INTERFACE DYNAMICS IN SEMILINEAR WAVE EQUATIONS

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ABSTRACT. We consider the wave equation $\varepsilon^2(-\partial_t^2+\Delta)u+f(u)=0$ for $0<\varepsilon\ll 1$, where f is the derivative of a balanced, double-well potential, the model case being $f(u)=u-u^3$. For equations of this form, we construct solutions that exhibit an interface of thickness $O(\varepsilon)$ that separates regions where the solution is $O(\varepsilon^k)$ close to ± 1, and that is close to a timelike hypersurface of vanishing *Minkowskian* mean curvature. This provides a Minkowskian analog of the numerous results that connect the Euclidean Allen-Cahn equation and minimal surfaces or the parabolic Allen-Cahn equation and motion by mean curvature. Compared to earlier results of the same character, we develop a new constructive approach that applies to a larger class of nonlinearities and yields much more precise information about the solutions under consideration.

1. Introduction

Consider the initial value problem

$$\begin{cases} \varepsilon^2\Box u \,+\, f(u) \;=0 \quad \text{in } [0,T]\times\mathbb{R}^n \\ u=u_0, \quad \partial_t u = u_1 \quad \text{in } \{0\}\times\mathbb{R}^n \end{cases} \tag{1.1}$$

where $\Box u=-\partial_t^2 u+\Delta_x u$ and $\Delta_x u=\sum_{i=1}^n \partial_{x_i}^2 u$. We are interested in nonlinearities of the form

$$f(s)=-W'(s)$$

where $W(s)$ is a "balanced double-well potential", namely a C^∞ even function such that

$$\begin{aligned} W(s) &>0 \quad \text{in } \mathbb{R}\setminus\{-1,1\} \\ W(\pm1) &= W'(\pm1) \,=\, 0, \\ W''(\pm1) &= a>0 \end{aligned} \tag{1.2}$$

A canonical example is the wave version of the Allen-Cahn equation

$$W(u)=\frac{1}{4}(1-u^2)^2$$

sometimes called the ϕ^4 model.

Since the mid 70's, it has been accepted in the physics and cosmology literature (see for example [17, 22, 26]) that under some circumstances, solutions of (1.1) should exhibit an interface, separating regions where $u\approx 1$ and $u\approx -1$, that approximately sweeps out a timelike minimal surface in Minkowski space. (The timelike Minkowskian minimal surface equation — the condition that the mean curvature, with respect to the Minkowski metric, vanishes identically — is a quasi-linear geometric wave equation described below.) Formal asymptotic arguments in support of the same picture have been known in the applied mathematics literature for about 20 years, see for example [21, 24]. The first rigorous verification of this scenario appeared only in rather recent work of the second author and collaborators [16, 14, 11], which constructed solutions of (1.1) with an interface concentrated near a timelike minimal surface.

In this paper, we revisit this problem, developing an entirely new approach that yields stronger results and is likely to be more robust and flexible. In doing so, we are largely motivated by the clear analogy between the problem we study and the numerous classical results concerning solutions of the elliptic Allen-Cahn equation

$$\varepsilon^2 \Delta u + f(u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n$$

with interfaces that concentrate near (Euclidean) minimal hypersurfaces in Ω. Many proofs in the elliptic setting fall into one of two large families:

- proofs involving Γ-convergence and related ideas, see for example [20, 25], which proceed by characterizing energy concentration when $0 < \varepsilon \ll 1$, and
- proofs involving Liapunov-Schmidt reduction or its variants, ultimately relying on a linearization of the equation about an approximate solution built around a minimal surface.

The latter family of arguments has a number of advantages over the former — it is capable of providing much more precise descriptions of the solutions being studied [23]; it is more readily adapted to studying solutions of finite (nonzero) Morse index; it can be used to build entire solutions of the Allen-Cahn equations [8], including counterexamples to the de Giorgi conjecture [9]; it can be used to study refined phenomena such as interface foliation [7, 1].

Prior rigorous work on timelike minimal surfaces and interfaces in solutions of (1.1) is more similar in spirit to the first family of elliptic results described above — all papers to date rely on weighted energy estimates to show that under suitable hypotheses, energy concentrates near a timelike minimal surface. In this paper, by contrast, we aim to adapt to the hyperbolic setting techniques from the second family of elliptic results — for example, linearization about a high-order approximate solution. Thus, our proofs may be loosely seen as hyperbolic analogs of those in [23, 8, 9]. Our results show that as with elliptic problems, this approach yields a much sharper description of the solutions constructed than appears to be available from energy estimates alone. A more detailed comparison of our results with earlier work is given in Section 1.3. Interfaces in the parabolic analog of Equation (1.1),

$$-\partial_t u + \Delta u + \varepsilon^{-2} f(u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

located near solutions of mean curvature flow for surfaces is also a subject that has been widely treated. See [6, 3, 13, 15]. See also [10] and references therein for the corresponding interface foliation problem.

1.1. **Some preliminaries.** Let J denote the $(n+1) \times (n+1)$ diagonal matrix

$$J := \begin{bmatrix} -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

We consider the standard Minkowski inner product

$$\langle a, b \rangle_m = a \cdot Jb, \quad a, b \in \mathbb{R}^{n+1},$$

where $\cdot$ denotes the standard Euclidean inner product.

We let ν be a Minkowki unit normal vector field along Γ. This means that ν satisfies $|\langle \nu, \nu \rangle_m| = 1$ and $\langle \nu, \tau \rangle_m = 0$ for all vectors τ tangent to Γ. Since J^2 is the identity, it is easy to see that $\nu = J\bar{\nu}$, where $\bar{\nu}$ is normal to Γ with respect to the Euclidean inner product.

An orientable hypersurface Γ in $\mathbb{R}^{n+1}$ with Minkowski normal vector field ν is said to be **time-like** if $\langle \nu, \nu \rangle_m > 0 \quad$ on Γ. Normalizing ν we will always assume

$$\langle \nu, \nu \rangle_m = 1.$$

A basic fact is that under assumptions (1.2) there is a unique solution to the problem

$$w''(\zeta) + f(w(\zeta)), \quad w(0) = 0, \quad w(\pm\infty) = \pm 1. \tag{1.3}$$

which is defined by the relation

$$\zeta = \int_0^{w(\zeta)} \frac{ds}{\sqrt{2W(s)}}.$$

$w(\zeta)$ is an odd function since W is even. It satisfies

$$w(\zeta) \to \pm 1 + O(e^{-a|\zeta|}) \quad \text{as} \quad \zeta \to \pm\infty,$$

and

$$D_\zeta^k w(\zeta) = O(e^{-a|\zeta|}) \quad \text{as} \quad \zeta \to \pm\infty \qquad \text{for all } k \in \mathbb{N}.$$

In the case of the Allen Cahn nonlinearity $f(u) = u(1-u^2)$, we explicitly have

$$w(\zeta) \ = \ \tanh\left(\frac{\zeta}{\sqrt{2}}\right).$$

We will need a standard fact for the quadratic form associated to the linearization of equation (1.3): there is a positive constant c such that for any $\psi \in H^1(\mathbb{R})$ with $\int_{\mathbb{R}} \psi w' = 0$ we have

$$Q(\psi) := \int_{\mathbb{R}} |\psi'|^2 - f'(w)\psi^2 \ \geq \ c \int_{\mathbb{R}} |\psi'|^2 + |\psi|^2. \tag{1.4}$$

This estimate follows from a direct compactness argument and the identity

$$Q(\psi) = \int_{\mathbb{R}} {w'}^2 |\rho'|^2, \quad \psi = \rho w'.$$

Using $w(\zeta)$ we can find a class of explicit examples to solutions with phase transition across *time-like planes*. Let Γ be a time-like hyperplane in $\mathbb{R}^{n+1}$ with Minkowski normal ν. Then for any $p \in \Gamma$, all its points can be described as the set

$$\Gamma = \{ Y = (x,t) \in \mathbb{R}^{n+1} \ / \ \big\langle \, Y - p \, , \, \nu \, \big\rangle_m \ = \ 0 \, \}, \quad \big\langle \nu, \nu \big\rangle = 1. \tag{1.5}$$

We observe that all points $(x,t) \in \mathbb{R}^{n+1}$ can be expressed as

$$(x,t) \ = \ p + z\nu, \quad (p,z) \in \Gamma \times \mathbb{R}.$$

Clearly we have $z = \big\langle (x,t) - p, \nu \big\rangle_m$. Let us consider the function

$$u(x,t) = w\left(\frac{z}{\varepsilon}\right), \quad z = \big\langle (x,t) - p, \nu \big\rangle_m \tag{1.6}$$

We quickly check that

$$\varepsilon^2 \Box u(x,t) + f(u(x,t)) = \langle \nu, \nu \rangle_m \, w''(\zeta) + f(w(\zeta)) = 0, \quad \zeta = \frac{z}{\varepsilon},$$

and hence u solves (1.1) with a sharp transition on Γ between the values -1 and $+1$, for suitable initial data.

1.2. **Statement of the Main Result.** Next we introduce the objects and notation necessary for the statement of our main result.

• We assume that Γ is a smooth, **time-like hypersurface** in $[0,T]\times\mathbb{R}^n$ that divides the space $[0,T]\times\mathbb{R}^n$ into two disjoint open components $\mathcal{O}^-$ and $\mathcal{O}^+$ with $\mathcal{O}^-$ being bounded.

• We assume in addition that Γ is a **Minkowski minimal surface** in $\mathbb{R}^{n+1}$ (in the sense of Definition 1 below), and that the velocity of Γ vanishes at $t=0$. We remark that the Cauchy problem for timelike minimal surfaces is studied in [2, 18, 19].

• We also assume that there exists some $\delta>0$ such that

$$(Y,z)\in\Gamma\times(-\delta,\delta)\quad\longrightarrow\quad(x,t)=Y+z\nu(Y)\quad\text{ is injective,}\tag{1.7}$$

where $\nu(Y)$ is a Minkowski normal vector field on Γ with

$$\langle\nu(Y),\nu(Y)\rangle_m=1.$$

Let us call $\mathcal{N}$ the set of all points of the form (1.7). For a function $\xi(x,t)$ defined on $\mathcal{N}$ sufficiently smooth we write $D_Y^jD_z^l\xi(Y,z)$ meaning iterated directional derivatives respectively on tangent directions to Γ at Y or in ν-direction. We choose ν to be the normal pointing towards $\mathcal{O}^+$. We let the limit phase function be

$$\mathbb{I}(x,t)=\begin{cases}-1 & \text{if } (x,t)\in\mathcal{O}^-\\ +1 & \text{if } (x,t)\in\mathcal{O}^+\end{cases}.\tag{1.8}$$

Our main theorem is the following.

Theorem 1. *For each $j\in\mathbb{N}$, there exist initial conditions u_0^ε, u_1^ε for a solution $u_\varepsilon(x,t)$ of problem (1.1) with the property that*

$$u_\varepsilon(x,t)\to\mathbb{I}(x,t)\quad\textit{as}\quad\varepsilon\to0\qquad\textit{in the } C^{j+1} \textit{ sense}$$

in compact subsets of $([0,T]\times\mathbb{R}^n)\setminus\mathcal{N}$. Inside $\mathcal{N}$ we have

$$u_\varepsilon(x,t)=w\left(\frac{z}{\varepsilon}\right)+\phi_\varepsilon(x,t),\quad(x,t)=Y+z\nu(Y).$$

and

$$|\phi_\varepsilon(x,t)|+|D_x^{j+1}\phi_\varepsilon(x,t)|+|D_x^j\partial_t\phi_\varepsilon(x,t)|\ \le\ C\varepsilon.\tag{1.9}$$

The proof of Theorem 1 involves various ingredients with a simple philosophy: First we obtain an expansion in powers of ε of a true solution that gives an arbitrarily algebraic high order of approximation in ε. After this approximation is built, estimates for the remainder with a sufficiently good control are found. This is a delicate step in which positivity of the one-variable quadratic form associated to the linearization of the equation (1.3) is essential, as well as designing well-adapted systems of coordinates.

The proof provides much more precise information about the solution. In fact, for a given number $k\ge1$ we can find a solution that near Γ takes the form

$$u_\varepsilon(x,t)=w\left(\frac{z}{\varepsilon}-h_\varepsilon^*(Y)\right)+\phi_\varepsilon^*(x,t)+\varphi_\varepsilon(x,t),\quad(x,t)=Y+z\nu(Y),\tag{1.10}$$

where $h^*_\varepsilon, \phi^*_\varepsilon$ are explicit functions with $h^*_\varepsilon = O(\varepsilon)$, $\phi^*_\varepsilon = O(\varepsilon^2)$ in smooth sense, and the remainder φ_ε satisfies

$$|\varphi_\varepsilon(x,t)| + |D_x^{j+1}\varphi_\varepsilon(x,t)| + |D_x^j \partial_t \varphi_\varepsilon(x,t)| \ \le\ C\varepsilon^k. \tag{1.11}$$

The solution described is stable in the sense that smooth perturbations of its initial condition with size $O(\varepsilon^m)$ and sufficiently large m produce a solution with the same qualitative features. This rules out exponential growth of small perturbations (which in general may happen).

1.3. **More about prior work.** As noted above, our proof of Theorem 1 constructs solutions u_ε whose behavior we are able to describe to arbitrary precision, in arbitrarily strong norms. The best (indeed, the only) prior results construct solutions u_ε that satisfy the weaker estimate

$$\|u_\varepsilon - \mathbb{I}\|_{L^2((0,T')\times\mathbb{R}^N)} \le C(T')\varepsilon^{1/2} \qquad \text{for any } T' < T,$$

together with some weighted estimates that quantify energy concentration around Γ. This was proved in [16] under the assumption that Γ_0 is a topological torus, but allowing rather general initial velocity for Γ — more general in fact than we consider here. The proof in [14] assumes that Γ_0 has zero initial velocity but allows it to be an arbitrary smooth connected compact manifold, among a number of generalizations of the results in [16].

It has been recently proved in [11] that when $n = 2$, one can extract from the weighted energy estimates in [16, 14] an estimate of the form

$$\|u_\varepsilon - u^*_\varepsilon\|_{L^2((0,T')\times\mathbb{R}^2)} + \varepsilon\|D(u_\varepsilon - u^*_\varepsilon)\|_{L^2((0,T')\times\mathbb{R}^2)} \le C(T')\varepsilon^{3/2} \text{ for any } T' < T$$

for some u^*_ε whose description is less explicit than the one that we construct in this paper. This seems to be the limits of the precision attainable by the strategy employed in prior work, and it also seems only to be available in 2 space dimensions.

Another drawback of the technique of [16, 14, 11] is that these results rely on standard well-posedness theory to provide solutions of (1.1). This imposes growth conditions that render these papers unable to address the canonical cubic nonlinearity $f(u) = u(1-u^2)$ in high dimensions. No such growth conditions are needed in this paper.

Related prior results on issues that we do not address include the following:

- In [14], equations like (1.1), but with asymmetric nonlinearities for which there is a bias toward one of the potential wells, are shown to have solutions with an interface that approximately sweeps out a timelike hypersurface of constant (nonzero) Minkowskian mean curvature.
- In [16], a Ginzburg-Landau wave equation — like (1.1), but for a complex-valued function u, with nonlinearity $f(u) = u(1-|u|^2)$ — is shown to have solutions for which energy concentrates near a codimension 2 surface of vanishing Minkowskian mean curvature.
- Results of a similar character are proved for the Abelian Higgs model in [5], a Ginzburg-Landau wave equation coupled to a wave equation for an electromagnetic potential, for certain values of a coupling constant appearing in the equations.
- A scattering result is proved in [4] for (1.1) in $\mathbb{R}^{1+3}$ for initial data $(u,u_t)|_{t=0}$ a small, very smooth perturbation of $(w(x^3),0)$. This can be seen as an

analog for (1.1) of results [18, 2] that establish scattering Minkowski minimal surfaces with initial data that is a small perturbation of a motionless hyperplane.

We believe that it should be possible to strengthen at least some of the above results by the methods that we introduce here.

2. Construction of an approximation

2.1. **The wave operator in Fermi coordinates.** Let us consider a general smooth, orientable n-dimensional manifold Γ embedded in $\mathbb{R}^{n+1}$ and let $\mathcal{N}$ be a small tubular neighborhood of Γ defined by relation (1.7). We will find an expression for the wave operator acting on functions $u(x,t)$ with $(x,t) \in \mathcal{N}$,

$$\Box u = -\partial_t^2 u + \Delta_x u \quad \text{in } \mathcal{N}$$

when u is expressed in *Minkowskian Fermi coordinates* that we introduce next. All points in $\mathcal{N}$ can be uniquely represented in the form

$$(x,t) = Y + z\nu(Y), \quad Y \in \Gamma, \quad |z| < \delta.$$

provided that δ is taken sufficiently small.

Let assume that Γ is compact and parametrized by a finite number of smooth maps

$$y \in \Lambda_i \subset \mathbb{R}^n \;\mapsto\; Y_i(y) \in \mathbb{R}^{n+1}, \quad i \in I,$$

so that

$$\Gamma = \bigcup_{i \in I} Y_i(\Lambda_i).$$

We also use the convention $y = (y_0, \dots, y_{n-1})$.

Define

$$I[u] = \iint_{\mathbb{R}^{n+1}} (|\nabla_x u(x,t)|^2 - |u_t(x,t)|^2)\, dx\, dt$$

where $u(x,t)$ is a smooth function supported sufficiently close to a compact portion of the manifold Γ. We write

$$\nabla u(x,t) = \begin{bmatrix} \partial_t u(x,t) \\ \nabla_x u(x,t) \end{bmatrix}.$$

Then

$$I[u] = \iint_{\mathbb{R}^{n+1}} \nabla u(x,t)^T J \nabla u(x,t)\, dx\, dt$$

Let us assume for the moment that u is supported close to one of the coordinate patches $Y_i(\Lambda_i)$. Let us omit the subindex i in the pair (Λ_i, Y_i) and consider local coordinates in a neighborhood of $Y(\Lambda) \subset \Gamma$ given by

$$(x,t) := Y(y) + z\nu(y), \quad y \in \Lambda \subset \mathbb{R}^n, \quad |z| < \delta$$

where we are just setting $\nu(y) := \nu(Y(y))$. We refer to (y,z) as *Fermi coordinates* associated to the local coordinate system $Y : \Lambda \to \Gamma$. Let us write

$$v(y,z) = u(x,t), \quad (x,t) = Y(y) + z\nu(y).$$

and

$$\nabla v(y,z) = \begin{bmatrix} \nabla_y v(y,z) \\ \partial_z v(y,z) \end{bmatrix}.$$

We use the following notation

$$Y_a = \partial_{y_a} Y + z\partial_{y_a}\nu, \quad = 0, 1, \ldots, n-1; \qquad Y_n = \nu$$

and

$$g_{\alpha\beta}(y,z) = \langle Y_\alpha, Y_\beta\rangle_m, \qquad \alpha, \beta = 0, \ldots, n.$$

We will call $g(y,z)$ the matrix of entries $[g(y,z)]_{\alpha\beta} = g_{\alpha\beta}(y,z)$. We will also denote

$$g^{\alpha\beta}(y,z) = [g(y,z)^{-1}]_{\alpha\beta}$$

and

$$g^0_{ab}(y) = g_{ab}(y,0), \qquad a, b = 0 \ldots, n-1.$$

Consistent with this, we will always tacitly assume that $\alpha, \beta, \ldots$ run from 0 to n, and $a, b, \ldots$ run from 0 to $n-1$, and we will sum over repeated indices.

We introduce the matrix

$$B = \big[Y_0 \cdots Y_n\big],$$

and we remark that $B^T JB = g$. The definition of the Minkowskian normal $\nu = Y_n$ directly imply the following basic property of Fermi coordinates:

$$\begin{aligned} g_{an} = g_{na} = \langle Y_a, \nu\rangle_m &= 0 \qquad \text{for } a = 0, \ldots, n-1, \\ g_{nn} = \langle \nu, \nu\rangle_m &= 1. \end{aligned} \tag{2.1}$$

From Chain's rule we find

$$\nabla u(x,t) \;=\; B^{-T}\,\nabla v(y,z), \quad (x,t) = Y(y) + z\nu(y).$$

and hence

$$(\nabla u)^T J \nabla u \;=\; (\nabla v)^T A^{-1}\,\nabla v$$

where

$$A = B^T JB = g \tag{2.2}$$

and hence (using (2.1))

$$A^{-1} = g^{-1} = (g^{\alpha\beta})^n_{\alpha,\beta=0} = \begin{bmatrix} (g^{ab})^{n-1}_{a,b=0} & 0 \\ 0 & 1 \end{bmatrix}.$$

A related observation is that

$$\sqrt{|\det g(y,z)|} \;=\; \big|\det\big[B\,\big|\,\nu\big]\big| \tag{2.3}$$

Indeed, we have

$$\big|\det\big[B\,\big|\,\nu\big]\big| \;=\; \sqrt{|\det A|}$$

where A is the matrix in (2.2) and (2.3) readily follows.

Then we find that

$$\begin{aligned} I[u] &= \iint_{\mathbb{R}^{n+1}} \nabla v^T g(y,z)^{-1} \nabla v \sqrt{\big|\det g(z)\big|}\, dy\, dz \\ &= \iint_{\mathbb{R}^{n+1}} g^{\alpha\beta}(y,z)\, \partial_\alpha v\, \partial_\beta v\, \sqrt{\big|\det g(z)\big|}\, dy\, dz. \end{aligned}$$

Taking a test function $\varphi(x,t)$ supported within the range of validity of these local coordinates we find

$$-\frac{1}{2}\frac{d}{d\lambda} I[u+\lambda\varphi]\big|_{\lambda=0} = \int_{\mathbb{R}^{n+1}} \Box u(x,t)\,\varphi(x,t)\, dt\, dx$$

hence letting $\psi(y,z)=\varphi(Y(y)+z\nu(y))$ we find

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}} \Box u(x,t)\,\varphi(x,t)\,dt\,dx &= \iint_{\mathbb{R}^{n+1}} \mathcal{L}[v](y,z)\,\psi(y,z)\,\sqrt{|\det g(y,z)|}\,dy\,dz\\ &= \iint_{\mathbb{R}^{n+1}} \mathcal{L}[v]\,\varphi\,dt\,dx\end{aligned}$$

where

$$\mathcal{L}[v]=\frac{1}{\sqrt{|\det g(y,z)|}}\partial_\alpha[\sqrt{|\det g(y,z)|}g^{\alpha\beta}(y,z)\partial_\beta v]. \tag{2.4}$$

Recalling the form of $(g^{\alpha\beta})$, this simplifies to

$$\mathcal{L}[v]=\frac{1}{\sqrt{|\det g(y,z)|}}\partial_a[\sqrt{|\det g(y,z)|}g^{ab}(y,z)\partial_b v]+\frac{1}{\sqrt{|\det g(y,z)|}}\partial_z(\sqrt{|\det g(y,z)|}\partial_z v)$$

and $g^{ab}(y,z)=[g(y,z)^{-1}]_{ab}$. The following definition is in order. For a sufficiently small z, the wave operator associated to a time-like manifold

$$\Gamma_z=\{Y+z\nu(Y)\ /\ Y\in\Gamma\}$$

is given by

$$\Box_{\Gamma_z} \;=\; \frac{1}{\sqrt{|\det g(y,z)|}}\,\partial_a[\sqrt{|\det g(y,z)|}g^{ab}(y,z)\,\partial_b\,]$$

which acts on functions of the local coordinate y for Γ_z. The *mean curvature* in the Minkowski sense of the manifold Γ_z at the point $Y(y)+z\nu(y)$ is defined by the quantity

$$H_{\Gamma_z}(y) \;=\; -\frac{1}{2}\frac{\partial}{\partial z}\log|\det g(y,z)|,$$

so that correspondingly the Minkowskian mean curvature of Γ at the point $Y=Y(y)$ is given by

$$H_\Gamma(Y) \;=\; -\frac{1}{2}\frac{\partial}{\partial z}\log|\det g(y,z)|\,\big|_{z=0} \tag{2.5}$$

For a function f defined on Γ we will write indistinctly $f(Y)$ or $f(y)$ when $Y=Y(y)$, with reference to local coordinates.

In summary, we have proven the validity of the formula

$$\Box=\Box_{\Gamma_z}+\partial_z^2-H_{\Gamma_z}\partial_z\ . \tag{2.6}$$

At this point we establish the key definition.

Definition 1. *A time-like hypersurface Γ in $\mathbb{R}^{n+1}$ is said to be* **minimal** *in the Minkowski sense if its Minkowski mean curvature given by (2.5) vanishes:*

$$H_\Gamma(Y) \;=\; 0 \quad \textit{for all} \quad Y\in\Gamma. \tag{2.7}$$

In what follows we will always assume that Γ is minimal. We can then write

$$H_{\Gamma_z}(y)=z\,\mathbf{a}_\Gamma(y)+z^2\mathbf{b}_\Gamma(y,z) \tag{2.8}$$

The justification of the notation $\Box_{\Gamma_z}$ comes from the fact that the matrix $g(y,z)$ defining the metric, and hence the operator in local coordinates has all positive eigenvalues except one which is negative. This is a consequence of the time-like character of the surface Γ. To see this, we check that in the case of a non-vertical time-like plane in $\mathbb{R}^{n+1}$. We can parametrize it in the form

$$(x,t)=(\alpha\cdot y,y),\quad y=(y_1,\dots,y_n).$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$. A normal vector to this plane is $(-1, \alpha)$ and the time-like character clearly corresponds to the relation $|\alpha|^2 > 1$. In this case we directly compute

$$(B^T JB)_{ij} = \delta_{ij} - \alpha_i \alpha_j.$$

We see that 1 is an eigenvalue of this matrix with multiplicity $n-1$, while its trace is negative because of the time-like condition. Hence in addition, exactly one negative eigenvalue is present. It follows from this fact and the compactness of Γ that (after shrinking δ if necessary)

$$\det g(y,z) \le -c < 0 \quad \text{everywhere in } \Gamma \times (-\delta, \delta). \tag{2.9}$$

We will introduce in §3.1 local coordinates under which $\Box_\Gamma$ truly becomes a wave operator.

2.2. **Shift of coordinates and construction.** Our purpose is to find a good approximate solution for the equation

$$S(u) := \varepsilon^2 \Box u + f(u) = 0 \tag{2.10}$$

valid in a small ε-independent neighborhood of the manifold Γ, that has a sharp transition layer near Γ. More precisely, let us consider a heteroclinic $w(\zeta)$ as defined in (1.3). Taking into account expression (2.6) for the wave operator and (2.8), we see that equation (2.10) can be written as

$$S(u) = \varepsilon^2 \partial_z^2 u + f(u) + \varepsilon^2 \Box_{\Gamma_z} u - \varepsilon^2 (\mathbf{a}_\Gamma + z \mathbf{b}_\Gamma) z \partial_z u = 0,$$

where we write

$$u(y,z) = u(x,t) \quad \text{for } (x,t) = Y(y) + z\nu(y).$$

We take as a first approximation, in the small neighborhood of Γ, $|z| < \delta$, $u_0(x,t) = w\left(\frac{z}{\varepsilon}\right)$. In that region we get

$$S(u_0) = -\varepsilon^2 (\mathbf{a}_\Gamma(y) + z\mathbf{b}_\Gamma(y,z)) \zeta \partial_\zeta w(\zeta)\,|_{\zeta = \varepsilon^{-1} z} = O(\varepsilon^2 e^{-\frac{a|z|}{\varepsilon}}).$$

To be observed is that the fact that Γ is a Minkowskian minimal surface yields that the order of approximation on the interface is ε times better. As we will see, more than this: is will be possible to slightly modify u_0 so that the order of approximation is $O(\varepsilon^k e^{-\frac{|z|}{\varepsilon}})$ for any given $k \ge 2$. Indeed, as we will see one can find a function $u^k(y,z)$ that achieves this property in the region $|z| < \delta$ for a small δ with the form

$$u^k(x,t) = v^k(y,\zeta), \qquad \zeta = \frac{z}{\varepsilon} - h^k(y), \qquad (x,t) = Y(y) + z\nu(y) \tag{2.11}$$

where

$$v^k(y,\zeta) = w(\zeta) + \phi^k(y,\zeta), \quad \phi^k(y,\zeta) = O(\varepsilon^2 e^{-|\zeta|}). \tag{2.12}$$

For a given, sufficiently small function h defined on Γ and a function $v(y,\zeta)$ of the form

$$u(x,t) = v(y, \varepsilon^{-1} z - h(y)), \quad (x,t) = Y(y) + z\nu(y).$$

we compute

$$S(u) \; = \varepsilon^2 \Box u(x,t) + f(u(x,t)) \; = \; S(v,h)$$

where

$$\begin{aligned} S(v,h) \; &:= \partial_\zeta^2 v(y,\zeta) + f(v(y,\zeta)) \; + \; \mathcal{L}_\varepsilon(z,h)[v](y,\zeta)\,\big|_{z = \varepsilon(\zeta + h(y))}, \\ \mathcal{L}_\varepsilon(z,h)[v] \; &= \varepsilon^2 \Box_{\Gamma_z} v - \varepsilon^2 \Box_{\Gamma_z} h\, \partial_\zeta v - \varepsilon z (\mathbf{a}_\Gamma + z\mathbf{b}_\Gamma) \partial_\zeta v \\ &\quad + \varepsilon^2 \langle \nabla_{\Gamma_z} h, \nabla_{\Gamma_z} h \rangle \partial_\zeta^2 v - 2\varepsilon^2 \langle \nabla_{\Gamma_z} \partial_\zeta v, \nabla_{\Gamma_z} h \rangle \end{aligned}$$

and we have denoted, for functions $h_1(y)$, $h_2(y)$,

$$\langle \nabla_{\Gamma_z} h_1(y), \nabla_{\Gamma_z} h_2(y)\rangle \;=\; g^{ab}(y,z)\partial_a h_1(y)\partial_b h_2(y).$$

To construct a first approximation, we let $v^0(y,\zeta) = w(\zeta) + \phi^0(y,\zeta)$. Choosing $h = 0$ we get

$$\begin{aligned} S(v^0,0) \;=\; & \partial_\zeta^2\phi^0 + f'(w)\phi^0 - \varepsilon^2 \mathbf{a}_\Gamma(y)\zeta w'(\zeta) - \varepsilon^3 \mathbf{b}_\Gamma(y,\varepsilon\zeta)\zeta^2 w'(\zeta) \\ & + \varepsilon^2 \Box_{\Gamma_{\varepsilon\zeta}}\phi^0 - \varepsilon^2\left(\mathbf{a}_\Gamma(y)\zeta - \varepsilon \mathbf{b}_\Gamma(y,\varepsilon\zeta)\zeta^2\right)\partial_\zeta\phi^0 + N(\phi^0), \end{aligned} \tag{2.13}$$

where

$$N(\phi) \;=\; f(w(\zeta)+\phi) - f(w(\zeta)) - f'(w(\zeta))\phi.$$

A basic property that we will use is that the equation

$$p''(\zeta) + f'(w(\zeta))p(\zeta) + q(\zeta) = 0, \quad \zeta \in [-R,R]$$

has the solution

$$p(\zeta) = \mathcal{T}[q(\zeta)] := w'(\zeta)\int_{-R}^{\zeta} w'(s)^{-2}\left(\int_{-R}^{s} q(\tau)w'(\tau)d\tau\right) ds. \tag{2.14}$$

We have that if

$$\int_{-R}^{R} q(\tau)w'(\tau)d\tau = 0, \tag{2.15}$$

and for $j \geq 0$

$$|D^j q(\zeta)| \;\leq\; (1+|\zeta|^m)e^{-a|\zeta|}, \quad \zeta \in [-R,R],$$

then

$$|D^j p(\zeta)| \;\leq\; C_j\,(1+|\zeta|^{m+1})e^{-a|\zeta|}, \quad \zeta \in [-R,R]. \tag{2.16}$$

with C uniform in all large R. At this point we observe that since w the heteroclinic is odd by assumption, $q(\zeta) = \zeta w'(\zeta)$ satisfies (2.15) for any $R > 0$. We now let

$$\phi^0(y,\zeta) = -\varepsilon^2 \mathbf{a}_\Gamma(y)\mathcal{T}[\zeta w'(\zeta)].$$

Using (2.16) we see that for $j, l \geq 0$ we have

$$|D_y^l D_\zeta^j \phi^0(y,\zeta)| \;\leq\; C_{jl}\varepsilon^2\,(1+|\zeta|)e^{-a|\zeta|}, \quad |\zeta| \leq \frac{\delta}{\varepsilon}.$$

Moreover, the first three terms in expansion (2.13) are identically cancelled and the resulting error gets one order smaller. Indeed, we directly check that

$$|D_y^l D_\zeta^j S(v^0,0)(y,\zeta)| \;\leq\; C_{jl}\varepsilon^3\,(1+|\zeta|^2)e^{-a|\zeta|}, \quad |\zeta| \leq \frac{\delta}{\varepsilon}\ .$$

This procedure can be continued inductively but involves adjusting the function $h(y)$ to get the orthogonality conditions (2.15) satisfied. As we will see, such an adjustment will involve an equation for h that involves the Jacobi-Minkowski operator of the minimal surface Γ. More precisely we need to solve equations on Γ of the form

$$\begin{aligned} J_\Gamma[h] := \Box_\Gamma h + \mathbf{a}_\Gamma(y)h \;&= g \quad \text{in } \Gamma \\ h \;=\; \partial_t h \;&= 0 \quad \text{on } \Gamma \cap \{t=0\}. \end{aligned} \tag{2.17}$$

Lemma 2.1. *Let g a function of class $C^\infty(\Gamma)$. Then Problem (2.17) has a unique solution h which is also of class $C^\infty(\Gamma)$. Moreover for each $j \geq 0$ there are numbers m_j, C_j such that*

$$\|D_y^j h\|_{L^\infty(\Gamma)} \leq C_j \sum_{l=0}^{m_j} \|D_y^l g\|_{L^\infty(\Gamma)}.$$

The proof consists of reducing the problem to one for a standard wave-like operator. We postpone it for the appendix. Our main result in this section is the following.

Proposition 2.1. *Given $k \geq 0$ there exists smooth functions $h^k(y)$ and $\phi^k(y,\zeta)$, with ϕ^k defined in the set*

$$\mathcal{D} = \{(y,\zeta) \ : \ y \in \Gamma, \ -\frac{\delta}{2\varepsilon} < \zeta < \frac{\delta}{2\varepsilon}\},$$

such that for all $j, l \geq 0$

$$|D_y^l D_\zeta^j \phi^k(y,\zeta)| \ \leq C_{jlk}\varepsilon^2 \, (1+|\zeta|)e^{-a|\zeta|}, \quad |D_y^l D_\zeta^j h^k(y,\zeta)| \ \leq \quad C_{jk}\varepsilon \tag{2.18}$$

and for $v^k(y,\zeta) = w(\zeta) + \phi^k(y,\zeta)$ we have

$$|D_y^l D_\zeta^j S(v^k, h^k)| \leq C_{ljk}\, \varepsilon^{k+3}(1+|\zeta|^{k+2})e^{-a|\zeta|} \quad \textit{in } \mathcal{D}. \tag{2.19}$$

Proof. We proceed by induction. The case $k = 0$ has just been dealt with with the choice $h^0 = 0$. Let us assume the existence of functions h^k, ϕ^k as in (2.18)-(2.19). We will make a choice for h^{k+1}, ϕ^{k+1}.

Let us consider two functions $h(y)$ and $\phi(y,\zeta)$ with the following properties: for a certain m and each numbers j, l, there are constants C_{ljk}, C_{jk} such that for all sufficiently small ε we have

$$|D_\zeta^l D_y^j \phi(y,\zeta)| \ \leq C_{ljk}\varepsilon^{k+3}(1+|\zeta|^{k+2})e^{-a|\zeta|}, \tag{2.20}$$

$$|D_y^j h(y)| \ \leq C_{jk}\varepsilon^{k+1}. \tag{2.21}$$

We explicitly find functions that satisfy constraints of this form such that $h^{k+1} = h^k + h$ and $\phi^{k+1} = \phi^k + \phi$ reduce the error, thus completing the induction step.

We expand in the region $\mathcal{D}$,

$$\begin{aligned}
\mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k + h), h^k + h)[v^k + \phi] \ &= \mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k), h^k)[v^k] \\
&\quad + \left(\mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k + h), h^k + h) - \mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k), h^k)\right)[v_k] \\
&\quad + \mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k + h), h^k + h)[\phi] \\
&= \ \ \mathcal{L}_\varepsilon(\varepsilon(\zeta + h^k), h^k)[v^k] + \ \varepsilon^2[\Box_\Gamma h \ + \mathbf{a}_\Gamma h]\partial_\zeta w \\
&\quad + \ \Theta_1(h,\phi)
\end{aligned}$$

where the remainder $\Theta_1(h,\phi)$ satisfies

$$|D_\zeta^l D_y^j \Theta_1(h,\phi)| \ \leq \ C_{ljk}\varepsilon^{k+4}(1+|\zeta|^{k+3})e^{-|\zeta|} \tag{2.22}$$

for some constants relabeled C_{lj}. Hence we find

$$\begin{aligned}
&S_\varepsilon(v^k + \phi, h^k + h) \\
&= \partial_\zeta^2 \phi + f'(w(\zeta))\phi + S_\varepsilon(v^k, h^k) + \varepsilon^2[\Box_\Gamma h \ + \mathbf{a}_\Gamma h]\partial_\zeta w + \Theta(h,\phi)
\end{aligned} \tag{2.23}$$

where Θ satisfies an estimate of the form (2.22). Next we choose the function h: We consider $h(y)$ such that the following relation holds.

$$\int_{-\frac{\delta}{2\varepsilon}}^{\frac{\delta}{2\varepsilon}} \mathcal{E}(y,\zeta)\,\partial_\zeta w(\zeta)\,d\zeta \;=\; 0 \quad \text{for all} \quad y\in\Gamma. \tag{2.24}$$

where

$$\mathcal{E}(y,\zeta) = S_\varepsilon(v^k,h^k)(y,\zeta) + \varepsilon^2[\Box_\Gamma h(y) \,+ \mathbf{a}(y)h(y)]\partial_\zeta w(\zeta),$$

We can write this equation in the form

$$J_\Gamma[h](y) = \Box_\Gamma h(y) \,+ \mathbf{a}_\Gamma(y)h(y) = g(y) \quad \text{on } \Gamma$$

where the function $g(y)$ satisfies that for each $j\geq 0$

$$|D_y^j g(y)| \;\leq\; C_{jk}\varepsilon^{k+1} \quad \text{in } \Gamma$$

Assuming the initial conditions $h=\partial_t h = 0$ on $\Gamma\cap\{t=0\}$ we see from Lemma 2.1 that a unique solution h of this problem exists which also satisfies a bound of the form (2.21). Now, we choose $\phi(y,\zeta)$ to be the solution of the equation

$$\partial_\zeta^2\phi + f'(w(\tau))\phi + \mathcal{E}(y,\zeta) = 0, \quad |\zeta|<\frac{\delta}{\varepsilon}$$

given by

$$\phi(y,\zeta) = \mathcal{T}[\mathcal{E}(y,\zeta)]$$

with $\mathcal{T}$ as in (2.14). Using Estimate (2.16) we get that ϕ satisfies the bounds

$$|D_y^l D_\zeta^j\phi(y,\zeta)| \;\leq\; C_{jlk}\varepsilon^{k+3}\,(1+|\zeta|^{k+2})e^{-|\zeta|}, \quad |\zeta|\leq\frac{\delta}{\varepsilon}.$$

With these choices of h and ϕ made, we indeed have the validity of (2.20). Hence setting $v^{k+1}=v^k+\phi, \quad h^{k+1}=h^k+h$ we get

$$S(v^{k+1},h^{k+1}) \;=\; \Theta(h,\phi)$$

which satisfies bounds (2.22). The induction is thus complete and the proposition follows. $\square$

2.3. **The global approximation.** We have built in Proposition 2.1 an approximation to a solution of $S(u)$ =0 of the form

$$u_\varepsilon^k(x,t) = w(\varepsilon^{-1}z - h_k(y)) + \phi^k(y,\varepsilon^{-1}z-h_k(y)), \quad (x,t)=Y(y)+z\nu(y),$$

which is only defined in the small neighborhood $\mathcal{N}$ of Γ. We can obtain a globally defined approximation by just interpolating with the function $\mathbb{I}$ defined in (1.8) as follows. Let us consider a smooth, nonnegative cut-off function $\eta(s)$ such that $\eta(s)=1$ for $s<1$ and $=0$ for $s>2$, and set

$$\chi_0(x,t) = \eta\left(\frac{|z|}{r}\right), \tag{2.25}$$

where $2r<\delta$ and this function is understood as zero whenever (x,t) is outside the neighborhood of Γ of points with coordinate $|z|<\delta$ and r is a sufficiently small number which we will specify at the beginning of Section 4, Then we define

$$u_\varepsilon^*(x,t) := \chi_0(x,t)u_\varepsilon^k(x,t) + (1-\chi_0(x,t))\mathbb{I}(x,t) \tag{2.26}$$

where the number k will be chosen sufficiently large.

3. Further coordinate systems

3.1. A canonical coordinate system in Γ. Let us consider a time-like manifold Γ endowed with local parametrizations

$$(Y_l, \Lambda_l), \quad l = 1, \dots, m.$$

The tangent space to Γ at the point $Y = Y_l(y)$ is the n-dimensional space

$$T_Y\Gamma \ = \ \mathrm{Span}\, \{\partial_i Y_l(y) \ / \ i = 0, \dots, n-1\}.$$

We denote Γ^t the t-section of Γ, namely

$$\Gamma^t = \Gamma \cap \{(x,t) \ / x \in \mathbb{R}^n\}$$

We claim that if Γ^t is nonempty, it is a $n-1$ dimensional smooth manifold. Indeed, let of Γ. Writing

$$Y_l(y) = (t_l(y), x_l(y)), \quad y \in \Lambda_l \subset \mathbb{R}^n.$$

Then Γ^t is locally parametrized by the equations $t_l(y) = t$. This set is a smooth manifold. In fact, $\nabla_y t_l(y) \neq 0$ In fact if $\nabla_y t(y) = 0$ we would have that $T_Y\Gamma$ at $Y = Y_l(y)$ is just $\{0\} \times \mathbb{R}^n$. Hence an Euclidean normal vector is $e_0 = (1, 0, \dots, 0)$. That contradicts the time-like condition. The section Γ_t has an $n-1$-dimensional tangent space $T_Y\Gamma_t$ contained in $\{0\} \times \mathbb{R}^n$. We consider a vector $E(Y) \in T_Y\Gamma$ which lies in the orthogonal to $T_Y\Gamma^t$. We make the unique choice of this vector with

$$E(Y) \cdot e_0 = 1.$$

The map $Y \in \Gamma \mapsto E(Y) \in T_Y\Gamma$ defines a smooth vector field on Γ which we will use to define a natural system of local coordinates that will be helpful for computations.

Natural coordinates on Γ are those associated to *flow lines* for the vector field E. These are the trajectories of the differential equation on the manifold Γ

$$\frac{dY}{ds}(s) = E(Y(s)), \quad Y(s) \in \Gamma. \tag{3.1}$$

The meaning of this equation is given by local coordinates as $Y(s) = Y_l(y(s))$, where $y(s) \in \Lambda_l \subset \mathbb{R}^n$ solves the system of equations

$$DY_l(y(s))[\frac{dy}{ds}] = E(Y_l(y(s)))$$

or equivalently the system of ODEs

$$\frac{dy}{ds}(s) = F(y(s))$$

where

$$F(y) = [(DY_l(y))^T(DY_l(y))]^{-1}(DY_l(y))^T E(Y_l(y)).$$

For each point $Y^0 = (0, x_0) \in \Gamma^0$, Equation (3.1) has a unique solution $Y(s)$ with $Y(0) = Y^0$ which we denote as $Y(s, x_0)$. To be observed is that by definition of E, this function has the form

$$Y(t, x_0) = (t, X(t; x_0))$$

where $X(0; x_0) = x_0$.

Using this map we can define local coordinates on Γ just based on coordinates on Γ^0 We regard Γ^0 as a $n-1$ dimensional manifold in $\{0\} \times \mathbb{R}^n$. We consider a family of smooth maps $X_l^0 : V_l \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ with the functions

$$y' = (y_1, \ldots, y_{n-1}) \in V_l \mapsto (0, X_l^0(y')) \in \Gamma^0$$

defining local coordinates for Γ^0. Then the following maps define local coordinates that parametrize entire Γ. We let $T > 0$ be any number such that Γ^s is nonempty for all $0 \le s \le T$ and define

$$\Lambda_l = [0, T] \times V_l, \quad l = 1, \ldots, m$$

and consider the maps Y_l defined as

$$Y_l(y_0, y') = Y(y_0, X_l^0(y')) = (y_0, X(y_0, X_l^0(y'))) \tag{3.2}$$

Let us consider the Minkowski metric $g^0(y)$ associated to this parametrization, defined on Γ as

$$g^0_{ab}(y) \;:=\; \langle \partial_a Y_l(y), \partial_b Y_l(y) \rangle, \quad a, b = 0, \ldots n-1. \tag{3.3}$$

Lemma 3.1. *The following properties of the metric g^0 defined above hold:*

$$g^0_{0a} \;= 0, \quad a = 1, \ldots n-1, \tag{3.4}$$

$$g^0_{00} \;< 0. \tag{3.5}$$

The matrix $\bar g^0$ with coefficients

$$[\bar g^0]_{ij} = g^0_{ij} \quad i, j = 1, \ldots n-1$$

is positive definite.

Proof. To be noticed is that for $i = 1, \ldots, n-1$ and $Y = Y_l(y)$ we have that $\partial_a Y_l \in T_Y \Gamma^t$ and $\partial_0 Y_l = E(Y)$.

Since $J\partial_a Y_l = \partial_a Y_l$, then by definition of E we have that

$$g_{0a} = \langle \partial_0 Y_l, \partial_i Y_l \rangle = 0, \quad i = 1, \ldots n-1.$$

Next, let us observe that for $i, j = 1, \ldots, n-1$ we have that

$$\langle \partial_i Y_l, \partial_j Y_l \rangle = \partial_i X_l \cdot \partial_j X_l$$

Besides the $n-1$ vectors $\xi_i(t) := \partial_i X_l(t, y')$ are linearly independent. This follows from the fact that all vectors $\xi_i(t)$ are solutions of a linear system of the form

$$\frac{d\xi_i}{dt}(t) = A(t)[\xi_i(t)].$$

They are linearly independent at $t = 0$ since they are associated to local coordinates for the manifold Γ^0, and that property is preserved in time. The matrix $\Xi(t)$ whose columns are $\xi_i(t)$ is therefore non-singular, hence the matrix

$$\bar g^0(y_0, y') = \Xi(y_0)^T \Xi(y_0)$$

is positive definite.

Finally, (3.4) and the positive definiteness of $\bar g^0$ implies that g_{00} and $\det g$ have the same sign, so (3.5) follows from (2.9).

The proof is concluded. □

A nice characteristic of the local coordinates built above is that they allow to express the $\Box_\Gamma$ operator in a clean way as a second order wave operator. That leads to a clean proof of Lemma 2.1

3.2. **Proof of Lemma 2.1.** We want to solve the equation

$$\begin{aligned} \Box_\Gamma h + \mathbf{a}_\Gamma h &= g \quad \text{in } \Gamma \\ h = \partial_t h &= 0 \quad \text{on } \Gamma \cap \{t = 0\}. \end{aligned} \tag{3.6}$$

for a given function g. In local coordinates around $\Lambda_l = [0,T] \times V_l$ the equation is expressed as

$$\frac{1}{\sqrt{|\det g^0(y)|}} \partial_a \big(\sqrt{|\det g^0(y)|} g^{0,ab}(y) \partial_b h\big) + \mathbf{a}_\Gamma(y))h \;=\; q(y) \tag{3.7}$$

As customary we write $g^{0,ab}(y)$ for the entries of the matrix $(g^0(y))^{-1}$. From the previous lemma, we see that

$$(g^0(y))^{-1} = \begin{bmatrix} (g^0_{00}(y))^{-1} & 0 \\ 0 & (\bar g^0(y))^{-1} \end{bmatrix}.$$

and hence, naturally relabeling $y = (t, y')$, (3.7) can be written in the coordinate patch $[0,T] \times V_l$ in the form

$$\begin{aligned} -\partial_t^2 h + a_{ij}(t,y')\partial_{ij} h + b_0(t,y')\partial_t h + b_i(t,y')\partial_i h + \bar{\mathbf{a}}_\Gamma(t,y')h = Q(t,y'), \\ (t,y') \in [0,T] \times V_l, \\ h(0,y') = h_t(0,y') = 0, \quad y' \in V_l. \end{aligned} \tag{3.8}$$

for certain coefficients b_α, where

$$\begin{aligned} a_{ij}(t,y') &= g_{00}(t,y')\, g^{0,ij}(t,y'), \\ \bar{\mathbf{a}}_\Gamma(t,y') &= g_{00}(t,y')\; \mathbf{a}_\Gamma(t,y'), \\ Q(t,y') &= g_{00}(t,y')\, q(t,y'). \end{aligned}$$

The matrix with entries $a_{ij}(t,y)$ is uniformly positive definite. If we consider a smooth bounded domain $\bar\Omega \subset V_l$ and restrict equation (3.8) to Ω with zero boundary conditions, the standard theory for linear wave equations based on energy estimates, as developed in [12], Section 7.2 yields existence and regularity with uniform controls in Sobolev spaces of arbitrary order in terms of corresponding norms of Q. Existence of a solution of the full problem (3.6) follows from a standard argument using a partition of unity on Γ, while uniqueness is a byproduct of energy identities. That solution clearly has uniform controls as stated thanks to Sobolev embeddings. The proof is complete.

3.3. **modified Fermi coordinates.** In our later arguments, we will need to glue together estimates close to and far from Γ. In order to do this, it is convenient to introduce a new coordinate system in a neighborhood of Γ. These will coincide with Fermi coordinates near Γ and farther from Γ, they will have the property that the timelike variable coincides with the t variable of standard (x,t) coordinates.

We will mostly[1] abuse notation somewhat and not distinguish between Fermi coordinates and modified Fermi coordinates. Thus we will continue to write the coordinates as (y, z), where $y = (y_0, y') = (y_0, \ldots, y_{n-1})$. Similarly, we will generally write $g_{\alpha\beta}$ to denote the metric tensor with respect to these coordinates.

To state the main properties of this coordinate system, we first need to introduce some notation. Let

$$Y_l : \Lambda_l \to \Gamma, \quad l = 1 \ldots, m \qquad \text{for } \Lambda_l = [0, T] \times V_l \tag{3.9}$$

be the canonical local parametrizations fixed in Section 3.1. With this notation, given $T_1 < T$, the modified Fermi coordinate system will be defined locally via

$$(x, t) = \Phi_l(y, z), \qquad (y, z) \in [0, T_1] \times V_l \times (-\delta_1, \delta_1)$$

for some map $\Phi_l : [0, T_1] \times V_l \times (-\delta_1, \delta_1) \to \mathbb{R}^{1+n}$ and some $\delta_1 \le \delta$, both constructed in the proof of Lemma 3.2 below. We will choose these maps to be independent of l in the sense that

$$\text{if } Y_l(y) = Y_k(\widetilde{y}) \text{ for } y \in \Lambda_l \text{ and } \widetilde{y} \in \Lambda_k, \text{ then } \Phi_l(y, z) = \Phi_k(\widetilde{y}, z). \tag{3.10}$$

This implies that $\{\Phi_l\}_l$ will induce a well-defined function

$$\Phi : [0, T_1] \times \Gamma^0 \times (-\delta_1, \delta_1) \to \mathbb{R}^{1+n}$$

defined by setting $\Phi(y_0, X_l^0(y'), z) := \Phi_l(y_0, y', z)$. We will abuse notation somewhat and write Φ to mean either this function or else its representative Φ_l with respect to a generic local parametrization $X_l^0 : V_l \to \Gamma^0$, depending on the context.

We will use the notation $\left[\ g_{\alpha\beta}\ \right]_{\alpha,\beta=0}^{n}$ for components of the metric tensor in local coordinates:

$$g_{\alpha\beta}(y, z) := \left\langle \frac{\partial \Phi_l}{\partial y_\alpha}, \frac{\partial \Phi_l}{\partial y_\beta} \right\rangle_m, \qquad \alpha, \beta = 0, \ldots, n, \text{ with } \frac{\partial}{\partial y_n} := \frac{\partial}{\partial z}.$$

Similarly, $\left[\ g^{\alpha\beta}\ \right]_{\alpha,\beta=0}^{n}$ denotes the inverse metric tensor.

Lemma 3.2. *There exists coordinates as described above and numbers $r_2 < r_1 \le \delta_1/2$ with the following properties.*

First, $\Phi(y, z) = Y(y) + z\nu(y)$ for $|z| < r_2$, where Y is the canonical parametrization of Γ from (3.2), and thus

$$\left.\begin{aligned} g_{0i} = g^{0i} &= O(|z|) \quad \text{for } i = 1, \ldots, n \\ g_{an} = g^{an} &= 0 \quad \text{for } i = 0, \ldots, n-1 \\ g_{nn} = g^{nn} &= 1 \end{aligned}\right\} \qquad \text{when } |z| < r_2 \tag{3.11}$$

and

$$\frac{\partial}{\partial z} \det(g(y, z))\Big|_{z=0} = 0\ . \tag{3.12}$$

Second,

$$y_0 = t \quad \text{when } (x, t) = \Phi(y, z), \quad \text{if } \begin{cases} |z| \ge r_1 & \text{or} \\ y_0 = 0. \end{cases} \tag{3.13}$$

Finally,

$$\left.\begin{array}{c} g_{00},\ g^{00} < 0 \\ \left[\ g_{ij}\ \right]_{i,j=1}^{n}, \left[\ g^{ij}\ \right]_{i,j=1}^{n} \text{ are positive definite} \end{array}\right\} \tag{3.14}$$

[1]except in the proof of Lemma 3.2 in Appendix A, in which we need to distinguish carefully between the different coordinate systems.

everywhere in $[0,T_1]\times V_l\times(-\delta_1,\delta_1)$ *for all* l.

We defer the proof to Appendix A. Conclusions (3.11) and (3.12) will be immediate from our construction and from properties of Fermi coordinates noted above. Properties (3.13) will be useful when we patch together energy estimates near and far from Γ. Condition (3.14) is the point in the proof that requires the most attention. It is needed to guarantee coercivity of energy estimates computed with respect to this coordinate system.

4. Linear theory

We are interested in linear estimates associated to the operator

$$L_\varepsilon[\varphi] := \Box\varphi + \frac{1}{\varepsilon^2}f'(u^*_\varepsilon)\varphi \tag{4.1}$$

obtained by linearizing (1.1) around the global approximate solution u^*_ε constructed in Sections 2.2 and 2.3. We first introduce some notation.

Recall that the construction of modified Fermi coordinates introduced induces a map $\Phi:[0,T_1]\times\Gamma^0\times(-\delta_1,\delta_1)\to\mathbb{R}^{1+n}$. For $s\in[0,T_1]$ we will write

$$\Sigma^{nr}_s := \{\Phi(s,y',z) : y'\in\Gamma^0, |z|<\delta_1\},$$

Our standing assumptions imply that $\Gamma_s=\{\Phi(s,y',0):y'\in\Gamma^0\}$ divides $\{s\}\times\mathbb{R}^n$ into two disjoint open components, say $\mathcal{O}^+_s$ and $\mathcal{O}^-_s$, with $\mathcal{O}^-_s$ being bounded. The same thus holds for $\Gamma_{s,z}:=\{\Phi(s,y',z):y'\in\Gamma^0\}$ whenever $r_1\le|z|<\delta_1$, since then (3.13) imples that $\Gamma_{s,z}$ is a subset of $\{s\}\times\mathbb{R}^n$ that retracts onto Γ_s. For $s\in[0,T_1]$ we define

$$\begin{aligned}
\Sigma^-_s &:= \text{ the bounded component of } (\{s\}\times\mathbb{R}^n)\setminus\Gamma_{s,-r_1}\\
\Sigma^+_s &:= \text{ the unbounded component of } (\{s\}\times\mathbb{R}^n)\setminus\Gamma_{s,r_1}\\
\Sigma^{far}_s &:= \Sigma^+_s\cup\Sigma^-_s\\
\Sigma_s &:= \Sigma^{nr}_s\cup\Sigma^{far}_s\\
\Sigma &:= \cup_{s\in[0,T_1]}\Sigma_s\,.
\end{aligned}$$

For $0<\rho\le\delta_1$ we will also use the notation

$$\mathcal{N}_\rho = \{\Phi(y,y',z) : (y_0,y',z)\in[0,T_1]\times\Gamma^0\times(-\rho,\rho)\}. \tag{4.2}$$

Next, we specify that the cutoff function χ_0 in the definition of u^*_ε satisfies

$$\chi_0=1 \text{ in } \mathcal{N}_{r_2/4}, \qquad \chi_0=0 \text{ in } \Sigma\setminus\mathcal{N}_{r_2/2}. \tag{4.3}$$

The exponential decay of the local approximate solution u_ε away from Γ implies that for every l, $\sup_{\mathcal{N}_{\delta_1}}|D^l(u_\varepsilon-u^*_\varepsilon)|\le Ce^{-c/\varepsilon}$ for suitable constants C,c (depending on l). Hence in $\mathcal{N}_{\delta_1}$, writing u^*_ε as a function of modified Fermi coordinates (y,z),

$$u^*_\varepsilon(y,z)=w_\varepsilon(y,z)+\phi(y,z), \qquad w_\varepsilon(y,z)=w\left(\frac{z}{\varepsilon}-h(y)\right) \tag{4.4}$$

where w is the heteroclinic (1.3) and h,ϕ satisfy (2.18). We will prove

Proposition 4.1. *Given a smooth function* $\eta\in L^2(\Sigma)$ *and smooth data* $(\varphi_0,\varphi_1)\in H^1\times L^2(\mathbb{R}^n)$*, there exists a smooth solution* $\varphi:\Sigma\to\mathbb{R}$ *to the initial value problem*

$$L_\varepsilon[\varphi]=\eta \quad \text{in } \Sigma, \qquad (\varphi,\partial_t\varphi)\Big|_{t=0}=(\varphi_0,\varphi_1). \tag{4.5}$$

In addition, there exist $C, \varepsilon_0 > 0$, depending only on Γ and T_1 and δ_1, such that and for every $s \in [0, T_1]$ and $\varepsilon \in (0, \varepsilon_0)$, we have the estimate

$$\begin{aligned}\int_{\Sigma_s} \varepsilon^2 \left(|\nabla_x \varphi|^2 + (\partial_t \varphi)^2]\right) + \varphi^2 \\ \le C \int_0^s \Big(\int_{\Sigma_\sigma} \eta^2 \Big) d\sigma + C \int_{\mathbb{R}^n} \left[|\nabla_x \varphi_0|^2 + |\varphi_1|^2 + \frac{1}{\varepsilon^2} \varphi_0^2 \right] dx.\end{aligned} \tag{4.6}$$

In (4.6), the integrals over Σ_s are with respect to the induced Euclidean n-dimensional volume. (In fact we will employ a variety of different n-forms in our arguments, but all of them are uniformly comparable to the Euclidean n-volume.)

The point of the proposition is the estimate; existence of a solution is standard and we will not discuss it.

Note that even when $\eta = 0$, the estimate as stated allows the terms on the left-hand side to be larger by a factor of ε^{-2} than the corresponding terms on the right-hand side. In fact our proof yields a sharper estimate, see Remark 4.1. However, (4.6) is sufficient for our later purposes.

Proof of Proposition 4.1. **1**. Our overall aim is to construct some quantity $E(s)$ that controls the left-hand side of (4.6), and that satisfies a differential inequality allowing for the application of Grönwall's inequality. This quantity will be constructed by integrating an energy density over Σ_s with respect to well-chosen n-forms. We will treat Σ_s^{nr} and Σ_s^{far} separately.

We start by describing the n-form we will employ on Σ_s^{nr}. First, fix a volume form on Γ^0, which we will write in local coordinates as $\omega^0(y')dy'$. Let $\chi^{nr} : \mathbb{R} \to [0, 1]$ be a smooth function such that

$$-C(r_1) \le \partial_z \chi^{nr} \le 0, \qquad \chi^{nr}(z) = 1 \text{ for } z \le r_1, \qquad \chi^{nr}(z) = 0 \text{ for } z \ge 2r_1$$

for r_1 defined in (3.13). Then we define ω_s^{nr} to be the n-form on Σ_s^{nr} written in local coordinates as

$$\omega_s^{nr} = \omega^0(y')\chi^{nr}(z) dy'\, dz.$$

Thus for any $s \in [0, T_1]$ and function $f = f(s, y', z)$ on $\Sigma_s^{nr} \cong \{s\} \times \Gamma^0 \times (-\delta_1, \delta_1)$,

$$\int_{\Sigma_s^{nr}} f \omega_s^{nr} = \int_{-\delta_1}^{\delta_1} \int_{y' \in \Gamma^0} f(s, y', z)\, \omega^0(y') dy'\, \chi^{nr}(z)\, dz.$$

We next define ω_s^{far} as the n-form on Σ_s^{far} written in (x, t) coordinates on as

$$\omega_s^{far} := \chi^{far}(s, x)\, dx$$

where $\chi^{far} \in C^\infty(\mathcal{O})$ satisfies

$$\chi^{far} = 1 \text{ outside of } \mathcal{N}_{\delta_1}$$

and in $\mathcal{N}_{\delta_1}$, writing χ^{far} as a function of modified Fermi coordinates (y, z),

$$\chi^{far}(y, z) = 1 - \chi^{nr}(z).$$

Finally we will define $\omega_s := \omega_s^{far} + \omega_s^{nr}$, an n-form on Σ_s that is uniformly comparable to the induced Euclidean n-volume.

2. We now derive an energy identity in modified Fermi coordinates near Γ. In doing so, we will write the solution φ of (4.5) as a function of modified Fermi coordinates (y, z), and we will identify $\frac{\partial}{\partial y_n}$ with $\frac{\partial}{\partial z}$. In these coordinates, we find[2] from (2.4) that (4.5) has the form

$$\frac{1}{\sqrt{|\det g|}}\frac{\partial}{\partial y_\alpha}\left(\sqrt{|\det g|}g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\right)+\frac{1}{\varepsilon^2}f'(u^*_\varepsilon)\varphi=\eta.$$

We rewrite this as

$$\frac{1}{\omega^0}\frac{\partial}{\partial y_\alpha}\left(\omega^0 g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\right)+b^\beta\frac{\partial\varphi}{\partial y_\beta}+\frac{1}{\varepsilon^2}f'(u^*_\varepsilon)\varphi=\eta,$$

for

$$b^\beta:=\frac{\omega^0}{\sqrt{|\det g|}}g^{\alpha\beta}\frac{\partial}{\partial y_\alpha}\left(\frac{\sqrt{|\det g|}}{\omega^0}\right).$$

We multiply this equation by $\partial_{y_0}\varphi$ and rewrite. This gives rise to a number of terms. An easy term is

$$\varepsilon^{-2}f'(u^*_\varepsilon)\varphi\frac{\partial\varphi}{\partial y_0}=\varepsilon^{-2}\frac{\partial}{\partial y_0}\left(\frac{1}{2}f'(u^*_\varepsilon)\varphi^2\right)-\varepsilon^{-2}\frac{\varphi^2}{2}\frac{\partial}{\partial y_0}(f'(u^*_\varepsilon)),$$

The error term $b^\beta\partial_\beta\varphi\,\partial_0\varphi$ we keep as it is. The leading term is rewritten as follows:

$$\begin{aligned}\frac{1}{\omega^0}\frac{\partial}{\partial y_\alpha}\left(\omega^0 g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\right)\frac{\partial\varphi}{\partial y_0}&=\frac{1}{\omega^0}\frac{\partial}{\partial y_\alpha}\left(\omega^0 g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)-g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\;\frac{\partial^2\varphi}{\partial y_\alpha\partial y_0}\\&=\frac{1}{\omega^0}\frac{\partial}{\partial y_\alpha}\left(\omega^0 g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)-\frac{1}{2}\frac{\partial}{\partial y_0}(g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_\alpha})\\&\qquad+\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial y_0}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}.\end{aligned}$$

We substitute these computations into the equation, write $\frac{\partial}{\partial y_\alpha}(\cdots)$ as $\frac{\partial}{\partial y_0}(\cdots)+\frac{\partial}{\partial y_i}(\cdots)$, where i runs from 1 to n, and rearrange to obtain

$$\begin{aligned}\frac{\partial}{\partial y_0}&\left(-g^{0\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}+\frac{1}{2}g^{\alpha\beta}\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta}-\frac{1}{2\varepsilon^2}f'(v)\varphi^2\right)\\&=\frac{1}{\omega^0}\frac{\partial}{\partial y_i}\left(\omega^0 g^{i\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)+\frac{1}{2}(\frac{\partial}{\partial y_0}g^{\alpha\beta})\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta}\\&\qquad-\frac{\varphi^2}{2\varepsilon^2}\frac{\partial}{\partial y_0}(f'(u^*_\varepsilon))+b^\beta\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}-\eta\frac{\partial\varphi}{\partial y_0}.\end{aligned}\tag{4.7}$$

We introduce the tensor $(a^{\alpha\beta})$, defined by

$$a^{00}=-g^{00},\qquad a^{0i}=a^{i0}=0\qquad a^{ij}=g^{ij}\tag{4.8}$$

for $i, j = 1, \ldots, n$. It is then easy to check that $(a^{\alpha\beta})$ is positive definite, and that

$$-g^{\alpha\beta}\xi_0\xi_\beta+\frac{1}{2}g^{\alpha\beta}\xi_\alpha\xi_\beta=\frac{1}{2}a^{\alpha\beta}\xi_\alpha\xi_\beta\qquad\text{for all }\xi.\tag{4.9}$$

Note also that the quadratic form $a^{ab}\partial_a\varphi\,\partial_b\varphi$ does not depend on the choice of local coordinates on Γ^0. We will write

$$e^{nr}_\varepsilon(\varphi)=\left(\frac{1}{2}a^{\alpha\beta}\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta}-f'(u^*_\varepsilon)\frac{\varphi^2}{2\varepsilon^2}\right).\tag{4.10}$$

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[2] In the discussion that contains (2.4), we were interested in Fermi coordinates, but (2.4) is completely general, and the particular choice of coordinates was used only later. In any case this is standard.

With this notation, (4.7) becomes

$$\frac{\partial}{\partial y_0}\Big(e_\varepsilon^{nr}(\varphi)\Big) = \frac{1}{\omega^0}\frac{\partial}{\partial y_i}\left(\omega^0 g^{i\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right) + \frac{1}{2}(\frac{\partial}{\partial y_0}g^{\alpha\beta})\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta} - \frac{\varphi^2}{2\varepsilon^2}\frac{\partial}{\partial y_0}(f'(u_\varepsilon^*)) + b^\beta\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0} - \eta\frac{\partial\varphi}{\partial y_0}. \tag{4.11}$$

We will also write

$$E_\varepsilon^{nr}(s;\varphi) = E_\varepsilon^{nr}(s) = \int_{\Sigma_s^{nr}} e_\varepsilon^{nr}(\varphi)\ \omega_s^{nr} = \int_{\Sigma_s^{nr}} e_\varepsilon^{nr}(\varphi)\omega^0(y')\chi^{nr}(z)dy'\,dz$$

3. We will next integrate (4.11) with respect to the n-form ω_s^{nr} over Σ_s^{nr}. First note that since ω^0 and χ^{nr} are independent of y_0,

$$\int_{\Sigma_s^{nr}}\frac{\partial}{\partial y_0}\Big(e_\varepsilon^{nr}(\varphi)\Big)\omega_s^{nr} = \int_{-\delta_1}^{\delta_1}\int_{\Gamma^0}\frac{\partial}{\partial y_0}\Big(e_\varepsilon^{nr}(\varphi)\Big)\chi^{nr}(z)\omega^0(y')\,dy'\,dz\Big|_{y_0=s} = \frac{d}{ds}E_\varepsilon^{nr}(s).$$

Next, for every $z\in(-\delta_1,\delta_1)$, let $X(\cdot;z)$ denote the vector field on Γ^0 whose ith component in local coordinates on Γ^0 is given by $X^i(y') = g^{i\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}(s,y',z)$. For every fixed z, the divergence of $X(\cdot,z)$ on Γ^0 with respect to the $n-1$- form $\omega^0(y')dy'$ is

$$\mathrm{div}_{\omega^0} X = \frac{1}{\omega^0}\sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}(\omega^0 X^i)$$

Since Γ_0 is a compact manifold without boundary, $\int_{\Gamma_0}(\mathrm{div}_{\omega^0} X)\ \omega^0(y')dy' = 0$. Thus

$$\int_{\Sigma_s^{nr}}\frac{1}{\omega^0}\frac{\partial}{\partial y_i}\left(\omega^0 g^{i\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)\omega_s^{nr} = \int_{-\delta_1}^{\delta_1}\int_{\Gamma^0}\frac{\partial}{\partial z}\left(\omega^0 g^{n\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)dy'\,\chi^{nr}(z)\,dz\Big|_{y_0=s} = -\int_{-\delta_1}^{\delta_1}\int_{\Gamma^0} g^{n\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}(\chi^{nr})'(z)\omega^0(y')dy'\ dz\Big|_{y_0=s}.$$

By integrating (4.11) we thus obtain

$$\begin{aligned}\frac{d}{ds}E_\varepsilon^{nr}(s) = &-\int_{\delta_1}^{\delta_1}\int_{\Gamma^0} g^{n\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}(\chi^{nr})'(z)\omega^0(y')dy'dz\Big|_{y_0=s}\\ &+\int_{\Sigma_s^{nr}}\left(-\frac{\partial}{\partial y_0}(f'(u_\varepsilon^*))\frac{\varphi^2}{2\varepsilon^2} + b^\beta\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}\right)\omega_s^{nr}\\ &+\int_{\Sigma_s^{nr}}\left(\frac{1}{2}(\partial_0 g^{\alpha\beta})\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta} - \eta\,\frac{\partial\varphi}{\partial y_0}\right)\omega^{nr}.\end{aligned} \tag{4.12}$$

4. We next derive a (completely standard) parallel identity far from Γ.

The counterpart of $(a^{\alpha\beta})$, defined as in (4.8), but starting from the Minkowski metric tensor J in standard (x,t) coordinates, is just the identity tensor (δ^{ab}). We thus define

$$e_\varepsilon^{far}(\varphi) := \frac{1}{2}\left[(\partial_t\varphi)^2 + |\nabla_x\varphi|^2 - f'(u_\varepsilon^*)\frac{\varphi^2}{\varepsilon^2}\right] = \frac{1}{2}\left[(\partial_t\varphi)^2 + |\nabla_x\varphi|^2 + \frac{\sigma}{\varepsilon^2}\varphi^2\right]$$

where $\sigma = -f'(\pm 1)$ and we have used the fact that $u^*_\varepsilon = \pm 1$ in Σ^{far}. We also set

$$E^{far}_\varepsilon(s;\varphi) := E^{far}_\varepsilon(s) = \int_{\Sigma^{far}_s} e^{far}_\varepsilon(\varphi)\ \omega^{far}_s.$$

Then arguments like those in the derivation of (4.12), but significantly easier, lead to the identity

$$\begin{aligned}\frac{d}{ds}E^{far}_\varepsilon(s) = -\int_{\Sigma^{far}_s} \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x_i}\frac{\partial\chi^{far}}{\partial x_i}dx + \int_{\Sigma^{far}_s} e^{far}_\varepsilon(\varphi)\frac{\partial\chi^{far}}{\partial t}\,dx \\ + \int_{\Sigma^{far}_s} -\eta\,\frac{\partial\varphi}{\partial y_0}\chi^{far}(s,x)\,dx.\end{aligned} \tag{4.13}$$

5. As mentioned earlier, we plan to construct a quantity $E(s)$ which satisfies a differential inequality allowing for the application of Grönwall's inequality. This will have the form

$$E(s) := E^{nr}_\varepsilon(s) + E^{far}_\varepsilon(s) + \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma(s,y')^2\omega^0(y')dy \tag{4.14}$$

where $\gamma : \Gamma \to \mathbb{R}$ is a function defined in (4.19) below, arising as a component in a decomposition of φ, and C is also fixed below. These are needed to guarantee that $E(s)$ bounds suitable norms of φ; this is not completely straightforward, since the quantity $-\varepsilon^{-2}f'(u^*_\varepsilon)\varphi^2$ appearing in $E^{nr}_\varepsilon(s)$ is negative in places.

We next derive the relevant bounds. In doing so, we will define γ and fix the constant C in (4.14).

It is convenient to decompose $E^{nr}_\varepsilon(s)$ into pieces. Recall from Lemma 3.2 that modified Fermi coordinates coincide with actual Fermi coordinates in $\mathcal{N}_{r_2}$. To take advantage of this, we fix $\chi_1 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$|\partial_z\chi_1| \le C(r_2), \qquad \chi_1(z) = 1 \text{ for } |z| \le \frac{1}{2}r_2\,, \qquad \chi_1(z) = 0 \text{ for } |z| \ge r_2\,.$$

We then split $E^{nr}(s)$ into two pieces as follows.

$$I_1 := \int_{\Sigma^{nr}_s} e^{nr}_\varepsilon(\varphi)\chi_1^2(z)\,\omega^{nr}_s, \qquad I_2 := \int_{\Sigma^{nr}_s} e^{nr}_\varepsilon(\varphi)(1-\chi_1^2(z))\,\omega^{nr}_s.$$

Concerning I_2, we only note that we have arranged in (2.26), (4.3) that $u^*_\varepsilon = \pm 1$, and hence $-f'(u^*_\varepsilon) = \sigma$ when $|z| \ge r_2/2$, so

$$I_2 = \frac{1}{2}\int_{\Sigma^{nr}_s}\left(a^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\varphi + \frac{\sigma}{\varepsilon^2}\varphi^2\right)(1-\chi_1^2(x))\omega^{nr}_s. \tag{4.15}$$

Next, it follows from (4.8) and properties of Fermi coordinates, see (3.11), that

$$I_1 = \frac{1}{2}\int_{\Gamma^0}\int_{-r_2}^{r_2}\left(a^{ab}\partial_a\varphi\partial_b\varphi + (\partial_z\varphi)^2 - \frac{1}{\varepsilon^2}f'(u^*_\varepsilon)\varphi^2\right)\chi_1^2(z)\,\omega^0(y')\,dz\,dy'.$$

(For the duration of the estimate of I_1, all integrals are evaluated at $y_0 = s$.) We define $\bar\varphi(y', z) := \varphi(y', z)\chi_1(z)$. Then

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\Gamma^0}\int_{-r_2}^{r_2} a^{ab}\partial_a\bar\varphi\ \partial_b\bar\varphi\ \omega^0(y')\,dz\,dy' \\ &\quad + \frac{1}{2}\int_{\Gamma^0}\int_{-r_2}^{r_2}\left((\partial_z\bar\varphi)^2 - \frac{1}{\varepsilon^2}f'(u_\varepsilon^*)\bar\varphi^2\right)\omega^0(y')\,dz\,dy' \\ &\quad - \int_{\Gamma^0}\int_{-r_2}^{r_2}\left(\frac{1}{2}(\chi_1')^2\varphi^2 + \chi_1\,\chi_1'\,\varphi\,\partial_z\varphi\right)\omega^0(y')\,dz\,dy' \\ &= I_{1,1} + I_{1,2} - I_{1,3}. \end{aligned} \tag{4.16}$$

It is clear that $I_{1,1}$ is positive definite. We write $\varphi\partial_z\varphi = \frac{1}{2}\partial_z(\varphi^2)$ and integrate by parts to obtain

$$|I_{1,3}| \le C\int_{\Gamma^0}\int_{-r_2}^{r_2}\varphi^2\omega^0(y')\,\chi^{nr}(z)\,dz\,dy'\ . \tag{4.17}$$

Since $\varphi^2 = \chi_1^2\varphi^2 + (1-\chi_1^2)\varphi^2 = \bar\varphi^2 + (1-\chi_1^2)\varphi^2$, it follows from (4.15) that

$$|I_{1,3}| \le C\left(\varepsilon^2 I_2 + \int_{\Gamma^0}\int_{-r_2}^{r_2}\bar\varphi^2\omega^0(y')\,dz\,dy'\right).$$

We now split $\bar\varphi$ as

$$\bar\varphi(y,z) := \bar\varphi^\perp(y,z) + \gamma(y)\partial_z w_\varepsilon(y,z), \tag{4.18}$$

where $w_\varepsilon(y,z) = w(\frac{z}{\varepsilon} - h(y))$ and

$$\gamma(y) := \frac{\varepsilon}{\Xi}\int_{\mathbb{R}}\bar\varphi(y,z)\partial_z w_\varepsilon(y,z)dz, \qquad\qquad \Xi := \int_{\mathbb{R}} w'^2(\zeta)d\zeta. \tag{4.19}$$

The definition implies that

$$\int_{\mathbb{R}}\bar\varphi^\perp(y,z)\partial_z w_\varepsilon(z)dz = 0 \tag{4.20}$$

for all y and hence that

$$\int_{\Sigma_s^{nr}}\bar\varphi^2\omega_s^{nr} = \int_{\Gamma^0}\int_{\mathbb{R}}(\bar\varphi^\perp)^2\omega^0(y')\,dz\,dy'\ + \frac{\Xi}{\varepsilon}\int_{\Gamma^0}\gamma^2\omega^0(y')dy'. \tag{4.21}$$

In terms of $\bar\varphi^\perp$ and γ, our above estimate of $I_{1,3}$ takes the form

$$|I_{1,3}| \le C\varepsilon^2 I_2 + C\int_{\Gamma^0}\int_{\mathbb{R}}(\bar\varphi^\perp)^2 dz\,\omega^0(y')\,dy' + \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma^2\omega^0(y')dy'\ . \tag{4.22}$$

Turning to $I_{1,2}$, and omitting "dz" and "$\omega^0(y')dy'$" when no confusion can result, we now set $f'(u_\varepsilon^*) = f'(\pm 1) = \sigma$ for $|z| \ge r_2$, and we rewrite

$$\begin{aligned} I_{1,2} &= \frac{1}{2}\int_{\Gamma^0}\int_{\mathbb{R}}(\partial_z\bar\varphi^\perp)^2 - \frac{1}{\varepsilon^2}f'(u_\varepsilon^*)(\bar\varphi^\perp)^2 \\ &\qquad + \int_{\Gamma^0}\int_{\mathbb{R}}\gamma\ \partial_z^2 w_\varepsilon\,\partial_z(\bar\varphi^\perp + \frac{\gamma}{2}\partial_z w_\varepsilon) - \frac{1}{\varepsilon^2}f'(u_\varepsilon^*)\gamma\partial_z w_\varepsilon(\bar\varphi^\perp + \frac{\gamma}{2}\partial_z w_\varepsilon). \\ &= I_{1,2,1} + I_{1,2,2}. \end{aligned}$$

We integrate by parts in the z variable and use the fact that $\partial_z^3 w_\varepsilon + \varepsilon^{-2}f'(w_\varepsilon)\partial_z w_\varepsilon = 0$ to find that

$$I_{1,2,2} = \int_{\Gamma^0}\int_{\mathbb{R}}\frac{1}{\varepsilon^2}(f'(w_\varepsilon) - f'(u_\varepsilon^*))\,\gamma\,\partial_z w_\varepsilon\,(\bar\varphi^\perp + \frac{\gamma}{2}\partial_z w_\varepsilon).$$

It follows from (4.4), (2.18) that $|f'(w_\varepsilon) - f'(u^*_\varepsilon)| \le C\varepsilon^2$ everywhere, and as a result,

$$|I_{1,2,2}| \le C\int_{\Gamma^0}\int_{\mathbb{R}}(|\gamma\partial_z w_\varepsilon|\,|\bar\varphi^\perp| + \gamma^2(\partial_z w_\varepsilon)^2) \le \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma^2 + C\int_{\Gamma^0}\int_{\mathbb{R}}(\bar\varphi^\perp)^2. \quad (4.23)$$

Next, because $\int \bar\varphi^\perp\partial_z w_\varepsilon = 0$, it follows from (1.4) that there exists some $c > 0$ such that

$$\int_{\mathbb{R}}(\partial_z\bar\varphi^\perp)^2 - \frac{1}{\varepsilon^2}f'(w_\varepsilon)(\bar\varphi^\perp)^2\,dz \ \ge c\int_{\mathbb{R}}(\partial_z\bar\varphi^\perp)^2 + \frac{1}{\varepsilon^2}(\bar\varphi^\perp)^2dz$$

for every y'. Arguing as with (4.23), we infer that

$$\begin{aligned} I_{1,2,1} \ge c\int_{\Gamma^0}\int_{\mathbb{R}}&(\partial_z\bar\varphi^\perp)^2 + \frac{1}{\varepsilon^2}(\bar\varphi^\perp)^2\,\omega^0(y')\,dz\,dy' \\ &+ \frac{1}{2}\int_{\Gamma^0}\int_{\mathbb{R}}\frac{1}{\varepsilon^2}(f'(w_\varepsilon) - f'(u^*_\varepsilon))(\bar\varphi^\perp)^2\,\omega^0(y')\,dz\,dy' \\ \ge c\int_{\Gamma^0}\int_{\mathbb{R}}&(\partial_z\bar\varphi^\perp)^2 + \frac{1}{\varepsilon^2}(\bar\varphi^\perp)^2\,\omega^0(y')\,dz\,dy' \end{aligned} \quad (4.24)$$

for ε sufficiently small.

Next, we consider $I_{1,1}$. By compactness, there exists some $c > 0$ such that, if we write $a_0^{ab}(y) := a^{ab}(y,z)$, then for every $\xi = (\xi_0,\dots,\xi_{n-1}) \in \mathbb{R}^n$,

$$ca_0^{ab}(y)\xi_a\xi_b \le a^{ab}(y,z)\xi_a\xi_b \le c^{-1}a_0^{ab}(y)\xi_a\xi_b \qquad \text{for all } (y,z) \in [0,T_1]\times\Gamma^0\times(-\delta_1,\delta_1).$$

Then noting that $\partial_b\partial_z w_\varepsilon(y,z) = -\frac{1}{\varepsilon}w''(\frac{z}{\varepsilon} - h(y))\partial_b h(y) = -\varepsilon\,\partial_{zz}w_\varepsilon\,\partial_b h$,

$$\begin{aligned} I_{1,1} \ge \frac{c}{2}\int_{\mathbb{R}}\int_{\Gamma_0} &a_0^{ab}(y)\left[\partial_a\bar\varphi^\perp\,\partial_b\bar\varphi^\perp + \partial_a\gamma\,\partial_b\gamma(\partial_z w_\varepsilon)^2 + \varepsilon^2\gamma^2\,(\partial_{zz}w_\varepsilon)^2\,\partial_a h\,\partial_b h\right]\omega^0(y')dy'dz \\ &+ c\int_{\mathbb{R}}\int_{\Gamma_0} a_0^{ab}(y)[\partial_a\bar\varphi^\perp\partial_b\,\gamma\,\partial_z w_\varepsilon - \varepsilon\,\partial_a\bar\varphi^\perp\gamma\,\partial_{zz}w_\varepsilon\,\partial_b h]\omega^0(y')dy'dz \\ &- c\int_{\Gamma_0}\int_{\mathbb{R}} a_0^{ab}\,\gamma\,\partial_a\gamma\partial_b h\varepsilon\,\partial_z w_\varepsilon\,\partial_{zz}w_\varepsilon\,\omega^0(y')dy'\,dz \end{aligned}$$

The last term vanishes because w is odd. In any coordinate chart we can differentiate the orthogonality condition (4.20) to find that

$$0 = \partial_a\int_{\mathbb{R}}\bar\varphi^\perp\partial_z w_\varepsilon\,dz = \int_{\mathbb{R}}\partial_a\bar\varphi^\perp\partial_z w_\varepsilon\,dz - \int_{\mathbb{R}}\varepsilon\bar\varphi^\perp\partial_{zz}w_\varepsilon\partial_a h\,dz.$$

Using this we can rewrite the middle term as

$$-c\int_{\Gamma_0}a_0^{ab}(y)\int_{\mathbb{R}}\varepsilon\,\partial_{zz}w_\varepsilon\partial_b h(\bar\varphi^\perp\partial_a\,\gamma\, + \partial_a\bar\varphi^\perp\gamma)\omega^0(y')dy'dz.$$

For any f, we will write $|D_y f|^2 = a^{ab}\partial_a f\,\partial_b f$. Since $\int_{\mathbb{R}}(\partial_{zz}w_\varepsilon)^2\,dz \le C\varepsilon^{-3}$ and $|D_y h| \le C\varepsilon$, the above is bounded in absolute value by

$$C\varepsilon\int_{\Gamma^0}\int_{\mathbb{R}}(\bar\varphi^\perp)^2 + |D_y\bar\varphi^\perp|^2\omega^0(y')dy'dz + C\int_{\Gamma^0}\gamma^2 + |D_y\gamma|^2\omega^0(y')dy'.$$

Clearly $|D_y\bar\varphi^\perp|^2 + (\partial_z\bar\varphi^\perp)^2 \approx |D\bar\varphi^\perp|^2 = (\partial_t\bar\varphi^\perp)^2 + |\nabla_x\bar\varphi^\perp|^2$, so we may combine the above terms from $I_{1,1}$ with other terms estimated in (4.22), (4.23), (4.24), to find that for ε sufficiently small,

$$\begin{aligned} I_1 \geq -C\varepsilon^2 I_2 - \frac{C}{\varepsilon}\int_{\Gamma^0} \gamma^2\,\omega^0(y')\,dy' + \frac{c}{\varepsilon}\int_{\Gamma^0} |D_y\gamma|^2\ \omega^0(y')\,dy' \\ + c\int_{\Gamma^0}\int_{\mathbb{R}} |D\bar\varphi^\perp|^2 + \frac{1}{\varepsilon^2}(\bar\varphi^\perp)^2\,\omega^0(y')\,dz\,dy'\ . \end{aligned} \tag{4.25}$$

In view of this and (4.15), we may fix a particular constant C such that if we define $E(s)$ as in (4.14), which we recall is

$$E(s) := E^{nr}_\varepsilon(s) + E^{far}_\varepsilon(s) + \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma^2(s,y')\omega^0(y')dy',$$

then for every $s\in[0,T_1]$, after adjusting c we have

$$\begin{aligned} E(s) \geq \frac{c}{\varepsilon}\int_{\Gamma^0}\left(\gamma^2 + |D_y\gamma|^2\right)\omega^0(y')dy'\Big|_{y_0=s} + c\int_{\Sigma_s}(1-\chi_1^2)\left(|D\varphi|^2 + \frac{1}{\varepsilon^2}\varphi^2\right)\omega_s \\ c\int_{\Gamma^0}\int_{\mathbb{R}}|D\bar\varphi^\perp|^2 + \frac{1}{\varepsilon^2}(\bar\varphi^\perp)^2\,\omega^0(y')\,dz\,dy'\ . \end{aligned} \tag{4.26}$$

for $\omega_s := \omega_s^{nr} + \omega_s^{far}$. (We have extended χ_1 to a function defined on all of Σ_s, and vanishing on Σ_s^{far}.)

6. We now want to show that

$$\frac{d}{ds}E(s) \leq CE(s).$$

We first consider terms arising from $\frac{d}{ds}E^{nr}_\varepsilon(s)$. We rewrite the localized energy identity (4.12) as

$$\frac{d}{ds}E^{nr}_\varepsilon(s) = \int_{\Sigma_s^{nr}}\mathcal{F}\,\omega_s^{nr} - \int_{\delta_1}^{\delta_1}\int_{\Gamma^0} g^{n\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}(\chi^{nr})'(z)\omega^0(y')dy'dz\Big|_{y_0=s}$$

where

$$\mathcal{F} := -\frac{\partial}{\partial y_0}(f'(u^*_\varepsilon))\frac{\varphi^2}{2\varepsilon^2} + b^\beta\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0} + \frac{1}{2}(\partial_0 g^{\alpha\beta})\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta} - \eta\,\frac{\partial\varphi}{\partial y_0}.$$

Recall that $\chi_1 = 0$ in $\operatorname{supp}(\chi^{nr})' \subset \{(y,z) : 2r_1 \leq z \leq 4r_1\}$. It follows that

$$\left|\int_{\delta_1}^{\delta_1}\int_{\Gamma^0} g^{n\beta}\frac{\partial\varphi}{\partial y_\beta}\frac{\partial\varphi}{\partial y_0}(\chi^{nr})'(z)\omega^0(y')dy'dz\right|_{y_0=s} \leq C\int_{\Sigma_s}(1-\chi_1^2)|D\varphi|^2\omega_s \leq CE(s).$$

Straightforward estimates show that

$$|\mathcal{F}| \leq C\left(|D\varphi|^2 + \frac{1}{\varepsilon^2}\varphi^2\right) + C\eta^2 \text{ on the support of } 1-\chi_1^2.$$

(In particular, the construction of u^*_ε implies that $\partial_0 f'(u^*_\varepsilon) = 0$ on this set.) We use (4.26) to deduce that

$$\frac{d}{ds}E^{nr}_\varepsilon(s) = \int_{\Sigma_s^{nr}}\mathcal{F}\ \chi_1^2(z)\omega_s^{nr} + C\int_{\Sigma_s^{nr}}(1-\chi_1^2)\eta^2\omega_s^{nr} + CE(s). \tag{4.27}$$

We now consider various terms in the first integral on the right-hand side above. First, again writing $\chi_1\varphi = \bar\varphi = \bar\varphi^\perp + \gamma\,\partial_z w_\varepsilon$, we decompose

$$\int_{\Sigma_s^{nr}}\partial_0(f'(u^*_\varepsilon))\frac{\varphi^2}{2\varepsilon^2}\chi_1^2\omega_s^{nr} = I_{3,1} + I_{3,2}$$

where

$$I_{3,1} = \frac{1}{2\varepsilon^2}\int_{\Sigma_s^{nr}} \partial_0(f'(u_\varepsilon^*))[(\bar\varphi^\perp)^2 + 2\bar\varphi^\perp\gamma\,\partial_z w_\varepsilon]\omega_s^{nr},$$
$$I_{3,2} = \frac{1}{2\varepsilon^2}\int_{\Sigma_s^{nr}} \partial_0(f'(u_\varepsilon^*))\gamma^2(\partial_z w_\varepsilon)^2\omega_s^{nr}.$$

It follows from (4.4), (2.18) that

$$\partial_0(f'(u_\varepsilon^*)) = (f''(w_\varepsilon)+O(\varepsilon^2))(\varepsilon\partial_z w_\varepsilon\,\partial_0 h + \partial_0\phi) = f''(w_\varepsilon)\varepsilon\partial_z w_\varepsilon\,\partial_0 h + O(\varepsilon^2). \tag{4.28}$$

Moreover, $|f''(w_\varepsilon)\varepsilon\partial_z w_\varepsilon\,\partial_0 h| \le C\varepsilon$. It follows that

$$\begin{aligned}|I_{3,1}| &\le \frac{C}{\varepsilon}\int_{\Sigma_s^{nr}} (\bar\varphi^\perp)^2 + |\bar\varphi^\perp|\,|\gamma\partial_z w_\varepsilon|)\omega_s^{nr} \le C\int_{\Sigma_s^{nr}} \frac{(\bar\varphi^\perp)^2}{\varepsilon^2} + \gamma^2(\partial_z w_\varepsilon)^2\omega_s^{nr}\\ &\le \frac{C}{\varepsilon^2}\int_{\Sigma_s^{nr}} (\bar\varphi^\perp)^2\omega_s^{nr} + \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma^2(s,y')\,\omega^0(y')dy'\\ &\overset{(4.26)}{\le} CE(s).\end{aligned}$$

We next use (4.28) to write

$$I_{3,2} = \frac{1}{2\varepsilon}\int_{\Sigma_s^{nr}} [f''(w_\varepsilon)\partial_z w_\varepsilon\,\partial_0 h + O(\varepsilon)]\gamma^2(\partial_z w_\varepsilon)^2\,\omega_s^{nr}\,.$$

Since f and w are odd, and $w_\varepsilon = w(\frac{z}{\varepsilon} - h(y))$, we find that $z \mapsto f''(w_\varepsilon)(\partial_z w_\varepsilon)^3$ is odd, modulo a translation by $h(y) = O(\varepsilon)$. Since it decays exponentially, its integral over $z \in (-r_2, r_2)$ is thus exponentially small. We then easily deduce that

$$|I_{3,2}| \le \frac{C}{\varepsilon}\int_{\Gamma^0}\gamma^2(s,y')\,\omega^0(y')dy'$$

and hence that

$$\left|\int_{\Sigma_s^{nr}} \partial_0(f'(u_\varepsilon^*))\frac{\varphi^2}{2\varepsilon^2}\chi_1^2\,\omega_s^{nr}\right| \le CE(s). \tag{4.29}$$

We next consider the convection term. Recall that by definition,

$$b^\beta := \frac{\omega^0}{\sqrt{|\det g|}}g^{\alpha\beta}\frac{\partial}{\partial y_\alpha}\left(\frac{\sqrt{|\det g|}}{\omega^0}\right).$$

In particular, since $g^{\alpha n} = \delta^{\alpha n}$ in $\operatorname{supp}(\chi_1)$ and ω^0 is independent of z,

$$b^n(y,z) = \frac{1}{\sqrt{|\det g(y,z)|}}\frac{\partial}{\partial z}\sqrt{|\det g(y,z)|}\,,$$

and thus it follows from (3.12) that

$$|b^n(y,z)| \le C|z| \qquad \text{in } \operatorname{supp}(\chi_1).$$

Therefore

$$\begin{aligned}\left|b^\gamma\frac{\partial\varphi}{\partial y_\gamma}\frac{\partial\varphi}{\partial y_0}\right|\chi_1^2 &\le C\left(a^{ab}\frac{\partial\bar\varphi}{\partial y_a}\frac{\partial\bar\varphi}{\partial y_b} + z^2(\frac{\partial\varphi}{\partial z})^2\chi_1^2\right)\\ &= C\left(a^{ab}\frac{\partial\bar\varphi}{\partial y_a}\frac{\partial\bar\varphi}{\partial y_b} + z^2(\frac{\partial\bar\varphi}{\partial z})^2 - z^2\chi_1'^2\varphi^2 - z^2\chi_1\chi_1'\frac{\partial}{\partial z}(\varphi^2)\right).\end{aligned}$$

We again write $\bar\varphi = \bar\varphi^\perp + \gamma(y)w_\varepsilon'(z)$. Using the fact that $\int_{\mathbb{R}} z^2 w_\varepsilon''^2 dz = \frac{C}{\varepsilon}$, and arguing as in the estimate of $I_{1,3}$ above, we find that

$$\left| \int_{\Sigma_s^{nr}} b^\gamma \frac{\partial\varphi}{\partial y_\gamma}\frac{\partial\varphi}{\partial y_0}\chi_1^2\,\omega_s^{nr}\right| \overset{(4.26)}{\leq} CE(s)\ .$$

Next, again because $g^{\alpha n} = \delta^{\alpha n}$ in $\mathrm{supp}(\chi_1)$, it is clear that

$$\left| \int_{\Sigma_s^{nr}} (\partial_0 g^{\alpha\beta})\frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta}\chi_1^2(z)\omega_s^{nr}\right| = \left| \int_{\Sigma_s^{nr}} (\partial_0 g^{ab})\frac{\partial\bar\varphi}{\partial y_a}\frac{\partial\bar\varphi}{\partial y_b}\omega_s^{nr}\right| \overset{(4.26)}{\leq} CE(s).$$

Finally, since $\frac{\partial}{\partial y_0}\bar\varphi = \frac{\partial}{\partial y_0}(\chi_1\varphi) = \chi_1\frac{\partial}{\partial y_0}\varphi$,

$$\begin{aligned}\left| \int_{\Sigma_s^{nr}} \frac{\partial\varphi}{\partial y_0}\eta\chi_1^2\ \omega_s^{nr}\right| &\leq \left| \int_{\Sigma_s^{nr}} (\frac{\partial\bar\varphi}{\partial y_0})^2\ + \chi_1^2\eta^2\,\omega_s^{nr}\right| \\ &\overset{(4.26)}{\leq} CE(s) + C\int_{\Sigma_s^{nr}} \chi_1^2\eta^2\omega_s^{nr}\end{aligned}$$

Putting the above estimates into (4.27), we obtain

$$\frac{d}{ds}E_\varepsilon^{nr}(s) \leq CE(s) + C\int_{\Sigma_s^{nr}} \eta^2\omega_s^{nr}.$$

7. Similar but easier arguments, using (4.13) and (4.26), lead to the estimate

$$\frac{d}{ds}E_\varepsilon^{far}(s) \leq CE(s) + C\int_{\Sigma_s^{far}} \eta^2\omega_s^{far}.$$

The point is that $\mathrm{supp}(D\chi^{far}) \subset \{\chi_1^2 = 0\}$. Thus terms in (4.13) containing derivatives of χ^{far} are easily estimated by $\int_{\Sigma_s}(1-\chi_1^2)\left(|D\varphi|^2 + \frac{1}{\varepsilon^2}\varphi^2\right)\omega_s \leq CE(s)$.

Similarly,

$$\frac{d}{ds}\frac{1}{\varepsilon}\int_{\Gamma^0}\gamma^2\omega^0(y')dy' \leq \frac{1}{\varepsilon}\int_{\Gamma^0}(\gamma^2 + (\partial_0\gamma)^2)\omega^0(y')dy' \leq CE(s).$$

Combining the last three inequalities, we conclude that

$$\frac{d}{ds}E(s) \leq CE(s) + C\int_{\Sigma_s}\eta^2\ \omega_s \tag{4.30}$$

Then it follows from Grönwall's inequality that

$$E(s) \leq e^{Cs}E(0) + C\int_0^s\int_{\Sigma_s} e^{C(s-\sigma)}\eta^2\,\omega_\sigma\,d\sigma \tag{4.31}$$

for all $0 \leq s < T_1$.

8. The conclusion (4.6) of the Proposition follows from (4.31). We sketch the straightforward verification.

Recall that the n-form ω_s is uniformly comparable to the induced Euclidean n-dimensional area in Σ_s, a fact that we will use repeatedly and without further mention. Thus for example it is immediate that

$$\int_0^s\int_{\Sigma_\sigma} e^{C(s-\sigma)}\eta^2\,\omega_\sigma\,d\sigma \leq C\int_0^s\left(\int_{\Sigma_s}\eta^2\right)d\sigma \qquad \text{for all } s\in[0,T_1],$$

where on the right-hand side, we implicity integrate with respect to the Euclidean area, as in (4.6). Next we claim that the left-hand side of (4.6) is bounded by $CE(s)$. Toward this end, first note from (4.21) that

$$\int_{\Sigma_s} \varphi^2 \omega_s = \int_{\Sigma_s} (1-\chi_1^2)\varphi^2 \omega_s + \int_{\Gamma^0}\int_{\mathbb{R}} (\bar\varphi^{\perp})^2 \omega^0(y')\, dz\, dy' \ + \frac{\Xi}{\varepsilon}\int_{\Gamma^0} \gamma^2 \omega^0(y') dy'. \tag{4.32}$$

Thus (4.26) implies that $\int_{\Sigma_s} \varphi^2 \omega_s \le CE(s)$.

Next, we write $|D\varphi|^2 = (1-\chi_1^2)|D\varphi|^2 + \chi_1^2 |D\varphi|^2$. The first term is immediately controlled $CE(s)$, due to (4.26). For the second term we may compute in modified Fermi coordinates, in which we have

$$\begin{aligned}\chi_1^2 |D\varphi|^2 &\le C\chi_1^2[a^{ab}\partial_a\varphi\,\partial_b\varphi + (\partial_z\varphi)^2] \\ &\le C\left[a^{ab}\partial_a\bar\varphi\,\partial_b\bar\varphi + (\partial_z\bar\varphi)^2 + \chi_1(z)\chi_1''(z)\varphi^2 - \partial_z(\chi_1\chi_1'\varphi^2)\right]\end{aligned}$$

Note that $\int_{\Sigma_s^{nr}} a^{ab}\partial_a\bar\varphi\,\partial_b\bar\varphi \omega_s^{nr}$ is exactly the term $I_{1,1}$ that appeared above in the lower bound for $E(s)$. There it was convenient to split it into several pieces (from which we obtained separate control over $(\partial_0\gamma)^2$, needed above), but if we keep that term as it is, then our earlier estimates of all the other contributions show that $I_{1,1} \le CE(s)$. The terms involving derivatives of χ_1 are handled as in our estimate of $I_{1,3}$. Finally, we write $(\partial_z\bar\varphi)^2 = (\partial_z\bar\varphi^{\perp} + \gamma\partial_{zz}w_\varepsilon)^2 \le 2(\partial_z\bar\varphi^{\perp})^2 + 2\gamma^2(\partial_{zz}w_\varepsilon)^2$. By integrating and combining with the above estimates, we finally conclude that $\varepsilon^2\int_{\Sigma_s}\chi_1^2|D\varphi|^2 \le CE(s)$, with the loss of a factor of ε^2 coming from the term $2\gamma^2(\partial_{zz}w_\varepsilon)^2$.

To finish the verification of (4.6), we must check that

$$\begin{aligned} E(0) &= \int_{\Sigma_0^{nr}} e_\varepsilon^{nr}(\varphi)\omega_0^{nr} + \int_{\Sigma_0^{far}} e_\varepsilon^{far}(\varphi)\omega_0^{far} + \frac{C}{\varepsilon}\int_{\Gamma_0}\gamma^2(0,y')\omega^0(y')dy' \\ &\le C\int_{\mathbb{R}^n}\left[|\nabla_x\varphi_0|^2 + |\varphi_1|^2 + \frac{1}{\varepsilon^2}\varphi_0^2\right] dx. \end{aligned}\tag{4.33}$$

Indeed, it is immediate from the definition (4.10) that

$$e_\varepsilon^{nr}(\varphi)(y,z) \le C\left[(\partial_t\varphi)^2 + |\nabla_x\varphi|^2 + \varepsilon^{-2}\varphi^2\right](\Phi(y,z))$$

Since $\Phi(y,z)\in\{0\}\times\mathbb{R}^n$ when $y_0 = 0$, see (3.13), the initial condition $(\varphi,\partial_t\varphi)|_{t=0} = (\varphi_0,\varphi_1)$ implies that

$$\int_{\Sigma_0^{nr}} e_\varepsilon^{nr}(\varphi)\omega_0^{nr} \le C\int_{\mathbb{R}^n}\left[|\nabla_x\varphi_0|^2 + |\varphi_1|^2 + \frac{1}{\varepsilon^2}\varphi_0^2\right] dx$$

The corresponding estimate for e_ε^{far} on Σ_0^{far} is immediate. We conclude from these facts and (4.32) that (4.33) holds.

□

Remark 4.1. *In view of* (4.26)*, it follows from* (4.31) *that*

$$\begin{aligned} \frac{c}{\varepsilon}\int_{\Gamma^0}\left(\gamma^2 + |D_y\gamma|^2\right)\omega^0(y')dy'\Big|_{y_0=s} &+ c\int_{\Sigma_s}(1-\chi_1^2)\left(|D\varphi|^2 + \frac{1}{\varepsilon^2}\varphi^2\right)\omega_s \\ &+ c\int_{\Gamma^0}\int_{\mathbb{R}} |D\bar\varphi^{\perp}|^2 + \frac{1}{\varepsilon^2}(\bar\varphi^{\perp})^2\,\omega^0(y')\,dz\,dy' \\ \le C\int_0^s\Big(\int_{\Sigma_\sigma}\eta^2\Big)d\sigma &+ C\int_{\mathbb{R}^n}\left[|\nabla_x\varphi_0|^2 + |\varphi_1|^2 + \frac{1}{\varepsilon^2}\varphi_0^2\right]dx. \end{aligned}$$

This is considerably stronger than (4.6)

Higher order estimates. We need estimates similar to (4.6) for higher order space derivatives. It is convenient to introducing the L^2-L^∞ norm for functions η defined on Σ,

$$\|\eta\|_{L^\infty L^2(\Sigma)} \ := \sup_{0\le s\le T_1} \|\eta\|_{L^2(\Sigma_s)}.$$

Then from (4.6) we get the L^∞-H^1-estimate for the solution φ of Problem (4.5)

$$\begin{aligned} \|\varphi\|_{L^\infty L^2(\Sigma)} + \varepsilon\|\varphi_t\|_{L^\infty L^2(\Sigma)} + \varepsilon\|D_x\varphi\|_{L^\infty L^2(\Sigma)} \ \le \\ C\Big[\|\eta\|_{L^\infty L^2(\Sigma)} + \frac{1}{\varepsilon}\big(\|\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|\varphi_1\|_{L^2(\mathbb{R}^n)}\big)\Big]. \end{aligned} \tag{4.34}$$

Assuming further smoothness on initial data and the right hand side we can derive higher order estimates as follows. Let us differentiate twice the equation. We write $D_i = \partial_{x_i}$, $D_{ij} = \partial^2_{x_i x_j}$. Then we get

$$\begin{aligned} L[D_{ij}\varphi] \ &= D_{ij}\eta - \varepsilon^{-2}\varphi D_{ij}(f'(u^*_\varepsilon)) - \varepsilon^{-2}(D_i\varphi D_j(f'(u^*_\varepsilon)) + D_j\varphi D_i(f'(u^*_\varepsilon))) \quad \text{in } \Sigma, \\ (D_{ij}\varphi, \partial_t D_{ij}\varphi)\Big|_{t=0} \ &= (D_{ij}\varphi_0, D_{ij}\varphi_1). \end{aligned}$$

Using estimate (4.34) and the facts $D_i(f'(u^*_\varepsilon)) = O(\varepsilon^{-1})$, $D_{ij}(f'(u^*_\varepsilon)) = O(\varepsilon^{-2})$ we get

$$\begin{aligned} &\|D^2_x\varphi\|_{L^\infty L^2(\Sigma)} + \varepsilon\|D^2_x\varphi_t\|_{L^\infty L^2(\Sigma)} + \varepsilon\|D^3_x\varphi\|_{L^\infty L^2(\Sigma)} \\ &\le \ \frac{C}{\varepsilon^4}\Big[\|\eta\|_{L^\infty L^2(\Sigma)} + \varepsilon^4\|D^2_x\eta\|_{L^\infty L^2(\Sigma)}\Big] \\ &\quad + \frac{C}{\varepsilon^5}\Big[\|\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|\varphi_1\|_{L^2(\mathbb{R}^n)}\Big] \\ &\quad + \frac{C}{\varepsilon}\Big[\|D^2_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D^3_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D^2_x\varphi_1\|_{L^2(\mathbb{R}^n)}\Big]. \end{aligned} \tag{4.35}$$

Let us consider for $m \ge 0$ the following L^∞-H^m norm

$$\|\eta\|_{L^\infty H^m(\Sigma)} := \sum_{j=0}^m \|D^j\eta\|_{L^\infty L^2(\Sigma)}.$$

Then from (4.35) and interpolating with bound (4.34), the following estimate readily follows:

$$\|\varphi\|_{L^\infty H^3(\Sigma)} \ \le \ \frac{C}{\varepsilon^5}\Big[\|\eta\|_{L^\infty H^2(\Sigma)} + \|\varphi_1\|_{H^2(\mathbb{R}^n)} + \varepsilon^{-1}\|\varphi_0\|_{H^3(\mathbb{R}^n)}\Big] \tag{4.36}$$

An induction argument (differentiating an even number of times m) yields

$$\begin{aligned} &\|D^m_x\varphi\|_{L^\infty L^2(\Sigma)} + \varepsilon\|D^m_x\varphi_t\|_{L^\infty L^2(\Sigma)} + \varepsilon\|D^{m+1}_x\varphi\|_{L^\infty L^2(\Sigma)} \\ &\le \ \frac{C}{\varepsilon^{2m}}\Big[\|\eta\|_{L^\infty L^2(\Sigma)} + \varepsilon^{2m}\|D^m_x\eta\|_{L^\infty L^2(\Sigma)}\Big] \\ &\quad + \frac{C}{\varepsilon^{2m+1}}\Big[\|\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|\varphi_1\|_{L^2(\mathbb{R}^n)}\Big] \\ &\quad + \frac{C}{\varepsilon}\Big[\|D^m_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D^{m+1}_x\varphi_0\|_{L^2(\mathbb{R}^n)} + \varepsilon\|D^m_x\varphi_1\|_{L^2(\mathbb{R}^n)}\Big]. \end{aligned}$$

As in (4.36) we finally find the estimate

Lemma 4.1. *The solution φ of Equation (4.5) satisfies the estimate*

$$\|\varphi\|_{L^\infty H^{m+1}(\Sigma)} \ \le\ \frac{C}{\varepsilon^{2m+1}}\Big[\|\eta\|_{L^\infty H^m(\Sigma)} + \|\varphi_1\|_{H^m(\mathbb{R}^n)} + \varepsilon^{-1}\|\varphi_0\|_{H^{m+1}(\mathbb{R}^n)}\Big] \quad (4.37)$$

for each even integer m.

5. The proof of Theorem 1

Now we have all the ingredients to proceed to the proof of Theorem 1. We look for a solution to Problem $S(u) = 0$ close the approximation u_ε^* given by (2.26), where the number k will be chosen sufficiently large. We look for a solution of the form

$$u(x,t) = u_\varepsilon^*(x,t) + \varphi(x,t).$$

In terms of φ the equation becomes

$$L_\varepsilon[\varphi] + \varepsilon^{-2}S(u_*) + \varepsilon^{-2}N(\varphi,x,t) = 0 \quad \text{in } \Sigma \quad (5.1)$$

where

$$N(\varphi,x,t) = f(u_\varepsilon^*(x,t)+\varphi) - f'(u_\varepsilon^*(x,t))\varphi - f(u_\varepsilon^*(x,t))$$
$$L_\varepsilon[\varphi] = \Box\varphi + \varepsilon^{-2}f'(u_\varepsilon^*)\varphi.$$

Let us consider the unique solution $\varphi = \mathcal{T}[\eta]$ of the linear problem

$$L_\varepsilon[\varphi] + \eta = 0 \quad \text{in } \Sigma, \qquad (\varphi,\partial_t\varphi)\Big|_{t=0} = (0,0). \quad (5.2)$$

which we have estimated in Proposition 4.1. Problem (5.1) with initial data $(\varphi,\partial_t\varphi)\Big|_{t=0} = (0,0)$ can then be written as the fixed point problem

$$\varphi = \mathcal{T}[\varepsilon^{-2}S(u_\varepsilon^*) + \varepsilon^{-2}N(\varphi,\cdot)] =: \mathcal{M}(\varphi), \quad \varphi \in \mathcal{B} \quad (5.3)$$

We will solve this problem by contraction mapping principle in a suitable space $\mathcal{B}$ of small functions defined on Σ. We consider the Banach space $L^\infty H^m(\Sigma)$ endowed with its natural norm and consider the region

$$\mathcal{B} = \{\varphi \in L^\infty H^m(\Sigma) \ / \ \|\varphi\|_{L^\infty H^m(\Sigma)} \le \varepsilon^{\frac{k}{2}}\}$$

where k is the number in the definition of u_ε^* in (2.26). Let us fix a number $m > n/2$. Among other consequences, this implies that $L^\infty H^m(\Sigma)$ is embedded into $L^\infty(\Sigma)$ and

$$\|\varphi\psi\|_{L^\infty H^m(\Sigma)} \le C\|\varphi\|_{L^\infty H^m(\Sigma)}\|\psi\|_{L^\infty H^m(\Sigma)}. \quad (5.4)$$

From Proposition 2.1 (specifically bound (2.19)), together with the fact that errors are exponentially small in ε where the cut-off is not constant, we find that

$$\|\varepsilon^{-2}S(u_\varepsilon^*)\|_{L^\infty H^m(\Sigma)} \ \le\ C\varepsilon^k.$$

Next, we claim that for $\varphi, \tilde\varphi \in \mathcal{B}$,

$$\|N(\varphi,\cdot) - N(\tilde\varphi,\cdot)\|_{L^\infty H^m(\Sigma)} \le C\varepsilon^{\frac{k}{2}-m}\|\varphi - \tilde\varphi\|_{L^\infty H^m(\Sigma)} \quad (5.5)$$

To prove this, observe that

$$N(\varphi) - N(\tilde\varphi) = \left(\int_0^1 [f'(u_\varepsilon^* + \sigma\varphi + (1-\sigma)\tilde\varphi) - f'(u_\varepsilon^*)]\, d\sigma\right)(\varphi - \tilde\varphi)\,.$$

In view of (5.4), to establish (5.5) it suffices to observe that

$$\|f'(u_\varepsilon^*(t,\cdot)+\psi) - f'(u_\varepsilon^*)\|_{L^\infty H^m(\Sigma)} \ \le\ C\varepsilon^{-m}\|\psi\|_{L^\infty H^m(\Sigma)} \ \le\ C\varepsilon^{\frac{k}{2}-m}, \qquad \text{for all } \psi \in \mathcal{B}.$$

which follows from a direct computation using Leibnitz rule, Sobolev embedding and the fact that $D_x^m u_\varepsilon^* = O(\varepsilon^{-m})$.

At this point we fix a number k with $k > 6m + 2$. Let us consider the operator $\mathcal{M}[\varphi]$ defined on $\mathcal{B}$ in formula (5.3). Estimates (4.37) and (5.5) lead to

$$\|\mathcal{M}[\varphi] - \mathcal{M}[\tilde\varphi]\|_{L^\infty H^m(\Sigma)} \ \le\ C\varepsilon^{\frac{k}{2}-3m-1}\|\varphi - \tilde\varphi\|_{L^\infty H^m(\Sigma)} \quad \text{for all} \quad \varphi, \tilde\varphi \in \mathcal{B}.$$

and

$$\|\mathcal{M}[0]\|_{L^\infty H^m(\Sigma)} \ \le\ C\varepsilon^{k-2m-1}.$$

It follows that for all sufficiently small ε we get that $\mathcal{M}(\mathcal{B}) \subset \mathcal{B}$ and that $\mathcal{M}$ is a contraction mapping in $\mathcal{B}$. Hence Problem (5.3) has a unique solution. The conclusion of Theorem 1 readily follows. □

This proof applies equally well to yield the stability assertion made at the end of §1.2 by just considering the operator $\mathcal{T}$ involving sufficiently small initial data.

Appendix A. Modified Fermi Coordinates

In this appendix we present the proof of Lemma 3.2.

Proof. For the proof, we will denote the modified Fermi coordinates as

$$(\mathtt{y}, \mathtt{z}) = (\mathtt{y}_0, \mathtt{y}', \mathtt{z}) = (\mathtt{y}_0, \ldots, \mathtt{y}_{n-1}, \mathtt{z})$$

(denoted (y, z) in the statement of the lemma and elsewhere in this paper), and we will reserve (y, z) for Fermi coordinates associated to a canonical local parametrization $(y, z) \in [0, T] \times V_l \times (-\delta, \delta) \mapsto Y_l(z) + z\nu(y)$, as constructed in Section 3.1. We specify that the relationship between modified and Fermi coordinates has the form

$$(y_0, y', z) = (y_0(\mathtt{y}, \mathtt{z}), \mathtt{y}', \mathtt{z})$$

where y_0 depends on $(\mathtt{y}, \mathtt{z})$ in a way to be described below. Thus they are related to (x, t) coordinates via

$$(x, t) = Y_l(y_0(\mathtt{y}, \mathtt{z}), \mathtt{y}', \mathtt{z}) + \mathtt{z}\nu(y_0(\mathtt{y}, \mathtt{z}), \mathtt{y}') =: \Phi_l(\mathtt{y}, \mathtt{z})$$

for

$$(\mathtt{y}, \mathtt{z}) \in [0, T_1] \times V_l \times (-\delta_1, \delta_1)$$

We will also write

$$\begin{aligned} \widetilde{Y}_a &:= \frac{\partial \Phi}{\partial \mathtt{y}_a} \qquad \text{for } a = 0, \ldots, n-1 \\ \widetilde{Y}_n &:= \frac{\partial \Phi}{\partial \mathtt{z}} \\ \mathtt{g}_{\alpha\beta} &:= \langle Y_\alpha, Y_\beta \rangle_m, \qquad \text{for } \alpha, \beta = 0, \ldots, n. \end{aligned}$$

To define $y_0(\mathtt{y}, \mathtt{z})$, fix some l and consider $Y_l : \Lambda_l \to \Gamma$ as in (3.9). We will often omit the subscript l, and we will write

$$(t(y, z), x(y, z)) = Y(y) + z\nu(y)$$

to indicate the dependence of (x, t) on (y, z). Recall that Y_l is constructed to that $t(y, 0) = t(y_0, y', 0) = y_0$, see (3.2), and hence $\frac{\partial t}{\partial y_0}(y, 0) = 1$ everywhere in Λ_l. Thus the Implicit Function Theorem implies that for any $T_1 < T$ there exists $\delta_1 < \delta$ and a function $\eta_0 : [0, T_1] \times V_l \times (-\delta_1, \delta_1) \to [0, T]$ such that

$$\eta_0(\mathtt{y}, 0) = \mathtt{y}_0, \qquad t(\eta_0(\mathtt{y}, \mathtt{z}), \mathtt{y}', \mathtt{z}) = \mathtt{y}_0 \qquad \text{everywhere in } [0, T_1] \times V_l \times (-\delta_1, \delta_1).$$

Here we are implicitly using our assumption that the velocity of Γ vanishes at $t = 0$. For $Y \in \Gamma^0$ and $|z| < \delta$, this implies that $Y + z\nu(Y)$ belongs to $\{0\} \times \mathbb{R}^n$, and hence that $t(0, \mathtt{y}', \mathtt{z}) = 0$ for all $(\mathtt{y}', \mathtt{z}) \in V_l \times (-\delta, \delta)$. It is this property that allows us to extend the domain of η_0 all the way to $\{y_0 = 0\}$, and it implies that $\eta_0(0, \mathtt{y}', \mathtt{z}) = 0$ for all $(\mathtt{y}', \mathtt{z})$.

We can choose δ_1 such that the above properties hold for all Λ_l, $l = 1, \dots, m$.

We will take $y_0(\mathtt{y}, \mathtt{z})$ to have the form

$$y_0(\mathtt{y}, \mathtt{z}) = \chi_0(\mathtt{z})\mathtt{y}_0 + (1 - \chi_0(\mathtt{z}))\eta_0(\mathtt{y}, \mathtt{z})$$

where χ_0 will be specified below in the proof of (3.14). It will be the case that

$$\chi_0(\mathtt{z}) = 1 \text{ if } |\mathtt{z}| < r_2, \qquad \chi_0(\mathtt{z}) = 0 \text{ if } |\mathtt{z}| > r_1,$$

for $0 < r_2 < r_1 < \delta_1$ also to be fixed below. We will require that r_1, r_2, χ_0 are chosen uniformly for $l = 1, \dots, m$, so that (3.10) holds.

It is immediate that $(\mathtt{y}, \mathtt{z}) = (y, z)$, and hence $\mathtt{g}_{\alpha\beta} = g_{\alpha\beta}$ for $(\mathtt{y}, \mathtt{z}) \in [0, T_1] \times V \times (-r_2, r_2)$. Thus (3.11) follows from the corresponding properties of Fermi coordinates, see (2.1). Similarly, (3.12) is a basic property of Fermi coordinates, together with the fact that Γ is minimal, see (2.5) and (2.8) Likewise, (3.13) is a straightforward consequence of the definition of $y_0(\mathtt{y}, \mathtt{z})$.

It remains only to prove (3.14). To do this we note that

$$\widetilde{Y}_0 = \frac{\partial}{\partial \mathtt{y}_0}(Y + \mathtt{z}\nu) = \frac{\partial}{\partial y_0}(Y + \mathtt{z}\nu)\frac{\partial y_0}{\partial \mathtt{y}_0} = Y_0 \frac{\partial y_0}{\partial \mathtt{y}_0}.$$

and similarly

$$\widetilde{Y}_i = Y_0 \frac{\partial y_0}{\partial \mathtt{y}_i} + Y_i \quad \text{ for } i = 1, \dots, n$$

where here and below, we sometimes write $\mathtt{z}$ as $\mathtt{y}_n$. It follows that

$$\begin{aligned} \mathtt{g}_{00} &= (\frac{\partial y_0}{\partial \mathtt{y}_0})^2 g_{00}, \\ \mathtt{g}_{0i} = \mathtt{g}_{i0} &= \frac{\partial y_0}{\partial \mathtt{y}_0}\frac{\partial y_0}{\partial \mathtt{y}_i} g_{00}, \\ \mathtt{g}_{ij} &= \frac{\partial y_0}{\partial \mathtt{y}_i}\frac{\partial y_0}{\partial \mathtt{y}_j} g_{00} + g_{ij} \end{aligned} \tag{A.1}$$

for $i, j = 1, \dots, n$.

The fact that $\mathtt{g}_{00} < 0$ everywhere now follows from (A.1) and the corresponding property of Fermi coordinates.

We next prove that $\left[\ \mathtt{g}_{ij}\ \right]_{i,j=1}^n$ is positive definite. For $|\mathtt{z}| \le r_2$ this follows from standard properties of Fermi coordinates. For $|\mathtt{z}| \ge r_1$ it is also straightforward. Indeed, in this set, for every fixed $\mathtt{y}_0$, the map $(\mathtt{y}', \mathtt{z}) \to Y + \mathtt{z}\nu$ is just a parametrization of a portion of the hypersurface $\{y_0\} \times \mathbb{R}^n$, on which the induced metric is simply the Euclidean metric. So in this set, the metric tensor $\left[\ \mathtt{g}_{ij}\ \right]_{i,j=1}^n$ is just the Euclidean metric rewritten with respect to a new coordinate system. Hence it is clearly positive definite.

We now consider $r_2 < |z| < r_1$. We start with the main point which, it turns out, is to fix r_1, r_2 and χ_0 so that $\mathtt{g}_{nn}$ is bounded away from zero. Since $g_{nn} = 1$, as recalled in the proof of (3.11),

$$\mathtt{g}_{nn} = 1 + (\frac{\partial y_0}{\partial \mathtt{z}})^2 g_{00}\ . \tag{A.2}$$

For fixed $\mathtt{y} = (\mathtt{y}_0, \mathtt{y}')$, consider the curve

$$\mathtt{z} \mapsto Y(\eta_0(\mathtt{y},\mathtt{z}), \mathtt{y}', \mathtt{z}) + \mathtt{z}\nu(\eta_0(\mathtt{y},\mathtt{z}), \mathtt{y}', \mathtt{z}) =: \Phi_0(\mathtt{y},\mathtt{z})$$

and let X denote the tangent vector $\partial\Phi_0/\partial\mathtt{z}$. The definition of η_0 implies that the image of the curve is contained in the hypersurface $\{\mathtt{y}_0\} \times \mathbb{R}^n$, and hence that X is spacelike, or in other words that $\langle X, X\rangle_m > 0$. By compactness, after poisbly shrinking δ_1 there exists some $c > 0$ such that $\langle X, X\rangle_m \geq c$ everywhere in $[0, T_1] \times V \times (-\delta_1, \delta_1)$. Writing out this inequality in coordinates, and again using the fact that $g_{nn} = 1$, we obtain

$$1 + (\frac{\partial\eta_0}{\partial\mathtt{z}}(\mathtt{y},\mathtt{z}))^2 \ g_{00}(\eta_0(\mathtt{y},\mathtt{z}), \mathtt{y}', \mathtt{z}) \geq c. \tag{A.3}$$

Next, since $\mathtt{y}_0 = \eta_0(\mathtt{y}_0, 0) = \eta_0(\mathtt{y},\mathtt{z}) - \mathtt{z}\partial_{\mathtt{z}}\eta_0(\mathtt{y},\mathtt{z}) + O(\mathtt{z}^2)$, we use the definition of $y_0(\mathtt{y},\mathtt{z})$ to compute

$$\begin{aligned}\frac{\partial y_0}{\partial\mathtt{z}}(\mathtt{y},\mathtt{z}) &= \chi_0'(\mathtt{z})(\mathtt{y}_0 - \eta_0(\mathtt{y},\mathtt{z})) + (1-\chi_0(\mathtt{z}))\frac{\partial\eta_0}{\partial\mathtt{z}}(\mathtt{y},\mathtt{z})\\ &= \frac{\partial\eta_0}{\partial\mathtt{z}}(\mathtt{y},\mathtt{z})(1-\chi_0(\mathtt{z}) - \mathtt{z}\chi_0'(\mathtt{z})) + O(\mathtt{z}^2\chi_0'(\mathtt{z})).\end{aligned}$$

We now take χ_0 of the form

$$\chi_0(\mathtt{z}) = \begin{cases} 1 & \text{if } \mathtt{z} \leq r_2 = r_1^2/(1+r_1)\\ r_1(\frac{r_1}{\mathtt{z}} - 1) & \text{if } r_2 \leq \mathtt{z} \leq r_1\\ 0 & \text{if } \mathtt{z} \geq r_1\end{cases}.$$

for $r_1 > 0$ to be chosen below. (More precisely, we take χ_0 to be a regularization of the function defined above, and satisfying essentially the same estimates. But for simplicity we will compute with the function defined above, which is merely Lipschitz.) With this choice, $-\chi_0(\mathtt{z}) - \mathtt{z}\chi_0'(\mathtt{z}) = r_1$ on the support of χ_0', so

$$\frac{\partial y_0}{\partial\mathtt{z}}(\mathtt{y},\mathtt{z}) = \partial_{\mathtt{z}}\eta_0(\mathtt{y},\mathtt{z})(1+r_1) + O(r_1^2).$$

We also observe that $|\eta_0(\mathtt{y},\mathtt{z}) - y_0(\mathtt{y},\mathtt{z})| \leq C|\mathtt{z}|$, because $\eta_0(\mathtt{y},0) = y_0(\mathtt{y},0) = \mathtt{y}_0$. It follows that

$$g_{00}(y_0(\mathtt{y},\mathtt{z}), \mathtt{y}', \mathtt{z}) = g_{00}(\eta_0(\mathtt{y},\mathtt{z}), \mathtt{y}', \mathtt{z}) + O(|\mathtt{z}|).$$

By combining these with (A.2), (A.3), we find that

$$\mathtt{g}_{nn}(\mathtt{y},\mathtt{z}) \geq c - Cr_1$$

for C depending on $\|g_{00}\|_{W^{1,\infty}}$ and $\|\partial_{\mathtt{z}}\eta_0\|_{L^\infty}$. It follows that

$$\mathtt{g}_{nn} \geq c/2 \qquad \text{at all points where } r_2 \leq z \leq r_1. \tag{A.4}$$

for all sufficiently small choices of r_1 (and hence r_2) in the definition of χ_0.

We next remark that since $\eta_0(\mathtt{y},0) = \mathtt{y}_0$ for all $\mathtt{y}$, it is clear that $\frac{\partial\eta_0}{\partial\mathtt{y}_i}(\mathtt{y},0) = 0$ for $i = 1,\dots,n-1$. It follows that

$$|\frac{\partial\eta_0}{\partial\mathtt{y}_i}(\mathtt{y},\mathtt{z})| \leq C|\mathtt{z}| \quad \text{for } i = 1,\dots,n-1, \qquad |\eta_0(\mathtt{y},\mathtt{z}) - \mathtt{y}_0| \leq C|\mathtt{z}|,$$

everywhere in its domain, and hence that the same properties hold for $y_0(\mathtt{y}, \mathtt{z})$. We then see from (A.1) that for $r_2 \leq |\mathtt{z}| \leq r_1$,

$$|\mathsf{g}_{ij} - g_{ij}| \leq Cr_1^2 \qquad \text{for } 1 \leq i, j \leq n-1\ , \quad \text{and}$$
$$|\mathsf{g}_{in}| = |\mathsf{g}_{ni}| \leq Cr_1 \qquad \text{for } 1 \leq i \leq n-1.$$

Since $\left[\ g_{ij}\ \right]_{i,j=1}^{n-1}$ is positive definite, we conclude from this and (A.4) that r_1 may be chosen so that $\left[\ \mathsf{g}_{ij}\ \right]_{i,j=1}^{n}$ is positive definite everywhere.

Finally, the facts that $\mathsf{g}_{00} < 0$ and $\left[\ \mathsf{g}_{ij}\ \right]_{i,j=1}^{n}$ is positive definite imply the same properties for g^{00} and $\left[\ \mathsf{g}^{ij}\ \right]_{i,j=1}^{n}$. This is a consequence of the general formula for the inverse of a matrix in block form

$$\begin{pmatrix} a & b \\ b^T & B \end{pmatrix}^{-1} = \begin{pmatrix} (a - bB^{-1}b^T)^{-1} & -a^{-1}b(B - b^Ta^{-1}b)^{-1} \\ -B^{-1}b^T(a - bB^{-1}b^T)^{-1} & (B - b^Ta^{-1}b)^{-1} \end{pmatrix},$$

where $a \in \mathbb{R}$ and b, B are $1 \times n$ and $n \times n$ matrices respectively. This formula can be checked by multiplying the right-hand side by $\begin{pmatrix} a & b \\ b^T & B \end{pmatrix}$. $\square$

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References

[1] O. Agudelo, M. del Pino, J. Wei, *Solutions with multiple catenoidal ends to the Allen-Cahn equation in* $\mathbb{R}^3$. J. Math. Pures Appl. (9) 103 (2015), no. 1, 142-218.

[2] S. Brendle. *Hypersurfaces in Minkowski space with vanishing mean curvature*, Comm. Pure Appl. Math., 55(10):1249–1279, 2002.

[3] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations,* J. Differential Equations 96 (1992), no. 1, 116–141.

[4] S. Cuccagna, *On asymptotic stability in 3D of kinks for the* ϕ^4 *model*, Trans. Amer. Math. Soc. 360 (2008), no. 5, 2581–2614.

[5] M. Czubak, R.L.Jerrard, *Topological defects in the abelian Higgs model*, Discrete Contin. Dyn. Syst. 35 (2015), no. 5, 1933-1968.

[6] P. de Mottoni and M. Schatzman. *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc., 347(5):1533–1589, 1995.

[7] M. del Pino, M. Kowalczyk, F. Pacard, J.Wei. *The Toda system and multiple-end solutions of autonomous planar elliptic problems.* Adv. Math. 224 (2010), no. 4, 1462–1516

[8] M. del Pino, M. Kowalczyk, J. Wei, *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature in* $\mathbb{R}^3$. J. Differential Geom. 93 (2013), no. 1, 67131.

[9] M. del Pino, M. Kowalczyk, J. Wei, *On De Giorgi's conjecture in dimension* $N = 9$. Ann. of Math. (2) 174 (2011), no. 3, 1485-1569.

[10] M. del Pino, K. Gkikas, *Ancient shrinking spherical interfaces in the Allen-Cahn flow.* Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), no. 1, 187-215.

[11] M. El Smaily, R.L. Jerrard, *A refined description of evolving interfaces in certain nonlinear wave equations.* NoDEA Nonlinear Differential Equations Appl. 25 (2018), no. 2, Art. 15, 21 pp.

[12] L. C. Evans, *Partial differential equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp. ISBN: 0–8218–0772–2.

[13] L. C. Evans, H. M. Soner and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature,* Comm. Pure Appl. Math. 45 (1992), no. 9, 1097–1123.

[14] B. Galvão Sousa and R. L. Jerrard. *Accelerating fronts in semilinear wave equations*, Rend. Circ. Mat. Palermo (2), 64(1):117–148, 2015.

[15] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, J. Differential Geom. 38 (1993), no. 2, 417–461. MR1237490 (94h:58051)
[16] R.L. Jerrard, *Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space.* Anal. PDE 4 (2011), no. 2, 285-340.
[17] T.W.B. Kibble, *Topology of cosmic domains and strings*, J. Phys. A: Math. Gen., 9 (1976), no. 8, 1387–1398.
[18] H. Lindblad, *A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time*, Proc. Amer. Math. Soc. 132 (2004), no. 4, 1095–1102 (electronic).
[19] O. Milbredt, *The Cauchy Problem for Membranes, Dissertation*, Freie Universität Berlin (2008).
[20] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. 98 (1987), no. 2, 123–142.
[21] J. C. Neu, *Kinks and the minimal surface equation in Minkowski space*, Phys. D **43** (1990), no. 2-3, 421–434.
[22] H.B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, Nuclear Phys. B, **61** (1973), 45-61.
[23] F. Pacard and M. Ritoré. *From constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom., 64(3):359–423, 2003.
[24] H. G. Rotstein and A. A. Nepomnyashchy, *Dynamics of kinks in two-dimensional hyperbolic models*, Phys. D **136** (2000), no. 3-4, 245–265.
[25] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal.**101** (1988), no. 3, 209–260.
[26] A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects*, Cambridge Univ. Press, Cambridge, 1994.

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