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RESEARCH ARTICLE

Information within coalitions: risk and ambiguity

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Abstract
We address economies with asymmetric information where agents are not perfectly aware of the informational structure for coalitions. Thus, when joining a coalition, each consumer considers the informational risk and may be uncertain about the prior relevant to her decision. In this context, we introduce cooperative solutions that we refer to as *risky core*, *ambiguous core*, and MEU-*core*. We provide existence results and a variety of properties of these concepts, including their coalitional incentive compatibility. We also formalize the intuition that the blocking power of coalitions is increasing with their information but decreasing with the degree of risk or ambiguity aversion faced by their members.

Keywords Differential information · Risky core · Ambiguous core · MEU-core

JEL Classification D71 · D81 · D82

1 Introduction

Within a complete information scenario, the core is a natural cooperative solution that features stability against the blocking of coalitions. In fact, a feasible allocation belongs to the core of a complete information economy if no coalition is able to improve it via an alternative that is available for its members by using their own resources.

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The extension of the core notion to an incomplete information setting does not lead to a unique concept since it is necessary to state the mechanism that allocates the information to agents joining a coalition. For instance, the approach underlying the coarse core, introduced by Wilson (1978), considers no sharing of information and agents in a coalition can only rely on common knowledge events to construct potential objections. However, if members of a coalition pool their information, then one obtains the notion of the fine core, also proposed by Wilson (1978). As pointed out by Wilson (1978) those two notions have existence and incentive compatibility problems. Specifically, the coarse core may not exist, the fine core exits, but it is not incentive compatible. Yannelis (1991) introduces the private core where the information of an individual is not modified when a coalition is formed, that is, each member keeps her private information regardless of the coalition she belongs to. This notion exists and it is incentive compatible as shown in Koutsougeras and Yannelis (1993). These cooperative concepts are particular cases of the solutions proposed by Maus (2003) and Allen (2006), where sharing rules determine the information that agents have in coalitions.

In all the previous cooperative solutions, the information that each member will have in a group is riskless. Nevertheless, the informational profile that will prevail when asymmetrically informed agents join a coalition requires a process of communication among them. Thus, the behavior of agents concerning the degree of revelation of their information may depend on the features of the others to which they join. For instance, trust or mutual distrust, the number of times the group has met, knowing or not the others, the inherent characteristics of the coalition itself, the techniques used in the communication systems, among other issues, usually play an important role regarding the information that is finally revealed. All these points may result in a complex mechanism of information transmission that in addition is subject to a variety of changes. Then, it is difficult to argue that the informational profiles for coalitions are certain. This leads individuals to face uncertainty, risk or ambiguity, about the information they will have or they can use when joining a coalition.

In this paper, we provide a variety of new cooperative solutions for differential information economies that we call the *risky core*, the *ambiguous core*, and the MEU-*core*. We analyze these concepts and obtain existence and incentive compatibility results, along with further properties. In the absence of risk, these new concepts include as particular cases the fine and coarse core that go back to by Wilson (1978), the private core introduced by Yannelis (1991), the cores defined in Maus (2003) and Allen (2006), and the core presented in Hervés-Beloso et al. (2014) in which the available information for coalitions may be in relation with their size.

We consider an economy where consumers commit to resource assignments before learning their private information. There is also uncertainty regarding the information that each individual will have when forming groups. In this way, agents have beliefs about the informational profile that will be followed when they join a coalition. When these beliefs are given by a probability distribution and the blocking mechanism considers expected utility functions over the informational scenarios, we obtain the *risky core*. When beliefs are given by a set of probability distributions, we get either the *ambiguous core* or the MEU-*core* depending on agents' attitude toward ambiguity. The former is based on the decision model stated by Klibanoff et al. (2005) while the later follows the maxmin expected utility model proposed by Gilboa and Schmeidler (1989). To analyze these cooperative solutions, we study their existence, incentive compatibility properties, and some comparative static consequences.

We follow different approaches that allow us to obtain independent existence results for the risky core: (i) recast the economy as a non-transferable utility game with a non-empty core; (ii) obtain the risky core as a larger set than the one of Walrasian expectation allocations; (iii) show that the risky core equates the set of equilibria of a coalitional exchange economy (see Del Mercato 2006). Furthermore, we state some boundaries to more general existence results by showing that, when the original private information is riskless, the non-emptiness of the risky core requires that members in a coalition do not give a large probability to share their information.

To study incentive compatibility properties of our core concepts, we extend to our setting the notion of *coalitional incentive compatibility* introduced by Koutsougeras and Yannelis (1993) and Krasa and Yannelis (1994). In this way, we show that, when utility functions are separable in the states of nature and there is no risk or ambiguity for the grand coalition, our cooperative solutions concepts satisfy the coalitional incentive compatibility, that is, when agents commit to an allocation in the risky, ambiguous, or MEU-core, coalitions will not have incentives to misreport the information after the realization of the state of nature.

Furthermore, we formalize the intuition that the blocking power of coalitions is increasing in information and decreasing in the degree of risk aversion faced by their members. Our comparative static results show that the risky core does not shrink when agents become either more risk averse or more optimistic about the amount of information that will be attained in a coalition and, analogously, the ambiguous core cannot shrink when agents become more ambiguity averse. Hence, existence results for both the ambiguous core and the MEU-core are obtained as byproducts of the non-emptiness for the risky core.

The rest of the paper is organized as follows. In Sect. 2 we describe the economy and in Sect. 3 we provide the risky core definition. In Sect. 4 we state a collection of existence results for the risky core, whereas in Sects. 5 and 6 we analyze its coalitional incentive compatibility and comparative static properties, respectively. In Sect. 7 we extend the analysis to include ambiguity. In Sect. 8, we conclude with some remarks. To facilitate the reading, all the proofs are relegated to a final appendix.

2 The economy

Let us consider a differential information exchange economy $\mathcal{E}$ with a finite set $N = \{1, \ldots, n\}$ of consumers. The economy extends over two time periods and there is uncertainty about the state of nature that will be realized at the second period, where consumption takes place.

Let Ω be the finite set of states of nature. A profile of information $\mathcal{P} = (P_i)_{i \in N}$ specifies a partition P_i of Ω to each consumer i and determines the information that every individual will have about the realization of the states of nature. In our model, there is uncertainty about the profile of information that will prevail and each consumer

i has beliefs that are given by a probability distribution $r_i = (r_i^{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}}$, where $\mathbb{P}$ denotes the set of informational profiles.

There is a finite number ℓ of commodities at each $\omega \in \Omega$ and, regardless of the informational profile, every agent i has endowments $e_i = (e_i(\omega))_{\omega\in\Omega} \in (\mathbb{R}^\ell_+)^k$, where k is the number of states of nature. Individuals' beliefs are compatible with their endowments in the sense that for any $\mathcal{P} = (P_j)_{j\in N}$ such that $r_i^{\mathcal{P}} > 0$, e_i is P_i-measurable.[1] That is, agents cannot have less information than the one revealed by their endowments.

Consumers commit to allocations before learning the information they will have. Hence, a consumption plan for an individual is given by a vector of commodity bundles at each state of nature for every informational profile, that is, it is a function from $\mathbb{P}$ to $(\mathbb{R}^\ell_+)^k$. Each agent i assigns to a consumption plan $x_i = (x_i^{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}}$ the utility level $U_i(x_i) = \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(x_i^{\mathcal{P}})$, where $u_i : (\mathbb{R}^\ell_+)^k \to \mathbb{R}_+$ is a continuous, increasing, and concave function.

A net trade z is given by vectors $z_i^{\mathcal{P}} = (z_i^{\mathcal{P}}(\omega))_{\omega\in\Omega}$ for each consumer i and every $\mathcal{P} \in \mathbb{P}$. A net trade is *feasible* if it is both physically and informationally feasible. *Physical feasibility* means $\sum_{i\in N} z_i^{\mathcal{P}}(\omega) \leq 0$, for all $(\mathcal{P}, \omega) \in \mathbb{P} \times \Omega$. *Informational feasibility* requires $z_i^{\mathcal{P}}$ to be P_i-measurable for each $\mathcal{P} = (P_i)_{i\in N}$ and $i \in N$. An allocation x assigns a commodity bundle $x_i^{\mathcal{P}}(\omega) \in \mathbb{R}^\ell_+$ to each consumer i, for every $(\mathcal{P}, \omega) \in \mathbb{P} \times \Omega$. The allocation x is *feasible* if $x_i^{\mathcal{P}}(\omega) = e_i(\omega) + z_i^{\mathcal{P}}(\omega)$ for some feasible net trade z. Let $\mathcal{F}$ denote the set of feasible allocations.

3 Coalitions and information: the risky core

In general terms, an allocation belongs to the core of an economy if it is feasible and it is not blocked by any coalition. Addressing differential information economies, to propose a cooperative solution concept like the core, one has to specify the information that coalitions have. In this work, we consider that individuals are not perfectly aware of the information that is going to be obtained when they join a coalition. In this way, each member in a coalition has beliefs given by a probability distribution on the set of profiles of information. To be precise, for each coalition S there is a set $\rho(S) = \{\rho_i(S) : i \in S\}$ of probability distributions $\rho_i(S) = (\rho_i^{\mathcal{P}}(S))_{\mathcal{P}\in\mathbb{P}}$ that capture the risk of the informational communication process, where $\rho_i^{\mathcal{P}}(S)$ denotes the probability that i gives to the implementation of $\mathcal{P}$ when S is formed.[2]

Within this framework, and in order to block an allocation, members of a coalition consider net trades for every possible profile of information. That is, given a coalition S, a vector $z = (z^{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}}$, where $z^{\mathcal{P}} = (z_i^{\mathcal{P}})_{i\in S}$, is a *net trade attainable for S* if (i) $z^{\mathcal{P}}$ is physically feasible for S, i.e., $\sum_{i\in S} z_i^{\mathcal{P}} \leq 0$, and (ii) for each

[1] Given a partition P of Ω, $x = (x(w))_{w\in\Omega} \in (\mathbb{R}^\ell_+)^k$ is said to be P-measurable when it is constant on the elements of the partition P. That is, $x(\omega) = x(\omega')$ for all states ω and ω' belonging to the same element of P.

[2] To unify notations, members of a coalition consider probability distributions on the set of informational profiles. However, the veto power of a coalition S does not depend on the information that agents in $N\backslash S$ have.

$i \in S$, $z_i^{\mathcal{P}}$ is P_i-measurable, for every $\mathcal{P} = (P_i)_{i\in N}$. In this way, $y = (y^{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}}$, where $y^{\mathcal{P}} = (y_i^{\mathcal{P}})_{i\in S}$, is an *allocation attainable for* S if there exists a net trade z attainable for S such that $y_i^{\mathcal{P}} = e_i + z_i^{\mathcal{P}} \geq 0$ for every $\mathcal{P} \in \mathbb{P}$ and $i \in S$. Let $\mathcal{F}(S)$ denote the set of attainable allocations for S.

Definition 1 (*Risky core*) Given $\rho = (\rho(S))_{S\subseteq N}$, an allocation x is blocked by a coalition $S \subseteq N$ if there is $y \in \mathcal{F}(S)$ such that $\sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)u_i(y_i^{\mathcal{P}}) > U_i(x_i)$ for every $i \in S$.[3] The risky core $\mathcal{C}_\rho(\mathcal{E})$ is the set of feasible allocations that are not blocked by any coalition.

In the definition of the veto mechanism that underlies the risky core, when an individual i joins a coalition S, her tastes for consumption are not altered and remain represented by u_i. However, her original beliefs r_i about the profile of information are updated and become $\rho_i(S)$, affecting both her welfare and the veto power of the coalition. To be precise, albeit the distributions $(r_i)_{i\in N}$ capture the beliefs about the information that the consumers will have when the contingent promises are delivered in the economy, they do not necessarily identify the information that a group of agents will have when they join a coalition with the aim to block an allocation. Thus, the informational risk determining the veto mechanism that leads to the risky core is given by the set of probability distributions $\rho(S) = \{\rho_i(S) : i \in S\}$ for each coalition S. Although the notion of risky core does not require particular relations among the different probability distributions, some conditions on them considered along the paper will allow us to state a variety of results and remarks.

In general, an allocation is said to be *Pareto optimal* or *efficient* if it is feasible and there is no other feasible allocation that improves every consumer. Within our approach, when $\rho_i(N) = r_i$ for every individual $i \in N$, a feasible allocation is efficient if and only if it is not blocked by the grand coalition, as it is the case for pure exchange economies with complete information. In addition, an allocation is said to be *individually rational* if every consumer becomes no worse than with her endowments. Note that if an allocation x is not blocked by the coalitions with just one member, then we have that $U_i(x_i) \geq \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(\{i\})u_i(e_i) = u_i(e_i) = \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i) = U_i(e_i)$, and then x is individually rational, regardless of the individual beliefs.

Along this paper we will use the following terminology and notation. Given partitions P and P' of Ω, we say that P is finer than P' (or, equivalently, P' is coarser than P) if for every $A \in P$ there is $B \in P'$ such that $A \subseteq B$. The join of a set $\{P_h : h \in H\}$ of partitions of Ω, denoted by $\bigvee_{h\in H} P_h$, is the coarsest partition that is finer than

[3] Following the related literature, we consider the strong veto condition requiring that every member in a blocking coalition becomes better off. Even with continuous and monotone utility functions, in differential information economies the weak and strong veto are not equivalent. To show this, consider an economy with three consumers, three states and one commodity. Private information structures are $P_1 = \{\{a, b\}, \{c\}\}$, $P_2 = \{\{a, c\}, \{b\}\}$ and $P_3 = \{\{a\}, \{b, c\}\}$. Endowments are $e_1 = (1, 1, 0)$ $e_2 = (1, 0, 1)$, and $e_3 = (0, 0, 0)$. The expected utility functions are $U_1(x_a, x_b, x_c) = (x_a + x_b)/4 + x_c/2$, $U_2(x_a, x_b, x_c) = (x_a + x_c)/4 + x_b/2$ and $U_3(x_a, x_b, x_c) = (x_b + x_c)/4 + x_a/2$. The endowment allocation is blocked in the weak sense by the big coalition via the allocation that assigns $(0, 0, 1)$ to agent 1, $(0, 1, 0)$ to agent 2 and $(1, 0, 0)$ to agent 3. However, there is no coalition that blocks the endowments in the strong sense.

P_h for every $h \in H$. The meet of $\{P_h : h \in H\}$, denoted by $\bigwedge_{h\in H} P_h$, is the finest partition that is coarser than P_h for every $h \in H$.[4]

We say that $\mathcal{P} = (P_i)_{i\in N}$ dominates $\mathcal{P}' = (P'_i)_{i\in N}$, and we write $\mathcal{P}' < \mathcal{P}$, if P'_i is not finer than P_i for any $i \in N$ and P'_j is strictly coarser than P_j for some $j \in N$. Note that $(\mathbb{P}, \leq)$ is a partially ordered set. Finally, $\widehat{\mathbb{P}} \subseteq \mathbb{P}$ is *comprehensive* if given $\mathcal{P} \in \widehat{\mathbb{P}}$, any $\mathcal{P}' < \mathcal{P}$ belongs also to $\widehat{\mathbb{P}}$.

We remark that a variety of notions of core for asymmetric information economies can be obtained as particular cases of our framework. To be precise, let us consider that every r_i is a degenerated probability distribution supported in an informational profile $\Pi = (\Pi_i)_{i\in N}$. Then, the economy $\mathcal{E}$ becomes a classical differential information economy $\mathcal{E}_{|\Pi}$ in which each agent i has private information given by Π_i. If, in addition, every member in a coalition S has associated the same probability distribution degenerated and supported in some informational profile $\mathcal{P}_S$, then the risky core becomes the core considered in Maus (2003) and Allen (2006).

Moreover, different specifications of the aforementioned profile $\mathcal{P}_S$ result in different well-known core solutions for the economy $\mathcal{E}_{|\Pi}$. Indeed, when $\mathcal{P}_S = (\bigwedge_{i\in S} \Pi_i, \ldots, \bigwedge_{i\in S} \Pi_i)$ the risky core $\mathcal{C}_\rho(\mathcal{E})$ is the coarse core denoted by $\mathcal{C}^\wedge(\mathcal{E}_{|\Pi})$, if $\mathcal{P}_S = \Pi$ we obtain the private core denoted by $\mathcal{C}^\circ(\mathcal{E}_{|\Pi})$, and if $\mathcal{P}_S = (\bigvee_{i\in S} \Pi_i, \ldots, \bigvee_{i\in S} \Pi_i)$ we get the fine core $\mathcal{C}^\vee(\mathcal{E}_{|\Pi})$ (see Yannelis 1991). Our model also captures situations where the information profiles depend on properties of the coalitions themselves. For instance, as it is considered in Hervés-Beloso et al. (2014), the final available information for coalitions may be in relation with their size.

4 Non-emptiness of the risky core

In this section, we provide existence results for the risky core. Our first non-emptiness result is based on requirements on the informational risk that individuals face when forming coalitions. These requirements extend to our framework the *trade boundedness* condition stated in Maus (2003) for an economy without informational risk within coalitions. For it, given a probability distribution φ over $\mathbb{P}$ let $\sigma(\varphi)$ denote its support.

Assumption A The following conditions hold:

(a) Given a coalition $S \subseteq N$ and a comprehensive set $\widehat{\mathbb{P}} \subseteq \mathbb{P}$,

$$\sum_{\mathcal{P}\in\widehat{\mathbb{P}}} r_i^{\mathcal{P}} \leq \sum_{\mathcal{P}\in\widehat{\mathbb{P}}} \rho_i^{\mathcal{P}}(S), \quad \forall i \in S.$$

(b) For each agent $i \in N$, the supports of the probability distributions r_i and $\rho_i(S)$, with $i \in S$, belong to a set $\prod_{j\in N} \mathbb{P}_j$, where every $\mathbb{P}_j$ is a completely ordered set of partitions of Ω.[5]

[4] That is, $A \in \bigvee_{h\in H} P_h$ if and only if $A = \bigcap_{h\in H} A_h$, with $A_h \in P_h$ for every h. In addition, $A \in \bigwedge_{h\in H} P_h$ if and only if for $A = \bigcup_{h\in H} A_h$, with $A_h \in P_h$ for every h.

[5] Given $P, P' \in \mathbb{P}_j$, P is either finer or coarser than P'.

(c) For each $S \subseteq N$, the probability distributions $\{\rho_i(S) : i \in S\}$ have a common support σ_S. In addition, if $\mathcal{P} \in \bigcup_{i\in S} \sigma(r_i)$, then the set $\{\mathcal{P}' \in \sigma_S | \mathcal{P}' \leq \mathcal{P}\}$ has a unique profile.

Condition (a) means that for every $i \in N$ the multivariate probability distribution r_i first-order stochastically dominates $\rho_i(S)$ for every $S \subseteq N$ such that $i \in S$. This dominance property and the fact that each $\mathbb{P}_i$ is completely ordered, as it is stated in condition (b), will allow us to show that the expected utility levels that individuals can attain when they join a coalition can also be attained via a feasible allocation in the economy $\mathcal{E}$.[6] Furthermore, conditions (a) and (c) imply that the possible profiles of information generated by the individuals' beliefs are not comparable: if a coalition gives positive probability to a profile of information, then any other profile in which every member is better informed has null probability. That is, in spite of the informational risk, agents in a coalition exhaust the possibilities to obtain information.

As we have already remarked, Assumption A is an extension of the *trade boundedness* condition required in Maus (2003) in a model that can be obtained from ours by considering that (i) there exists $\bar{\mathcal{P}} \in \mathbb{P}$ such that $r_i^{\bar{\mathcal{P}}} = \rho_i^{\bar{\mathcal{P}}}(N) = 1$ for all $i \in N$; and (ii) for each coalition $S \neq N$, there is $\mathcal{P}(S) \in \mathbb{P}$ such that $\rho_i^{\mathcal{P}(S)}(S) = 1$ for all $i \in S$. In this case, Assumption A implies that $\mathcal{P}(S) \leq \bar{\mathcal{P}}$ for every coalition $S \neq N$, which ensures the aforementioned trade boundedness condition.

Note that the probability distributions defining the coarse and the private cores satisfy this property. However, Assumption A does not hold for the probabilities that define the fine core.

Theorem 1 *Under Assumption A the risky core is non-empty.*

To show this existence result we first prove the non-emptiness of the core of an auxiliary NTU game following analogous arguments to those in Maus (2003, Theorem 5). The existence for the core of the NTU game allows us to obtain an allocation that belongs to $\mathcal{C}_\rho(\mathcal{E})$ (see "Appendix").

Under alternative assumptions, we can obtain further existence results for the risky core without restricting the support of individuals beliefs. First, when there is no uncertainty about the original information profile and agents cannot refine their information when forming coalitions. Second, when individual beliefs do not depend on the coalition that is formed.

Theorem 2 *The risky core of $\mathcal{E} = \mathcal{E}_{|\Pi}$ is non-empty whenever the following conditions hold:*

(a) $\rho_i(S) = \rho_S$ *for every* $i \in S$, *and*
(b) $\sigma(\rho_S) \subseteq \{\mathcal{P} \in \mathbb{P} : \mathcal{P} \leq \Pi\}$.

Theorem 3 $\mathcal{C}_\rho(\mathcal{E})$ *is non-empty whenever* $\rho_i(S) = r_i$ *for all* $i \in S$ *and* $S \subseteq N$.

To prove Theorem 2 we use the fact that the Walrasian expectation equilibrium allocations of the economy $\mathcal{E}_{|\Pi}$ belong to the private core $\mathcal{C}^\circ(\mathcal{E}_{|\Pi})$. On the other hand,

[6] Assumption A(b) implies that the support of possible individual beliefs can be identified with a subset $\mathbb{R}^n$. Thus, we can apply the properties of *multivariate* first-order stochastic dominance in Levhari et al. (1975).

Theorem 3 is a consequence of the non-emptiness of the core in *coalitional exchange economies* (see Del Mercato 2006).

The following example illustrates the relevance of the assumptions in the previous results on existence of the risky core.

Example 1 Let $\mathcal{E}$ be an economy with two consumers, one commodity, and two states of nature at the second period, $\Omega = \{\omega_1, \omega_2\}$. In this context, and to describe individual beliefs, consider the following informational profiles $\mathcal{P}_a = (P_1, P_0)$, $\mathcal{P}_b = (P_0, P_1)$, and $\mathcal{P}_c = (P_1, P_1)$, where $P_0 = \{\Omega\}$ and $P_1 = \{\{\omega_1\}, \{\omega_2\}\}$ are partitions of Ω.

We assume that both agents have one unit of commodity at each state and their beliefs are given by $r_1^{\mathcal{P}_c} = r_2^{\mathcal{P}_c} = 1$. In addition, $U_1(x_1) = \sqrt{x_{1,\omega_1}^{\mathcal{P}_c}} + \sqrt{x_{1,\omega_2}^{\mathcal{P}_c}}$ and $U_2(x_2) = \sqrt{x_{2,\omega_1}^{\mathcal{P}_c}} + \sqrt{x_{2,\omega_2}^{\mathcal{P}_c}}$, where $x_{i,\omega}^{\mathcal{P}}$ denotes the consumption of agent i at state w when the information available is determined by the profile $\mathcal{P}$. The risk ρ is given by $\rho_1^{\mathcal{P}_a}(S) = \rho_2^{\mathcal{P}_b}(S) = 1$ for all $S \subseteq \{1, 2\}$.

Let us show that the risky core is empty. Indeed, since any allocation $x = (x_1, x_2) \in \mathcal{C}_\rho(\mathcal{E})$ is individually rational, the physical feasibility implies that $x_{1,w_1}^{\mathcal{P}_c} = x_{1,\omega_2}^{\mathcal{P}_c} = x_{1,\omega_2}^{\mathcal{P}_c} = x_{2,\omega_2}^{\mathcal{P}_c} = 1$, and then $U_1(x_1) = U_2(x_2) = 2$. However, the grand coalition $\{1, 2\}$ blocks x via any allocation y such that $(y_1^{\mathcal{P}_a}, y_2^{\mathcal{P}_a}) = ((2, 2), (0, 0))$ and $(y_1^{\mathcal{P}_b}, y_2^{\mathcal{P}_b}) = ((0, 0), (2, 2))$.

Since this example verifies Assumptions A(a) and A(b), we have that A(c) cannot be dispensed with in Theorem 1. Moreover, neither (a) in Theorem 2 nor the requirement in Theorem 3 can be deleted. We also remark that although Assumption A(c) holds for the case $r_1^{\mathcal{P}_a} = r_2^{\mathcal{P}_a} = 1$, the risky core is still empty. Hence, Assumption A(a) cannot be dropped in Theorem 1. □

The next example shows that each of our three existence results enlarges the set of economies whose risky core is not empty.

Example 2 Consider an economy with two consumers and where $\{u_i : i \in \{1, 2\}\}$ are continuous, increasing, and concave. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Denote by $P = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}\}$ the information corresponding to distinguish no state and consider also the following partitions:

$$
\begin{aligned}
P_1 &= \{\{\omega_1\}, \{\omega_2, \omega_4\}, \{\omega_3\}\},\ P_1' = \{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}\},\\
P_2 &= \{\{\omega_1, \omega_3\}, \{\omega_2\}, \{\omega_4\}\},\ P_2' = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}.
\end{aligned}
$$

It follows that:

- If agent i's beliefs satisfy $r_i^{(P_1,P)} = r_i^{(P,P_2)} = 0.5$ and $\rho_i^{(P_1',P)}(S) = \rho_i^{(P,P_2')}(S) = 0.5$, then Assumption A holds and the risky core in non-empty by Theorem 1. However, the requirements of Theorems 2 and 3 are not fulfilled.
- If for every i we have that $r_i^{(P_1,P_1)} = 1$ and $\rho_i^{(P_1,P_1)}(S) = \rho_i^{(P_1',P_1')}(S) = 1/2$, then all the conditions in Theorem 2 are verified, but neither Theorems 1 nor 3 can be applied.

– If $r_i^{(P_1,P_1)} = r_i^{(P_2,P_2)} = 0.5$ and $\rho_i^{(P_1,P_1)}(S) = \rho_i^{(P_2,P_2)}(S) = 0.5$ for all $i \in \{1,2\}$, then the risky core is non-empty as a consequence of Theorem 3. However, the assumptions neither in Theorems 1 nor 2 hold.

□

The fine core of a differential information economy $\mathcal{E}_{|\Pi}$ may be empty, because coalitions increase their veto power if blocking allocations are just required to be compatible with the shared information. The next proposition points out an analogous property for the risky core and states a boundary for more general existence results. To be precise, we show that when the fine core associated to a profile $\Pi = (\Pi_i)_{i\in N}$ is empty, the non-emptiness of $\mathcal{C}_\rho(\mathcal{E}_{|\Pi})$ requires that members of any coalition S do not assign large probabilities to the profile $\Pi_S^\vee := (\bigvee_{i\in S}\Pi_i, \ldots, \bigvee_{i\in S}\Pi_i)$.

Proposition 1 *$\mathcal{C}_\rho(\mathcal{E}_{|\Pi})$ is empty whenever the fine core $\mathcal{C}^\vee(\mathcal{E}_{|\Pi})$ is empty and $\min_{S\subseteq N}\min_{i\in S}\rho_i^{\Pi_S^\vee}(S)$ is large enough.*

The variety of situations for which we have obtained existence of the risky core imply non-emptiness for the core notions that have already been analyzed in the literature, except for the fine core. In spite of this, assumptions required in our existence results state limits to the heterogeneity of the possible informational profiles for coalitions. Indeed, we identify some conditions under which the risky core is empty.

5 Incentive compatibility of the risky core

In this section, we analyze incentive compatibility properties of the risky core. For this, by considering that utility functions are separable in states of nature, we extend to our framework the notion of coalitional incentive compatibility in Koutsougeras and Yannelis (1993) and Krasa and Yannelis (1994).

Assumption B For every agent $i \in N$, $u_i(x_i) = \sum_{\omega\in\Omega}\gamma_i(\omega)v_i(x_i^{\mathcal{P}}(\omega))$, where $v_i : \mathbb{R}_+^\ell \to \mathbb{R}_+$ is strictly increasing and $\gamma_i(\omega) > 0$ is the probability that i gives to the realization of $\omega \in \Omega$.

Given a partition P of Ω, let $P(\omega)$ be the set of states of nature that are indistinguishable from ω under the information P. The states of nature ω, ω' are indistinguishable by S under $\mathcal{P} = (P_i)_{i\in N}$ if $\omega' \in P_i(\omega)$ for every $i \in S$.

Definition 2 (*Coalitional incentive compatibility*) A feasible allocation x is coalitionally incentive compatible if it is not possible to find a coalition S, a profile of information $\mathcal{P} = (P_i)_{i\in N}$, and states of nature ω, ω' such that:

(a) $(r_i^{\mathcal{P}})_{i\in N} \gg 0$.
(b) ω and ω' are indistinguishable by S^c under $\mathcal{P}$.
(c) $P_i(\omega) \in \bigwedge_{j\in S} P_j$, for all $i \in S$.
(d) $v_i(e_i(\omega) + x_i^{\mathcal{P}}(\omega') - e_i(\omega')) > v_i(x_i^{\mathcal{P}}(\omega))$ for all $i \in S$.[7]

[7] Note that implicitly $e_i(\omega) + x_i^{\mathcal{P}}(\omega') - e_i(\omega')$ is required to be in $\mathbb{R}_+^\ell$ for every $i \in S$.

The coalitional incentive compatibility of a feasible allocation x ensures that coalitions cannot use their informational advantage, if any, to manipulate the implementation of the contingent contracts determined by x. That is, for any informational profile $\mathcal{P}$ to which all give positive probability, and after the realization of a state of nature ω, the individuals that distinguish ω from ω' have no incentive to announce ω' to the rest of the agents.

Next result states conditions on individual beliefs that suffice to obtain coalitional incentive compatibility for allocations in the risky core. As a consequence we also obtain the compatibility of incentives of the cores analyzed by Maus (2003) and Allen (2006).

Theorem 4 *In addition to Assumption B, suppose that for every $\mathcal{P} = (P_i)_{i\in N}$ with $(r_i^{\mathcal{P}})_{i\in N} \gg 0$, and each $\omega \in \Omega$, we have $P_j(\omega) = \{\omega\}$ for some $j \in N$. Then, any efficient allocation $x \gg 0$ is coalitionally incentive compatible. Therefore, when $\rho_i(N) = r_i$ for every $i \in N$, any allocation $x \gg 0$ in the risky core is coalitionally incentive compatible.*

Thus, the compatibility of incentives of an interior allocation in $\mathcal{C}_\rho(\mathcal{E})$ is guaranteed whenever the formation of the grand coalition does not affect individual beliefs and, regardless of the realized profile of information, each state of nature is distinguished by at least one agent.[8]

Theorem 4 ensures the *ex-post* incentive compatibility of the allocations in the risky core. In a model without informational risk, Allen (2003, 2006) shows that when both feasibility allocations and blocking allocations are restricted to be *ex-ante* incentive compatible the core may be empty. This gives light to the fact that when defining incentive compatibility, the timing of information revelation is an important issue.

6 Further properties of the risky core

This section includes some comparative static results that state how the risky core changes under certain modifications on either beliefs about informational profiles or preferences for consumption. Basically, we formalize the intuition that the blocking power of coalitions is increasing with their information but decreasing with the degree of risk aversion faced by their members.

Definition 3 (*First-order stochastic dominance*) Given risks described by $\bar{\rho} = (\bar{\rho}(S))_{S\subseteq N}$ and $\rho = (\rho(S))_{S\subseteq N}$, we say that $\bar{\rho}$ first-order stochastically dominates ρ when for any coalition S and comprehensive set $\widehat{\mathbb{P}} \subseteq \mathbb{P}$ we have

$$\sum_{\mathcal{P}\in\widehat{\mathbb{P}}} \bar{\rho}_i^{\mathcal{P}}(S) \leq \sum_{\mathcal{P}\in\widehat{\mathbb{P}}} \rho_i^{\mathcal{P}}(S), \quad \forall i \in S.$$

[8] The requirement of Theorem 4 implies that any efficient interior allocation can be obtained by a net trade that satisfies the physical feasibility restriction as an equality. This condition can be dropped in a model without free disposal.

Note that the above definition means that every member of a coalition assigns larger probabilities under $\bar{\rho}$ than under ρ to more informative informational profiles. If it is the case, the next result shows that the risky core $\mathcal{C}_{\bar{\rho}}(\mathcal{E})$ is contained in $\mathcal{C}_{\rho}(\mathcal{E})$.

Proposition 2 *Let $\bar{\rho} = (\bar{\rho}(S))_{S\subseteq N}$ and $\rho = (\rho(S))_{S\subseteq N}$ be risks verifying one of the next conditions:*

(a) *For every $i \in S$ the supports of $\bar{\rho}_i(S)$ and $\rho_i(S)$ belong to a totally ordered subset of $(\mathbb{P}, \leq)$.*
(b) *Assumptions A(b) and A(c) hold for both $\bar{\rho}$ and ρ.*

If $\bar{\rho}$ first-order stochastically dominates ρ, then $\mathcal{C}_{\bar{\rho}}(\mathcal{E}) \subseteq \mathcal{C}_{\rho}(\mathcal{E})$.

To illustrate our results, we adapt our risky core to the Example 5.1 in Koutsougeras and Yannelis (1993) and, by analyzing the outcomes, we provide new insights to the blocking mechanism under asymmetric information.

Example 3 Consider an economy $\mathcal{E}_{|\Pi}$ with one commodity, three consumers, and three states of nature, where initial information and endowments are

$$\Pi_1 = \{\{\omega_1, \omega_2\}, \omega_3\}, \quad \Pi_2 = \{\{\omega_1, \omega_3\}, \omega_2\}, \quad \Pi_3 = \{\{\omega_2, \omega_3\}, \omega_1\};$$
$$e_1 = (10, 10, 0), \quad e_2 = (10, 0, 10), \quad e_3 = (0, 0, 0).$$

Each agent $i \in \{1, 2, 3\}$ has a utility function

$$U_i(x_i) \equiv u_i(x_i^{\Pi}) = \sqrt{x_i^{\Pi}(\omega_1)} + \sqrt{x_i^{\Pi}(\omega_2)} + \sqrt{x_i^{\Pi}(\omega_3)}.$$

To analyze the risky core within this example, we assume no free disposal as Koutsougeras and Yannelis (1993) do for the private core. Let us consider that, before join a coalition, each agent believes that she can obtain the full information available in the coalition with probability κ. Thus, given $\kappa \in [0, 1]$, let ρ_κ be the informational risk satisfying $(\rho_i^{\Pi_S^{\vee}}(S), \rho_i^{\Pi}(S)) = (\kappa, 1-\kappa)$ for every coalition $S \subseteq \{1, 2, 3\}$ and for each agent $i \in S$.

Note that $\mathcal{C}_{\rho_0}(\mathcal{E}_{|\Pi})$ is actually the private core $\mathcal{C}^{\circ}(\mathcal{E}_{|\Pi})$. Moreover, applying Proposition 2, we deduce that the risky core $\mathcal{C}_{\rho_\kappa}(\mathcal{E}_{|\Pi})$ shrinks when κ increases, converging to $\mathcal{C}_{\rho_1}(\mathcal{E}_{|\Pi}) = \mathcal{C}^{\vee}(\mathcal{E}_{|\Pi})$, which is an empty set (see Koutsougeras and Yannelis 1993). Furthermore, as $\min_{S\subseteq N} \min_{i\in S} \rho_i^{\Pi_S^{\vee}}(S) = \kappa$, Proposition 1 implies that $\mathcal{C}_{\rho_\kappa}(\mathcal{E}_{|\Pi})$ is empty when κ is large enough.

That is, in this example, when the veto mechanism entails informational risk in the process of forming coalition, every allocation in the private core of $\mathcal{E}_{|\Pi}$ end up being blocked whenever the probability κ of sharing information becomes sufficiently large. For instance, as pointed out by Koutsougeras and Yannelis (1993), $\widetilde{x} = (\widetilde{x}_1^{\Pi}, \widetilde{x}_2^{\Pi}, \widetilde{x}_3^{\Pi}) = ((8, 8, 2), (8, 2, 8), (4, 0, 0))$ belongs to $\mathcal{C}^{\circ}(\mathcal{E}_{|\Pi})$; however, if $\kappa > 0.7$, then $\widetilde{x}$ can be blocked by the coalition $S = \{1, 2\}$ with any $y \in \mathcal{F}(S)$ such that $(y_1^{\Pi}, y_2^{\Pi}) = (e_1, e_2)$ and $(y_1^{\Pi_S^{\vee}}, y_2^{\Pi_S^{\vee}}) = ((10, 8, 2), (10, 2, 8))$.

Therefore, unlike it happens with the private core, outcomes in the risky core $\mathcal{C}_{\rho_\kappa}(\mathcal{E}_{|\Pi})$ do not necessarily result in an advantage for a consumer who is the unique able to distinguish some state but is endowed with no good. In fact, for the agent 3 to act as an intermediary and be rewarded for it, it is necessary that consumers 1 and 2 give a small probability to share their information when they deviate to form a blocking coalition. □

To analyze the impact that preferences for consumption have in the risky core, denote by $\mathcal{C}_\rho((u_i)_{i\in N})$ the core when agent i's utility function is given by $U_i(x_i) = \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(x_i^{\mathcal{P}})$. In this context, consider the following partial ordering: $(u_i)_{i\in N} \preceq (\bar{u}_i)_{i\in N}$ if and only if $\bar{u}_i = f_i \circ u_i$ for every $i \in N$, where $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and strictly increasing function.

Proposition 3 *If $\mathcal{E} = \mathcal{E}_{|\Pi}$ and $(u_i)_{i\in N} \preceq (\bar{u}_i)_{i\in N}$, then $\mathcal{C}_\rho((u_i)_{i\in N}) \subseteq \mathcal{C}_\rho((\bar{u}_i)_{i\in N})$.*

Following Kihlstrom and Mirman (1974), $(u_i)_{i\in N} \preceq (\bar{u}_i)_{i\in N}$ means that $\bar{u}_i$ is at least as risk averse as u_i for every $i \in N$. In this way, Proposition 3 states that the risky core becomes larger when the risk aversion increases. That is, the higher agents' risk aversion, the lower the blocking power of coalitions.

7 Extensions: ambiguous core and MEU-core

We provide notions of core reflecting *ambiguity in the veto mechanism*, in the sense that individuals are uncertain about the priors relevant to their decisions. That is, when forming coalitions, agents do not behave considering only one distribution of probability about the informational structures.

Our first core concept follows the model of preferences provided by Klibanoff et al. (2005). This approach allows us to separate the attitude that individuals have regarding the risk about the informational profiles from the attitude that they have toward the ambiguity about the distribution of probability representing this risk.

Given a coalition $S \subseteq N$ and an agent $i \in S$, let $\pi_i(S) = (\pi_i(S,\theta))_{\theta\in\Theta}$ be the individual i's subjective beliefs about the probability distribution that describes the informational risk. These beliefs are supported in a finite set Θ of signals.[9] When $\theta \in \Theta$ is realized, i behaves in accordance with a subjective prior $\tau_i(S,\theta)$ about the profile of information that may prevail when S is formed. Denote $\pi(S) = (\pi_i(S); i \in S)$ and $\tau(S) = (\tau_i(S); i \in S)$, where $\tau_i(S) = (\tau_i(S,\theta))_{\theta\in\Theta}$.

Definition 4 (*Ambiguous core*) Given the ambiguity $(\pi,\tau) = (\pi(S),\tau(S))_{S\subseteq N}$, an allocation x is blocked by a coalition $S \subseteq N$ if there is $y \in \mathcal{F}(S)$ such that

$$\Phi^{-1}\left(\sum_{\theta\in\Theta} \pi_i(S,\theta)\ \Phi\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\right)\right) > U_i(x_i), \quad \text{for every } i \in S,$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, strictly increasing, and concave function satisfying $\Phi(0) = 0$. The ambiguous core $\mathscr{C}_\tau^\pi(\mathcal{E},\Phi)$ is the set of feasible allocations that are not blocked by any coalition.

[9] To simplify, the set Θ is assumed to be finite. However, the results we obtain can be recast without this assumption by adding the adequate topological properties on the set of signals.

We remark that Φ characterizes ambiguity aversion in the sense of Klibanoff et al. (2005). As in the risky core, u_i represents individual i's preferences on consumption and incorporates her attitude toward risk. Note that, before joining a coalition S to deviate via an allocation $y \in \mathcal{F}(S)$, each member $i \in S$ determines her potential expected gains in two stages. First, i estimates the expected utility $\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}})$ for every risky scenario $\theta \in \Theta$. Then, using her beliefs $\pi_i(S)$ about the realization of the different informational risks, the previous utility levels are updated with the function Φ, which measures her ambiguity attitude. Therefore, the blocking mechanism that leads to the ambiguous core allows us to separate the *tastes* of agent i about risk and ambiguity, represented by u_i and Φ, respectively, from her *beliefs* about the informational risk and informational ambiguity, represented by $\tau_i(S)$ and $\pi_i(S)$, respectively.

Each ambiguity $\alpha = (\pi, \tau)$ induces a risk ρ_α that is obtained by the reduction of first and second-order probabilities about the informational outcome, i.e., $\rho_{\alpha,i}^{\mathcal{P}}(S) = \sum_{\theta\in\Theta} \pi_i(S,\theta)\tau_i^{\mathcal{P}}(S,\theta)$. Hence, when agents are ambiguity neutral, i.e., $\Phi(t) = at$, with $a > 0$, the ambiguous core coincides the risky core $\mathcal{C}_{\rho_\alpha}(\mathcal{E})$.

The attitude toward ambiguity can be partially ordered as follows: $\Phi \preceq \bar{\Phi}$ if and only if $\bar{\Phi} = f \circ \Phi$ for some concave and strictly increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$. That is, $\Phi \preceq \bar{\Phi}$ means that agents are more *ambiguity averse* under $\bar{\Phi}$ than under Φ (see Definition 5 and Theorem 2 in Klibanoff et al. 2005).

Next theorem shows that the blocking power of coalitions is non-increasing with respect the individuals' ambiguity aversion, that is, the ambiguous core shrinks when the ambiguity aversion becomes smaller. Also, it guarantees the non-emptiness of the ambiguous core whenever the risky core defined by ρ_α is not empty (see Theorems 1, 2, 3).

Theorem 5 *Given* $\alpha = (\pi, \tau)$ *we have* $\mathcal{C}_{\rho_\alpha}(\mathcal{E}) \subseteq \mathscr{C}_\tau^\pi(\mathcal{E}, \Phi) \subseteq \mathscr{C}_\tau^\pi(\mathcal{E}, \bar{\Phi})$ *whenever* $\Phi \preceq \bar{\Phi}$.

Our second core concept is focused on the maxmin expected utility model (MEU) introduced by Gilboa and Schmeidler (1989). In this case, individuals are pessimistic and they required to become better off in the worst scenario in order to block an allocation.

Definition 5 (MEU-*core*) Given $\tau = (\tau(S))_{S\subseteq N}$ an allocation x is blocked by a coalition S if there is $y \in \mathcal{F}(S)$ such that

$$\min_{\theta\in\Theta} \sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}}) > U_i(x_i), \quad \text{for every } i \in S.$$

The MEU-core $\mathscr{C}_\tau(\mathcal{E})$ is the set of feasible allocations that are not blocked by any coalition.

Note that, for each $\theta \in \Theta$, τ induces the informational risk τ_θ described by the set of probability distributions $\tau_\theta(S) = \{\tau_i(S,\theta) : i \in S\}$ for each coalition $S \subseteq N$. The following result describes relationships of the MEU-core with both the ambiguous core and the risky core. In this way, as a consequence of our previous existence theorems, we obtain sufficient conditions to ensure the non-emptiness of the MEU-core.

Theorem 6 *Given an ambiguity* (π, τ), *we have* $\mathcal{C}_{\tau_\theta}(\mathcal{E}) \cup \mathscr{C}^{\pi}_{\tau}(\mathcal{E}, \Phi) \subseteq \mathscr{C}_{\tau}(\mathcal{E})$ *for every* $\theta \in \Theta$.

As it occurs for the risky core, the MEU-core is increasing with the degree of individuals' risk aversion. This can be shown following the proof of Proposition 3.

The notion of coalitional incentive compatibility is based on ex-post behaviors and does not depend on the characteristics of risk or ambiguity faced by individuals (see Definition 2). Therefore, if the utility functions are separable in the states of nature and there is no ambiguity for the grand coalition, then the allocations either in the ambiguous core or in the MEU-core are incentive compatible. To be precise, as an immediate consequence of the proof of Theorem 4 we obtain the following result.

Theorem 7 *Suppose that Assumption B and the following conditions are satisfied:*

(a) *Given* $\theta \in \Theta$, $\tau_i(N, \theta) = r_i$ *for every* $i \in N$.
(b) *For each* $\mathcal{P} = (P_i)_{i \in N}$ *and* $\omega \in \Omega$, $P_j(\omega) = \{\omega\}$ *for some* $j \in N$ *whenever* $(r_i^{\mathcal{P}})_{i \in N} \gg 0$.

Then, any feasible allocation $x \gg 0$ *in* $\mathscr{C}^{\pi}_{\tau}(\mathcal{E}, \Phi) \cup \mathscr{C}_{\tau}(\mathcal{E})$ *is coalitionally incentive compatible.*

Analogous to risk theory, given a twice continuously differentiable and strictly increasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, let $\mathrm{AP}(\Phi, t) = -\Phi''(t)/\Phi'(t)$ be the Arrow-Pratt measure of *absolute ambiguity aversion* at point t. When ambiguity aversion increases to infinity, Klibanoff et al. (2005) show that agents asymptotically exhibit a maxmin expected utility behavior *á la* Gilboa and Schmeidler (1989). This result for preferences leads us to prove that the MEU-core can be characterized in terms of the limit of a sequence of ambiguous cores.

Proposition 4 *Let* $\{\Phi_n\}_{n \in \mathbb{N}}$ *be a sequence of functions such that:*

(a) $\Phi_n : \mathbb{R}_+ \to \mathbb{R}_+$ *is twice continuously differentiable and strictly increasing.*
(b) $0 \leq \mathrm{AP}(\Phi_n, t) \leq \mathrm{AP}(\Phi_{n+1}, t)$ *for all* $t \in \mathbb{R}_+$.
(c) $\Phi_n(0) = 0$ *and* $\lim_{n \to +\infty} \inf_{t \geq 0} \mathrm{AP}(\Phi_n, t) = +\infty$.

Then, the increasing sequence $\{\mathscr{C}^{\pi}_{\tau}(\mathcal{E}, \Phi_n)\}_{n \in \mathbb{N}}$ *satisfies*[10]

$$\mathscr{C}_{\tau}(\mathcal{E}) = \overline{\lim_{n \to \infty} \mathscr{C}^{\pi}_{\tau}(\mathcal{E}, \Phi_n)}.$$

To analyze the effect of ambiguity on the veto power of coalitions, we have applied two decision models, namely, the differentiable model of Klibanoff et al. (2005) and the maxmin expected utility model of Gilboa and Schmeidler (1989). Alternative approaches to ambiguity are the Choquet expected utility (CEU) model of Schmeidler (1989) and the α-maxmin expected utility (α-MEU) model of Ghirardato et al. (2004). Although the study of the properties of the core concepts induced by these alternative models is a matter of future research, one may expect some analogous results to those obtained in this paper.

[10] Recall that $\lim_{n \to \infty} A_n = \bigcup_{n \in \mathbb{N}} A_n$ for any increasing sequence of sets $\{A_n\}_{n \in \mathbb{N}}$.

8 Some final remarks

To define the new cooperative solutions provided in this paper, by considering risk and ambiguity regarding the information when forming coalitions, the feasibility of allocations in the economy and the attainable allocations for coalitions entail measurability requirements. As it is known, measurability restrictions reduce the possibilities of trading and as a consequence avoid first best efficient outcomes. In fact, the conflict that may appear between efficiency and incentive compatibility becomes an issue in mechanism design within an asymmetric information framework.

To overcome the aforementioned weakness, recent works consider that agents adopt a pessimistic behavior, maximizing interim expected utility functions taking into account the worse potential scenario to occur. Following this approach, we find papers on the core with maximin preferences where no information measurability restrictions are made (see de Castro et al. 2011) and guaranteeing that any efficient allocation is incentive compatible (see de Castro and Yannelis 2010). Also, within this context, any individually rational and efficient allocation can be implementable as a maximin equilibrium by a direct revelation mechanism (see de Castro et al. 2017a, b; Liu 2016). The ideas in the present work could also be used to recast the risky and the ambiguous core notions in those settings and also further implementability results should be pursued as future research.

9 Appendix

Proof of Theorem 1 Consider the NTU game (N, V) where the correspondence $V : 2^N \backslash \{\emptyset\} \twoheadrightarrow \mathbb{R}^n$ associates to each coalition $S \neq N$ the set

$$V(S) = \{a \in \mathbb{R}^n \mid \text{there exists a net trade } (z_i^{\mathcal{P}})_{i \in S, \mathcal{P} \in \mathbb{P}} \text{ attainable for } S \\ \text{such that, } (e_i + z_i^{\mathcal{P}})_{\mathcal{P} \in \mathbb{P}} \geq 0 \\ \text{and } a_i \leq \textstyle\sum_{\mathcal{P} \in \mathbb{P}} \rho_i^{\mathcal{P}}(S) u_i(e_i + z_i^{\mathcal{P}}), \text{ for every } i \in S\},$$

and associates to the grand coalition the set

$$V(N) = \{a \in \mathbb{R}^n \mid \text{there exists a feasible net trade } (z_i^{\mathcal{P}})_{i \in N, \mathcal{P} \in \mathbb{P}} \text{ such that,} \\ (e_i + z_i^{\mathcal{P}})_{\mathcal{P} \in \mathbb{P}} \geq 0 \\ \text{and } a_i \leq \textstyle\sum_{\mathcal{P} \in \mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + z_i^{\mathcal{P}}), \text{ for every } i \in N\}.$$

By Scarf (1967), to prove that (N, V) has a non-empty core, i.e., $V(N) \backslash \bigcup_{S \subseteq N S \neq \emptyset} \text{int}(V(S)) \neq \emptyset$, it suffices to show that (N, V) is balanced, that is, $\bigcap_{S \in \mathcal{S}} V(S) \subseteq V(N)$ for any balanced set of coalitions $\mathcal{S}$.

Let $\mathcal{S}$ be a balanced set of coalitions and let $(a_i)_{i \in N} \in \bigcap_{S \in \mathcal{S}} V(S)$. It follows that, for every coalition $S \in \mathcal{S}$ there exists a net trade $(z_i^{\mathcal{P}}(S))_{i \in S, \mathcal{P} \in \mathbb{P}}$ attainable for S such that $(e_i + z_i^{\mathcal{P}}(S))_{\mathcal{P} \in \mathbb{P}} \geq 0$ and $a_i \leq \sum_{\mathcal{P} \in \mathbb{P}} \rho_i^{\mathcal{P}}(S) u_i(e_i + z_i^{\mathcal{P}}(S))$, $\forall i \in S$. Moreover, for any $S \neq N$, the net trade can be chosen verifying $u_i(e_i + z_i^{\mathcal{P}}(S)) \leq u_i(e_i + z_i^{\mathcal{P}'}(S))$ whenever $\mathcal{P} < \mathcal{P}'$ and $\mathcal{P}, \mathcal{P}' \in \sigma(r_i) \cup \sigma_S$. To see this, note that if $\mathcal{P} \in \sigma_S$, then

Assumptions A(a) and A(c) guarantee that there is no $\mathcal{P}' \in \sigma_S$ such that either $\mathcal{P}' < \mathcal{P}$ or $\mathcal{P} < \mathcal{P}'$. Therefore, to ensure the monotonicity of the mapping $\mathcal{P} \to u_i(e_i + z_i^{\mathcal{P}}(S))$ on $\sigma(r_i) \cup \sigma_S$, we can assume that for every $\mathcal{P} \in \sigma(r_i) \setminus \sigma_S$ the net trades verify $z_j^{\mathcal{P}}(S) = z_j^{\bar{\mathcal{P}}}(S)$, being $\bar{\mathcal{P}} \in \sigma_S$ the unique informational profile satisfying $\bar{\mathcal{P}} \leq \mathcal{P}$ whose existence is guaranteed by Assumption A(c).

Without loss of generality, define $z_i^{\mathcal{P}}(S) = 0$ for every $i \notin S$. Hence, $\sum_{i\in N} z_i^{\mathcal{P}}(S) = \sum_{i\in S} z_i^{\mathcal{P}}(S) \leq 0$. Since $\mathcal{S}$ is balanced, there exists a function $\alpha : \mathcal{S} \to (0, 1]$ such that $\sum_{S\in\mathcal{S}:i\in S} \alpha(S) = 1$ for every $i \in N$. Given $i \in N$ and $\mathcal{P} = (P_i)_{i\in N} \in \mathbb{P}$, let $z_i^{\mathcal{P}} := \sum_{S\in\mathcal{S}:i\in S} \alpha(S) z_i^{\mathcal{P}}(S) = \sum_{S\in\mathcal{S}} \alpha(S) z_i^{\mathcal{P}}(S)$. Note that $z_i^{\mathcal{P}}$ is P_i-measurable, $e_i + z_i^{\mathcal{P}} \geq 0$, and $\sum_{i\in N} z_i^{\mathcal{P}} \leq 0$. Furthermore, the concavity of utility functions implies that, for every $i \in N$,

$$\begin{aligned}\sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + z_i^{\mathcal{P}}) &= \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i\left(\sum_{S\in\mathcal{S}, i\in S} \alpha(S)(e_i + z_i^{\mathcal{P}}(S))\right)\\ &\geq \sum_{S\in\mathcal{S}, i\in S} \alpha(S) \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + z_i^{\mathcal{P}}(S))\\ &\geq \min_{S\in\mathcal{S}, i\in S} \left\{\sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + z_i^{\mathcal{P}}(S))\right\}.\end{aligned}$$

For any coalition S, by conditions (a) and (b) in Assumption A and the monotonicity of $\mathcal{P} \to u_i(e_i + z_i^{\mathcal{P}}(S))$ one obtains that $\sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + z_i^{\mathcal{P}}(S)) \geq \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S) u_i(e_i + z_i^{\mathcal{P}}(S)) \geq a_i$ (see Levhari et al. 1975). Therefore, $\sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i (e_i + z_i^{\mathcal{P}}) \geq a_i$, which implies that $(a_i)_{i\in N} \in V(N)$. We conclude that the core of (N, V) is non-empty.

Consider a vector a in the core of (N, V). Since $a \in V(N) \setminus \bigcup_{S\subseteq N\, S\neq\emptyset} \mathrm{int}(V(S))$, there is a feasible allocation $x = (x_i)_{i\in N}$ such that

$$a \leq (U_i(x_i))_{i\in N}, \left(\{(U_i(x_i))_{i\in N}\} + \mathbb{R}^n_{++}\right) \cap V(N) = \emptyset,$$

and x is not blocked by any coalition $S \neq N$. To prove that x belongs to the risky core it remains to ensure that it cannot be blocked by the grand coalition. By contradiction, suppose that there is a feasible net trade $(\tilde{z}_i^{\mathcal{P}})_{i\in N,\mathcal{P}\in\mathbb{P}}$ such that $(e_i + \tilde{z}_i^{\mathcal{P}})_{\mathcal{P}\in\mathbb{P}} \geq 0$ and $\sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(N) u_i(e_i + \tilde{z}_i^{\mathcal{P}}) > U_i(x_i)$ for every $i \in N$. As before, $(\tilde{z}_i^{\mathcal{P}})_{i\in N,\mathcal{P}\in\mathbb{P}}$ can be chosen verifying $u_i(e_i + \tilde{z}_i^{\mathcal{P}}) \leq u_i(e_i + \tilde{z}_i^{\mathcal{P}'})$ whenever $\mathcal{P} < \mathcal{P}'$ and $\mathcal{P}, \mathcal{P}' \in \sigma(r_i) \cup \sigma_N$. Hence, (a) and (b) in Assumption A guarantee that,

$$\sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}} u_i(e_i + \tilde{z}_i^{\mathcal{P}}) \geq \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(N) u_i(e_i + \tilde{z}_i^{\mathcal{P}}) > U_i(x_i), \quad \forall i \in N,$$

which is a contradiction with the fact that $\left(\{(U_i(x_i))_{i\in N}\} + \mathbb{R}^n_{++}\right) \cap V(N) = \emptyset$. □

Proof of Theorem 2 Under continuity, concavity, and locally non-satiability of utility functions, the set of Walrasian expectations equilibrium allocations is non-empty and it is included in the private core $\mathcal{C}^\circ(\mathcal{E}_{|\Pi})$. Let $x \in \mathcal{C}^\circ(\mathcal{E}_{|\Pi})$ such that $x \notin \mathcal{C}_\rho(\mathcal{E})$. Then, we can find a coalition S and $y \in \mathcal{F}(S)$ such that $\sum_{\mathcal{P}\in\mathbb{P}} \rho_S^{\mathcal{P}}\ u_i(y_i^{\mathcal{P}}) > u_i(x_i^{\Pi})$ for every $i \in S$. Since, by condition (b), agents are unable to get more information than the given by Π we have that $y_i^{\mathcal{P}}$ is Π_i-measurable. The concavity of the functions u_i implies that $u_i\left(\sum_{\mathcal{P}\in\mathbb{P}} \rho_S^{\mathcal{P}}\ y_i^{\mathcal{P}}\right) > u_i(x_i^{\Pi})$ for every $i \in S$. Condition (a) allows us to obtain that the allocation that assigns to each $i \in S$ the bundle $\sum_{\mathcal{P}\in\mathbb{P}} \rho_S^{\mathcal{P}}\ y_i^{\mathcal{P}}$ is physically attainable for S. This is in contradiction with the fact that $x \in \mathcal{C}^\circ(\mathcal{E}_{|\Pi})$. □

Proof of Theorem 3 Our economy can be recast as a *coalitional exchange economy* (see Del Mercato 2006). In this framework, neither utility functions nor consumption sets depend on coalitions. Therefore, the non-emptiness of the risky core is a consequence of Theorem 5 in Del Mercato (2006). □

Proof of Proposition 1 The continuity of utility functions, the compactness of sets $\mathcal{F}$ and $\{\mathcal{F}(S)\}_{S\subseteq N}$, and the emptiness of the fine core $\mathcal{C}^\vee(\mathcal{E}_{|\Pi})$ ensure that the continuous mapping

$$(x,\theta) \in \mathcal{F}\times[0,1] \longrightarrow \Phi(x,\theta) := \max_{S\subseteq N}\ \max_{y\in\mathcal{F}(S)}\ \min_{i\in S}\left(\theta u_i(y_i^{\Pi_S^\vee}) - u_i(x_i^{\Pi})\right)$$

satisfies the following properties: (i) for any $\theta', \theta \in [0,1]$, if $\theta' > \theta$ and $\Phi(x,\theta) > 0$, then $\Phi(x,\theta') > \Phi(x,\theta)$; and (ii) there exists $a > 0$ such that $\Phi(x,1) \geq a, \forall x \in \mathcal{F}$.[11] Given $x \in \mathcal{F}$, let $\Theta(x) = \{\theta \in [0,1] : \Phi(x,\theta) \geq \bar{a}\}$, where $\bar{a} \in (0,a)$. It is not difficult to verify that Θ is a continuous correspondence with non-empty and compact values.[12] Therefore, the Berge's Maximum Theorem guarantees that $x \in \mathcal{F} \longrightarrow \min\{\theta : \theta \in \Theta(x)\}$ is a continuous function and it has values strictly lower than one. We conclude that there exists $\kappa \in (0,1)$ such that $\Phi(x,\theta) > 0, \forall x \in \mathcal{F}, \forall\theta \in [\kappa,1]$.

Notice that $\mathcal{C}_\rho(\mathcal{E}_{|\Pi}) = \{x \in \mathcal{F} : \Omega(\rho,x) \leq 0\}$ where

$$\Omega(\rho,x) := \max_{S\subseteq N}\ \max_{y\in\mathcal{F}(S)}\ \min_{i\in S}\left(\sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)\, u_i(y_i^{\mathcal{P}}) - u_i(x_i^{\Pi})\right).$$

Therefore, $x \notin \mathcal{C}_\rho(\mathcal{E}_{|\Pi})$ if and only if $\Omega(\rho,x) > 0$.

Since u_i takes non-negative values, $\Omega(\rho,x) \geq \Phi\left(x, \min_{S\subseteq N}\ \min_{i\in S} \rho_i^{\Pi_S^\vee}(S)\right)$. Therefore, if $\min_{S\subseteq N}\ \min_{i\in S} \rho_i^{\Pi_S^\vee}(S) \geq \kappa$, then $\mathcal{C}_\rho(\mathcal{E}_{|\Pi}) = \emptyset$. □

[11] Since $\mathcal{C}^\vee(\mathcal{E}_{|\Pi}) = \emptyset$, we have that $\Phi(x,1) > 0, \forall x \in \mathcal{F}$. Thus, (ii) is a direct consequence of the compactness of $\mathcal{F}$ and the continuity of Φ.

[12] For any $x \in \mathcal{F}$, $1 \in \Theta(x)$. Also, the continuity of Φ guarantees that $\Theta(x)$ is a closed subset of [0, 1]. Therefore, Θ has non-empty and compact values. Θ is upper hemicontinuous, because has closed graph and compact codomain.
Since $x \twoheadrightarrow \dot{\Theta}(x) := \{\theta \in [0,1] : \Phi(x,\theta) > \bar{a}\}$ has open graph, its closure is lower-hemicontinuous. In addition, given $x \in \mathcal{F}$ and $\theta \in \Theta(x) \cap [0,1)$, it follows from (i) that for any $n \in \mathbb{N}$ we have that $\Phi(x, n^{-1} + (1-n^{-1})\theta) > \Phi(x,\theta) \geq \bar{a}$. Hence $\{n^{-1} + (1-n^{-1})\theta\}_{n\in\mathbb{N}} \subseteq \dot{\Theta}(x)$ and converges to θ. This ensures that $\Theta(x) \subseteq \overline{\dot{\Theta}}(x)$. We conclude that $\Theta = \overline{\dot{\Theta}}$, which implies in the lower hemicontinuity of Θ.

Proof of Theorem 4 Let $x \gg 0$ be an efficient allocation. Thus, there is a feasible net trade z given by vectors $z_i^{\mathcal{P}} = (z_i^{\mathcal{P}}(\omega))_{\omega \in \Omega}$ such that $x_i^{\mathcal{P}}(\omega) = e_i(\omega) + z_i^{\mathcal{P}}(\omega)$, for each consumer i and every $\mathcal{P} \in \mathbb{P}$. If x is not coalitionally incentive compatible, then one can find a coalition S, a profile $\mathcal{P} = (P_i)_{i \in N}$, and states of nature ω, ω' for which conditions (a)–(d) in Definition 2 hold.

Since $\omega' \in \bigcap_{i \notin S} P_i(\omega)$, one has $\sum_{i \notin S} z_i^{\mathcal{P}}(\omega) = \sum_{i \notin S} z_i^{\mathcal{P}}(\omega')$. By the assumptions in the statement of the theorem and monotonicity of preferences, we can take z such that $\sum_{i \in N} z_i^{\mathcal{P}}(\omega) = 0$ and $\sum_{i \in N} z_i^{\mathcal{P}}(\omega') = 0$. We can deduce that $\sum_{i \in S} z_i^{\mathcal{P}}(\omega) = \sum_{i \in S} z_i^{\mathcal{P}}(\omega')$. For each $i \in S$ and $\kappa \in \Omega$, consider the net trade given by

$$\bar{z}_i^{\mathcal{P}}(\kappa) = \begin{cases} z_i^{\mathcal{P}}(\kappa) \text{ if } \kappa \notin P_i(\omega), \\ z_i^{\mathcal{P}}(\omega') \text{ if } \kappa \in P_i(\omega). \end{cases}$$

By construction, $\bar{z}_i^{\mathcal{P}}$ is P_i-measurable for every $i \in S$ and, by condition (c) in Definition 2, one has $\sum_{i \in S} \bar{z}_i^{\mathcal{P}}(\kappa) + \sum_{i \notin S} z_i^{\mathcal{P}}(\kappa) = 0$, for every $\kappa \in \Omega$. Consider the feasible allocation $\bar{x}$ defined as

$$\bar{x}_i^{\mathcal{P}'} = \begin{cases} x_i^{\mathcal{P}'} \text{ if } \mathcal{P}' \neq \mathcal{P} \text{ or } i \notin S, \\ e_i + \bar{z}_i^{\mathcal{P}}, \text{ otherwise.} \end{cases}$$

Note that $\bar{x}_i^{\mathcal{P}}(\omega) = e_i(\omega) + z_i^{\mathcal{P}}(\omega')$ for every $i \in S$. From Definition 2(d), it follows that $v_i(\bar{x}_i^{\mathcal{P}}(\omega)) > v_i(x_i^{\mathcal{P}}(\omega))$ for every $i \in S$. Definition 2(a) implies that $U_i(\bar{x}_i) > U_i(x_i)$, for every $i \in S$, and $U_i(\bar{x}_i) = U_i(x_i)$, otherwise.

Fix an agent $h \in S$. By continuity of v_h there is a non-null vector $\varepsilon \in \mathbb{R}_+^{\ell}$ such that $v_h(\bar{x}_h^{\mathcal{P}}(\omega) - \varepsilon) > v_h(x_h^{\mathcal{P}}(\omega))$. By construction of $\bar{x}$, since x is an interior allocation we can take ε such that $U_h(\hat{x}_h) > U_h(x_h)$, where $\hat{x}_h$ is given by $\hat{x}_h^{\mathcal{P}}(\kappa) = \bar{x}_h^{\mathcal{P}}(\kappa) - \varepsilon$ for every $\kappa \in \Omega$. Let m be the number of members in S^c and consider the allocation $\tilde{x}$ given by $\tilde{x}_h = \hat{x}_h,\ \tilde{x}_i = \bar{x}_i$ for every $i \in S \setminus \{h\}$, and for each $i \notin S$ and $\kappa \in \Omega$:

$$\tilde{x}_i^{\mathcal{P}}(\kappa) = \begin{cases} x_i^{\mathcal{P}'}(\kappa) \text{ if } \mathcal{P}' \neq \mathcal{P}, \\ x_i^{\mathcal{P}}(\kappa) + \frac{\varepsilon}{m}, \text{ otherwise.} \end{cases}$$

By strict monotonicity of the functions v_i we have $U_i(\tilde{x}_i) > U_i(x_i)$ for every $i \in N$. A contradiction with the efficiency of x. □

The following auxiliary result will be used to prove Proposition 2.

Lemma *Suppose that the probability distribution $\hat{\nu}$ given by the vector of probabilities $(\hat{\nu}^1, \ldots, \hat{\nu}^m)$ over the ordered set $\{1, \ldots, m\}$ first-order stochastically dominates $\nu = (\nu^1, \ldots, \nu^m)$. Then, for each $k \in \{1, \ldots, m\}$ and $h \in \{1, \ldots, k\}$ there exists $a_{k,h} \geq 0$ verifying*

$$\sum_{h=k}^{m} a_{h,k} = \nu^k, \quad \sum_{h=1}^{k} a_{k,h} = \hat{\nu}^k, \quad \forall k \in \{1, \ldots, m\}.$$

Proof We show it by induction. When $m = 1$ there is no uncertainty and the result trivially holds. Assume that the result is true for $m = t$ and let us prove that it is also true for $m = t+1$. Notice that $\hat{\nu}_* = (\hat{\nu}^1, \dots, \hat{\nu}^{t-1}, \hat{\nu}^t + \hat{\nu}^{t+1})$ first-order stochastically dominates $\nu_* = (\nu^1, \dots, \nu^{t-1}, \nu^t + \nu^{t+1})$. Therefore, it follows from the induction hypothesis that, for each $k \in \{1, \dots, t\}$ and $h \in \{1, \dots, k\}$ there exists $a^*_{k,h} \geq 0$ verifying $\sum_{h=k}^{t} a^*_{h,k} = \nu^k_*$ and $\sum_{h=1}^{k} a^*_{k,h} = \hat{\nu}^k_*$.

Given $k \in \{1, \dots, t+1\}$ and $h \in \{1, \dots, k\}$, define

$$a_{k,h} = \begin{cases} a^*_{k,h}, & h \leq k < t; \\ a^*_{k,h} - \alpha_h, & h < k = t; \\ a^*_{k,h} - \nu^{t+1} - \alpha_h, & h = k = t; \\ \alpha_h, & h < k = t+1; \\ \nu^{t+1}, & h = k = t+1, \end{cases}$$

where $(\alpha_h)_{1 \leq h \leq t} \geq 0$ satisfies $\sum_{h=1}^{t} \alpha_h = \hat{\nu}^{t+1} - \nu^{t+1}$.

It follows that $\sum_{h=k}^{t+1} a_{h,k} = \nu^k$ and $\sum_{h=1}^{k} a_{k,h} = \hat{\nu}^k$, for all $k \in \{1, \dots, t+1\}$. □

Proof of Proposition 2 Let S be a coalition that blocks $x \notin \mathcal{C}_\rho(\mathcal{E})$. That is, there exists $y \in \mathcal{F}(S)$ such that $\sum_{\mathcal{P} \in \mathbb{P}} \rho_i^{\mathcal{P}}(S)\ u_i(y_i^{\mathcal{P}}) > U_i(x_i), \forall i \in S$.

Since (a) holds, the supports of $\bar{\rho}_i(S)$ and $\rho_i(S)$ are contained in a finite totally ordered set $\psi_i(S) \subset \mathbb{P}$. Denoting by $m(S,i)$ the cardinality of $\psi_i(S)$, we write $\psi_i(S) = \{\mathcal{P}_i^1, \dots, \mathcal{P}_i^{m(S,i)}\} \subset \mathbb{P}$, where $\mathcal{P}_i^h = (P_{i,j}^h, j \in S)$ and $P_{i,j}^h \leq P_{i,j}^{\tilde{h}}$, for every $h \leq \tilde{h}$ and every $j \in S$. Furthermore, as $\bar{\rho}$ first-order stochastically dominates ρ, it follows from the previous Lemma that, for each $k \in \{1, \dots, m(S,i)\}$ and $h \in \{1, \dots, k\}$ there exists $a_{k,h}^i \geq 0$ such that,

$$\sum_{h=k}^{m(S,i)} a_{h,k}^i = \rho_i^{\mathcal{P}_i^k}(S), \qquad \sum_{h=1}^{k} a_{k,h}^i = \bar{\rho}_i^{\mathcal{P}_i^k}(S), \qquad \forall k \in \{1, \dots, m(S,i)\}.$$

Therefore, the fact that $y \in \mathcal{F}(S)$ ensures that the allocation $\hat{y}$ characterized by

$$\hat{y}_i^{\mathcal{P}_i^k} = \sum_{h=1}^{k} \frac{a_{k,h}^i}{\bar{\rho}_i^{\mathcal{P}_i^k}(S)} y_i^{\mathcal{P}_i^h}, \qquad \forall i \in S,\ \forall k \in \{1, \dots, m(S,i)\},$$

is attainable for S. Furthermore, the concavity of the functions u_i implies that

$$\begin{aligned} \sum_{k=1}^{m(S,i)} \bar{\rho}_i^{\mathcal{P}_i^k}(S)\ u_i(\hat{y}_i^{\mathcal{P}_i^k}) &\geq \sum_{h=1}^{m(S,i)} \left(\sum_{k=h}^{m(S,i)} a_{k,h}^i \right) u_i(y_i^{\mathcal{P}_i^h}) \\ &= \sum_{h=1}^{m(S,i)} \rho_i^{\mathcal{P}_i^h}(S)\ u_i(y_i^{\mathcal{P}_i^h}) > U_i(x_i), \quad \forall i \in S. \end{aligned}$$

We conclude that $x \notin \mathcal{C}_{\bar{\rho}}(\mathcal{E})$.

If *(b)* holds, as in the proof of Theorem 1, the blocking allocation y at the beginning of this proof can be chosen verifying $u_i(y_i^{\mathcal{P}}(S)) \leq u_i(y_i^{\mathcal{P}'}(S))$ whenever $\mathcal{P} < \mathcal{P}'$ for every $i \in S$. Indeed, given $\bar{\mathcal{P}} \in \mathbb{P}$ such that $\rho_i^{\bar{\mathcal{P}}}(S) > 0$ for some $i \in S$, by Assumption A(c) one has $(\rho_j^{\mathcal{P}}(S))_{j\in N} = 0$ for any $\mathcal{P} \in \mathbb{P}$ satisfying $\mathcal{P} < \bar{\mathcal{P}}$ or $\bar{\mathcal{P}} < \mathcal{P}$. Then, to ensure the monotonicity of the mapping $\mathcal{P} \to u_i(y_i^{\mathcal{P}}(S))$ on the set $\{\mathcal{P} \in \mathbb{P} : \mathcal{P} \leq \bar{\mathcal{P}} \vee \bar{\mathcal{P}} \leq \mathcal{P}\}$, it suffices to define $y_j^{\mathcal{P}}(S) := 0$ when $\mathcal{P} < \bar{\mathcal{P}}$ and $y_j^{\mathcal{P}}(S) := y_j^{\bar{\mathcal{P}}}(S)$ when $\bar{\mathcal{P}} < \mathcal{P}$.

By Assumption A(b), the support of $\bar{\rho}_i(S)$ (resp., $\rho_i(S)$) is contained in a cartesian product $\bar{\mathbb{H}}^i = \prod_{j\in N} \bar{\mathbb{H}}_j^i$ (resp., $\mathbb{H}^i = \prod_{j\in N} \mathbb{H}_j^i$) of completely ordered sets of partitions of Ω. Hence, for every $i \in S$, the supports of $\bar{\rho}_i(S)$ and $\rho_i(S)$ are contained in $\prod_{j\in N} \mathbb{P}_j$, where $\mathbb{P}_j$ is the completely ordered set of partitions of Ω given by the j-projection of $\bar{\mathbb{H}}^i \cup \mathbb{H}^i$. It follows from Levhari et al. (1975) that the first-order stochastic dominance of $\bar{\rho}$ over ρ joint with the monotonicity of the mappings $\mathcal{P} \to u_i(y_i^{\mathcal{P}}(S))$, imply that

$$\sum_{\mathcal{P}\in\mathbb{P}} \bar{\rho}_i^{\mathcal{P}}(S)\ u_i(y_i^{\mathcal{P}}) \geq \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)\ u_i(y_i^{\mathcal{P}}), \quad \forall i \in S.$$

As in the previous case, we conclude that $x \notin \mathcal{C}_{\bar{\rho}}(\mathcal{E})$. □

Proof of Proposition 3 Let S be a coalition that blocks $x \notin \mathcal{C}_{\rho}((\overline{u}_i)_{i\in N})$. That is, there exists $y \in \mathcal{F}(S)$ such that $\sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)\,\overline{u}_i(y_i^{\mathcal{P}}) > \sum_{\mathcal{P}\in\mathbb{P}} r_i^{\mathcal{P}}\,\overline{u}_i(x_i^{\mathcal{P}}) = \overline{u}_i(x_i^{\Pi}), \forall i \in S$. Since $(u_i)_{i\in N} \preceq (\overline{u}_i)_{i\in N}$, for each agent i there is a concave and strictly increasing function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\overline{u}_i = f_i \circ u_i$. The concavity of functions f_i implies that

$$\begin{aligned} f_i\left(\sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S) u_i(y_i^{\mathcal{P}})\right) &\geq \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)\ f_i \circ u_i(y_i^{\mathcal{P}}) \\ &= \sum_{\mathcal{P}\in\mathbb{P}} \rho_i^{\mathcal{P}}(S)\,\overline{u}_i(y_i^{\mathcal{P}}) > \overline{u}_i(x_i^{\Pi}) = f_i(u_i(x_i^{\Pi})). \end{aligned}$$

The strict monotonicity of f_i guarantee that f_i^{-1} is well-defined and strictly increasing. Thus, we conclude that $x \notin \mathcal{C}_{\rho}((u_i)_{i\in N})$. □

Proof of Theorem 5 Let $x \notin \mathscr{C}_{\tau}^{\pi}(\mathcal{E}, \bar{\Phi})$. Then, there exist a coalition S and $y \in \mathcal{F}(S)$ such that $\sum_{\theta\in\Theta} \pi_i(S,\theta)\ \bar{\Phi}\big(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\big) > \bar{\Phi}(U_i(x_i))$ for every $i \in S$. Since $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, concave, and strictly increasing function such that $\Phi \preceq \overline{\Phi}$, then

$$\sum_{\theta\in\Theta} \pi_i(S,\theta)\ f \circ \Phi\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\right) > f \circ \Phi(U_i(x_i)), \quad \text{for every } i \in S,$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and strictly increasing mapping satisfying $\bar{\Phi} = f \circ \Phi$. The concavity and invertibility of f imply that

$$\sum_{\theta \in \Theta} \pi_i(S,\theta)\, \Phi\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\right) > \Phi(U_i(x_i))$$

for all $i \in S$. Therefore, $x \notin \mathscr{C}_\tau^\pi(\mathcal{E}, \Phi)$ and the concavity and monotonicity of Φ imply that

$$\sum_{\mathcal{P}\in\mathbb{P}} \left(\sum_{\theta \in \Theta} \pi_i(S,\theta)\tau_i^{\mathcal{P}}(S,\theta)\right) u_i(y_i^{\mathcal{P}}) > U_i(x_i), \quad \text{for every } i \in S.$$

We conclude that x does not belong to the risky core $\mathcal{C}_{\rho_\alpha}(\mathcal{E})$. □

Proof of Theorem 6 If $x \notin \mathscr{C}_\tau(\mathcal{E})$, then there is a coalition $S \subseteq N$ and $y \in \mathcal{F}(S)$ such that $\min_{\theta\in\Theta} \sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}}) > U_i(x_i)$ for all $i \in S$. Therefore, for any $\theta \in \Theta$ and $i \in S$ we have that $\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}}) > U_i(x_i)$, which implies that $x \notin \mathcal{C}_{\tau_\theta}(\mathcal{E})$. Furthermore, for each $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing we have that

$$\begin{aligned}\sum_{\theta \in \Theta} \pi_i(S,\theta)\, \Phi\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\right) &\geq \Phi\left(\min_{\theta\in\Theta} \sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta) u_i(y_i^{\mathcal{P}})\right)\\ &> \Phi(U_i(x_i)), \ \text{for every } i \in S.\end{aligned}$$

Since Φ is continuous, strictly increasing and $\Phi(0) = 0$ we have that Φ^{-1} is well-defined in $\mathbb{R}_+$ and strictly increasing. Then, we deduce that $x \notin \mathscr{C}_\tau^\pi(\mathcal{E}, \Phi)$. □

Proof of Proposition 4 Assumption (a) implies that

$$\overline{\bigcup_{n\in\mathbb{N}} \mathscr{C}_\tau^\pi(\mathcal{E}, \Phi_n)} = \bigcap_{\varepsilon>0} \bigcup_{n\in\mathbb{N}} \mathscr{C}_\tau^\pi[\Phi_n, \epsilon],$$

where $\mathscr{C}_\tau^\pi[\Phi_n, \varepsilon]$ is the set of allocations $x \in \mathcal{F}$ such that

$$\max_{S\subseteq N} \max_{y\in\mathcal{F}(S)} \min_{i\in S} \left(\Phi_n^{-1}\left(\sum_{\theta\in\Theta} \pi_i(S,\theta)\, \Phi_n\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}})\right)\right) - U_i(x_i)\right) \leq \varepsilon.$$

Given $\varepsilon > 0$, if $x \notin \bigcup_{n\in\mathbb{N}} \mathscr{C}_\tau^\pi[\Phi_n, \varepsilon]$, then for each $n \in \mathbb{N}$ there exist a coalition $S_n \subseteq N$ and a feasible allocation $y_n \in \mathcal{F}(S_n)$ such that

$$\Phi_n^{-1}\left(\sum_{\theta\in\Theta} \pi_i(S_n,\theta)\, \Phi_n\left(\sum_{\mathcal{P}\in\mathbb{P}} \tau_i^{\mathcal{P}}(S_n,\theta)u_i(y_{n,i}^{\mathcal{P}})\right)\right) - U_i(x_i) > \varepsilon, \quad \forall i \in S_n.$$

Since there are finitely many coalitions and sets of attainable allocations are compact, there exist $S \subseteq N$ and $\{y_{n_k}\}_{k\in\mathbb{N}} \subseteq \{y_n\}_{n\in\mathbb{N}}$ belonging to $\mathcal{F}(S)$ and converging to some $\overline{y}$ such that,

$$\Phi_{n_k}^{-1}\left(\sum_{\theta\in\Theta}\pi_i(S,\theta)\ \Phi_{n_k}\left(\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(y_{n_k,i}^{\mathcal{P}})\right)\right) - U_i(x_i) > \varepsilon, \quad \forall(i,k)\in S\times\mathbb{N}.$$

Given $k \in \mathbb{N}$, it follows from assumption (b) that for any $k' > k$ we have that $\Phi_{n_{k'}} = f \circ \Phi_{n_k}$ for some strictly increasing and concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ (see Proposition 6.C.2 in Mas-Colell et al. 1995). Hence, for every $i \in S$ and $k' > k$ the following property holds

$$\begin{aligned}&\Phi_{n_k}^{-1}\left(\sum_{\theta\in\Theta}\pi_i(S,\theta)\ \Phi_{n_k}\left(\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(y_{n_{k'},i}^{\mathcal{P}})\right)\right) - U_i(x_i)\\ &\quad\geq \Phi_{n_{k'}}^{-1}\left(\sum_{\theta\in\Theta}\pi_i(S,\theta)\ \Phi_{n_{k'}}\left(\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(y_{n_{k'},i}^{\mathcal{P}})\right)\right) - U_i(x_i) > \varepsilon.\end{aligned}$$

Taking the limit as k' goes to infinity we conclude that

$$\Phi_{n_k}^{-1}\left(\sum_{\theta\in\Theta}\pi_i(S,\theta)\ \Phi_{n_k}\left(\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(\overline{y}_i^{\mathcal{P}})\right)\right) - U_i(x_i) \geq \varepsilon, \ \forall(i,k)\in S\times\mathbb{N}.$$

Assumptions (a)–(c) and Lemma 8 in Klibanoff et al. (2005) imply that, by taking the limit as k goes to infinity, we have that $\min_{\theta\in\Theta}\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(\overline{y}_i^{\mathcal{P}}) - U_i(x_i) \geq \varepsilon$ for every $i \in S$. Hence $x \notin \mathscr{C}_\tau(\mathcal{E})$ and by Proposition 3 and Theorem 6 we have

$$\lim_{n\to\infty}\mathscr{C}_\tau^\pi(\mathcal{E},\Phi_n) = \bigcup_{n\in\mathbb{N}}\mathscr{C}_\tau^\pi(\mathcal{E},\Phi_n) \subseteq \mathscr{C}_\tau(\mathcal{E}) \subseteq \bigcap_{\varepsilon>0}\bigcup_{n\in\mathbb{N}}\mathscr{C}_\tau^\pi[\Phi_n,\varepsilon].$$

Since $\mathscr{C}_\tau(\mathcal{E})$ is closed,[13] to conclude the proof it remains to take the closure of the above sets. □

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[13] Note that $\mathscr{C}_\tau(\mathcal{E}) = \{x \in \mathcal{F} : \Upsilon(x) \leq 0\}$, where $\Upsilon(x) = \max_{S\subseteq N}\max_{y\in\mathcal{F}(S)}\min_{i\in S}\left(\min_{\theta\in\Theta}\sum_{\mathcal{P}\in\mathbb{P}}\tau_i^{\mathcal{P}}(S,\theta)u_i(y_i^{\mathcal{P}}) - U_i(x_i)\right)$ is a continuous function in $\mathcal{F}$.

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