SOME UNIVERSAL SOLUTIONS FOR A CLASS OF INCOMPRESSIBLE ELASTIC BODY THAT IS NOT GREEN ELASTIC: THE CASE OF LARGE ELASTIC DEFORMATIONS

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[Received 30 January 2020. Revise 30 January 2020. Accepted 18 February 2020]

Summary

Some universal solutions are studied for a new class of elastic bodies, wherein the Hencky strain tensor is assumed to be a function of the Kirchhoff stress tensor, considering in particular the case of assuming the bodies to be isotropic and incompressible. It is shown that the families of universal solutions found in the classical theory of nonlinear elasticity, are also universal solutions for this new type of constitutive equation.

1. Introduction

New classes of constitutive equations and relations have been proposed in the recent years in order to study the behaviour of elastic and inelastic bodies, see, for example, (**1**–**12**). Using the ideas presented in (**8**, **10**), Srinivasa (**13**) proposed a Gibbs potential that depends on the Kirchhoff stress tensor, in order to obtain an expression for the Hencky strain tensor as a function of the derivative of such Gibbs potential in the Kirchhoff stress.[1] The expression that is obtained is originally an implicit relation (see Eq. 9 of (**13**)), because of the presence of the determinant of the deformation gradient J in the definition of the Kirchhoff stress tensor. In the case that the body is incompressible, that is, when $J = 1$, the Kirchhoff stress tensor becomes the Cauchy stress tensor and the above implicit relation becomes an explicit constitutive equation, for the Hencky strain tensor in terms of the Cauchy stress tensor.

In the present article, we use the constitutive equation described above, and we find restrictions on the Gibbs potential for the body to be incompressible, showing that the Hencky stress tensor only depends on the deviatoric part of the Cauchy stress tensor (see (**14**)). That property is used in order to find universal solutions, in the sense that such solutions of the boundary value problems are valid for any Gibbs potential for isotropic incompressible bodies.

This communication is divided in the following sections: in Section 2, some basic equations of the theory of nonlinear elasticity are presented. In Section 3, the constitutive relation based on the Gibbs potential is studied for the particular case of incompressible bodies. In Section 4, some boundary value problems with homogeneous distributions of stresses and strains are considered. In Section 5, we study the problem of inflation and extension of a cylindrical annulus, comparing the solution

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[1] See also the recent and deeper work of Průša *et al.* (**14**) on the use of such Gibbs free energy for the modelling of elastic bodies.

Q. Jl Mech. Appl. Math, Vol. 73. No. 2
Advance Access publication 4 April 2020. doi:10.1093/qjmam/hbaa006

with the case of considering the classical theory of elasticity with the Green elastic body (see (**15**)). In Section 6, the well-known families of universal solutions of the classical theory of nonlinear elasticity (**16**, **17**) are analysed, and it is found that they are indeed also solutions for this new class of elastic body. Some final comments are given in Section 7.

2. Basic equations

For a body $\mathscr{B}$ the reference and current configurations are denoted $\boldsymbol{\kappa}_{\mathrm{r}}(\mathscr{B})$ and $\boldsymbol{\kappa}_{\mathrm{t}}(\mathscr{B})$, respectively. The positions of a point $X \in \mathscr{B}$ in the reference and current configurations are denoted $\mathbf{X}$ and $\mathbf{x}$, respectively, and it is assumed that there exists a one-to-one function $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, where t is time. The deformation gradient, the left stretch tensor and the left Cauchy-Green tensor are

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} = \mathbf{VR}, \quad \text{where} \quad \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{1}$$

respectively, where $J = \det \mathbf{F} > 0$, and the body it is said to be incompressible if $J = 1$ for any deformation.

The Cauchy stress tensor is denoted $\mathbf{T}$ and for the remaining of this article we assume quasi-static deformations without body forces, therefore, the stress $\mathbf{T}$ must satisfy the equation of equilibrium

$$\operatorname{div} \mathbf{T} = \mathbf{0}. \tag{2}$$

In Sections 5 and 6, different boundary value problems are studied considering the use of cylindrical and spherical coordinates, therefore, here (2) is presented in such coordinate systems. In the case of cylindrical coordinates, we have

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \tag{3}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r}T_{r\theta} = 0, \tag{4}$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} = 0, \tag{5}$$

while in spherical coordinates (2) becomes

$$\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\phi}}{\partial \theta} + \frac{3}{r}T_{r\phi} + \frac{\cos\phi}{r\sin\phi}(T_{\phi\phi} - T_{\theta\theta}) = 0, \tag{6}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{3}{r}T_{r\theta} + \frac{2}{r}\frac{\cos\phi}{\sin\phi}T_{\phi\theta} = 0, \tag{7}$$

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\cos\phi}{r\sin\phi}T_{r\phi} + \frac{1}{r}(2T_{rr} - T_{\phi\phi} - T_{\theta\theta}) = 0. \tag{8}$$

More details about the kinematics of continuum media and the equations of equilibrium can be found, for example, in (**18**).

Let us end this section showing the constitutive equation for an incompressible isotropic Green elastic body (see, for example, Section 49 of (**17**)):

$$\mathbf{T} = -p\mathbf{I} + \gamma_1 \mathbf{B} + \gamma_2 \mathbf{B}^2, \tag{9}$$

where p is a scalar function, $-p\mathbf{I}$ is the part of the stress that does not do any work with any deformation compatible with the constraint, and γ_1, γ_2 are scalar functions that depend on the invariants of $\mathbf{B}$, and for Green elastic bodies are given in terms of derivatives of the energy function.

3. Constitutive relations and equations

In (**13**), Srinivasa assumed the existence of a Gibbs potential G that depends on the Kirchhoff stress tensor $\boldsymbol{\tau}$ defined as

$$\boldsymbol{\tau} = J\mathbf{T}, \tag{10}$$

from where he obtained the relation

$$\ln \mathbf{V} = \frac{\partial G}{\partial \boldsymbol{\tau}}, \tag{11}$$

where $\ln \mathbf{V} = \frac{1}{2} \ln \mathbf{B}$ is the Hencky strain tensor (see, for example, (**19**) and references therein). Equation (11) is an implicit relation, because from (10) we have that $\boldsymbol{\tau}$ depends on J that is a part of the deformation of the body.

It is possible to show that (see, for example, (**13**))

$$\ln J = \operatorname{tr} \ln \mathbf{V} = \operatorname{tr} \left(\frac{\partial G}{\partial \boldsymbol{\tau}} \right). \tag{12}$$

In the case that G is an isotropic function, we have that $G = G(I_1, I_2, I_3)$, where the invariants I_i, $i = 1, 2, 3$ are defined as

$$I_1 = \operatorname{tr} \boldsymbol{\tau}, \quad I_2 = \frac{1}{2} \operatorname{tr}(\boldsymbol{\tau}^2), \quad I_3 = \frac{1}{3} \operatorname{tr}(\boldsymbol{\tau}^3), \tag{13}$$

and from (11) we obtain

$$\ln \mathbf{V} = G_1 \mathbf{I} + G_2 \boldsymbol{\tau} + G_3 \boldsymbol{\tau}^2, \tag{14}$$

where $G_i = \frac{\partial G}{\partial I_i}$, $i = 1, 2, 3$.

If τ_i and λ_i, $i = 1, 2, 3$ represent the principal values of $\boldsymbol{\tau}$ and the principal stretches, respectively, an alternative representation for G in the case it is an isotropic function is $G = G(\tau_1, \tau_2, \tau_3)$, where $G(\tau_1, \tau_2, \tau_3) = G(\tau_2, \tau_1, \tau_3) = G(\tau_1, \tau_3, \tau_2) = G(\tau_3, \tau_2, \tau_1)$, and from (11) we have

$$\ln \lambda_i = \frac{\partial G}{\partial \tau_i}, \quad i = 1, 2, 3. \tag{15}$$

In the case the body is incompressible $J = 1$, then $\boldsymbol{\tau} = \mathbf{T}$ and from (12), (14) we have

$$\operatorname{tr} \ln \mathbf{V} = \ln J = 0 = 3G_1 + G_2 I_1 + 2G_3 I_2, \tag{16}$$

which is a first-order linear partial differential equation, whose solution is (see (16)–(18) in (**20**))

$$G = \bar{G}(\bar{I}_1, \bar{I}_2), \quad \text{where} \quad \bar{I}_1 = I_2 - \frac{I_1^2}{6}, \quad \bar{I}_2 = I_3 + \frac{2I_1^3}{27} - \frac{2I_1 I_2}{3}, \tag{17}$$

where in this case from (13)

$$I_1 = \mathrm{tr}\,\mathbf{T}, \quad I_2 = \frac{1}{2}\mathrm{tr}(\mathbf{T}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}(\mathbf{T}^3). \tag{18}$$

From (17) and (11), we have

$$\ln \mathbf{V} = \left(\mathbf{T} - \frac{I_1}{3}\mathbf{I}\right)\frac{\partial \bar{G}}{\partial \bar{I}_1} + \left[2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2\right]\frac{\partial \bar{G}}{\partial \bar{I}_2}. \tag{19}$$

Similar representations can be obtained using (15) instead of (14), but in this article only the case of (14) is studied in detail.

It is important to mention here that the structure of the partial differential equation (16) and its solution (17), is much simpler than the case of considering the alternative constitutive equation $\mathbf{B} = \frac{\partial \Omega}{\partial \mathbf{T}}$, which was studied in detail by Bustamante *et al.* in (**21**) for incompressible bodies. In that work, $\Omega = \Omega(\mathbf{T})$ was a scalar potential (derived from an implicit relation) used to obtain the left Cauchy-Green tensor. We can compare, for example, (16) and (17) with (24) and (60) of (**21**).

Let us decompose the Cauchy stress tensor in a spherical part σ_{S} and a deviatoric part $\mathbf{T}_{\mathrm{D}}$ as

$$\mathbf{T} = -\sigma_{\mathrm{S}}\mathbf{I} + \mathbf{T}_{\mathrm{D}}, \tag{20}$$

where $\sigma_{\mathrm{S}} = -\frac{\mathrm{tr}\,\mathbf{T}}{3}$, $\mathbf{T}_{\mathrm{D}} = \mathbf{T} - \frac{(\mathrm{tr}\,\mathbf{T})}{3}\mathbf{I}$ and $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$. The invariants I_{D_2} and I_{D_3} of the deviatoric part of the stress are defined as

$$I_{\mathrm{D}_2} = \frac{1}{2}\mathrm{tr}(\mathbf{T}_{\mathrm{D}}^2), \quad I_{\mathrm{D}_3} = \frac{1}{3}\mathrm{tr}(\mathbf{T}_{\mathrm{D}}^3), \tag{21}$$

where $I_{\mathrm{D}_1} = \mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$. It is possible to show that (see (**3**)) $\bar{I}_1 = I_{\mathrm{D}_2}$, $\bar{I}_2 = I_{\mathrm{D}_3}$, as a result for incompressible bodies we have $G = \hat{G}(I_{\mathrm{D}_2}, I_{\mathrm{D}_3})$, and from (19) we obtain

$$\ln \mathbf{V} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T}_{\mathrm{D}} + \alpha_2\mathbf{T}_{\mathrm{D}}^2, \tag{22}$$

where we have defined (see (**14**))

$$\alpha_0 = \alpha_0(\mathbf{T}_{\mathrm{D}}) = -\frac{2I_{\mathrm{D}_2}}{3}\frac{\partial \hat{G}}{\partial I_{\mathrm{D}_3}}, \quad \alpha_1 = \alpha_1(\mathbf{T}_{\mathrm{D}}) = \frac{\partial \hat{G}}{\partial I_{\mathrm{D}_2}}, \quad \alpha_2 = \alpha_2(\mathbf{T}_{\mathrm{D}}) = \frac{\partial \hat{G}}{\partial I_{\mathrm{D}_3}}. \tag{23}$$

From (1) we have $\ln \mathbf{B} = 2\ln \mathbf{V}$, as a result an alternative expression for (22) is

$$\frac{1}{2}\ln \mathbf{B} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T}_{\mathrm{D}} + \alpha_2\mathbf{T}_{\mathrm{D}}^2. \tag{24}$$

From (22), (24) it is easy to obtain the linear universal relations

$$\mathbf{T}_{\mathrm{D}}\ln(\mathbf{V}) = \ln(\mathbf{V})\mathbf{T}_{\mathrm{D}} \quad \Leftrightarrow \quad \mathbf{T}_{\mathrm{D}}\ln(\mathbf{B}) = \ln(\mathbf{B})\mathbf{T}_{\mathrm{D}}. \tag{25}$$

If $\overset{(j)}{\mathbf{v}}$ are the eigenvectors of $\mathbf{V}$ we have the spectral representations $\mathbf{V} = \sum_{j=1}^{3}\lambda_j \overset{(j)}{\mathbf{v}} \otimes \overset{(j)}{\mathbf{v}}$, $\mathbf{B} = \sum_{j=1}^{3}\lambda_j^2 \overset{(j)}{\mathbf{v}} \otimes \overset{(j)}{\mathbf{v}}$ and (22) becomes

$$\sum_{j=1}^{3}\ln \lambda_j \overset{(j)}{\mathbf{v}} \otimes \overset{(j)}{\mathbf{v}} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T}_{\mathrm{D}} + \alpha_2\mathbf{T}_{\mathrm{D}}^2. \tag{26}$$

In Section 4 of (**22**), alternative expressions for the natural logarithm of second-order tensors are presented, which are more general than (26).

Finally, let us consider (22), imagine that we know $\mathbf{V}$ and that we need to find $\mathbf{T}_{\mathrm{D}}$ solving that equation. It is possible to show that such stress is traceless. Let us take the trace of the whole equation (22), we obtain $\mathrm{tr}\,\ln\mathbf{V} = 0 = 3\alpha_0 + \alpha_1 \mathrm{tr}\,\mathbf{T}_{\mathrm{D}} + \alpha_2 \mathrm{tr}(\mathbf{T}_{\mathrm{D}}^2)$, but from $(21)_1$ we have that $\mathrm{tr}(\mathbf{T}_{\mathrm{D}}^2) = 2I_{\mathrm{D}_2}$ and considering $(23)_{1,3}$ we obtain $0 = \frac{\partial \hat{G}}{\partial I_{\mathrm{D}_2}} \mathrm{tr}\,\mathbf{T}_{\mathrm{D}}$, so if $\frac{\partial \hat{G}}{\partial I_{\mathrm{D}_2}} \neq 0$ then $\mathrm{tr}\,\mathbf{T}_{\mathrm{D}} = 0$, which is an essential condition to be satisfied for a deviatoric stress.

4. Problems with homogeneous distributions of stresses and strains

If $\mathbf{T}$ is constant in $\mathbf{x}$ the equation of equilibrium (2) is satisfied automatically. In this section, we study briefly some problems where the stresses are constant.

In the problems studied in this section and in Sections 5 and 6, the semi-inverse method is used in order to solve boundary value problems, assuming at the same time simplified expressions for the stresses $\mathbf{T}$ and the deformations $\mathbf{x}$. This is the simplest method, which has been developed so far, to solve boundary value problems for these new constitutive theories, and it has been used, for example, in (**23**–**25**) for unconstrained solids. A possible alternative approach would be to propose simplified expressions only for the stresses, and to use the compatibility equations for the strains in order to determine the final expression for such stresses. However, such an approach is not convenient among other reasons due to the highly complicated structure of the compatibility equations in the case of large deformations (see, for example, Section 34 of (**18**), and (**2**) in the case of considering small strains).

4.1 *Uniform tension/compression of a cylinder*

Let us consider a cylinder defined in the reference configuration in cylindrical coordinates as

$$0 \leq R \leq R_{\mathrm{o}}, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L. \tag{27}$$

Let us assume that the cylinder deforms due to the presence of the homogeneous stress distribution

$$\mathbf{T} = T_{rr_o}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta_o}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz_o}\mathbf{e}_z \otimes \mathbf{e}_z, \tag{28}$$

where T_{rr_o}, $T_{\theta\theta_o}$ and T_{zz_o} are constants.

Let us suppose that due to the application of the above load the cylinder deforms as

$$r = CR, \quad \theta = \Theta, \quad z = \lambda Z, \tag{29}$$

where C and λ are constants. The deformation gradient is $\mathbf{F} = C\mathbf{e}_r \otimes \mathbf{E}_R + C\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda\mathbf{e}_z \otimes \mathbf{E}_Z$ and if the body is incompressible this means that

$$C = \frac{1}{\sqrt{\lambda}}. \tag{30}$$

From (28) the deviatoric stress is given as $\mathbf{T} = T_{\mathrm{D}_{rr_o}}\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{\theta\theta_o}}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz_o}}\mathbf{e}_z \otimes \mathbf{e}_z$, where $T_{\mathrm{D}_{rr_o}}$, $T_{\mathrm{D}_{\theta\theta_o}}$ and $T_{\mathrm{D}_{zz_o}}$ are constants, and

$$T_{rr_o} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{rr_o}}, \quad T_{\theta\theta_o} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{\theta\theta_o}}, \quad T_{zz_o} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{zz_o}}. \tag{31}$$

From (26), (30) we obtain

$$-\frac{1}{2}\ln\lambda = C = \alpha_0 + \alpha_1 T_{\mathrm{D}_{rr_o}} + \alpha_2 T^2_{\mathrm{D}_{rr_o}}, \tag{32}$$

$$-\frac{1}{2}\ln\lambda = C = \alpha_0 + \alpha_1 T_{\mathrm{D}_{\theta\theta_o}} + \alpha_2 T^2_{\mathrm{D}_{\theta\theta_o}}, \tag{33}$$

$$\lambda = \alpha_0 + \alpha_1 T_{\mathrm{D}_{zz_o}} + \alpha_2 T^2_{\mathrm{D}_{zz_o}}. \tag{34}$$

From (32), (33) it is easy to see that $T_{\mathrm{D}_{\theta\theta_o}} = T_{\mathrm{D}_{rr_o}}$. If $T_{\mathrm{D}_{zz_o}}$ is given, from (32) and (34) λ and $T_{\mathrm{D}_{rr_o}}$ can be obtained. Let us assume that there is no lateral load in the cylinder, then $T_{rr_o} = 0$ and from $(31)_1$ we have $\sigma_{\mathrm{S}} = T_{\mathrm{D}_{rr_o}}$, and from $(31)_3$ the external load to be applied on the cylinder in the axial direction is $T_{zz_o} = T_{\mathrm{D}_{zz_o}} - T_{\mathrm{D}_{rr_o}}$.

4.2 *Biaxial load on a block*

Consider the block defined in the reference configuration as

$$-\frac{L_i}{2} \le X_i \le \frac{L_i}{2}, \quad i = 1, 2, 3. \tag{35}$$

It is assumed that this block deforms due to the application of the stress

$$\mathbf{T} = T_{11_o}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{22_o}\mathbf{e}_2 \otimes \mathbf{e}_2, \tag{36}$$

and the current configuration for deformed slab is supposed to be

$$x_i = \lambda_i X_i, \quad \text{(there is no sum in } i\text{, and } i = 1, 2, 3), \tag{37}$$

where $\lambda_i > 0$, $i = 1, 2, 3$ are constants. The deformation gradient is $\mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{E}_i$, and since the body is incompressible

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2}. \tag{38}$$

From (36), the deviatoric stress is of the form

$$\mathbf{T}_{\mathrm{D}} = \sum_{i=1}^{3} T_{\mathrm{D}_{ii_o}} \mathbf{e}_i \otimes \mathbf{e}_i, \tag{39}$$

where $T_{\mathrm{D}_{ii_o}}$, $i = 1, 2, 3$ are constants, where

$$T_{11_o} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{11_o}}, \quad T_{22_o} = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{22_o}}, \quad 0 = -\sigma_{\mathrm{S}} + T_{\mathrm{D}_{33_o}}, \tag{40}$$

from where $\sigma_{\mathrm{S}} = T_{\mathrm{D}_{33_o}}$.

Using the above expressions in (26), (38) we obtain

$$\ln\lambda_1 = \alpha_0 + \alpha_1 T_{\mathrm{D}_{11_o}} + \alpha_2 T^2_{\mathrm{D}_{11_o}}, \tag{41}$$

$$\ln\lambda_2 = \alpha_0 + \alpha_1 T_{D_{22_o}} + \alpha_2 T^2_{D_{22_o}}, \tag{42}$$

$$-\ln\lambda_1 - \ln\lambda_2 = \ln\lambda_3 = \alpha_0 + \alpha_1 T_{D_{33_o}} + \alpha_2 T^2_{D_{33_o}}. \tag{43}$$

From (41)–(43) if $T_{D_{11_o}}$ and $T_{D_{22_o}}$ are given, we can obtain λ_1, λ_2 and $T_{D_{33_o}}$. Since $\sigma_S = T_{D_{33_o}}$, then from $(40)_{1,2}$ the loads to be applied in the directions 1 and 2 are

$$T_{11_o} = T_{D_{11_o}} - T_{D_{33_o}}, \quad T_{22_o} = T_{D_{22_o}} - T_{D_{33_o}}. \tag{44}$$

4.3 *Uniform shear of a block*

Consider the same block defined in (35), but where we apply the stress

$$\mathbf{T} = T_{12_o}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{45}$$

where T_{12_o} is constant. Let us assume that the block deforms as

$$x_1 = \lambda_A X_1 + \gamma_A X_2, \quad x_2 = \lambda_B X_2 + \gamma_B X_1, \quad x_3 = \lambda_C X_3, \tag{46}$$

where λ_A, λ_B, λ_C, γ_A and γ_B are constants. The deformation gradient and the left Cauchy-Green tensors are

$$\mathbf{F} = \lambda_A \mathbf{e}_1 \otimes \mathbf{E}_1 + \gamma_A \mathbf{e}_1 \otimes \mathbf{E}_2 + \gamma_B \mathbf{e}_2 \otimes \mathbf{E}_1 + \lambda_B \mathbf{e}_2 \otimes \mathbf{E}_2 + \lambda_C \mathbf{e}_3 \otimes \mathbf{E}_3, \tag{47}$$

$$\mathbf{B} = (\lambda_A^2 + \gamma^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + (\lambda_A\gamma_B + \lambda_B\gamma_A)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + (\lambda_B^2 + \gamma_B^2)\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_C^2 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{48}$$

The slab is incompressible, as a result

$$\lambda_C = \frac{1}{(\lambda_A\lambda_B - \gamma_A\gamma_B)}. \tag{49}$$

The principal stretches are

$$\lambda_1 = \sqrt{\frac{\lambda_a - \lambda_b}{2}}, \quad \lambda_2 = \sqrt{\frac{\lambda_a + \lambda_b}{2}}, \quad \lambda_3 = \lambda_C, \tag{50}$$

where

$$\lambda_a = \lambda_A^2 + \lambda_B^2 + \gamma_A^2 + \gamma_B^2, \tag{51}$$

$$\lambda_b = \sqrt{[(\lambda_A - \lambda_B)^2 + (\gamma_A + \gamma_B)^2][(\lambda_A + \lambda_B)^2 + (\gamma_A - \gamma_B)^2]}, \tag{52}$$

and the principal directions are

$$\overset{(1)}{\mathbf{v}} = \overset{(1)}{v}_1 \mathbf{e}_1 + \overset{(1)}{v}_2 \mathbf{e}_2, \quad \overset{(2)}{\mathbf{v}} = \overset{(2)}{v}_1 \mathbf{e}_1 + \overset{(2)}{v}_2 \mathbf{e}_2, \quad \overset{(3)}{\mathbf{v}} = \mathbf{e}_3, \tag{53}$$

where

$$\overset{(i)}{v}_1 = \frac{1}{\sqrt{1+\frac{(\lambda_i^2-\lambda_A^2-\gamma_A^2)^2}{(\lambda_A\gamma_B+\lambda_B\gamma_A)^2}}}, \quad \overset{(i)}{v}_2 = \frac{1}{\sqrt{1+\frac{(\lambda_A\gamma_B+\lambda_B\gamma_A)^2}{(\lambda_i^2-\lambda_A^2-\gamma_A^2)^2}}}, \quad i=1,2. \tag{54}$$

From (45), it is possible to see that $\sigma_S = 0$ and that $\mathbf{T}_D = \mathbf{T}$, then using (50), (53) and (45) in (26) we have

$$\ln\lambda_1\left(\overset{(1)}{v}_1\right)^2 + \ln\lambda_2\left(\overset{(2)}{v}_1\right)^2 = \alpha_0 + \alpha_2 T_{12_o}^2, \tag{55}$$

$$\ln\lambda_1\left(\overset{(1)}{v}_2\right)^2 + \ln\lambda_2\left(\overset{(2)}{v}_2\right)^2 = \alpha_0 + \alpha_2 T_{12_o}^2, \tag{56}$$

$$-\ln(\lambda_A\lambda_B - \gamma_A\gamma_B) = \ln\lambda_3 = \alpha_0, \tag{57}$$

$$\ln\lambda_1 \overset{(1)}{v}_1 \overset{(1)}{v}_2 + \ln\lambda_2 \overset{(2)}{v}_1 \overset{(2)}{v}_2 = \alpha_1 T_{12_o}. \tag{58}$$

From (55) and (56), we obtain

$$\ln\lambda_1\left(\overset{(1)}{v}_1\right)^2 + \ln\lambda_2\left(\overset{(2)}{v}_1\right)^2 = \ln\lambda_1\left(\overset{(1)}{v}_2\right)^2 + \ln\lambda_2\left(\overset{(2)}{v}_2\right)^2. \tag{59}$$

Let us assume that T_{12_o} is given, then from (55) and (57)–(59) we can obtain λ_A, λ_B, γ_A and γ_B.

In the next section, we study a problem with non-homogeneous distributions for the stresses and in particular for the strains, studying in detail how incompressibility can be used to simplify the boundary value problem within the context of this new constitutive theory (22), (24), (26).

5. The problem of the inflation and extension of a circular annulus

As in (**15**), in this section we study the problem of the inflation and axial extension of a circular annulus, and we compare that with the case of studying that boundary value problem within the context of the classical theory of nonlinear elasticity using (9). The problem studied here is a special case of the one studied in Section 6.3.

5.1 *The case of considering the classical theory of nonlinear elasticity*

In the reference configuration in cylindrical coordinates, the annulus is defined as

$$R_i \le R \le R_o, \quad 0 \le \Theta \le 2\pi, \quad 0 \le Z \le L. \tag{60}$$

Following Rivlin (see Chapter A of Volume I in (**26**) and also Section 57 of (**17**)), the deformation $\mathbf{x}$ is assumed to be of the form

$$r = f(R), \quad \theta = \Theta, \quad z = \lambda Z, \tag{61}$$

where λ is a constant. From (61) we obtain

$$\mathbf{F} = f'(R)\mathbf{e}_r \otimes \mathbf{E}_R + \frac{f(R)}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{62}$$

and as a result

$$\mathbf{B} = f'(R)^2 \mathbf{e}_r \otimes \mathbf{e}_r + \frac{f(R)^2}{R^2} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda^2 \mathbf{e}_z \otimes \mathbf{e}_z. \tag{63}$$

For an incompressible body we need $J = 1$, which implies that $f'(R)\frac{f(R)}{R}\lambda = 1$, whose solution is

$$r = f(R) = \sqrt{\frac{R^2 - R_\mathrm{i}^2}{\lambda} + r_\mathrm{i}^2}, \tag{64}$$

where $r_\mathrm{i} = f(R_\mathrm{i})$ is the inner radius in the current configuration. Let us introduce the notation $\tilde{\mathbf{T}} = \gamma_1 \mathbf{B} + \gamma_2 \mathbf{B}^2$, then from (63) and (64) we have that γ_1 and γ_2 only depends on the radial position, while from (9) the Cauchy stress has only normal components. Using such stress in (3)–(9), we can show that $p = p(r)$, and from (3), (9) we obtain

$$\frac{d\tilde{T}_{rr}}{dr} - \frac{dp}{dr} + \frac{1}{r}(\tilde{T}_{rr} - \tilde{T}_{\theta\theta}) = 0, \tag{65}$$

which can be solved easily for $p(r)$. If on the inner surface of the annulus we have a normal load P, and on the outer surface of the annulus we assume there is no external traction, we have

$$P = \int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\xi}[\tilde{T}_{rr}(\xi) - \tilde{T}_{\theta\theta}(\xi)]\, d\xi, \tag{66}$$

where $r_\mathrm{o} = \sqrt{\frac{R_\mathrm{o}^2 - R_\mathrm{i}^2}{\lambda} + r_\mathrm{i}^2}$. The above equation can be used to obtain r_i for a given P.

The above solution (61) is universal because is valid for any expression for the functions γ_1 and γ_2 in (9). It is an explicit solution for the deformation $\mathbf{x}$ (in this case in particular for the function $f(R)$) up to a constant r_i, which should be found, in general numerically, by solving (66). Replacing (63) in (9), we also obtain explicit expressions for the components of $\mathbf{T}$. Finally, the above solution was obtained using the semi-inverse method assuming only a simplified expression for the deformation field.

5.2 *The case of considering the new constitutive equation*

In this section, we study the same annulus described in (60), but the first difference with the classical approach presented in the previous section, is that we use the semi-inverse method assuming some simplified expressions for the stresses and the deformation field. Then, let us assume that such an annulus (60) deforms due to the presence of the stress

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{67}$$

This stress is supposed to deform the body as (61), which produces the deformation gradient (62). For later use $f'(R)$ is presented here as

$$f'(R) = \frac{R}{r\lambda}, \quad \text{where} \quad R = \sqrt{\lambda(r^2 - r_\mathrm{i}^2) + R_\mathrm{i}^2}. \tag{68}$$

From (67) the deviatoric stress is assumed to be of the form

$$\mathbf{T} = T_{\mathrm{D}_{rr}}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{\theta\theta}}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz}}(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{69}$$

The deviatoric stress tensor is diagonal. Let us consider the decomposition (20) for this case

$$T_{rr}(r) = -\sigma_{\mathrm{S}}(r) + T_{\mathrm{D}_{rr}}(r), \quad T_{\theta\theta}(r) = -\sigma_{\mathrm{S}}(r) + T_{\mathrm{D}_{\theta\theta}}(r), \quad T_{zz}(r) = -\sigma_{\mathrm{S}}(r) + T_{\mathrm{D}_{zz}}(r). \tag{70}$$

Considering (69) from (3)–(5) and (70), we obtain

$$\frac{dT_{rr}}{dr} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0 \quad \Leftrightarrow \quad \frac{dT_{rr}}{dr} + \frac{1}{r}(T_{\mathrm{D}_{rr}} - T_{\mathrm{D}_{\theta\theta}}) = 0, \tag{71}$$

from where we have (see (**15**))

$$T_{rr}(r) = \int_{r_{\mathrm{i}}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\, d\xi + T_{rr}(r_{\mathrm{i}}), \tag{72}$$

where $T_{rr}(r_{\mathrm{i}})$ is the radial component of the stress evaluated at $r = r_{\mathrm{i}}$.

Let us assume that on the inner surface of the annulus there is an external radial load P, whereas on the outer surface of the annulus there is no external traction (as in the case of the problem presented in Section 5.1), therefore

$$T_{rr}(r_{\mathrm{i}}) = -P, \quad T_{rr}(r_{\mathrm{o}}) = 0, \tag{73}$$

as a result from $(70)_1$, (72) and $(73)_1$ we obtain

$$\sigma_{\mathrm{S}}(r) = T_{\mathrm{D}_{rr}}(r) + P + \int_{r_{\mathrm{i}}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{rr}}(\xi) - T_{\mathrm{D}_{\theta\theta}}(\xi)]\, d\xi. \tag{74}$$

From (26) considering (63), (69) the constitutive equations become (compare with the results presented in (**15**))

$$\ln[f'(R)] = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{rr}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{rr}}(r)^2, \tag{75}$$

$$\ln\left[\frac{f(R)}{R}\right] = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{\theta\theta}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{\theta\theta}}(r)^2, \tag{76}$$

$$\ln\lambda = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{zz}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{zz}}(r)^2, \tag{77}$$

where from (64) and (68) we see that on the left side we have functions that depend on the radial position in the current configuration, since $f'(R) = \frac{\sqrt{\lambda(r^2-r_{\mathrm{i}}^2)+R_{\mathrm{i}}^2}}{r\lambda}$ and $\frac{f(R)}{R} = \frac{r}{\sqrt{\lambda(r^2-r_{\mathrm{i}}^2)+R_{\mathrm{i}}^2}}$.

From (72) and $(73)_2$, we have

$$P = \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \frac{1}{\xi}[T_{\mathrm{D}_{rr}}(\xi) - T_{\mathrm{D}_{\theta\theta}}(\xi)]\, d\xi, \tag{78}$$

where $r_{\mathrm{o}} = \sqrt{\frac{R_{\mathrm{o}}^2-R_{\mathrm{i}}^2}{\lambda} + r_{\mathrm{i}}^2}$.

Let us study and compare the above solution with the case presented in Section 5.1. From (64), we have the same explicit solution for the deformation field $\mathbf{x}$ (in particular the function $f(R)$) up to a constant r_{i}. Unlike the classical solution studied in Section 5.1, here we do not have explicit expressions for the components of the stress. From (75)–(77), we have expressions from where we can find the components of the deviatoric stress $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$ and $T_{\mathrm{D}_{zz}}(r)$ implicitly, that is, we have implicit solutions for $\mathbf{T}_{\mathrm{D}}$, and as a result from (70) and (74) we have implicit solutions for $T_{rr}(r)$, $T_{\theta\theta}(r)$ and $T_{zz}(r)$. From (75)–(77), (64) and (68) these implicit solutions for $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$ and $T_{\mathrm{D}_{zz}}(r)$ depend on r_{i}, and that constant can be found from (78). In general not only r_{i} should be found numerically, but the same happens with the components of the deviatoric stress tensor $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$ and $T_{\mathrm{D}_{zz}}(r)$ from (75)–(77). These solutions (64) and (75)–(77) are valid for any function $\hat{G}$, therefore, they are universal solutions.

Finally, (75)–(77) are in general nonlinear, and there exists the possibility that more than one set of solutions can be found for $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$ and $T_{\mathrm{D}_{zz}}(r)$. Such possible non-uniqueness is not studied in this article.

6. Other universal solutions

In this section, we study briefly the different universal solutions that are well known from the literature in nonlinear elasticity (see, (**16**, **26**) and Section 57 of (**17**)), which have been obtained for the classical constitutive equation (9), showing that they are also solutions for the case of considering the new type of constitutive equation (26).

6.1 *Bending, stretching and shearing of a rectangular block*

Let us consider the block defined in the reference configuration (Cartesian coordinates) as

$$X_1 \leq X \leq X_2, \quad -Y_{\mathrm{o}} \leq Y \leq Y_{\mathrm{o}}, \quad -Z_{\mathrm{o}} \leq Z \leq Z_{\mathrm{o}}. \tag{79}$$

Let us assume that the above block deforms due to the application of the stress field (in cylindrical coordinates)

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z + T_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta), \tag{80}$$

and that due to the application of the above stress the deformation is

$$r = \sqrt{2AX}, \quad \theta = BY, \quad z = \frac{Z}{AB} - BCY, \tag{81}$$

where A, B and C are constants and $AB \neq 0$. In such a case the deformation gradient is

$$\mathbf{F} = \frac{A}{r}\mathbf{e}_r \otimes \mathbf{E}_R + rB\mathbf{e}_\theta \otimes \mathbf{E}_\Theta - BC\mathbf{e}_z \otimes \mathbf{E}_\Theta + \frac{1}{AB}\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{82}$$

and the left Cauchy-Green tensor is

$$\mathbf{B} = \frac{A^2}{r^2}\mathbf{e}_r \otimes \mathbf{e}_r + r^2B^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta - rB^2C(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta) + \frac{1}{A^2B^2}\mathbf{e}_z \otimes \mathbf{e}_z. \tag{83}$$

The principal stretches are

$$\lambda_1 = \frac{A}{r}, \quad \lambda_2 = \frac{\sqrt{\lambda_{\mathrm{a}} - \lambda_{\mathrm{b}}}}{\sqrt{2}AB}, \quad \lambda_3 = \frac{\sqrt{\lambda_{\mathrm{a}} + \lambda_{\mathrm{b}}}}{\sqrt{2}AB}, \tag{84}$$

where we have defined

$$\lambda_{\mathrm{a}} = 1 + A^2B^2(C^2 + r^2), \quad \lambda_{\mathrm{b}} = \sqrt{1 + 2A^2B^4(C^2 - r^2) + A^4B^8(C^2 + r^2)^2}. \tag{85}$$

On the other hand, the corresponding eigenvectors of $\mathbf{V}$ are

$$\overset{(1)}{\mathbf{v}} = \mathbf{e}_r, \quad \overset{(k)}{\mathbf{v}} = \overset{(k)}{v}_2 \, \mathbf{e}_\theta + \overset{(k)}{v}_3 \, \mathbf{e}_z, \quad k = 2, 3, \tag{86}$$

where

$$\overset{(k)}{v}_2 = \frac{1}{\sqrt{1 + \frac{1}{C^2}\left(r - \frac{\lambda_k^2}{rB^2}\right)^2}}, \quad \overset{(k)}{v}_3 = \frac{\frac{1}{C}\left(r - \frac{\lambda_k^2}{rB^2}\right)}{\sqrt{1 + \frac{1}{C^2}\left(r - \frac{\lambda_k^2}{rB^2}\right)^2}}, \quad k = 2, 3. \tag{87}$$

From (80), the deviatoric stress is given as

$$\mathbf{T}_{\mathrm{D}} = T_{\mathrm{D}_{rr}}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{\theta\theta}}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz}}(r)\mathbf{e}_z \otimes \mathbf{e}_z + T_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta). \tag{88}$$

Considering (84), (86) and (88) from (26), we obtain

$$\ln\left(\frac{A}{r}\right) = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{rr}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{rr}}(r)^2, \tag{89}$$

$$\ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} - \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\left(\overset{(2)}{v}_2\right)^2 + \ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} + \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\left(\overset{(3)}{v}_2\right)^2 = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{\theta\theta}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})[T_{\mathrm{D}_{\theta\theta}}(r)^2 + T_{\theta z}(r)^2], \tag{90}$$

$$\ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} - \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\left(\overset{(2)}{v}_3\right)^2 + \ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} + \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\left(\overset{(3)}{v}_3\right)^2 = \alpha_0(\mathbf{T}_{\mathrm{D}}) + \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\mathrm{D}_{zz}}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})[T_{\theta z}(r)^2 + T_{\mathrm{D}_{zz}}(r)^2], \tag{91}$$

$$\ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} - \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\overset{(2)}{v}_2\overset{(2)}{v}_3 + \ln\left[\frac{\sqrt{\lambda_{\mathrm{a}} + \lambda_{\mathrm{b}}}}{\sqrt{2}AB}\right]\overset{(3)}{v}_2\overset{(3)}{v}_3 = \alpha_1(\mathbf{T}_{\mathrm{D}})T_{\theta z}(r) + \alpha_2(\mathbf{T}_{\mathrm{D}})T_{\theta z}(r)[T_{\mathrm{D}_{\theta\theta}}(r) + T_{\mathrm{D}_{zz}}(r)]. \tag{92}$$

The solution (81) is explicit up to the constants A, B and C. From (89)–(92), we can find implicitly $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$, $T_{\mathrm{D}_{zz}}(r)$ and $T_{\theta z}(r)$ in terms of (81) and the constants A, B and C.

If (80) is replaced in (3)–(5), we obtain the same expression for $T_{rr}(r)$ as in (72). We can consider the surfaces $X = X_1$ and $X = X_2$ free of external traction, therefore, from (72) taking into account that $r_{\mathrm{i}} = \sqrt{2AX_1}$ and $r_{\mathrm{o}} = \sqrt{2AX_2}$, we obtain

$$\int_{\sqrt{2AX_1}}^{\sqrt{2AX_2}} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\, d\xi = 0, \tag{93}$$

which is an equation that could be used to find, for example, the constant A, recalling that from (77)–(87), (89), (92) the components of the deviatoric stress depend on the constants A, B and C. In this case from (3), (20) the expression for the spherical component of the stress is

$$\sigma_{\mathrm{S}}(r) = T_{\mathrm{D}_{rr}}(r) + \int_{r_{\mathrm{i}}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{rr}}(\xi) - T_{\mathrm{D}_{\theta\theta}}(\xi)]\, d\xi. \tag{94}$$

Let us study briefly the boundary conditions for the other surfaces of the body. For the surfaces $Y = \pm Y_{\mathrm{o}}$, we have that the normal unit vector to those surfaces is $\mathbf{n} = \mathbf{e}_\theta$, and from (80) we obtain $\hat{\mathbf{t}} = \mathbf{Tn} = T_{\theta\theta}\mathbf{e}_\theta + T_{\theta z}\mathbf{e}_z$. From the two components of this stress vector, we can calculate the external total bending moment $\mathcal{M}$ and total shear force $\mathcal{S}$, which are necessary to deform the block as $\mathcal{M} = 2BCY_{\mathrm{o}} \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} rT_{\theta\theta}(r)\, dr$ and $\mathcal{S} = 2BCY_{\mathrm{o}} \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} T_{\theta z}(r)\, dr$, respectively.

In the case of the surfaces $Z = \pm Z_{\mathrm{o}}$, the normal unit vector is $\mathbf{n} = \frac{1}{\sqrt{C^2+r^2}}(-C\mathbf{e}_\theta + r\mathbf{e}_z)$, and from (80), we obtain $\hat{\mathbf{t}} = \mathbf{Tn} = \frac{1}{\sqrt{C^2+r^2}}[(-CT_{\theta\theta}(r) + rT_{\theta z}(r))\mathbf{e}_\theta + (-CT_{\theta z}(r) + rT_{zz}(r))\mathbf{e}_z]$, therefore, the normal component σ_{N} of that stress vector (to that surface) is $\sigma_{\mathrm{N}} = \hat{\mathbf{t}} \cdot \mathbf{n}$, while the shear component τ is given by $\tau = \sqrt{|\hat{\mathbf{t}}|^2 - \sigma_{\mathrm{N}}^2}$. These two components can be used to determine the total stretching force and shear force (in those planes) to deform the body.

6.2 *Straightening, stretching and shearing of a sector of a hollow cylinder*

Consider the sector of a hollow cylinder defined in the reference configuration as

$$R_1 \le R \le R_2, \quad -\Theta_{\mathrm{o}} \le \Theta \le \Theta_{\mathrm{o}}, \quad -Z_{\mathrm{o}} \le Z \le Z_{\mathrm{o}}. \tag{95}$$

It is supposed that this sector deforms and becomes a block due to the presence of the stress

$$\mathbf{T} = \sum_{i=1}^{3} T_{ii}(x)\mathbf{e}_i \otimes \mathbf{e}_i + T_{23}(x)(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \tag{96}$$

where the deformation is assumed to be

$$x = \frac{1}{2}AB^2R^2, \quad y = \frac{\Theta}{AB}, \quad z = \frac{Z}{B} + \frac{C\Theta}{AB}, \tag{97}$$

where A, B and C are constants and $AB \neq 0$. In this case from (97) we have

$$\mathbf{F} = AB^2R\mathbf{e}_1 \otimes \mathbf{E}_R + \frac{1}{ABR}\mathbf{e}_2 \otimes \mathbf{E}_\Theta + \frac{1}{B}\mathbf{e}_3 \otimes \mathbf{E}_Z + \frac{C}{ABR}\mathbf{e}_3 \otimes \mathbf{E}_\Theta, \tag{98}$$

and

$$\mathbf{B} = A^2B^4R^2\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{A^2B^2R^2}\mathbf{e}_2 \otimes \mathbf{e}_2 + \left(\frac{C^2}{A^2B^2R^2} + \frac{1}{B^2}\right)\mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{C}{A^2B^2R^2}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2). \tag{99}$$

From (99), the principal stretches are

$$\lambda_1 = AB^2R, \quad \lambda_2 = \frac{1}{ABR}\sqrt{\frac{\lambda_a - \lambda_b}{2}}, \quad \lambda_3 = \frac{1}{ABR}\sqrt{\frac{\lambda_a + \lambda_b}{2}}, \tag{100}$$

where we have defined

$$\lambda_a = 1 + C^2 + A^2R^2, \quad \lambda_b = \sqrt{(1 + C^2)^2 + 2A^2(C^2 - 1)R^2 + A^4R^4}. \tag{101}$$

The above principal stretches can be expressed in terms of x from $(97)_1$ using

$$R = \frac{1}{B}\sqrt{\frac{2x}{A}}. \tag{102}$$

The principal directions of $\mathbf{V}$ are

$$\overset{(1)}{\mathbf{v}} = \mathbf{e}_1, \quad \overset{(k)}{\mathbf{v}} = \overset{(k)}{v}_2\, \mathbf{e}_2 + \overset{(k)}{v}_3\, \mathbf{e}_3, \quad k = 2, 3, \tag{103}$$

where

$$\overset{(k)}{v}_2 = \frac{1}{\sqrt{1 + \frac{1}{C^2}\left(\lambda_k^2A^2B^2R^2 - 1\right)^2}}, \quad \overset{(k)}{v}_3 = \frac{1}{\sqrt{\frac{C^2}{\left(\lambda_k^2A^2B^2R^2 - 1\right)^2} + 1}}, \quad k = 2, 3. \tag{104}$$

Regarding the deviatoric part of the stress from (96), we have

$$\mathbf{T}_\mathrm{D} = \sum_{i=1}^{3} T_{\mathrm{D}_{ii}}(x)\mathbf{e}_i \otimes \mathbf{e}_i + T_{23}(x)(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2). \tag{105}$$

Using (100) and (103) (considering (105)) from (26), we obtain

$$\ln\left(AB^2R\right) = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{11}}(x) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{11}}(x)^2, \tag{106}$$

$$\ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a - \lambda_b}{2}}\right]\left(\overset{(2)}{v}_2\right)^2 + \ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a + \lambda_b}{2}}\right]\left(\overset{(3)}{v}_2\right)^2 = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{22}}(x) + \alpha_2(\mathbf{T}_\mathrm{D})[T_{\mathrm{D}_{22}}(x)^2 + T_{23}(x)^2], \tag{107}$$

$$\ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a-\lambda_b}{2}}\right]\left(\overset{(2)}{v}_3\right)^2+\ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a+\lambda_b}{2}}\right]\left(\overset{(3)}{v}_3\right)^2=\alpha_0(\mathbf{T}_D)$$
$$+\alpha_1(\mathbf{T}_D)T_{D_{33}}(x)+\alpha_2(\mathbf{T}_D)[T_{23}(x)^2+T_{33}(x)^2], \tag{108}$$

$$\ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a-\lambda_b}{2}}\right]\overset{(2)}{v}_2\overset{(2)}{v}_3+\ln\left[\frac{1}{ABR}\sqrt{\frac{\lambda_a+\lambda_b}{2}}\right]\overset{(3)}{v}_2\overset{(3)}{v}_3=\alpha_1(\mathbf{T}_D)T_{23}(x)$$
$$+\alpha_2(\mathbf{T}_D)T_{23}(x)[T_{D_{22}}(x)+T_{33}(x)]. \tag{109}$$

The above four equations can be used to obtain (implicitly) the four independent components of the deviatoric part of the stress (105), in terms of x (see (100)–(102)) and the constants A, B and C.

Regarding the spherical part of the stress, from (96), (105) and (20) in the equilibrium equation we obtain

$$\frac{dT_{11}}{dx}=0 \quad\Leftrightarrow\quad -\frac{d\sigma_S}{dx}+\frac{dT_{D_{11}}}{dx}=0, \tag{110}$$

which leads to $T_{11}(x)=C_o$ and $\sigma_S(x)=T_{D_{11}}(x)+\tilde{C}_o$, where C_o and $\tilde{C}_o$ are constants. Let us define $x_1=\frac{1}{2}AB^2R_1^2$, $x_2=\frac{1}{2}AB^2R_2^2$, then if the surfaces $x=x_1$ and $x=x_2$ (that are equivalent to the surfaces $R=R_1$ and $R=R_2$) are free of traction, then $T_{11}=0$, as a result $C_o=0$ and $\tilde{C}_o=0$ and

$$\sigma_S(x)=T_{D_{11}}(x). \tag{111}$$

Let us study briefly the other surfaces for the deformed body. Let us use the notation $y_o=\frac{\Theta_o}{AB}$ for the surface $\Theta=\Theta_o$. For such surfaces the vector $\hat{\mathbf{t}}=\mathbf{Tn}$ is equal to $\hat{\mathbf{t}}=T_{22}\mathbf{e}_2+T_{23}\mathbf{e}_3$, and the total bending moment and shear forces are $\mathcal{M}=\frac{2Z_o}{B}\int_{x_1}^{x_2}xT_{22}(x)\,dx$ and $\mathcal{S}=\frac{2Z_o}{B}\int_{x_1}^{x_2}T_{23}(x)\,dx$, respectively.

Regarding the slanted faces (see pp. 189 in (**17**)) $z-Cy=constant$, the normal vector is $\mathbf{n}=\frac{1}{\sqrt{1+C^2}}(-C\mathbf{e}_2+\mathbf{e}_3)$, and as a result $\hat{\mathbf{t}}=\frac{1}{\sqrt{1+C^2}}[(T_{23}-CT_{22})\mathbf{e}_2+(T_{33}-CT_{23})\mathbf{e}_3]$, from where a normal σ_N and a shear components τ (to the slanted faces) can be obtained in the same manner as in the previous example (see end of Section 6.1).

6.3 *Inflation, bending, torsion, extension and shearing of an annular wedge*

Consider the annular wedge defined in the reference configuration as

$$R_1\le R\le R_2,\quad 0\le\Theta\le\Theta_o,\quad 0\le Z\le L. \tag{112}$$

Let us assume the above wedge deforms due to the application of a stress tensor of the form (80) as (see Section 57 of (**17**))

$$r=\sqrt{AR^2+B},\quad \theta=C\Theta+DZ,\quad z=E\Theta+FZ, \tag{113}$$

where A, B, C, D, E and F are constants, where $A(CF-DE)=1$. The case studied in Section 5 is recovered when $C=1$, $D=0$ and $E=0$. The deformation gradient is given by

$$\mathbf{F}=\frac{AR}{r}\mathbf{e}_r\otimes\mathbf{E}_R+\frac{rC}{R}\mathbf{e}_\theta\otimes\mathbf{E}_\Theta+rD\mathbf{e}_\theta\otimes\mathbf{E}_Z+\frac{E}{R}\mathbf{e}_z\otimes\mathbf{E}_\Theta+F\mathbf{e}_z\otimes\mathbf{E}_Z, \tag{114}$$

as a results the left Cauchy-Green tensor is

$$\mathbf{B} = \frac{A^2R^2}{r^2}\mathbf{e}_r \otimes \mathbf{e}_r + r^2\left(D^2 + \frac{C^2}{R^2}\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + r\left(DF + \frac{CE}{R^2}\right)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta) + \left(F^2 + \frac{E^2}{R^2}\right)\mathbf{e}_z \otimes \mathbf{e}_z, \quad (115)$$

from where the principal stretches can be calculated as

$$\lambda_1 = \frac{AR}{r}, \quad \lambda_2 = \frac{1}{R}\sqrt{\frac{\lambda_\mathrm{a} - \lambda_\mathrm{b}}{2}}, \quad \lambda_3 = \frac{1}{R}\sqrt{\frac{\lambda_\mathrm{a} + \lambda_\mathrm{b}}{2}}, \quad (116)$$

where

$$\lambda_\mathrm{a} = E^2 + C^2r^2 + R^2(F^2 + D^2r^2), \quad (117)$$

$$\lambda_\mathrm{b} = \sqrt{[E^2 + C^2r^2 + (F^2 + D^2r^2)R^2]^2 - 4(DE - CF)^2r^2R^2}. \quad (118)$$

The principal directions are given in the same manner as in (86), where in this case

$$\overset{(k)}{v}_2 = \frac{1}{\sqrt{1 + \frac{[\lambda_k^2 - r^2(D^2 + C^2/R^2)]^2}{r^2(DF + CE/R^2)^2}}}, \quad \overset{(k)}{v}_3 = \frac{1}{\sqrt{1 + \frac{r^2(DF + CE/R^2)^2}{[\lambda_k^2 - r^2(D^2 + C^2/R^2)]^2}}}, \quad k = 2, 3. \quad (119)$$

In this problem, the deviatoric stress has the same form as in (88). Using that and (116), (119), (86) in (26), we obtain

$$\ln\left(\frac{AR}{r}\right) = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{rr}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{rr}}(r)^2, \quad (120)$$

$$\ln\left[\frac{\sqrt{\lambda_\mathrm{a} - \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\left(\overset{(2)}{v}_2\right)^2 + \ln\left[\frac{\sqrt{\lambda_\mathrm{a} + \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\left(\overset{(3)}{v}_2\right)^2 = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{\theta\theta}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})[T_{\mathrm{D}_{\theta\theta}}(r)^2 + T_{\theta z}(r)^2], \quad (121)$$

$$\ln\left[\frac{\sqrt{\lambda_\mathrm{a} - \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\left(\overset{(2)}{v}_3\right)^2 + \ln\left[\frac{\sqrt{\lambda_\mathrm{a} + \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\left(\overset{(3)}{v}_3\right)^2 = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{zz}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})[T_{\theta z}(r)^2 + T_{\mathrm{D}_{zz}}(r)^2], \quad (122)$$

$$\ln\left[\frac{\sqrt{\lambda_\mathrm{a} - \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\overset{(2)}{v}_2\overset{(2)}{v}_3 + \ln\left[\frac{\sqrt{\lambda_\mathrm{a} + \lambda_\mathrm{b}}}{\sqrt{2}R}\right]\overset{(3)}{v}_2\overset{(3)}{v}_3 = \alpha_1(\mathbf{T}_\mathrm{D})T_{\theta z}(r) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\theta z}(r)[T_{\mathrm{D}_{\theta\theta}}(r) + T_{\mathrm{D}_{zz}}(r)]. \quad (123)$$

The above four equations (120)–(123) can be used to find the components $T_{\mathrm{D}_{rr}}(r)$, $T_{\mathrm{D}_{\theta\theta}}(r)$, $T_{\theta z}(r)$ and $T_{\mathrm{D}_{zz}}(r)$ of the deviatoric stress. Regarding σ_S and T_{rr} they are given by $(70)_1$, (72).

Let us study briefly the boundary conditions for this problem. From (113), we can observe that the constants A and B are related with the inflation/deflation and evertion of the wedge, while the constant C is connected with the angular opening or closure of the wedge, depending if $C \geq 1$ or $0 < C < 1$, respectively. The constant D is related with torsion, the constant E would produce a sort of azimuthal shear (see Section 57 of (**17**)), and F is connected with an axial uniform extension/compression for the wedge.

The surfaces $R = constant$ have normal vectors $\mathbf{n} = \mathbf{e}_r$, therefore, $\mathbf{Tn} = T_{rr}\mathbf{e}_r$. If $A > 0$ the wedge is inflated, let us assume the boundary conditions $T_{rr}(r_\mathrm{i}) = -P$ and $T_{rr}(r_\mathrm{o}) = 0$, that is, there is a radial traction applied on the inner surface of the wedge $r = r_\mathrm{i}$ ($R = R_\mathrm{i}$), and on the outer surface $r = r_\mathrm{o}$ ($R = R_\mathrm{o}$) there is no external traction, where $r_\mathrm{i} = \sqrt{AR_\mathrm{i}^2 + B}$ and $r_\mathrm{o} = \sqrt{AR_\mathrm{o}^2 + B} \geq r_\mathrm{i}$. From the above boundary conditions, we obtain the same expression for P given in (78). In the case there is no external traction on the inner surface $T_{rr}(r_\mathrm{i}) = 0$ and on the outer surface $T_{rr}(r_\mathrm{o}) = -P$, we obtain $-P = \int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\, d\xi$. In the above calculations from $r_\mathrm{i} = \sqrt{AR_\mathrm{i}^2 + B}$ we obtain $B = r_\mathrm{i}^2 - AR_\mathrm{i}^2$, as a result $r = \sqrt{A(R^2 - R_\mathrm{i}^2) + r_\mathrm{i}^2}$, and $r_\mathrm{o} = \sqrt{A(R_\mathrm{o}^2 - R_\mathrm{i}^2) + r_\mathrm{i}^2}$. If the wedge is everted then $A < 0$, therefore, $r_\mathrm{o} = \sqrt{A(R_\mathrm{o}^2 - R_\mathrm{i}^2) + r_\mathrm{i}^2} \leq r_\mathrm{i}$. Let us assume that in this situation there is no traction on the outer and inner surfaces $r = r_\mathrm{i}$, $r = r_\mathrm{o}$ of the wedge, we obtain $\int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\xi}[T_{\mathrm{D}_{rr}}(\xi) - T_{\mathrm{D}_{\theta\theta}}(\xi)]\, d\xi = 0$.

The surfaces $\Theta = constant$ have normal unit vectors $\mathbf{n} = \frac{1}{\sqrt{\frac{F^2}{r^2}+D^2}}\left(\frac{F}{r}\mathbf{e}_\theta + D\mathbf{e}_z\right)$, as a result $\hat{\mathbf{t}} = \mathbf{Tn} = \frac{1}{\sqrt{\frac{F^2}{r^2}+D^2}}\left[\left(\frac{F}{r}T_{\theta\theta}(r) + DT_{\theta z}(r)\right)\mathbf{e}_\theta + \left(\frac{F}{r}T_{\theta z}(r) + DT_{zz}(r)\right)\mathbf{e}_z\right]$, from where we can calculate the normal and shear components (to the above surfaces) $\sigma_\mathbf{N} = \hat{\mathbf{t}}\cdot\mathbf{n}$ and $\tau = \sqrt{|\hat{\mathbf{t}}|^2 - \sigma_\mathrm{N}^2}$, respectively, from where we can determine the total normal load and total shear force as $\mathcal{N} = FL\int_{r_\mathrm{i}}^{r_\mathrm{o}} \sigma_\mathrm{N}(r)\, dr$ and $\mathcal{S} = FL\int_{r_\mathrm{i}}^{r_\mathrm{o}} \tau(r)\, dr$, respectively.

Finally, for the surfaces $Z = constant$, the normal unit vector is $\mathbf{n} = \frac{1}{\sqrt{\frac{E^2}{r^2}+C^2}}\left(-\frac{E}{r}\mathbf{e}_\theta + C\mathbf{e}_z\right)$, from where $\hat{\mathbf{t}} = \mathbf{Tn} = \frac{1}{\sqrt{\frac{E^2}{r^2}+C^2}}\left[\left(-\frac{E}{r}T_{\theta\theta}(r) + CT_{\theta z}(r)\right)\mathbf{e}_\theta + \left(-\frac{E}{r}T_{\theta z}(r) + CT_{zz}(r)\right)\mathbf{e}_z\right]$ is obtained, and if $\sigma_\mathbf{N} = \hat{\mathbf{t}}\cdot\mathbf{n}$ and $\tau = \sqrt{|\hat{\mathbf{t}}|^2 - \sigma_\mathrm{N}^2}$ and the total normal load and twisting moment can be calculated.

6.4 *Inflation or eversion of a sector of a spherical shell*

Consider the sector of a spherical shell defined as

$$R_\mathrm{i} \leq R \leq R_\mathrm{o}, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq \Phi \leq \Phi_\mathrm{o} \leq \pi, \tag{124}$$

which we assume deforms due to the presence of the stress field

$$\mathbf{T} = T_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\phi\phi}(r)\mathbf{e}_\phi \otimes \mathbf{e}_\phi. \tag{125}$$

The deformation of the body is supposed be (see (**17**))

$$r = \sqrt[3]{A \pm R^3}, \quad \theta = \pm\Theta, \quad \phi = \Phi, \tag{126}$$

where A is a constant. The inflation is related with the $(+)$ sign and the eversion with the $(-)$ sign in the above expressions.

Considering (125) the deviatoric stress tensor is

$$\mathbf{T}_\mathrm{D}(r) = T_{\mathrm{D}_{rr}}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{\theta\theta}}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz}}(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{127}$$

The left Cauchy-Green tensor is

$$\mathbf{B} = \frac{R^4}{r^4}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{r^2}{R^2}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi), \tag{128}$$

and from (26) we obtain

$$\ln\left(\frac{R^2}{r^2}\right) = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{rr}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{rr}}(r)^2, \tag{129}$$

$$\ln\left(\frac{r}{R}\right) = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{\theta\theta}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{\theta\theta}}(r)^2, \tag{130}$$

$$\ln\left(\frac{r}{R}\right) = \alpha_0(\mathbf{T}_\mathrm{D}) + \alpha_1(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{\phi\phi}}(r) + \alpha_2(\mathbf{T}_\mathrm{D})T_{\mathrm{D}_{\phi\phi}}(r)^2. \tag{131}$$

From (130) and (131) it is possible to see that $T_{\mathrm{D}_{\theta\theta}}(r) = T_{\mathrm{D}_{\phi\phi}}(r)$, then (129) and (130) can be used to obtain $T_{\mathrm{D}_{rr}}(r)$ and $T_{\mathrm{D}_{\theta\theta}}(r)$, which from (126) depend on A. If (125) is replaced in (6)-(8) considering (20) and (127) we obtain $T_{rr}(r) = 2\int_{r_\mathrm{i}}^{r} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\,d\xi + C$, where C is a constant. In the case of inflation, if on the inner surface of the spherical sector we assume the application of the radial traction P, and on the outer surface we assume there is no external traction, we obtain

$$P = 2\int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\,d\xi. \tag{132}$$

The above equation can be used to find the constant A in (126). In the case we consider the problem of eversion, assuming that on the inner and outer surfaces of the spherical sector there is no external traction, instead of (132) we obtain $\int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\xi}[T_{\mathrm{D}_{\theta\theta}}(\xi) - T_{\mathrm{D}_{rr}}(\xi)]\,d\xi = 0$, from where A can be found as well.

6.5 *Azimuthal shear of a cuboid*

The last problem to be studied corresponds to the azimuthal shear of a cuboid or annular wedge, which was presented by Singh and Pipkin in (**27**, **28**). The cuboid is defined in the reference configuration as

$$R_\mathrm{i} \leq R \leq R_\mathrm{o}, \quad -\Theta_\mathrm{o} \leq \Theta \leq \Theta_\mathrm{o}, \quad 0 \leq Z \leq L. \tag{133}$$

In this problem, it is assumed that the stress tensor that deforms the body is of the form (see the decomposition (20))

$$\mathbf{T} = -\sigma_\mathrm{S}(r,\theta)\mathbf{I} + \mathbf{T}_\mathrm{D}(r), \tag{134}$$

where the deviatoric part of the stress is supposed to be

$$\mathbf{T}_\mathrm{D}(r) = T_{\mathrm{D}_{rr}}(r)\mathbf{e}_r \otimes \mathbf{e}_r + T_{\mathrm{D}_{r\theta}}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + T_{\mathrm{D}_{\theta\theta}}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + T_{\mathrm{D}_{zz}}(r)\mathbf{e}_z \otimes \mathbf{e}_z. \tag{135}$$

The cuboid (133) is assumed to deform due to the presence of the stress (134) as

$$r = AR, \quad \theta = B\ln R + C\Theta, \quad z = \frac{Z}{A^2C}, \tag{136}$$

where A, B and C are constants. In this case the deformation gradient is

$$\mathbf{F} = A\mathbf{e}_r \otimes \mathbf{E}_R + AB\mathbf{e}_\theta \otimes \mathbf{E}_R + AC\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \frac{1}{A^2C}\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{137}$$

and as a result the left Cauchy-Green tensor is

$$\mathbf{B} = A^2\mathbf{e}_r \otimes \mathbf{e}_r + A^2B(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + A^2(B^2 + C^2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{1}{A^4C^2}\mathbf{e}_z \otimes \mathbf{e}_z, \tag{138}$$

from where we obtain the principal stretches

$$\lambda_1 = A\sqrt{\frac{\lambda_a - \lambda_b}{2}}, \quad \lambda_2 = A\sqrt{\frac{\lambda_a + \lambda_b}{2}}, \quad \lambda_3 = \frac{1}{A^2C}, \tag{139}$$

where

$$\lambda_a = 1 + B^2 + C^2, \quad \lambda_b = \sqrt{(1+B^2)^2 + 2(B^2-1)C^2 + C^4}. \tag{140}$$

The principal directions of $\mathbf{V}$ are

$$\overset{(q)}{\mathbf{v}} = \overset{(q)}{v}_1\, \mathbf{e}_r + \overset{(q)}{v}_2\, \mathbf{e}_\theta, \quad q = 1, 2, \quad \overset{(3)}{\mathbf{v}} = \mathbf{e}_z, \tag{141}$$

where

$$\overset{(q)}{v}_1 = \frac{1}{\sqrt{1 + \frac{1}{A^4B^2}(\lambda_q^2 - A^2)^2}}, \quad \overset{(q)}{v}_2 = \frac{1}{\sqrt{1 + \frac{A^4B^2}{(\lambda_q^2 - A^2)^2}}}. \tag{142}$$

If (134) is replaced in the equations of equilibrium (3)–(5), we obtain

$$-\frac{\partial \sigma_S}{\partial r} + \frac{dT_{D_{rr}}}{dr} + \frac{1}{r}(T_{D_{rr}} - T_{D_{\theta\theta}}) = 0, \quad \frac{dT_{D_{r\theta}}}{dr} - \frac{1}{r}\frac{\partial \sigma_S}{\partial \theta} + \frac{2T_{D_{r\theta}}}{r} = 0. \tag{143}$$

The solution of $(143)_1$ is $\sigma_S = T_{D_{rr}}(r) + \int_{r_i}^{r} \frac{1}{\xi}[T_{D_{rr}}(\xi) - T_{D_{\theta\theta}}(\xi)]\, d\xi + C(\theta)$, and using this in $(143)_2$ we obtain $C(\theta) = \frac{\theta}{r}\frac{d}{dr}(r^2 T_{D_{r\theta}}(r)) + C_o$, where C_o is a constant, therefore, the spherical part of the stress is

$$\sigma_S = T_{D_{rr}}(r) + \int_{r_i}^{r} \frac{1}{\xi}[T_{D_{rr}}(\xi) - T_{D_{\theta\theta}}(\xi)]\, d\xi + \frac{\theta}{r}\frac{d}{dr}(r^2 T_{D_{r\theta}}(r)) + C_o. \tag{144}$$

On the other hand, using (139)–(141) in (26) considering (135) we have

$$\ln\left[A\sqrt{\frac{\lambda_a-\lambda_b}{2}}\right]\left(\overset{(1)}{v}_1\right)^2+\ln\left[A\sqrt{\frac{\lambda_a+\lambda_b}{2}}\right]\left(\overset{(2)}{v}_1\right)^2=\alpha_0(\mathbf{T}_D)+\alpha_1(\mathbf{T}_D)T_{D_{rr}}(r)$$

$$+\alpha_2(\mathbf{T}_D)(T_{D_{rr}}(r)^2+T_{D_{r\theta}}(r)^2), \quad (145)$$

$$\ln\left[A\sqrt{\frac{\lambda_a-\lambda_b}{2}}\right]\left(\overset{(1)}{v}_2\right)^2+\ln\left[A\sqrt{\frac{\lambda_a+\lambda_b}{2}}\right]\left(\overset{(2)}{v}_2\right)^2=\alpha_0(\mathbf{T}_D)+\alpha_1(\mathbf{T}_D)T_{D_{\theta\theta}}(r)$$

$$+\alpha_2(\mathbf{T}_D)(T_{D_{r\theta}}(r)^2+T_{D_{\theta\theta}}(r)^2), \quad (146)$$

$$\ln\left(\frac{1}{A^2C}\right)=\alpha_0(\mathbf{T}_D)+\alpha_1(\mathbf{T}_D)T_{D_{zz}}(r)$$

$$+\alpha_2(\mathbf{T}_D)T_{D_{zz}}(r)^2, \quad (147)$$

$$\ln\left[A\sqrt{\frac{\lambda_a-\lambda_b}{2}}\right]\overset{(1)}{v}_1\overset{(1)}{v}_2+\ln\left[A\sqrt{\frac{\lambda_a+\lambda_b}{2}}\right]\overset{(2)}{v}_1\overset{(2)}{v}_2=\alpha_1(\mathbf{T}_D)T_{D_{r\theta}}(r)$$

$$+\alpha_2(\mathbf{T}_D)T_{D_{r\theta}}(r)(T_{D_{rr}}(r)+T_{D_{\theta\theta}}(r)). \quad (148)$$

These four equations can be used to find $T_{D_{rr}}(r)$, $T_{D_{r\theta}}(r)$, $T_{D_{\theta\theta}}(r)$ and $T_{D_{zz}}(r)$. From (140), it is possible to see that the left side of (145)–(148) do not depend on r, therefore, the above components of the deviatoric stress are constants, as a result from (144) the spherical part of the stress is given as

$$\sigma_S(r)=(T_{D_{rr}}-T_{D_{\theta\theta}})\ln r+2\theta T_{D_{r\theta}}+\tilde{C}_o, \quad (149)$$

where $\tilde{C}_o$ is a constant.

Let us finish this section studying briefly the boundary conditions for the different surfaces of the cuboid. In the case of the surfaces $R=$ *constant* the unit normal vector is $\mathbf{n}=\mathbf{e}_r$, as a results from (134) (considering (135), (149)) we obtain $\hat{\mathbf{t}}=[(T_{D_{\theta\theta}}-T_{D_{rr}})\ln r-2\theta T_{D_{r\theta}}-\tilde{C}_o+T_{D_{rr}}]\mathbf{e}_r+T_{D_{r\theta}}\mathbf{e}_\theta$, from where we can calculate the total normal force $\mathcal{N}$ as $\mathcal{N}=\frac{2Z_o}{A^2C}\int_{-C\Theta_o+B\ln R_i}^{C\Theta_o+B\ln R_i}\hat{t}_r(r_i)r_i\,d\theta$ for the inner surface of the cuboid, where $r_i=AR_i$, and $\mathcal{N}=\frac{2Z_o}{A^2C}\int_{-C\Theta_o+B\ln R_o}^{C\Theta_o+B\ln R_o}\hat{t}_r(r_o)r_o\,d\theta$, for the outer surface of the cuboid, where $r_o=AR_o$. Taking into account that the components of the deviatoric stress are constants we obtain $\mathcal{N}_i=\frac{4Z_oR_i\Theta_o}{A}[(T_{D_{rr}}-T_{D_{\theta\theta}})\ln r_i+\tilde{C}_o+T_{D_{rr}}-2T_{D_{r\theta}}B\ln R_i]$ and $\mathcal{N}_o=\frac{4Z_oR_o\Theta_o}{A}[(T_{D_{rr}}-T_{D_{\theta\theta}})\ln r_o+\tilde{C}_o+T_{D_{rr}}-2T_{D_{r\theta}}B\ln R_o]$. Something similar can be done with the component $\hat{t}_\theta$ of $\hat{\mathbf{t}}$ above, from where we have the total shear forces $\mathcal{S}_i=-\frac{4Z_o}{A}T_{D_{r\theta}}\Theta_oR_i$ and $\mathcal{S}_o=-\frac{4Z_o}{A}T_{D_{r\theta}}\Theta_oR_o$, for the inner and outer surfaces, respectively.

In the case of the surfaces $Z=$ *constant* we have $\mathbf{n}=\mathbf{e}_z$, and from (134) taking into account (135), (149) we obtain $\hat{\mathbf{t}}=[(T_{D_{\theta\theta}}-T_{D_{rr}})\ln r-2\theta T_{D_{r\theta}}-\tilde{C}_o+T_{D_{zz}}]\mathbf{e}_z$, from where we can calculate the total axial load as $\mathcal{N}=C\Theta_o\Big\{(T_{D_{\theta\theta}}-T_{D_{rr}})(r_o^2\ln r_o-r_i^2\ln r_i-r_o-r_i)-2(T_{D_{zz}}-\tilde{C}_o)(r_o-r_i)4T_{D_{r\theta}}B\left[r_o\left(1-\ln\left(\frac{r_o}{A}\right)\right)-r_i\left(1-\ln\left(\frac{r_i}{A}\right)\right)\right]\Big\}$.

In the case of the surfaces $\Theta=$ *constant*, the normal unit vector is $\mathbf{n}=\frac{1}{\sqrt{1+B^2}}(-B\mathbf{e}_r+\mathbf{e}_\theta)$ from where we obtain $\hat{\mathbf{t}}=\frac{1}{\sqrt{1+B^2}}\{[B(\sigma_S-T_{D_{rr}})+T_{D_{r\theta}}]\mathbf{e}_r+[-BT_{D_{r\theta}}+T_{D_{\theta\theta}}-\sigma_S]\mathbf{e}_\theta\}$. As in some of the boundary value problems studied before, normal and shear components σ_{N}, τ (to the surface with that normal $\mathbf{n}$) can be calculated as $\sigma_{\mathrm{N}} = \hat{\mathbf{t}} \cdot \mathbf{n}$ and $\tau = \sqrt{|\hat{\mathbf{t}}|^2 - \sigma_{\mathrm{N}}^2}$, from where total external loads can be calculated, but for brevity are not shown here.

7. Final remarks

In this work, some universal solutions have been studied for a new class of constitutive equations for nonlinear elastic incompressible isotropic bodies described in (**13**). The universal solutions analysed correspond to the classical solutions found in the nonlinear theory of elasticity (**16**, **17**, **29**), that is, it has been shown that the same universal solutions are valid for these new constitutive theories. An hybrid semi-inverse method was used to solve the boundary value problems, assuming simplified expressions for the stresses and the deformation, and the solutions that were found are explicit for the deformation field (up to some constants), and implicit for the components of the deviatoric stress. Some of the constants associated with the deformations can be found from the boundary conditions, which were studied in some detail in this communication.

We have not looked for all the universal solutions for (22) (see (24) and (26)). To do that we would need to repeat an analysis similar to the work by Ericksen (**16**), recalling that in the classical theory of nonlinear elasticity for isotropic incompressible elastic bodies (9), the problem of finding all the universal solutions is not closed (**30**, **31**). Another important issue to be considered in future works is the possible lack of uniqueness for the components of the deviatoric stress tensor, when solving the algebraic equations (26).

The possibility of finding universal solutions for these new classes of constitutive theories is very important, in particular when looking for specific expressions for the constitutive equations for some specific materials.

Acknowledgements

R. Bustamante would like to express his gratitude for the financial support provided by FONDECYT (Chile) under grant no. 1160030.

References

1. R. Bustamante, Some topics on a new class of elastic bodies, *Proc. R. Soc. A* **465** (2009) 1377–1392.
2. R. Bustamante and K. R. Rajagopal, A note on plane strain and plane stress problems for a new class of elastic bodies, *Math. Mech. Solids* **15** (2010) 229–238.
3. R. Bustamante and K. R. Rajagopal, A review of implicit constitutive theories to describe the response of elastic bodies, *Constitutive Modelling of Solid Continua* (eds J. Merodio & R. W. Ogden; Springer International Publishing, Cham, Switzerland 2019) 187–230.
4. K.R. Rajagopal, On implicit constitutive theories, *Appl. Math.* **48** (2003) 279–319.
5. K. R. Rajagopal, The elasticity of elasticity, *Z. Angew. Math. Phys.* **58** (2007) 309–317.
6. K. R. Rajagopal and A. R. Srinivasa, On the response of non-dissipative solids, *Proc. R. Soc. A* **463** (2007) 357–367.
7. K. R. Rajagopal and A. R. Srinivasa, On a class of non-dissipative solids that are not hyperelastic, *Proc. R. Soc. A* **465** (2009) 493–500.

8. K. R. Rajagopal and A. R. Srinivasa, A Gibbs-potential-based formulation for obtaining the response functions for a class of viscoelastic materials, *Proc. R. Soc. A* **467** (2011) 39–58.

9. K. R. Rajagopal, Conspectus of concepts of elasticity, *Math. Mech. Solids* **16** (2011) 536–562.

10. K. R. Rajagopal and A. R. Srinivasa, An implicit thermomechanical theory based on a Gibbs potential formulation for describing the response of thermoviscoelastic solids, *Int. J. Eng. Sci.* **70** (2013) 15–28.

11. K. R. Rajagopal and A. R. Srinivasa, Inelastic response of solids described by implicit constitutive relations with nonlinear small strain elastic response, *Int. J. Plasticity* **71** (2015) 1–9.

12. K. R. Rajagopal and A. R. Srinivasa, An implicit three-dimensional model for describing the inelastic response of solids undergoing finite deformation, *Z. Angew. Math. Phys.* **67** (2016) 86.

13. A. R. Srinivasa, On a class of Gibbs potential-based nonlinear elastic models with small strains, *Acta Mech.* **226** (2015) 571–583.

14. V. Průša, K. R. Rajagopal and K. Tůma, Gibbs free energy based representation formula within the context of implicit constitutive relations for elastic solids, *Int. J. Nonlin. Mech.* **121** (2020) 103433.

15. R. Bustamante, Some universal solutions for incompressible elastic bodies that are not Green elastic, *Int. J. Eng. Sci.* **149** (2020) 103223.

16. J. L. Ericksen, Deformations possible in every isotropic, incompressible, perfectly elastic body. *Z. Angew. Math. Phys.* **5** (1954) 466-488.

17. C. A. Truesdell and W. Noll, *The Non-linear Field Theories of Mechanics*, 3rd edn. (ed. S. S Antman; Springer, Berlin, Germany 2004).

18. C.A. Truesdell and R. Toupin, The classical field theories, *Handbuch der Physik*, Vol. III/1 (ed. S. Flügge; Springer, Berlin 1960) 226–902.

19. J. E. Fitzgerald, A tensorial Hencky measure of strain and strain rate for finite deformations, *J. Appl. Phys.* **51** (1980) 5111–5115.

20. R. Bustamante and K. R. Rajagopal, On the consequences of the constraint of incompressibility with regard to a new class of constitutive relations for elastic bodies: Small displacement gradient approximation, *Continuum Mech. Therm.* **28** (2016) 293–303.

21. R. Bustamante, O. Orellana, R. Meneses and K. R. Rajagopal, Large deformations of a new class of incompressible elastic bodies, *Z. Angew. Math. Phys.* **67** (2016) 47.

22. C. S. Jog, The explicit determination of the logarithm of a tensor and its derivatives, *J. Elasticity* **93** (2008) 141–148.

23. R. Bustamante and K. R. Rajagopal, Solutions of some boundary value problems for a new class of elastic bodies undergoing small strains. Comparison with the predictions of the classical theory of linearized elasticity: Part I. Problems with cylindrical symmetry, *Acta Mech.* **226** (2015) 1815–1838.

24. R. Bustamante and K. R. Rajagopal, Solutions of some boundary value problems for a new class of elastic bodies undergoing small strains. Comparison with the predictions of the classical theory of linearized elasticity: Part II. A problem with spherical symmetry, *Acta Mech.* **226** (2015) 1807–1813.

25. R. Bustamante, Solutions of some boundary value problems for a class of constitutive relation for nonlinear elastic bodies that is not Green elastic, *Q. J. Mech. Appl. Math.* **69** (2016) 257–279.

26. G. I. Barenblatt and D. D. Joseph, *R.S. Rivlin, Collected Papers* (Springer, New York 1997).

27. M. Singh and A. C. Pipkin, Note on Ericksen's problem, *Z. Angew. Math. Phys.* **16** (1965) 706–709.

28. M. Singh and A. C. Pipkin, Controllable states of elastic dielectrics, *Arch. Rat. Mech. Anal.* **21** (1966) 169–210.

29. J. L. Ericksen, Deformations possible in every isotropic compressible, perfectly elastic material. *J. Math. Phys.* **34** (1955) 126–128.

30. J. M. Hill and D. J. Arrigo, A note on Ericksen's problem for radially symmetric deformations, *Math. Mech. Solids* **4** (1999) 395–405.

31. G. Saccomandi, Universal results in finite elasticity. *Nonlinear Elasticity. Theory and Applications*, London Mathematical Society Lecture Note Series 283 (eds Y. B. Fu & R. W. Ogden; Cambridge University Press, Cambridge, United Kingdom 2001) 97–134.