J. Math. Anal. Appl. 457 (2018) 1017–1037

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Invariant sets and Lyapunov pairs for differential inclusions with maximal monotone operators ☆

Samir Adly [a,*], Abderrahim Hantoute [b], Bao Tran Nguyen [a,b]

[a] *XLIM UMR-CNRS 7252, Université de Limoges, 87060 Limoges, France*
[b] *Center for Mathematical Modeling (CMM), Universidad de Chile, Santiago, Chile*

ARTICLE INFO

Article history:
Received 29 September 2016
Available online 28 April 2017
Submitted by G. Moscariello

Keywords:
Lyapunov stability
lsc Lyapunov pairs and functions
Invariant sets
Differential inclusions
Maximal monotone operators
Variational and nonsmooth analysis

ABSTRACT

We give different conditions for the invariance of closed sets with respect to differential inclusions governed by a maximal monotone operator defined on Hilbert spaces, which is subject to a Lipschitz continuous perturbation depending on the state. These sets are not necessarily weakly closed as in [3,4], while the invariance criteria are still written by using only the data of the system. So, no need to the explicit knowledge of neither the solution of this differential inclusion, nor the semi-group generated by the maximal monotone operator. These invariant/viability results are next applied to derive explicit criteria for a-Lyapunov pairs of lower semi-continuous (not necessarily weakly-lsc) functions associated to these differential inclusions. The lack of differentiability of the candidate Lyapunov functions and the consideration of general invariant sets (possibly not convex or smooth) are carried out by using techniques from nonsmooth analysis.

1. Introduction

We provide sufficient and, in many different interesting situations, necessary criteria for the invariance property of closed subsets with respect to the following differential inclusion, given in a Hilbert space H,

$$\dot{x}(t) \in f(x(t)) - Ax(t), \;\; x(0) = x_0 \in \overline{\operatorname{dom} A}, \;\text{a.e. } t \geq 0 \tag{1}$$

where A is a maximal monotone operator which is subject to a Lipschitzian perturbation f. Equivalently, we establish many primal and dual explicit criteria for a-Lyapunov pairs and functions associated to the differential inclusion above. The current work extends and improves some of the results given in [3,4] on weakly closed invariant sets and weakly lower semi-continuous a-Lyapunov pairs.

☆ Research partially supported by Conicyt-Fondecyt grant no. 1151003, Conicyt-Pcha/Doctorado Nacional/2014-63140104, and Conicyt-Redes no. 150040.
* Corresponding author.
E-mail addresses: samir.adly@unilim.fr (S. Adly), ahantoute@dim.uchile.cl (A. Hantoute), nguyenbaotran31@gmail.com (B.T. Nguyen).

http://dx.doi.org/10.1016/j.jmaa.2017.04.059

The domain of A does not need to be closed, nor the values of A are supposed to be bounded or even nonempty. Thus, the scope of the equation above goes beyond the differential inclusions treated in [6,7,12,14,15], where the right-hand side is generally represented by a cusco set-valued mapping (in particular, with nonempty and weak*-compact multi-valued operator). It is the monotonicity of A which compensates the lack of compacity in our differential inclusion, while the maximality of this operator guaranties, among other properties, the existence and the regularity of solutions. These two facts are also essential when checking the invariance of closed sets.

In front of the lack to a direct access to the explicit calculus of either the solution of the inclusion above or to the semi-group generated by A, the current work aims at finding weaker conditions for the invariance of closed sets, which only appeal to the fresh input data, namely the maximal monotone operator and the Lipschitz mapping. These conditions are applicable to a large variety of closed sets which do not need to be convex or smooth. Our approach fits the general scope and the main ideas behind Lyapunov's stability, which consists of looking for an adjacent function to the system described by the inclusion above; namely, an energy-like function which decreases along the trajectories and, so, under some extra usual conditions, forces the system to converge towards its equilibrium state and to remain there. Since our analysis allows to deal with extended-real valued functions, the invariance of a set occurs as long as the associated indicator function is a Lyapunov's function. However, our approach is more geometric since we first establish criteria for the invariance property and next deduce the adequate conditions for Lyapunov pairs and functions.

Invariant sets associated to general differential inclusions/equations have been the subject of extensive research during the last decades; namely, in relation with differential inclusions involving cusco mappings in their right-hand side (see, e.g., [6]). First results dealing with Lyapunov pairs and functions associated to the differential inclusions above have been first established in [19,20] in the case of homogeneous systems; that is, $f \equiv 0$. Pazy's criteria for a-Lyapunov pairs are given by means of directional-like derivative using the Moreau–Yoshida approximation of the operator A. This result has been extended to the general inclusion above in [11,17], with the use of implicit criteria depending heavily on the semi-group generated by the maximal monotone operator A. Recently, different criteria for weakly lower semi-continuous a-Lyapunov pairs have been investigated in [3,4].

The need of more explicit conditions, not depending on the semi-group generated by A, is of utmost importance for many reasons, one of which is that the inclusion above is sometimes evoked as a companion tool to analyze other differential inclusions. In that case, the operator A may not be known explicitly, and this fact makes the access to its semi-group more complicated. For instance, in our work [5] we have investigated the existence of solutions to a differential inclusion governed by the normal cone to a prox-regular set [21], by rewriting it in the form of (1) with A being some intrinsic maximal monotone operator to this prox-regular set. Such an operator A is not known explicitly but it processes enough information in order to check the invariance of the involved prox-regular set with respect to (1). This was sufficient to get the desired existence results; for more details, we refer the reader to [5].

Invariant sets are also referred to in the wide literature as viable sets [6–8], and are of crucial use in many domains, as in economic, renewable resources, biology, diseases propagation, control processes of species and so on. It is manifest, in recent papers [16,24], that the investigation of certain algebraic varieties is sufficient to characterize invariant sets forced by symmetries. Lyapunov pairs and functions are used extensively in dynamic systems and control theory, among many other applications; see, e.g., [1,10].

In this work, we provide different criteria to characterize those sets which are invariant with respect to the differential inclusion (1). Only the data, A and f, will be appealed to and no need to solve explicitly the equation. These invariant results are then rewritten as criteria for a-Lyapunov pairs, which are crucial for Lyapunov stability of (1). Because the sets we consider are not necessary convex or smooth, and the candidate Lyapunov functions are not necessarily sufficiently regular, we use techniques of nonsmooth analysis (e.g. [14,18,23]).

The organization of the paper is as follows. After an introductory section to present the main notations and tools which are used through this work, we give in Section 3 the main invariance criterion in Theorem 4, using the normal cone to the nominal set. Other corollaries follow in order to simplify this invariance criterion and provide equivalent primal and dual conditions. In Section 4, we apply the previous invariance result to investigate a-Lyapunov pairs associated to differential inclusion (1).

2. Notation and preliminary results

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a Hilbert space, with origin θ. Given a set $S \subset H$, by $\overline{S}$ and S^* we denote the *closure* of S and the *polar* of S, respectively, where

$$S^* := \{x^* \in H \mid \langle x^*, x\rangle \leq 0 \text{ for all } x \in S\}.$$

The *indicator* and the *distance functions* are respectively given by

$$\mathrm{I}_S(x) := 0 \text{ if } x \in S; +\infty \text{ if } x \notin S, \text{ and } d_S(x) := \inf\{\|x-y\| : \ y \in S\}$$

(in the sequel we shall adopt the convention $\inf_\emptyset = +\infty$). For $\delta \geq 0$, we denote P_S^δ the (*orthogonal*) *δ-projection mapping* onto S defined as

$$P_S^\delta(x) := \{y \in S : \ \|x-y\|^2 \leq d_S^2(x) + \delta^2\};$$

for $\delta = 0$, we simply write $P_S(x) := P_S^0(x)$. It is known that P_S is nonempty-valued on a dense subset of $H \setminus S$ [14]. For an extended-real valued function $\varphi : H \to \overline{\mathbb{R}} := (-\infty, +\infty]$, we denote $\operatorname{dom}\varphi := \{x \in H \mid \varphi(x) < +\infty\}$ and $\operatorname{epi}\varphi := \{(x,\alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}$. Function φ is lower semi-continuous (lsc, for short) if $\operatorname{epi}\varphi$ is closed. The *contingent directional derivative* of φ at $x \in \operatorname{dom}\varphi$ in the direction $v \in H$ is

$$\varphi'(x; v) := \liminf_{t\to 0^+, w\to v} \frac{\varphi(x+tw) - \varphi(x)}{t}.$$

A vector $\xi \in H$ is called a *proximal subgradient* of φ at $x \in H$, written $\xi \in \partial_P\varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y-x\rangle - \sigma\|y-x\|^2, \ \ \forall\, y \in B_\rho(x),$$

where $B_\rho(x)$ $(=: B(x,\rho))$ is the closed ball centred at $x \in H$ of radius $\rho > 0$. The vector ξ is called a *Fréchet subgradient* of φ at x, written $\xi \in \partial_F\varphi(x)$, if

$$\varphi(y) \geq \varphi(x) + \langle \xi, y-x\rangle + o(\|y-x\|), \ \forall\, y \in H;$$

and a *basic (or Limiting) subgradient* of φ at x, written $\xi \in \partial_L\varphi(x)$, if there exist sequences $(x_k)_k$ and $(\xi_k)_k$ such that

$$x_k \xrightarrow{\varphi} x, \ \xi_k \in \partial_P\varphi(x_k), \ \xi_k \rightharpoonup \xi,$$

where $\rightharpoonup$ refers to the weak convergence in H, and $x_k \xrightarrow{\varphi} x$ means that $x_k \to x$ together with $\varphi(x_k) \to \varphi(x)$.

If $x \notin \operatorname{dom}\varphi$, we write $\partial_P\varphi(x) = \partial_F\varphi(x) = \partial_L\varphi(x) = \emptyset$. If S is a closed set and $s \in S$, we define the *proximal normal cone* to S at s as $\mathrm{N}_S^P(s) = \partial_P\mathrm{I}_S(s)$, the *Fréchet normal* cone to S at s as $\mathrm{N}_S^F(s) = \partial_F\mathrm{I}_S(s)$, the *limiting normal cone* to S at s as $\mathrm{N}_S^L(s) = \partial_L\mathrm{I}_S(s)$, and *the Clarke normal cone* to S at s as $\mathrm{N}_S^C(s) =$

$\overline{\mathrm{co}}\mathrm{N}_S^L(s)$. Equivalently, we have that $\mathrm{N}_S^P(s) = \mathrm{cone}(P_S^{-1}(s) - s)$, where $P_S^{-1}(s) := \{x \in H \mid s \in P_S(x)\}$. The *Bouligand tangent cone to* S at x is defined as

$$T_S(x) := \left\{v \in H \;\mid \exists\, x_k \in S, \exists\, t_k \to 0, \text{ st. } t_k^{-1}(x_k - x) \to v \text{ as } k \to +\infty\right\}.$$

We also define the *Clarke subgradients* of φ at x as the vectors $\xi \in H$ such that $(\xi, -1) \in \mathrm{N}_{\mathrm{epi}\,\varphi}^C(x, \varphi(x))$, and denote $\partial_C\varphi(x)$ the Clarke subdifferential of φ at x. The *singular subdifferential* of φ at x, written $\partial_\infty\varphi(x)$, is the set of vectors $\xi \in H$ for which there are sequences $x_k \xrightarrow{\varphi} x$, $\xi_k \in \partial_P\varphi(x_k)$ and $\lambda_k \to 0^+$ such that $\lambda_k\xi_k \rightharpoonup \xi$; equivalently, $\xi \in \partial_\infty\varphi(x)$ iff $(\xi, 0) \in \mathrm{N}_{\mathrm{epi}\,\varphi}^L(x, \varphi(x))$ (see [18, Theorem 2.38]). It is known that every $\xi \in H$ such that $(\xi, 0) \in \mathrm{N}_{\mathrm{epi}\,\varphi}^P(x, \varphi(x))$ belongs to $\partial_\infty\varphi(x)$ and, moreover, there exist sequences as in the definition before but with $\lambda_k\xi_k \to \xi$ instead of $\lambda_k\xi_k \rightharpoonup \xi$ (see [18, Lemma 2.37]). Observe that $\partial_P\varphi(x) \subset \partial_F\varphi(x) \subset \partial_L\varphi(x) \subset \partial_C\varphi(x)$. For all these concepts and properties we refer to [18,23].

We shall use the following version of Gronwall's Lemma:

Lemma 1. (Gronwall's Lemma [2]) *Let $T > 0$ and $a, b \in L^1(t_0, t_0+T; \mathbb{R})$ such that $b(t) \geq 0$ a.e. $t \in [t_0, t_0+T]$. If an absolutely continuous function $w : [t_0, t_0 + T] \to \mathbb{R}_+$ satisfies, for $0 \leq \alpha < 1$,*

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \quad a.e.\ t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0)e^{\int_{t_0}^t a(\tau)d\tau} + \int_{t_0}^{t} e^{\int_s^t a(\tau)d\tau}b(s)ds,\ \forall t \in [t_0, t_0 + T].$$

Next, we review some facts about monotone and maximal monotone operators. Given a set-valued operator $A : H \rightrightarrows H$, which we identify with its graph, we denote its *domain* by $\mathrm{dom}\, A := \{x \in H \mid Ax \neq \emptyset\}$. Operator A is *monotone* if

$$\langle x_1 - x_2, y_1 - y_2\rangle \geq 0 \text{ for all } (x_1, y_1),\ (x_2, y_2) \in A.$$

We say that A is *maximal monotone* if A is monotone and coincides with every monotone operator containing its graph. In such a case, it is known that Ax is convex and closed for every $x \in H$; moreover, for every $\lambda > 0$ there exists a unique vector $J_\lambda x \in (\mathrm{id} + \lambda A)^{-1}(x)$, which is the *resolvent* of the (maximal monotone) operator A, while $A_\lambda x := \frac{x - J_\lambda x}{\lambda}$ is the *Moreau–Yoshida approximation* of A. If $S \subset H$ is a closed convex set, we denote $S^0 := \{y \in S \mid \|y\| = \min_{z \in S}\|z\|\}$; in particular, we write $A^0x := (Ax)^\circ$, $x \in \mathrm{dom}\, A$.

Associated with a maximal monotone operator $A : H \rightrightarrows H$ we consider the differential inclusion given in (1):

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \text{ a.e. } t \geq 0,\ x(0) = x_0 \in \overline{\mathrm{dom}\, A},$$

where $f : H \to H$ is a given (L-)Lipschitz continuous mapping. Every solution of differential inclusion (1) will be denoted by $x(\cdot; x_0)$.

We introduce the concept of invariant sets (see, e.g., [6,13,14]):

Definition 1. A set $S \subset \overline{\mathrm{dom}\, A}$ is said to be invariant for (1) provided that $x(t; x_0) \in S$ for every $x_0 \in S$ and every $t \geq 0$.

We also recall the following result on the existence of solutions of (1); for more details, we refer to [9].

Proposition 2. *For any $x_0 \in \overline{\operatorname{dom} A}$ and $T > 0$, system (1) has a unique continuous solution, which is the uniform limit on $[0, T]$ of $x_\lambda(\cdot; x_0)$ (as $\lambda \downarrow 0$), where $x_\lambda(\cdot; x_0)$ is the solution of the differential equation*

$$\dot{x}_\lambda(t) = f(x_\lambda(t)) - A_\lambda(x_\lambda(t)),\ x_\lambda(0) = x_0.$$

Moreover, the following holds:

(i) *For all $s, t \geq 0$ and all $y_0 \in \overline{\operatorname{dom} A}$ we have that*

$$x(s; x(t; x_0)) = x(t + s; x_0),\quad \|x(t; x_0) - x(t; y_0)\| \leq e^{Lt} \|x_0 - y_0\|.$$

(ii) *If $x(t_0, x_0) \in \operatorname{dom} A$ for some $t_0 \geq 0$, then*

$$\frac{d^+ x(t_0; x_0)}{dt} = (f(x(t_0; x_0)) - Ax(t_0; x_0))^0.$$

(iii) *The function $t \to \frac{d^+ x(t;x_0)}{dt}$ is right-continuous at every $t \geq t_0$, where $t_0 \geq 0$ is such that $x(t_0; x_0) \in \operatorname{dom} A$, and we have*

$$\left\| \frac{d^+ x(t; x_0)}{dt} \right\| \leq e^{L(t-t_0)} \left\| \frac{d^+ x(t_0; x_0)}{dt} \right\|.$$

3. Invariant sets

In this section, we achieve our first goal to characterize those closed sets in the Hilbert space H, which are invariant with respect to differential inclusion (1):

$$\dot{x}(t) \in f(x(t)) - A(x(t)),\quad t \in [0, \infty),\ x(0) = x_0 \in \overline{\operatorname{dom} A};$$

the unique solution of this inclusion is written $x(\cdot; x_0)$.

It is worth observing that whenever differential inclusion (1) possesses a strong solution starting from S ($x_0 \in S$), which is an absolutely continuous function such that $x(t; x_0) \in \operatorname{dom} A$ for all $t > 0$, each invariant closed set $S \subset \overline{\operatorname{dom} A}$ satisfies the condition

$$S = \overline{\operatorname{dom} A \cap S}. \tag{2}$$

However, this condition may not be true when only weak solutions exist. This is why we shall assume in what follows that our invariance candidate sets satisfy this "almost necessary" condition.

Remark 1. Theorem 4 below gives the main invariance criterion, stated in (3), for closed sets with respect to differential inclusion (1), using only the data in (1) which are the operator A and the mapping f. Hence, explicit calculus of either the solution or the semigroup generated by A are not required. Criterion (3) extends and adapts some of the results given in [3,4] on weakly closed invariant sets. Its geometric meaning is very similar to the classical ones established in [12,14] for differential inclusions of the form

$$\dot{x}(t) \in F(x(t)),$$

with a w^*-compact, nonempty and convex multifunction F. In our case, condition (3) takes into account that the right-hand side in (1), which is governed by a general maximal monotone operator, may have empty or unbounded values. As well, another crucial difference between (1) and the last inclusion above is that our analysis also allows the initial condition in (1) to start from the larger set $\overline{\mathrm{dom}\,A}$. Thus, the scope of our analysis goes beyond the differential inclusions treated in [6,7,12,14,15]. First invariance criteria for differential inclusions involving maximal monotone operators have been given in [19] (see, also, [9]) without considering the Lipschitzian perturbation. Such results have been extended in [11,17] to maximal monotone operators which are subject to Lipschitz perturbations, using criteria which depend on the semi-group of contractions generated by $-A$. Compared to [11,17] (see, also, references therein), condition (3) relies exclusively on the geometry of C as in [12,14].

Before we state the main theorem of this section, Theorem 4 below, we give the following lemma.

Lemma 3. *Given a closed set $S \subset H$ and an $m \geq 0$, we denote*

$$S_m := \left\{x \in S \cap \mathrm{dom}\,A \mid \left\|(f(x) - Ax)^0\right\| \leq m\right\}.$$

Then the set S_m is closed.

Proof. Take a sequence $(x_k)_k \subset S_m$ such that $x_k \to x$ $(\in S)$. Without loss of generality, and taking into account the norm-weak upper semi-continuity of the maximal monotone operator A, we conclude that the sequence $(P_{Ax_k}(f(x_k)))_k$ weakly converges to some $z \in Ax$. Then

$$\begin{aligned}\left\|(f(x) - Ax)^0\right\| &\leq \|f(x) - z\| \\ &\leq \liminf_{k\to\infty} \|f(x_k) - P_{Ax_k}(f(x_k))\| \\ &= \liminf_{k\to\infty} \left\|(f(x_k) - Ax_k)^0\right\| \leq m,\end{aligned}$$

so that $x \in S_m$. $\square$

Theorem 4. *Given a closed set $S \subset \overline{\mathrm{dom}\,A \cap S}$, we assume that for every $x \in S \cap \mathrm{dom}\,A$ there exist $m, r > 0$ such that $\|P_{Ax}(f(x))\| \leq m$ and*

$$\sup_{\xi\in \mathrm{N}^P_{S_m}(y)} \min_{y^*\in Ay\cap B(\theta,m)} \langle \xi, f(y) - y^*\rangle \leq 0 \quad \textit{for all } y \in B(x, r). \tag{3}$$

Then S is invariant for (1).

Proof. We fix $x_0 \in S \cap \mathrm{dom}\,A$ and $\varepsilon > 0$. Let $m, r > 0$ be as in the current assumption (with $x = x_0$), and choose an $M > 0$ such that

$$f(y) - Ay \cap B(\theta, m) \subset B(\theta, M) \quad \text{for all } y \in K := S_m \cap B(x_0, r). \tag{4}$$

We also choose sufficiently small numbers $\bar{t}, \delta > 0$ and a sufficiently large integer N such that

$$\max\{6M^2\bar{t}^2, 8\delta^2\} < \frac{r^2}{2},\ \delta < \frac{\bar{t}}{N}, \tag{5}$$

$$\max\left\{\frac{(M^2 + 4M + 1)\bar{t}^2}{N}, \frac{M^2\bar{t}^2}{N^2} + 2\delta^2\right\} < \frac{\varepsilon^2}{4}. \tag{6}$$

We denote by $\pi := \{t_0, t_1, ..., t_N\}$ the uniform partition of the interval $[0, \bar{t}]$. We put $d(\pi) := \max\limits_{0\leq i\leq N-1}(t_{i+1} - t_i) = \frac{\bar{t}}{N}$ and, by (4), we choose an element $s_0^* \in f(x_0) - A(x_0)$ such that $\|s_0^*\| \leq M$. We consider the function $z_0(t),\ t \in [t_0, t_1]$ such that

$$\begin{cases} \dot{z}_0(t) = s_0^*, \ t \in [t_0, t_1], \\ z_0(0) = x_0, \end{cases}$$

and denote $z_1 := x_0 + s_0^* t_1$. We pick $\hat{s}_1 \in P_K^\delta(z_1)$. Then there exists a pair (y_1, s_1) such that $s_1 \in K$, $y_1 - s_1 \in \mathrm{N}_K^P(s_1)$ and (see, e.g., [13,22])

$$\max\{\|y_1 - z_1\|, \|s_1 - \hat{s}_1\|\} \leq \delta, \ \|(y_1 - s_1) - (z_1 - \hat{s}_1)\| \leq 2\delta,$$

as well as (see [5, Lemma 4])

$$\|s_1 - x_0\|^2 \leq 6\|z_1 - x_0\|^2 + 8\delta^2 = 6t_1^2\|s_0^*\|^2 + 8\delta^2 < 6\bar{t}^2 M^2 + 8\delta^2 < r^2;$$

hence, $s_1 \in \mathrm{int}(B(x_0, r))$ and, so, $\mathrm{N}_K^P(s_1) = \mathrm{N}_{S_m}^P(s_1)$. Consequently, by the current assumption of the theorem, we find $s_1^* \in (f(s_1) - A(s_1)) \cap B(\theta, M)$ such that

$$\langle y_1 - s_1, s_1^* \rangle \leq 0.$$

With this vector s_1^* in hand, we consider the function $z_1(t)$, $t \in [t_1, t_2]$, such that

$$\begin{cases} \dot{z}_1(t) = s_1^*, \ t \in [t_1, t_2] \\ z_1(t_1) = z_1. \end{cases}$$

By repeating the arguments used above, for each $i \in \overline{2, N-1}$, we consider the function $z_i(t), \ t \in [t_i, t_{i+1}]$, such that

$$\begin{cases} \dot{z}_i(t) = s_i^*, \ t \in [t_i, t_{i+1}] \\ z_i(t_i) = z_{i-1}(t_i) =: z_i, \end{cases}$$

and the corresponding elements $(\hat{s}_i, y_i, s_i, s_i^*)$ such that $\hat{s}_i \in P_K^\delta(z_i)$, $y_i - s_i \in \mathrm{N}_K^P(s_i) = \mathrm{N}_{S_m}^P(s_i)$, $s_i^* \in [f(s_i) - A(s_i)] \cap B(\theta, M)$,

$$\langle y_i - s_i, s_i^* \rangle \leq 0,$$
$$\max\{\|y_i - z_i\|, \|s_i - \hat{s}_i\|\} \leq \delta, \ \|(y_i - s_i) - (z_i - \hat{s}_i)\| \leq 2\delta.$$

Now, we are going to prove that the absolute continuous trajectory $z(\cdot)$, defined on $[0, \bar{t}\,]$ as $z(t) := z_i(t) = z_i + (t - t_i)s_i^*$ for $t \in [t_i, t_{i+1}]$, satisfies

$$d_S(z(t)) \leq \varepsilon, \ \forall t \in [0, \bar{t}\,], \tag{7}$$

$$\|s_i - z(t)\| \leq 2\varepsilon, \ \forall t \in [t_i, t_{i+1}]. \tag{8}$$

Indeed, for any $1 \leq i \leq N - 1$, one has

$$\begin{aligned} d_K^2(z_{i+1}) &\leq \|z_{i+1} - \hat{s}_i\|^2 = \|z_{i+1} - z_i\|^2 + \|z_i - \hat{s}_i\|^2 + 2\langle z_{i+1} - z_i, z_i - \hat{s}_i \rangle \\ &= \|(t_{i+1} - t_i)s_i^*\|^2 + d_K^2(z_i) + \delta^2 + 2d(\pi)\langle s_i^*, z_i - \hat{s}_i \rangle \\ &\leq M^2 d^2(\pi) + d_K^2(z_i) + \delta^2 + 2d(\pi)\langle s_i^*, y_i - s_i \rangle \\ &\quad + 2d(\pi)\langle s_i^*, (z_i - \hat{s}_i) - (y_i - s_i) \rangle \\ &\leq d_K^2(z_i) + (M^2 + 4M + 1)d(\pi)(t_{i+1} - t_i), \end{aligned}$$

which gives us

$$
\begin{aligned}
d_K^2(z_{i+1}) &\le d_K^2(z_1) + (M^2+4M+1)d(\pi)(t_{i+1}-t_1) \\
&\le \|z_1 - x_0\|^2 + (M^2+4M+1)d(\pi)(t_{i+1}-t_1) \\
&\le (M^2+4M+1)d(\pi)\bar{t} \le \frac{(M^2+4M+1)\bar{t}^2}{N} < \frac{\varepsilon^2}{4}.
\end{aligned} \tag{9}
$$

This shows that, for every $t \in [t_i, t_{i+1}]$,

$$
\begin{aligned}
d_S^2(z(t)) \le d_K^2(z(t)) &= d_K^2(z_i(t))) = d_K^2(z_i(t_i) + (t-t_i)s_i^*) \\
&\le 2d_K^2(z_i) + 2(t-t_i)^2M^2 \le \frac{\varepsilon^2}{2} + 2d^2(\pi)M^2 \le \varepsilon^2
\end{aligned}
$$

and (7) follows. Inequality (8) also follows since that for every $t \in [t_i, t_{i+1}]$

$$
\begin{aligned}
\|s_i - z(t)\|^2 &\le 2\|z(t) - z_i\|^2 + 2\|s_i - z_i\|^2 \\
&\le 2(t-t_i)^2M^2 + 4\|s_i - \hat{s}_i\|^2 + 4\|z_i - \hat{s}_i\|^2 \\
&\le 2(t-t_i)^2M^2 + 4d_K^2(z_i) + 8\delta^2 \\
&\le 2d^2(\pi)M^2 + \varepsilon^2 + 8\delta^2 \le 2\varepsilon^2,
\end{aligned}
$$

where in the last inequality we used (9).

Now, let $x(t)$ be the (strong) solution of (1) starting at x_0, and denote $l_i(t) := s_i - z(t)$, $t \in [t_i, t_{i+1}]$, so that $\dot{z}(t) = s_i^* \in f(s_i) - A(s_i) = f(z(t) + l_i(t)) - A(z(t) + l_i(t))$. Hence, by using the monotonicity of A we get

$$
\langle f(z(t) + l_i(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t), z(t) + l_i(t) - x(t)\rangle \ge 0,
$$

which leads us, using (7) and (8) together with the L-Lipschitzianity of f, to

$$
\begin{aligned}
\langle \dot{z}(t) - \dot{x}(t), z(t) - x(t)\rangle &\le 2\varepsilon \left\| f(z(t) + l_i(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t)\right\| \\
&\quad + \|z(t) - x(t)\| \left\| f(z(t) + l_i(t)) - f(x(t))\right\| \\
&\le 2\varepsilon \|\dot{z}(t) - \dot{x}(t)\| + 2\varepsilon L \|z(t) + l_i(t) - x(t)\| \\
&\quad + L\|z(t) - x(t)\| \|z(t) + l_i(t) - x(t)\|.
\end{aligned}
$$

So, if C is any constant such that $\|\dot{z}(t) - \dot{x}(t)\| \le C$ for all $t \in [0, \bar{t}\,]$ (as $\|\dot{z}(t)\| \le M$, and $x(\cdot)$ is Lipschitz on $[0, \bar{t}\,]$), we get

$$
\langle \dot{z}(t) - \dot{x}(t), z(t) - x(t)\rangle \le 2\varepsilon C + 4\varepsilon L \|z(t) - x(t)\| + L\|z(t) - x(t)\|^2 + 4\varepsilon^2 L.
$$

Next, by applying Lemma 1 to the function $\|z(\cdot) - x(\cdot)\|^2 + \frac{2\varepsilon C + 4\varepsilon^2 L}{L}$ we get, for all $t \in [0, \bar{t}\,]$

$$
\|z(t) - x(t)\| \le \left(\frac{4\varepsilon^2 L + 2\varepsilon C}{L}\right)^2 e^{Lt} + 4\varepsilon(e^{Lt} - 1),
$$

implying that, in view of (7) and (8),

$$
d_S(x(t)) \le d_S(z(t)) + \|z(t) - x(t)\| \le \left(\frac{4\varepsilon^2 L + 2\varepsilon C}{L}\right)^2 e^{L\bar{t}} + 4\varepsilon e^{L\bar{t}}.
$$

Consequently, by the arbitrariness of ε we conclude that $x(t) \in S$ for every $t \in [0, \bar{t}\,]$. Moreover, as $x(\bar{t}; x_0) \in S \cap \operatorname{dom} A$, by the same argument as above we find $\hat{t} > 0$ such that for every $t \in [0, \hat{t}\,]$ (recall Proposition 2)

$$x(t + \bar{t}; x_0) = x(t; x(\bar{t}; x_0)) \in S \cap \operatorname{dom} A;$$

that is, $x(t) \in S$ for every $t \in [0, \bar{t} + \hat{t}\,]$. This proves that $x(t) \in S$ for every $t \geq 0$. Finally, if $x_0 \in \overline{S \cap \operatorname{dom} A}$, we take a sequence $(x_k) \subset S \cap \operatorname{dom} A$ such that $x_k \to x_0$. As we have just shown, for every $k \geq 1$ we have that $x(t; x_k) \in S$ for every $t \geq 0$. Thus, since S is closed, as $k \to +\infty$ we deduce that $x(t; x_0) \in S$ for every $t \geq 0$. $\square$

The proof of Theorem 4 shows actually the following:

Corollary 5. *Given a closed set $S \subset \overline{\operatorname{dom} A \cap S}$ and $x_0 \in S \cap \operatorname{dom} A$, we assume that for some $m, r > 0$ such that $\|P_{Ax}(f(x_0))\| \leq m$ it holds*

$$\sup_{\xi \in \mathrm{N}^P_{S_m}(y)} \min_{y^* \in Ay \cap B(\theta, m)} \langle \xi, f(y) - y^* \rangle \leq 0 \quad \textit{for all } y \in B(x_0, r).$$

Then there exists $\bar{t} > 0$ such that $x(t; x_0) \in S$ for all $t \in [0, \bar{t}\,]$.

As we show in the corollary below the criterion of Theorem 4 becomes necessary if the maximal monotone operator A has a minimal norm section, which is locally bounded relative to its domain. As typical examples of such operators there are normal cones to closed convex sets, and the subdifferential mapping of lsc convex functions, which are Lipschitz relative to their domains. To fix this concept we say that the operator A is locally minimally bounded on S, if for every $x \in S \cap \operatorname{dom} A$ there exist $m, r > 0$ such that

$$\left\| A^0 y \right\| \leq m \text{ for all } y \in S \cap \operatorname{dom} A \cap B(x, r). \tag{10}$$

This condition is less restrictive compared with the local boundedness of A relative to S, which means that for every $x \in S \cap \operatorname{dom} A$ there exist $m, r > 0$ such that

$$\|y^*\| \leq m,\ \forall y^* \in Ay,\ \ y \in S \cap \operatorname{dom} A \cap B(x, r). \tag{11}$$

Obviously every locally bounded operator is locally minimally bounded.

Then the following result gives necessary and sufficient simpler criteria for the invariance of closed sets with respect to differential inclusion (1), using the normal cone mapping to S, N_S, which stands for either the proximal normal cone N^P_S or the Fréchet normal cone N^F_S.

Corollary 6. *Let $S \subset H$ be a closed set satisfying (2). Then the following statements are equivalent, provided that A is locally minimally bounded on S,*

(i) *S is an invariant set for (1);*

(ii) *for every $x \in S \cap \operatorname{dom} A$*

$$f(x) - P_{Ax}(f(x)) \in T_S(x);$$

(iii) *for every $x \in S \cap \operatorname{dom} A$*

$$\sup_{\xi \in \mathrm{N}_S(x)} \langle \xi, f(x) - P_{Ax}(f(x)) \rangle \leq 0;$$

(iv) *for every* $x \in S \cap \operatorname{dom} A$ *and every* $m \geq \|f(x) - P_{Ax}(f(x))\|$

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in (f(x)-Ax) \cap B(\theta,m)} \langle \xi, x^* \rangle \leq 0;$$

and the following assertion, when A *is locally bounded relative to* S*,*

(v) *for every* $x \in S \cap \operatorname{dom} A$

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in f(x)-Ax} \langle \xi, x^* \rangle \leq 0.$$

Proof. We fix $x \in S \cap \operatorname{dom} A$. The implication (iii) $\Longrightarrow$ (iv) is immediate, while the implication (ii) $\Longrightarrow$ (iii) follows because $T_S(x) \subset (\mathrm{N}_S(x))^*$. In the same line, implication (i) $\Rightarrow$ (ii) follows easily by observing that

$$(f(x) - A(x))^0 = \frac{d^+ x(\cdot; x)}{dt}(0) = \lim_{t \downarrow 0} \frac{x(t;x) - x}{t} \in T_S(x).$$

Thus, we only need to prove that (iv) $\Rightarrow$ (i). If (iv) holds, by the current local boundedness assumption of A^0 on $S \cap \operatorname{dom} A$ we pick $m, r > 0$ such that $\left\|(f(y) - Ay)^0\right\| \leq m$ for all $y \in B(x, 2r) \cap S \cap \operatorname{dom} A$. Hence,

$$B(x, 2r) \cap S \cap \operatorname{dom} A = S_m \cap B(x, 2r),$$

and, since $S = \overline{S \cap \operatorname{dom} A}$, for every $y \in B(x, r) \cap S_m$,

$$\mathrm{N}_{S_m}(y) = \mathrm{N}_{S_m \cap B(x,2r)}(y) = \mathrm{N}_{S \cap \operatorname{dom} A \cap B(x,2r)}(y) = \mathrm{N}_{S \cap \operatorname{dom} A}(y) = \mathrm{N}_S(y).$$

So (iv) gives us, for every $y \in B(x, r) \cap S_m$,

$$\sup_{\xi \in \mathrm{N}_{S_m}(y)} \inf_{x^* \in (f(y)-Ay) \cap B(\theta,m)} \langle \xi, x^* \rangle \leq 0,$$

and (i) follows, according to Theorem 4.

Suppose now that A is locally bounded on $S \cap \operatorname{dom} A$, and consider the intermediate assertion

(iv)$'$ for every $x \in S \cap \operatorname{dom} A$ and every large enough $m \geq \left\|(f(x) - Ax)^0\right\|$ we have that

$$\sup_{\xi \in \mathrm{N}_S(x)} \inf_{x^* \in (f(x)-Ax) \cap B(\theta,m)} \langle \xi, x^* \rangle \leq 0.$$

As we see from the proof above (namely, the implication (iv) $\Rightarrow$ (i)), we have that (iv)$'$ $\Rightarrow$ (i), so that (v) $\Rightarrow$ (iv)$'$ $\Rightarrow$ (i). The proof of the corollary is finished because the implication (iv) $\Longrightarrow$ (v) is immediate. □

In the following corollary we deduce another sufficient condition for the invariance of closed sets, using the Moreau–Yoshida approximations of A. Observe that we do not require here that set S satisfies condition (2).

Corollary 7. *Given a closed set* $S \subset H$*, we suppose that for every bounded subsets* B *of* S

$$\liminf_{\lambda \downarrow 0} \sup_{y \in B} \sup_{\xi \in \mathrm{N}_S^P(y)} \langle \xi, f(y) - A_\lambda y \rangle \leq 0.$$

Then S *is invariant set for* (1).

Proof. Fix an $x \in S$ and let $x(\cdot; x)$ be the corresponding solution of (1). Given an $r > 0$ we let λ_k, $k \geq 1$, be such that $\lambda_k \downarrow 0$ and

$$\sup_{\xi \in \mathrm{N}_S^P(y)} \langle \xi, f(y) - A_{\lambda_k} y \rangle \leq 0 \text{ for all } k \geq 1 \text{ and } y \in B(x,r) \cap S. \tag{12}$$

If $\varepsilon < \frac{r}{4}$ and $\bar{t} > 0$ are such that $x(t;x) \in B(x, \frac{r}{4})$ for all $t \in [0, \bar{t}]$, then for large enough $k \geq 1$ the solution $x_{\lambda_k}(\cdot; x)$ of the differential equation $\dot{x}(t) = f(x(t)) - A_{\lambda_k}(x(t)),\ x(0) = x$, satisfies (see Proposition 2)

$$\|x(t;x) - x_{\lambda_k}(t;x)\| \leq \varepsilon < \frac{r}{4}; \tag{13}$$

hence, $x_{\lambda_k}(t;x) \in B(x, \frac{r}{2})$ for all $t \in [0, \bar{t}]$. On the other hand, since A_{λ_k} is Lipschitz continuous, for large enough $m > 0$ we have $B(x,r) \cap S = \{z \in B(x,r) \cap S \mid \|A_{\lambda_k} z\| \leq m\}$. So, according to Corollary 5, (12) ensures that for some $\hat{t} > 0$, say $\hat{t} \in (0, \bar{t})$, it holds $x_{\lambda_k}(t;x) \in S$ for all $t \in [0, \hat{t}]$. Since $x_{\lambda_k}(t;x) \in B(x, \frac{r}{2})$ for all $t \in [0, \bar{t}]$, we infer that $x_{\lambda_k}(t;x) \in B(x, \frac{r}{2}) \cap S$ for all $t \in [0, \bar{t}]$. Consequently, by (13) we get $d_S(x(t;x)) \leq \varepsilon$ for all $t \in [0, \bar{t}]$. Then, as $\varepsilon \to 0$, we deduce that $x(t;x) \in S$ for all $t \in [0, \bar{t}]$. Finally, the invariance of S follows by using the semi-group property of the solution $x(\cdot; x)$ (see again Proposition 2). □

We consider now the special case where $f \equiv \theta$, so that our differential inclusion (1) takes the simpler form

$$\dot{x}(t) \in -Ax(t), \quad x(0) = x_0 \in \overline{\operatorname{dom} A}. \tag{14}$$

In this case, the criterion of Theorem 4 becomes also necessary as the following corollary shows. Here too N_{S_m} stands for either $\mathrm{N}_{S_m}^P$ or $\mathrm{N}_{S_m}^F$.

Corollary 8. *Let $S \subset H$ be a closed set satisfying (2). Then the following statements are equivalent:*

(i) *S is an invariant set of (14);*
(ii) *for every $x \in S \cap \operatorname{dom} A$*

$$-A^0 x \in T_{S_m}(x) \quad \textit{for all } m \geq \left\|A^0 x\right\|;$$

(iii) *for every $x \in S \cap \operatorname{dom} A$ and for every $m \geq \left\|A^0 x\right\|$*

$$\sup_{\xi \in \mathrm{N}_{S_m}(x)} \langle \xi, -A^0 x \rangle \leq 0;$$

(iv) *for any $x \in S \cap \operatorname{dom} A$ and every $m \geq \left\|A^0 x\right\|$*

$$\sup_{\xi \in \mathrm{N}_{S_m}(x)} \inf_{x^* \in (-Ax) \cap B(\theta, m)} \langle \xi, x^* \rangle \leq 0.$$

Proof. As in the proof of Corollary 6, the implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^F$) $\Longrightarrow$ (iv with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^P$) are immediate. For the implication (i) $\Longrightarrow$ (ii), we assume that S is an invariant set of (14). If $x \in S \cap \operatorname{dom} A$, then for a given $m \geq \left\|A^0 x\right\|$ we have

$$\left\|A^0 x(t;x)\right\| = \left\|\frac{d^+ x(t;x)}{dt}\right\| \leq \left\|\frac{d^+ x(0;x)}{dt}\right\| = \left\|A^0 x\right\| \leq m, \text{ for all } t \geq 0.$$

Hence, $x(t;x) \in S_m$ for all $t \geq 0$ and we deduce that $-A^0 x = \frac{d^+ x(0;x)}{dt} \in T_{S_m}(y)$, yielding (ii). Finally, the implication (iv with $\mathrm{N}_{S_m} = \mathrm{N}_{S_m}^P$) $\Longrightarrow$ (i) is direct from Theorem 4. □

To show how can our Theorem 4 be applied we consider the following example, which is treated in details in [5] in order to study the existence and the stability of solutions of differential inclusions involving the normal cone to a prox-regular set.

Recall that a closed set $C \subset H$ is said to be uniformly r-prox-regular ($r > 0$) if for every $x \in C$ and $\xi \in \mathrm{N}_C^P(x) \cap B(\theta, 1)$ we have [21]

$$\langle \xi, y - x \rangle \leq \frac{1}{2r} \|y - x\|^2 \text{ for all } y \in C.$$

Example 1. Let $C \subset H$ be a uniformly r-prox-regular set and consider the associated differential inclusion

$$\dot{x}(t) \in g(x(t)) - \mathrm{N}_C(x(t)), \text{ a.e. } t \in [0, T], \ x(0) = x_0 \in C, \tag{15}$$

where g is a Lipschitz mapping on H. According to [5, Lemma 6(c)], let $T : H \rightrightarrows H$ be a maximal monotone operator such that for some $m \geq 0$ it holds, for all $y \in C$,

$$\mathrm{N}_C(y) \cap B(0, m) + \frac{m}{r} y \subset T(y) \subset \mathrm{N}_C(y) + \frac{m}{r} y,$$

and consider the associated differential inclusion

$$\dot{x}(t) \in g(x(t)) + \frac{m}{r} x(t) - Tx(t), \text{ a.e. } t \in [0, T], \ x(0) = x_0 \in C \ (\subset \operatorname{dom} T). \tag{16}$$

This inclusion perfectly fits the form of differential inclusion (1). Then we make appeal to Theorem 4 to prove that the set C is invariant for (1), so that

$$\dot{x}(t) \in g(x(t)) + \frac{m}{r} x(t) - Tx(t) \subset g(x(t)) - \mathrm{N}_C(x(t)),$$

providing us with a solution for (15). We refer to [5] for more details.

4. Lyapunov pairs and functions

In this section, we apply the results of the previous section to derive different criteria for a-Lyapunov pairs with respect to differential inclusion (1):

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad t \in [0, \infty), \ x(0) = x_0 \in \overline{\operatorname{dom} A},$$

whose unique solution is written $x(\cdot; x_0)$. Similar criteria to ours have been established recently in [3,4] in the case of weakly lsc Lyapunov pairs.

Definition 2. We say that a pair (V, W) of proper lsc functions $V, W : H \to \overline{\mathbb{R}}$ with $W \geq 0$, is (or forms) an a-Lyapunov pair ($a \geq 0$) with respect to system (1) if, for every $x_0 \in \overline{\operatorname{dom} A}$,

$$e^{at} V(x(t; x_0)) + \int_s^t W(x(\tau; x_0)) d\tau \leq e^{as} V(x(s; x_0)), \quad \text{for all } t \geq s \geq 0.$$

Observe that (V, W) is an a-Lyapunov pair with respect to system (1) iff for every $x_0 \in \overline{\operatorname{dom} A}$ there exists a $t > 0$ such that (see, e.g., [3, Proposition 3.2])

$$e^{as}V(x(s;x_0)) + \int_0^s W(x(\tau;x_0))d\tau \leq V(x_0), \quad \text{for all } s \in [0,t]\,.$$

We may assume without loss of generality that W is Lipschitz continuous on every bounded set (see, e.g., [3, Lemma 3.1] or [14, Theorem 1.5.1]). While, concerning function V, one need to suppose the following condition

$$V(x) = \liminf_{y \overset{\mathrm{dom}\, A}{\to} x} V(y) \quad \text{for every } x \in \mathrm{dom}\, V, \tag{17}$$

which is in fact necessary for V to be a Lypunov function in many important cases (for instance, when differential inclusion (1) possesses a strong solution).

Theorem 9. *Given two proper lsc functions $V : H \to \overline{\mathbb{R}}$ satisfying (17), $W : H \to \overline{\mathbb{R}}_+$, and a real number $a \geq 0$, we assume that for every $x \in \mathrm{dom}\, V \cap \mathrm{dom}\, A$ there are $m, r > 0$ such that $\|P_{Ax}(f(x))\| \leq m$ and, for all $y \in B(x,r)$,*

$$\sup_{\xi \in \partial_P (V + \mathrm{I}_{A_m})(y)} \; \inf_{y^* \in Ay \cap B(\theta,m)} \langle \xi, f(y) - y^* \rangle + aV(x) + W(x) \leq 0.$$

Then (V,W) forms an a-Lyapunov pair with respect to system (1).

Proof. We fix $T > 0$ and $x_0 \in \mathrm{dom}\, V \cap \mathrm{dom}\, A$. Following the discussion made before the current theorem we may suppose without loss of generality that W is Lipschitz continuous on every bounded set containing the trajectory $\{x(t;x_0), t \in [0,T]\}$.

Let us define the maximal monotone operator $\hat{A} : H \times \mathbb{R}^4 \rightrightarrows H \times \mathbb{R}^4$ and the Lipschitz function $\hat{f} : H \times \mathbb{R}^4 \to H \times \mathbb{R}^4$ as

$$\hat{A}(x,\mu) := (Ax, \theta_{\mathbb{R}^4}), \;\; \hat{f}(x,\mu) := (f(x), 1, 0, 1, 0),$$

and, given a fixed $\mu_0 \in \mathbb{R}^4$, consider the associated differential inclusion given in $H \times \mathbb{R}^4$ by

$$\dot{y}(t) \in \hat{f}(y(t)) - \hat{A}(y(t)), \;\; \text{a.e. } t \in [0,T]; \quad y(0) = (x_0, \mu_0), \tag{18}$$

whose unique solution is $y(t) := (x(t), t, 0, t, 0) + (\theta, \mu_0)$, $t \in [0,T]$ (with $x(t) := x(t;x_0)$).

For each $n \geq 1$, we consider the lsc function $V_n : H \times \mathbb{R}^3 \to \overline{\mathbb{R}}$ defined as

$$V_n(x,\alpha,\beta,\gamma) := e^{a\gamma}V(x) + (\alpha - \beta)g_n(\alpha) + \frac{l}{2}(\alpha - \beta)^2, \tag{19}$$

where g_n is an l-Lipschitz extension of the function $W(x(\cdot;x_0)) - \frac{1}{n}$ from $[0,T]$ to $[-1,T+1]$; hence,

$$\partial_C g_n(\alpha) \subset B(0,l) \;\; \text{for all } \alpha \in [0,T+1]. \tag{20}$$

We denote

$$S := \mathrm{epi}\, V_n,$$

so that $S = \overline{S \cap \mathrm{dom}\, \hat{A}}$, by (17), and

$$\mathrm{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3}) = S \cap \hat{A}_m =: S_m. \tag{21}$$

We also denote $y_0 := (x_0, \theta_{\mathbb{R}^3}, V(x_0)) \in S \cap \operatorname{dom} \hat{A}$. Let $m, r > 0$ be as in the current assumption, corresponding to x_0, and choose $\bar{r} < r$ small enough such that for all $(x, \alpha, \beta, \gamma) \in B((x_0, \theta_{\mathbb{R}^3}), \bar{r})$

$$g_n(\alpha) - e^{a\gamma} W(x) + 2l\,|\alpha - \beta| \leq \frac{-1}{2n}. \tag{22}$$

Take $y := (y_1, \mu_1) \in B(y_0, \bar{r}) \cap S_m$, with $y_1 := (x_1, \alpha_1, \beta_1, \gamma_1)$, and pick $(\xi, -\kappa) \in \mathrm{N}^P_{S_m}(y)$. Due to (21) and [14, Exercise 1.2.1],

$$(\xi, -\kappa) \in \mathrm{N}^P_{S_m}(y) = \mathrm{N}^P_{\operatorname{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})}(y) \subset \mathrm{N}^P_{\operatorname{epi}(V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})}(y_1, V_n(y_1))\,;$$

hence, $\kappa \geq 0$. If $\kappa > 0$, say $\kappa = 1$ for simplicity, then $\xi \in \partial_P (V_n + \mathrm{I}_{A_m \times \mathbb{R}^3})(y_1)$ and, thanks to (19), we find $\xi_1 \in \partial_P (V + \mathrm{I}_{A_m})(x_1)$ and $\varsigma \in \partial_P g_n(\alpha_1) \subset \partial_C g_n(\alpha_1)$ such that

$$\xi \in (e^{a\gamma_1}\xi_1, g_n(\alpha_1) + (\alpha_1 - \beta_1)(\varsigma + l), -g_n(\alpha_1) + l(\beta_1 - \alpha_1), a e^{a\gamma_1} V(x_1))\,.$$

Since $y \in B(y_0, \bar{r}) \cap S_m$ we have that $x_1 \in B(x_0, \bar{r}) \cap A_m \cap \operatorname{dom} V$ and, so, by the current assumption, there exists an $x_1^* \in Ax_1 \cap B(\theta, m)$ (this last set being weak*-compact) such that

$$\langle \xi_1, f(x_1) - x_1^* \rangle + aV(x_1) + W(x_1) \leq 0.$$

Then we obtain (recall (20) and (22))

$$\begin{aligned}
\langle (\xi, -1), (f(x_1) - x_1^*, 1, 0, 1, 0) \rangle &= \langle e^{a\gamma_1}\xi_1, f(x_1) - x_1^* \rangle + g_n(\alpha_1) \\
&\quad + (\alpha_1 - \beta_1)(\varsigma + l) + a e^{a\gamma_1} V(x_1) \\
&= e^{a\gamma_1} \left(\langle \xi_1, f(x_1) - x_1^* \rangle + aV(x_1) + W(x_1) \right) \\
&\quad + g_n(\alpha_1) \; - e^{a\gamma_1} W(x_1) + (\alpha_1 - \beta_1)(\varsigma + l) \\
&\leq g_n(\alpha_1) - e^{a\gamma_1} W(x_1) + 2l\,|\alpha_1 - \beta_1| \leq \frac{-1}{2n}.
\end{aligned} \tag{23}$$

If $\kappa = 0$, then thanks to (19) we find $\xi_2 \in H$ such that $\xi = (\xi_2, \theta_{\mathbb{R}^3})$, with the property that there are sequences $\lambda_k \downarrow 0$, $z_k \overset{V + \mathrm{I}_{A_m}}{\longrightarrow} x_1$, $\zeta_k \in \partial_P (V + \mathrm{I}_{A_m})(z_k)$ such that $\lambda_k \zeta_k \to \xi_2$ as $k \to \infty$. By the current assumption, for each large enough k so that $z_k \in B(x_0, r)$ there exists $z_k^* \in Az_k \cap B(\theta, m)$ such that

$$\langle \zeta_k, f(z_k) - z_k^* \rangle + aV(z_k) + W(z_k) \leq 0.$$

Because A is maximal monotone and $(z_k^*)_k$ is bounded, we can find an $x_2^* \in Ax_1 \cap B(\theta, m)$ such that $\langle \xi_2, f(x_1) - x_2^* \rangle \leq 0$; hence, by multiplying the last inequality above by λ_k and taking the limit as $k \to \infty$,

$$\langle (\xi, 0), (f(x_1) - x_2^*, 1, 0, 1, 0) \rangle = \langle \xi, f(x_1) - x_2^* \rangle \leq 0. \tag{24}$$

According to Corollary 5, (23) and (24) imply the existence of some $\bar{t} := \bar{t}(n) \in (0, T]$ such that for every $t \in [0, \bar{t}\,]$,

$$(x(t), t, 0, t, V(x_0)) \in S;$$

in other words, $e^{at} V(x(t)) + t g_n(t) + \frac{l}{2} t^2 \leq V(x_0)$ and, so, for every $t \in [0, \bar{t}\,]$

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \le e^{at}V(x(t)) + \int_0^t (g(t) + l(t-\tau))d\tau + \frac{t}{n} \le V(x_0) + \frac{t}{n}. \tag{25}$$

Now, we claim that for all $t \in [0, T]$

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \le V(x_0) + \frac{e^{(1+a)t}}{n}. \tag{26}$$

To prove this claim we define

$$t^* := \sup\{t \in [0, T] \mid \text{inequality (26) holds on } [0, t]\}.$$

Indeed, from (25) and the lsc of V, it follows that (26) holds at t^*. If $t^* < T$, we denote $y^* := (x(t^*), \theta_{\mathbb{R}^3}, V(x(t^*)))$ and we easily check that $y^* \in S \cap \operatorname{dom} \hat{A}$. Then, arguing as with y_0 above, we arrive at a relation which is similar to (25); that is, there is some $\hat{t} > 0$ such that for all $t \in [0, \hat{t}\,]$

$$e^{at}V(x(t; x(t^*))) + \int_0^t W(x(\tau; x(t^*)))d\tau \le V(x(t^*)) + \frac{t}{n}. \tag{27}$$

Hence,

$$\begin{aligned}
&e^{a(t+t^*)}V(x(t+t^*)) + \textstyle\int_0^{t+t^*} W(x(\tau))d\tau \\
&\quad\le e^{a(t+t^*)}V(x(t+t^*)) + \textstyle\int_0^{t+t^*} W(x(\tau))d\tau + (e^{at^*} - 1)\int_0^t W(x(\tau+t^*))d\tau \\
&\quad= e^{at^*}(e^{at}V(x(t+t^*)) + \textstyle\int_0^t W(x(\tau+t^*))d\tau - \frac{t}{n}) + \int_0^{t^*} W(x(\tau))d\tau + \frac{e^{at^*}t}{n} \\
&\quad\le e^{at^*}V(x(t^*)) + \textstyle\int_0^{t^*} W(x(\tau))d\tau + \frac{e^{at^*}t}{n} \\
&\quad\le V(x_0) + \frac{e^{(1+a)t^*}}{n} + \frac{e^{at^*}t}{n}.
\end{aligned}$$

Consequently, due to the inequality $e^{\gamma} \ge 1 + \gamma$, we obtain that for all $t \in [0, \hat{t}\,]$

$$e^{a(t+t^*)}V(x(t+t^*)) + \int_0^{t+t^*} W(x(\tau))d\tau \le V(x_0) + \frac{e^{(1+a)(t+t^*)}}{n},$$

leading us to a contradiction with the definition of t^*.

Now, the claim being true, we take the limit in (26) as n goes to $+\infty$ to obtain that

$$e^{at}V(x(t)) + \int_0^t W(x(\tau))d\tau \le V(x_0) \text{ for all } t \in [0, T].$$

Finally, if $x_0 \in \operatorname{dom} V$, then by the current assumption (17), there exists a sequence $(x_k)_{k\ge1} \subset \operatorname{dom} V \cap \operatorname{dom} A$ such that $x_k \xrightarrow{V} x_0$. Thus, from the last inequality above we conclude that

$$e^{at}V(x(t; x_k)) + \int_0^t W(x(\tau; x_k))d\tau \le V(x_k) \text{ for all } t \in [0, T] \text{ and all } k \ge 1.$$

Hence, as k goes to $+\infty$, the lsc of V and Proposition 2 ensure that

$$e^{at}V(x(t;x_0)) + \int_0^t W(x(\tau;x_0))d\tau \leq V(x_0) \text{ for all } t \in [0,T],$$

showing that (V,W) is an a-Lyapunov pair. □

As in the case of the invariance of closed sets, the criterion of Theorem 9 takes a more simpler form when the maximal monotone operator A, or its minimal norm section, A^0, is locally bounded (see (10)). Here, ∂V stands for either $\partial_P V$ or $\partial_F V$.

Corollary 10. *Given two proper lsc functions $V, W : H \to \overline{\mathbb{R}}$, such that $W \geq 0$ and (17) holds, and a number $a \geq 0$, we assume that A is minimally locally bounded relative to* $\operatorname{dom} V$. *Then the following statements are equivalent.*

(i) *(V,W) is an a-Lyapunov pair for (1);*

(ii) *for any $x \in \operatorname{dom} V \cap \operatorname{dom} A$*

$$\sup_{\xi\in\partial V(x)} \langle \xi, (f(x) - Ax)^0\rangle + aV(x) + W(x) \leq 0;$$

(iii) *for any $x \in \operatorname{dom} V \cap \operatorname{dom} A$*

$$V'(x;(f(x) - Ax)^0) + aV(x) + W(x) \leq 0;$$

Moreover, if in addition, (11) holds, then the above statements are also equivalent to

(iv) *for any $x \in \operatorname{dom} V \cap \operatorname{dom} A$*

$$\sup_{\xi\in\partial V(x)} \inf_{x^*\in Ax} \langle \xi, f(x) - x^*\rangle + aV(x) + W(x) \leq 0;$$

(v) *for any $x \in \operatorname{dom} V \cap \operatorname{dom} A$*

$$\inf_{v\in Ax} V'(x; f(x) - v) + aV(x) + W(x) \leq 0.$$

Proof. First, the implications (iii) (with $\partial = \partial_F$) ⇒ (iii) (with $\partial = \partial_P$) ⇒ (ii) follow since that $\partial_P \subset \partial_F$ and $\sigma_{\partial_F V(x)} \leq V'(x;\cdot)$.

(i) ⇒ (iii). Fix $x_0 \in \operatorname{dom} V \cap \operatorname{dom} A$. Since (V,W) is an a-Lyapunov for (1), we have that for all $t > 0$

$$\frac{V(x(t;x_0)) - V(x_0)}{t} + \frac{e^{at}-1}{t}V(x(t;x_0)) + \frac{1}{t}\int_0^t W(x(\tau;x_0))d\tau \leq 0,$$

while Proposition 3 ensures that

$$\lim_{t\downarrow 0}\frac{x(t;x_0) - x_0}{t} = \frac{d^+x(0;x_0)}{dt} = (f(x_0) - Ax_0)^0.$$

Hence, using the lsc of V together with the continuity of $x(\cdot,x_0)$,

$$V'(x_0;(f(x_0) - Ax_0)^0) \leq \liminf_{t\downarrow 0} \frac{V(x(t;x_0)) - V(x_0)}{t} \leq -aV(x_0) - W(x_0), \tag{28}$$

leading us to (ii).

(ii) (with$\partial = \partial_P$) $\Rightarrow$ (i). We fix $x_0 \in \operatorname{dom} V \cap \operatorname{dom} A$. From the one hand, by the boundedness assumption of A^0, for a large $m \geq 0$ there exists an $r > 0$ such that

$$B(x_0, r) \cap \operatorname{dom} V \cap \operatorname{dom} A \subset A_m. \tag{29}$$

On the other hand, we have that

$$\partial_P(V + \mathrm{I}_{A_m})(x) \subset \partial_P V(x) \;\text{ for all } x \in B(x_0, \frac{r}{2}). \tag{30}$$

Indeed, if $\xi \in \partial_P(V + I_{A_m})(x)$ for $x \in B(x_0, \frac{r}{2})$, there exist $\delta > 0$ and $\rho \in (0, \frac{r}{2})$ such that

$$(V + \mathrm{I}_{A_m})(z) \geq V(x) + \langle \xi, z - x\rangle - \delta\|z - x\|^2 \quad \forall\, z \in B(x, \rho).$$

Take $z \in B(x, \frac{\rho}{4}) \cap \operatorname{dom} V$ $(\subset B(x_0, r))$. By (17) together with (29), there exists a sequence $(z_n)_n \subset B(x, \rho) \cap \operatorname{dom} V \cap A_m$ such that $z_n \to z$ and $V(z_n) \to V(z)$. Since each z_n satisfies the last inequality above, by taking the limit as $n \to \infty$ we arrive at $V(z) \geq V(x) + \langle \xi, z - x\rangle - \delta\|z - x\|^2$ and the inclusion (30) follows.

At this stage, from (29) and the Lipschitzianity of f there exists some $M \geq m$ such that, for all $x \in B(x_0, r)$,

$$\|P_{Ax}(f(x))\| \leq \|f(x)\| + \|A^\circ x\| \leq \|f(x)\| + m \leq M,$$

which shows that $(f(x) - Ax)^0 \in f(x) - Ax \cap B(\theta, M)$. Since $\partial_P(V + \mathrm{I}_{A_M}) \subset \partial_P(V + \mathrm{I}_{A_m})$, in view of (30), assumption (ii) (with$\partial = \partial_P$) implies that, for every $x \in B(x_0, \frac{r}{2})$

$$\sup_{\xi \in \partial_P(V + \mathrm{I}_{A_M})(x)} \inf_{x^* \in Ax \cap B(\theta, M)} \langle \xi, f(x) - x^*\rangle + aV(x) + W(x) \leq$$
$$\sup_{\xi \in \partial_P V(x)} \langle \xi, (f(x) - Ax)^0\rangle + aV(x) + W(x) \leq 0.$$

Thus, (i) follows from Theorem 9.

Finally, if A is locally bounded on $\operatorname{dom} V$, then from the first part of the proof one only needs to verify the implication (iv) $\Longrightarrow$ (i), the proof of which is similar to the one of "(ii) $\Rightarrow$ (i)" that we did above. $\square$

In the following corollary we provide criteria for a-Lyapunov pairs, which use the Moreau–Yoshida approximation of A.

Corollary 11. *Let V, W and a be as in Corollary 10, and let ∂ be such that $\partial_P \subset \partial \subset \partial_C$. If there exist $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$*

$$\sup_{\xi \in \partial V(x)} \langle \xi, f(x) - A_\lambda x\rangle + aV(x) + W(x) \leq 0, \forall\, x \in \operatorname{dom} V,$$

then (V, W) is an a-Lyapunov pair for (1).

Proof. Fix $x_0 \in \operatorname{dom} V$ and $t \geq 0$. If $x_\lambda(\cdot; x_0)$ is the solution of the differential equation

$$\dot{x}_\lambda(t) = f(x_\lambda(t)) - A_\lambda(x_\lambda(t)),\; x_\lambda(0) = x_0 \;(\lambda \in (0, \lambda_0]), \tag{31}$$

then, according to Corollary 10(ii), the pair (V, W) is an a-Lyapunov pair of (31); that is,

$$e^{at}V(x_\lambda(t)) + \int_0^t W(x_\lambda(\tau))d\tau \le V(x_0) \text{ for all } t \ge 0.$$

Hence, the conclusion follows as $\lambda \downarrow 0$. $\square$

We consider now the case when $f \equiv 0$ so that differential inclusion (1) reads

$$\dot{x}(t) \in -A(x(t)), x(0) = x_0 \in \overline{\operatorname{dom} A}. \tag{32}$$

In the following theorem ∂ stands for either ∂_P or ∂_F.

Corollary 12. *Let $V, W : H \to \overline{\mathbb{R}}$ be two proper lsc functions, such that $W \ge 0$ and (17) holds, and let $a \ge 0$. Then the following statements are equivalent:*

(i) *(V, W) is an a-Lyapunov pair for (32);*
(ii) *for every $x \in \operatorname{dom} V \cap \operatorname{dom} A$ and every $m \ge \|A^0 x\|$*

$$\sup_{\xi \in \partial(V + \mathrm{I}_{A_m})(x)} \langle \xi, -A^0 x\rangle + aV(x) + W(x) \le 0;$$

(iii) *for every x and m as in* (ii)

$$\sup_{\xi \in \partial(V + \mathrm{I}_{A_m})(x)} \inf_{x^* \in -Ax \cap B(\theta, m)} \langle \xi, x^*\rangle + aV(x) + W(x) \le 0;$$

(iv) *for every x and m as in* (ii)

$$(V + \mathrm{I}_{A_m})'(x; -A^0 x) + aV(x) + W(x) \le 0;$$

(v) *for every x and m as in* (ii)

$$\inf_{v \in -Ax \cap B(\theta, m)} (V + \mathrm{I}_{A_m})'(x; v) + aV(x) + W(x) \le 0.$$

Proof. The implications (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), (iv) $\Rightarrow$ (ii), and (v) $\Rightarrow$ (iii) are immediate. To prove that (i) $\Rightarrow$ (iv), we fix $x_0 \in \operatorname{dom} V \cap \operatorname{dom} A$ and $m \ge \|A^0 x_0\|$. According to Proposition 2, for any $t \ge 0$ we have that

$$\left\|-A^0(x(t; x_0))\right\| = \left\|\frac{d^+x(t; x_0)}{dt}\right\| \le \left\|\frac{d^+x(0; x_0)}{dt}\right\| = \left\|-A^0 x_0\right\| \le m;$$

that is, $x(t, x_0) \in A_m$ for all $t \ge 0$. Hence, since $\frac{x(t,x_0)-x_0}{t} \to -A^0 x_0$ as $t \downarrow 0$, provided that (V, W) is an a-Lyapunov pair for (32) we obtain, by arguing as in the proof of (28)),

$$\begin{aligned}(V + \mathrm{I}_{A_m})'(x_0; -A^0 x_0) &\le \liminf_{t \downarrow 0} \frac{(V + \mathrm{I}_{A_m})(x(t; x_0)) - (V + \mathrm{I}_{A_m})(x_0)}{t} \\ &= \liminf_{t \downarrow 0} \frac{V(x(t; x_0)) - V(x_0)}{t} \le -aV(x) - W(x),\end{aligned}$$

giving rise to (iv).

Finally, the conclusion of the corollary follows because the implication (iii) $\Rightarrow$ (i) holds according to Theorem 9. $\square$

We obtain the following corollary, which can be find in [17]; the original version of this result was established in [19]

Corollary 13. *Let $V, W : H \to \overline{\mathbb{R}}$ be two proper lsc functions, such that $W \geq 0$, and let $a \geq 0$. If condition (17) and, for every $x \in \operatorname{dom} V$,*

$$\liminf_{\lambda \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{\lambda} + aV(x) + W(x) \leq 0,$$

then (V, W) is an a-Lyapunov pair for (32).

Proof. We fix $x \in \operatorname{dom} V \cap A_m$ for some large $m \geq 1$. Since $A_\lambda x \in A(J_\lambda x)$ and $\|A_\lambda x\| \leq \|A^\circ x\| \leq m$, we infer that $J_\lambda x \in A_m$ and, so, using the current assumption,

$$(V + \mathrm{I}_{A_m})'(x_0; -A^0 x_0) \leq \liminf_{t \downarrow 0} \frac{V(J_\lambda(x)) - V(x)}{t} \leq -aV(x) - W(x).$$

The conclusion follows then from Corollary 12(iv). □

Corollary 10 obviously covers the case when A is the null operator, where (1) becomes a usual differential equation stated in the Hilbert space H as

$$\dot{x}(t) = f(x(t)), \text{ a.e. } t \geq 0, \quad x(0) = x_0 \in H. \tag{33}$$

The following characterization is known when ∂ is the viscosity subdifferential as defined in [17, Definition 2.7], while the case of weakly lsc a-Lyapunov pairs can be found in [3].

Corollary 14. *Let V, W and a be as in Corollary 10, and let ∂ be such that $\partial_P \subset \partial \subset \partial_C$. Then the following statements are equivalent:*

(i) *(V, W) is an a-Lyapunov pair for differential equation (33),*
(ii) *for every $x \in \operatorname{dom} V$*

$$\sup_{\xi \in \partial V(x)} \langle \xi, f(x) \rangle + aV(x) + W(x) \leq 0, \tag{34}$$

(iii) *for every $x \in \operatorname{dom} V$*

$$V'(x; f(x)) + aV(x) + W(x) \leq 0.$$

Proof. In view of Corollary 10, we only need to check that (i) $\Longrightarrow$ (ii) (with $\partial = \partial_C$), and this easily follows from the relation $\partial_C V = \overline{\mathrm{co}}\{\partial_L V + \partial_\infty V\}$. Indeed, assume that (i) holds and take $\xi \in \partial_L V(x)$ and $\zeta \in \partial_\infty V(x)$. By the definition of $\partial_L V(x)$ we choose sequences $\xi_k \in \partial_P V(x_k)$ such that $x_k \xrightarrow{V} x$ and $\xi_k \rightharpoonup \xi$. Then, by (i),

$$\langle \xi_k, f(x_k) \rangle + aV(x_k) + W(x_k) \leq 0 \text{ for all } k \geq 1,$$

and, so, as $k \to \infty$, we deduce that $\langle \xi, f(x) \rangle + aV(x) + W(x) \leq 0$. Similarly, we choose sequences $x_k \xrightarrow{V} x$ and $\lambda_k \downarrow 0$ such that $\zeta_k \in \partial_P V(x_k)$ and $\lambda_k \zeta_k \rightharpoonup \zeta$. Then, by arguing as above we deduce that $\langle \zeta, f(x) \rangle \leq 0$, which in turn yields

$$\langle \xi + \zeta, f(x) \rangle + aV(x) + W(x) \leq 0,$$

and this gives us (ii) (with $\partial = \partial_C$) by convexification. □

We close this section by analyzing a typical example of Lyapunov pairs.

Example 2. Assume that a function $V : H \to \overline{\mathbb{R}}$ is a proper, convex and lsc, and consider the differential inclusion

$$\dot{x}(t) \in -\partial V(x(t)).$$

Then the pair $(V, \left\|(\partial V)^0\right\|^2)$ is a Lyapunov pair, so that for every $x_0 \in \overline{\operatorname{dom} V}$

$$V(x(t, x_0)) + \int_0^t \left\|\dot{x}(\tau, x_0)\right\|^2 d\tau \leq V(x_0) \text{ for all } t > 0.$$

To see this fact we fix $x \in \operatorname{dom} V \cap \operatorname{dom} \partial V$. Since $A_\lambda(x) \in A(J_\lambda(x))$ for every $\lambda > 0$ $(A = \partial V)$, condition (17) holds and one has that

$$V(J_\lambda(x)) - V(x) \leq -\langle A_\lambda(x), x - J_\lambda(x) \rangle = -\frac{1}{\lambda} \left\|x - J_\lambda(x)\right\|^2.$$

Hence,

$$\liminf_{\lambda \downarrow 0} \tfrac{V(J_\lambda(x)) - V(x)}{\lambda} + \left\|A^0 x\right\|^2 \leq \liminf_{\lambda \downarrow 0} \left(\tfrac{V(J_\lambda(x)) - V(x)}{\lambda} + \tfrac{1}{\lambda^2} \left\|x - J_\lambda(x)\right\|^2 \right) \leq 0,$$

and Corollary 13 (together with Proposition 2) applies.

5. Conclusion and further research

We gave different conditions for the invariance of closed sets, which only involve the input data, represented by the maximal monotone operator and the Lipschitz mapping. These conditions are applicable to a large variety of closed sets which do not need to be convex or smooth. The current work extends and improves some of the results given in [3,4] and dealing with weakly closed invariant sets and weakly lower semi-continuous a-Lypunov pairs. It will be our aim in a forthcoming work to apply the current results to specific differential equations/inclusions where the underlying maximal monotone operator is not known explicitly. This will make the access to the corresponding semi-group more easier, namely regarding the behaviour at infinity of trajectories.

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