Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

(ω, c)-Pseudo periodic functions, first order Cauchy problem and Lasota–Wazewska model with ergodic and unbounded oscillating production of red cells

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Abstract

In this paper we study a new class of functions, which we call (ω, c)-pseudo periodic functions. This collection includes pseudo periodic, pseudo anti-periodic, pseudo Bloch-periodic, and unbounded functions. We prove that the set conformed by these functions is a Banach space with a suitable norm. Furthermore, we show several properties of this class of functions as the convolution invariance. We present some examples and a composition result. As an application, we prove the existence and uniqueness of (ω, c)-pseudo periodic mild solutions to the first order abstract Cauchy problem on the real line. Also, we establish some sufficient conditions for the existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska equation with unbounded oscillating production of red cells.

MSC: 34C25; 30D45; 47D06

Keywords: Anti-periodic; Periodic; (ω, c)-pseudo periodic functions; Convolution invariance; Completeness

1 Introduction

Let $\omega > 0$ and $c \in \mathbb{C} \setminus \{0\}$. Consider the c-mean of h given by

$$\mathcal{M}_c(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c^{-\sigma/\omega} h(\sigma)\, d\sigma,$$

whenever the limit exists. For example, for $h_1(t) = c^{t/\omega}$ and $h_2(t) = c^{t/\omega}e^{it}$, we have that $\mathcal{M}_c(h_1) = 1$ and $\mathcal{M}_c(h_2) = 0$. Furthermore, $\mathcal{M}_c$ is a linear and continuous operator. Indeed, if $c^{-t/\omega}h_n(t) \to c^{-t/\omega}h(t)$ uniformly as $n \to \infty$, then $\mathcal{M}_c(h_n) \to \mathcal{M}_c(h)$ as $n \to \infty$. Also, note that when $c = 1$ we have the mean of h, $\mathcal{M}(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(\sigma)\, d\sigma$. Other properties of $\mathcal{M}_c$ appear in Sect. 2.

Let us define the c-ergodic space

$$AA_{0,c}(X) = \{h \in C(\mathbb{R}, X) : \mathcal{M}_c(\|h\|) = 0\}.$$

Note that when $c = 1$ we recover the ergodic space of Zhang (see [29, 30])

$$AA_0(X) := \{h \in C(\mathbb{R}, X) : \mathcal{M}(\|h\|) = 0\}.$$

We say that f is a (ω, c)-periodic function if there is a pair (ω, c), $c \in (\mathbb{C} \setminus \{0\})$, $w > 0$ such that $f(t + \omega) = cf(t)$ for all $t \in \mathbb{R}$ (see [22]). It represents periodic functions with $c = 1$, anti-periodic functions with $c = -1$, Bloch waves with $c = e^{ik/\omega}$, and unbounded functions for $|c| \neq 1$. Linear systems with periodic coefficients produce, by Floquet's theorem, (ω, c)-periodic solutions. This is the case of the famous Hill's and Mathieu's equations (see [17, 31])

$$\frac{d^2y}{dt^2} + [a - 2q\cos(2t)]y = 0.$$

Mathieu's equation is a linearized model of an inverted pendulum, where the pivot point oscillates periodically in the vertical direction (see [19]). According to Floquet's theorem, these equations admit a complex valued basis of solutions of the form $y(t) = e^{\mu t}p(t)$, $t \in \mathbb{R}$, where μ is a complex number and p is a complex valued function which is ω-periodic (see [5, Ch. 8, Sect. 4]). We can observe that the solution is not periodic, but

$$y(t + \omega) = cy(t), \quad c = e^{\mu\omega}, t \in \mathbb{R}. \tag{1.1}$$

In fluid dynamics, we can find many examples of waves being described by Mathieu's equation. The research of Faraday surface waves is very active (see [4, 9, 23]).

Several properties of (ω, c)-periodic functions have been obtained in [3]. Also, this class of functions appears for example when the method of Bloch wave decomposition is used in order to obtain the homogenization of self-adjoint elliptic operators in arbitrary domains with periodically oscillating coefficients (see [6, 20] and the references therein).

Now, we are ready to introduce the space of (ω, c)-pseudo periodic functions. A continuous function f is said to be a (ω, c)-pseudo periodic function if it can be written as $f = g + h$, where g is a (ω, c)-periodic function and $h \in AA_{0,c}(X)$. Note that when $c = 1$ we obtain the space of pseudo periodic functions defined in [28, Definition 2 p. 873] (see also [16]), and when $c = -1$ we obtain the space of pseudo anti-periodic functions defined in [27]. When $c = e^{ik/\omega}$, we will call this set of functions pseudo Bloch-periodic functions. Also, it should be noted that the space of (ω, c)-periodic functions, asymptotically Bloch periodic functions (see [12, 13]), and the space of (ω, c)-asymptotically periodic functions (which basically are sums of (ω, c)-periodic functions with continuous functions h such that $c^{-t/\omega}h(t)$ goes to 0 as t goes to ∞, see [2, Definition 2.5]) are contained in the space of (ω, c)-pseudo periodic functions. For other works related to pseudo periodicity, see [10, 14, 25, 26].

We give several properties of (ω, c)-pseudo periodic functions including a characterization in terms of the pseudo periodic functions, uniqueness of the decomposition, and algebraic properties. Also, we prove a convolution theorem and that the space of (ω, c)-pseudo periodic functions is a Banach space with the norm $\|\cdot\|_{p\omega c}$ defined below (see Theorem 2.18). Furthermore, we prove that the range of these functions is relatively compact with this norm. A composition result is given and a variety of examples are showed. We point out that the pseudo periodic, pseudo anti-periodic, and pseudo Bloch-periodic functions are defined as a subspace of $BC(X)$, while our results include unbounded functions on $\mathbb{R}$ in both periodic and ergodic parts, that is, the cases $|c| < 1$ and $|c| > 1$.

The previous results allow us to show the existence and uniqueness of (ω, c)-pseudo periodic mild solutions for the following class of semilinear abstract integral and differential equations in Banach spaces:

$$u(t) = \int_{-\infty}^{t} R(t,s) f\bigl(s, u(s)\bigr)\, ds, \tag{1.2}$$

where f and the family R satisfy certain hypotheses. In particular, we obtain (ω, c)-pseudo periodic mild solutions for the semilinear first order problem

$$u'(t) = Au(t) + f\bigl(t, u(t)\bigr), \quad t \in \mathbb{R},$$

where A is a closed linear and densely defined operator on a Banach space X which generates an exponentially bounded C_0-semigroup $\{T(t)\}_{t \geq 0}$. The results can be extended to delayed systems, see Sect. 4.

Furthermore, we prove the existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t) e^{-a(t) y(t-\tau)}, \quad t \geq 0. \tag{1.3}$$

Wazewska–Czyzewska and Lasota [24] proposed this model to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red cells bloods in the time t, $\delta > 0$ is the probability of death of a red blood cell, $a(t)$ is a continuous and positive function which is related to the production of red blood cells by unity of time, τ is the time required to produce a red blood cell, $h(t)$ is a continuous and positive function which describes the generation of red blood cells per unit time.

This paper is organized as follows. In Sect. 2, we formalize the (ω, c)-pseudo periodic functions and give some important properties. Also, we show that the space of (ω, c)-pseudo periodic functions is a Banach space with a suitable norm and the fact that the range of this class of functions is relatively compact with this norm. Convolution and composition theorems will be proved. Several interesting examples are given. In Sect. 3, we prove the existence and uniqueness of (ω, c)-pseudo periodic solutions to the first order abstract Cauchy problem on $\mathbb{R}$. Finally, in Sect. 4, we prove the existence of positive (ω, c)-pseudo periodic solutions to the Lasota–Wazewska model with (ω, c)-pseudo periodic coefficients. Also, we show that the solution is exponentially stable.

2 (ω, c)-Pseudo periodic functions

Throughout the paper, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a complex Banach space with norm $\|\cdot\|$, $\Omega \subset X$, and we will denote the space of continuous functions as

$$C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}\},$$

the space of ergodic functions as

$$AA_0(X) := \bigl\{h \in C(\mathbb{R}, X) : \mathcal{M}\bigl(\|h\|\bigr) = 0\bigr\},$$

and

$$AA_0(\Omega,X) := \big\{h \in C(\mathbb{R}\times\Omega,X) : \mathcal{M}\big(\|h(\cdot,x)\|\big) = 0 \\ \text{for all } x \text{ in any compact subset of } \Omega\big\}.$$

Definition 2.1 ([3]) A function $g \in C(\mathbb{R},X)$ is said to be (ω,c)-periodic if $g(t+\omega) = cg(t)$ for all $t \in \mathbb{R}$. ω is called the c-period of g. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}(\mathbb{R},X)$. When $c = 1$ (ω-periodic case), we write $P_\omega(\mathbb{R},X)$ in spite of $P_{\omega 1}(\mathbb{R},X)$. Using the principal branch of the complex logarithm (i.e., the argument in $(-\pi,\pi]$), we define $c^{t/\omega} := \exp((t/\omega)\operatorname{Log}(c))$. Also, we will use the notation $c^\wedge(t) := c^{t/\omega}$ and $|c|^\wedge(t) := |c^\wedge(t)| = |c|^{t/\omega}$.

The following proposition gives a characterization of the (ω,c)-periodic functions.

Proposition 2.2 ([3]) *Let $f \in C(\mathbb{R},X)$. Then f is (ω,c)-periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) = c^{t/\omega}, \tag{2.1}$$

where $u(t)$ is a ω-periodic complex X-valued function.

In view of (2.1), for any $f \in P_{\omega c}(\mathbb{R},X)$, we say that $c^\wedge(t)u(t)$ is the c-factorization of f.

Remark 2.3 From Proposition 2.2, we can write all $f \in P_{\omega c}(\mathbb{R},X)$ as

$$f(t) = c^\wedge(t)u(t),$$

where $u(t)$ is ω-periodic on $\mathbb{R}$. We will call $u(t)$ *the periodic part of f*. With this convention, an anti-periodic function f can be written as $f(t) = (-1)^{t/\omega}u(t)$, where u is ω-periodic. For example, $f(t) = \sin t$ can be considered as an anti-periodic function, with $\omega = \pi$. As $\operatorname{Log}(-1) = i\pi$, f has the decomposition $f(t) = c^\wedge(t)u(t)$, where

$$c^\wedge(t) = (-1)^{t/\pi} = e^{ti} = [\cos t + i\sin t],$$

and

$$u(t) = \sin t(\cos t - i\sin t),$$

which is periodic with period π.

Let $c = e^{2\pi i/k}$ for some natural number $k \geq 2$, and let f be a (ω,c)-periodic function. Then f is a periodic function with period $k\omega$ but, in general, it can be written as $f(t) = e^{2\pi ti/k\omega}u(t)$, where u is a complex periodic function with period ω. In particular, if $k = 4$, a $(\omega, e^{\pi i/2})$-periodic function f can be at the same time a Bloch wave: $f(t+\omega) = e^{\pi i/2}f(t)$, an anti-periodic function with antiperiod 2ω: $f(t+2\omega) = -f(t)$, and a 4ω-periodic function: $f(t+4\omega) = f(t)$.

Definition 2.4 A function $h \in C(\mathbb{R}, X)$ is said to be c-ergodic if $c^\wedge(-t)h(t) \in AA_0(X)$, that is,

$$\mathcal{M}_c(\|h\|) = 0.$$

The collection of those functions will be denoted by $AA_{0,c}(X)$. Analogously, a function $h \in C(\mathbb{R} \times \Omega, X)$ is said to be c-ergodic if $c^\wedge(-t)h(t,x) \in AA_0(\Omega, X)$, that is,

$$\mathcal{M}_c(\|h(\cdot, x)\|) = 0$$

for all x in any compact subset of Ω. The collection of those functions will be denoted by $AA_{0,c}(\Omega, X)$.

Note that $C_{0,c}(X) := \{g \in C(\mathbb{R}, X) : \lim_{|t|\to\infty} g(t) = 0\}$ is contained in $AA_{0,c}(X)$. However, note that L^p-integrable functions belong to $AA_{0,c}(X)$ but there exist functions L^p-integrable that do not belong to $C_{0,c}(X)$.

Definition 2.5 A function $f \in C(\mathbb{R}, X)$ is said to be (ω, c)-pseudo periodic if $f = g + h$ where $g \in P_{\omega c}(\mathbb{R}, X)$ and $h \in AA_{0,c}(X)$. The collection of those functions (with the same c-period ω for the first component) will be denoted by $PP_{\omega c}(X)$.

Remark 2.6 The preceding collection includes the pseudo periodic functions $PP_{\omega 1}(X) := \{f \in C(\mathbb{R}, X) : f = g + h, g \in P_{\omega 1}(\mathbb{R}, X), h \in AA_0(X)\}$, the pseudo anti-periodic functions $PP_{\omega(-1)}(X) := \{f \in C(\mathbb{R}, X) : f = g + h, g \in P_{\omega(-1)}(\mathbb{R}, X), h \in AA_0(X)\}$, and pseudo (ω, c)-Bloch-periodic functions $PP_{\omega e^{ik\omega}}(X) := \{f \in C(\mathbb{R}, X) : f = g + h, g \in P_{\omega e^{ik\omega}}(\mathbb{R}, X), h \in AA_0(X)\}$.

Example 2.7 Let $\phi(t) = \max_{k\in\mathbb{Z}}\{e^{-(t\pm k^2)^2}\}$, $t \in \mathbb{R}$. It follows from [15, Example 2.5] that $\phi \in AA_0(\mathbb{R}, \mathbb{R})$. Let

$$f_1(t) = \sin t + \phi(t), \quad t \in \mathbb{R}.$$

Then f_1 is pseudo periodic because $g(t) = \sin t$ is periodic with period 2π and pseudo anti-periodic because $g(t) = \sin t$ is anti-periodic (with antiperiod π). Analogously, the function

$$f_2(t) = e^{ikt} + \phi(t), \quad t \in \mathbb{R}$$

belongs to $PP_{\omega e^{ik\omega}}(\mathbb{R}, \mathbb{R})$. The same is true for any $\phi \in AA_{0,c}(\mathbb{R})$.

The following proposition gives a characterization of the (ω, c)-pseudo periodic functions.

Proposition 2.8 *Let $f \in C(\mathbb{R}, X)$. Then f is (ω, c)-pseudo periodic if and only if*

$$f(t) = c^\wedge(t)u(t), \quad c^\wedge(t) = c^{t/\omega}, u \in PP_\omega(X). \tag{2.2}$$

Proof It is clear that if f satisfies (2.2) then f is a (ω, c)-pseudo periodic function. In order to show the inverse statement, let $f \in PP_{\omega c}(X)$. Then there exist $g \in P_{\omega c}(\mathbb{R}, X)$ and $h \in AA_{0,c}(X)$ such that $f = g + h$. If we write $u(t) := c^\wedge(-t)f(t) = c^{-t/\omega}f(t)$, then

$$u(t) = c^\wedge(-t)g(t) + c^\wedge(-t)h(t) =: F_1(t) + F_2(t).$$

It follows from [3, Proposition 2.5] that $F_1 \in P_\omega(\mathbb{R}, X)$ and by definition of $AA_{0,c}(X)$ we have that $F_2 \in AA_0(X)$. Hence $u \in PP_\omega(X)$. □

Remark 2.9 The decomposition in Definition 2.5 is unique, that is, there exist a unique $g \in P_{\omega c}(\mathbb{R}, X)$ and a unique $h \in AA_{0,c}(X)$ such that $f = g + h$. Indeed, suppose that

$$f(t) = g_1(t) + h_1(t) = g_2(t) + h_2(t), \quad g_1, g_2 \in P_{\omega c}(\mathbb{R}, X), h_1, h_2 \in AA_{0,c}(X), t \in \mathbb{R}.$$

Then

$$u(t) := c^\wedge(-t)f(t) = c^\wedge(-t)g_1(t) + c^\wedge(-t)h_1(t) = c^\wedge(-t)g_2(t) + c^\wedge(-t)h_2(t)$$

belongs to $PP_\omega(X)$ by Proposition 2.2. By the unique representation of the functions in this space, we have that $c^\wedge(-t)g_1(t) = c^\wedge(-t)g_2(t)$ and $c^\wedge(-t)h_1(t) = c^\wedge(-t)h_2(t)$, and consequently $g_1(t) = g_2(t)$ and $h_1(t) = h_2(t)$ for all $t \in \mathbb{R}$.

Remark 2.10 Note that if $|c| \geq 1$ then $AA_0(X) \subset AA_{0,c}(X)$, and consequently $P_{\omega c}(\mathbb{R}, X) + AA_0(X) \subset PP_{\omega c}(X)$.

As a consequence of Proposition 2.8, we have the following basic properties.

Lemma 2.11 *Let* $\alpha \in \mathbb{C}$. *Then*

(a) $(f + g) \in PP_{\omega c}(X)$ *and* $\alpha h \in PP_{\omega c}(X)$ *whenever* $f, g, h \in PP_{\omega c}(X)$.
(b) *If* $\tau \in \mathbb{R}$, *then* $f_\tau(t) = f(t + \tau) \in PP_{\omega c}(X)$ *whenever* $f \in PP_{\omega c}(X)$.

Proof The proof of (a) is a consequence of the definition. (b) follows from the invariant property of the space $P_{\omega c}(\mathbb{R}, X)$ and Lemma 2.16. □

Example 2.12 Let $\varphi(t) := t|\sin t|^{t^N}$ for $N > 6$. From [1, Example p. 1143] we have that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(s)|\, ds = 0$$

and $\varphi(t) \to \infty$ at the points $t = \frac{1}{2} + k$ as $|k| \to \infty$. Let

$$f(t) = 2^t \sin t + b^t \varphi(t), \quad t \in \mathbb{R}, |b| \leq 2.$$

Then $f \in PP_{\pi -2^\pi}(\mathbb{R})$. Indeed, note that $g(t) := 2^t \sin t$ is $(\pi, -2^\pi)$-periodic. Let us prove that $h(t) := b^t \varphi(t)$ belongs to $AA_{0,-2^\pi}(\mathbb{R})$.

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|(-2^\pi)^\wedge(-s)h(s)\right| ds &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\left(\frac{b}{2}\right)^s \varphi(s)\right| ds \\ &\leq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(s)|\, ds = 0. \end{aligned}$$

Hence f is a (ω, c)-pseudo periodic function which is not a (ω, c)-asymptotically periodic function.

Example 2.13 Let $X = \mathbb{C}$, $|b| \leq 2$. Consider

$$f(t) = 2^t \sin t + b^t h(t), \quad t \in \mathbb{R},$$

where h satisfies one of the following conditions:

(a) is integrable, or

(b) L^p-integrable for $1 < p < \infty$, or

(c) asymptotic at t in $-\infty$ and ∞.

Then f is a $(\pi, -2^\pi)$-pseudo periodic function. Since $c^\wedge(t) = \exp(\frac{t}{\pi} \operatorname{Log}(-2^\pi)) = 2^t e^{it}$, then by Proposition 2.2 we have that

$$g(t) = 2^t e^{it} u_1(t),$$

where

$$u_1(t) = \sin t(\cos t - i \sin t)$$

is periodic with period $\omega = \pi$. Analogously,

$$b^t h(t) = 2^t e^{it} u_2(t),$$

where

$$u_2(t) = \left(\frac{b}{2}\right)^t h(t)(\cos t - i \sin t)$$

belongs to $AA_0(X)$. Hence f has the decomposition

$$f(t) = 2^t \sin t + b^t h(t) = 2^t(\cos t + i \sin t)\left[\sin t(\cos t - i \sin t) + \left(\frac{b}{2}\right)^t h(t)(\cos t - i \sin t)\right].$$

Example 2.14 Let $u : \mathbb{R} \to X$ be a X-valued periodic function with period ω and $v : \mathbb{R} \to X$ in $AA_0(X)$. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a function with the semigroup property, that is, $\phi(t + s) = \phi(s)\phi(t)$ for all $t, s \in \mathbb{R}$ and such that $\phi(\omega) \neq 0$. Then

$$z(t) = \phi(t)u(t) + \phi(t)v(t), \quad t \in \mathbb{R},$$

is a $(\omega, \phi(\omega))$-asymptotically periodic function if $\varphi(t) := [\phi(\omega)]^\wedge(-t)\phi(t)$ is bounded. As a particular case, we take $\phi(t) = e^{ikt}$ and obtain the pseudo periodic Bloch functions.

Remark 2.15 In general, if u is a (ω, c)-pseudo periodic function and ϕ is a function with the semigroup property such that $\phi(\omega) \neq 0$, then $z(t) := \phi(t)u(t)$ is a $(\omega, c\phi(\omega))$-pseudo periodic if $\varphi(t) := [\phi(\omega)]^\wedge(-t)\phi(t)$ is bounded. Moreover, let $(u_k)_{k\in\mathbb{N}}$ be a sequence of (ω, c)-pseudo periodic functions and $(\phi_k)_{k\in\mathbb{N}}$ be a sequence of functions with the semigroup property and such that $\phi_k(\omega) = p \neq 0$ for all $k \in \mathbb{N}$. Assume that

$$\sum_{k=1}^{\infty} \phi_k(t)u_k(t)$$

is a uniformly convergent series on $\mathbb{R}$. Then

$$f(t) = \sum_{k=1}^{\infty} \phi_k(t)u_k(t)$$

is a (ω, cp)-pseudo periodic function if $\varphi_k(t) := p^\wedge(-t)\phi_k(t)$ is bounded for $k \in \mathbb{N}$.

Lemma 2.16 *$AA_{0,c}(X)$ is translation invariant, and for every $h \in AA_0(X)$, we have that $\mathcal{M}_c(g + h) = \mathcal{M}_c(g)$ for all $g \in C(\mathbb{R}, X)$.*

Proof Let $h \in AA_{0,c}(X)$ and $\tau \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} &\frac{1}{2T}\int_{-T}^{T} \left\| c^\wedge(-\sigma)h(\sigma - \tau)\right\| d\sigma \\ &\quad = \frac{1}{2T}\int_{-T-\tau}^{T+\tau} \left\| c^\wedge(-u-\tau)h(u)\right\| du \\ &\quad \le \frac{c^\wedge(-\tau)}{2T}\int_{-T-|\tau|}^{T+|\tau|} \left\| c^\wedge(-u)h(u)\right\| du \\ &\quad = \frac{c^\wedge(-\tau)(T+|\tau|)}{T}\left[\frac{1}{2(T+|\tau|)}\int_{-T-|\tau|}^{T+|\tau|} \left\| c^\wedge(-u)h(u)\right\| du\right] \to 0 \end{aligned}$$

as $T \to \infty$. The last assertion follows from the linearity of $\mathcal{M}$. □

We recall (see [3]) that the norm in the space $P_{\omega c}(\mathbb{R}, X)$ is given by

$$\|f\|_{\omega c} := \sup_{t\in[0,\omega]} \left\| |c|^\wedge(-t)f(t)\right\|.$$

Proposition 2.17 *Let $f \in P_{\omega c}(\mathbb{R}, X)$. Then the range $\{c^\wedge(-t)f(t) : t \in \mathbb{R}\}$ is relatively compact in X, that is, given $\epsilon > 0$, for all $t \in \mathbb{R}$, there exist $x_1, \ldots, x_k$ in X such that $\|c^\wedge(-t)f(t) - x_i\| < \epsilon$ for some $i = 1, \ldots, k$.*

The following result guarantees that $PP_{\omega c}(X)$ is a Banach space with the norm defined below.

Theorem 2.18 *$PP_{\omega c}(X)$ is a Banach space with the norm*

$$\|f\|_{p\omega c} := \sup_{t\in\mathbb{R}} \left\| |c|^\wedge(-t)f(t)\right\|.$$

Proof Let (f_n) be a Cauchy sequence in $PP_{\omega c}(X)$. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \ge N$, we have

$$\|f_n - f_m\|_{p\omega c} < \epsilon.$$

Since $f_m, f_n \in PP_{\omega c}(X)$, Proposition 2.8 implies that there exist $u_m, u_n \in PP_\omega(X)$ such that $f_m(t) = c^\wedge(t)u_m(t)$ and $f_n(t) = c^\wedge(t)u_n(t)$. Now, note that for $m, n \geq N$

$$\begin{aligned}
\|u_m - u_n\|_{p\omega} &= \sup_{t\in\mathbb{R}} \big\| u_m(t) - u_n(t) \big\| \\
&= \sup_{t\in\mathbb{R}} \big\| c^\wedge(-t)f_m(t) - c^\wedge(-t)f_n(t) \big\| \\
&= \sup_{t\in\mathbb{R}} \big\| |c|^\wedge(-t)\big[f_m(t) - f_n(t)\big] \big\| \\
&= \|f_n - f_m\|_{p\omega c} < \epsilon.
\end{aligned}$$

It follows that (u_n) is a Cauchy sequence in $PP_\omega(X)$. Since $PP_\omega(X)$ is complete, then there exists $u \in PP_\omega(X)$ such that $\|u_n - u\|_{p\omega} \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t)u(t)$. We claim that $\|f_n - f\|_{p\omega c} \to 0$ as $n \to \infty$. Indeed,

$$\begin{aligned}
\|f_n - f\|_{p\omega c} &= \sup_{t\in\mathbb{R}} \big\| |c|^\wedge(-t)\big[f_n(t) - f(t)\big] \big\| \\
&= \sup_{t\in\mathbb{R}} \big\| |c|^\wedge(-t)c^\wedge(t)u_n(t) - |c|^\wedge(-t)c^\wedge(t)u(t) \big\| \\
&= \sup_{t\in\mathbb{R}} \big\| u_m(t) - u(t) \big\| \to 0 \quad (n \to \infty).
\end{aligned}$$

Hence $PP_{\omega c}(X)$ is a Banach space with the norm $\|\cdot\|_{p\omega c}$. □

We recall the following convolution result.

Theorem 2.19 ([3, Theorem 2.7]) *Let $f \in P_{\omega c}(\mathbb{R}, X)$ with $f(t) = c^\wedge(t)p(t)$, $p \in P_\omega(\mathbb{R}, X)$. If $k^\sim(t) := c^\wedge(-t)k(t) \in L^1(\mathbb{R})$, then $(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s) \in P_{\omega c}(\mathbb{R}, X)$.*

Lemma 2.20 *Assume that $k^\sim(t) := c^\wedge(-t)k(t) \in L^1(\mathbb{R})$. Then $h \in AA_{0,c}(X)$ implies that $k * h \in AA_{0,c}(X)$.*

Proof It is clear that the convolution $k * h$ is a continuous function. Then

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \big\| c^\wedge(-t)(k * h)(t) \big\| \, dt &\leq \frac{1}{2T}\int_{-T}^{T} |c|^\wedge(-t) \int_{-\infty}^{\infty} \big|k(t-s)\big| \big\|h(s)\big\| \, ds \, dt \\
&= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} \big|k^\sim(t-s)\big| \big\|c^\wedge(-s)h(s)\big\| \, ds \, dt \\
&= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} \big|k^\sim(s)\big| \big\|c^\wedge\big(-(t-s)\big)h(t-s)\big\| \, ds \, dt \\
&= \int_{-\infty}^{\infty} \big|k^\sim(s)\big| \left(\frac{1}{2T}\int_{-T}^{T} \big\|c^\wedge\big(-(t-s)\big)h(t-s)\big\| \, dt \right) ds \\
&= \int_{-\infty}^{\infty} \big|k^\sim(s)\big| \Phi_T(s) \, ds,
\end{aligned}$$

where $\Phi_T(s) := \frac{1}{2T}\int_{-T}^{T} \|c^\wedge(-(t-s))h(t-s)\| dt$. Since $AA_{0,c}(X)$ is translation invariant by Lemma 2.16, then $\Phi_T(s) \to 0$ as $T \to \infty$. Next, since Φ_T is bounded ($\|\Phi_T\| \leq \|h\|_{p\omega c}$) and

$k^{\sim} \in L^1(\mathbb{R})$, using the dominated convergence theorem, it follows that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \left|k^{\sim}(s)\right| \Phi_T(s)\, ds = 0.$$

Hence $k * h \in AA_{0,c}(X)$. □

We are ready to present the convolution theorem for (ω, c)-pseudo periodic functions.

Theorem 2.21 *Let $f \in PP_{\omega c}(X)$ with $f(t) = c^{\wedge}(t)p(t)$, $p \in PP_{\omega}(X)$. If for some $k(t)$ we have that $k^{\sim}(t) := c^{\wedge}(-t)k(t) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\, ds = c^{\wedge}(t)\big(k^{\sim} * p\big)(t).$$

In particular, $(k * f)(t) \in PP_{\omega c}(X)$.

Proof Since $p \in PP_{\omega}(X)$, then there exist $p_1 \in P_{\omega}(\mathbb{R}, X)$ and $p_2 \in AA_0(X)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$ where $f_1(t) = c^{\wedge}(t)p_1(t) \in P_{\omega c}(\mathbb{R}, X)$ and $f_2(t) = c^{\wedge}(t)p_2(t) \in AA_{0,c}(X)$. We have

$$\begin{aligned}
(k * f)(t) &= \int_{-\infty}^{\infty} k(t-s)f(s)\, ds \\
&= \int_{-\infty}^{\infty} k(t-s)f_1(s)\, ds + \int_{-\infty}^{\infty} k(t-s)f_2(s)\, ds \\
&= (k * f_1)(t) + (k * f_2)(t) =: I_1(t) + I_2(t).
\end{aligned}$$

From Theorem 2.19 we have that $I_1 \in P_{\omega c}(\mathbb{R}, X)$. Next, by Lemma 2.20 we have that $I_2 \in AA_{0,c}(X)$. Now, from the definition of f we have that $(k * f)(t) = c^{\wedge}(t)(k^{\sim} * p)(t)$. Hence $(k * f) \in PP_{\omega c}(X)$. □

Example 2.22 Consider the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & t > 0, x \in \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

Let $u(x,t)$ be a regular solution with $u(x,0) = f(x)$. Then it is known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\, ds, \quad t > 0, x \in \mathbb{R}.$$

Fix $t_0 > 0$ and assume that $f(x)$ is (ω, c)-pseudo periodic. Then, by Theorem 2.21, we have that $u(x, t_0)$ is (ω, c)-pseudo periodic with respect to x.

The next lemma is analogous to [15, Lemma 2.1].

Lemma 2.23 *Let $h \in C(\mathbb{R}, X)$ such that $\sup_{t\in\mathbb{R}} \|c^{\wedge}(-t)h(t)\| < \infty$. Then $h \in AA_{0,c}(X)$ if and only if*

$$(\forall \epsilon > 0) \quad \lim_{T\to\infty} \frac{1}{2T} \operatorname{meas}\big(M_{T,\epsilon}(h)\big) = 0, \tag{2.3}$$

where

$$M_{T,\epsilon}(h) := \{t \in [-T,T] : \|c^\wedge(-t)h(t)\| \geq \epsilon\}.$$

Proof Assume that $h \in AA_{0,c}(X)$ and suppose that there exists $\epsilon_0 > 0$ such that $\frac{1}{2T} \times \text{meas}(M_{T,\epsilon}(h))$ does not converge to zero when $T \to \infty$. That is, there exists $\delta > 0$ such that, for $n \in \mathbb{N}$,

$$\frac{1}{2T_n}\,\text{meas}\big(M_{T_n,\epsilon_0}(h)\big) \geq \delta \quad \text{for}\, T_n > n.$$

Then

$$\begin{aligned}
&\frac{1}{2T_n}\int_{-T_n}^{T_n} \|c^\wedge(-t)h(t)\|\,dt \\
&\quad = \frac{1}{2T_n}\int_{M_{T_n,\epsilon_0}} \|c^\wedge(-t)h(t)\|\,dt + \frac{1}{2T_n}\int_{[-T_n,T_n]\setminus M_{T_n,\epsilon_0}} \|c^\wedge(-t)h(t)\|\,dt \\
&\quad \geq \frac{1}{2T_n}\int_{M_{T_n,\epsilon_0}} \|c^\wedge(-t)h(t)\|\,dt \\
&\quad \geq \frac{1}{2T_n}\,\text{meas}\big(M_{T_n,\epsilon_0}(h)\big)\cdot\epsilon_0 \geq \delta\epsilon_0,
\end{aligned}$$

which is a contradiction.

Now, assume (2.3). We prove that $h \in AA_{0,c}(X)$. By (2.3) we have that there exists $M > 0$ such that $\|c^\wedge(-t)h(t)\| \leq M$, and for all $\epsilon > 0$ there exists $T_0 > 0$ such that $T > T_0$ implies that

$$\frac{1}{2T}\,\text{meas}\big(M_{T,\epsilon}(h)\big) < \frac{\epsilon}{M+1}.$$

Then

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \|c^\wedge(-t)h(t)\|\,dt &= \frac{1}{2T}\int_{M_{T,\epsilon}} \|c^\wedge(-t)h(t)\|\,dt + \frac{1}{2T}\int_{[-T,T]\setminus M_{T,\epsilon}} \|c^\wedge(-t)h(t)\|\,dt \\
&\leq \frac{1}{2T}M\,\text{meas}\big(M_{T,\epsilon}(h)\big) + \frac{1}{2T}\epsilon\big(2T - \text{meas}\big(M_{T,\epsilon}(h)\big)\big) \\
&< \frac{(M-1)\epsilon}{M+1} + \epsilon < 2\epsilon.
\end{aligned}$$

Hence $h \in AA_{0,c}(X)$. □

Next, we have the following composition result. The idea of the proof follows from [15, Theorem 2.4].

Theorem 2.24 *Let $f(t,x) = g(t,x) + h(t,x)$ where $g(t+\omega, cx) = cg(t,x)$ and $h \in AA_{0,c}(X,X)$. Assume*

(a) $\sup_{t\in\mathbb{R}} \|c^\wedge(-t)f(t,x)\| < \infty$ *for all $x \in X$.*
(b) *$f_t(z) := c^\wedge(-t)f(t,c^\wedge(t)z)$ is uniformly continuous for z in any bounded subset $K \subset X$ uniformly in $t \in \mathbb{R}$; that is, given $\epsilon > 0$ and $K \subset X$ bounded, there exists $\delta > 0$ such that $x,y \in K$ and $\|x-y\| < \delta$ imply that $\|f_t(x) - f_t(y)\| \leq \epsilon$ for all $t \in \mathbb{R}$.*

(c) $h_t(x) := c^\wedge(-t)h(t, c^\wedge(t)x)$ *is uniformly continuous for* x *in any bounded set of* X *uniformly in* $t \in \mathbb{R}$ *and*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|h_t(x)\|\, dt = 0$$

for x *in any bounded subset of* X.

If $\varphi \in PP_{\omega c}(X)$, *then* $f(\cdot, \varphi(\cdot)) \in PP_{\omega c}(X)$.

Proof Let $\varphi(t) = \alpha(t) + \beta(t)$ with $\alpha \in P_{\omega c}(\mathbb{R}, X)$ and $\beta \in AA_{0,c}(X)$. Then we have

$$f\big(t, \varphi(t)\big) = \big[f\big(t, \varphi(t)\big) - f\big(t, \alpha(t)\big)\big] + g\big(t, \alpha(t)\big) + h\big(t, \alpha(t)\big) =: F(t) + G(t) + H(t).$$

By [3, Theorem 2.11] we have that $G(t) = g(t, \alpha(t))$ belongs to $P_{\omega c}(\mathbb{R}, X)$. On the other hand, note that $\phi(t) := c^\wedge(-t)\varphi(t)$ and $\phi_1(t) := c^\wedge(-t)\alpha(t)$ are bounded by definition and Proposition 2.17 respectively. From here we can choose $K \subset X$ bounded such that $\phi([-T,T]), \phi_1([-T,T]) \subset K$. Under assumption (b), $c^\wedge(-t)f(t, c^\wedge(t)\cdot)$ is uniformly continuous on the bounded set K uniform for $t \in [-T,T]$, so given $\epsilon > 0$, there exists $\delta := \delta_{\epsilon,K}$ such that $\|\phi(t) - \phi_1(t)\| = \|c^\wedge(-t)\varphi(t) - c^\wedge(-t)\alpha(t)\| = \|c^\wedge(-t)\beta(t)\| \le \delta$ implies that

$$\|c^\wedge(-t)f\big(t, \varphi(t)\big) - c^\wedge(-t)f\big(t, \alpha(t)\big)]\| = \big\|c^\wedge(-t)F(t)\big\| \le \epsilon$$

for all $t \in [-T,T]$. Then we have that

$$\lim_{T\to\infty} \frac{1}{2T} \operatorname{meas}\big(M_{T,\epsilon}(F)\big) \le \lim_{T\to\infty} \frac{1}{2T} \operatorname{meas}\big(M_{T,\delta}(\beta)\big).$$

Since $\beta \in AA_{0,c}(X)$, Lemma 2.3 yields for the above δ that

$$\lim_{T\to\infty} \frac{1}{2T} \operatorname{meas}\big(M_{T,\delta}(\beta)\big) = 0.$$

From here we can conclude that $F \in AA_{0,c}(X)$.

Finally, we prove that $H \in AA_{0,c}(X)$. Let $\phi(t) := c^\wedge(-t)\alpha(t)$ and $I = \phi([-T,T])$. Then ϕ is uniformly continuous in $[-T,T]$, and therefore I is compact in X. Let $\epsilon > 0$. Then, for every $\delta = \delta(\epsilon) > 0$, there exist finite open balls O_k $(k = 1, 2, \ldots, m)$ with centers in $x_k \in I$ respectively such that $I \subset \bigcup_{k=1}^m O_k$. Then, by the uniform continuity of $c^\wedge(-t)h(t, c^\wedge(t)\cdot)$, we have that

$$\big\|c^\wedge(-t)h\big(t, \alpha(t)\big) - c^\wedge(-t)h\big(t, c^\wedge(t)x_k\big)\big\| < \frac{\epsilon}{2}, \quad t \in [-T,T]. \tag{2.4}$$

The set $B_k := \{t \in [-T,T] : \phi(t) \in O_k\}$ is open in $[-T,T]$ and $[-T,T] = \bigcup_{k=1}^m B_k$. Let

$$E_1 = B_1, \qquad E_k := B_k \setminus \bigcup_{j=1}^{k-1} B_j \quad (k = 2, \ldots, m).$$

Then $E_i \cap E_j = \emptyset$ when $i \neq j$, $1 \leq i,j \leq m$. Note that

$$\begin{aligned}
&\bigl\{t \in [-T,T] : \bigl\| c^\wedge(-t)h\bigl(t,\alpha(t)\bigr)\bigr\| \geq \epsilon\bigr\} \\
&\quad\subset \bigcup_{k=1}^{m} \bigl\{t \in [-T,T] : \bigl\| c^\wedge(-t)h\bigl(t,\alpha(t)\bigr) - c^\wedge(-t)h\bigl(t,c^\wedge(t)x_k\bigr)\bigr\| \\
&\qquad+ \bigl\| c^\wedge(-t)h\bigl(t,c^\wedge(t)x_k\bigr)\bigr\| \geq \epsilon\bigr\} \\
&\quad\subset \bigcup_{k=1}^{m} \biggl(\biggl\{t \in [-T,T] : \bigl\| c^\wedge(-t)\bigl[h\bigl(t,\alpha(t)\bigr) - h\bigl(t,c^\wedge(t)x_k\bigr)\bigr]\bigr\| \geq \frac{\epsilon}{2}\biggr\} \\
&\qquad\cup \biggl\{t \in [-T,T] : \bigl\| c^\wedge(-t)h\bigl(t,c^\wedge(t)x_k\bigr)\bigr\| \geq \frac{\epsilon}{2}\biggr\}\biggr).
\end{aligned}$$

It follows from (2.4) that $\{t \in [-T,T] : \|c^\wedge(-t)[h(t,\alpha(t)) - h(t,c^\wedge(t)x_k)]\| \geq \frac{\epsilon}{2}\}$ are empty for $k = 1,\dots,m$. Therefore

$$\frac{1}{2T}\operatorname{meas}\bigl(M_{T,\epsilon}\bigl(h\bigl(t,\alpha(t)\bigr)\bigr)\bigr) \leq \sum_{k=1}^{m} \frac{1}{2T}\operatorname{meas}\bigl(M_{T,\frac{\epsilon}{2}}\bigl(h\bigl(t,c^\wedge(t)x_k\bigr)\bigr)\bigr).$$

Since $h(t,c^\wedge(t)x_k) \in AA_{0,c}(X,X)$ by (*c*), we have that

$$\lim_{T\to\infty} \frac{1}{2T}\operatorname{meas}\bigl(M_{T,\epsilon/2}\bigl(h\bigl(t,c^\wedge(t)x_k\bigr)\bigr)\bigr) = 0 \quad \text{for all } k = 1,\dots,m;$$

and therefore

$$\lim_{T\to\infty} \frac{1}{2T}\operatorname{meas}\bigl(M_{T,\epsilon}\bigl(h\bigl(t,\alpha(t)\bigr)\bigr)\bigr) = 0,$$

that is, $H \in AA_{0,c}(X)$. To summarize, we have proved that $f(\cdot,\varphi(\cdot)) \in PP_{\omega c}(X)$. □

Next, we present another composition theorem.

Theorem 2.25 *Let $f(t,x) = g(t,x) + h(t,x)$, where $g(t+\omega,cx) = cg(t,x)$ and $h \in AA_{0,c}(X,X)$. Assume the following*:

(a) *$h_t(x) := c^\wedge(-t)h(t,c^\wedge(t)x)$ is uniformly continuous for x in any bounded set of X uniformly in $t \in \mathbb{R}$ and*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \bigl\| c^\wedge(-t)h\bigl(t,c^\wedge(t)x\bigr)\bigr\|\,dt = 0$$

for x in any bounded subset of X.

(b) *There exists a nonnegative bounded function $L_f(t)$ such that*

$$\bigl\| f(t,x) - f(t,y)\bigr\| \leq L_f(t)\|x-y\|, \quad t \in \mathbb{R}, x,y \in X. \tag{2.5}$$

If $\varphi \in PP_{\omega c}(X)$, then $f(\cdot,\varphi(\cdot)) \in PP_{\omega c}(X)$.

Proof Let $\varphi(t) = \alpha(t) + \beta(t)$ with $\alpha \in P_{\omega c}(\mathbb{R}, X)$ and $\beta \in AA_{0,c}(X)$. Then we have

$$f\big(t, \varphi(t)\big) = \big[f\big(t, \varphi(t)\big) - f\big(t, \alpha(t)\big)\big] + g\big(t, \alpha(t)\big) + h\big(t, \alpha(t)\big) =: F(t) + G(t) + H(t).$$

Note that

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)F(t)\big\|\,dt &= \frac{1}{2T}\int_{-T}^{T}|c|^{\wedge}(-t)\big\|f\big(t, \varphi(t)\big) - f\big(t, \alpha(t)\big)\big\|\,dt \\
&\leq \frac{1}{2T}\int_{-T}^{T}|c|^{\wedge}(-t)L_f(t)\big\|\varphi(t) - \alpha(t)\big\|\,dt \\
&= L_f\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)\beta(t)\big\| \to 0, \quad T \to \infty,
\end{aligned}$$

where we have used that $L_f(t) \leq L_f$ and the fact that $\beta \in AA_{0,c}(X)$. It follows that $F \in AA_{0,c}(X)$. On the other hand, by [3, Theorem 2.11] we have that $G(t) = g(t, \alpha(t))$ belongs to $P_{\omega c}(\mathbb{R}, X)$. Finally, we prove that $H \in AA_{0,c}(X)$. From Proposition 2.17 we have that $K := \{c^{\wedge}(-t)\alpha(t) : t \in \mathbb{R}\}$ is relatively compact in X. Let $\epsilon > 0$. Then, for every $\delta > 0$, there exist $x_1, \ldots, x_k \in I$ such that

$$\big\{c^{\wedge}(-t)\alpha(t) : t \in \mathbb{R}\big\} \subset \bigcup_{j=1}^{k} B(x_j, \delta). \tag{2.6}$$

Consequently, given $t \in \mathbb{R}$ we can choose $j \in \{1, \ldots, k\}$ such that

$$\big\|c^{\wedge}(-t)\alpha(t) - x_j\big\| < \delta. \tag{2.7}$$

Since $h_t(\cdot) = c^{\wedge}(-t)h(t, c^{\wedge}(t)\cdot)$ is uniformly continuous on K uniformly for $t \in \mathbb{R}$, then taking $\delta = \delta(\frac{\epsilon}{2})$ we obtain that

$$\big\|c^{\wedge}(-t)\big[h\big(t, c^{\wedge}(t)c^{\wedge}(-t)\alpha(t)\big) - h\big(\cdot, c^{\wedge}(t)x_j\big)\big]\big\| < \frac{\epsilon}{2}, \tag{2.8}$$

uniformly $t \in \mathbb{R}$. From here, we can conclude that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)\big[h\big(t, \alpha(t)\big) - h\big(t, c^{\wedge}(t)x_j\big)\big]\big\|\,dt < \frac{\epsilon}{2}.$$

On the other hand, since

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t, c^{\wedge}(t)\cdot\big)\big\|\,dt = 0$$

on the bounded subsets of X, then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t, c^{\wedge}(t)x_j\big)\big\|\,dt = 0.$$

Thus there exists $N \in \mathbb{N}$ such that for all $t \geq N$ we have that

$$\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t, c^{\wedge}(t)x_j\big)\big\|\,dt < \frac{\epsilon}{2}. \tag{2.9}$$

Next, for all $t \geq N$ and some $j = 1, 2, \ldots, k$, we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t,\alpha(t)\big)\big\|\,dt &\leq \frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)\big[h\big(t,\alpha(t)\big)-h\big(t,c^{\wedge}(t)x_j\big)\big]\big\| \\ &\quad + \frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t,c^{\wedge}(t)x_j\big)\big\|\,dt \\ &< \epsilon.\end{aligned}$$

Hence

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)H(t)\big\|\,dt = 0.$$

Consequently, $f(\cdot,\varphi(\cdot)) \in PP_{\omega c}(X)$. □

3 Applications to abstract integral and differential equations in Banach spaces

Consider the integral equation (see [21])

$$u(t) = \int_{-\infty}^{t} R(t,s)f\big(s,u(s)\big)\,ds, \tag{3.1}$$

where f and R satisfy the following hypotheses.

(H1) $f(t,x) = g(t,x) + h(t,x)$, where $g(t+\omega, cx) = cg(t,x)$ and $h \in AA_{0,c}(X,X)$ and satisfies

$$\big\|f(t,x) - f(t,y)\big\| \leq L_f\|x-y\|, \quad t \in \mathbb{R}, x, y \in X,$$

where $L_f > 0$.

(H2) $h_t(x) := c^{\wedge}(-t)h(t,c^{\wedge}(t)x)$ is uniformly continuous for x in any bounded set of X uniformly in $t \in \mathbb{R}$ and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big\|c^{\wedge}(-t)h\big(t,c^{\wedge}(t)x\big)\big\|\,dt = 0$$

for x in any bounded subset of X.

(H3) The kernel R satisfies the inequality

$$\big\|R(t,s)\big\| \leq Mk(t-s), \quad t \geq s, M > 0,$$

where $k^{\sim}(t) = c^{\wedge}(-t)k(t) \in L^1([0,\infty))$.

(H4) $R(t,s)$ is *bi*-periodic in the sense of

$$R(t+\omega, s+\omega) = R(t,s), \quad t \geq s. \tag{3.2}$$

Note that, for an arbitrary periodic function a, the kernel defined by the following relation

$$R(t,s) := \exp\biggl(\int_{s}^{t} a(r)\,dr\biggr)$$

satisfies hypothesis (H4).

Theorem 3.1 *Assume that* $(H1)$–$(H4)$ *hold. Then, if* $L_f M\|k^{\sim}\|_1 < 1$, *the integral equation* (3.1) *has a unique* (ω,c)*-pseudo periodic solution.*

Proof We define $\mathcal{G} : PP_{\omega c}(X) \to PP_{\omega c}(X)$ by

$$(\mathcal{G}u)(t) = \int_{-\infty}^{t} R(t,s)f\bigl(s,u(s)\bigr)\,ds$$

for $u \in PP_{\omega c}(X)$ and $t \in \mathbb{R}$.

First, we prove that operator $\mathcal{G}$ is well defined. Indeed, let $\varphi(\cdot) = f(\cdot,u(\cdot))$. By Theorem 2.25, we have that $\varphi \in PP_{\omega c}(X)$. Then

$$\begin{aligned}
\|\mathcal{G}u\|_{p\omega c} &\le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \bigl\|c^\wedge(-t)R(t,s)\varphi(s)\bigr\|\,ds \\
&\le M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \bigl|c^\wedge\bigl(-(t-s)\bigr)k(t-s)\bigr|\bigl\|c^\wedge(-s)\varphi(s)\bigr\|\,ds \\
&\le M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \bigl|\tilde{k}(t-s)\bigr|\bigl\|c^\wedge(-s)\varphi(s)\bigr\|\,ds \\
&\le M\|\varphi\|_{p\omega c} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \bigl|\tilde{k}(t-s)\bigr|\,ds < \|\varphi\|_{p\omega c}\|\tilde{k}\|_1 < \infty.
\end{aligned}$$

Now, since $\varphi \in PP_{\omega c}(X)$, there exist functions $\varphi_1 \in P_{\omega c}(\mathbb{R},X)$ and $\varphi_2 \in AA_{0,c}(X)$ such that $\varphi = \varphi_1 + \varphi_2$. Then we can split $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ where

$$(\mathcal{G}_1 u)(t) := \int_{-\infty}^{t} R(t,s)\varphi_1(s)\,ds, \qquad (\mathcal{G}_2 u)(t) := \int_{-\infty}^{t} R(t,s)\varphi_2(s)\,ds.$$

First, we prove that $\mathcal{G}_1 \in P_{\omega c}(\mathbb{R},X)$. By (H4) we have that

$$\begin{aligned}
(\mathcal{G}_1 u)(t+\omega) &:= \int_{-\infty}^{t+\omega} R(t+\omega,s)\varphi_1(s)\,ds \\
&= \int_{-\infty}^{t} R(t+\omega,s+\omega)\varphi_1(s+\omega)\,ds \\
&= c\int_{-\infty}^{t} R(t,s)\varphi_1(s)\,ds = c(\mathcal{G}_1 u)(t).
\end{aligned}$$

It follows that $\mathcal{G}_1 \in P_{\omega c}(\mathbb{R},X)$.

Next, we prove that $\mathcal{G}_2 \in AA_{0,c}(X)$, that is,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bigl\|c^\wedge(-t)(\mathcal{G}_2 u)(t)\bigr\|\,dt = 0.$$

By (H3) we have that

$$\begin{aligned}
\bigl\|c^\wedge(-t)(\mathcal{G}_2 u)(t)\bigr\| &\le \int_{-\infty}^{t} |c|^\wedge\bigl(-(t-s)\bigr)\bigl\|R(t,s)\bigr\|\bigl\|c^\wedge(-s)\varphi_2(s)\bigr\|\,ds \\
&\le M\int_{-\infty}^{t} |c|^\wedge\bigl(-(t-s)\bigr)k(t-s)\bigl\|c^\wedge(-s)\varphi_2(s)\bigr\|\,ds
\end{aligned}$$

$$= M \int_{-\infty}^{t} k^{\sim}(t-s) \| c^{\wedge}(-s)\varphi_2(s) \| \, ds.$$

Since $\varphi_2 \in AA_{0,c}(X)$, the conclusion follows from convolution Theorem 2.21.

Therefore $\mathcal{G}(PP_{\omega c}(X)) \subset PP_{\omega c}(X)$. Now, if $u, v \in PP_{\omega c}(X)$, we have

$$\begin{aligned}
\| \mathcal{G}(u) - \mathcal{G}(v) \|_{p\omega c} &= \sup_{t\in\mathbb{R}} \left\| |c|^{\wedge}(-t) \int_{-\infty}^{t} R(t,s) \big[f\big(s,u(s)\big) - f\big(s,v(s)\big) \big] \, ds \right\| \\
&\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \big\| c^{\wedge}\big(-(t-s)\big) R(t,s) \big\| \cdot L_f \cdot \big\| c^{\wedge}(-s) \big[u(s) - v(s) \big] \big\| \, ds \\
&\leq M L_f \| u - v \|_{p\omega c} \int_{0}^{\infty} k^{\sim}(s) \, ds \\
&\leq M L_f \| u - v \|_{\omega c} \big\| k^{\sim} \big\|_1 .
\end{aligned}$$

It follows from the Banach fixed point theorem that there exists a unique $u \in PP_{\omega c}(X)$ such that $\mathcal{G} u = u$, that is, $u(t) = \int_{-\infty}^{t} R(t,s) f(s,u(s)) \, ds$. □

The previous results can be applied to obtain (ω, c)-pseudo periodic solutions to the semilinear evolution equation

$$u'(t) = Au(t) + f\big(t, u(t)\big), \quad t \in \mathbb{R}. \tag{3.3}$$

We assume the following condition.

(H5) The operator A generates an exponentially stable C_0- semigroup $(T(t))_{t\geq 0}$, that is, there exist constants $M > 0$ and $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for each $t \geq 0$ and $c > e^{-\alpha}$.

Thus, we have the following theorem.

Theorem 3.2 *Assume that* (*H*1) *and* (*H*5) *hold. Then* (3.3) *has a unique* (ω, c)*-pseudo periodic solution whenever* $ML_f < |k^{\sim}|^{-1}$, *where* $k^{\sim}(t) = c^{\wedge}(-t)e^{-\alpha t}$.

4 Lasota–Wazewska model with unbounded oscillating and ergodic production of red cells

The theory presented above can be extended to the semilinear abstract problem with delay

$$\begin{cases} y'(t) = Ay(t) + f(t, y(t-\tau)), & t \geq 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

where $\tau > 0$ and for which a mild solution is a solution of the integral equation

$$y(t) = T(t)y(0) + \int_0^t T(t-s) f\big(s, y(s-\tau)\big) \, ds, \quad t \geq 0.$$

Here, we need to know a history φ. Note that $y(t-\tau) = \varphi(t-\tau)$ for $t \in [0, \tau]$, and if y is (ω, c)-pseudo periodic, then $y(t-\tau)$ also is. As an example, we study the important Lasota–Wazewska model with (ω, c)-pseudo periodic variable coefficients.

The Lasota–Wazewska model is an autonomous differential equation of the form

$$y'(t) = -\delta y(t) + he^{-\gamma y(t-\tau)}, \quad t \geq 0. \tag{4.1}$$

Wazewska–Czyzewska and Lasota [24] proposed this model to describe the survival of red blood cells in the blood of an animal. In this equation, $y(t)$ describes the number of red cells bloods in the time t, $\delta > 0$ is the probability of death of a red blood cell, h and γ are positive constants related to the production of red blood cells by unity of time, and τ is the time required to produce a red blood cell.

In this section, we study the following model:

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0, \tag{4.2}$$

where $\tau > 0$, $h(t)$ and $a(t)$ are continuous and positive functions. Equation (4.2) models several situations in the real life, see, for example, [7, 8, 11, 18] and the references therein. We are looking for positive (ω, c)-pseudo periodic solutions for certain $\omega > 0$, $c > 0$. Let $f(t,y) = h(t)e^{-a(t)y}$ and assume:

(a) $\tau \leq \omega$;
(b) h is (ω, c)-pseudo periodic;
(c) a is $(\omega, \frac{1}{c})$-pseudo periodic;
(d) $c > e^{-\delta\omega}$;
(e) $\|ah\|_\infty < \delta$.

Remember that $y(\cdot) \in PP_{\omega c}(X)$ implies that $y(\cdot - \tau) \in PP_{\omega c}(X)$.

By (d) and (e) we have that $f(t,y) = h(t)e^{-a(t)y}$ satisfies the hypotheses of Theorem 3.2 since

$$|f(t,y_1) - f(t,y_2) \leq \bigl|a(t)h(t)\bigr||y_1 - y_2| \tag{4.3}$$

for $y_1, y_2 > 0$, and its (ω, c)-pseudo periodic part g satisfies

$$g(t+\omega, cy) = cg(t,y). \tag{4.4}$$

By the variation of constant formula

$$y(t) = e^{-\delta t}y(0) + \int_0^t e^{-\delta(t-s)} f\bigl(s, y(s-\tau)\bigr)\, ds, \tag{4.5}$$

and hence $y(0) > 0$ implies that $y(t) > 0$. Note that condition (d) is necessary for positive c-periodic solutions y. In fact, (4.5) and $h(t) > 0$ imply $y(t) > e^{-\delta t}y(0)$, which evaluated at $t = \omega$ implies (d) since $[c - e^{-\delta\omega}]y(0) > 0$.

Moreover, taking $y(0) = \int_{-\infty}^0 e^{\delta s} f(s, y(s-\tau))\, ds$, which is well defined, we have that y satisfies

$$y(t) = \int_{-\infty}^t e^{-\delta(t-s)} f\bigl(s, y(s-\tau)\bigr)\, ds. \tag{4.6}$$

Then by Theorem 3.2 we have that (4.6) has a unique solution y^* which belongs to $PP_{\omega c}(X)$. Hence y^* is also a solution of type $PP_{\omega c}(X)$ of equation (4.2). Moreover, y^* is exponentially stable. Indeed, for any solution y of (4.2), $z = y - y^*$ satisfies

$$\begin{aligned} z' &= -\delta z + f(t,y) - f\bigl(t,y^*\bigr) \\ &= -\delta z + f\bigl(t,y^* + z\bigr) - f\bigl(t,y^*\bigr). \end{aligned}$$

Note that

$$\bigl|f\bigl(t,y^* + z\bigr) - f\bigl(t,y^*\bigr)\bigr| \le \bigl|a(t)h(t)\bigr||z|.$$

Then, taking $\|ah\|_\infty < \delta$, z verifies that

$$\bigl|z(t)\bigr| \le e^{-\alpha(t-t_0)} \sup_{t_0-\tau\le s\le t_0} \bigl|z(s)\bigr|$$

for $t \ge t_0 \ge 0$ and $0 < \alpha < \delta - \|ah\|_\infty$.

We have proved the following theorem.

Theorem 4.1 *Assume that conditions* (*a*) *to* (*e*) *hold. Then the Lasota–Wazewska model has a unique* (ω,c)*-pseudo periodic solution which is exponentially stable.*

Acknowledgements
Thanks to Colciencias, Diubb, and Fondecyt.

Funding
E. Alvarez is partially supported by Colciencias, Grant Number 121556933876, S. Castillo is partially supported by Diubb 164408 3/R, and M. Pinto is partially supported by Fondecyt Grant Number 1170466.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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Publisher's Note

Received: 8 March 2019 Accepted: 27 May 2019 Published online: 19 June 2019

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