

Multiple solutions for periodic perturbations of a delayed autonomous system near an equilibrium

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Abstract

Small non-autonomous perturbations around an equilibrium of a non-linear delayed system are studied. Under appropriate assumptions, it is shown that the number of T -periodic solutions lying inside a bounded domain $\Omega \subset \mathbb{R}^N$ is, generically, at least $|\chi \pm 1| + 1$, where χ denotes the Euler characteristic of Ω . Moreover, some connections between the associated fixed point operator and the Poincaré operator are explored.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. An elementary result from the theory of ODEs establishes that if a smooth function $G : \overline{\Omega} \rightarrow \mathbb{R}^N$ is inwardly pointing over $\partial\Omega$, that is

$$\langle G(x), \nu(x) \rangle < 0 \quad x \in \partial\Omega, \quad (1)$$

where $\nu(x)$ denotes the outer normal at x , then the solutions of the autonomous system of ordinary differential equations

$$u'(t) = G(u(t))$$

with initial data $u(0) = u_0 \in \overline{\Omega}$ are defined and remain inside Ω for all $t > 0$.

Now, let us denote the space of T -periodic continuous functions as

$$C_T := \{u \in C(\mathbb{R}, \mathbb{R}^N) : u(t+T) = u(t)\}$$

and, for given $p \in C_T$, consider the non-autonomous system

$$u'(t) = G(u(t)) + p(t).$$

If $\overline{\Omega}$ has the fixed point property, then the above system has at least one T -periodic orbit, provided that $\|p\|_\infty$ is small. This is a straightforward consequence of the fact that the time-dependent vector field $G(x) + p(t)$ is still inwardly pointing for all t ; hence, the set $\overline{\Omega}$ is invariant for the associated flow and thus the Poincaré operator given by $Pu_0 := u(T)$ is well defined for $u_0 \in \overline{\Omega}$ and satisfies $P(\overline{\Omega}) \subset \overline{\Omega}$.

More generally, observe that, when (1) is assumed, the homotopy defined by $h(x, s) := sG(x) - (1 - s)\nu(x)$ with $s \in [0, 1]$ does not vanish on $\partial\Omega$; whence

$$\deg_B(G, \Omega, 0) = \deg_B(-\nu, \Omega, 0),$$

where \deg_B stands for the Brouwer degree. Thus, it follows from [3] that $\deg_B(G, \Omega, 0) = (-1)^N \chi(\Omega)$, where $\chi(\Omega)$ denotes the Euler characteristic of Ω .

It is worthy to recall (see *e.g.*, [11]) that if $\overline{\Omega}$ has the fixed point property, then $\chi(\Omega)$ is different from 0. This follows easily in the present setting from the fact that if $\chi(\Omega) = 0$ then one can construct a field G satisfying (1) that does not vanish in Ω . If $\overline{\Omega}$ has the fixed point property, then there exist (non-constant) T -periodic solutions of all periods which, in turn, implies that G vanishes, a contradiction. Interestingly, the converse of the result in [11] is not true; that is, one can easily find Ω with nonzero Euler characteristic such that $\overline{\Omega}$ has not the fixed point property. For such a domain, the Poincaré map has obviously a fixed point (because G vanishes in Ω). This yields the conclusion that a fixed point-free map in $C(\overline{\Omega}, \overline{\Omega})$ cannot belong to the closure of the set of all the Poincaré maps associated to the homotopy class of $-\nu$.

Now suppose, independently of the value of $\chi(\Omega)$, that G vanishes at some point $e \in \Omega$, namely, that e is an equilibrium point of the autonomous system. It is well known that if $M := DG(e)$ is nonsingular, then the degree of G over any small neighbourhood V of e is well defined and coincides with $s(M)$, where

$$s(M) := \operatorname{sgn}(\det(M)). \quad (2)$$

Thus, if $s(M)$ is different from $(-1)^N \chi(\Omega)$, then the excision property of the degree implies that the system has at least another equilibrium point in $\Omega \setminus \overline{V}$. Furthermore, it follows from Sard's lemma that, for almost all values \overline{p} in a neighbourhood of $0 \in \mathbb{R}^N$, the mapping $G + \overline{p}$ has at least Γ different zeros in Ω , with

$$\Gamma = \Gamma(M) := |\chi(\Omega) - (-1)^N s(M)| + 1. \quad (3)$$

Thus, one might expect that if $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T -periodic and $\|p\|_\infty$ is small, then the number of T -periodic solutions of the non-autonomous system is generically greater or equal to Γ . Here, 'generically' should be understood in the sense of Baire category, that is, the property is valid for all p (close to the origin) in the space of continuous T -periodic except for a meager set. It can be shown, indeed, that the fixed point index of the Poincaré map P at e is equal to $(-1)^N s(M)$ and, moreover, a homotopy argument shows that the degree of

P over Ω is equal to $\chi(\Omega)$. Details are omitted because the result follows from the main theorem of the present paper.

For several reasons, the situation is different for the delayed system

$$u'(t) = g(u(t), u(t - \tau)) \quad (4)$$

where, for simplicity, we shall assume that $g : \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}^N$ is continuously differentiable. In the first place observe that, due to the delay, the condition that the field $G(x) := g(x, x)$ is inwardly pointing does not necessarily avoid that solutions with initial data $x_0 := \phi \in C([-\tau, 0], \overline{\Omega})$ may eventually abandon $\overline{\Omega}$. However, taking into account that

$$|u(t_0 - \tau) - u(t_0)| \leq \tau \max_{t \in [t_0 - \tau, t_0]} |u'(t)|,$$

it follows that the flow-invariance property, now over the set $C([-\tau, 0], \overline{\Omega})$, is retrieved under the stronger assumption

$$\langle g(x, y), \nu(x) \rangle < 0 \quad (x, y) \in \mathcal{A}_\tau(\Omega) \quad (5)$$

where

$$\mathcal{A}_\tau(\Omega) := \{(x, y) \in \partial\Omega \times \overline{\Omega} : |y - x| \leq \tau \|g\|_\infty\}.$$

In the second place, the previous considerations regarding the Poincaré map become less obvious, since the latter is now defined not over $\overline{\Omega}$ but over the metric space $C([-\tau, 0], \overline{\Omega})$. In connection with this fact, we recall that the characteristic equation for the autonomous linear delayed systems is transcendental (also called quasipolynomial equation), so there exist typically infinitely many complex characteristic values.

Throughout the paper, we shall assume as before that system (4) has an equilibrium point $e \in \Omega$, that is, such that $g(e, e) = 0$. This necessarily occurs when $\chi(\Omega) \neq 0$, although this latter condition shall not be imposed.

Denote by $A, B \in \mathbb{R}^{N \times N}$ the respective matrices $D_x g(e, e)$ and $D_y g(e, e)$. Again, if $A+B$ is nonsingular and $s(A+B)$ is different from $(-1)^N \chi(\Omega)$, then the system has at least one extra equilibrium point in Ω ; furthermore, the number of equilibria in Ω is generically greater or equal to Γ . This is readily verified by writing the set of all the functions $g \in C^1(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}^N)$ satisfying (5) as the union of the closed sets

$$X_n := \left\{ g \in C^1(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}^N) : \langle g(x, y), \nu(x) \rangle \leq -\frac{1}{n} \quad \text{for } (x, y) \in \mathcal{A}_\tau(\Omega) \right\}$$

and noticing that $X_n \cap \mathcal{C}$ is nowhere dense, where \mathcal{C} denotes the set of those functions g such that 0 is a critical value of the corresponding G .

Our goal in this work is to extend the preceding ideas for non-autonomous periodic perturbations of (4), namely the problem

$$u'(t) = g(u(t), u(t - \tau)) + p(t) \quad (6)$$

with $p \in C_T$.

As a basic hypothesis, we shall assume that the linearisation at the equilibrium, that is, the system

$$u'(t) = Au(t) + Bu(t - \tau) \quad (7)$$

has no nontrivial T -periodic solutions. This clearly implies, in particular, the above-mentioned condition that $A + B$ is invertible. From the Floquet theory for DDEs, it is known that the latter condition is also sufficient for nearly all positive values of T (i.e., except for at most a countable set). For the sake of completeness, this specific consequence of the Floquet theory shall be shown below (see Remark 2.2).

Our main result reads as follows.

Theorem 1.1 *Let the equilibrium e and the matrices A and B be as before and assume that the linear system (7) has no nontrivial T -periodic solutions. Then:*

- (a) *There exists $r > 0$ such that for any $p \in C_T$ with $\|p\|_\infty < r$ the non-autonomous problem (6) has at least one T -periodic solution.*
- (b) *If moreover (5) holds and $s(A+B) \neq (-1)^N \chi(\Omega)$ with s defined as in (2), then (6) has at least two T -periodic solutions.*
- (c) *Furthermore, there exists a residual set $\Sigma_r \subset C_T$ such that if $p \in \Sigma_r \cap B_r(0)$, then the number of T -periodic solutions is at least $\Gamma(A+B)$, where Γ is given by (3).*

The next result is an immediate consequence of Theorem 1.1 combined with the preceding comments.

Corollary 1.1 *Let e, A and B be as before and assume that $A+B$ is invertible. Then for nearly all $T > 0$ there exists $r = r(T) > 0$ such that if $p \in C_T$ with $\|p\|_\infty < r$ then the non-autonomous problem (6) has at least one T -periodic solution. If moreover (5) holds and $s(A+B) \neq (-1)^N \chi(\Omega)$, then the number of T -periodic solutions is at least 2 and generically $\Gamma(A+B)$.*

For small delays, the condition that (7) has no nontrivial T -periodic solutions can be formulated explicitly in terms of the matrix $A + B$:

Corollary 1.2 *Let e, A and B be as before and assume that $\frac{2k\pi}{T}i$ is not an eigenvalue of the matrix $A + B$ for all $k \in \mathbb{N}_0$. Then for each τ small enough there exists $r = r(\tau)$ such that the non-autonomous problem (6) has at least one T -periodic solution for any $p \in C_T$ with $\|p\|_\infty < r$. If moreover (1) holds for $G(x) := g(x, x)$ and $s(A+B) \neq (-1)^N \chi(\Omega)$, then (6) has at least two T -periodic solutions and generically $\Gamma(A+B)$.*

It is worthy mentioning that if Ω is for example a ball, then the condition $s(A+B) \neq (-1)^N \chi(\Omega)$ implies that the equilibrium is unstable. As we shall see, this can be regarded as a consequence of the fact that the Leray-Schauder

index of the fixed point operator defined in the proof of our main theorem is $(-1)^{N+1}$. This connection can be deduced from a version of the Krasnoselskii relatedness principle, which implies that the mentioned index coincides except for a $(-1)^N$ factor with that of the Poincaré operator. As shown in Proposition 5.1, this implies, in turn, that the equilibrium cannot be stable.

The paper is organised as follows. In the next section, we prove some basic facts concerning the linearised problem (7); in particular, we give a necessary and sufficient condition in order to ensure that it has no nontrivial T -periodic solutions. In section 3 we present a proof of Theorem 1.1 by means of an appropriate fixed point operator. The next two sections are devoted to a proof in section 4, we give a proof Corollary 1.2. In section 5, we make some considerations on the stability of the equilibrium and the indices, on the one hand, of the fixed point operator defined in section 3 and of the Poincaré map, on the other hand. Finally, a simple application of the main results for a singular system is introduced in section 6.

2 Linearised system

In this section, we shall prove some basic facts concerning the linear system (7). To this end, let us introduce some notation. For $k \in \mathbb{N}_0$, define

$$\lambda_k := \frac{2k\pi}{T}$$

and

$$\varphi_k(t) := \cos(\lambda_k t) \quad \psi_k(t) := \sin(\lambda_k t).$$

It is readily verified that

$$\varphi_k(t - \tau) = \varphi_k(t)\varphi_k(\tau) + \psi_k(t)\psi_k(\tau)$$

$$\psi_k(t - \tau) = \psi_k(t)\varphi_k(\tau) - \varphi_k(t)\psi_k(\tau)$$

and

$$\varphi'_k = -\lambda_k \psi_k, \quad \psi'_k = \lambda_k \varphi_k.$$

For an element $u \in C_T$, we may consider its Fourier series, namely

$$u = a_0 + \sum_{k=1}^{\infty} (\varphi_k a_k + \psi_k b_k)$$

in the L^2 sense, with $a_k, b_k \in \mathbb{R}^N$. Furthermore, recall that if u is smooth (*e.g.*, of class C^2) then the series and its term-by-term derivative converge uniformly to u and u' respectively.

Lemma 2.1 *Let $u \in C_T$ and define*

$$X_k := A + \varphi_k(\tau)B, \quad Y_k := \lambda_k I + \psi_k(\tau)B. \quad (8)$$

Then u is a solution of (7) if and only if

$$\begin{pmatrix} X_k & -Y_k \\ Y_k & X_k \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (9)$$

for all $k \in \mathbb{N}_0$.

Proof: Since $\varphi'_k(t), \varphi_k(t-\tau), \psi'_k(t)$ and $\psi_k(t-\tau)$ belong to $\text{span}\{\varphi_k(t), \psi_k(t)\}$, it follows that u is a solution of (7) if and only if

$$(A+B)a_0 = 0$$

and

$$\varphi'_k(t)a_k + \psi'_k(t)b_k = A(\varphi_k(t)a_k + \psi_k(t)b_k) + B(\varphi_k(t-\tau)a_k + \psi_k(t-\tau)b_k)$$

for all $k > 0$. The latter identity, in turn, is equivalent to

$$\begin{aligned} \lambda_k b_k &= [A + \varphi_k(\tau)B]a_k - \psi_k(\tau)Bb_k \\ -\lambda_k a_k &= \psi_k(\tau)Ba_k + [A + \varphi_k(\tau)B]b_k, \end{aligned}$$

that is,

$$X_k a_k - Y_k b_k = Y_k a_k + X_k b_k = 0.$$

Because $X_0 = A+B$ and $Y_0 = 0$, we deduce that u is a solution of (7) if and only if (9) holds for all $k \in \mathbb{N}_0$. □

Corollary 2.1 (7) has no nontrivial T -periodic solutions if and only if

$$h_k := \det \begin{pmatrix} X_k & -Y_k \\ Y_k & X_k \end{pmatrix} \neq 0 \quad (10)$$

for all $k \in \mathbb{N}_0$.

Remark 2.2

1. Because $A+B$ is invertible, it is clear that for nearly all $T > 0$ condition (10) is satisfied for all k . Indeed, it suffices to observe that h_k , regarded as a function of $T \in (0, +\infty)$, is an analytic function and, consequently, it has at most a countable number of zeros.
2. It can be shown that $h_k \geq 0$; in particular, its roots have even multiplicity. The proof is straightforward when A and B commute, since in this case

$$\det \begin{pmatrix} X_k & -Y_k \\ Y_k & X_k \end{pmatrix} = \det(X_k^2 + Y_k^2).$$

The conclusion then follows, because for any pair of square real matrices X, Y such that $XY = YX$ it is verified that

$$\det(X^2 + Y^2) = \det[(X + iY)(X - iY)] = \det(X + iY)\overline{\det(X + iY)} \geq 0.$$

A proof for the non-commutative case is given below in section 3, step 3.

It is noticed that (10) may hold for non-invertible matrices X_k and Y_k : for instance, observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^2 = I.$$

3. Since $\lambda_k \rightarrow +\infty$ it follows, for k large, that

$$h_k = \det(Y_k)\det(Y_k + X_k Y_k^{-1} X_k) \simeq \lambda_k^{2N} > 0.$$

In particular, there exists k_0 such that if u is a T -periodic solution of (7) then $a_k = b_k = 0$ for $k > k_0$. This means that u is a (vector) trigonometric polynomial. Incidentally, observe that, because the family $\{\varphi_k, \psi_k\}$ is uniformly bounded, the constant k_0 may be chosen independent of τ .

In other words, if we consider the linear operator $L : C_T \rightarrow C_T$, given by $Lu(t) := u'(t) - Au(t) - Bu(t - \tau)$, then $\ker(L) \subset \mathbf{span}\{\varphi_k, \psi_k\}_{0 \leq k \leq k_0}$. Observe furthermore that $\text{Im}(L)$ consists of all the Fourier series $a_0 + \sum_{k>0}(\varphi_k a_k + \psi_k b_k)$ such that $a_0 \in \text{Im}(A + B)$ and $(a_k, b_k) \in \text{Im}(M_k)$, where M_k is the matrix defined in (9). This yields a direct proof of the well-known fact that L is a zero-index Fredholm operator. Moreover, it is verified that $(a_k, b_k) \in \ker(M_k) \iff (-b_k, a_k) \in \ker(M_k)$, a fact that will be of relevance in the proofs of our results.

3 Proof of the main theorem

For convenience, a little of extra notation shall be introduced. For a function $u \in C_T$, let us write

$$\mathcal{I}u(t) := \int_0^t u(s) ds, \quad \bar{u} := \frac{1}{T}\mathcal{I}u(T).$$

Moreover, denote by \mathcal{N} the Nemitskii operator associated to the problem, namely

$$\mathcal{N}u(t) := g(u(t), u(t - \tau)).$$

Without loss of generality we may assume $e = 0$ and fix $T > 0$ such that (7) has no nontrivial T -periodic solutions. For simplicity, we shall assume from the beginning that all the assumptions are satisfied; it shall be easy for the reader to deduce the existence of one solution near the equilibrium when (5) is not satisfied.

Define the open bounded set $U = \{u \in C_T : u(t) \in \Omega \text{ for all } t\}$ and the compact operator $K : \overline{U} \rightarrow C_T$ defined by

$$Ku(t) := \overline{u} - t\overline{\mathcal{N}u} + \mathcal{I}\mathcal{N}u(t) - \overline{\mathcal{I}\mathcal{N}u}.$$

We shall prove that the Leray-Schauder degree of $I - K$ is equal to $(-1)^N \chi(\Omega)$ over U and to $s(A + B)$ over $B_\rho(0)$ for small values of $\rho > 0$.

To this end, let us proceed in several steps:

1. Let $K_0u := \overline{u} - \frac{T}{2}\overline{\mathcal{N}u}$ and define, for $s \in [0, 1]$, the operator given by $K_s := sK + (1 - s)K_0$. We claim that K_s has no fixed points on ∂U . Indeed, for $s > 0$ it is clear that $u \in \overline{U}$ is a fixed point of K_s if and only if $u'(t) = s\mathcal{N}u(t)$, that is:

$$u'(t) = sg(u(t), u(t - \tau)).$$

Suppose there exists t_0 such that $u(t_0) \in \partial\Omega$, then we deduce, as before,

$$|u(t_0 - \tau) - u(t_0)| \leq \tau \max_{t \in [t_0 - \tau, t_0]} |u'(t)| \leq \tau \|g\|_\infty$$

and by (5) we obtain

$$0 = \langle u'(t_0), \nu(u(t_0)) \rangle = s \langle g(u(t_0), u(t_0 - \tau)), \nu(u(t_0)) \rangle < 0,$$

a contradiction. On the other hand, we observe that the range of K_0 is contained in the set of constant functions, which can be identified with \mathbb{R}^N ; thus, the Leray-Schauder degree of $I - K_0$ can be computed as the Brouwer degree of its restriction to $\overline{U} \cap \mathbb{R}^N = \overline{\Omega}$.

Furthermore, for $u(t) \equiv u \in \overline{\Omega}$ it is clear that $K_0u = u - \frac{T}{2}G(u)$, which does not vanish on $\partial\Omega = \partial U \cap \mathbb{R}^N$. By the homotopy invariance of the degree, we conclude that

$$\deg(I - K, U, 0) = \deg\left(\frac{T}{2}G, \Omega, 0\right) = (-1)^N \chi(\Omega).$$

2. Let K_L be the operator associated to the linearised problem, defined by

$$K_Lu(t) := \overline{u} - t\overline{\mathcal{N}_Lu} + \mathcal{I}\mathcal{N}_Lu(t) - \overline{\mathcal{I}\mathcal{N}_Lu},$$

with $\mathcal{N}_Lu(t) := Au(t) + Bu(t - \tau)$. As before, it is seen that $K_Lu = u$ if and only if u is a solution of (7); hence, it follows from the assumptions that K_L has no nontrivial fixed points.

Furthermore, the degree of $I - K_L$ coincides with the degree of $I - K$ on $B_\rho(0)$ when ρ is small. This is a well-known fact but, for the reader's convenience, a simple proof is sketched as follows.

Since the degree is locally constant, we may assume that g is of class C^2 near $(0, 0)$, then for some $C > 0$,

$$\|Kv - K_Lv\|_\infty \leq C\|\mathcal{N}v - \mathcal{N}_Lv\|_\infty = o(\rho).$$

Because K_L is compact, it is verified that, for some $\theta > 0$,

$$\|v - K_L v\|_\infty \geq \theta \rho$$

for all $v \in \partial B_\rho(0)$. Indeed, due to linearity, it suffices to prove the claim for $\rho = 1$. By contradiction, suppose there exists a sequence $\{v_n\} \subset \partial B_1(0)$ such that $\|v_n - K_L v_n\|_\infty \rightarrow 0$, then passing to a subsequence we may assume that $\{K_L v_n\}$ converges to some v . Then $v_n \rightarrow v$ which, in turn, implies that $\|v\|_\infty = 1$ and $v = K_L v$, a contradiction. It follows that if $\rho > 0$ is small then $sK + (1-s)K_L$ has no fixed points on $\partial B_\rho(0)$ for $s \in [0, 1]$ because

$$\|v - sKv - (1-s)K_L v\|_\infty \geq \|v - K_L v\|_\infty - \|K_L v - Kv\|_\infty \geq \theta \rho - o(\rho) > 0$$

for $v \in \partial B_\rho(0)$. Thus, the degree of $I - K$ is well defined and coincides with the degree of $I - K_L$ over $B_\rho(0)$.

3. Claim: $\deg(I - K_L, B_\rho(0), 0) = s(A + B)$.

Indeed, for u as before it is seen by direct computation that

$$u - K_L u = \tilde{a}_0 + \sum_{k \geq 1} (\varphi_k \tilde{a}_k + \psi_k \tilde{b}_k)$$

where

$$\tilde{a}_0 = \mathcal{M}_0 a_0$$

and

$$\begin{pmatrix} \tilde{a}_k \\ \tilde{b}_k \end{pmatrix} = \mathcal{M}_k \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

with

$$\mathcal{M}_0 := \frac{T}{2}(A + B) \quad \text{and} \quad \mathcal{M}_k := \frac{1}{\lambda_k} \begin{pmatrix} Y_k & X_k \\ -X_k & Y_k \end{pmatrix} \quad \text{for } k > 0.$$

Hence, the degree coincides with the sign of the determinant of the block matrix

$$\begin{pmatrix} \mathcal{M}_0 & 0 & 0 & \dots & 0 \\ 0 & \mathcal{M}_1 & 0 & \dots & 0 \\ 0 & 0 & \mathcal{M}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathcal{M}_J \end{pmatrix}$$

for J sufficiently large. Thus, the proof follows in a straightforward manner from the fact that $\det(\mathcal{M}_k) > 0$ for all $k > 0$. We remark that the latter property holds even when A and B do not commute (see Remark 2.2).

Indeed, identifying the pairs $(a, b) \in \mathbb{R}^N \times \mathbb{R}^N$ with vectors $a + ib \in \mathbb{C}^N$, a matrix of the form $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ may be called a \mathbb{C} -linear matrix. Thus, we need to prove that if \mathcal{M} is an arbitrary invertible \mathbb{C} -linear matrix,

then the algebraic multiplicity of each eigenvalue $\sigma < 0$ of \mathcal{M} is even. It is known that this value can be computed as the dimension of the kernel of the matrix $(\mathcal{M} - \sigma I)^m$, where m is the minimum integer such that $\ker(\mathcal{M} - \sigma I)^m = \ker(\mathcal{M} - \sigma I)^{m+1}$. Now observe that the set of \mathbb{C} -linear matrices is a subring of $\mathbb{R}^{2N \times 2N}$; thus, $(\mathcal{M} - \sigma I)^m$ is again a \mathbb{C} -linear matrix. In particular, if $(a, b) \in \ker(\mathcal{M} - \sigma I)^m$ then $(-b, a) \in \ker(\mathcal{M} - \sigma I)^m$ and the result follows.

4. *Existence of two solutions for small p .*

From the previous steps and the fact that the degree is locally constant we deduce that

$$\deg(I - K, U, \hat{p}) = (-1)^N \chi(\Omega), \quad \deg(I - K, B_\rho(0), \hat{p}) = s(A + B)$$

when $\|\hat{p}\|_\infty$ is small. Now the excision property of the Leray-Schauder degree implies

$$\deg(I - K, B_\rho(0), \hat{p}) = s(A + B) \neq 0,$$

and

$$\deg(I - K, U \setminus B_\rho(0), \hat{p}) = (-1)^N \chi(\Omega) - s(A + B) \neq 0.$$

Thus, there exists $\hat{r} > 0$ such that the equation $(I - K)u = \hat{p}$ has at least two solutions for $\|\hat{p}\|_\infty < \hat{r}$. Finally, for each $p \in C_T$ define

$$\hat{p}(t) := \mathcal{I}p(t) - \overline{\mathcal{I}p} - t\bar{p},$$

then clearly $\|\hat{p}\|_\infty \leq c\|p\|_\infty$ for some $c > 0$. The result is then deduced from the fact that if $u - Ku = \hat{p}$, then u is a T -periodic solution of (6).

$$u'(t) = g(u(t), u(t - \tau)) + p(t).$$

5. *Genericity.*

The last part of the proof follows as a consequence of the following particular case of the Sard-Smale Theorem [9]:

Theorem 3.1 *Let $\mathcal{F} : X \rightarrow Y$ be a C^1 Fredholm map of index 0 between Banach manifolds, i.e. such that $D\mathcal{F}(x) : T_x X \rightarrow T_{\mathcal{F}(x)} Y$ is a Fredholm operator of index 0 for every $x \in X$. Then the set of regular values of \mathcal{F} is residual in Y .*

At this point, we notice that the argument is a bit subtle: when applied to $\mathcal{F} := I - K$, the Sard-Smale Theorem implies the existence of a residual set $\Sigma \subset C_T$ such that the mapping $\mathcal{F} - \hat{p}$ has at least $\Gamma - 1$ zeros in $U \setminus B_\rho(0)$ for $\hat{p} \in \Sigma \cap B_{\hat{r}}(0)$.

Indeed, it is readily seen that K is of class C^1 and $DK(u)$ is compact for all u . Thus, $\mathcal{F} = I - K$ is a zero-index Fredholm operator. If \hat{p} is a regular

value, that is, $D\mathcal{F}(u)$ is surjective for every preimage $u \in \mathcal{F}^{-1}(\hat{p})$ then, since the index is 0, it is also injective and from the open mapping theorem we conclude that $D\mathcal{F}(u)$ is an isomorphism. Hence, the number of such preimages in $U \setminus B_\rho(0)$ is greater or equal than $|\deg(I - K, U \setminus B_\rho(0), 0)|$. This follows by taking small neighbourhoods N_u around each of these values u such that $\mathcal{F} : N_u \rightarrow \mathcal{F}(N_u)$ is a diffeomorphism. Because there are no other zeros of $\mathcal{F} - \hat{p}$ in $U \setminus B_\rho(0)$, the degree is the sum of the degrees d_u over each of these neighbourhoods. The claim then follows from the fact that $d_u = \pm 1$ for each u .

However, although the mapping $p \mapsto \hat{p}$ defined before establishes an isomorphism $J : C_T \rightarrow C_T^1$, it might happen that $J^{-1}(\Sigma \cap C_T^1)$ is not a residual set. The difficulty is overcome for example by considering the same operator K as before, now defined over the set

$$\hat{U} := \{u \in C_T^1 : u(t) \in \Omega, \|u'\|_\infty < \|g\|_\infty + 1\} \subset C_T^1.$$

Details are left to the reader.

Remark 3.2 *Notice that*

1. *The existence of a solution near the equilibrium can be also proved in a direct way by the Implicit Function Theorem.*
2. *Condition (5) alone implies the existence of generically $|\chi(\Omega)|$ solutions.*
3. *Analogous conclusions are obtained if the sign of (5) is reversed. In this case, G is homotopic to ν and hence $\deg(I - K, U, 0) = \chi(\Omega)$. However, in this latter case the considerations about the Poincaré operator become less clear, because it is not guaranteed that solutions with initial values ϕ with $\phi(t) \in \overline{\Omega}$ remain inside Ω .*

4 Small delays

As mentioned in the introduction, condition (5) implies that the vector field $G(x) = g(x, x)$ is inwardly pointing over $\partial\Omega$, although the converse is not true; the need of a condition stronger than (1) is due to the presence of the delay. However, if only (1) is assumed, then Theorem 1.1 is still valid for all $\tau < \tau^*$, where τ^* depends only on $\|g\|_\infty$. More precisely, by continuity we may fix $\varepsilon > 0$ such that (5) holds for all $x \in \partial\Omega$ and all $y \in \overline{\Omega}$ with $|y - x| < \varepsilon$ and take $\tau^* := \frac{\varepsilon}{\|g\|_\infty}$.

In this section, we show that the problem for small τ can be seen as a perturbation of the non-delayed case, thus giving the explicit sufficient condition for the non-existence of nontrivial T -periodic solutions of (7) expressed in Corollary 1.2. We shall make use of the following lemmas:

Lemma 4.1 *1 is a Floquet multiplier of the system $u'(t) = Mu(t)$ if and only if $-\lambda_k^2$ is an eigenvalue of M^2 for some $k \in \mathbb{N}_0$, that is, if and only if $\pm i\lambda_k$ are eigenvalues of M for some k .*

Proof: The result follows by direct computation, or from Lemma 2.1 with $\tau = 0$.
 \square

For example, when M is triangularizable (or, equivalently, when all its eigenvalues are real), 1 is not an eigenvalue of the system $u'(t) = Mu(t)$ if and only if M is nonsingular; in this particular case, the conclusion follows directly, because the system uncouples and the result is obviously true for a scalar equation.

Lemma 4.2 *Assume that 1 is not a Floquet multiplier of the linear ODE system $u'(t) = (A + B)u(t)$. Then the DDE system (7) has no nontrivial T -periodic solutions, provided that τ is small.*

Proof: Suppose that $u_n \in C_T$ is a nontrivial solution for $\tau_n \rightarrow 0$. Without loss of generality, it may be assumed that $\|u_n\|_\infty = 1$ and hence $\|u_n'\|_\infty \leq C$ for some constant C . Thus, we may assume that u_n converges uniformly to some $u \in C_T$ with $\|u\|_\infty = 1$. Because $\|u_n(t - \tau_n) - u_n(t)\| \leq C\tau_n \rightarrow 0$, it becomes clear that u_n' converges uniformly to $(A + B)u$ which, in turn, implies $u' = (A + B)u$, a contradiction. \square

Remark 4.3 *A more direct proof of Lemma 4.2 follows just by considering Remark 2.2.3 and Lemma 4.1. Indeed, in the context of Lemma 2.1 it suffices to check that $h_k \neq 0$ only for a finite number of values of k . By continuity, this is true for small τ , because $\det[(A + B)^2 + \lambda_k^2 I] \neq 0$ for all k . However, the previous proof has an interest in its own because it can be extended in a straightforward manner to the non-autonomous case.*

Proof of Corollary 1.2: As a consequence of the preceding lemma, the conclusions of Theorem 1.1 hold for small τ , provided that the linearisation has no nontrivial T -periodic solutions for the non-delayed case. Thus, in view of Lemma 4.1, the proof is complete. \square

5 Poincaré operator

In this section, we shall make some considerations regarding the Poincaré operator associated to the system. Let us firstly observe that if $\chi(\Omega) = 1$ (for example, if Ω is homeomorphic to a ball), then the condition $s(A + B) \neq (-1)^N \chi(\Omega)$ in Theorem 1.1 simply reads $(-1)^N \det(A + B) < 0$. This, in turn, implies that the equilibrium is unstable. Indeed, consider the characteristic function $h(\lambda) = \det(\lambda I - A - Be^{-\lambda\tau})$, then $h(0) = (-1)^N \det(A + B) < 0$ and $h(\lambda) = \lambda^N$ for $|\lambda| \gg 0$. In particular, this implies the existence of a characteristic value $\lambda > 0$.

We shall show that, in the present context, the instability of the equilibrium when $(-1)^N \det(A + B) < 0$ is due to the fact, proved in section 3, that the index of the fixed point operator K at e (i. e. the degree of $I - K$ over small balls around e) is equal to $(-1)^{N+1}$. When $\tau = 0$, this can be regarded as a direct consequence of the following properties:

1. $\deg(I - K, B_\rho(e), 0)$ with $B_\rho(e) \subset C_T$ is equal to $(-1)^N \deg_B(I - P, B_\rho(e), 0)$ with $B_\rho(e) \subset \mathbb{R}^N$, where P is the Poincaré map.

2. If the equilibrium is stable, then the index of P is 1.

The first property is a particular case of a *relatedness principle* due to Krasnoselskii (see [6]). The second property is well-known and can be found for example in [5]. For more details see [7], where sufficient conditions for the validity of the converse statement are also obtained.

Our goal in this section consists in understanding the connections between the instability of the equilibrium and the index of the fixed point operator defined in the proof of the main theorem.

With this aim, let us define the Poincaré operator for the delayed case as follows. Let $\tau \leq T$ and consider a general autonomous system

$$u'(t) = F(u_t) \tag{11}$$

with $F : C([-\tau, 0]) \rightarrow \mathbb{R}^N$ locally Lipschitz, *i.e.*: for all $R > 0$ there exists a constant L such that

$$\|F(\phi) - F(\psi)\| \leq L\|\phi - \psi\|_\infty$$

for all $\phi, \psi \in \overline{B_R(0)} \subset C([-\tau, 0], \mathbb{R}^n)$. The notation u_t expresses, as usual, the mapping defined by $u_t(\theta) := u(t + \theta)$ for $\theta \in [-\tau, 0]$.

Denote by $\text{dom}(P) \subset C([-\tau, 0])$ the set of those functions ϕ such that the unique solution $u = u(\phi)$ of the problem with initial condition ϕ is defined up to $t = T$, then $P : \text{dom}(P) \rightarrow C([-\tau, 0])$ is defined by

$$P\phi(s) := u(T + s).$$

Clearly, the T -periodic solutions of the problem can be identified with the fixed points of P . We shall see that, as in the non-delayed case, if the linearisation has no nontrivial T -periodic solutions then the index $i(P)$ of the operator P at a stable equilibrium is equal to 1.

To this end, assume without loss of generality that $e = 0$ and observe that stability implies that $\text{dom}(P)$ is a neighbourhood of 0. It is worth noticing that, in the general setting, extra conditions are required in order to prove the compactness of P (see *e.g.* [4]), so the Leray-Schauder degree may be not well defined; however, it is verified that the stability assumption implies that P is compact over small neighbourhoods of 0. More precisely:

Lemma 5.1 *Let F be as before and assume that for some open $U \subset C([-\tau, 0])$ there exists $R > 0$ such that if $\phi \in U$ then the solution u with initial condition ϕ is defined and satisfies $|u(t)| < R$ for all $t \in [0, T]$. Then P is well defined and compact over U .*

Proof:

Let $B \subset U$ be bounded and observe, in the first place, that $P(B)$ is bounded. Moreover, if u is a solution with initial condition $\phi \in B$, then

$$u(t) = \phi(0) + \int_0^t F(u_s) ds.$$

Enlarging R if necessary, we may assume $B \subset B_R(0)$, then $\|u_s\|_\infty < R$ for all $s \in [0, T]$. Given $t_1 < t_2$ in $[-\tau, 0]$, since $\tau \leq T$ it is verified that

$$|P\phi(t_2) - P\phi(t_1)| \leq \int_{T+t_1}^{T+t_2} |F(u_s)| ds.$$

Let L be the Lipschitz constant corresponding to R , then

$$|F(\phi)| \leq |F(0)| + L\|\phi\|_\infty \leq C + LR,$$

where $C := |F(0)|$. Hence $|P(t_2) - P(t_1)| \leq (C + LR)(t_2 - t_1)$ and the result follows from the Arzelà-Ascoli Theorem. \square

Remark 5.2 *For example, the assumptions of the previous lemma are satisfied if F has linear growth, that is*

$$|F(\phi)| \leq \gamma\|\phi\|_\infty + \delta.$$

Furthermore, extra assumptions are required to ensure the non-existence of nontrivial periodic solutions near 0; this is why we shall impose this fact as an extra condition (see Proposition 5.1 below), which is clearly satisfied for example when the stability is asymptotic. For simplicity, we shall also assume that F is Fréchet differentiable at 0, that is,

$$F(\phi) = D_\phi(0)\phi + \mathcal{R}(\phi)$$

with $\|\mathcal{R}(\phi)\|_\infty \leq o(\|\phi\|_\infty)\|\phi\|_\infty$. Thus, it is readily verified that the linearisation of P at the origin coincides with the Poincaré operator associated to the linearised system $u'(t) = D_\phi(0)u_t$.

Proposition 5.1 *In the previous setting, assume that 0 is a stable equilibrium of (11) such that its linearisation has no nontrivial T -periodic solutions. Then $i(P) = 1$.*

Proof: Without loss of generality, we may assume that P is compact on \overline{V} for some neighbourhood V of 0. It follows from the assumptions that the index of P is well defined and coincides with the index of its linearisation P_L . According to Theorem 13.8 in [1], $\deg(I - P_L, B_\rho(0), 0)$ is equal to $(-1)^\alpha$, where α is the sum of the (finite) algebraic multiplicities of the (finitely many) eigenvalues σ of P_L satisfying $\sigma > 1$.

If $\deg(I - P_L, B_\rho(0), 0) = -1$, then P_L has an eigenfunction ϕ with eigenvalue $\sigma > 1$. If u is the corresponding solution of the linearised problem with initial condition $u = \phi$ on $[-\tau, 0]$ then u can be extended to \mathbb{R} in a (T, σ) -periodic fashion, that is, with $u(t + T) = \sigma u(t)$ for all t (see [8]). In particular, $u(t)$ is unbounded for $t > 0$. In other words, 0 is unstable for the linearised problem which, in turn, implies that it cannot be stable for the original problem (see *e.g.* [2]). \square

In order to complete the picture for system (4), it would be interesting to prove that, indeed, the index of the Poincaré operator at the equilibrium when the linearisation has no nontrivial solutions is $(-1)^N s(A + B) = (-1)^N i(K)$. Here, we shall simply verify that the claim holds when the delay is small; the analysis of the general case and a version of the Krasnoselskii relatedness principle for delayed systems shall be the subject of a forthcoming paper.

To this end, let us start with a direct computation for the non-delayed case:

Lemma 5.3 *Let $M \in \mathbb{R}^{N \times N}$ and let P_M be the Poincaré operator associated to the linear ODE system $u'(t) = Mu(t)$ for some fixed T . If 1 is not a Floquet multiplier, then*

$$\deg_B(I - P_M, V, 0) = (-1)^N s(M)$$

for any neighbourhood $V \subset \mathbb{R}^N$ of the origin.

Proof:

By definition,

$$(I - P_M)(u) = (I - e^{TM})u.$$

Write M in its (possibly complex) Jordan form $M = C^{-1}JC$, where J is upper triangular. Then

$$\det(I - e^{TM}) = \det(I - e^{TJ}) = \prod_{j=1}^N (1 - e^{\lambda_j T}),$$

where λ_j are the eigenvalues of M . Now observe that if $\lambda = a + ib \notin \mathbb{R}$, then

$$(1 - e^{\lambda T})(1 - e^{\bar{\lambda} T}) = 1 + e^{aT}(e^{aT} - 2\cos(bT)) > 0.$$

Thus, complex eigenvalues do not affect the sign of $\det(I - e^{TM})$, as well as it happens with the sign of $\det(M)$ because $\lambda\bar{\lambda} = |\lambda|^2$. The result follows now from the fact that, for $\lambda \in \mathbb{R}$,

$$\text{sgn}(1 - e^{\lambda T}) = -\text{sgn}(\lambda).$$

□

Remark 5.4 *An alternative (somewhat exotic) proof follows from the relatedness principle. Indeed, we may consider the operator K_L in the proof of Theorem 1.1 with $A = M$ and $B = 0$, then $\deg_B(I - P, V, 0) = (-1)^N \deg(I - K_L, V, 0) = (-1)^N s(M)$.*

The conclusion for small τ is obtained now by a continuity argument. Indeed, fix $r > 0$ and P_L as before. The solutions of (7) with initial value $\phi \in B_r(0)$ are uniformly bounded; thus, by Gronwall's lemma we deduce that $\|P - P_0\| = O(\tau)$, where the operator P_0 is defined by $P_0(\phi)(t) \equiv v(T)$, with v the unique solution of the system $v'(t) = (A + B)v(t)$ satisfying $v(0) = \phi(0)$. Moreover, recall that if τ is small then P_L is homotopic to P_0 ; thus, the result follows from Lemma 5.3.

6 Example: a system of DDEs with singularities

A simple example is presented here in order to illustrate our main results. Let $0 \leq J_0 \leq J \neq 0$ and

$$g(x, y) := -dx + |y|^2 \left(\sum_{j=1}^{J_0} a_j \frac{x - v_j}{|x - v_j|^{\alpha_j}} + \sum_{j=J_0+1}^J a_j \frac{y - v_j}{|y - v_j|^{\alpha_j}} \right)$$

where $d, a_j > 0$, $\alpha_j > 2$ and $v_j \in \mathbb{R}^N \setminus \{0\}$ are pairwise different vectors. A simple computation shows that

$$\langle g(x, x), x \rangle < 0 \quad |x| \gg 0$$

and

$$\langle g(x, x), v_j - x \rangle < 0 \quad |x - v_j| \ll 1$$

for $j = 1, \dots, J$. Moreover, $g(0, 0) = 0$ and

$$A = D_x g(0, 0) = -dI, \quad B = D_y g(0, 0) = 0.$$

Thus, taking $\Omega := B_R(0) \setminus \cup_{j=1}^J B_\eta(v_j)$ where $R \gg 0$ and $\eta \ll 1$, Corollary 1.2 applies. Since $\chi(\Omega) = 1 - J < 1 = (-1)^N s(A + B)$, we conclude that the number of T -periodic solutions of (6) for small τ and $\|p\|_\infty$ is generically $J + 1$.

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References

- [1] R. F. Brown, *A Topological Introduction to Nonlinear Analysis*. First edition, Birkhäuser (2004).
- [2] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York (1993).
- [3] H. Hopf, *Vektorfelder in n-dimensionalen Mannigfaltigkeiten*. Math. Ann. 96 (1926/1927), pp. 225–250.
- [4] J. Liu, G. N'Guérékata and Nguyen Van Minh, *Topics on Stability and Periodicity in Abstract Differential Equations*. World Scientific, Singapore (2008).

- [5] M. A. Krasnoselskii, *The operator of translation along the trajectories of differential equations*. Amer. Math. Soc., Providence, RI, (1968).
- [6] M. A. Krasnoselskii, P.P. Zabreiko, *Geometrical methods of nonlinear analysis*. Springer-Verlag, Berlin (1984).
- [7] R. Ortega, *Topological degree and stability of periodic solutions for certain differential equations*, J. London Math. Soc. (2) 42 (1990), pp. 505–516.
- [8] M. Pinto, *Pseudo-almost periodic solutions of neutral integral and differential equations with applications*. Nonlinear Anal. 72(12) (2010), pp. 4377–4383.
- [9] S. Smale, *An infinite dimensional version of Sard's theorem*. American Journal of Mathematics 87 (1965), pp. 861–866.
- [10] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer-Verlag, New York (2011).
- [11] F. Wecken, *Fixpunktklassen*. III, Math. Ann. 118-119 (1941-1943), pp. 544–577.

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