values for the stress σ_o, the heat flux h_o, the rate of the heat flux h_{o_v} the temperature gradient $\aleph$ and temperature θ_o, such that we can define the dimensionless quantities

$$\frac{1}{\sigma_o}\mathbf{S}, \quad \frac{1}{h_o}\mathbf{h}_{\mathrm{r}}, \quad \frac{1}{h_{o_v}}\dot{\mathbf{h}}_{\mathrm{r}}, \quad \frac{1}{\aleph}\nabla_{\mathrm{r}}\theta, \quad \frac{\theta}{\theta_o}. \tag{20}$$

In the following sections we do not use a different notation for these dimensionless variables.

4. Isotropic bodies

The following is the list of invariants for an isotropic function or relation that depends on the tensors $\mathbf{S}$, $\mathbf{E}$, the vectors $\mathbf{h}_{\mathrm{r}}$, $\dot{\mathbf{h}}_{\mathrm{r}}$, $\nabla_{\mathrm{r}}\theta$ and the scalar field θ (see [27,28]).[1].

$$I_1 = \mathrm{tr}\mathbf{S}, \quad I_2 = \frac{1}{2}\mathrm{tr}(\mathbf{S}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}(\mathbf{S}^3), \tag{21}$$

$$I_4 = \mathrm{tr}\mathbf{E}, \quad I_5 = \frac{1}{2}\mathrm{tr}(\mathbf{E}^2), \quad I_6 = \frac{1}{3}\mathrm{tr}(\mathbf{E}^3), \tag{22}$$

$$I_7 = \mathbf{h}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}}, \quad I_8 = \dot{\mathbf{h}}_{\mathrm{r}}\cdot\dot{\mathbf{h}}_{\mathrm{r}}, \quad I_9 = \nabla_{\mathrm{r}}\theta\cdot\nabla_{\mathrm{r}}\theta, \tag{23}$$

$$I_{10} = \mathrm{tr}(\mathbf{SE}), \quad I_{11} = \mathrm{tr}(\mathbf{S}^2\mathbf{E}), \quad I_{12} = \mathrm{tr}(\mathbf{SE}^2), \quad I_{13} = \mathrm{tr}(\mathbf{S}^2\mathbf{E}^2) \tag{24}$$

$$I_{14} = \mathbf{h}_{\mathrm{r}}\cdot(\mathbf{S}\mathbf{h}_{\mathrm{r}}), \quad I_{15} = \mathbf{h}_{\mathrm{r}}\cdot(\mathbf{S}^2\mathbf{h}_{\mathrm{r}}), \quad I_{16} = \mathbf{h}_{\mathrm{r}}\cdot(\mathbf{E}\mathbf{h}_{\mathrm{r}}), \quad I_{17} = \mathbf{h}_{\mathrm{r}}\cdot(\mathbf{E}^2\mathbf{h}_{\mathrm{r}}), \tag{25}$$

$$I_{18} = \dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}\dot{\mathbf{h}}_{\mathrm{r}}), \quad I_{19} = \dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}^2\dot{\mathbf{h}}_{\mathrm{r}}), \quad I_{20} = \dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}\dot{\mathbf{h}}_{\mathrm{r}}), \quad I_{21} = \dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}^2\dot{\mathbf{h}}_{\mathrm{r}}), \tag{26}$$

$$I_{22} = \nabla_{\mathrm{r}}\theta\cdot(\mathbf{S}\nabla_{\mathrm{r}}\theta), \quad I_{23} = \nabla_{\mathrm{r}}\theta\cdot(\mathbf{S}^2\nabla_{\mathrm{r}}\theta),$$
$$I_{24} = \nabla_{\mathrm{r}}\theta\cdot(\mathbf{E}\nabla_{\mathrm{r}}\theta), \quad I_{25} = \nabla_{\mathrm{r}}\theta\cdot(\mathbf{E}^2\nabla_{\mathrm{r}}\theta), \tag{27}$$

$$I_{26} = (\mathbf{h}_{\mathrm{r}}\cdot\dot{\mathbf{h}}_{\mathrm{r}})^2, \quad I_{27} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)^2, \quad I_{28} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)^2, \tag{28}$$

$$I_{29} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}})\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}\mathbf{h}_{\mathrm{r}}), \quad I_{30} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}})\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}^2\mathbf{h}_{\mathrm{r}}), \tag{29}$$

$$I_{31} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}})\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}\mathbf{h}_{\mathrm{r}}), \quad I_{32} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}})\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}^2\mathbf{h}_{\mathrm{r}}), \tag{30}$$

$$I_{33} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\mathbf{h}_{\mathrm{r}}\cdot(\mathbf{S}\nabla_{\mathrm{r}}\theta), \quad I_{34} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\mathbf{h}_{\mathrm{r}}\cdot(\mathbf{S}^2\nabla_{\mathrm{r}}\theta), \tag{31}$$

$$I_{35} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\mathbf{h}_{\mathrm{r}}\cdot(\mathbf{E}\nabla_{\mathrm{r}}\theta), \quad I_{36} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\mathbf{h}_{\mathrm{r}}\cdot(\mathbf{E}^2\nabla_{\mathrm{r}}\theta), \tag{32}$$

$$I_{37} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}\nabla_{\mathrm{r}}\theta), \quad I_{38} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{S}^2\nabla_{\mathrm{r}}\theta), \tag{33}$$

$$I_{39} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}\nabla_{\mathrm{r}}\theta), \quad I_{40} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{E}^2\nabla_{\mathrm{r}}\theta), \tag{34}$$

$$I_{41} = \mathbf{h}_{\mathrm{r}}\cdot(\mathbf{SE}\mathbf{h}_{\mathrm{r}}), \quad I_{42} = \dot{\mathbf{h}}_{\mathrm{r}}\cdot(\mathbf{SE}\dot{\mathbf{h}}_{\mathrm{r}}), \quad I_{43} = \nabla_{\mathrm{r}}\theta\cdot(\mathbf{SE}\nabla_{\mathrm{r}}\theta), \tag{35}$$

$$I_{44} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\mathbf{h}_{\mathrm{r}})\dot{\mathbf{h}}_{\mathrm{r}}\cdot[(\mathbf{SE} - \mathbf{ES})\mathbf{h}_{\mathrm{r}}], \quad I_{45} = (\mathbf{h}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\mathbf{h}_{\mathrm{r}}\cdot[(\mathbf{SE} - \mathbf{ES})\nabla_{\mathrm{r}}\theta], \tag{36}$$

$$I_{46} = (\dot{\mathbf{h}}_{\mathrm{r}}\cdot\nabla_{\mathrm{r}}\theta)\dot{\mathbf{h}}_{\mathrm{r}}\cdot[(\mathbf{SE} - \mathbf{ES})\nabla_{\mathrm{r}}\theta], \quad I_{47} = \theta. \tag{37}$$

Considering the above list of invariants, in the case of an isotropic relation we obtain for $\mathfrak{F}$ (see [27,28])

$$\begin{aligned}\alpha_0\mathbf{I} \;&+\; \alpha_1\mathbf{S} + \alpha_2\mathbf{S}^2 + \alpha_3\mathbf{E} + \alpha_4\mathbf{E}^2 + \alpha_5(\mathbf{ES} + \mathbf{SE})\\ &+\; \alpha_6(\mathbf{E}^2\mathbf{S} + \mathbf{SE}^2) + \alpha_7(\mathbf{S}^2\mathbf{E} + \mathbf{ES}^2)\\ &+\; \alpha_8(\mathbf{S}^2\mathbf{E}^2 + \mathbf{E}^2\mathbf{S}^2) = \mathbf{0},\end{aligned} \tag{38}$$

where the scalar functions $\alpha_k = \alpha_k(I_j)$, $k = 0, 1, 2, \ldots, 8$ depend on the invariants I_l, $l = 1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 47$.

In the case of the vector relation (19), for the sake of mathematical simplicity we assume that there exists a scalar function[2] $\Xi = \Xi(\mathbf{S}, \mathbf{E}, \mathbf{h}_{\mathrm{r}}, \dot{\mathbf{h}}_{\mathrm{r}}, \nabla_{\mathrm{r}}\theta, \theta)$ such that

$$\mathfrak{g}(\mathbf{S}, \mathbf{E}, \mathbf{h}_{\mathrm{r}}, \dot{\mathbf{h}}_{\mathrm{r}}, \nabla_{\mathrm{r}}\theta, \theta) = \frac{\partial \Xi}{\partial \mathbf{h}_{\mathrm{r}}} + \frac{\partial \Xi}{\partial \dot{\mathbf{h}}_{\mathrm{r}}} + \frac{\partial \Xi}{\partial \nabla_{\mathrm{r}}\theta}. \tag{39}$$

On considering the invariants (21)–(37) from (39) we obtain

$$\begin{aligned}&\beta_0\nabla_{\mathrm{r}}\theta + \beta_1\mathbf{S}\nabla_{\mathrm{r}}\theta + \beta_2\mathbf{S}^2\nabla_{\mathrm{r}}\theta + \beta_3\mathbf{E}\nabla_{\mathrm{r}}\theta + \beta_4\mathbf{E}^2\nabla_{\mathrm{r}}\theta + \beta_5\mathbf{h}_{\mathrm{r}} + \beta_6\mathbf{S}\mathbf{h}_{\mathrm{r}} + \beta_7\mathbf{S}^2\mathbf{h}_{\mathrm{r}}\\ &+\beta_8\mathbf{E}\mathbf{h}_{\mathrm{r}} + \beta_9\mathbf{E}^2\mathbf{h}_{\mathrm{r}} + \beta_{10}\dot{\mathbf{h}}_{\mathrm{r}} + \beta_{11}\mathbf{S}\dot{\mathbf{h}}_{\mathrm{r}} + \beta_{12}\mathbf{S}^2\dot{\mathbf{h}}_{\mathrm{r}} + \beta_{13}\mathbf{E}\dot{\mathbf{h}}_{\mathrm{r}} + \beta_{14}\mathbf{E}^2\dot{\mathbf{h}}_{\mathrm{r}}\beta_{15}(\mathbf{SE} + \mathbf{ES})\nabla_{\mathrm{r}}\theta\\ &+\beta_{16}(\mathbf{SE} + \mathbf{ES})\mathbf{h}_{\mathrm{r}} + \beta_{17}(\mathbf{SE} + \mathbf{ES})\dot{\mathbf{h}}_{\mathrm{r}} = \mathbf{0},\end{aligned} \tag{40}$$

where the functions β_j, $j = 0, 1, 2, \ldots, 17$ depend on the invariants given in (21)–(37) and are given in terms of the derivatives of Ξ, which as in the previous case for brevity are not shown here.

[1] In a series of recent works Shariff (see, for example, [29,30]) has proposed some new classes of invariants, which have clearer physical meanings than the classical invariants by Rivlin and Spencer (see, for example, [27,28]). In the present work we use the classical invariants instead such new invariants, as the invariants used in [27,28] have been used commonly for a long time, and suffice for the purpose of illustrating our ideas.

[2] Here it is necessary to indicate that the function Ξ does not necessarily have any physical meaning, and that the representation (39) has been assumed only for the sake of simplicity, considering that explicit expressions for an implicit vector function, which depends on two tensor and three vector functions, have not been found in the literature on invariants.

4.1. The sub-class $|\nabla_r \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$

In this case let us assume that $|\nabla_r \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, therefore we have the approximations $\nabla_r \mathbf{u} \approx \nabla \mathbf{u}$, $\mathbf{E} \approx \boldsymbol{\varepsilon}$, $\mathbf{S} \approx \mathbf{T}$, $\nabla_r \theta \approx \nabla \theta$, $\mathbf{h}_r \approx \mathbf{h}$ and $\dot{\mathbf{h}}_r \approx \dot{\mathbf{h}}$, where $\nabla$is the gradient with respect to the current configuration. In this case we have also the approximation

$$\alpha_i \approx \alpha_i(\mathbf{T}, \mathbf{0}, \theta) + \left.\frac{\partial \alpha_i}{\partial E_{\alpha\beta}}\right|_{(\mathbf{T},\mathbf{0},\theta)} \varepsilon_{\alpha\beta}, \tag{41}$$

and from (38) neglecting terms of order δ^r, $r \geq 2$, after some manipulations we obtain the following equation

$$\mathfrak{A}_{ijkl}(\mathbf{T}, \theta)\varepsilon_{kl} = \mathfrak{G}_{ij}(\mathbf{T}, \theta), \tag{42}$$

where $\mathfrak{A}(\mathbf{T}, \theta)$ and $\mathfrak{G}(\mathbf{T}, \theta)$ are a fourth order tensor and a second order tensor functions, respectively. If $\mathfrak{A}$ is invertible, then (42) can be expressed in general in the form

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}, \theta), \tag{43}$$

which in the case $\mathfrak{f}$ is an isotropic function becomes

$$\boldsymbol{\varepsilon} = \omega_0 \mathbf{I} + \omega_1 \mathbf{T} + \omega_2 \mathbf{T}^2, \tag{44}$$

where the functions ω_i, $i = 0, 1, 2$ depend on the invariants[3] I_1, I_2, I_3 and I_{47} from (21), $(37)_2$.

Since in this case $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}, \theta)$ then in (19) the relation $\mathfrak{g}$ does not depend directly on $\mathbf{E} \approx \boldsymbol{\varepsilon}$, therefore (40) becomes

$$\xi_0 \nabla\theta + \xi_1 \mathbf{T}\nabla\theta + \xi_2 \mathbf{T}^2\nabla\theta + \xi_3 \mathbf{h} + \xi_4 \mathbf{T}\mathbf{h} + \xi_5 \mathbf{T}^2\mathbf{h} + \xi_6 \dot{\mathbf{h}} + \xi_7 \mathbf{T}\dot{\mathbf{h}} + \xi_5 \mathbf{T}^2\dot{\mathbf{h}} = \mathbf{0}, \tag{45}$$

where the functions ξ_i, $i = 0, 1, \ldots, 5$ depend on the invariants I_i, i= 1, 2, 3, 7, 8, 9, 14, 15, 18, 19, 22, 23, 26–30, 33, 34, 37, 38 and 47 from (21)–(37) (replacing $\mathbf{S}$, $\mathbf{h}_r$, $\nabla_r\theta$ and $\dot{\mathbf{h}}_r$ by $\mathbf{T}$, $\mathbf{h}$, $\nabla\theta$ and $\dot{\mathbf{h}}$, respectively).

A special class of constitutive relation is obtained from (43), if we assume that ω_i, $i = 0, 1, 2$ in (44) are expressed in terms of a scalar function $\Pi = \Pi(\mathbf{T}, \theta) = \Pi(I_1, I_2, I_3, \theta)$ as

$$\boldsymbol{\varepsilon} = \frac{\partial \Pi}{\partial \mathbf{T}} = \Pi_1 \mathbf{I} + \Pi_2 \mathbf{T} + \Pi_3 \mathbf{T}^2, \tag{46}$$

where $\Pi_j = \frac{\partial \Pi}{\partial I_j}$, $j = 1, 2, 3$.

The constitutive Eq. (43) can be very important in its own right, as a generalization of $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T})$, which as indicated, for example, in [9], could be a very interesting model to describe the behaviour of materials such as rock, concrete, gum metal, and also in the fracture analysis of brittle bodies.

4.1.1. The subclass wherein $|\mathbf{h}| \sim O(\delta)$, $\delta \ll 1$

In this case let us assume that $|\mathbf{h}| \sim O(\delta)$, $\delta \ll 1$, then from (45) following a procedure similar to the one considered in the previous section we obtain a relation of the form

$$\mathbf{h} = -\mathfrak{j}(\mathbf{T}, \dot{\mathbf{h}}, \nabla\theta, \theta), \tag{47}$$

which in the case that $\mathfrak{j}$ is an isotropic function becomes

$$\mathbf{h} = -(\zeta_0 \nabla\theta + \zeta_1 \mathbf{T}\nabla\theta + \zeta_2 \mathbf{T}^2\nabla\theta + \zeta_3 \dot{\mathbf{h}} + \zeta_4 \mathbf{T}\dot{\mathbf{h}} + \zeta_5 \mathbf{T}^2\dot{\mathbf{h}}), \tag{48}$$

where ζ_i, $i = 0, 1, 2$ depend on the invariants I_k, k= 1, 2, 3, 8, 9, 18, 19, 22, 23, 28, 37, 38, 47 and from (8) this function must satisfy the inequality for all $\mathbf{T}$, $\dot{\mathbf{h}}$, $\nabla\theta$, θ

$$-(\zeta_0 \nabla\theta + \zeta_1 \mathbf{T}\nabla\theta + \zeta_2 \mathbf{T}^2\nabla\theta + \zeta_3 \dot{\mathbf{h}} + \zeta_4 \mathbf{T}\dot{\mathbf{h}} + \zeta_5 \mathbf{T}^2\dot{\mathbf{h}})\cdot\nabla\left(\frac{1}{\theta}\right) \geq 0. \tag{49}$$

An important and interesting special case can be obtained from (48), assuming that $\zeta_4 = 0$, $\zeta_5 = 0$ and that ζ_i, $i = 0, 1, 2, 3$ do not depend on $\dot{\mathbf{h}}$ and $\nabla\theta$. From (48) we obtain

$$\mathbf{h} + \zeta_4 \dot{\mathbf{h}} = -(\zeta_0 \nabla\theta + \zeta_1 \mathbf{T}\nabla\theta + \zeta_2 \mathbf{T}^2\nabla\theta). \tag{50}$$

In [2,26] some models have been presented to study the heat transfer for conduction, where unlike the Fourier's model, the speed of propagation of heat waves is finite. One of such models (see, for example, [26]) can be derived as a special case from (50) if one assumes in (50) that $\zeta_1 = \zeta_2 = 0$ and ζ_0 does not depend on $\mathbf{T}$; in such a case (50) becomes $\mathbf{h} + \zeta_4 \dot{\mathbf{h}} = -\zeta_0 \nabla\theta$ that is the model proposed, for example, in [1]. The Fourier's model appears if we further assume that $\zeta_4 = 0$.

4.1.2. The subclass wherein $|\nabla\theta| \sim O(\delta)$, $\delta \ll 1$

Let us study here the alternative situation where $|\nabla\theta| \sim O(\delta)$, $\delta \ll 1$ and $|\mathbf{h}|$ can be arbitrarily large. From (45) it is possible to show that that relation becomes

$$\nabla\theta = \mathfrak{p}(\mathbf{T}, \mathbf{h}, \dot{\mathbf{h}}, \theta), \tag{51}$$

which in the case $\mathfrak{p}$ is an isotropic function becomes

$$\nabla\theta = \iota_0 \mathbf{h} + \iota_1 \mathbf{T}\mathbf{h} + \iota_2 \mathbf{T}^2\mathbf{h} + \iota_3 \dot{\mathbf{h}} + \iota_4 \mathbf{T}\dot{\mathbf{h}} + \iota_5 \mathbf{T}^2\dot{\mathbf{h}}, \tag{52}$$

where ι_i, $i = 0, 1, 2, \ldots, 6$ are scalar functions that depends on the invariants I_k, k= 1, 2, 3, 7, 8, 14, 15, 18, 19, 26, 29, 30 and 47 from (21)–(37).

4.2. The sub-class $|\mathbf{S}| \sim O(\delta)$, $\delta \ll 1$

In this section we want to obtain the classical constitutive relations for a nonlinear thermoelastic body in terms of the strains, starting from the general implicit relations (38). If $|\mathbf{S}| \sim O(\delta)$, $\delta \ll 1$ and $|\nabla_r \mathbf{u}|$ is arbitrarily large, it is easy to show (following the same procedure as presented as in Section 4.1) that from (38) we obtain a relation of the form

$$\mathbf{S} = \mathfrak{K}(\mathbf{E}, \theta), \tag{53}$$

where if $\mathfrak{K}$ is an isotropic function it becomes

$$\mathbf{S} = \varpi_0 \mathbf{I} + \varpi_1 \mathbf{E} + \varpi_2 \mathbf{E}^2, \tag{54}$$

where the scalar functions ϖ_i, $i = 0, 1, 2$ depend on the invariants I_4, I_5, I_6, I_{47} from (21)–(37). Eq. (53) is the classical constitutive equations for a nonlinear thermoelastic body, see, for example, Equation 96b.26 of [31].

Taking into account the previous considerations, from (19) $\mathfrak{g}$ would depend on $\mathbf{E}$, $\nabla_r\theta$, $\mathbf{h}_r$, $\dot{\mathbf{h}}_r$ and θ, and (40) becomes

$$\nu_0 \nabla_r\theta + \nu_1 \mathbf{E}\nabla_r\theta + \nu_2 \mathbf{E}^2\nabla_r\theta + \nu_3 \mathbf{h}_r + \nu_4 \mathbf{E}\mathbf{h}_r + \nu_5 \mathbf{E}^2\mathbf{h}_r + \nu_5 \dot{\mathbf{h}}_r + \nu_4 \mathbf{E}\dot{\mathbf{h}}_r + \nu_5 \mathbf{E}^2\dot{\mathbf{h}}_r = \mathbf{0}, \tag{55}$$

where the scalar functions ν_i depend on the invariants I_i, i= 4–9, 16, 17, 20, 21, 24–28, 31, 32, 35, 36, 39 and 40 from (21)–(37).

If we assume again that $|\mathbf{h}_r| \sim O(\delta)$, $\delta \ll 1$, them from (55) we would obtain a relation of the form

$$\mathbf{h}_r = -\mathfrak{q}(\mathbf{E}, \dot{\mathbf{h}}_r, \nabla_r\theta, \theta). \tag{56}$$

In the case $\mathfrak{q}$ is an isotropic function it becomes

$$\mathbf{h}_r = -(\vartheta_0 \nabla_r\theta + \vartheta_1 \mathbf{E}\nabla_r\theta + \vartheta_2 \mathbf{E}^2\nabla_r\theta + \vartheta_3 \dot{\mathbf{h}}_r + \vartheta_4 \mathbf{E}\dot{\mathbf{h}}_r + \vartheta_5 \mathbf{E}^2\dot{\mathbf{h}}_r), \tag{57}$$

where the functions ϑ_i, $i = 0, 1, 2$ depend on the invariants I_j, j= 4–9, 20, 21, 24, 25, 28, 39, 40, 47 from (21)–(37). The classical Fourier model for heat transfer appears if we assume $\vartheta_i = 0$, $i = 1, 2, 3, 4, 5$ and $\vartheta_0 = \vartheta_0(\theta)$, in which case (57) becomes $\mathbf{h}_r = -\vartheta_0(\theta)\nabla_r\theta$.

4.3. Another alternative simplification

Let us assume that in (38) the functions α_i, $i = 4, 5, .., 8$ are equal to zero, $\alpha_3 \neq 0$ and $\alpha_j = \alpha_j(\mathbf{S}, \theta)$, $j = 0, 1, 2, 3$, therefore from (38) we obtain a relation of the form

$$\mathbf{E} = \mathfrak{H}(\mathbf{S}, \theta), \tag{58}$$

[3] In the definitions of the invariants it is necessary to replace $\mathbf{S}$ by $\mathbf{T}$.

which if $\mathfrak{H}$ is an isotropic function becomes

$$\mathbf{E} = \varphi_0\mathbf{I} + \varphi_1\mathbf{S} + \varphi_2\mathbf{S}^2, \tag{59}$$

where the functions φ_i, $i = 0, 1, 2$ depend on the invariants I_1, I_2 and I_3 (see (21)).

In the case of (19), the final expression is similar to (45), replacing $\mathbf{T}$ by $\mathbf{S}$, $\nabla\theta$ by $\nabla_r\theta$ and $\mathbf{h}$ by $\mathbf{h}_r$.

4.4. The linearized theory of thermoelasticity for isotropic bodies

Linearized constitutive equations for thermoelastic bodies can be derived, for example, from (44) in the following manner. Let us assume that[4] $|\theta - 1| \sim O(\delta)$, $\delta \ll 1$. Then from (44) we have the approximations $\omega_i(\mathbf{T}, \theta) \approx \omega_i(\mathbf{T}, 1) + \frac{\partial\omega_i}{\partial\theta}\Big|_{(\mathbf{T},1)}(\theta - 1)$, and after some manipulations (neglecting terms of order δ^r, $r \geq 2$) from (44) we would obtain something of the form

$$\boldsymbol{\varepsilon} = \varpi_0\mathbf{I} + \varpi_1\mathbf{T} + \varpi_2\mathbf{T}^2 + (\widehat{\omega}_0\mathbf{I} + \widehat{\omega}_1\mathbf{T} + \widehat{\omega}_2\mathbf{T}^2)(\theta - 1), \tag{60}$$

where the functions ϖ_i, $\widehat{\omega}_i$, $i = 0, 1, 2$ only depend on $\mathbf{T}$.

Let us assume additionally that $|\mathbf{T}| \sim O(\delta)$, $\delta \ll 1$, then from (60) we have the approximation (in index notation) $\varpi_i(\mathbf{T}, 1) \approx \varpi_i(\mathbf{0}, 1) + \frac{\partial\varpi_i}{\partial T_{jk}}\Big|_{(\mathbf{0},1)} T_{jk}$ and $\widehat{\omega}_i(\mathbf{T}, 1) \approx \widehat{\omega}_i(\mathbf{0}, 1) + \frac{\partial\widehat{\omega}_i}{\partial T_{jk}}\Big|_{(\mathbf{0},1)} T_{jk}$. Replacing in (60) and neglecting terms of order $O(\delta^r)$, $r \geq 2$ (and using the assumption that $|\theta - 1| \sim O(\delta)$, $\delta \ll 1$), after some simplifications we obtain.[5]

$$\boldsymbol{\varepsilon} = \widetilde{\omega}_0\mathrm{tr}(\mathbf{T})\mathbf{I} + \widetilde{\omega}_1\mathbf{T} + \widetilde{\omega}_2\mathbf{I}(\theta - 1), \tag{61}$$

where $\widetilde{\omega}_0$, $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are constant.

5. Boundary value problems

In this section we study some boundary value problems for the particular subclass (46)considering[6] (50). From (46) we have

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2, \tag{62}$$

where $\Pi = \Pi(I_1, I_2, I_3, \theta)$ and $\Pi_i = \frac{\partial\Pi}{\partial I_i}$, $i = 1, 2, 3$.

From (50) we have

$$\mathbf{h} + \zeta_4\dot{\mathbf{h}} = -(\zeta_0\nabla\theta + \zeta_1\mathbf{T}\nabla\theta + \zeta_2\mathbf{T}^2\nabla\theta), \tag{63}$$

where we have assumed that $\zeta_j = \zeta_j(\mathbf{T}, \theta) = \zeta_j(I_1, I_2, I_3, \theta)$, $j = 0, 1, 2, 4$.

Regarding (18) we assume that $\psi = \psi(\mathbf{T}, \theta) = \psi(I_1, I_2, I_3, \theta)$, then it becomes (interchanging $\mathbf{S}$ and $\mathbf{T}$)

$$\rho\left[\left(\frac{\partial\psi}{\partial I_1} + \theta\frac{\partial^2\psi}{\partial I_1\partial\theta}\right)\mathrm{tr}\dot{\mathbf{T}} + \left(\frac{\partial\psi}{\partial I_2} + \theta\frac{\partial^2\psi}{\partial I_2\partial\theta}\right)\mathrm{tr}(\mathbf{T}\dot{\mathbf{T}}) + \left(\frac{\partial\psi}{\partial I_3} + \theta\frac{\partial^2\psi}{\partial I_3\partial\theta}\right)\mathrm{tr}(\mathbf{T}^2\dot{\mathbf{T}}) - \theta\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}\right] - \mathrm{tr}(\mathbf{T}\dot{\boldsymbol{\varepsilon}}) = \mathrm{div}\mathbf{h} + \rho \mathrm{r}. \tag{64}$$

Finally we recall the equation of motion (2).

$$\rho\ddot{\mathbf{u}} = \mathrm{div}\mathbf{T} + \rho\mathbf{b}. \tag{65}$$

Considering that $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$ from (62)–(65) we have 13 equations for the components of $\mathbf{T}$ (6 unknowns), $\mathbf{u}$ (3 components), $\mathbf{h}$ (3 components) and θ (one unknown), i.e., in total for 13 unknowns. In the following sections we study some simple boundary value problems analyzing the above equations.

[4] This would be equivalent to say that $|\theta - \theta_o| \ll 0$, i.e., that the temperature of the body does not change much from the reference temperature θ_o.

[5] We assume that if $\mathbf{T} = \mathbf{0}$ and $\theta = 1$ then $\boldsymbol{\varepsilon} = \mathbf{0}$.

[6] For brevity we do not consider the other classes of constitutive relations presented in Section 4.

5.1. Uniform stress and temperature distributions

In the problems presented in this section we assume that

$$\mathbf{T} = \mathbf{T}_o(t), \quad \theta = \theta_o(t), \tag{66}$$

i.e., that the stress and the temperature are uniform but may depend on time. In general, as shown below, this would not satisfy the equation of motion (65). From (62) we would have

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T}_o + \Pi_3\mathbf{T}_o^2, \tag{67}$$

where $\Pi_i = \Pi_i(\mathbf{T}_o(t), \theta_o(t))$, which means that $\boldsymbol{\varepsilon}$ would only depend on the time t. If $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(t)$ a possible solution $\mathbf{u}$ for $(1)_5$ would be of the form

$$\mathbf{u} = \mathbf{A}(t)\mathbf{x} + \mathbf{u}_o(t), \tag{68}$$

where $\mathbf{A}$ is a symmetric tensor, and its 6 independent components can be found from $\mathbf{A} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T}_o + \Pi_3\mathbf{T}_o^2$. Since $\mathrm{div}\mathbf{T}_o(\mathrm{t}) = \mathbf{0}$ the equation of motion (65) becomes

$$\rho[\ddot{\mathbf{A}}(t)\mathbf{x} + \ddot{\mathbf{u}}_o(t)] = \rho\mathbf{b}. \tag{69}$$

It is possible to see that the above equation would not be satisfied in general. For example, for the case $\mathbf{b} = \mathbf{0}$ we would need $\ddot{\mathbf{A}} = \mathbf{0}$ and $\ddot{\mathbf{u}}_o = \mathbf{0}$, which would imply that $\Pi_1\mathbf{I} + \Pi_2\mathbf{T}_o + \Pi_3\mathbf{T}_o^2 = c_1 t + c_0$, where c_1 and c_0 are constants. The above condition may not be satisfied always.

In the examples presented in the following subsections we assume that $\mathbf{b} = \mathbf{0}$ and $\mathbf{u}_o = \mathbf{0}$, and that $\mathbf{T}_o(t)$ is small enough such that $\ddot{\mathbf{A}}(t)\mathbf{x} = \mathbf{0}$ would be satisfied approximately.

If $\mathbf{T} = \mathbf{T}_o(t)$ and $\theta = \theta_o(t)$ we have that $\nabla\theta = \mathbf{0}$ and $\zeta_4 = \zeta_4(t) = \zeta_4(\mathbf{T}_o(t), \theta_o(t))$, therefore (63) becomes

$$\mathbf{h} + \zeta_4(t)\dot{\mathbf{h}} = \mathbf{0}, \tag{70}$$

whose solution is

$$\mathbf{h}(t) = \mathbf{c}e^{-\int_0^t \frac{1}{\zeta_4(\mathbf{T}_o(\xi), \theta_o(\xi))}\mathrm{d}\xi}, \tag{71}$$

where $\mathbf{c}$ is a constant vector. In the particular case when $\mathbf{T}_o$ and θ_o do not depend on time, we would have

$$\mathbf{h}(t) = \mathbf{c}e^{-\frac{t}{\zeta_4}}. \tag{72}$$

Finally since $\mathrm{div}\mathbf{h} = 0$, assuming additionally that[7] $r = 0$ from (64) we have

$$\rho\left[\left(\frac{\partial\psi}{\partial I_1} + \theta_o\frac{\partial^2\psi}{\partial I_1\partial\theta}\right)\mathrm{tr}\dot{\mathbf{T}}_o + \left(\frac{\partial\psi}{\partial I_2} + \theta_o\frac{\partial^2\psi}{\partial I_2\partial\theta}\right)\mathrm{tr}(\mathbf{T}_o\dot{\mathbf{T}}_o) + \left(\frac{\partial\psi}{\partial I_3} + \theta_o\frac{\partial^2\psi}{\partial I_3\partial\theta}\right)\mathrm{tr}(\mathbf{T}_o^2\dot{\mathbf{T}}_o) - \theta_o\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}_o\right] - \mathrm{tr}(\mathbf{T}_o\dot{\mathbf{A}}) = 0, \tag{73}$$

where I_i would be calculated with $\mathbf{T}_o$ and ψ and its derivatives depend on $\mathbf{T}_o$ and θ_o. It is possible to see that the above equation would impose a restriction on $\mathbf{T}_o$ and θ_o, i.e., they would not be independent.

5.1.1. Uniform traction of a cylinder

In this problem we consider a cylinder defined in cylindrical coordinates as

$$0 \leq r \leq r_o, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq z \leq L. \tag{74}$$

We assume that this cylinder is under the influence of the stress distribution

$$\mathbf{T}_o(t) = \sigma_z(t)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{75}$$

We assume that the above stress and temperature distributions generate a displacement field of the form

$$u_r = c_o r, \quad u_\phi = 0, \quad u_z = (\lambda - 1)z, \tag{76}$$

[7] For all the problem to be presented in the following section we assume that the internal heat generation is zero.

where $c_o = c_o(t) > 0$ and $\lambda = \lambda(t)$. In this case $\epsilon_{rr} = c_o = \epsilon_{\phi\phi}$, $\epsilon_{zz} = \lambda - 1$ and $\epsilon_{ij} = 0$ $i \neq j$.

From (21) we have

$$I_1 = \sigma_z, \quad I_2 = \frac{1}{2}\sigma_z^2, \quad I_3 = \frac{1}{3}\sigma_z^3. \tag{77}$$

Substituting the above expressions in (62) we have

$$c_o = \Pi_1, \quad \lambda - 1 = \Pi_1 + \Pi_2\sigma_z + \Pi_3\sigma_z^2, \tag{78}$$

where $\Pi_i = \Pi_i(\sigma_z(t), \theta_o(t))$, $i = 1, 2, 3$.

For $\mathbf{h}$ the solution is presented in (71), where we can assume that $\mathbf{c} = h_o\mathbf{e}_z$. Finally, in this problem Eq. (73) becomes

$$\rho\left[\left(\frac{\partial\psi}{\partial I_1} + \theta_o\frac{\partial^2\psi}{\partial I_1\partial\theta}\right)\dot{\sigma}_z + \left(\frac{\partial\psi}{\partial I_2} + \theta_o\frac{\partial^2\psi}{\partial I_2\partial\theta}\right)\sigma_z\dot{\sigma}_z + \left(\frac{\partial\psi}{\partial I_3} + \theta_o\frac{\partial^2\psi}{\partial I_3\partial\theta}\right)\sigma_z^2\dot{\sigma}_z - \theta_o\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}_o\right] - \sigma_z\dot{\lambda} = 0, \tag{79}$$

where ψ and its derivatives depend on $\sigma_z(t)$ and $\theta_o(t)$. This last equation could be used to find, for example, $\sigma_z(t)$ in term of $\theta_o(t)$ or viceversa.

5.1.2. Uniform shear of a slab

Let us consider the slab defined through

$$-\frac{L_i}{2} \leq x_i \leq \frac{L_i}{2}, \quad i = 1, 2, 3, \tag{80}$$

which we assume is deforming under the influence of the stress tensor

$$\mathbf{T}_o = \tau(t)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{81}$$

We assume that under the effect of the above stress tensor the slab deforms as

$$u_1 = (\lambda_a - 1)x_1 + \kappa x_2, \quad u_2 = (\lambda_b - 1)x_2, \quad u_3 = (\lambda_c - 1)x_3, \tag{82}$$

where $\lambda_a = \lambda_a(t)$, $\lambda_b = \lambda_b(t)$ and $\lambda_c = \lambda_c(t)$ are positive and $\kappa = \kappa(t)$.

Using the above assumptions from (21) we have $I_1 = I_3 = 0$ and $I_2 = \tau^2(t)$, whereas $\epsilon_{11} = \lambda_a - 1$, $\epsilon_{22} = \lambda_b - 1$, $\epsilon_{33} = \lambda_c - 1$, $\epsilon_{12} = \frac{\kappa}{2}$ and $\epsilon_{13} = \epsilon_{23} = 0$. From (62) we have

$$\lambda_a - 1 = \lambda_b - 1 = \Pi_1 + \Pi_3\tau^2, \quad \lambda_c - 1 = \Pi_1, \quad \frac{\kappa}{2} = \Pi_2\tau, \tag{83}$$

where $\Pi_i = \Pi_i(\tau, \theta_o)$, $i = 1, 2, 3$.

Regarding $\mathbf{h}$ we have the solution (71), where now $\mathbf{c}$ can be assumed to be of the form $\mathbf{c} = h_{o_1}\mathbf{e}_1 + h_{o_2}\mathbf{e}_2$. Finally, regarding (73) in the present problem that equation becomes

$$\rho\left[\left(\frac{\partial\psi}{\partial I_2} + \theta_o\frac{\partial^2\psi}{\partial I_2\partial\theta}\right)2\tau\dot{\tau} - \theta_o\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}_o\right] - \tau\dot{\kappa} = 0, \tag{84}$$

which could be used again to find, for example, θ_o in terms of τ or viceversa, recalling that here ψ and its derivatives depend on τ and θ_o.

5.2. Non uniform distributions for the stress and temperature

In this section we study two simple boundary value problems, where the different variables can depend on the position and time.

5.2.1. A one-dimensional rod

Let us consider the one-dimensional rod defined in the un-deformed configuration (cylindrical coordinates) as

$$0 \leq r \leq r_o, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty. \tag{85}$$

In this problem we assume that the stress and temperature are of the form

$$\mathbf{T} = \sigma_z(z, t)\mathbf{e}_z \otimes \mathbf{e}_z, \quad \theta = \theta(z, t). \tag{86}$$

Furthermore, we suppose that under the influence of the above stress and temperature distributions that the displacement field is approximately of the form

$$\mathbf{u} = u_z(z, t)\mathbf{e}_z. \tag{87}$$

Finally, regarding the heat flux $\mathbf{h}$, we assume that through the surface $r = r_o$ there is no heat transfer and so

$$\mathbf{h} = h_z(z, t)\mathbf{e}_z. \tag{88}$$

In virtue of the above assumptions, the equation of motion (65) (if $\mathbf{b} = \mathbf{0}$) becomes

$$\frac{\partial\sigma_z}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}. \tag{89}$$

Regarding $(1)_5$ we have that $\epsilon_{zz} = \frac{\partial u_z}{\partial z}$ and from (87) and (62) we obtain

$$\frac{\partial u_z}{\partial z} = \Pi_1 + \Pi_2\sigma_z + \Pi_3\sigma_z^2. \tag{90}$$

It is necessary to recognize that from (62) we also have that $\epsilon_{rr} = \epsilon_{\phi\phi} = \Pi_1$, which in general is not zero, therefore we do have a radial displacement besides the axial displacement presented in (87). But if we assume that r_o is very small in comparison with the axial dimension of the rod, we neglect such displacement for the present problem. In (62) $\Pi_i = \Pi_i(\sigma_z(z, t), \theta(z, t))$, $i = 1, 2, 3$.

From (86) and (88) we find that Eq. (63) becomes

$$h_z + \zeta_4\frac{\partial h_z}{\partial t} = -(\zeta_0 + \zeta_1\sigma_z + \zeta_2\sigma_z^2)\frac{\partial\theta}{\partial z}, \tag{91}$$

where $\zeta_i = \zeta_i(\sigma_z(z, t), \theta(z, t))$, $i = 0, 1, 2, 4$.

Finally, considering (86), (88) we have that (64) is of the form

$$\rho\left[\left(\frac{\partial\psi}{\partial\sigma_z} + \theta\frac{\partial^2\psi}{\partial\theta\partial\sigma_z}\right)\frac{\partial\sigma_z}{\partial t} - \theta\frac{\partial^2\psi}{\partial\theta^2}\frac{\partial\theta}{\partial t}\right] - \sigma_z\frac{\partial^2 u_z}{\partial z\partial t} = \frac{\partial h_z}{\partial z}. \tag{92}$$

We are interested now in studying the above equations within the context of the following especial expressions for the different functions

$$\sigma_z(z, t) = \Sigma(p), \quad \theta(z, t) = \Theta(p), \quad u_z(z, t) = U(p), \quad h_z(z, t) = H(p), \tag{93}$$

where

$$p = kz + \lambda t, \tag{94}$$

where $k \neq 0$ and $\lambda \neq 0$ are constants. Considering the above expressions, it is easy to show that

$$\frac{\partial\sigma_z}{\partial z} = k\frac{\mathrm{d}\Sigma}{\mathrm{d}p}, \quad \frac{\partial\sigma_z}{\partial t} = \lambda\frac{\mathrm{d}\Sigma}{\mathrm{d}p}, \tag{95}$$

and similar expressions can be found for the derivatives of the other functions in (93). Using this in (89)–(92) we obtain the following system of ordinary differential equations (for the functions $\Sigma(p)$, $\Theta(p)$, $U(p)$ and $H(p)$):

$$k\frac{\mathrm{d}\Sigma}{\mathrm{d}p} = \rho\lambda^2\frac{\mathrm{d}^2U}{\mathrm{d}p^2}, \tag{96}$$

$$k\frac{\mathrm{d}U}{\mathrm{d}p} = \Pi_1(\Sigma, \Theta) + \Pi_2(\Sigma, \Theta)\Sigma + \Pi_3(\Sigma, \Theta)\Sigma^2, \tag{97}$$

$$H + \lambda\zeta_4(\Sigma, \Theta)\frac{\mathrm{d}H}{\mathrm{d}p} = -k[\zeta_0(\Sigma, \Theta) + \zeta_1(\Sigma, \Theta)\Sigma + \zeta_2(\Sigma, \Theta)\Sigma^2]\frac{\mathrm{d}\Theta}{\mathrm{d}p}, \tag{98}$$

$$\rho\lambda\left[\left(\frac{\partial\psi}{\partial\Sigma} + \Theta\frac{\partial^2\psi}{\partial\Theta\partial\Sigma}\right)\frac{\mathrm{d}\Sigma}{\mathrm{d}p} - \Theta\frac{\partial^2\psi}{\partial\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}p}\right] - k\lambda\Sigma\frac{\mathrm{d}^2U}{\mathrm{d}p^2} = k\frac{\mathrm{d}H}{\mathrm{d}p}. \tag{99}$$

From (96) and (97) we obtain

$$\frac{k^2}{\rho\lambda^2}\Sigma = \Pi_1(\Sigma, \Theta) + \Pi_2(\Sigma, \Theta)\Sigma + \Pi_3(\Sigma, \Theta)\Sigma^2 + c_0, \tag{100}$$

$$U(p) = \frac{1}{k}\int_0^p [\Pi_1(\Sigma(s), \Theta(s)) + \Pi_2(\Sigma(s), \Theta(s))\Sigma(s) + \Pi_3(\Sigma(s), \Theta(s))\Sigma^2(s)]\mathrm{d}s + c_1, \tag{101}$$

where c_0 and c_1 are constants.

With regard to (98), we have the solution

$$H(p) = e^{-\frac{1}{\lambda}\int_1^p \frac{1}{\zeta_4(\Sigma(s),\Theta(s))}\mathrm{d}s}\Bigg\{c_2 - \frac{k}{\lambda}\int_1^p \frac{e^{\frac{1}{\lambda}\int_1^l \frac{1}{\zeta_4(\Sigma(s),\Theta(s))}\mathrm{d}s}}{\zeta_4(\Sigma(l), \Theta(l))}[\zeta_0(\Sigma(l), \Theta(l)) + \zeta_1(\Sigma(l), \Theta(l))\Sigma(l) + \zeta_2(\Sigma(l), \Theta(l))\Sigma^2(l)]\frac{\mathrm{d}\Theta}{\mathrm{d}p}(l)\,\mathrm{d}l\Bigg\}, \tag{102}$$

where c_2 is a constant.

Finally, it follows from (99) and (96) that $\frac{\mathrm{d}^2U}{\mathrm{d}p^2} = \frac{k}{\rho\lambda^2}\frac{\partial\Sigma}{\partial p}$, and therefore (99) becomes

$$\lambda\rho\left[\left(\frac{\partial\psi}{\partial\Sigma} + \Theta\frac{\partial^2\psi}{\partial\Theta\partial\Sigma}\right)\frac{\mathrm{d}\Sigma}{\mathrm{d}p} - \Theta\frac{\partial^2\psi}{\partial\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}p}\right] - \frac{k^2}{2\rho\lambda}\frac{\mathrm{d}}{\mathrm{d}p}(\Sigma^2) = k\frac{\mathrm{d}H}{\mathrm{d}p}. \tag{103}$$

Let us use the notation

$$\mathcal{D}(\Sigma, \Theta) = \frac{e^{\frac{1}{\lambda}\int_1^l \frac{1}{\zeta_4(\Sigma(s),\Theta(s))}\mathrm{d}s}}{\zeta_4(\Sigma(l), \Theta(l))}[\zeta_0(\Sigma, \Theta) + \zeta_1(\Sigma, \Theta)\Sigma + \zeta_2(\Sigma, \Theta)\Sigma^2], \tag{104}$$

Integrating Eq. (103) becomes

$$\lambda\rho\int_1^p\left\{\left[\frac{\partial\psi}{\partial\Sigma}(\Sigma(s), \Theta(s)) + \Theta(s)\frac{\partial^2\psi}{\partial\Theta\partial\Sigma}(\Sigma(s), \Theta(s))\right]\frac{\mathrm{d}\Sigma}{\mathrm{d}s}(s) - \Theta(s)\frac{\partial^2\psi}{\partial\Theta}(\Sigma(s), \Theta(s))\frac{\mathrm{d}\Theta}{\mathrm{d}p}(s)\right\}\mathrm{d}s - \frac{k^2}{2\rho\lambda}\Sigma^2(p) = ke^{-\frac{1}{\lambda}\int_1^p \frac{1}{\zeta_4(\Sigma(s),\Theta(s))}\mathrm{d}s}\left[c_2 - \frac{k}{\lambda}\int_1^p \mathcal{D}(\Sigma(s), \Theta(s))\frac{\mathrm{d}\Theta}{\mathrm{d}s}\,\mathrm{d}s\right] + c_3, \tag{105}$$

where c_3 is a constant. Therefore, in order to obtain closed solutions for this boundary value problem, we would need to solve in parallel the general nonlinear algebraic Eq. (100) and the integral Eq. (105), in order to find $\Sigma(p)$ and $\Theta(p)$.

5.2.2. Inflation of an infinitely long cylindrical tube

In this last problem we are interested in analyzing the case of a cylindrical tube deforming under the influence of mechanical loading and a temperature field, which depend only on the radial position (in a cylindrical coordinates system). In the un-deformed configuration the tube is described by

$$r_i \le r \le r_o, \quad 0 \le \phi \le 2\pi, \quad 0 \le z \le L, \tag{106}$$

and we assume that in this tube we have stress and temperature distributions of the form

$$\mathbf{T} = \sigma_r(r, t)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\phi(r, t)\mathbf{e}_\phi \otimes \mathbf{e}_\phi + \sigma_z(r, t)\mathbf{e}_z \otimes \mathbf{e}_z, \quad \theta = \theta(r, z). \tag{107}$$

We assume that the above stress and temperature distributions can cause the following displacement field and heat flux

$$\mathbf{u} = u_r(r, t)\mathbf{e}_r + (\lambda - 1)z\mathbf{e}_z, \quad \mathbf{h} = h_r(r, t)\mathbf{e}_r + h_z(r, t)\mathbf{e}_z, \tag{108}$$

where λ is a constant. From $(1)_5$ and $(108)_1$ we have

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \lambda - 1. \tag{109}$$

Considering the above assumptions the equation of motion (65) becomes

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\phi) = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{110}$$

while from (62) we have

$$\frac{\partial u_r}{\partial r} = \Pi_1 + \Pi_2\sigma_r + \Pi_3\sigma_r^2, \quad \frac{u_r}{r} = \Pi_1 + \Pi_2\sigma_\phi + \Pi_3\sigma_\phi^2, \quad \lambda - 1 = \Pi_1 + \Pi_2\sigma_z + \Pi_3\sigma_z^2. \tag{111}$$

On the other hand from (63) we obtain

$$h_r + \zeta_4\frac{\partial h_r}{\partial t} = -(\zeta_0 + \zeta_1\sigma_r + \zeta_2\sigma_r^2)\frac{\partial\theta}{\partial r}, \quad h_z + \zeta_4\frac{\partial h_z}{\partial t} = 0, \tag{112}$$

and finally (64) becomes

$$\rho\Bigg[\left(\frac{\partial\psi}{\partial I_1} + \theta\frac{\partial\psi}{\partial I_1\partial\theta}\right)\left(\frac{\partial\sigma_r}{\partial t} + \frac{\partial\sigma_\phi}{\partial t} + \frac{\partial\sigma_z}{\partial t}\right) + \left(\frac{\partial\psi}{\partial I_2} + \theta\frac{\partial\psi}{\partial I_2\partial\theta}\right)\left(\sigma_r\frac{\partial\sigma_r}{\partial t} + \sigma_\phi\frac{\partial\sigma_\phi}{\partial t} + \sigma_z\frac{\partial\sigma_z}{\partial t}\right) + \left(\frac{\partial\psi}{\partial I_3} + \theta\frac{\partial\psi}{\partial I_3\partial\theta}\right)\left(\sigma_r^2\frac{\partial\sigma_r}{\partial t} + \sigma_\phi^2\frac{\partial\sigma_\phi}{\partial t} + \sigma_z^2\frac{\partial\sigma_z}{\partial t}\right) - \theta\frac{\partial^2\psi}{\partial\theta^2}\frac{\partial\theta}{\partial t}\Bigg] - \sigma_r\frac{\partial^2 u_r}{\partial r\partial t} - \frac{\sigma_\phi}{r}\frac{\partial u_r}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(rh_r). \tag{113}$$

In (111)–(113) the functions Π_i, $i = 1, 2, 3$, ζ_j, $j = 0, 1, 2, 4$ and ψ depend on the invariants $I_1 = \sigma_r + \sigma_\phi + \sigma_z$, $I_2 = \frac{1}{2}(\sigma_r^2 + \sigma_\phi^2 + \sigma_z^2)$, $I_3 = \frac{1}{3}(\sigma_r^3 + \sigma_\phi^3 + \sigma_z^3)$ and θ.

The 7 Eqs. (111)–(113) must be solved to find the 7 functions $\sigma_r(r, t)$, $\sigma_\phi(r, t)$, $\sigma_z(r, t)$, $h_r(r, t)$, $h_z(r, t)$, $\theta(r, t)$ and $u_r(r, t)$.

6. Constraints

In this section we study how the kinematical constraints of incompressibility and inextensibility can be imposed for one of the subclasses of constitutive equations presented in Section 4.1, namely the case $\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}, \theta)$. The analysis presented here follows closely the studies published in [32,33] for the purely elastic case.

6.1. An isotropic thermoelastic body, which is mechanically incompressible but thermally compressible. The case $|\nabla_r\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$

Let us study the incompressibility constraint for this new class of constitutive relations (44). We are interested in modelling the behaviour of a body, which is mechanically incompressible, but whose volume can change due to variations in the temperature, i.e., where

$$J = \det\mathbf{F} = f(\theta) \quad \text{where} \quad f(1) = 1 \quad \text{and} \quad f(\theta) > 0, \tag{114}$$

where notice again that the volume of the body is only affected by a change in the temperature (from the reference temperature). Two additional restrictions on f are: $\theta < 1$ then $0 < f(\theta) < 1$, and if $\theta \ge 1$ then $f(\theta) \ge 1$.

Under the assumption $|\nabla_r\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ the above constraint can be written in terms of $\boldsymbol{\varepsilon}$ as

$$\mathrm{tr}\boldsymbol{\varepsilon} = \mathrm{g}(\theta), \tag{115}$$

where we have defined $g(\theta) = \frac{f^2(\theta) - 1}{2}$ and if $\theta < 1$ then $-\frac{1}{2} < g(\theta) < 0$ and if $\theta \ge 1$ then $0 \le g(\theta)$.

Replacing (46) in (115) we obtain the first order linear partial differential equation

$$3\Pi_1 + I_1\Pi_2 + 2I_2\Pi_3 = g(\theta), \tag{116}$$

whose solution is

$$\Pi(I_1, I_2, I_3, \theta) = g(\theta)\frac{I_1}{3} + \overline{\Pi}(\bar{I}_1, \bar{I}_2, \theta), \tag{117}$$

where we have defined (see [33])

$$\bar{I}_1 = I_1 - \frac{I_1^2}{6}, \quad \bar{I}_2 = I_3 - \frac{2}{3}I_1I_2 + \frac{2}{27}I_1^3. \tag{118}$$

Using (117) in (46) we finally obtain

$$\boldsymbol{\varepsilon} = g(\theta)\mathbf{I} + \left(\mathbf{T} - \frac{I_1}{3}\mathbf{I}\right)\frac{\partial\overline{\Pi}}{\partial\bar{I}_1} + \left[2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2\right]\frac{\partial\overline{\Pi}}{\partial\bar{I}_2} \tag{119}$$

If we define $p = -\frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{I}$, the stress tensor can be decomposed as $\mathbf{T} = \mathbf{T}_o - p\mathbf{I}$, where $\mathbf{T}_o = \mathbf{T} - (\mathrm{tr}\mathbf{T})\mathbf{I}$. As shown in [33], it is easy to prove that $\bar{I}_1$ and $\bar{I}_2$ are the same if they are calculated using $\mathbf{T}$ or $\mathbf{T}_o$, moreover $\mathbf{T} - \frac{I_1}{3}\mathbf{I} = \mathbf{T}_o - \frac{I_{o1}}{3}\mathbf{I}$ and $2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2 = 2\left(\frac{I_{o1}^2}{9} - \frac{I_{o2}}{3}\right)\mathbf{I} - \frac{2I_{o1}}{3}\mathbf{T}_o + \mathbf{T}_o^2$, where $I_{o_1} = \mathrm{tr}\mathbf{T}_o$ and $I_{o_2} = \frac{1}{2}\mathrm{tr}(\mathbf{T}_o^2)$, therefore from (119) we have

$$\boldsymbol{\varepsilon}(\mathbf{T}, \theta) = \boldsymbol{\varepsilon}(\mathbf{T}_o, \theta). \tag{120}$$

6.2. An intextensible transversely isotropic thermoelastic body

Let us study the behaviour of a transversely isotropic body, where we have a matrix filled with inextensible fibres (if the temperature is constant and $\theta = 1$). Let us assume that both the body and the fibres can react to changes in the temperature, in particular the length of such fibres is affected by the temperature. Considering this, if $\mathbf{a}_0$ is the unit vector field describing the directions of the family of fibres in the reference configuration, we have the constraint

$$\mathbf{a}_0\cdot(\mathbf{C}\mathbf{a}_0) = f(\theta) \quad \Leftrightarrow \quad \mathbf{a}_0\cdot(\mathbf{E}\mathbf{a}_0) = g(\theta) = \frac{f(\theta) - 1}{2}, \tag{121}$$

where $f(1) = 1$ that implies that $g(1) = 0$. In the case $|\nabla\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ we have $\mathbf{E} \approx \boldsymbol{\varepsilon}$ and the constraint can be written as

$$\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) = g(\theta), \tag{122}$$

where $\mathbf{a} = \mathbf{F}\mathbf{a}_0 \approx \mathbf{a}_0$.

Assuming again that there exists a function $\Pi = \Pi(\mathbf{T}, \theta, \mathbf{a})$ such that $\boldsymbol{\varepsilon} = \frac{\partial\Pi}{\partial\mathbf{T}}$, for a transversely isotropic function we have $\Pi = \Pi(I_1, I_2, I_3, I_4, I_5, \theta)$, where I_1, I_2 and I_3 have been defined in $(21)_{1,2,3}$ and

$$I_4 = \mathbf{a}\cdot(\mathbf{T}\mathbf{a}), \quad I_5 = \mathbf{a}\cdot(\mathbf{T}^2\mathbf{a}). \tag{123}$$

Using (21) and (123) in $\boldsymbol{\varepsilon} = \frac{\partial\Pi}{\partial\mathbf{T}}$ we obtain

$$\boldsymbol{\varepsilon} = \Pi_1\mathbf{I} + \Pi_2\mathbf{T} + \Pi_3\mathbf{T}^2 + \Pi_4\mathbf{a}\otimes\mathbf{a} + \Pi_5[\mathbf{a}\otimes(\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a})\otimes\mathbf{a}], \tag{124}$$

where $\Pi_i = \frac{\partial\Pi}{\partial I_i}$, $i = 1, 2, 3, 4, 5$.

Replacing (124) in (122) we obtain the first order linear partial differential equation

$$\Pi_1 + \Pi_2 I_4 + \Pi_3 I_5 + \Pi_4 + 2\Pi_5 I_4 = g(\theta). \tag{125}$$

One solution of the above equation is (see [32])

$$\Pi(\mathbf{T}, \theta, \mathbf{a}) = I_4 g(\theta) + \overline{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \theta), \tag{126}$$

where we have defined

$$\bar{I}_1 = I_4 - I_1, \quad \bar{I}_2 = \frac{1}{2}I_1^2 + I_2 - I_1I_4, \quad \bar{I}_3 = I_1^2 - 2I_1I_4 + I_5, \tag{127}$$

$$\bar{I}_4 = -\frac{1}{3}I_1^3 + I_3 + I_1^2I_4 - I_1I_5. \tag{128}$$

Using this in (115) we obtain.[8]

$$\begin{aligned}\boldsymbol{\varepsilon} \;=\;& g(\theta)\mathbf{a}\otimes\mathbf{a} + \frac{\partial\overline{\Pi}}{\partial\bar{I}_1}(\mathbf{a}\otimes\mathbf{a} - \mathbf{I}) + \frac{\partial\overline{\Pi}}{\partial\bar{I}_2}(-\bar{I}_1\mathbf{I} + \mathbf{T} - I_1\mathbf{a}\otimes\mathbf{a})\\ &+\frac{\partial\overline{\Pi}}{\partial\bar{I}_3}[-2\bar{I}_1\mathbf{I} - 2I_1\mathbf{a}\otimes\mathbf{a} + \mathbf{a}\otimes(\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a})\otimes\mathbf{a}]\\ &+\frac{\partial\overline{\Pi}}{\partial\bar{I}_4}\{-\bar{I}_3\mathbf{I} + \mathbf{T}^2 + I_1^2\mathbf{a}\otimes\mathbf{a} - I_1[\mathbf{a}\otimes(\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a})\otimes\mathbf{a}]\}.\end{aligned} \tag{129}$$

If we decompose the stress tensor as $\mathbf{T} = \mathbf{T}_o + q\mathbf{a}\otimes\mathbf{a}$, where $\mathbf{a}\cdot(\mathbf{T}_o\mathbf{a}) = 0$ it is easy to show that $\bar{I}_k = \bar{I}_{o_k}$, where $\bar{I}_{o_k}$ are the invariants presented in (127), (128) defined in terms of $\mathbf{T}_o$. More generally from (129) it is possible to show (see [32]) that

$$\boldsymbol{\varepsilon}(\mathbf{T}, \theta) = \boldsymbol{\varepsilon}(\mathbf{T}_o, \theta). \tag{130}$$

If the constraint only affects the fibres in traction then (122) is replaced by

$$\mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) \le g(\theta), \tag{131}$$

and considering the above results we have for this case

$$\Pi(\mathbf{T}, \theta, \mathbf{a}) = \begin{cases}\Pi(I_1, I_2, I_3, I_4, I_5, \theta) & \text{if} \quad \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) < g(\theta),\\ I_4 g(\theta) + \overline{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \theta) & \text{if} \quad \mathbf{a}\cdot(\boldsymbol{\varepsilon}\mathbf{a}) = g(\theta).\end{cases} \tag{132}$$

7. Final remarks

In the present paper we have studied the extension of the implicit constitutive theories proposed by Rajagopal and his co-workers [4–8], within the context of thermoelasticity. Some subclasses of such implicit constitutive theories, where the strains are given as nonlinear functions of the stresses and the temperature, could be very interesting for the potential applications in the modelling of some metal alloys and rock (see, for example, [9] and the references cited therein). It has been found that such implicit constitutive theories and in particular some of its subclasses, can be very useful and appropriate for addressing some problems such as the incorporation of kinematical constraints into the constitutive equations. In future works some boundary value problems will be studied considering some specific expressions for the constitutive functions and relations, as well as this, the incremental equations will be obtained, to study how small thermal and mechanical waves propagate, especially for the case of incompressible bodies.

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[8] Interestingly, there is a second solution for (125) of the form $\Pi(\mathbf{T}, \theta, \mathbf{a}) = I_1 g(\theta) + \overline{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \theta)$, in which case the expression for the strain is

$$\begin{aligned}\boldsymbol{\varepsilon} \;=\;& g(\theta)\mathbf{I} + \frac{\partial\overline{\Pi}}{\partial\bar{I}_1}(\mathbf{a}\otimes\mathbf{a} - \mathbf{I}) + \frac{\partial\overline{\Pi}}{\partial\bar{I}_2}(-\bar{I}_1\mathbf{I} + \mathbf{T} - I_1\mathbf{a}\otimes\mathbf{a})\\ &+\frac{\partial\overline{\Pi}}{\partial\bar{I}_3}[-2\bar{I}_1\mathbf{I} - 2I_1\mathbf{a}\otimes\mathbf{a} + \mathbf{a}\otimes(\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a})\otimes\mathbf{a}]\\ &+\frac{\partial\overline{\Pi}}{\partial\bar{I}_4}\{-\bar{I}_3\mathbf{I} + \mathbf{T}^2 + I_1^2\mathbf{a}\otimes\mathbf{a} - I_1[\mathbf{a}\otimes(\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a})\otimes\mathbf{a}]\}.\end{aligned}$$

From (126) and this solution, taking into account that (125) is a linear partial differential equation, we could consider a more general solution of the form

$$\Pi(\mathbf{T}, \theta, \mathbf{a}) = (\alpha I_1 + \beta I_4)g(\theta) + \overline{\Pi}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \theta),$$

where α and β are constants and $\alpha + \beta = 1$.

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