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Walrasian equilibrium as limit of competitive equilibria without divisible goods*

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Abstract

This paper investigates the limit properties of a sequence of competitive outcomes existing for economies where all commodities are indivisible, as indivisibility vanishes. The nature of this limit depends on whether the "strong survival assumption" is assumed or not in the limit economy, a standard convex economy. If this condition holds, then the equilibrium sequence converges to a Walras equilibrium for the convex economy; otherwise it converges to a hierarchic equilibrium, a competitive outcome existing in this economy despite the fact that a Walras equilibrium might not exist.

Keywords: competitive equilibrium, indivisible goods, convergence.

JEL Classification: C62, D50, E40

1 Introduction

The "discrete economy" proposed by Florig and Rivera [10] is a private ownership economy where the indivisibility of consumption goods matters at an individual level, but is negligible at the level of the entire economy. The continuum of individuals that participate in this economy is partitioned into a finite a number of "types of agents". Individually, consumption and production sets are discrete sets (the same subset for agents of the same type), while their aggregate by type of agent is the convex hull of the individual set. Consumers of a given type are identical, except for a continuum parameter with which we initially endow them. This parameter could be identified as fiat money (see Drèze and Müller [6]), whose sole role is to facilitate the trade of indivisible goods.

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Despite the fact that fiat money has no intrinsic value whatsoever, since it does not enter into consumer preferences, it plays a fundamental role in the assignment of resources.[1]

We now observe that a standard economy,[2] with a finite number of agents and polyhedral consumption and production sets, can be approximated by a sequence of discrete economies. The types of agents of these economies are the agents of the standard economy, and the corresponding individual consumption and production sets are discrete subsets, such that their convex hulls converge to the polyhedral sets of the standard economy.[3] Hence, we say that the standard economy is the limit of a sequence of discrete economies, as indivisibilities vanishes.

The "rationing equilibrium" is a competitive outcome existing for discrete economies, with a strictly positive price for fiat money (Theorem 4.1 in Florig and Rivera [10]).[4] This outcome is a refinement of a Walras equilibrium (with a strictly positive price of fiat money), and they coincide under the condition that different consumers are initially endowed with a different amount of fiat money –dispersedness of fiat money–. Nonetheless, fiat money may continue having a positive price even in case all consumption goods are perfectly divisible. This occurs, for instance, when local satiation holds for some consumers, or when some rigidities of prices are present. In such cases, the rationing equilibrium becomes a dividend equilibrium, or equilibrium with slack, a generalized notion of a Walras equilibrium with money, where fiat money has a strictly positive price (see Kajii [12] and Mas-Colell [13]). Otherwise, when prices are flexible and local non-satiation holds, fiat money becomes worthless for a standard economy.

Despite the fact that the price of fiat money is zero in the standard convex economy above, the result does not inform of the properties of the rationing equilibrium allocation when indivisibilities vanished. In fact, due to the close relationship existing between rationing equilibrium and the Walras equilibrium for discrete economies, and from the approximation of a standard economy by discrete economies, one would reasonably expect the existence of some relationship between such allocation and a Walras equilibrium in the standard economy. This paper aims to prove such a relationship. In doing so, by the approximation of economies, we need to assume that consumption and production sets of the standard economy are polyhedron (not a restrictive condition), and then we investigate the limit properties of a rationing equilibrium sequence that arises from the sequence of discrete economies. In other words, we want to provide an answer to the question of what a rationing equilibrium allocation becomes as indivisibilities vanishes.

[1]Fiat money should not be confused with "commodity money" (also known as inside money), yet another continuum parameter widely employed in the literature to assure the existence of a Walras equilibrium when consumption goods are indivisible (see Bobzin [4] for a review of general equilibrium models with indivisible goods). Contrary to fiat money, commodity money satisfies overriding desirability, i.e. it is so desirable by the agents that an adequate amount of it could replace the consumption of any bundle of indivisible goods.

[2]A standard economy is an economy with a finite number of agents, where both consumption sets and production sets are convex (see Arrow and Debreu [1]). A standard economy with fiat money is a standard economy where consumers are initially endowed with fiat money. Throughout this paper, convex economy and standard economy are used indistinctively.

[3]The Kuratowski – Painlevé set convergence notion is used in this paper. See Rockafellar and Wets [17].

[4]The efficiency and core equivalence properties of a rationing equilibrium are studied in Florig and Rivera [9]

Our results show that the nature of this limit depends on whether the "strong survival condition" holds in the standard economy or not. If this condition is satisfied, under mild conditions, we show that this limit is a Walras equilibrium (without money) for the standard economy, and therefore the indivisibility of consumption goods becomes irrelevant when it is small. However, this situation could be quite different when the initial endowment of resources to each consumer does not belong to the interior of the respective consumptions set. In this case, the indivisibility of consumption goods could matter, regardless of how small it is. It may then occur that not all consumers have access to all goods, i.e. a good may be so expensive that some consumers who do not own expensive goods cannot buy a single unit by selling their entire initial endowment. When the consumption goods become "more divisible", i.e. if the minimal unit per commodity decreases, then the equilibrium price may react so that the situation persists.

Following Gay [11], based on a generalized concept of price, several authors have proposed generalizations of the Walras equilibrium existing in the convex case even when the Walras equilibrium does not exist due to a failure of the strong survival assumption (see, for instance, Danilov and Sotskow [5], Marakulin [14] and Mertens [15]). Supported by several examples, Florig [7] proposes an interpretation of those generalized prices in terms of small indivisibilities, introducing the concept of "hierarchic equilibrium". In the case of linear preferences, Florig [8] shows that a hierarchic equilibrium is the limit of standard competitive equilibria of economies with discrete consumption sets converging to the positive octant.[5] See also Piccioni and Rubinstein [16] for an alternative point of view of the hierarchic equilibrium.

When strong survival does not hold in the standard economy, we show then that rationing equilibria converge to a "hierarchic equilibrium", a competitive outcome in this economy. This result formalizes the interpretation of hierarchic equilibria in terms of small indivisibilities given in Florig [7].

This work is organized as follows. Section 2 introduces preliminary concepts and notations, while Section 3 presents the model of economies and equilibria notions used in this paper. There we also define the notion of convergence of a sequence of discrete economies to a convex economy. In Section 4 we present the main contributions of this paper, the convergence of equilibrium results (namely, Proposition 4.2 when the strong survival condition holds, and Theorem 4.1 for a general case). Finally, most of the proofs are provided in the Appendix, i.e. Section 5.

2 Notation and some concepts

In what follows, 0_m is the origin of $\mathbb{R}^m$, x^{t} is the transpose of $x \in \mathbb{R}^m$, whose Euclidean norm is $\|x\|$; the inner product between x, $y \in \mathbb{R}^m$ is $x \cdot y = x^{\mathrm{t}} y$, and the open ball centered at x with radius $\varepsilon > 0$ is $\mathbb{B}(x, \varepsilon)$. For a couple of sets K_1, $K_2 \subseteq \mathbb{R}^m$, $\xi \in \mathbb{R}$ and $p \in \mathbb{R}^m$, we denote $\xi K_1 = \{\xi x,\, x \in K_1\}$, $p \cdot K_1 = \{p \cdot x,\, x \in K_1\}$ and $K_1 \pm K_2 = \{x_1 \pm x_2, x_1 \in K_1,\, x_2 \in K_2\}$, while the set-difference

[5] As local satiation cannot hold in the case of discrete consumption, dividend equilibria are employed.

between them is denoted $K_1 \setminus K_2$. Furthermore, $\mathrm{cl}\, K_1$, $\mathrm{int}\, K_1$ and $\mathrm{conv}\, K_1$ denote, respectively, the closure, interior and the convex hull of K_1.

Let $\lambda(\cdot)$ the standard Lebesgue measure in the underlying space, and for a couple of sets $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subset \mathbb{R}^k$, $L^1(K_1, K_2)$ is the subset of Lebesgue integrable functions from K_1 to K_2.

We follow Rockafellar and Wets [17] to denote

$$\mathbb{N}_\infty = \{\mathbf{N} \subseteq \mathbb{N} : \mathbb{N} \setminus \mathbf{N} \text{ is finite}\} \qquad \text{and} \qquad \mathbb{N}_\infty^* = \{\mathbf{N} \subset \mathbb{N} : \mathbf{N} \text{ is infinite}\}.$$

For $\mathbf{N} \in \mathbb{N}_\infty^*$, the subset of accumulation points of $\{x_n\}_{n\in\mathbf{N}}$ is[6]

$$\mathrm{acc}\,\{x_n\}_{n\in\mathbf{N}} = \{x \in \mathbb{R}^m : \exists \mathbf{N}' \subset \mathbf{N},\, \mathbf{N}' \in \mathbb{N}_\infty^*,\, x_n \to_{\mathbf{N}'} x\}.$$

We also recall that the outer limit of a sequence of subsets $\{K_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^m$ is the subset

$$\limsup_{n\to\infty} K_n \;=\; \{x \in \mathbb{R}^m : \exists \mathbf{N} \in \mathbb{N}_\infty^*,\; \exists x_n \in K_n,\, n \in \mathbf{N}, \text{ with } x_n \to_{\mathbf{N}} x\},$$

while the inner limit is

$$\liminf_{n\to\infty} K_n \;=\; \{x \in \mathbb{R}^m : \exists \mathbf{N} \in \mathbb{N}_\infty,\; \exists x_n \in K_n, n \in \mathbf{N},\; \text{with } x_n \to_{\mathbf{N}} x\}.$$

The sequence of subsets $\{K_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^m$ converges in the sense of Kuratowski – Painlevé to the subset $K \subseteq \mathbb{R}^m$ if

$$\limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n = K,$$

which is case we write $\lim\limits_{n\to\infty} K_n = K$.

Finally, the outer limit of a correspondence $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^m$ is

$$\limsup_{x\to\bar{x}}\; \Psi(x) = \bigcup_{\{x_n\to\bar{x}\}} \limsup_{n\to\infty} \Psi(x_n).$$

3 The model

3.1 Economies and related concepts

The "limit economy" (or the "convex economy" by conditions below) is a standard private ownership economy à la Arrow-Debreu, where we also consider that consumers are initially endowed with a positive parameter called "fiat money", whose sole role is to facilitate the exchange among consumptions goods, with no intrinsic value whatsoever. For this economy, we set I, J and L as the set of consumers and firms, and the number of consumption goods, respectively. Firm $j \in J$

[6] For $\mathbf{N} \in \mathbb{N}_\infty$ or $\mathbf{N} \in \mathbb{N}_\infty^*$, and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of vectors of $\mathbb{R}^m$, we write $x_n \to_{\mathbf{N}} x$ when $\lim_{n\to\infty, n\in\mathbf{N}} x_n = x$; in case $\mathbf{N} = \mathbb{N}$ we put $x_n \to x$.

is characterized by a production set $Y_j \subset \mathbb{R}^L$, and consumer $i \in I$ by a consumption set $X_i \subset \mathbb{R}^L$, endowments $e_i \in \mathbb{R}^L$, and a strict preference correspondence $P_i : X_i \rightrightarrows X_i$. Let $m_i \in \mathbb{R}_{++}$ be the amount of fiat money with which consumers $i \in I$ is initially endowed. The vector of total initial resources is $e = \sum_{i \in I} e_i \in \mathbb{R}^L$, and for $(i,j) \in I \times J$, $\theta_{ij} \in [0,1]$ is the consumer i's share in firms j. As usual, we assume that for every $j \in J$, $\sum_{i \in I} \theta_{ij} = 1$. The limit economy is the collection

$$\mathcal{E} \quad = \quad \left(\{X_i, P_i, e_i\}_{i \in I}, \{Y_j\}_{j \in J}, \{\theta_{ij}\}_{(i,j) \in I \times J}, \{m_i\}_{i \in I}, \right).$$

Next conditions are part of our standing assumptions throughout this paper.

Assumption C. *For all $i \in I$, X_i is a compact, convex polyhedron, and $P_i : X_i \rightrightarrows X_i$ is irreflexive, transitive and has an open graph in $X_i \times X_i$.*

Assumption P. *For all $j \in J$, Y_j is a compact, convex polyhedron.*

As a counterpart for the convex economy, we now introduce the model of a“ discrete economy”, which follows Florig and Rivera [10] when they model an economy where the indivisibility of consumption goods matters at individual level, but is negligible at the level of of the entire economy. By convenience, when we later introduce a sequence of such economies, we use index $n \in \mathbb{N}$ to denote a discrete economy. The "types" of agents of the discrete economy are the agents of the convex economy, such that the type of consumer $i \in I$ and the type of producer $j \in J$ are conformed by a continuum of individuals indexed, respectively, by compacts subsets $T_i \subset \mathbb{R}$ and $T_j \subset \mathbb{R}$, pairwise disjoint. The subset of consumers and firms is respectively denoted by

$$\mathcal{I} = \bigcup_{i \in I} T_i \qquad \text{and} \qquad \mathcal{J} = \bigcup_{j \in J} T_j.$$

The type of producer $t \in \mathcal{J}$ is $j(t) \in J$, and firms of type $j \in J$ are characterized by a discrete production set $Y_j^n \subseteq \mathbb{R}^L$. The aggregate production set of these firms is the convex hull of $\lambda(T_j) Y_j^n$. A production plan for a firm $t \in \mathcal{J}$ is denoted by $y(t) \in Y_{j(t)}^n$, and the set of admissible production plans is

$$Y^n = \left\{ y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j^n) : y(t) \in Y_{j(t)}^n \text{ a.e. } t \in \mathcal{J} \right\}.$$

The type of consumer $t \in \mathcal{I}$ is $i(t) \in I$, and each consumer of type $i \in I$ is characterized by a discrete consumption set $X_i^n \subseteq \mathbb{R}^L$, an initial endowment of resources $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i^n : X_i^n \rightrightarrows X_i^n$. A consumption plan of individual $t \in \mathcal{I}$ is denoted by $x(t) \in X_{i(t)}^n$, and the set of admissible consumption plans is

$$X^n = \left\{ x \in L^1(\mathcal{I}, \cup_{i \in I} X_i^n) : x(t) \in X_{i(t)}^n \text{ a.e. } t \in \mathcal{I} \right\}.$$

The total initial resources of the economy is $e = \sum_{i \in I} \lambda(T_i)\, e_i \in \mathbb{R}^L$, and for $(i,j) \in I \times J$, $\theta_{ij} \geq 0$ is the consumer of type i's share in firms of type j. For every $j \in J$, we assume that

$\sum_{i\in I}\lambda(T_i)\theta_{ij}=1$. In addition, we also assume that each consumer $t\in\mathcal{I}$ is initially endowed with an amount of fiat money $m(t)\in\mathbb{R}_+$, where $m\in L^1(\mathcal{I},\mathbb{R}_+)$.

A discrete economy $\mathcal{E}^n$ is the collection

$$\mathcal{E}^n \quad = \quad \left(\{X_i^n, P_i^n, e_i\}_{i\in I}, \{Y_j^n\}_{j\in J}, \{\theta_{ij}\}_{(i,j)\in I\times J}, m, \{T_i\}_{i\in I}, \{T_j\}_{j\in J}\right),$$

and the feasible consumption-production plans of $\mathcal{E}^n$ are the elements of (see Aubin and Frankowska [3] for the definition of integral of a correspondence)

$$A(\mathcal{E}^n) \quad = \quad \left\{(x,y)\in X^n\times Y^n : \int_{\mathcal{I}} x(t)dt = \int_{\mathcal{J}} y(t)dt + e\right\}.$$

We now define supply and demand concepts for economy $\mathcal{E}^n$, readily extendible to economy $\mathcal{E}$. In the following, $p\in\mathbb{R}^L$, $q\in\mathbb{R}_+$ and K stands for a "salient cone" of $\mathbb{R}^L$, whose family is $\mathcal{C}_L$.[7] The "profit", the "Walras supply" and the "rationing supply" of a type $j\in J$ firm are, respectively,

$$\pi_j^n(p)=\lambda(T_j)\sup_{z\in Y_j^n} p\cdot z, \quad S_j^n(p)=\operatorname*{arg\,max}_{z\in Y_j^n}\ p\cdot z$$

and $\sigma_j^n(p,K) \;=\; \left\{z\in S_j^n(p)\,:\, p\neq 0_L \Rightarrow (Y_j-\{z\})\cap K=\{0_L\}\right\}$. In addition, the "income" of consumer $t\in\mathcal{I}$ is

$$w_t^n(p,q)=p\cdot e_{i(t)}+qm(t)+\sum_{j\in J}\theta_{i(t)j}\pi_j^n(p),$$

whose "budget set" is $B_t^n(p,q)=\left\{\xi\in X_{i(t)}:p\cdot\xi\le w_t^n(p,q)\right\}$. The "Walras demand", the "weak demand" and "rationing demand" for consumer $t\in\mathcal{I}$ are, respectively,

$$d_t^n(p,q)=\left\{\xi\in B_t^n(p,q):B_t^n(p,q)\cap P_{i(t)}^n(\xi)=\emptyset\right\}, \quad D_t^n(p,q)=\limsup_{(p',q')\to(p,q)} d_t^n(p',q'),$$

and $\delta_t^n(p,q,K)=\left\{\xi\in D_t^n(p,q):P_{i(t)}^n(\xi)-\{\xi\}\subseteq K\right\}$.

Remark 3.1. *As we shall ensure that $d_t^n(\cdot)$ is closed valued and locally bounded, Theorem 5.19 in Rockafellar and Wets [17] implies that $D_t^n(\cdot)$ is upper hemi-continuous while $d_t^n(\cdot)$ may fail to be upper hemi-continuous. Notice also that, by definition, $d_t^n(p,q)\subseteq D_t^n(p,q)$ and $\delta_t^n(p,q,K)\subseteq D_t^n(p,q)$. See Florig and Rivera [10] for more details on these concepts.*

Remark 3.2. *The next characterization of weak demand is a straightforward consequence of Proposition 3.1 in Florig and Rivera [10].*

Proposition 3.1. *Let $n\in\mathbb{N}$, $(p,q)\in\mathbb{R}^L\times\mathbb{R}_{++}$ and assume $m(t)>0$. Then, the following holds:*

$$D_t^n(p,q) \quad = \quad \left\{\xi\in B_t^n(p,q):\inf\left\{p\cdot P_{i(t)}^n(\xi)\right\}\ge w_t^n(p,q),\ \xi\notin \operatorname{conv} P_{i(t)}^n(\xi)\right\}.$$

[7] We recall a convex set $K\subset\mathbb{R}^L$ is a convex cone if $0_L\in K$ and $\xi K\subset K$ for all $\xi>0$; a convex cone K is said to be "salient" if $K\cap -K=\{0_L\}$.

3.2 Equilibrium notions

Some of the equilibrium notions that are used in this paper are presented in next definition. They are presented for a discrete economy $\mathcal{E}^n$, and can be straightforwardly extended to the convex economy $\mathcal{E}$.

Definition 3.1. *Given $(x_n, y_n, p_n, q_n) \in A(\mathcal{E}^n) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K_n \in \mathcal{C}_L$, we call*

(a) (x_n, y_n, p_n, q_n) a "Walras equilibrium with fiat money" of $\mathcal{E}^n$, if for a.e. $t \in \mathcal{I}$, $x_n(t) \in d_t^n(p_n, q_n)$ and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}^n(p_n)$,

(b) (x_n, y_n, p_n, q_n) a "weak equilibrium" of $\mathcal{E}^n$, if for a.e. $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n, q_n)$ and for a.e. $t \in \mathcal{J}$, $y_n(t) \in S_{j(t)}^n(p_n)$,

(c) $(x_n, y_n, p_n, q_n, K_n)$ a "rationing equilibrium" of $\mathcal{E}^n$, if for a.e. $t \in \mathcal{I}$, $x_n(t) \in \delta_t^n(p_n, q_n, K_n)$ and for a.e. $t \in \mathcal{J}$, $y_n(t) \in \sigma_t^n(p_n, K_n)$.[8]

We end this part introducing the "hierarchic equilibrium" concept, a competitive outcome for economy $\mathcal{E}^n$. Hereinafter, vectors of $\mathbb{R}^L$ are supposed to be columns, and for $k \in \mathbb{N}$, $[p_1, \ldots, p_k] \in \mathbb{R}^{L\times k}$ is the matrix whose columns are $p_1, \ldots, p_k \in \mathbb{R}^L$.

A "hierarchic price" for consumption goods is $\mathcal{P} = [p_1, \ldots, p_k]^{\mathrm{t}} \in \mathbb{R}^{k\times L}$, and the "hierarchic value" of $\xi \in \mathbb{R}^L$ is $\mathcal{P}\xi = (p_1 \cdot \xi, \ldots, p_k \cdot \xi)^{\mathrm{t}} \in \mathbb{R}^k$. Moreover, denoting by $\sup_{lex}$ the supremum with respect to $\leq_{lex}$, the lexicographic order[9] on $\mathbb{R}^L$, the "hierarchic supply" and the "hierarchic profit" of a firm of type $j \in J$ of economy $\mathcal{E}^n$ at $\mathcal{P}$ are

$$S_j^n(\mathcal{P}) = \{z \in Y_j^n : \forall z' \in Y_j^n,\ \mathcal{P}z' \leq_{lex} \mathcal{P}z\} \qquad \text{and} \qquad \pi_j^n(\mathcal{P}) = \lambda(T_j)\sup\nolimits_{lex}\{\mathcal{Q}z : z \in Y_j^n\},$$

respectively, and given $\mathcal{Q} \in \mathbb{R}_+^k$, the hierarchic budget set of consumer $t \in \mathcal{I}$ is

$$B_t^n(\mathcal{P}, \mathcal{Q}) \;=\; \mathrm{cl}\left\{\xi \in X_{i(t)}^n : \mathcal{P}\xi \leq_{lex} \mathcal{P}e_{i(t)} + m(t)\,\mathcal{Q} + \sum_{j\in J}\theta_{i(t)j}\pi_j^n(\mathcal{P})\right\}.$$

Based in Florig [7], we introduce the next equilibrium concept.[10]

Definition 3.2. *A collection $(x_n, y_n, \mathcal{P}_n, \mathcal{Q}_n) \in A(\mathcal{E}^n) \times \mathbb{R}^{k\times L} \times \mathbb{R}_+^k$ is a "hierarchic equilibrium" of the economy $\mathcal{E}^n$ if:*

[8] The salient cone K in the rationing equilibrium definition is determined endogenously as part of the equilibrium, and summarizes the information that each consumer needs to have in addition to market prices (and their own characteristics) in order to formulate a demand, leading to a stable economic situation, in the sense that no further trading can take place making all participants in a second round of trading strictly better off. Under general conditions over the economy, the existence of such an equilibrium is proved in Florig and Rivera [10].

[9] For $(s,t) \in \mathbb{R}^m \times \mathbb{R}^m$, we recall $s \leq_{lex} t$, if $s_r > t_r$, $r \in \{1, \ldots, m\}$ implies that $\exists \rho \in \{1, \ldots, r-1\}$ such that $s_\rho < t_\rho$. We write $s <_{lex} t$ if $s \leq_{lex} t$, but not $t \leq_{lex} s$. The maximum and the argmax with respect to this order are denoted by $\max_{lex}$ and argmax_{lex}, respectively (similarly for $\min_{lex}$ and argmin_{lex}).

[10] Marakulin [14] introduced a similar notion for exchange economies, using non-standard analysis.

(a) for a.e. $t \in \mathcal{J}$, $y_n(t) \in S^n_{j(t)}(\mathcal{P}_n)$,

(b) for a.e. $t \in \mathcal{I}$, $x_n(t) \in B^n_t(\mathcal{P}_n, \mathcal{Q}_n)$ *and* $P^n_{i(t)}(x_n(t)) \cap B^n_t(\mathcal{P}_n, \mathcal{Q}_n) = \emptyset$.

The number $k \in \mathbb{N}$ in last expressions will be determined at the equilibrium. When $k = 1$, the hierarchic equilibrium reduces to a Walras equilibrium (with fiat money).

3.3 Approximation of convex economies

In order to define a sequence of discrete economies $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ that approximates the economy $\mathcal{E}$, it seems to be natural assume the conditions for all $i \in I$, $m(t) = m_i \in \mathbb{R}_+, t \in T_i$, and $\lambda(T_i) = \lambda(T_j) = 1, (i, j) \in I \times J$. However, these conditions are not needed in proving our results, and they could be considered just for interpretative purposes. The only relevant aspect in defining that sequence is the way by mean of which individual consumption and production sets of the discrete economies approximates corresponding subsets of the convex economy. To do so, in view of Assumptions **C** and **P** we just need to consider sequences of discrete sets whose convex hulls converge to the consumption and production sets of economy $\mathcal{E}$. For the sequel we use given sequences $\nu_h : \mathbb{N} \to \mathbb{N}$, $h = 1, \ldots, L$, such that $\lim_{n\to\infty} \nu_h(n) = \infty$, for all h. The family of subsets $\{M^n\}_{n\in\mathbb{N}}$ with

$$M^n \quad = \quad \left\{\xi = (\xi_1, \ldots, \xi_L) \in \mathbb{R}^L : (\nu_1(n)\xi_1, \ldots, \nu_L(n)\xi_L) \in \mathbb{Z}^L\right\},\ n \in \mathbb{N},$$

then converges in the sense of Kuratowski-Painlevé to $\mathbb{R}^L$. Let $\{X^n_i\}_{n\in\mathbb{N}}, i \in I$, and $\{Y^n_j\}_{n\in\mathbb{N}}, j \in J$, such that

$$Y^n_j = Y_j \cap M^n \neq \emptyset \quad \text{and} \quad X^n_i = X_i \cap M^n \neq \emptyset,$$

and let $P^n_i : X^n_i \rightrightarrows X^n_i$ be the restriction of P_i to X^n_i. Using these concepts, for $n \in \mathbb{N}$ we have that the discrete economy $\mathcal{E}^n$ is the following collection:

$$\mathcal{E}^n \quad = \quad \left((X^n_i, P^n_i, e_i)_{i\in I}, (Y^n_j)_{j\in J}, (\theta_{ij})_{(i,j)\in I\times J}, m, \{T_i\}_{i\in I}, \{T_j\}_{j\in J}\right).$$

4 Hypotheses and convergence results

In addition to the standing conditions above, the next are assumptions that are used at different parts of this paper, they depending on the convergence result to be established.

Assumption M. $m : \mathcal{I} \to \mathbb{R}_+$ *is bounded and for a.e.* $t \in \mathcal{I}$, $m(t) > 0$.

Assumption S. *For all* $i \in I$, $e_i \in \left(X_i - \sum_{j\in J} \theta_{ij}\lambda(T_j)\, Y_j\right)$.

Assumption SA. *For all* $i \in I$ $e_i \in \operatorname{int}\left(X_i - \sum_{j\in J} \theta_{ij}\lambda(T_j)\, Y_j\right)$.

Assumption A. *For all* $n \in \mathbb{N}$, $i \in I$ *and all* $j \in J$, $X_i = \operatorname{conv} X^n_i$ *and* $Y_j = \operatorname{conv} Y^n_j$.

Assumption F. *For all $i \in I$ and each face F of X_i such that*[11]

$$\left(\{e_i\} + \sum_{j \in J} \theta_{ij} \lambda(T_j) Y_j\right) \cap X_i \subseteq F,$$

the sequence $\{F \cap X_i^n\}_{n \in \mathbb{N}}$ converges in the sense of Kuratowski-Painlevé to F.

Assumption **F** requires that X_i^n restricted to the affine subspace for which the interiority assumption holds converges to X_i restricted to that affine subspace. This is important to ensure that the budget set for a sequence of equilibria of the economies $\mathcal{E}^n$ converges to a budget set of the economy $\mathcal{E}$ for some limit of the price sequence considered.

The following proposition is an immediate consequence of Theorem 4.1 in Florig and Rivera [10]. For the proof it is enough to check that Assumption **C** on the economy $\mathcal{E}$ implies that the consumption and production sets of any economy of sequence $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ that approximates $\mathcal{E}$ are finite (i.e., the number of its elements is finite). The proposition ensures that the sequence of equilibria for which we study convergence do actually exist.

Proposition 4.1. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ be a sequence of economies approximating $\mathcal{E}$. For each $n \in \mathbb{N}$, there exists a rationing equilibrium $(x_n, y_n, p_n, q_n, K_n)$, with $q_n > 0$, for economy $\mathcal{E}^n$.*

4.1 Convergence under the "strong survival assumption"

In the next proposition, the survival assumption **SA** plays an important role in establishing the convergence to a Walras equilibrium. While this hypothesis is widely used, it is unrealistic, because it states that every consumer is initially endowed with a strictly positive quantity of every existing good. Typically, most consumers have a single good to sell (usually, their labor). In fact, it implies that all agents have the same level of income at equilibrium in the sense that they have all access to the same goods.

Proposition 4.2. *Suppose $\mathcal{E}$ satisfies Assumptions* **C**, **P**, **M** *and* **SA**, *and let $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ be a sequence of economies approximating $\mathcal{E}$ satisfying Assumption* **A**. *For each $n \in \mathbb{N}$, let (x_n, y_n, p_n, q_n) be a weak equilibrium of $\mathcal{E}^n$, with $q_n > 0$ and $\| (p_n, q_n) \| = 1$. Then, there exists $\mathbf{N} \in \mathbb{N}^*_\infty$, such that the following hold:*

(a) $(p_n, q_n) \to_{\mathbf{N}} (p^*, q^*)$,

(b) there is $(x^*, y^*) \in A(\mathcal{E})$, such that for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n \in \mathbf{N}}$, and for a.e. $t' \in \mathcal{J}$, $y^*(t') \in \operatorname{acc}\{y_n(t')\}_{n \in \mathbf{N}}$, with (x^*, y^*, p^*, q^*) a Walras equilibrium with fiat money for $\mathcal{E}$.

[11] For a convex compact polyhedron $P \subset \mathbb{R}^m$, a face is a set $F \subseteq P$ such that there exists $\psi \in \mathbb{R}^m$ with $F = \operatorname{argmax} \psi \cdot P$.

Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{cl} P_{i(t)}(x^*(t))$, *then* (x^*, y^*, p^*) *is a Walras equilibrium for* $\mathcal{E}$.

Proof. First note that $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ approximating $\mathcal{E}$ implies that for all $i \in I$, $\lim_{n\to\infty} X_i^n = X_i$. By Assumption **SA**, the smallest face of X_i containing

$$\left(\{e_i\} + \sum_{j\in J} \theta_{ij}\lambda(T_j)Y_j\right) \cap X_i$$

is X_i, which implies that Assumption **F** is satisfied. Therefore, all the assumptions of Theorem 4.1 below are satisfied. Assumption **SA** implies that for a hierarchic equilibrium $(x, y, \mathcal{P}, \mathcal{Q})$ with $\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}} \in \mathbb{R}^{k\times L}$ and $\mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}} \in \mathbb{R}^k_+$ (see definition in next section), such that $(x^*, y^*, \mathrm{p}_1, \mathrm{q}_1)$ is a Walras equilibrium with fiat money (cf Florig [7]). Moreover, if for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{cl} P_{i(t)}(x^*(t))$, then standard arguments imply that $\mathrm{q}_1 = 0$, this concluding the proof. □

4.2 The general case

We now replace assumption **SA** by a more realistic one, assuming that every consumer could decide not to exchange anything. We will not assume however that he could survive for very long without exchanging anything. In such a case the limit of a sequence of rationing equilibria will not necessarily be a Walras equilibrium, it will be a hierarchic equilibrium, which is a competitive equilibrium with a segmentation of individuals according to their level of wealth. When this segmentation consists of just one group, the hierarchic equilibrium reduces to a Walras equilibrium.

The next theorem, a generalization of Proposition 4.2, is the main result of this paper. The proof is given in the Appendix.

Theorem 4.1. *Suppose* $\mathcal{E}$ *satisfies Assumptions* **C**, **P**, **M** *and* **S**, *and let* $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ *be a sequence of economies that approximates* $\mathcal{E}$ *and satisfying Assumptions* **A** *and* **F**. *For each* $n \in \mathbb{N}$, *let* (x_n, y_n, p_n, q_n) *be a weak equilibrium of* $\mathcal{E}^n$, *with* $q_n > 0$ *and* $\| (p_n, q_n) \| = 1$. *Then, there exists a hierarchic equilibrium* $(x^*, y^*, \mathcal{P}, \mathcal{Q})$ *for economy* $\mathcal{E}$, *with* $\mathcal{P} = [\mathrm{p}_1, \ldots, \mathrm{p}_k]^{\mathrm{t}}$, $\mathcal{Q} = (\mathrm{q}_1, \ldots, \mathrm{q}_k)^{\mathrm{t}}$, $k \in \{1, \ldots, L\}$, *such that for some* $\mathbf{N} \in \mathbb{N}^*_\infty$ *the following hold:*

(i) for each $n \in \mathbf{N}$, $p_n = \sum_{r=1}^k \varepsilon_r(n)\, \mathrm{p}_r$, *with* $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathbf{N}} 0$,

(ii) for a.e. $t \in \mathcal{I}$, $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n\in\mathbf{N}}$, *and for a.e.* $t \in \mathcal{J}$, $y^*(t) \in \operatorname{acc}\{y_n(t)\}_{n\in\mathbf{N}}$.

Remark 4.1. *Since a rationing equilibrium is a weak equilibrium (see Definition 3.1), it follows that Theorem 4.1 remains valid when using a sequence of rationing equilibria instead of a sequence of weak equilibria as stated.*

5 Appendix: the proofs

The proof of Theorem 4.1 requires some additional definitions and technical results, presented in § 5.1. This Theorem is proved in § 5.2.

5.1 Preliminary results

Definition 5.1 and Lemma 5.1 are taken borrowed from Florig and Rivera [10].

Definition 5.1. *For integer $k \in \{1, \ldots, m\}$, a set of orthonormal vectors $\{\psi_1, \ldots, \psi_k\} \subset \mathbb{R}^m$ coupled with sequences $\varepsilon_r : \mathbb{N} \to \mathbb{R}_{++}$, $r \in \{1, \ldots, k\}$, is called a lexicographic decomposition of a sequence $\psi : \mathbb{N} \to \mathbb{R}^m$, if there exists $\mathbf{N} \in \mathbb{N}^*_\infty$ such that following hold:*

(a) for all $r \in \{1, \ldots, k-1\}$, $\varepsilon_{r+1}(n)/\varepsilon_r(n) \to_{\mathbf{N}} 0$,

(b) for all $n \in \mathbf{N}$, $\psi(n) = \sum_{r=1}^k \varepsilon_r(n)\psi_r$.

The lexicographic decomposition of $\psi : \mathbb{N} \to \mathbb{R}^m$ is denoted as $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathbf{N}\}$.

Lemma 5.1. *Every sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$ admits a lexicographic decomposition.*

For the lexicographic decomposition above and $1 \le r \le k$, we set

$$\Psi(r) = [\psi_1, \ldots, \psi_r] \in \mathbb{R}^{m \times r}.$$

For $z \in \mathbb{R}^m$ we denote $\Psi(r)z = (\psi_1 \cdot z, \ldots, \psi_r \cdot z)^{\mathrm{t}} \in \mathbb{R}^r$, and for $Z \subseteq \mathbb{R}^m$ we also define

$$\Psi(r)Z = \{\Psi(r)z : z \in Z\}.$$

The next lemmata refers to a sequence $\psi : \mathbb{N} \to \mathbb{R}^m \setminus \{0_m\}$, whose lexicographic decomposition is $\{\{\psi_r, \varepsilon_r\}_{r=1}^k, \mathbf{N}\}$. Parts (i) and (ii) of this result are proved in Florig and Rivera [10], while part (iii) is a direct consequence of part (ii) coupled with the observation that for any $\xi \in \mathbb{R}^m$ and finite set of points $Z \subset \mathbb{R}^m$, $\operatorname{conv} \operatorname{argmax} \xi \cdot Z = \operatorname{argmax} \xi \cdot \operatorname{conv} Z$.

Lemma 5.2.

(i) For all $z \in \mathbb{R}^m$ there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\Psi(k)\, z \le_{lex} 0_k \quad \Longleftrightarrow \quad \psi(n) \cdot z \le 0.$$

(ii) If $Z \subseteq \mathbb{R}^m$ is a finite set, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \;\; = \;\; \operatorname{argmax}\, \psi(n) \cdot Z.$$

(iii) If $Z \subseteq \mathbb{R}^m$ is a convex and compact polyhedron, then there exists $\bar{n} \in \mathbb{N}$ such that for all $n > \bar{n}$ with $n \in \mathbf{N}$:

$$\operatorname{argmax}_{lex} \Psi(k)\, Z \;\; = \;\; \operatorname{argmax}\, \psi(n) \cdot Z.$$

Notice that both parts (ii) and (iii) in Lemma 5.2 remain valid when replacing $\operatorname{argmax}_{lex}$ by $\operatorname{argmin}_{lex}$.

The next lemmata is a key property used in the proof of our main result.

Lemma 5.3. *Let $Z \subset \mathbb{R}^m$ be a convex and compact polyhedron, and define*

$$\rho = \max\left\{r \in \{0,\ldots,k\} : \min_{lex} \Psi(r)Z = 0_{\max\{1,r\}}\right\} \quad \text{and} \quad \mathcal{F} = \operatorname{argmin}_{lex} \Psi(\rho)Z.$$

The following holds:

(i) $\limsup_{n\to\infty}\{z \in Z : \psi(n)\cdot z \leq 0\} \subseteq \operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\}$.

Suppose now that $\min_{lex} \Psi(k)Z <_{lex} 0_k$, *and let* $\{Z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ *such that*

$$\lim_{n\to\infty} Z_n = Z \quad \text{and} \quad \lim_{n\to\infty} (Z_n \cap \mathcal{F}) = Z \cap \mathcal{F}.$$

Then the following holds:

(ii) $\operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\} \subset \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} : \psi(n)\cdot z < 0\}$.

Proof. For $\mathbf{N} \in \mathbb{N}^*_\infty$ and $n' \in \mathbb{N}$, we set $\mathbf{N}_{n'} = \{n \in \mathbf{N} : n > n'\}$.

For the proof of part (i), let $\bar{z} \in \operatorname{argmin}_{lex} \Psi(k)Z$ and assume that $\limsup_{n\to\infty}\{z \in Z : \psi(n)\cdot z \leq 0\} \neq \emptyset$, since otherwise the result is trivial. Hence, for z^* in that subset, there is $\overline{\mathbf{N}} \in \mathbb{N}^*_\infty$ and $\{z_n\}_{n\in\overline{\mathbf{N}}} \subset Z$ such that $z_n \to_{\overline{\mathbf{N}}} z^*$ and for all $n \in \overline{\mathbf{N}}$, $\psi(n)\cdot z_n \leq 0$. By Lemma 5.2, part (iii), there exists $n_1 \in \mathbb{N}$ such that for all $n \in \overline{\mathbf{N}}_{n_1}$, we have

$$\operatorname{argmin}_{lex} \Psi(k)\, Z = \operatorname{argmin} \psi(n)\cdot Z.$$

As for all $n \in \overline{\mathbf{N}}_{n_1}$,

$$\psi(n)\cdot\bar{z} = \min \psi(n)\cdot Z \leq \psi(n)\cdot z_n \leq 0,$$

we have by part (i) of Lemma 5.2 that $\Psi(k)\,\bar{z} \leq_{lex} 0_k$.

Let $\sigma = \max\left\{r \in \{0,\ldots,k\} : \Psi(r)z^* = 0_{\max\{1,r\}}\right\}$. If $\Psi(k)z^* \leq_{lex} 0_k$, then the conclusion is trivial. Therefore, we assume $\Psi(k)z^* >_{lex} 0_k$, which implies that $\sigma < k$ and $\psi_{\sigma+1}\cdot z^* = \delta > 0$. At this stage, two cases must be considered.

Case 1. $\rho < \sigma$.

As $\rho < \sigma$, we have $\rho < k$, $\Psi(\rho+1)\,\bar{z} <_{lex} 0_{\rho+1}$ and $\Psi(\rho+1)\,z^* = 0_{\rho+1}$. Therefore, for all $\mu \in [0,1[$,

$$\Psi(\rho+1)\,(\mu\bar{z} + (1-\mu)z^*) <_{lex} 0_{\rho+1}.$$

Hence $\Psi(k)\,(\mu\bar{z} + (1-\mu)z^*) <_{lex} 0_k$, implying that $z^* \in \operatorname{cl}\{z \in Z : \Psi(k)z \leq_{lex} 0_k\}$.

Case 2. $\rho \geq \sigma$.

As $\rho \geq \sigma$, for all $r \in \{1, \ldots, \sigma\}$, $\psi_r \cdot \bar{z} = \psi_r \cdot z^* = 0$. Then $\{\bar{z}, z^*\} \subseteq \operatorname{argmin}_{lex} \Psi(\sigma) Z$. For $n \in \overline{\mathbf{N}}$ we set

$$\psi^*(n) = \sum_{r=1}^{\sigma} \varepsilon_r(n) \psi_r,$$

with $\psi^*(n) = 0$ when $\sigma = 0$. By part (ii) in Lemma 5.2 there exists $n_2 > n_1$ such that for all $n \in \overline{\mathbf{N}}_{n_2}$, $0 = \psi^*(n) \cdot \bar{z} = \psi^*(n) \cdot z^* \leq \psi^*(n) \cdot z_n$. For $n \in \overline{\mathbf{N}}$, we set

$$a_n = \sum_{r=1}^{\sigma+1} \varepsilon_r(n)\, \psi_r \cdot z_n \quad \text{and} \quad b_n = \frac{\varepsilon_{\sigma+2}(n)}{\varepsilon_{\sigma+1}(n)} \sum_{r=\sigma+2}^{k} \frac{\varepsilon_r(n)}{\varepsilon_{\sigma+2}(n)}\, \psi_r \cdot z_n,$$

with $b_n = 0$ if $\sigma + 1 = k$. By the fact that $\{z_n\}_{n \in \overline{\mathbf{N}}}$ remains in a compact set, there exists $n_3 > n_2$ such that for all $n \in \overline{\mathbf{N}}_{n_3}$, on the one hand

$$a_n \geq \varepsilon_{\sigma+1}(n)\, \psi_{\sigma+1} \cdot z_n > \varepsilon_{\sigma+1}(n) \frac{\delta}{2},$$

and, on the other hand, since b_n converges to zero,

$$b_n \in \frac{1}{4}[-\delta, \delta].$$

Therefore, for all $n \in \overline{\mathbf{N}}_{n_3}$,

$$0 \geq \psi(n) \cdot z_n = a_n + \varepsilon_{\sigma+1}(n) b_n \geq \varepsilon_{\sigma+1}(n) \frac{\delta}{4},$$

contradicting $\delta > 0$, hence concluding the proof of part (i).

In order to prove the part (ii), let $\bar{z} \in \operatorname{argmin}_{lex} \Psi(k) Z$. By the fact that $\min_{lex} \Psi(k) Z <_{lex} 0_k$, we have $\rho < k$ and $\psi_{\rho+1} \cdot \bar{z} < 0$. Let $\zeta \in \operatorname{cl}\{z \in Z : \Psi(k) z \leq_{lex} 0_k\}$. Then, for $\varepsilon \in]0, 1]$ there exists $\zeta_\varepsilon \in \mathbb{B}(\zeta, \varepsilon/2) \cap Z$ such that $\Psi(k)\zeta_\varepsilon \leq_{lex} 0_k$. By the convexity of Z, for $\mu \in]0, \varepsilon/2[$ it follows that

$$z_\varepsilon = (1 - \mu)\zeta_\varepsilon + \mu \bar{z} \in Z \cap \mathbb{B}(\zeta, \varepsilon),$$

and then $\Psi(k)\bar{z} \leq_{lex} \Psi(k) z_\varepsilon \leq_{lex} \Psi(k)\zeta_\varepsilon \leq_{lex} 0_k$.

The definition of ρ implies $\Psi(\rho)\bar{z} = 0_{\max\{1,\rho\}}$ and therefore we have also

$$\Psi(\rho) z_\varepsilon = \Psi(\rho)\zeta_\varepsilon = 0_{\max\{1,\rho\}}.$$

This last result coupled with the fact that $\rho < k$ implies

$$\psi_{\rho+1} \cdot \bar{z} \leq \psi_{\rho+1} \cdot z_\varepsilon \leq \psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0 \quad \text{and} \quad \psi_{\rho+1} \cdot \bar{z} < 0.$$

Since $\psi_{\rho+1} \cdot \zeta_\varepsilon \leq 0$, we also have $\delta = \psi_{\rho+1} \cdot z_\varepsilon < 0$. Therefore $\Psi(\rho + 1) z_\varepsilon <_{lex} 0_{\rho+1}$. Hence, we have established that $\Psi(k)z_\varepsilon <_{lex} 0$, $z_\varepsilon \in \mathcal{F}$ and $z_\varepsilon \in \mathbb{B}(\zeta,\varepsilon)$. Let us now consider $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \cap Z_n$ with $z_n \to_{\mathbb{N}} z_\varepsilon$. We observe that

$$\psi(n)\cdot z_n = \sum_{r=1}^{k} \varepsilon_r(n)\psi_r \cdot z_n = \varepsilon_{\rho+1}(n)\,(\alpha_n + \beta_n),$$

where

$$\alpha_n = \frac{1}{\varepsilon_{\rho+1}(n)} \sum_{r=1}^{\rho+1} \varepsilon_r(n)\psi_r \cdot z_n \quad \text{and} \quad \beta_n = \frac{1}{\varepsilon_{\rho+1}} \sum_{r=\rho+2}^{k} \varepsilon_r(n)\psi_r \cdot z_n,$$

with $\beta_n = 0$ if $\rho+1 = k$. Given that, for all $n \in \mathbb{N}$, $\Psi(\rho)z_n = 0_{\max\{1,\rho\}}$, and as β_n converges to 0 and $\delta < 0$, there exists $\bar{n}$ such that for all $n \in \mathbb{N}$ with $n > \bar{n}$, $\alpha_n < \delta/2$ and $\beta_n < -\delta/4$ and therefore $\alpha_n + \beta_n < \delta/4 < 0$. All of this implies that for all $n \in \mathbb{N}$ with $n > \bar{n}$,

$$\psi(n)\cdot z_n = \varepsilon_{\rho+1}(n)\,(\alpha_n+\beta_n) < \varepsilon_{\rho+1}(n)\,\delta/4 < 0.$$

Therefore, for $\zeta \in \mathrm{cl}\{z \in Z \,:\, \Psi(k)z \leq_{lex} 0_k\}$ and $\varepsilon \in]0,1]$, we have that

$$z_\varepsilon \in \mathbb{B}(\zeta,\varepsilon) \cap \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \,:\, \psi(n)\cdot z < 0\},$$

and then, since the lim inf above is a closed set,[12] $\zeta \in \liminf_{n\to\infty}\{z \in Z_n \cap \mathcal{F} \,:\, \psi(n)\cdot z < 0\}$. □

5.2 Proof of Theorem 4.1

Proof. In the following, we use a sequence $(x_n, y_n, p_n, q_n)_{n\in\mathbb{N}}$ of weak equilibria with $q_n > 0$ of the economy $\mathcal{E}^n$ (which exists by Proposition 4.1, considering that a rationing equilibrium is a weak equilibrium). We can assume without loss of generality that for all $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n, q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S_j^n(p_n)$.[13]

In the remaining we split the proof of theorem into six steps.

Step 1 . *Hierarchic price.*

Since $\|(p_n,q_n)\| = 1$, $n \in \mathbb{N}$, from Lemma 5.1 there exist $\{\{(\mathrm{p}_r,\mathrm{q}_r),\varepsilon_r\}_{r=1,\dots,k},\mathbf{N}\}$, a lexicographic decomposition of the sequence $\{\psi(n) = (p_n,q_n)\}_{n\in\mathbb{N}}$. In the sequel, without loss of generality, we identify that subset $\mathbf{N}$ with $\mathbb{N}$, and we denote

$$\mathcal{P} = [\mathrm{p}_1,\dots,\mathrm{p}_k]^{\mathrm{t}} \quad \text{and} \quad \mathcal{Q} = (\mathrm{q}_1,\dots,\mathrm{q}_k)^{\mathrm{t}},$$

and for $r \in \{1,\dots,k\}$, we set $\mathcal{P}(r) = [\mathrm{p}_1,\dots,\mathrm{p}_r]^{\mathrm{t}}$ and $\mathcal{Q}(r) = (\mathrm{q}_1,\dots,\mathrm{q}_r)^{\mathrm{t}}$.

[12]See, for example, Proposition 4.4 in Rockafellar and Wets [17].

[13]Since a countable union of negligible sets is negligible, we could restrict the sequel to an appropriate subset of full Lebesgue measure. Here, as the consumption and production sets are finite for each $n \in \mathbb{N}$, we could also adjust the sequence (x_n, y_n) such that for all $t \in \mathcal{I}$, $x_n(t) \in D_t^n(p_n,q_n)$ and all $t \in \mathcal{J}$, $y_n(t) \in S_j^n(p_n)$ while maintaining $(x_n,y_n) \in A(\mathcal{E}^n)$.

Step 2. *Supply: for all $t \in \mathcal{J}$, $\limsup_{n\to\infty} S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P})$.*

As for all $j \in J$, by Lemma 5.2 there exists $n_j \in \mathbb{N}$ such that for all $n > n_j$,

$$S_j(p_n) = S_j(\mathcal{P}) = \text{argmax}_{lex}\mathcal{P}Y_j.$$

For all $n \in \mathbb{N}$ and all $j \in J$, $\text{conv}Y^n_j = Y_j$, $S^n_j(p_n) \subseteq S_j(p_n) = \text{conv}S^n_j(p_n)$, implying that for all $n > n_J = \max\{n_j,\, j = 1, \dots, J\}$, and all $t \in \mathcal{J}$,

$$S^n_{j(t)}(p_n) \subseteq S_{j(t)}(\mathcal{P}) = \text{conv}\, S^n_{j(t)}(p_n),$$

hence concluding the proof of this Step.

Step 3. *Income.*

For the sequel, for all $j \in J$, let $\zeta_j \in \text{argmax}_{lex}\mathcal{P}Y_j$, and for all $i \in I$, we set $z_i = e_i + \sum_{j\in J}\theta_{ij}\lambda(T_j)\zeta_j$. By Step 2, for all $t \in \mathcal{I}$ and all $n > n_J$, $w_t(p_n, q_n) = p_n \cdot z_{i(t)} + q_n m(t)$.

Step 4. *Budget: For all $t \in \mathcal{I}$, $\limsup_{n\to\infty} B_t(p_n, q_n) \subseteq B_t(\mathcal{P}, \mathcal{Q})$. Moreover, if $m(t) > 0$ then*

$$B_t(\mathcal{P}, \mathcal{Q}) \quad\subseteq\quad \liminf_{n\to\infty} \left\{ x \in X^n_{i(t)} \,:\, p_n \cdot x < w_t(p_n, q_n) \right\}.$$

Using z_i from Step 3, the first inclusion is a straightforward consequence of part (i) of Lemma 5.3 applied to $Z = \big(X_{i(t)} - z_i\big) \times \{-m(t)\}$. Indeed, note that for all $n \in \mathbb{N}$, $n > n_J$, and all $x'_n(t) \in B_t(p_n, q_n)$ we have $\psi_n \cdot z_n \le 0$ with $z_n = (x'_n(t) - z_{i(t)}, -m(t))$ and $\psi(n) = (p_n, q_n)$.

For the second inclusion, for $t \in \mathcal{I}$ and $n \in \mathbb{N}$, we set $\mathcal{F} = \min_{lex}\mathcal{P}(\rho)X_{i(t)}$ and

$$\rho = \max\left\{ r \in \{0, \dots, k\} \,:\, \min_{lex}\mathcal{P}(r)X_{i(t)} = 0_{\max\{1,r\}} \right\}.$$

Assumption **S** coupled with the observation $m(t)\mathcal{Q} >_{lex} 0_k$ implies

$$\min_{lex}\mathcal{P}(\big(X_{i(t)} - z_{i(t)}\big) - m(t)\mathcal{Q} <_{lex} 0_k \quad\text{and}\quad m(t)\mathcal{Q}(\rho) = 0_{\max\{1,\rho\}}.$$

Therefore, producers profit maximization and Assumption **S** implies

$$\left(\{e_{i(t)}\} + \sum_{j\in J}\theta_{i(t)j}\lambda(T_j)\, Y_j \right) \cap X_{i(t)} \subseteq \mathcal{F}.$$

By part (iii) of Lemma 5.2 we observe that $\mathcal{F}$ is a face of $X_{i(t)}$, and then, by Assumption **F** it follows that

$$\lim_{n\to\infty} X^n_{i(t)} \cap \mathcal{F} = X_{i(t)} \cap \mathcal{F}.$$

By part (ii) of Lemma 5.3

$$B_t(\mathcal{P},\mathcal{Q}) \quad \subseteq \quad \liminf_{n\to\infty} \left\{ x \in X^n_{i(t)} \cap \mathcal{F} \,:\, p_n \cdot (x - z_{i(t)}) < q_n m(t) \right\},$$

and since

$$\liminf_{n\to\infty} \left\{ x \in X^n_{i(t)} \cap \mathcal{F} \,:\, p_n \cdot (x - z_{i(t)}) < q_n m(t) \right\} \quad \subseteq \quad \liminf_{n\to\infty} \left\{ x \in X^n_{i(t)} \,:\, p_n \cdot x < w_t(p_n, q_n) \right\}$$

the second inclusion holds true.

Step 5. *Demand: for all* $t \in \mathcal{I}$ *with* $m(t) > 0$ *and all* $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n\in\mathbb{N}}$,

$$P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P},\mathcal{Q}) = \emptyset.$$

Let $t \in \mathcal{I}$ such that $m(t) > 0$ and choose $\mathbf{N}(t) \in \mathbb{N}^*_\infty$ such that $x_n(t) \to_{\mathbf{N}(t)} x^*(t)$ and for all $n \in \mathbf{N}(t)$, $n > n_J$. By contraposition, assume that there is $\xi \in P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P},\mathcal{Q})$. Then, by Step 5 there exists $\bar{n}_1 > n_J$ and $\xi_n \to_{\mathbb{N}} \xi$ such that for all $n > \bar{n}_1$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)}.$$

As the graph of $P_{i(t)}$ is open, there exists $\bar{n}_2 > \bar{n}_1$ such that for all $n > \bar{n}_2$ with $n \in \mathbb{N}$,

$$p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0 \quad \text{and} \quad \xi_n \in X^n_{i(t)} \cap P_{i(t)}(x^*(t)),$$

and, again, as the graph of $P_{i(t)}$ is open, we can choose $\bar{n}_3 > \bar{n}_2$ such that for all $n > \bar{n}_3$ with $n \in N(t)$, we have $p_n \cdot (\xi_n - z_{i(t)}) - q_n m(t) < 0$ and $\xi_n \in P^n_{i(t)}(x_n(t))$. As $q_n m(t) > 0$, the last fact contradicts $x_n(t) \in D^n_t(p_n, q_n)$ for all $n > \bar{n}_3$ with $n \in N(t)$ (see Proposition 3.1).

Step 6. *Equilibrium allocation.*

Using Fatou's lemma in Artstein [2], there exists $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$, $x^*(t) \in \operatorname{acc}\{x_n(t)\}_{n\in\mathbb{N}}$ and $y^*(t') \in \operatorname{acc}\{y_n(t')\}_{n\in\mathbb{N}}$. By Step 2, for a.e. $t \in \mathcal{J}$, $y^*(t) \in S_{j(t)}(\mathcal{P})$, and by Steps 4 and 5, for a.e. $t \in \mathcal{I}$, $x^*(t) \in B_t(\mathcal{P},\mathcal{Q})$ and $P_{i(t)}(x^*(t)) \cap B_t(\mathcal{P},\mathcal{Q}) = \emptyset$. $\square$

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