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INTERIOR REGULARITY RESULTS FOR ZEROTH ORDER OPERATORS APPROACHING THE FRACTIONAL LAPLACIAN

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ABSTRACT

In this article we are interested in interior regularity results for the solution $u_\epsilon \in C(\bar{\Omega})$ of the Dirichlet problem

$$\begin{cases} -\mathcal{I}_\epsilon(u) = f_\epsilon & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where Ω is a bounded, open set and $f_\epsilon \in C(\bar{\Omega})$ for all $\epsilon \in (0,1)$. For some $\sigma \in (0,2)$ fixed, the operator $\mathcal{I}_\epsilon$ is explicitly given by

$$\mathcal{I}_\epsilon(u,x) = \int_{\mathbb{R}^N} \frac{[u(x+z)-u(x)]dz}{\epsilon^{N+\sigma}+|z|^{N+\sigma}},$$

which is an approximation of the well-known fractional Laplacian of order σ, as ϵ tends to zero. The purpose of this article is to understand how the interior regularity of u_ϵ evolves as ϵ approaches zero. We establish that u_ϵ has a modulus of continuity which depends on the modulus of f_ϵ, which becomes the expected Hölder profile for fractional problems, as $\epsilon \to 0$. This analysis includes the case when f_ϵ deteriorates its modulus of continuity as $\epsilon \to 0$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain and $\epsilon \in (0,1)$. In this paper we are interested in understanding interior regularity of solutions u_ϵ to the Dirichlet problem

$$\begin{cases} -\mathcal{I}_\epsilon(u) = f_\epsilon & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1}$$

where $\mathcal{I}_\epsilon$ is the non-local operator

$$\mathcal{I}_\epsilon(u,x) = \int_{\mathbb{R}^N} [u(x+z)-u(x)]K_\epsilon(z)dz, \tag{2}$$

with kernel $K_\epsilon : \mathbb{R}^N \to \mathbb{R}$ explicitly given by

$$K_\epsilon(z) = \frac{1}{\epsilon^{N+\sigma}+|z|^{N+\sigma}}, \tag{3}$$

for some $\sigma \in (0,2)$ fixed. Here we also assume $f_\epsilon \in C(\bar{\Omega})$ for each $\epsilon \in (0,1)$ and the family $\{f_\epsilon\}$ is uniformly bounded, that is, there exists $\Lambda > 0$ such that

$$||f_\epsilon||_{L^\infty(\bar{\Omega})} \leq \Lambda, \quad \text{for all } \epsilon \in (0,1). \tag{4}$$

The characteristic feature of the non-local operators like $\mathcal{I}_\epsilon$ is the integrability of the kernel K_ϵ defining it. In the literature, this fact leads to say that $\mathcal{I}_\epsilon$ is a **zeroth order non-local operator**.

On the other hand, of main importance in this paper is the role of the fractional Laplacian of order σ, defined as

$$(-\Delta)^{\sigma/2}u(x) = -C_{N,\sigma} \text{ P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)]|z|^{-(N+\sigma)}dz,$$

where P.V. is the Cauchy Principal Value and $C_{N,\sigma} > 0$ is a normalizing constant, see [10]. Notice that in this case, up to the normalizing constant $C_{N,\sigma}$, the kernel defining $(-\Delta)^{\sigma/2}$ can be formally identified with the limit case of K_ϵ, when $\epsilon = 0$ in (3), that is $K_0(z) = |z|^{-(N+\sigma)}$ for $z \neq 0$, which is a non-integrable around the origin. This non-integrability of the kernel determines a deep qualitative contrast between zeroth order problems like (1) and fractional non-local problems with $\mathcal{I}_\epsilon$ replaced by the fractional Laplacian in (1).

The purpose of this paper is to contribute to the analysis of regularity of the solution u_ϵ of (1) in the passage to the limit as $\epsilon \to 0$. As $\mathcal{I}_\epsilon$ is a zeroth order operator, it does not have a regularizing effect, and thus u_ϵ is merely continuous when f_ϵ is continuous. However, when $\epsilon = 0$ the solution u_0 is Hölder continuous, even of class $C^{1,\alpha}$ when $\sigma > 1$. The question is: How does the regularity of u_ϵ improve as ϵ approaches zero?

Concerning the zeroth order Dirichlet problem (1), the integrability of the kernel defining $\mathcal{I}_\epsilon$ allows the application of the Fixed Point Theorem on $C(\bar{\Omega})$ to obtain the existence and uniqueness of a solution $u_\epsilon \in C(\bar{\Omega})$ of (1), see [8, 2, 12]. The integrability of the kernel of the operator has two remarkable effects on u_ϵ. First, in general u_ϵ does not attain the boundary value on $\partial\Omega$, that is, $u_\epsilon \neq 0$ for some points on $\partial\Omega$; and second, u_ϵ has no more regularity than the one exhibited by f_ϵ, showing a lack of the regularizing effect of classical elliptic equations.

In [12] and [11], two of the authors addressed the question of how the modulus of continuity of the solution of (1), inherited from the modulus of continuity of f_ϵ, evolves as $\epsilon \to 0$. A systematical use of the comparison principle satisfied by $\mathcal{I}_\epsilon$ in the full range of $\epsilon \in (0,1)$ provides a control of the jump discontinuity of u_ϵ on $\partial\Omega$, which in turn allows to construct suitable barriers for the modulus of continuity of u_ϵ on $\bar{\Omega}$, only depending on the modulus of continuity of $\{f_\epsilon\}$. This result can be expressed as the existence of a constant $C > 0$ depending on Λ in (4) but not depending on ϵ such that

$$|u_\epsilon(x) - u_\epsilon(y)| \leq C|f_\epsilon(x) - f_\epsilon(y)|, \quad \text{for all } x, y \in \bar{\Omega}. \tag{5}$$

In particular, if the family $\{f_\epsilon\}$ is equicontinuous in $\bar{\Omega}$, so is $\{u_\epsilon\}$ and this provides robust stability properties as $\epsilon \to 0$ on the closed set $\bar{\Omega}$.

However, the above result does not seem to be optimal in view of the available regularity results for fractional problems. For instance, given $f_0 \in L^\infty(\bar{\Omega})$, any bounded viscosity solution to problem

$$\begin{cases} (-\Delta)^{\sigma/2} u = f_0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{6}$$

enjoys interior C^α estimates for some $\alpha \in (0,1)$. Moreover, in the case $\sigma > 1$, interior $C^{1,\alpha}$ estimates for solutions of this problems hold, see [5, 6, 16]. In this direction, we also remark the recent regularity results up to the boundary for (6) provided by Ros-Oton and Serra in [15]. These Hölder regularity results for problem (6) are a consequence of the elliptic regularity effect of the fractional Laplacian, closely related to classical elliptic regularity results for second-order problems.

In view of the discussion above, it is natural to ask if the modulus of continuity of the solution to (1) actually improves as $\epsilon \to 0$, at least locally in Ω, reaching the known Hölder regularity results for fractional problems described above. Furthermore, of particular interest is the case in which the family $\{f_\epsilon\}$ is not equicontinuous in $\bar{\Omega}$ and therefore its modulus of continuity may worsen in the passage to the limit. To this end, a different argument than the one based on comparison principles presented in [12] must be applied in order to capture the gain of ellipticity of the operator $\mathcal{I}_\epsilon$, when ϵ approaches zero.

Our first result on regularity of u_ϵ is the following:

Theorem 1.1: *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\sigma \in (0,2)$. For $C_0 > 0$ and $\epsilon \in (0,1)$ let u_ϵ be continuous in $\bar{\Omega}$, satisfying the inequalities*

$$||u_\epsilon||_{L^\infty(\mathbb{R}^N)} \leq C_0, \quad \textit{for all } \epsilon \in (0,1) \tag{7}$$

and

$$-C_0 \leq \mathcal{I}_\epsilon(u_\epsilon) \leq C_0 \quad \textit{in } \Omega. \tag{8}$$

Then for all $\Omega' \subset\subset \Omega$ there exist $\epsilon_0, \alpha \in (0,1)$ and $L, l > 0$, independent of ϵ, such that for each $\epsilon \in (0, \epsilon_0)$

$$|u_\epsilon(x) - u_\epsilon(y)| \leq L(|x-y|^\alpha \mathbf{1}_{B^c_{l_\epsilon}}(x-y) + m_\epsilon(|x-y|)\mathbf{1}_{B_{l_\epsilon}}(x-y)),$$

for all $x, y \in \Omega'$. Here m_ϵ denotes a modulus of continuity for u_ϵ in B_1 and the constants ϵ_0, α, L and l depend on N, σ, C_0 and $\mathrm{dist}(\Omega', \Omega^c)$.

The proof of this theorem is obtained by controlling the oscillation of the solutions to (8), following the ideas of the classical approach for Hölder regularity by De Giorgi for local equations in divergence form and by Krylov and Safonov for local equations in non-divergence form. Actually, we adapt the approach by Silvestre in [16] in the non-local framework. Despite the lack of homogeneity of our non-local operator $\mathcal{I}_\epsilon$, we are still able to scale it, see Lemma 2.3, and then to diminish the oscillation of the solution u_ϵ through an inductive procedure, but to a limited number of steps depending on ϵ. Basically, we obtain the Hölder profile of the solution corrected by the modulus of continuity of u_ϵ in a small ball, as shown in the theorem.

Our second result provides a different expression for the modulus continuity of the solution u_ϵ:

Theorem 1.2: *Let $\sigma \in (0,2)$, $\epsilon \in (0,1)$ and let $u_\epsilon \in C(\bar{\Omega})$ satisfying (7) and (8). Then, for each $\Omega' \subset\subset \Omega$, $\alpha \in (0, \min\{1,\sigma\})$, $\gamma \in (0,1)$ and $\beta \in (0, 1-\gamma)$, there exist $\epsilon_0 \in (0,1)$ and a constant $C > 0$ such that for all $\epsilon \in (0,\epsilon_0)$ we have*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq C|x-y|^\alpha + m_\epsilon(\epsilon^\beta |x-y|^\gamma), \quad \textit{for all } x, y \in \Omega',$$

where m_ϵ is a modulus of continuity of u_ϵ on $\bar{\Omega}$ and the constants C, ϵ_0 depend on N, σ, C_0 and $\mathrm{dist}(\Omega', \Omega^c)$.

This result is obtained through a non-local version of the Ishii–Lions method given by Barles, Chasseigne and Imbert in [3], see also [1, 13]. The method is based on the adequate use of the expected modulus of continuity as test function for the solution u_ϵ, and uses in a significant way the accumulation of mass of K_ϵ around zero in order to extract valuable information on the concavity of this test function. This is possible in a crucial way by the consideration of the m_ϵ-term and the radiality of K_ϵ.

Before continuing, we would like to discuss how Theorems 1.1 and 1.2 apply to our model problem (1.1). Assuming (4) and that the boundary $\partial\Omega$ is of class C^1 and satisfies the uniform exterior ball condition in case $\sigma > 1$, then (5) and (7) hold, see [12]. Thus we can apply both theorems to obtain the regularity results for u_ϵ given above.

It is interesting to observe that when $f_\epsilon \in L^\infty(\bar\Omega)$ has a deteriorating modulus of continuity such that, for example,

$$m_\epsilon(s) = m(\epsilon^{-\beta'} s), \quad \forall s > 0,$$

with $\beta' \in (0,1)$ and m is a fixed modulus of continuity, then still the correction term approaches zero as $\epsilon \to 0$ in both cases. This shows the interplay between the worsening of the continuity of f_ϵ and the improvement of the ellipticity of the operator $\mathcal{I}_\epsilon$.

Finally, we obtain a higher interior regularity result for the solution to (1) when $\sigma \in (1,2)$.

Theorem 1.3: *Let $\sigma \in (1,2)$, $\epsilon \in (0,1)$ and let $u_\epsilon \in C(\bar\Omega)$ satisfying (7), (8) and having a uniform modulus of continuity m. Then, for each $\Omega' \subset\subset \Omega$, there exist $\alpha \in (0,\sigma-1)$, $\epsilon_0 \in (0,1)$, $c > 0$ and $C > 0$, independent of ϵ, such that for each $\epsilon \in (0,\epsilon_0)$ there exist linear functions $l_\epsilon(x) = a_\epsilon + b_\epsilon \cdot x$ with $a_\epsilon \to a_0, b_\epsilon \to b_0$ as $\epsilon \to 0$, such that*

$$|u_\epsilon(x) - u_\epsilon(y) - l_\epsilon(x-y)| \leq C|x-y|^{1+\alpha}\mathbf{1}_{B_{c_\epsilon(y)}}(x) + m(|x-y|)\mathbf{1}_{B_{c_\epsilon}(y)}(x)$$

for all $x, y \in \Omega'$, where $c_\epsilon = cm(\epsilon)^{1/\sigma}$ and the constants ϵ_0, α, c and C depend on N, σ, C_0 and $\mathrm{dist}(\Omega', \Omega^c)$.

The proof of this theorem follows the ideas of Theorem 5.2 by Caffarelli and Silvestre in [6], where an inductive construction of the approximating linear functions is presented for the case of non-local operators, extending previous fundamental work by Caffarelli for second-order problems in [7]. As in the proof of Theorem 1.1, the idea is to exploit the gain of homogeneity of the non-local operator through scaling and a limited inductive process as $\epsilon \to 0$. This allows an approximation procedure to a tangent equation that has a priori $C^{1,\alpha}$ regularity bounds. In doing this, a modified version of Theorem 1.1 for unbounded functions must be applied in order to control the tails of the integral in the evaluation of unbounded functions in the non-local operator $\mathcal{I}_\epsilon$, see Corollary 2.4.

We finish this introduction making some remarks about the concept of modulus (see [14, 9]): we say that a function $m : [0,+\infty) \to \mathbb{R}$ is a modulus (or modulus of continuity) if it is continuous, non-decreasing, and satisfies $m(0) = 0$

and $m(r+s) \leq m(r) + m(s)$ for $r, s \geq 0$. The last condition implies that m is a concave function, from which we can assume without loss of generality that $m \in C^2(0, +\infty)$ and it satisfies

$$t \leq m(t), \quad \text{for all } t > 0. \tag{9}$$

2. Proof of Theorem 1.1

We prove a slightly simpler version of the interior regularity that allows us to get Theorem 1.1 as a consequence.

THEOREM 2.1: *There exists $\alpha, \theta_0 \in (0,1)$ such that, for all $\epsilon \in (0, 1/2)$ and $u_\epsilon : \mathbb{R}^N \to \mathbb{R}$ continuous in B_1 (with modulus of continuity m_ϵ) such that $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \leq 1/2$ and satisfying*

$$-\theta_0 \leq \mathcal{I}_\epsilon(u_\epsilon) \leq \theta_0 \quad \text{in } B_1, \tag{10}$$

then we have that

$$|u_\epsilon(x) - u_\epsilon(0)| \leq 2^\alpha |x|^\alpha \mathbf{1}_{B_\epsilon^c}(x) + m_\epsilon(|x|)\mathbf{1}_{B_\epsilon}(x),$$

for all $x \in B_{1/2}$. The constants α, θ_0 depend only on N and σ.

The way to obtain Theorem 1.1 is rather standard and we leave the details until the end of this section. Hence, in what follows we concentrate on the proof of Theorem 2.1.

We start with the following technical lemma which is going to play a key role. We would like to mention that this result is a consequence of the zeroth order nature of the operator more than its proximity to the fractional order, in the sense that we just require positive mass around the origin in K_ϵ.

LEMMA 2.2: *Let $\epsilon \in (0,1)$. For all $\delta > 0$, there exist $c_\delta > 0$ (just depending on N, σ and δ) such that, for all $\theta, \eta, \gamma \in (0, c_\delta)$ and each bounded, upper semicontinuous continuous function $u : \mathbb{R}^N \to \mathbb{R}$ satisfying*

1. *$-\mathcal{I}_\epsilon(u) \leq \theta$ in B_1,*
2. *$u(x) \leq 1$ in B_1,*
3. *$u(x) \leq 2|2x|^\eta - 1$ in B_1^c, and*
4. *$\delta \leq |A_0|$, where $A_0 := \{x \in B_1 : u(x) \leq 0\}$,*

then we have $u \leq 1 - \gamma$ in $B_{1/2}$.

Proof. Denote $\beta(x) = (1-x^2)^2$ and consider the function

$$b(x) = \begin{cases} \beta(|x|) & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_1. \end{cases}$$

For $k > 0$ to be fixed small enough, denote

$$\gamma = k\Big(\beta\Big(\frac{1}{2}\Big) - \beta\Big(\frac{3}{4}\Big)\Big).$$

Assume there exists $x_0 \in B_{1/2}$ such that $u(x_0) > 1-\gamma$. Then, it is clear that $u(x_0) + kb(x_0) > 1$, and using the radial monotony of b together with (2) we can write

$$u(x_0) + kb(x_0) > u(y) + kb(y), \quad \text{for all } y \in B_1 \setminus B_{3/4}.$$

Thus there exists $x_1 \in B_{3/4}$ such that

$$u(x_1) + kb(x_1) = \max_{B_1}(u(x) + kb(x)) > 1. \tag{11}$$

Now we write

$$-\mathcal{I}_\epsilon(u + kb, x_1) = I_1 + I_2,$$

where after a change of variables we have denoted

$$I_1 = \int_{B_1^c} [(u+kb)(x_1) - (u+kb)(z)] K_\epsilon(z - x_1) dz, \quad \text{and}$$
$$I_2 = \int_{B_1} [(u+kb)(x_1) - (u+kb)(z)] K_\epsilon(z - x_1) dz.$$

To continue we estimate I_1, I_2 from below. For I_1, we use that $b = 0$ in B_1^c, (11) and (3) to get the inequality

$$I_1 \geq \int_{B_1^c} (2 - 2|2z|^\eta) K_\epsilon(z - x_1) dz,$$

and since $x_1 \in B_{3/4}$ we arrive at

$$I_1 \geq 2 \int_{B_1^c} \frac{1 - |2z|^\eta}{|z|^\sigma - (3/4)^\sigma} dz.$$

Then, by the Dominated Convergence Theorem we can find $\eta > 0$ small enough in terms of N, σ and θ to conclude that

$$I_1 \geq -\theta.$$

For I_2, we notice that by (11) the integrand is non-negative. Then we write

$$I_2 \geq \int_{A_0} [(u+kb)(x_1) - (u+kb)(z)] K_\epsilon(z - x_1) dz,$$

and using again (11) together with (4) and taking $k \leq 1/2$ we arrive at

$$I_2 \geq \frac{1}{2} \int_{A_0} K_\epsilon(z - x_1) dz \geq \frac{1}{2} \frac{1}{2^{N+\sigma} + \epsilon^{N+\sigma}} \delta.$$

Since $\epsilon \in (0, 1)$, we conclude the existence of a universal constant $c > 0$ not depending on ϵ or δ such that

$$I_2 \geq c\delta.$$

Putting together the above estimates, using the linearity of $\mathcal{I}_\epsilon$ and (1) we get

$$-\theta + c\delta \leq -\mathcal{I}_\epsilon(u, x_1) - \mathcal{I}_\epsilon(kb, x_1) \leq \theta_0 - \mathcal{I}_\epsilon(kb, x_1) \leq \theta - k\mathcal{I}_\epsilon(kb, x_1),$$

leading us to

$$c\delta \leq 2\theta - k\mathcal{I}_\epsilon(b, x_1),$$

but since b is bounded in $\mathbb{R}^N$ and smooth (with C^2 estimates which depend only on N), there exists a constant $C_1 > 0$ just depending on N and σ for which the following inequality holds:

$$c\delta \leq 2\theta + C_1 k.$$

Then there exists $\bar{c} > 0$, just depending on N and σ, such that if $\theta, k \leq \bar{c}\delta$, the last inequality is not possible, concluding the result. ∎

Remark 2.1: Replacing u by $-u$ in the last lemma we can get analogous lower bounds in B_1 for functions whose evaluation in $\mathcal{I}_\epsilon$ is not too negative, satisfying lower bounds in B_1 and at infinity, and whose positive set in B_1 is non-trivial.

As we mentioned in the introduction, the lack of homogeneity of the operator $\mathcal{I}_\epsilon$ creates difficulties in the application of De Giorgi–Krylov–Safonov iterative techniques. To deal with them, we start with a simple scaling property of $\mathcal{I}_\epsilon$ which is going to be useful in the arguments to come.

LEMMA 2.3: *Let $\epsilon, \lambda, \theta > 0$ and $u : \mathbb{R}^N \to \mathbb{R}$ satisfying the pointwise set of inequalities*

$$-\theta \leq \mathcal{I}_\epsilon(u) \leq \theta \quad \text{in } B_\lambda.$$

Then $u_\lambda(x) := u(\lambda x)$ satisfies

$$-\lambda^\sigma \theta \leq \mathcal{I}_{\frac{\epsilon}{\lambda}}(u_\lambda) \leq \lambda^\sigma \theta \quad \text{in } B_1.$$

Proof. Let $x \in B_1$. From the very definition of the non-local operator and u_λ we see that

$$\begin{aligned}\mathcal{I}_{\frac{\epsilon}{\lambda}}(u_\lambda, x) =& \int_{\mathbb{R}^N} \frac{[u(\lambda x + \lambda z) - u(\lambda x)]}{(\frac{\epsilon}{\lambda})^{N+\sigma} + |z|^{N+\sigma}} dz \\ =&\lambda^{N+\sigma} \int_{\mathbb{R}^N} \frac{[u(\lambda x + \lambda z) - u(\lambda x)]}{\epsilon^{N+\sigma} + |\lambda z|^{N+\sigma}} dz.\end{aligned}$$

Then performing the change of variables $y = \lambda z$ in the last integral we obtain that

$$\mathcal{I}_{\frac{\epsilon}{\lambda}}(u_\lambda, x) = \lambda^\sigma \mathcal{I}_\epsilon(u, \lambda x),$$

from which the result follows from the inequality satisfied by u. ∎

Now we are in position to provide the proof of Theorem 2.1. The argument follows the ideas presented in [16], using scaling arguments, but with the main difference that the corresponding inductive procedure stops after a finite number of steps depending on ϵ. This gives rise to the correction term $m_\epsilon \mathbf{1}_{B_\epsilon}$ in the statement of the theorem.

Proof of Theorem 2.1. We start proving a discrete version of the theorem. Let u_ϵ satisfy (10) with $\|u_\epsilon\|_{L^\infty(\mathbb{R}^N)} \leq 1/2$, then define $q = q_\epsilon \in \mathbb{N}$ such that

$$2^{-(q+1)} \leq \epsilon < 2^{-q}; \tag{12}$$

then for all $k \leq q_\epsilon$ we have

$$|u_\epsilon(x) - u_\epsilon(0)| \leq 2^{-k\alpha} \quad \text{for all } x \in B_{2^{-k}}. \tag{13}$$

From here we get Theorem 2.1 from (13) by noticing that for each $x \in B_{1/2} \setminus B_\epsilon$ we can consider $k \leq q$ with q as in (12) such that

$$2^{-(k+1)} \leq |x| \leq 2^{-k},$$

and if (13) holds we see that

$$|u_\epsilon(x) - u_\epsilon(0)| \leq 2^{-k\alpha} = 2^\alpha 2^{-(k+1)\alpha} \leq 2^\alpha |x|^\alpha.$$

Now, since by the definition of m_ϵ we have that $|u_\epsilon(x) - u_\epsilon(0)| \leq m_\epsilon(|x|)$ for all $x \in B_1$ we arrive at the inequality stated in Theorem 2.1.

Hence we concentrate on proving that (13) holds. Let $\delta = |B_1|/2$ and from now on we consider c_δ as in Lemma 2.2 (which depends only on N, σ by the choice of δ), and we consider $\theta, \eta, \gamma \in (0, c_\delta)$. We start asking $\theta_0 < \theta/2$, $\alpha \leq \eta$.

The proof of (13) consists in exhibiting sequences $(a_k),(b_k)$, non-increasing and non-decreasing respectively, such that

$$\begin{cases} b_k \le u_\epsilon \le a_k \text{ in } B_{2^{-k}} \\ a_k - b_k = 2^{-\alpha k} \end{cases} \quad \text{for all } 0 \le k \le q_\epsilon. \tag{14}$$

We construct such a sequence recursively. For $k = 0$, we define $b_0 = -1/2$ and $a_0 = 1/2$. Since $|u_\epsilon| \le 1/2$ in B_1 we have the above requirements for the case $k = 0$. Next, we consider $k < q_\epsilon$ and assume that there exist a_i and b_i satisfying the desired properties for all $i \le k$.

Defining $m = m_k = (a_k + b_k)/2$, by the monotony of the sequences $(a_k),(b_k)$ we see that $-1/2 \le m \le 1/2$ and by the inductive hypothesis (14) we see that

$$|u_\epsilon - m| \le 2^{-k\alpha-1} \quad \text{in } B_{2^{-k}}. \tag{15}$$

Now consider the function

$$\bar{u}_\epsilon(x) = 2^{k\alpha+1}(u_\epsilon(2^{-k}x) - m),$$

and we next prove that $\bar{u}_\epsilon$ satisfies the assumptions to apply Lemma 2.2. Without loss of generality we can assume assumption (4) in that lemma holds, that is

$$|\{x \in B_1 : \bar{u}_\epsilon \le 0\}| \ge |B_1|/2 = \delta,$$

since the case in which the set $\{x \in B_1 : \bar{u}_\epsilon \ge 0\}$ covers more than half of the ball follows similar arguments as those presented below taking into account Remark 2.1.

In view of (15) and the definition of $\bar{u}_\epsilon$ we see that $|\bar{u}_\epsilon| \le 1$ in B_1. By the choice of θ_0 and using that u_ϵ satisfies (10), applying Lemma 2.3 one can directly verify that

$$-\mathcal{I}_{\epsilon/2^{-k}}(\bar{u}_\epsilon, x) \le 2^{k(\alpha-\sigma)+1}\theta_0 \le \theta \quad \text{in } B_1,$$

and by the definition of q_ϵ in (12) we see that $\epsilon/2^{-k} < 1$. Thus conditions (1) and (2) in Lemma 2.2 hold.

Now consider $x \in B_1^c$ and the corresponding $j = j(x) \ge 0$ such that $2^j \le |x| < 2^{j+1}$. If $j \ge k$, using that the sequence (b_k) is non-decreasing we see that

$$\bar{u}_\epsilon(x) \le 2^{k\alpha+1}(1/2 - b_k + b_k - m) \le 2^{k\alpha+1}(1 + (b_k - a_k)/2),$$

from which we get that

$$\bar{u}_\epsilon(x) \leq 2^{k\alpha+1}(1 - 2^{-k\alpha-1}) \leq 2^{j\alpha+1} - 1 \leq 2|x|^\alpha - 1,$$

by (14) and the definition of j. If $j < k$ we see that $2^{-k}x \in B_{2^{j+1-k}}$, and by the definition of $\bar{u}_\epsilon$ and m, the induction hypothesis (14) and the monotony of (b_k) we can write

$$\begin{aligned}\bar{u}_\epsilon(x) &\leq 2^{k\alpha+1}\Big(2^{-\alpha(k-j-1)} + b_{k-j-1} - \frac{a_k + b_k}{2}\Big)\\ &\leq 2^{(j+1)\alpha+1} + 2^{k\alpha+1}\frac{b_k - a_k}{2} \leq 2|2x|^\alpha - 1.\end{aligned}$$

Hence we get that $\bar{u}_\epsilon(x) \leq 2|2x|^\alpha - 1$ for $x \in B_1^c$, and therefore we have verified condition (4) in Lemma 2.2. Thus there exists $\gamma > 0$ just depending on N and σ such that

$$\bar{u}_\epsilon(x) \leq 1 - \gamma \quad \text{in } B_{1/2},$$

and coming back to u_ϵ we conclude that

$$u_\epsilon \leq 2^{-k\alpha-1}(1-\gamma) + m \quad \text{in } B_{2^{-(k+1)}}.$$

We notice that by the definition of m, from the above inequality we can get

$$u_\epsilon \leq b_k + 2^{-k\alpha-1}(2-\gamma) \quad \text{in } B_{2^{-(k+1)}},$$

from which, taking $\alpha > 0$ small enough to have $2-\gamma \leq 2^{1-\alpha}$, we define $b_{k+1} = b_k$ and $a_{k+1} = b_k + 2^{-(k+1)\alpha}$ to conclude the result. ∎

Proof of Theorem 1.1. Let $\Omega' \subset\subset \Omega$ and denote $\rho = \mathrm{dist}(\Omega', \Omega^c)/2$, and without loss of generality we may assume that $\rho \leq 1$. Then define

$$\bar{c} = \frac{\min\{1, \theta_0\}}{\max\{1, C_0, 2||u_\epsilon||_{L^\infty(\mathbb{R}^N)}\}}, \tag{16}$$

where θ_0 is given by Theorem 2.1. Then for $y \in \Omega'$ we consider the function

$$v_\epsilon(z) := \bar{c}u_\epsilon(y + \rho z), \quad z \in \mathbb{R}^N.$$

It is straightforward to see that $||v_\epsilon||_{L^\infty(\mathbb{R}^N)} \leq 1/2$, and using the linearity and translation invariance of the non-local operator together with Lemma 2.3 we get that

$$-\theta_0 \leq \mathcal{I}_{\epsilon/\rho}(v_\epsilon) \leq \theta_0 \quad \text{in } B_1.$$

Hence considering $\epsilon \in (0, \rho/2)$ we are in position to use Theorem 2.1 to conclude that

$$|v_\epsilon(z) - v_\epsilon(0)| \leq 2^\alpha |z|^\alpha \mathbf{1}_{B_\epsilon}(z) + m_\epsilon(|z|)\mathbf{1}_{B_\epsilon^c}(z) \quad \text{for all } z \in B_{1/2}.$$

Using the sub-linearity of m_ϵ and the definition of v_ϵ, the above inequality can be translated in terms of u_ϵ as

$$|u_\epsilon(y+x) - u_\epsilon(y)| \leq \bar{C}(|x|^\alpha \mathbf{1}_{B_{\rho\epsilon}}(x) + m_\epsilon(|x|)\mathbf{1}_{B_{\rho\epsilon}^c}(x)), \quad \text{for all } x \in B_{\rho/2},$$

for some constant $\bar{C}$ depending on $\bar{c}$ and ρ. Since the argument is independent of $y \in \Omega'$, the proof follows. ∎

We continue with a corollary of Theorem 1.1 related to unbounded solutions which is going to be important in the proof of Theorem 1.3.

Corollary 2.4: *Assume the hypotheses of Theorem 1.1 hold, but with a function $u_\epsilon : \mathbb{R}^N \to \mathbb{R}$ such that there exists $\tilde{C} > 0$ and $0 < \sigma' < \sigma$ satisfying $|u_\epsilon(x)| \leq \tilde{C}(1 + |x|)^{\sigma'}$ for all $x \in \mathbb{R}^N$. Then there exists $\alpha > 0$ such that, for each $\Omega' \subset\subset \Omega$, there exist $L, l, \epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ we have*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq L(|x-y|^\alpha \mathbf{1}_{B_{l\epsilon}^c}(x-y) + m_\epsilon(|x-y|)\mathbf{1}_{B_{l\epsilon}}(x-y)),$$

for all $x, y \in \Omega'$. Moreover, there exists $C > 0$ not depending on ϵ or R such that $L \leq C\mathrm{diam}(\Omega)^{\sigma'}$.

Proof. Consider a function $\eta \in C^2(\mathbb{R}^N)$ with $||\eta||_{C^2} < +\infty$ and such that $\eta = 1$ in B_1, $\eta = 0$ in B_2^c. Thus for each $R > 1$ and $x \in \mathbb{R}^N$ we denote $\eta_R(x) = \eta(x/R)$ and if we assume the existence of $\tilde{C}$ such that $|u_\epsilon(x)| \leq \tilde{C}|x|^{\sigma'}$ for $|x| > 1$ with $0 < \sigma' < \sigma$, then we replace u_ϵ by

$$\tilde{u}_\epsilon(x) = \eta_R(x)u_\epsilon(x), \quad x \in \mathbb{R}^N,$$

for some $R > 0$ such that $\Omega \subset B_{R/2}$. This function is bounded (in terms of $\tilde{C}$, R and σ'), continuous in $\bar{\Omega}$, and if u_ϵ satisfies (8) then we see that for each $x \in \Omega$ we have

$$\begin{aligned}\mathcal{I}_\epsilon(\tilde{u}_\epsilon, x) =& \mathcal{I}_\epsilon(u_\epsilon, x) + \int_{\mathbb{R}^N} (u_\epsilon(x+z)(\eta_R(x+z) - 1))K_\epsilon(z)dz \\ \leq& C_0 + \tilde{C}\int_{B_{R/2}^c} (R + |x|)^{\sigma'}|z|^{-(N+\sigma)}dz,\end{aligned}$$

from which we conclude that

$$\mathcal{I}_\epsilon(\tilde{u}_\epsilon) \leq C_0 + CR^{\sigma'-\sigma},$$

where C_0 is the constant in (8) and $C > 0$ does not depend on R, for all $\epsilon \in (0,1)$. An analogous lower bound can be established and applying Theorem 1.1 we conclude the result. The upper bound for the constant L can be obtained by the growth assumption over u_ϵ and the scaling given in (16). ∎

Remark 2.2: In Theorem 1.1 and Corollary 2.4, assume $\Omega = B_R$ for some $R > 2$. Then it is possible to take $\rho = 1$ in the proof above to conclude regularity estimates in $B_{R/2}$ with constants $l, \epsilon_0 > 0$ independent of R.

3. Proof of Theorem 1.2

As in the previous section, we first prove a simpler theorem and then we get Theorem 1.2 as a consequence. We state it now.

Theorem 3.1: *Let $\sigma \in (0,2)$, $\epsilon \in (0,1)$ and $u_\epsilon : \mathbb{R}^N \to \mathbb{R}$ bounded, with $|u_\epsilon| \leq 1$ in $\bar{B}_1$, $u_\epsilon \in C(\bar{B}_1)$ and satisfying (10) for some $\theta_0 > 0$.*

Then for all $\alpha \in (0, \min\{1, \sigma\})$, $\gamma \in (0,1)$ and $\beta \in [0, 1-\gamma)$, there exists $L > 0$ not depending on ϵ such that

$$|u_\epsilon(x) - u_\epsilon(0)| \leq L|x|^\alpha + m_\epsilon(\epsilon^\beta |x|^\gamma) \quad \text{for all } x \in B_{1/2},$$

where m_ϵ is a modulus of continuity for u_ϵ in $\bar{B}_1$.

We start by introducing some notation. For $\bar{x} \in \mathbb{R}^N \setminus \{0\}$, $\rho > 0$ and $\eta \in (0,1)$ we consider the set

$$\mathcal{C} = \mathcal{C}_{\rho,\eta}(x) := \{z \in B_\rho : (1-\eta)|z||x| \leq |\langle x, z\rangle|\}. \tag{17}$$

Notice that $\mathcal{C} = -\mathcal{C}$ and that there exists a universal constant $c_1 > 0$ such that

$$|\mathcal{C}_{1,\eta}| \geq c_1(1-\eta). \tag{18}$$

Given $D \subset \mathbb{R}^N$ we denote

$$\mathcal{I}[D](\psi, x) = \int_D [\psi(x+z) - \psi(x)] K_\epsilon(z) dz$$

for ψ for which the integral makes sense.

The following lemma is the key technical estimate to get Theorem 1.2. Its proof follows closely the lines of [3, 1] but we provide here a proof for the sake of completeness.

Lemma 3.2: *Let $m \in C^2(0,1) \cap C[0,1]$ be a function such that $m' \geq 0$ and $m'' \leq 0$ in $(0,1)$. For $\gamma \in (0,1)$ consider the function $\psi(x) = m(|x|^\gamma)$ defined for $x \in B_1$. Then, there exist $\rho_0, \eta \in (0,1/2)$ such that, for all $\beta \in (0,1)$ there exists $\underline{c} > 0$ small satisfying*

$$\mathcal{I}[\mathcal{C}_x](\psi, x) \leq -\underline{c}\; m'((1+\rho_0)|x|)|x|^{\gamma-\sigma} \quad \text{for all } \epsilon^\beta < |x| < 1/2,$$

where $\mathcal{C}_x := \mathcal{C}_{\rho_0|x|,\eta}$ is defined as in (17). The constant $\underline{c}$ does not depend on ϵ or x.

Proof. We fix x such that $\epsilon^\beta < |x| < 1/2$ and from now on write $\mathcal{C} = \mathcal{C}_x$. By the symmetry of $\mathcal{C}$ and K_ϵ we can write

$$\mathcal{I}[\mathcal{C}](\psi, x) = \frac{1}{2}\int_{\mathcal{C}} [\psi(x+z) + \psi(x-z) - 2\psi(x)] K_\epsilon(z) dz. \tag{19}$$

Notice that since $x \in B_{1/2}$ and $\rho_0 < 1/2$ we have that $x+z, x-z \in B_1$ for all $z \in \mathcal{C}$. Since $x \neq 0$ we can perform a Taylor expansion with integral reminder to get

$$\mathcal{I}[\mathcal{C}](\psi, x) = \frac{1}{2}\int_0^1 (1-s) \int_{\mathcal{C}} \langle D^2\psi(x+sz)z, z\rangle K_\epsilon(z) dz\; ds. \tag{20}$$

In what follows, our interest is to provide an upper bound for the term $\langle D^2\psi(x+sz)z, z\rangle$. A direct computation leads us to

$$D^2\psi(x) = \gamma^2 m''(|x|^\gamma)|x|^{2\gamma-2}\widehat{x}\otimes\widehat{x} + \gamma m'(|x|^\gamma)|x|^{\gamma-2}((\gamma-2)\widehat{x}\otimes\widehat{x} + I_N),$$

and just by applying that $m'' \leq 0$ we arrive at

$$D^2\psi(x) \leq \gamma m'(|x|^\gamma)|x|^{\gamma-2}((\gamma-2)\widehat{x}\otimes\widehat{x} + I_N).$$

By the definition of $\mathcal{C} = \mathcal{C}_x$ we see that for each $z \in \mathcal{C}$ and $s \in (0,1)$ we have

$$\begin{gathered}(1-\rho_0)|x| \leq |x+sz| \leq (1+\rho_0)|x|, \quad \text{and} \\ |\langle x+sz, z\rangle| \geq (1-\eta-\rho_0)|x||z|.\end{gathered} \tag{21}$$

Using the above inequalities and that $m' \geq 0$ we can write

$$\begin{aligned}&\langle D^2\psi(x+sz)z, z\rangle \\ &\quad\leq \gamma m'(|x+sz|)|x+sz|^{\gamma-2}((\gamma-2)(1-\eta-\rho_0)^2/(1+\rho_0)^2 + 1)|z|^2.\end{aligned}$$

Now, since $\gamma < 1$ we have the existence of ρ_0, η small (in terms of $1-\gamma$) for which

$$(\gamma-2)(1-\eta-\rho_0)|^2/(1+\rho_0)^2 + 1 \leq (\gamma-1)/2,$$

from which we arrive at

$$\langle D^2\psi(x+sz)z,z\rangle \leq \gamma(\gamma-1)m'(|x+sz|)|x+sz|^{\gamma-2}|z|^2/2,$$

and noticing that $m'' \leq 0$ implies m' is non-increasing, we conclude that

$$\langle D^2\psi(x+sz)z,z\rangle \leq -cm'((1+\rho_0)|x|)|x|^{\gamma-2}|z|^2,$$

where $c = \gamma(1-\gamma)(1+\rho_0)^{\gamma-2}/2 > 0$. Then we replace this expression on (20) to conclude that

$$\mathcal{I}[\mathcal{C}](\psi,x) \leq -cm'((1+\rho_0)|x|)|x|^{\gamma-2}\int_{\mathcal{C}}|z|^2K_\epsilon(z)dz. \tag{22}$$

From now on our interest is to provide a lower bound for the integral term in (22). Using polar coordinates and since the $N-1$ dimensional measure of the angle coordinates of $\mathcal{C}$ are comparable to $|\partial B_1|$ (see (18)), there exists a constant $c > 0$ small (depending on γ, ρ_0, η and N) such that

$$\int_{\mathcal{C}}|z|^2K_\epsilon(z)dz \geq c\int_0^\rho r^{N+1}(\epsilon^{N+\sigma}+r^{N+\sigma})^{-1}dr,$$

and performing the change of variables $s = r/\epsilon$ in the last integral we get

$$\int_{\mathcal{C}}|z|^2K_\epsilon(z)dz \geq \epsilon^{2-\sigma}\int_0^{\rho_0|x|/\epsilon}\frac{s^{N+1}}{1+s^{N+\sigma}}ds.$$

Using that $|x| > \epsilon^\beta$ with $\beta \in (0,1)$ we see that $\rho_0|x|/\epsilon \to \infty$ as $\epsilon \to 0$. Thus there exists a constant $c > 0$ not depending on x such that, for all ϵ small (just depending on N, σ, ρ_0 and β), we have

$$\int_{\mathcal{C}}|z|^2K_\epsilon(z)dz \geq c\epsilon^{2-\sigma}(\rho_0|x|/\epsilon)^{2-\sigma} = c\rho_0^{2-\sigma}|x|^{2-\sigma}.$$

Substituting this into (22) we arrive at

$$\mathcal{I}[\mathcal{C}](\psi,x) \leq -cm'((1+\rho_0)|x|)|x|^{\gamma-\sigma},$$

for some $c > 0$ depending on the data, γ, θ, η and ρ_0 but not on ϵ. This concludes the proof. ∎

Now we are in position to provide the

Proof of Theorem 3.1. We start considering a bounded, non-negative, smooth function $\phi : \mathbb{R}^N \to \mathbb{R}$, with uniformly bounded first and second derivatives in $\mathbb{R}^N$ and such that $\phi(0) = 0$, $\phi(x) > 0$ for each $x \neq 0$ and such that $\phi > 3$ in $B^c_{1/2}$.

We also define the function ω_ϵ as

$$\omega_\epsilon(t) = m_\epsilon(\epsilon^\beta\ t^\gamma), \quad t \geq 0, \tag{23}$$

and for $L > 0$ consider the function $\Phi : \bar{B}_1 \times \bar{B}_1 \to \mathbb{R}$ defined as follows:

$$\Phi(x, y) = u_\epsilon(x) - u_\epsilon(y) - L|x - y|^\alpha - \omega_\epsilon(|x - y|) - \phi(x).$$

We prove that for L fixed large in terms of the data, $\Phi \leq 0$ in $\bar{B}_1^2$. We initially ask that $(2/L)^{1/\alpha} < 1/4$.

For a contradiction we assume the contrary, which can be translated as the existence of a point $(\bar{x}, \bar{y}) \in \bar{B}_1^2$ such that

$$\Phi(\bar{x}, \bar{y}) = \max_{\bar{B}_1^2}\{\Phi\} > 0. \tag{24}$$

This inequality implies that

$$\phi(\bar{x}) \leq 2||u_\epsilon||_\infty,$$

and since $||u_\epsilon||_\infty \leq 1$, the definition of ϕ leads to $\bar{x} \in B_{1/2}$. By a similar argument, using inequality (24), the boundedness of u_ϵ, the definition of Φ and the non-negativity of ϕ we can also write

$$|\bar{x} - \bar{y}| \leq (2/L)^\alpha, \tag{25}$$

and by the initial choice of L we conclude that $\bar{y} \in B_{3/4}$. Moreover, inequality (24) also implies that

$$\omega_\epsilon(|\bar{x} - \bar{y}|) \leq u_\epsilon(\bar{x}) - u_\epsilon(\bar{y}),$$

and recalling the definition of ω_ϵ in (23) and the fact that m_ϵ is a modulus of continuity for u_ϵ we get that

$$\epsilon^{\beta/(1-\gamma)} \leq |\bar{x} - \bar{y}|, \tag{26}$$

for each $\epsilon \in (0, 1)$. Notice that $\beta/(1 - \gamma) < 1$.

For convenience we next define

$$\phi_\epsilon(x, y) = L|x - y|^\alpha + \omega_\epsilon(|x - y|) + \phi(x)$$

and we see that, since $(\bar{x}, \bar{y})$ is a maximum point of Φ, for all $z \in \mathbb{R}^N$ we have the following inequalities:

$$\begin{aligned} u_\epsilon(\bar{x} + z) - u_\epsilon(\bar{x}) \leq& u_\epsilon(\bar{y} + z) - u_\epsilon(\bar{y}) + \phi(\bar{x} + z) - \phi(\bar{x}), \\ u_\epsilon(\bar{y} + z) - u_\epsilon(\bar{y}) \geq& -\phi_\epsilon(\bar{x}, \bar{y} + z) + \phi_\epsilon(\bar{x}, \bar{y}), \\ u_\epsilon(\bar{x} + z) - u_\epsilon(\bar{x}) \leq& \phi_\epsilon(\bar{x} + z, \bar{y}) - \phi_\epsilon(\bar{x}, \bar{y}). \end{aligned} \tag{27}$$

Using the equation for u_ϵ at $\bar{x}$ and $\bar{y}$ and substracting them, we get

$$-\mathcal{I}_\epsilon(u_\epsilon,\bar{x})+\mathcal{I}_\epsilon(u_\epsilon,\bar{y})\leq 2\theta_0. \tag{28}$$

From now on we denote $\bar{a}=\bar{x}-\bar{y}$ and denote $\mathcal{C}=\mathcal{C}_{\rho_0|\bar{a}|,\eta}$ defined as in (17) for some ρ_0,η to be fixed later.

Inserting the set of inequalities (27) into (28) we arrive at

$$-\mathcal{I}_\epsilon[\mathcal{C}](\phi_\epsilon(\cdot,\bar{y}),\bar{x})-\mathcal{I}_\epsilon[\mathcal{C}](\phi_\epsilon(\bar{x},\cdot),\bar{y})-\mathcal{I}[\mathcal{C}^c](\phi,\bar{x})\leq 2\theta_0,$$

and using the definition of ϕ_ϵ and the symmetry of $\mathcal{C}$ and K_ϵ we obtain

$$-\mathcal{I}_\epsilon(\phi,\bar{x})\leq 2I_1+2I_2+2\theta_0, \tag{29}$$

where

$$I_1=L\mathcal{I}_\epsilon[\mathcal{C}](|\cdot-\bar{y}|^\alpha,\bar{x})\quad\text{and}\quad I_2=\mathcal{I}_\epsilon[\mathcal{C}](\omega_\epsilon(|\cdot-\bar{y}|),\bar{x}).$$

In what follows, we estimate each term arising in (29). Using the symmetry of K_ϵ and the domain of integration we can express the integrand in $\mathcal{I}_\epsilon(\phi,\bar{x})$ as a finite second-order difference (see (19)) to conclude that

$$\mathcal{I}_\epsilon(\phi,\bar{x})\leq C, \tag{30}$$

where $C>0$ depends only on N and σ.

Now we deal with I_1 and I_2 in (29). For I_1 we use Lemma 3.2 with $x=\bar{a}$, $m(t)=t$, $\gamma=\alpha$ and $\theta=\beta/(1-\gamma)$ (see (26)) to write

$$I_1\leq -cL|\bar{a}|^{\alpha-\sigma},$$

while for I_2 we use Lemma 3.2 with the above choices of parameters up to the choice $m(t)=m_\epsilon(\epsilon t)$ to simply write

$$I_2\leq 0.$$

Thus inserting (30) and the above estimates for I_1 and I_2 into (29) we arrive at

$$cL|\bar{a}|^{\alpha-\sigma}\leq 2\theta_0+C.$$

In view of (25) we arrive at a contradiction by taking L large enough in terms of the data. This concludes the proof. ∎

4. Proof of Theorem 1.3

Following the presentation of the previous sections, we write a simpler version of Theorem 1.3. The most general statement is carried out in the same way as in the proof of Theorem 1.1 as a consequence of Theorem 2.1.

THEOREM 4.1: *Let m be a modulus of continuity. There exists $\alpha, \epsilon_0, C > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$ and for each $u_\epsilon : \mathbb{R}^N \to \mathbb{R}$ with $||u_\epsilon||_{L^\infty(\mathbb{R}^N)} \leq 1$, satisfying* (10) *and*

$$|u_\epsilon(x) - u_\epsilon(y)| \leq m(|x-y|) \quad \text{for all } x, y \in B_1,$$

there exist linear functions

$$l_\epsilon(x) = a_\epsilon + b_\epsilon \cdot x$$

with $a_\epsilon \to a$, $b_\epsilon \to b$ as $\epsilon \to 0$, and $c_\epsilon > 0$ with $c_\epsilon \to 0$ as $\epsilon \to 0$ such that

$$|u_\epsilon(x) - u_\epsilon(0) - \mathbf{1}_{B^c_{c_\epsilon}}(x) l_\epsilon(x)| \leq C|x|^{1+\alpha} \mathbf{1}_{B^c_{c_\epsilon}}(x) + m(|x|)\mathbf{1}_{B_{c_\epsilon}}(x),$$

for all $x \in B_{1/2}$.

Next we present the basic approximation result which plays the role of Lemma 2.2 in the proof of Theorem 2.1. To state it precisely, we consider the notion of a **local modulus of continuity** given by a function

$$m : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$$

such that for each $R > 0$, the function $m(\cdot, R)$ is a modulus of continuity. We also consider the notation $a \vee b = \max\{a, b\}$ for $a, b \in \mathbb{R}$.

LEMMA 4.2: *Given constants $\rho, M, \gamma > 0$ with $\gamma < \sigma$, and a local modulus of continuity m, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $w : \mathbb{R}^N \to \mathbb{R}$ satisfying*

(1) $-\epsilon_0 \leq \mathcal{I}_\epsilon(w, x) \leq \epsilon_0$ *in* B_1,
(2) $|w(x) - w(y)| \leq m(|x-y|, |x| \vee |y|)$ *for all* $x, y \in \mathbb{R}^N$,
(3) $|w(x)| \leq M(1 + |x|)^\gamma$,

and $v : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$-\epsilon_0 \leq -\mathcal{I}_0(v) \leq \epsilon_0 \quad \text{in } B_1; \quad v = w \quad \text{in } B_1^c,$$

then

$$\sup_{x \in B_1} |w(x) - v(x)| \leq \rho.$$

Proof. For a contradiction, assume the existence of $M,\rho > 0$, $\gamma \in (0,\sigma)$, a local modulus of continuity m, and sequences $0 < \epsilon_k' < \epsilon_k \to 0$, $w_k : \mathbb{R}^N \to \mathbb{R}$ satisfying (2), (3) (with w replaced by w_k) with

$$|\mathcal{I}_{\epsilon_k'}(w_k)| \leq \epsilon_k \quad \text{in } B_1$$

for all k large enough, and such that

$$||w_k - v_k||_{L^\infty(B_1)} > \rho, \tag{31}$$

where $v_k : \mathbb{R}^N \to \mathbb{R}$ solves

$$-\epsilon_k' \leq \mathcal{I}_0(v_k) \leq \epsilon_k' \quad \text{in } B_1; \quad v_k = w_k \quad \text{in } B_1^c.$$

Notice that (2), (3) imply that w_k converges (up to a subsequence) locally uniform in $\mathbb{R}^N$ to some function $\bar{w} \in C(\mathbb{R}^N)$ satisfying (3). From now on we argue over the convergent subsequence of w_k which we denote as w_k as well.

Notice that (3) implies that the family $\{v_k\}$ is uniformly bounded in $L^\infty_{loc}(\mathbb{R}^N)$ by the comparison result in [3], and locally equicontinuous in Ω by the interior elliptic estimates of [5]. Hence $\{u_k\}, \{v_k\}$ converge locally uniform in Ω, and by the application of standard stability results in viscosity theory (see [4]) we see that both sequences converge locally uniform in $\mathbb{R}^N$ to the unique viscosity solution to the problem

$$-\mathcal{I}_0(u) = 0 \quad \text{in } B_1, \quad u = \bar{w} \quad \text{in } B_1^c,$$

providing a contradiction with (31). ∎

The core of the proof of Theorem 4.1 is contained in the following

Proposition 4.3: *Let $\sigma \in (1,2)$ and m be a modulus of continuity. There exist $\alpha \in (0,\sigma-1)$, $\epsilon_0', \lambda \in (0,1)$ small and $C, R > 1$ large enough such that for all $\epsilon \in (0,\epsilon_0')$ and each $u_\epsilon : \mathbb{R}^N \to \mathbb{R}$ such that $||u_\epsilon||_{L^\infty(\mathbb{R}^N)} \leq 1$ and further satisfying*

$$-\epsilon_0' \leq \mathcal{I}_\epsilon(u_\epsilon) \leq \epsilon_0' \quad \textit{in } B_{4R} \tag{32}$$

and

$$|u_\epsilon(x) - u_\epsilon(y)| \leq m(|x-y|) \quad \textit{for all } x,y \in B_R, \tag{33}$$

then, for each $k \leq q_\epsilon$, with $q_\epsilon \in \mathbb{N}$ defined as

$$q_\epsilon = \sup\{k \in \mathbb{N} : \ m(\epsilon) \leq \lambda^{\sigma k}\}, \tag{34}$$

there exist linear functions $\ell_k(x) = a_k + b_k \cdot x$ satisfying

$$\begin{gathered} |a_{k+1} - a_k| \leq C\lambda^{(1+\alpha)k}, \quad |b_{k+1} - b_k| \leq C\lambda^{\alpha k}, \quad \textit{and} \\ \sup_{B_{\lambda^k}} |u_\epsilon - \ell_k| \leq C\lambda^{(1+\alpha)k}. \end{gathered} \tag{35}$$

Proof. We fix $0 < \bar{\alpha} < \sigma - 1$ such that interior $C^{1,\bar{\alpha}}$ regularity estimates for fractional harmonic functions hold (see [5, 6]), that is, there exist universal constants $\bar{A}, \bar{\alpha} > 0$ such that, for each function $h : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\mathcal{I}_0(h) = 0 \quad \text{in } B_1, \quad \text{with} \quad |h(x)| \leq (2 + |x|)^{1+\bar{\alpha}}, \quad x \in B_1^c,$$

we have

$$\begin{aligned} &|h| \leq \bar{A} \text{ in } B_1; \quad |Dh(0)| \leq \bar{A}, \quad \text{and} \\ &|h(x) - h(0) - Dh(0)x| \leq \bar{A}|x|^{1+\bar{\alpha}} \quad \text{for } x \in B_{1/2}. \end{aligned} \tag{36}$$

By reasons that will be clear later, we fix the parameter λ as

$$\lambda = \min\{1/2, (12(\bar{A} + 1))^{1/(\bar{\alpha} - \alpha)}\}. \tag{37}$$

We construct the sequence (ℓ_k) recursively. For $k = 0$ we define $\ell_0 = 0$ and then we define $w_k, v_k : \mathbb{R}^N \to \mathbb{R}$ as

$$w_k(x) = \frac{(u_\epsilon - \ell_k)(\lambda^k x)}{\lambda^{k(1+\alpha)}}; \quad \text{and} \quad \begin{cases} \mathcal{I}_0(v_k) = 0 & \text{in } B_1, \\ v_k = w_k & \text{in } \mathbb{R}^N \setminus B_1. \end{cases} \tag{38}$$

Then we can consider $\bar{a}_k = v_k(0)$, $\bar{b}_k = Dv_k(0)$ and from this we denote $\bar{\ell}_k(x) = \bar{a}_k + \bar{b}_k x$. Finally, we define

$$\ell_{k+1}(x) = \ell_k(x) + \lambda^{k(1+\alpha)} \bar{\ell}_k\Big(\frac{x}{\lambda^k}\Big), \tag{39}$$

that is, $a_{k+1} = a_k + \lambda^{k(1+\alpha)} \bar{a}_k$, $b_{k+1} = b_k + \lambda^{k\alpha} \bar{b}_k$. The choice of λ above and ϵ_0' small enough makes it possible to prove inductively that the following two inequalities hold for each $k \leq q_\epsilon$:

$$|\bar{a}_k|, \ |\bar{b}_k| \leq \bar{A}, \tag{40}$$

$$|w_k| \leq (1 + |x|)^{1+\bar{\alpha}} \quad \text{in } \mathbb{R}^N. \tag{41}$$

To conclude the proof of this proposition, we assume these two inequalities hold. We notice that (41) implies $|w_k| \leq 2^{1+\bar{\alpha}}$ in B_1, from which the third inequality in (35) holds in view of the definition of w_k. On the other hand, since $a_{k+1} - a_k = \lambda^{k(1+\alpha)} \bar{a}_k$ and $b_{k+1} - b_k = \lambda^{k\alpha} \bar{b}_k$, if (40) holds then the first two inequalities in (35) hold. ∎

We devote the rest of the section to a proof of (40) and (41), which is carried out through an inductive procedure developed in various lemmas.

We start with a compactness result for the family $\{w_k\}$ under the assumption of (40) and (41).

Lemma 4.4: *Assume u_ϵ bounded satisfies* (32) *for some* $\epsilon_0' > 0$ *and* $R > 1$, *and such that* (33) *holds for some modulus of continuity* m.

For some $\lambda \in (0, 1/2)$, *let* w_k, ℓ_k *be defined as in* (38) *and* (39), *respectively, and further assume that* (40), (41) *hold. Then there exists* $C_R > 0$ *and a modulus of continuity* $\bar{m}$ *independent of* R, *such that for all* ϵ_0' *small and all* $\epsilon \in (0, \epsilon_0')$ *we have*

$$|w_k(x) - w_k(y)| \leq C_R\, \bar{m}(|x-y|) \quad \text{for } x, y \in B_{2R},$$

for all $k \leq q_\epsilon$.

Proof. Since $\sigma > 1$, then linear functions can be classicaly evaluated on $\mathcal{I}_\epsilon$ for all $\epsilon \geq 0$ and this evaluation is null. Then, by definition of w_k, the problem solved by u_ϵ and Lemma 2.3 we see that

$$-\lambda^{k(\sigma-1-\alpha)}\epsilon_0' \leq \mathcal{I}_{\lambda^{-k}\epsilon}(w_k, x) \leq \lambda^{k(\sigma-1-\alpha)}\epsilon_0' \quad \text{in } B_{\lambda^{-k}4R}.$$

Noticing that by the choice of λ in (37) we have $\lambda \leq 1/2$, and using that $\sigma - 1 - \alpha > 0$ and $k \geq 1$ we conclude that

$$-\epsilon_0' \leq \mathcal{I}_{\lambda^{-k}\epsilon}(w_k) \leq \epsilon_0' \quad \text{in } B_{\lambda^{-k}4R}. \tag{42}$$

Let $\beta \in (0,1)$ such that $\beta\sigma - 1 \geq 0$. Using that m is a modulus that can be assumed to satisfy (9), for all $k \leq q_\epsilon$ we can write

$$\lambda^{-k}\epsilon \leq \lambda^{-k} m(\epsilon)^\beta \epsilon^{1-\beta} \leq \lambda^{k(-1+\beta\sigma)k}\epsilon^{1-\beta} \leq \epsilon^{1-\beta},$$

from which, taking ϵ_0' small with respect to ϵ_0 in Corollary 2.4 (which does not depend on R, see Remark 2.2), we conclude that

$$|w_k(x) - w_k(y)| \leq C_R\{|x-y|^\alpha \mathbf{1}_{B^c_{\lambda^{-k}\epsilon}}(x-y) + m_{w_k}(|x-y|)\mathbf{1}_{B_{\lambda^{-k}\epsilon}}(x-y)\} \tag{43}$$

for all $x, y \in B_{2R}$, where m_{w_k} is the modulus of continuity of w_k, and C_R depends on R and $\bar{\alpha}$ but not on λ, ϵ or k. In what follows we show that the term in brackets in the last inequality represents a modulus of continuity independent of k, ϵ or R. For this, we start by noticing that since $\lambda \leq 1/2$ it is easy to see from the definition of a_k, b_k in terms of $\bar{a}_k, \bar{b}_k$ and the assumption (40) that

$$|a_k| \leq \bar{A}, \quad |b_k| \leq \bar{A}/(2^\alpha - 1) \tag{44}$$

for all $k \leq q_\epsilon$. Using the latter, for $t \leq \lambda^{-k}\epsilon$ and by the definition of w_k in (38) we can write

$$m_{w_k}(t) \leq \frac{m(\lambda^k t)}{\lambda^{(1+\alpha)k}} + \frac{|b_k| t}{\lambda^{\alpha k}},$$

where m is given by (33). Then, by the definition of q_ϵ, (9) and observing that $\lambda^k t \le t$, for all $\beta \in (0,1)$ we see that

$$\begin{aligned} m_{w_k}(t) \le& \lambda^{-(1+\alpha)k} m^\beta(\epsilon) m^{1-\beta}(t) + \bar{A}\lambda^{-k\alpha} t^\beta t^{1-\beta}/(2^\alpha - 1) \\ \le& (\lambda^{-(1+\alpha)k/\beta} m(\epsilon))^\beta m^{1-\beta}(t) + C(\lambda^{-k\alpha/\beta} t)^\beta t^{1-\beta} \\ \le& (\lambda^{-(1+\alpha)k/\beta} m(\epsilon))^\beta m^{1-\beta}(t) + C(\lambda^{-k(\alpha+\beta)/\beta} m(\epsilon))^\beta t^{1-\beta}, \end{aligned}$$

where $C > 0$ depends only on $\bar{A}$ and α. At this point, since $\alpha < \sigma$ we can fix β close to 1 in order to have $(1+\alpha)/\beta < \sigma$, and since $k \le q_\epsilon$ we arrive at

$$m_{w_k}(|x-y|) \le m^{1-\beta}(|x-y|) + C|x-y|^{1-\beta}$$

for all $x, y \in B_{2R}$ with $|x-y| \le \lambda^{-k}\epsilon$, and where C, β do not depend on k, ϵ or R. Inserting this into (43) we get the result with

$$\bar{m}(t) = m^{1-\beta}(t) + Ct^{1-\beta} + t^\alpha, \quad t > 0,$$

where $C > 0$ does not depend on ϵ, k or R. ∎

This result allows us to get the following

Lemma 4.5: *Let $\lambda \in (0, 1/2)$. There exists $\epsilon_0' > 0$ small and $R > 2$ large enough such that if the bounded function u_ϵ satisfies* (32) *and* (33), *and w_k, v_k defined as in* (38) *satisfy* (40), (41), *then we have*

$$\sup_{B_1} |w_k - v_k| \le \lambda^{1+\alpha}, \tag{45}$$

for all $k \le q_\epsilon$.

Proof. The main idea is to apply Lemma 4.2 to w_k and v_k. In fact, we argue over the function

$$\tilde{w}_k = \eta_R w_k,$$

where η_R is a suitable cut-off function as defined in Corollary 2.4. In view of the assumed property (41) we see that

$$|\tilde{w}_k(x)| \le (1+|x|)^{\bar{\alpha}} \quad \text{for all } x \in \mathbb{R}^N.$$

Now, Lemma 4.4 together with the upper bound for the constant L asserted in Corollary 2.4 tells us that by taking ϵ_0' small enough we have

$$|\tilde{w}_k(x) - \tilde{w}_k(y)| \le \bar{C} R^{\bar{\alpha}}\ \bar{m}(|x-y|) \quad \text{for all } x, y \in B_{2R}$$

for some $\bar{C} > 0$ not depending on R, ϵ or k. In view of the definition of $\tilde{w}_k$, we can re-state the last inequality as

$$|\tilde{w}_k(x) - \tilde{w}_k(y)| \leq \bar{C}(|x| \vee |y|)^{\bar{\alpha}} \bar{m}(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^N. \tag{46}$$

The above expression represents a local modulus of continuity for the family $\{\tilde{w}_k\}$ which is independent of k, ϵ or R.

Using (42) and assumption (41) together with the arguments in Corollary 2.4 we can see that

$$-\epsilon_0' - CR^{\bar{\alpha}-\sigma} \leq \mathcal{I}_{\lambda^{-k}\epsilon}(\tilde{w}_k) \leq \epsilon_0' + CR^{\bar{\alpha}-\sigma} \quad \text{in } B_1$$

for some universal constant $C > 0$.

Finally, using a similar argument as in the previous discussion together with the definition of v_k in (38), we see that the function $\tilde{v}_k$ defined as

$$\tilde{v}_k = \eta_R v_k$$

satisfies $\tilde{v}_k = \tilde{w}_k$ in B_1^c and

$$-CR^{\bar{\alpha}-\sigma} \leq (-\Delta)^{\sigma/2}\tilde{v} \leq CR^{\bar{\alpha}-\sigma} \quad \text{in } B_1.$$

Hence we consider $\rho = \lambda^{1+\alpha}$, $M = 1$, $\gamma = \bar{\alpha}$ and m the local modulus of continuity given in (46) in Lemma 4.2 and the corresponding ϵ_0 given in that lemma associated to this data, and fix $\epsilon_0' \leq \epsilon_0/2$ and R large enough to have $CR^{\bar{\alpha}-\sigma} \leq \epsilon_0/2$ and conclude that

$$\sup_{B_1} |\tilde{w}_k - \tilde{v}_k| \leq \lambda^{1+\alpha},$$

which leads to (45) by definition of $\tilde{w}_k, \tilde{v}_k$. ∎

Finally, we are in position to provide the

Proof of (40) & (41). We prove (40), (41) inductively. Notice that the basic case $k = 0$ is satisfied because $w_0 = u_\epsilon$ with $||u_\epsilon||_\infty \leq 1$, and $\ell_0 = 0$. Suppose now that the induction hypothesis is true for all $j \in \mathbb{N}$ with $j \leq k < q_\epsilon$, and let us prove it is true for $k+1$. Without loss of generality we assume $k \geq 1$. We see that (40) for $k+1$ is a consequence of (41) for $k+1$ since $\bar{a}_{k+1} = v_{k+1}(0)$, $\bar{b}_{k+1} = Dv_{k+1}(0)$ with v_{k+1} the fractional harmonic function in B_1 equal to w_{k+1} in B_1^c. Thus (40) for $k+1$ follows from the combination of the previous discussion and (36).

Now, for (41) we observe that by the definition of w_k and $\bar{\ell}_k$ we can write equivalently

$$w_{k+1}(x) = \frac{(w_k - \bar{\ell}_k)(\lambda x)}{\lambda^{1+\alpha}}, \quad x \in \mathbb{R}^N.$$

We split the analysis depending on the size of x relative to λ. If $\lambda|x| \geq 1$, using (41) and (44) we see that

$$\begin{aligned}|w_{k+1}(x)| \leq& \lambda^{-(1+\alpha)}((1 + |\lambda x|)^{1+\bar{\alpha}} + \bar{A} + \bar{A}\lambda|x|)\\ \leq& 2^{1+\bar{\alpha}}\lambda^{-(1+\alpha)}(\lambda^{1+\bar{\alpha}}|x|^{1+\bar{\alpha}} + \bar{A}\lambda|x|)\\ \leq& 2^{1+\bar{\alpha}}\lambda^{\bar{\alpha}-\alpha}(1 + \bar{A})|x|^{1+\bar{\alpha}},\end{aligned}$$

from which, in view of the choice of λ in (37), we get (41) for $k+1$ in this case. If $1/2 \leq \lambda|x| < 1$, we use the uniform bound for v_k in B_1 given by the first inequality in (36) together with (45) to write

$$\begin{aligned}|w_{k+1}(x)| \leq& \lambda^{-(1+\alpha)}(\lambda^{1+\alpha} + |v_k(\lambda x)| + |\bar{\ell}_k(\lambda x)|)\\ \leq& \lambda^{-(1+\alpha)}(\lambda^{1+\alpha} + 3\bar{A})\\ \leq& 1 + 3\bar{A}\lambda^{-(1+\alpha)}2^{1+\bar{\alpha}}(\lambda|x|)^{1+\bar{\alpha}},\end{aligned}$$

and since $\bar{\alpha} < 1$, by the choice of λ in (37) we get (41) for $k+1$ in this case. Finally, if $\lambda|x| < 1/2$ we use (45) and the $C^{1,\bar{\alpha}}$ estimates given by the last inequality in (36) to write

$$\begin{aligned}|w_{k+1}(x)| \leq& \lambda^{-(1+\alpha)}(\lambda^{1+\alpha} + |v_k(\lambda x) - \bar{\ell}_k(\lambda x)|)\\ \leq& 1 + \bar{A}\lambda^{\bar{\alpha}-\alpha}|x|^{1+\bar{\alpha}}\end{aligned}$$

and, as before, by the choice of λ, we conclude (41) for this case, completing (41) for $k+1$. This finishes the proof. ∎

Now we can prove our simplified version of Theorem 1.3.

Proof of Theorem 4.1. Recalling that m is a modulus of continuity for u_ϵ in B_1, we consider $R > 2$ given in Proposition 4.3 and the function

$$\tilde{u}_\epsilon(x) = u_\epsilon\Big(\frac{x}{4R}\Big), \quad x \in \mathbb{R}^N.$$

By monotony of the modulus, notice that $\tilde{u}_\epsilon$ has the same modulus of continuity m in B_{4R} since

$$|\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)| \leq m(|y - y|/(4R)) \leq m(|x - y|),$$

and using Lemma 2.3 we see that

$$-R^{-\sigma}\theta_0 \le \mathcal{I}_{R\epsilon}(\tilde{u}_\epsilon) \le R^\sigma \theta_0 \quad \text{in } B_{4R}.$$

Then, by taking θ_0 and ϵ_0 small enough in order to apply Proposition 4.3 over $\tilde{u}_\epsilon$, we consider the linear functions asserted in that proposition, which we still denote by ℓ_k for all $k \le q_\epsilon$ defined in (34). From this point, we omit the superscript "$\sim$" and argue directly over u_ϵ.

We recall that q_ϵ defined in (34) depends only on λ, m, ϵ and σ and is such that $q_\epsilon \to +\infty$ as $\epsilon \to 0$. With this in mind, we define

$$a_\epsilon = u_\epsilon(0) + a_{q_\epsilon}, \quad b_\epsilon = b_{q_\epsilon}, \quad \text{and} \quad c_\epsilon = \lambda^{q_\epsilon},$$

from which $l_\epsilon(x) = a_\epsilon + b_\epsilon x$. By Proposition 4.3 and the equicontinuity of u_ϵ we see that $a_\epsilon \to a$, $b_\epsilon \to b$ as $\epsilon \to 0$ for some $a \in \mathbb{R}, b \in \mathbb{R}^N$. Let $x \in B_{1/2}$ and let $k = k(x) \in \mathbb{N}$ such that $\lambda^{k+1} \le |x| \le \lambda^{k+1}$. If $x \in B_{c_\epsilon}$ then we write

$$|u_\epsilon(x) - u_\epsilon(0)| \le m(|x|),$$

and this is the statement for which the theorem reduces by the asserted equicontinuity assumption over u_ϵ.

If $x \in B^c_{c_\epsilon}$ we write

$$|u_\epsilon(x) - u_\epsilon(0) - l_\epsilon(x)| \le |u_\epsilon(x) - l_k(x)| + \sum_{j=k}^{q_\epsilon - 1} |l_{j+1}(x) - l_j(x)|,$$

and since necessarily $k \le q_\epsilon$ in this case, by Proposition 4.3 we can write

$$\begin{aligned} |u_\epsilon(x) - u_\epsilon(0) - l_\epsilon(x)| \le & C|x|^{(1+\alpha)k} + C\sum_{j=k}^{+\infty} \lambda^{(1+\alpha)j} + \lambda^{\alpha j}|x| \\ \le & C(|x|^{(1+\alpha)k} + \lambda^{(1+\alpha)k} + \lambda^{\alpha k}|x|). \end{aligned}$$

This leads to the result because $\lambda^{k+1} \le |x|$. ∎

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