TRUNCATION IN DUALITY AND INTERTWINING KERNELS

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Abstract. We study properties of truncations in the dual and intertwining process in the monotone case. The main properties are stated for the time-reversed process and the time of absorption of the truncated intertwining process.

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1. Introduction

In this work we study truncation of stochastic kernels and their relation to duality and intertwining.

In next section we recall these concepts, a main one being that when one starts from an stochastic and positive recurrent matrix and one construct a dual, then the associated intertwining matrix is associated to the time-reversed matrix of the original one.

Thus, many of the concepts that are proven for the original matrices need to be shown for the time-reversed matrix. This gives rise to a problem when dealing with monotonicity because it is not necessarily invariant by time-reversing. On the other hand monotonicity property plays a central role in duality and intertwining because a nonnegative dual kernel exists for the Siegmund dual function, and in this case several properties can be stated between the original kernel and the dual and intertwining associated kernels.

In this framework our first result is to extend the Pollak-Siegmund relation stated for the monotone process to their time-reversed process. In terms of truncation this result asserts that whatever is the truncation, when the level increases, the quasi-stationary distribution converges to the stationary distribution. This is done in Proposition 6 in Section 5.1.

In Section 5.2 we introduce the truncation given by the mean expected value, which satisfies that the truncation up to N preserves the stationary distribution up to $N-1$. In Proposition 8 it is shown that it has a nonnegative dual, and in Proposition 9 it is proven that the time of attaining the absorbing state is increasing with the level, either for the dual and the intertwining matrices. For the Diaconis-Fill coupling this means that the time for attaining the stationary distribution is stochastically increasing with the truncation of the reversed process.

In Proposition 13 we compare the quasi-stationary behavior for the Diaconis-Fill coupling with respect to the intertwining kernel, when in both processes one avoids the absorbing state.

We recall some of the main general results in duality and intertwining in Section 3 and for the monotone case and the Siegmund kernel this is done in Section 4. The strong stationary times and some of its properties are given in Section 6, and the Diaconis-Fill coupling is given in Section 7.

2. Duality and Intertwining

2.1. **Notation.** Let I be a countable set and $\mathbf{1}$ be the unit function defined on I. A nonnegative matrix $P=(P(x,y): x,y\in I)$ is called a kernel on I. A substochastic kernel is such that $P\mathbf{1}\le \mathbf{1}$, it is stochastic when $P\mathbf{1}=\mathbf{1}$, and strictly substochastic if it is substochastic and there exists some $x\in I$ such that $P\mathbf{1}(x)<1$.

A kernel P is irreducible if for any pair $x,y\in I$ there exists $n>0$ such that $P^n(x,y)>0$. A point $x_0\in I$ is absorbing for P when $P(x_0,y)=\delta_{y,x_0}$ for $y\in I$ ($\delta_{x_0,x_0}=1$ and $\delta_{y,x_0}=0$ otherwise).

Let P be substochastic. Then, there exists a Markov chain $X=(X_n: n<\mathcal{T}^X)$ uniquely defined in distribution, taking values in I, with lifetime $\mathcal{T}^X$ and with transition kernel P. Take $\partial\not\in I$ and define $X_n=\partial$ for all $n\ge \mathcal{T}^X$. P acts on the set of bounded or nonnegative real functions by $Pf(x)=\mathbb{E}(f(X_1)\,\mathbf{1}(\mathcal{T}^X>1))$ and $P^nf(x)=\mathbb{E}(f(X_n)\,\mathbf{1}(\mathcal{T}^X>n))$ for $n\ge 1$, $x\in I$.

In the sequel, we introduce three kernels, P, $\widehat{P}$, $\widetilde{P}$; defined respectively on the countable sets I, $\widehat{I}$, $\widetilde{I}$. When they are substochastic the associated Markov chains are respectively denoted by X $\widehat{X}$, $\widetilde{X}$ with lifetimes $\mathcal{T},\widehat{\mathcal{T}},\widetilde{\mathcal{T}}$. For the chain X we note by $\tau_K=\inf\{n\ge 0: X_n\in K\}$ the hitting time of $K\subseteq I$ and put $\tau_a=\tau_{\{a\}}$ for $a\in I$. Analogously, we define for $\widehat{X}$ (respectively for $\widetilde{X}$) the hitting times $\widehat{\tau}_{\widehat{K}}$ and $\widehat{\tau}_{\widehat{a}}$ (respectively $\widetilde{\tau}_{\widetilde{K}}$ and $\widetilde{\tau}_{\widetilde{a}}$).

2.2. **Definitions.** We recall the duality and the intertwining relations. As usual M' denotes the transpose of the matrix M, so $M'(x,y)=M(y,x)$ for $x,y\in I$.

Definition 1. *Let P and $\widehat{P}$ be two kernels defined on the countable sets I and $\widehat{I}$, and let $H=(H(x,y): x\in I, y\in \widehat{I})$ be a matrix. Then $\widehat{P}$ is said to be a $H-$dual of P, and H is called a duality function between $(P,\widehat{P})$, if it is satisfied*

$$H\widehat{P}'=PH. \tag{1}$$

□

Duality is symmetric because if $\widehat{P}$ is a $H-$dual of P, then P is a $H'-$dual of $\widehat{P}$. We only consider nonnegative duality functions H. Note that if H is a duality function between $(P,\widehat{P})$, then cH also is for all $c>0$. We assume that no row and no column of H vanishes completely.

If $\widehat{P}$ is a $H-$dual of P, then $\widehat{X}$ is a $H-$dual of X, in the following sense

$$\forall x \in I,\, y \in \widehat{I},\, \forall n \geq 0 : \mathbb{E}_x(H(X_n, y)) = \mathbb{E}_y(H(x, \widehat{X}_n)),$$

where H is extended to $(I \cup \{\partial\}) \times (\widehat{I} \cup \{\partial\})$ by putting $H(x, \partial) = H(\partial, y) = H(\partial, \partial) = 0$ for all $x \in I$, $y \in \widehat{I}$.

This notion of duality (1) coincides with the one between Markov processes found in references [13], [14] and [5], among others. Let us now introduce intertwining as in [4].

Definition 2. *Let P and $\widetilde{P}$ be two kernels defined on the countable sets I and $\widetilde{I}$ and let $\Lambda = (\Lambda(y, x) : y \in \widetilde{I}, x \in I)$ be a stochastic matrix. We say that $\widetilde{P}$ is a $\Lambda-$intertwining of P, and Λ is called a link between $(P, \widetilde{P})$, if it is satisfied*

$$\widetilde{P}\Lambda = \Lambda P.$$

□

Intertwining is not necessarily symmetric because Λ' is not necessarily stochastic. If Λ is doubly stochastic then $\widetilde{P}$ a $\Lambda-$ intertwining of P implies P is a $\Lambda'-$intertwining of $\widetilde{P}$. If $\widetilde{P}$ is a $\Lambda-$intertwining of P, then $\widetilde{X}$ is said to be a $\Lambda-$intertwining of X.

Throughout this paper we consider $I = \mathbb{N} = \{1, 2, ...\}$, the set of nonnegative integers in the infinite case or $I = I_N = \{1, \cdots, N\}$ with $N \geq 2$ in the finite case.

We note by $\mathbf{e}_a$ a column vector with 0 entries except for its $a-$th entry which is 1.

For a vector $\rho \in \mathbf{R}^I$ we denote by D_ρ the diagonal matrix with diagonal entries $(D_\rho)(x, x) = \rho(x)$, $x \in I$.

Assumption. From now on, $P = (P(x, y) : x, y \in I)$ is assumed to be an irreducible positive recurrent stochastic kernel and its stationary distribution is noted by $\pi = (\pi(i) : i \in I) > 0$.

Let $\overleftarrow{P}$ be the time-reversed transition kernel of P, so $\overleftarrow{P}' = D_\pi P D_\pi^{-1}$ Since $\overleftarrow{P}$ is also irreducible positive recurrent with stationary distribution π', we can exchange the roles of P and $\overleftarrow{P}$.

Remark 1. *Let $\widetilde{P}$ be a $\Lambda-$intertwining of P. Since Λ is stochastic, when $\widetilde{\beta}'$ is a stationary probability measure of $\widetilde{P}$ then $\widetilde{\beta}'\Lambda$ is a stationary probability measure of P. Therefore, if $\widetilde{a}$ is an absorbing state for $\widetilde{P}$, then we necessarily have,*

$$\pi' = \mathbf{e}'_{\widetilde{a}}\Lambda. \tag{2}$$

□

3. Relations

Below we supply Theorem 1 shown in [10] which summarizes several relations on duality and intertwining.

Theorem 1. *Let P be an irreducible positive recurrent stochastic kernel with stationary distribution π'. Assume $\widehat{P}$ is a kernel which is $H-$dual of P, $H\widehat{P}' = PH$. Then:*

(i) $\widehat{P}H'D_\pi = H'D_\pi\overleftarrow{P}$;

(ii) $\varphi := H'\pi$ *is strictly positive and satisfies* $\widehat{P}\varphi = \varphi$*;*

When $\widehat{P}$ *is stochastic and irreducible then* $\varphi = c\mathbf{1}$ *for some* $c > 0$ *and* $\widetilde{P} = \widehat{P}$*.*

(iii) $\widetilde{P} = D_\varphi^{-1}\widehat{P}D_\varphi$ *is a stochastic kernel (defined on* $\widetilde{I} = \widehat{I}$*) and the matrix* $\Lambda := D_\varphi^{-1}H'D_\pi$ *is stochastic. Moreover* $\widetilde{P}$ *is a* $\Lambda-$*intertwining of* $\overleftarrow{P}$*. Hence, it holds*

$$\widetilde{P}\Lambda = \Lambda\overleftarrow{P} \;\; \textit{satisfying } \widetilde{P}\mathbf{1} = \mathbf{1} = \Lambda\mathbf{1}.$$

Now assume I *and* $\widehat{I}$ *are finite sets.*

(iv) If $\widehat{P}$ *is strictly substochastic then it is not irreducible.*

(v) If $\widehat{P}$ *is strictly substochastic and has a unique stochastic class* $\widehat{I}_\ell$*, then*

$$\frac{\varphi(x)}{\varphi(y)} = \mathbb{P}_x(\widehat{\tau}_{\widehat{I}_\ell} < \widehat{\mathcal{T}}) \textit{ for any } y \in \widehat{I}_\ell,$$

and the intertwined Markov chain $\widetilde{X}$ *is given by the Doob transform*

$$\mathbb{P}_x(\widetilde{X}_1 = y_1, \cdots, \widetilde{X}_k = y_k) = \mathbb{P}_x(\widehat{X}_1 = y_1, \cdots, \widehat{X}_k = y_k \,|\, \widehat{\tau}_{\widehat{I}_\ell} < \widehat{\mathcal{T}}). \tag{3}$$

(vi) If $\widehat{a}$ *is an absorbing state for* $\widehat{P}$ *then* $\widehat{a}$ *is an absorbing state for* $\widetilde{P}$ *and the relation (2)* $\pi' = \mathbf{e}'_{\widehat{a}}\Lambda$*, is satisfied.*

$\Box$

One has

$$\Lambda(x,y) = \varphi(x)^{-1}H(y,x)\pi(y),$$

in particular $\Lambda(x,y) = 0$ if and only if $H(y,x) = 0$.

Notice that when $\varphi > 0$, the equality $\widetilde{P} = D_\varphi^{-1}\widehat{P}D_\varphi$ implies that the sets of absorbing points for $\widehat{P}$ and $\widetilde{P}$ coincide.

Remark 2. *When the starting equality between stochastic kernels is the intertwining relation* $\widetilde{P}\Lambda = \Lambda\overleftarrow{P}$*, then we have the duality relation* $H\widehat{P}' = PH$ *with* $H = D_\pi^{-1}\Lambda'$ *and* $\widehat{P} = \widetilde{P}$*. In this case* $\varphi = \mathbf{1}$*.* $\Box$

The next result of having a constant column appears as a condition in the study of sharp duals, see Remark 6 in Section 6.

Proposition 2. *Assume* H *is nonsingular and has a strictly positive constant column, that is*

$$\exists \widehat{a} \in \widehat{I} : H\mathbf{e}_{\widehat{a}} = c\mathbf{1} \textit{ for some } c > 0.$$

Then:

(i) $\widehat{a}$ *is an absorbing state for* $\widehat{P}$ *(so* $\{\widehat{a}\}$ *is a stochastic class).*

(ii) Under the hypotheses of Theorem 1, $\pi' = \mathbf{e}'_{\widehat{a}}\Lambda$ *holds and if* $\widehat{P}$ *is strictly substochastic and* $\{\widehat{a}\}$ *is the unique stochastic class then* $\mathbb{P}_y(\widehat{\tau}_{\widehat{a}} < \widehat{\mathcal{T}}) = \varphi(y)/\varphi(\widehat{a})$ *and the relation (3) is satisfied.* $\Box$

Remark 3. *Duality functions with constant columns appear in the following situations. If* $x_0 \in I$ *is an absorbing point of the kernel* P *and* $\widehat{P}$ *is a substochastic kernel that is a* $H-$*dual of* P*, then* $h(y) := H(x_0, y)$*,* $y \in \widehat{I}$*, is a nonnegative* $\widehat{P}-$*harmonic function. So, when* H *is bounded and* $\widehat{P}$ *is a stochastic recurrent kernel, the* x_0-*row* $H(x_0, \cdot)$ *is constant.* $\Box$

Remark 4. *The time-reversed transition kernel $\overleftarrow{P}$ can always be put as a Doob transform $\overleftarrow{P} = D_\varphi P' D_\varphi^{-1}$ with $\varphi(x) = \mathbb{P}_x(\tau > 1)$ for some stopping time τ. In this case $\varphi = \pi$ so we must only define τ. For the Markov chain $X = (X_n)$ with transition matrix P it exists a set of random function $(U_{n,x} : n \geq 1, i \in I)$ taking values in I and independent for different n, such that $X_{n+1} = U_{n+1,X_n}$ for all $n \geq 0$. Now take a collection of independent identically distributed random variables $(\mathcal{J}_n : n \geq 1)$ distributed as π, and define the Markov chain $Y = (Y_n : n \geq 0)$ by*

$$Y_{n+1} = \mathcal{J}_{n+1} \cdot \mathbf{1}(U_{n+1,\mathcal{J}_{n+1}} = Y_n) + \partial \cdot \mathbf{1}(U_{n+1,\mathcal{J}_{n+1}} \neq Y_n),\, n \geq 0,$$

where $\partial \notin I$ is an absorbing state for Y. For $x, y \in I$ we have $\mathbb{P}(Y_{n+1} = y, Y_n = x) = \pi(y)\mathbb{P}(U_{n+1,y} = x) = \pi(y)P(y,x)$. Let $\tau = \inf(n \geq 1 : Y_n = \partial)$, then we have $\mathbb{P}_x(\tau > 1) = \pi(x)$ and so $\mathbb{P}_x(Y_1 = y \,|\, \tau > 1) = \pi(y)P(y,x)\pi(x)^{-1}$. Then $(Y_n : n \geq 0)$ is a Markov chain such that for all $n \geq 1$ one has $\mathbb{P}_x(Y_n = y \,|\, \tau > n) = \overleftarrow{P}^{(n)}(x,y)$.

4. Monotonicity and the Siegmund kernel

The Siegmund kernel, see [16], is defined by

$$H^S(x,y) = \mathbf{1}(x \leq y), x, y \in I.$$

This kernel is nonsingular and $(H^S)^{-1} = Id - R$ with $R(x,y) = \mathbf{1}(x+1 = y)$ (which is a strictly substochastic kernel because the $N-$th row vanishes), so $(H^S)^{-1}(x,y) = \mathbf{1}(x = y) - \mathbf{1}(x+1 = y)$ and $H^S = (Id - R)^{-1}$ is the potential matrix associated to R. In [16] the Siegmund duality was used to show the equivalence between absorbing and reflecting barrier problems for stochastically monotone chains.

Let $\widehat{P}$ be a substochastic kernels such that $H^S \widehat{P}' = PH^S$. Since $(H^S \widehat{P}')(x,y) = \sum_{z \geq x} \widehat{P}(y,z)$ and $(PH^S)(x,y) = \sum_{z \leq y} P(x,z)$, they must satisfy

$$\widehat{P}(y,x) = \sum_{z \geq x} \widehat{P}(y,z) - \sum_{z > x} \widehat{P}(y,z) = \sum_{z \leq y} (P(x,z) - P(x+1,z)),\; x, y \in I. \tag{4}$$

In particular, the condition $\widehat{P} \geq 0$ requires the monotonicity of P,

$$\forall y \in I \,:\, \sum_{z \leq y} P(x,z) \text{ decreases in } x \in I.$$

From $P(1,1) < 1$ one gets that $\widehat{P}$ loses mass through 1. Moreover, if r is the smallest integer such that $\sum_{z \leq r} P(1,z) = 1$, then $\widehat{P}$ loses mass through $\{x < r\}$ and it does not lose mass through $\{r, \cdots, N\}$. By applying Theorem 1 one gets

$$\varphi(x) = (H^S)'\pi(x) = \sum_{y \in I} \mathbf{1}(y \leq x)\pi(y) = \sum_{y \leq x} \pi(y) =: \pi^c(x),$$

the cumulative distribution of π, which is not constant because $\pi > 0$.

Consider the finite case with $I = \widehat{I} = \widetilde{I} = I_N$, $N \geq 2$. If P is substochastic then the equality $\widehat{P}(N,x) = \sum\limits_{z \leq N} (P(x,z) - P(x+1,z))$ implies $\widehat{P}(N,x) = \delta_{x,N}$, so N is an absorbing state for $\widehat{P}$. It can be checked that N is the unique absorbing state for $\widehat{P}$. We can summarize the above analysis by the following result.

Corollary 3. *Let H^S be the Siegmund kernel, P be a monotone irreducible positive recurrent stochastic kernel with stationary distribution π'. Let $H^S \widehat{P}' = PH^S$ with $\widehat{P} \geq 0$. Then:*

(i) $\varphi = \pi^c$ and the stochastic intertwining kernel Λ satisfies

$$\Lambda(x,y) = \mathbf{1}(x \geq y) \frac{\pi(y)}{\pi^c(x)}. \tag{5}$$

The intertwining matrix $\widetilde{P}$ of $\overleftarrow{P}$, that verifies $\widetilde{P}\Lambda = \Lambda \overleftarrow{P}$, is given by

$$\widetilde{P}(x,y) = \widehat{P}(x,y) \frac{\pi^c(y)}{\pi^c(x)}, \; x, y \in I. \tag{6}$$

Now assume $I = I_N$, then,

(ii) $\widehat{P}$ is strictly substochastic and loses mass through 1, and parts (iv), (v) and (vi) of Theorem 1 hold.

(iii) N is the unique absorbing state for $\widehat{P}$ (and for $\widetilde{P}$), Theorem 1 parts (v) and (vi) are fulfilled with $\widehat{I}_\ell = \{N\}$ and $\widehat{a} = N$. In particular $\pi' = \mathbf{e}'_N \Lambda$ holds.

(iv) The following relation is satisfied

$$\Lambda \mathbf{e}_N = \pi(N) \mathbf{e}_N \, . \tag{7}$$

□

5. Truncation for Monotone kernels

The purpose of this section is to see how the truncations behaves with the duality relation.

5.1. The Pollak-Siegmund limit for the reversed chain.

Since P has stationary distribution π' it holds $\lim\limits_{n\to\infty} \mathbb{P}_x(X_n = y) = \pi(y)$ for $y \in \mathbb{N}$. Pollak and Siegmund proved in [15] that if P is also monotone then,

$$\forall x, y \in \mathbb{N} : \quad \lim_{n,N\to\infty} \mathbb{P}_x(X_n = y \,|\, \tau_{(N)} > n) = \pi(y), \tag{8}$$

where $\tau_{(N)}$ is the hitting times of the domain $R_N = \{z \geq N\}$ by the chain X. So, a truncation at a sufficiently high level, will have a stationary distribution close to the one of the original process.

Now, in the framework of Theorem 1 the intertwining relation $\widetilde{P}\Lambda = \Lambda \overleftarrow{P}$ is constructed from a duality relation, and so $\overleftarrow{P}$ plays the role of P. This leads us to show the Pollak-Siegmund relation for the reversed kernel $\overleftarrow{P}(x,y) = \pi(x)^{-1} P(y,x) \pi(y)$.

First note that for any path $x_0, ..., x_n$ it holds

$$\prod_{k=0}^{n-1} \overleftarrow{P}^n(x_k, x_{k+1}) = \pi(x_0)^{-1} \left(\prod_{k=0}^{n-1} P^n(x_{k+1}, x_k) \right) \pi(x_n). \tag{9}$$

Let $\overleftarrow{\tau}_{(N)}$ be the hitting times of $R_N = \{z \geq N\}$ by the chain $\overleftarrow{X}$. From (9) one gets

$$\mathbb{P}_x(\overleftarrow{X}_n = z, \overleftarrow{\tau}_{(N)} > n) = \pi(x)^{-1}\pi(z)\mathbb{P}_z(\tau_{(N)} > n, X_n = x) \text{ and}$$
$$\mathbb{P}_x(\overleftarrow{\tau}_{(N)} > n) = \pi(x)^{-1}\sum_{z\in\mathbb{N}}\pi(z)\mathbb{P}_z(\tau_{(N)} > n, X_n = x).$$

Then,

$$\begin{aligned}(10)\, \mathbb{P}_x(\overleftarrow{X}_n = y \,|\, \overleftarrow{\tau}_{(N)} > n) \quad &= \quad \frac{\pi(y)\mathbb{P}_y(\tau_{(N)} > n, X_n = x)}{\sum_{z\in\mathbb{N}}\pi(z)\mathbb{P}_z(\tau_{(N)} > n, X_n = x)} \\ &= \quad \frac{\pi(y)\mathbb{P}_y(X_n = x \,|\, \tau_{(N)} > n)}{\sum_{z\in\mathbb{N}}\pi(z)\mathbb{P}_z(X_n = x \,|\, \tau_{(N)} > n)\frac{\mathbb{P}_z(\tau_{(N)}>n)}{\mathbb{P}_y(\tau_{(N)}>n)}}.\end{aligned}$$

We recall P monotone means that for $z \geq y$ all $N \geq 1$ it is satisfied $\mathbb{P}_y(X_1 < N) \geq \mathbb{P}_z(X_1 < N)$ for all $N \geq 1$. This implies for all $r \geq 1$ we have $\mathbb{P}_y(X_r < N) \geq \mathbb{P}_z(X_r < N)$ when $y \leq z$. Monotonicity is also equivalent to the fact that for all decreasing bounded function $h : \mathbb{N} \to \mathbb{R}$ one has $\mathbb{E}_y(h(X_1)) \geq \mathbb{E}_z(h(X_1))$ when $y \leq z$.

Lemma 4. *Assume P is monotone. Then, for all $N \geq 1$ one has*

$$(11) \qquad \forall y \leq z \leq N, \forall n \geq 1 : \quad \mathbb{P}_y(\tau_{(N)} > n) \geq \mathbb{P}_z(\tau_{(N)} > n).$$

Proof. Let N be fixed. We will show by recurrence on $k \geq 0$ that for all $n > k$ one has

$$(12) \qquad \mathbb{P}_y(X_{n-k} < N, .., X_n < N) \geq \mathbb{P}_z(X_{n-k} < N, .., X_n < N).$$

The inequality for $k = 0$, $\mathbb{P}_y(X_n < N) \geq \mathbb{P}_z(X_n < k)$ holds for all $n > 0$ because P is monotone and $y \leq z$. Let us show it for $k \geq 1$ and all $n > k$. So, we may assume we have shown it up to $k-1$ and all $n > k-1$. From the Markov property we have

$$\mathbb{P}_x(X_{n-k} < N, .., X_n < N) = \sum_{u<N}\mathbb{P}_u(X_1 < N, ..X_k < N)\mathbb{P}_x(X_{n-k} = u).$$

We claim that the function h defined by $h(u) = \mathbb{P}_u(X_1 < N, .., X_k < N)$ for $u < N$ and $h(u) = 0$ for $u \geq N$, is decreasing in u. In fact this is exactly the induction hypothesis for $k-1$ when one takes $n = k$ in (12). Then $\mathbb{E}_y(h(X_{n-1}) \geq \mathbb{E}_z(h(X_{n-1})$, which is the inequality we want to prove: $\sum_{u<N}\mathbb{P}_u(X_1 < N)\mathbb{P}_y(X_{n-1} = u) \geq \sum_{u<N}\mathbb{P}_u(X_1 < N)\mathbb{P}_z(X_{n-1} = u)$.

Then, relation (11) is shown because (12) for $k = n-1$ is equivalent to

$$\mathbb{P}_y(\tau_{(N)}>n)=\mathbb{P}_y(X_{n-k}<N, .., X_n<N) \geq \mathbb{P}_z(X_{n-k}<N, .., X_n<N)=\mathbb{P}_z(\tau_{(N)}>n).$$

□

Let us now show a ratio limit result.

Lemma 5. *Let P be a monotone irreducible positive recurrent stochastic kernel with stationary distribution π'. We have*

$$(13) \qquad \forall y, z \in \mathbb{N} : \quad \lim_{n,N\to\infty}\frac{\mathbb{P}_y(\tau_{(N)} > n)}{\mathbb{P}_z(\tau_{(N)} > n)} = 1.$$

Proof. We will use a recurrence on $n \in \mathbb{N}$. We will also use the following remark that follows from Lemma 4: if $\lim\limits_{n,N\to\infty} \mathbb{P}_y(\tau_{(N)} > n)/\mathbb{P}_z(\tau_{(N)} > n) = 1$ for $y < z$, then

$$\forall\, y < x < z: \quad \lim_{n,N\to\infty} \frac{\mathbb{P}_x(\tau_{(N)} > n)}{\mathbb{P}_y(\tau_{(N)} > n)} = \lim_{n,N\to\infty} \frac{\mathbb{P}_x(\tau_{(N)} > n)}{\mathbb{P}_z(\tau_{(N)} > n)} = 1 \tag{14}$$

We claim that

$$\forall x \in \mathbb{N}: \quad \sum_{z\le x} P(x,z) < 1. \tag{15}$$

In fact, if $\sum_{z\le x} P(x,z) = 1$ holds for some x, then the monotone property implies $\sum_{z\le x} P(y,z) = 1$ for all $y \le x$ and so the set of points $\{1,..,x\}$ is a closed set of P contradicting the irreducibility property. Then, the claim holds and for all $x \in \mathbb{N}$ there exists some $z > x$ such that $P(x,z) > 0$.

Let $x \in \mathbb{N}$. For $\epsilon > 0$ there exists a bounded $L = L(\epsilon)$ such that $\sum_{z\in L} \pi_z > 1 - (\epsilon/2)$. From (8) we get that there exists n_0, N_0 such that for $n \ge n_0, N \ge N_0$ one has

$$\lim_{n,N\to\infty} \mathbb{P}_x(X_n \le L \,|\, \tau_{(N)} > n) > 1 - \epsilon.$$

Then,

$$\forall\, x \in \mathbb{N}: \quad \lim_{n,N\to\infty} \mathbb{P}_x(\tau_{(N)} > n \,|\, \tau_{(N)} > n-1) = 1. \tag{16}$$

For the induction we use the equality

$$\mathbb{P}_x(\tau_{(N)} > n) = \sum_{z\in\mathbb{N}} P(x,z)\mathbb{P}_z(\tau_{(N)} > n-1),$$

so

$$1 = \sum_{z\in\mathbb{N}} P(x,z) \frac{\mathbb{P}_z(\tau_{(N)} > n-1)}{\mathbb{P}_x(\tau_{(N)} > n-1)} \frac{1}{\mathbb{P}_x(\tau_{(N)} > n \,|\, \tau_{(N)} > n-1)}. \tag{17}$$

We define:

$$\begin{aligned} &\text{Property Prop}(x) \text{ at } x \text{ is}: \\ &\forall y \le x: \ \lim_{n,N\to\infty} \mathbb{P}_x(\tau_{(N)} > n)/\mathbb{P}_y(\tau_{(N)} > n) = 1 \text{ and} \\ &\exists\, z > x \text{ such that } \lim_{n,N\to\infty} \mathbb{P}_x(\tau_{(N)} > n)/\mathbb{P}_z(\tau_{(N)} > n) = 1. \end{aligned} \tag{18}$$

Note that (14) implies that if Prop(x) holds then we can always assume $z = x+1$ in (18) and so that

$$\forall\, y \le x: \quad \lim_{n,N\to\infty} \mathbb{P}_x(\tau_{(N)} > n)/\mathbb{P}_{x+1}(\tau_{(N)} > n) = 1.$$

Let us prove that Prop(1) holds. We need only to show that for some $z > x$ one has $\lim\limits_{n,N\to\infty} \mathbb{P}_1(\tau_{(N)} > n)/\mathbb{P}_z(\tau_{(N)} > n) = 1$. From (17) we have

$$1 = \frac{P(1,1)}{\mathbb{P}_1(\tau_{(N)} > n \,|\, \tau_{(N)} > n-1)} + \sum_{z>1} P(1,z) \frac{\mathbb{P}_z(\tau_{(N)} > n-1)}{\mathbb{P}_1(\tau_{(N)} > n-1)} \frac{1}{\mathbb{P}_1(\tau_{(N)} > n \,|\, \tau_{(N)} > n-1)}.$$

From (16), (11) we deduce that $\lim\limits_{n,N\to\infty} \mathbb{P}_z(\tau_{(N)} > n-1)/\mathbb{P}_1(\tau_{(N)} > n-1) = 1$ for all $z > 1$ such that $P(1,z) > 0$. From (15) we get Prop(1).

Let $x>1$. We assume Prop(y) holds up to $y=x-1$ and let us show Prop(x) is satisfied. We have

$$1=\sum_{z\in\mathbb{N}} P(x,z)\frac{\mathbb{P}_z(\tau_{(N)}>n-1)}{\mathbb{P}_x(\tau_{(N)}>n-1)}\frac{1}{\mathbb{P}_x(\tau_{(N)}>n\,|\,\tau_{(N)}>n-1)}. \tag{19}$$

From (19), (16), (11) we get that $\lim\limits_{n,N\to\infty}\mathbb{P}_z(\tau_{(N)}>n-1)/\mathbb{P}_1(\tau_{(N)}>n-1)=1$ for all $z>1$ such that $P(x,z)>0$. From (15) we get Prop(x). Then, the result is shown. □

Let us state the Pollak-Siegmund limit relation (8) for the reversed chain.

Proposition 6. *We have*

$$\forall x,y\in\mathbb{N}: \quad \lim_{n,N\to\infty}\mathbb{P}_x(\overleftarrow{X}_n=y\,|\,\overleftarrow{\tau}_{(N)}>n)=\pi(y).$$

Proof. From condition (11) and (8), (13), we can use the dominated convergence theorem in (10) to get

$$\lim_{n,N\to\infty}\mathbb{P}_x(\overleftarrow{X}_n=y\,|\,\overleftarrow{\tau}_{(N)}>n)=\frac{\pi(y)\pi(x)}{\sum_{z\in\mathbb{N}}\pi(z)\pi(x)}=\pi(y).$$

□

5.2. **The truncation of the mean expected value have nonnegative dual.** We assume P is monotone on $\mathbb{N}$. Then, we can consider that a truncation of P at level N is a kernel P_N taking values in I_N that satisfies

$$\text{for } x<N:\ P_N(x,y)=P(x,y) \text{ if } y<N \text{ and } P_N(x,N)=\sum_{z\geq N}P(x,z). \tag{20}$$

The unique degree of freedom is to define the redistribution of mass at N. In the truncation P_N we will define, we do it in such a way that the stationary distribution is preserved in a very specific way. Let $\overline{\pi}$ be the tail of π, $\overline{\pi}(x)=\sum_{z\geq x}\pi(z)$, in particular $\overline{\pi}(N)=\sum_{z\geq N}\pi(z)$. We define P_N by (20) and such that

$$P_N(N,y)=\frac{1}{\overline{\pi}(N)}\sum_{z\geq N}\pi(z)P(z,y),\quad P_N(N,N)=\frac{1}{\overline{\pi}(N)}\sum_{z\geq N}\pi(z)(\sum_{u\geq N}P(z,u)).$$

Let π_N be given by $\pi_N(x)=\pi(x)$ for $x<N$ and $\pi_N(N)=\overline{\pi}(N)$. Let us check that π_N is the stationary distribution of P_N. For $y<N$ one has

$$\sum_{x=1}^{N}\pi_N(x)P_N(x,y)=\sum_{x<N}\pi(x)P_N(x,y)+(\sum_{z\geq N}\pi(z)P(z,y))=\pi(y)=\pi_N(y),$$

and for $y=N$,

$$\begin{aligned}\sum_{x=1}^{N}\pi_N(x)P_N(x,N) &= \sum_{x<N}\pi(x)(\sum_{u\geq N}P(x,u))+\sum_{z\geq N}\pi(z)(\sum_{u\geq N}P(z,u))\\ &= \sum_{x\in\mathbb{N}}\pi(x)\sum_{u\geq N}P(x,u)=\overline{\pi}(N)=\pi_N(N).\end{aligned}$$

This truncation can be written as the action of a mean expected operator. Let $\mathbb{E}_N$ be the mean expected operator on $L^1(\mathbb{N},\pi)$ with respect to the $\sigma-$field induced by the partition $\alpha_N=\{\{x\}: x<N\}\cup\{x\ge N\}$. So,

$$\mathbb{E}_N f(x)=f(x) \text{ if } x<N \text{ and } \mathbb{E}_N f(x)=\frac{1}{\overline{\pi}(N)}\sum_{y\ge N}\pi(y)f(y) \text{ if } x\ge N.$$

Since $\mathbb{E}_N f(x)$ is constant for $x>N$ we can identify α_N with I_N, the atom $\{x\}$ is identified with x when $x<N$ and the atom $\{x\ge N\}$ is identified with N.

Proposition 7. *The truncation P_N satisfies*

$$P_N=\mathbb{E}_N P, \tag{21}$$

where $\mathbb{E}_N$ is the mean expected operator defined as above.

Proof. Let P be a stochastic kernel. Since $\mathbb{E}_N$ is stochastic then $\mathbb{E}_N P$ is also stochastic. It satisfies

$$\mathbb{E}_N Pf(x)=Pf(x) \text{ if } x<N \text{ and}$$
$$\mathbb{E}_N Pf(x)=\frac{1}{\overline{\pi}(N)}\sum_{z\ge N}\pi(z)(Pf)(z)=\frac{1}{\overline{\pi}(N)}\sum_{z\ge N}\pi(z)\sum_{y\in\mathbb{N}}P(z,y)f(y) \text{ if } x\ge N.$$

For $y<N$ one has

$$\mathbb{E}_N(P\mathbf{1}_{\{y\}})(x)=P\mathbf{1}_{\{y\}}(x)=P(x,y) \text{ if } x<N \text{ and}$$
$$\mathbb{E}_N(P\mathbf{1}_{\{y\}})(x)=\frac{1}{\overline{\pi}(N)}\sum_{z\ge N}\pi(y)P\mathbf{1}_{\{y\}}(z)=\frac{1}{\overline{\pi}(N)}\sum_{z\ge N}\pi(y)P(z,y) \text{ if } x\ge N.$$

Since $P\mathbf{1}_{\{y\}}(x)=P(x,y)$ we have proven

$$P_N(x,y)=\mathbb{E}_N P(x,y) \text{ for } y<N, x\in\mathbb{N}. \tag{22}$$

Since $\mathbb{E}_N P\mathbf{1}=\mathbf{1}=\sum_{y=1}^N \mathbf{1}_{\{y\}}$, and since P_N and $E_N P$ are stochastic operators, from (22) we conclude $P_N(x,N)=(\mathbb{E}_N P)(x,N)$. Hence, (21) follows. $\square$

Let us now prove that these truncations have a nonnegative dual.

Proposition 8. *let P be a monotone kernel on $\mathbb{N}$. Then, P_N is a monotone kernel having a nonnegative Siegmund dual $\widehat{P}_N$ with values in I_N and such that N is an absorbing state.*

Proof. We claim that the monotone property on P implies the monotonicity of P_N. Firstly, for all $v,w\in I_N$ we have $\sum\limits_{z\le N}P_N(v,z)=1=\sum\limits_{z\le N}P_N(w,z)$. Now, let $x<N$. For $v\le w<N$ one has

$$\sum_{z\le x}P_N(v,z)=\sum_{z\le x}P(v,z)\le\sum_{z\le x}P(w,z)=\sum_{z\le x}P_N(w,z),$$

and for $v<N$ it holds

$$\begin{aligned}\sum_{z\le x}P_N(N,z) &= \frac{1}{\overline{\pi}(N)}(\sum_{z\le x}\sum_{u\ge N}\pi(u)P(u,z))=\frac{1}{\overline{\pi}(N)}(\sum_{u\ge N}\pi(u)\sum_{z\le x}P(u,z))\\ &\le \frac{1}{\overline{\pi}(N)}(\sum_{u\ge N}\pi(u)\sum_{z\le x}P(v,z))=\sum_{z\le x}P(v,z),\end{aligned}$$

where the monotonicity of P was used to state the last $\leq$ relation. Then the claim holds, that is P_N is monotone.

Hence P_N has a nonnegative Siegmund dual $\widehat{P}_N$ with values in I_N and that following (4) it satisfies, $\widehat{P}_N(x,y) = \sum_{z \leq x} (P_N(y,z) - P_N(y+1,z))$ for $x, y \in I_N$. This gives:

$$\widehat{P}_N(x,y) = \sum_{z \leq x} (P(y,z) - P(y+1,z)) = \widehat{P}(x,y) \text{ for } x < N, y < N-1, \tag{23}$$

and for $x = N$ we get

$$\widehat{P}_N(N,y) = 0 \text{ for } y \leq N-1.$$

Note that,

$$\widehat{P}_N(x,N-1) = \sum_{z \leq x} P(N-1,z) - \sum_{z \leq x} P_N(N,z) \geq 0 \text{ for } x < N. \tag{24}$$

Finally

$$\widehat{P}_N(x,N) = \sum_{z \leq x} P_N(N,z) = \frac{1}{\overline{\pi}(N)} \sum_{z \leq x} (\sum_{u \geq N} \pi(u) P(u,z)) \text{ for } x < N; \tag{25}$$

and

$$\widehat{P}_N(N,N) = \sum_{z \leq N} P_N(N,z) = 1.$$

Hence, $\widehat{P}_N$ is a Siegmund dual of P_N with values in I_N and N is an absorbing state for $\widehat{P}_N$. $\square$

Notice that for $x < N$ one has that the difference between the kernels $\widehat{P}_N(x,y)$ and $\widehat{P}(x,y)$ only happens at $y = N-1$ and $y = N$. We have

$$\widehat{P}_N(x,N-1) + \widehat{P}_N(x,N) = \sum_{z \leq x} P(N-1,z) \text{ and}$$

$$\widehat{P}(x,N-1) + \widehat{P}(x,N) = \sum_{z \leq x} P(N-1,z) + \sum_{z \leq x} P(N+1,z),$$

and so

$$(\widehat{P}(x,N-1) + \widehat{P}(x,N)) - (\widehat{P}_N(x,N-1) + \widehat{P}_N(x,N)) = \sum_{z \leq x} P(N+1,z) \geq 0.$$

Therefore, if $\widehat{P}$ loses mass through $x < N$ then $\widehat{P}_N$ also does.

On the other hand it is straightforward to check that the truncation of the reversed kernel $\overleftarrow{P}_N$ is the reversed of P_N with respect to π_N, that is it satisfies

$$\overleftarrow{P}_N(x,y) = \pi_N(x)^{-1} P_N(y,x) \pi_N(y), \; x, y \leq N.$$

From (5) we can define the intertwining matrix $\Lambda(x,y) = \mathbf{1}(x \geq y)(\pi_N(y)/\pi^c_N(x))$ for $x, y \in I_N$ where $\pi_N(y) = \pi(y)$ for $y < N$ and $\pi_N(N) = \overline{\pi}(N)$. The intertwined matrix $\widetilde{P}_N$ of $\overleftarrow{P}_N$ which satisfies $\widetilde{P}_N \Lambda_N = \Lambda_N \overleftarrow{P}_N$, is given by (6). It is $\widetilde{P}_N(x,y) = \widehat{P}_N(x,y)(\pi^c_N(y)/\pi^c_N(x))$ for $x, y \in I_N$. Note that $\pi^c_N(y) = \pi^c(y)$ for $y < N$ and $\pi^c_N(N) = 1$.

Proposition 9. *Let us consider two truncations as above, P_N and P_{N+1} at levels N and $N+1$, respectively. Then, the time $\widetilde{\tau}_N^N$ of hitting N by $\widetilde{P}_N$ is stochastically smaller than the time $\widetilde{\tau}_{N+1}^{N+1}$ of hitting $N+1$ by $\widetilde{P}_{N+1}$.*

Proof. We will note by $\widehat{X}^N = (\widehat{X}_n^N)$ and $\widehat{X}^{N+1} = (\widehat{X}_n^{N+1})$ the Markov chains associated to kernels $\widehat{P}_N$ and $\widehat{P}_{N+1}$, respectively.
From (24), for every $x < N-1$ one has

$$\widehat{P}_N(x, N-1) - \widehat{P}_{N+1}(x, N-1) = \sum_{z \le x} P(N, z) - \frac{1}{\overline{\pi}(N)}(\sum_{u \ge N} \pi(u) \sum_{z \le x} P(u, z)).$$

Similarly,

$$\widehat{P}_{N+1}(x, N) = \sum_{z \le x} P(N, z) - \frac{1}{\overline{\pi}(N+1)}(\sum_{u \ge N+1} \pi(u) \sum_{z \le x} P(u, z)).$$

Then, by monotonicity

$$\widehat{P}_N(x, N-1) - \widehat{P}_{N+1}(x, N-1) \le \widehat{P}_{N+1}(x, N).$$

So,

$$\begin{aligned}&(\widehat{P}_{N+1}(x, N-1) + \widehat{P}_{N+1}(x, N-1)) - \widehat{P}_N(x, N)\\ &= \frac{1}{\overline{\pi}(N)}(\sum_{u \ge N} \pi(u) \sum_{z \le x} P(u, z)) - \frac{1}{\overline{\pi}(N+1)}(\sum_{u \ge N+1} \pi(u) \sum_{z \le x} P(u, z)).\end{aligned}$$

from (25) we also have

$$\begin{aligned}&0 \le \widehat{P}_N(x, N) - \widehat{P}_{N+1}(x, N+1)\\ &= \frac{1}{\overline{\pi}(N)}(\sum_{u \ge N} \pi(u) \sum_{z \le x} P(u, z)) - \frac{1}{\overline{\pi}(N+1)}(\sum_{u \ge N+1} \pi(u) \sum_{z \le x} P(u, z)),\end{aligned}$$

where the nonnegativity follows from monotonicity of P. Then,

$$\widehat{P}_{N+1}(x, N-1) + \widehat{P}_{N+1}(x, N) + \widehat{P}_{N+1}(x, N+1) = \widehat{P}_N(x, N-1) + \widehat{P}_N(x, N).$$

From the above equalities and inequalities and by using (23) at every step when we are in some state $y < N-1$, we can make a coupling between both chains $\widehat{X}_N$ and $\widehat{X}_{N+1}$ such that when both chains start from $x < N-1$ we have

$$\begin{aligned}&\widehat{X}_{N+1} \in \{N, N+1\} \Leftrightarrow \widehat{X}_N = N \text{ and}\\ &\forall y \le N-1,\ \widehat{X}_{N+1} = y \Leftrightarrow \widehat{X}_N = y.\end{aligned}$$

On the other hand from (23) we get

$$\widehat{P}_N(N-1, y) = \sum_{z \le x}(P(y, z) - P(y+1, z)) = \widehat{P}(x, y) = \widehat{P}_{N+1}(N-1, y),$$

so the distribution to $y \le N-1$ is the same for the two kernels. Moreover

$$\widehat{P}_N(N-1, N) = \sum_{z \le N-1} P(N, z),$$

and

$$\begin{aligned}&\widehat{P}_{N+1}(N-1,N)+\widehat{P}_{N+1}(N-1,N+1)\\&=\sum_{z\le N-1}P(N,z)-\sum_{z\le N-1}P_{N+1}(N+1,z)+\sum_{z\le N-1}P_{N+1}(N+1,z).\end{aligned}$$

Therefore

$$\widehat{P}_N(N-1,N)=\widehat{P}_{N+1}(N-1,N)+\widehat{P}_{N+1}(N-1,N+1).$$

Hence, once both chains start from $N-1$, they can be coupled to return to some state $y\le N-1$, or, if not, the rest of the mass of kernel $\widehat{P}_N$ moves to the absorbing state N, and for $\widehat{P}_{N+1}$ part of this mass moves to the absorbing state $N+1$ while the rest goes to N.

We have shown that the absorption time $\widehat{\tau}_N^N$ of $\widehat{P}_N$ at level N, is smaller than the absorption time $\widehat{\tau}_{N+1}^{N+1}$ of $\widehat{P}_{N+1}$ at level $N+1$, that is

$$\forall x\le N-1,k>0:\ \mathbb{P}_x(\widehat{\tau}_N^N\le k)\ge\mathbb{P}_x(\widehat{\tau}_{N+1}^{N+1}\le k).$$

Now for $x\le N-1$ one has, $\pi_N^c(x)=\pi^c(x)=\pi_{N+1}^c(x)$ and $\pi_N^c(N)=1=\pi_{N+1}^c(N+1)$, so

$$\mathbb{P}_x(\widetilde{X}_1^N=x_1,..,\widetilde{X}_k^N=N)=\frac{1}{\pi^c(x)}\mathbb{P}_x(\widehat{X}_1^N=x_1,..,\widehat{X}_k^N=N),$$

$$\mathbb{P}_x(\widetilde{X}_1^{N+1}=x_1,..,\widetilde{X}_k^{N+1}=N)=\frac{1}{\pi^c(x)}\mathbb{P}_x(\widehat{X}_1^{N+1}=x_1,...,\widehat{X}_k^{N+1}=N).$$

Then,

$$\mathbb{P}_x(\widetilde{\tau}_N^N=j)=\frac{1}{\pi^c(x)}\mathbb{P}_x(\widehat{\tau}_N^N=j)\text{ and }\mathbb{P}_x(\widetilde{\tau}_{N+1}^{N+1}=j)=\frac{1}{\pi^c(x)}\mathbb{P}_x(\widehat{\tau}_{N+1}^{N+1}=j).$$

Hence, we conclude

$$\forall x\le N-1,k>0:\ \mathbb{P}_x(\widetilde{\tau}_N^N\le k)\ge\mathbb{P}_x(\widetilde{\tau}_{N+1}^{N+1}\le k),$$

and so $\widetilde{\tau}_N^N$ is stochastically smaller than $\widetilde{\tau}_{N+1}^{N+1}$. □

6. Strong Stationary Times

Let π_0' be the initial distribution of X_0, so $\pi_n'=\pi_0'P^n$ is the distribution of X_n. A stopping time is noted T_ρ when $X_{T_\rho}\sim\rho$. We recall $\tau_a=\inf\{n\ge 0:X_n=a\}$ for $a\in I$. When one wants to emphasize the initial distribution π_0' of X_0, these times are written by $T_\rho^{\pi_0}$ and $\tau_a^{\pi_0}$, respectively.

A stopping time T is called a strong stationary time if $X_T\sim\pi'$ and it is independent of T, see [1]. The separation discrepancy is defined by

$$\text{sep}\,(\pi_n,\pi):=\sup_{y\in I}\left[1-\frac{\pi_n(y)}{\pi(y)}\right].$$

In Proposition 2.10 in [1] it was proven that every strong stationary time T satisfies

$$\forall\, n\ge 0:\quad \text{sep}\,(\pi_n,\pi)\le\mathbb{P}_{\pi_0}(T>n)\ . \tag{26}$$

In Proposition 3.2 in [1] it was shown that there exists a strong stationary time T, called sharp, that satisfies equality in (26),

$$\forall\, n\ge 0:\quad \text{sep}\,(\pi_n,\pi)=\mathbb{P}_{\pi_0}(T>n)\ .$$

Assume we are in the framework of Theorem 1, so $\widetilde{P}\Lambda = \Lambda \overleftarrow{P}$. A random time for $\widetilde{X}$ is noted by $\widetilde{T}$ and we use similar notations as those introduced for random times T for X. The initial distributions of $\overleftarrow{X}_0$ and $\widetilde{X}_0$ are respectively noted by $\overleftarrow{\pi}'_0$ and $\widetilde{\pi}'_0$. We assume they are linked, this means:

$$\overleftarrow{\pi}'_0 = \widetilde{\pi}'_0 \Lambda. \tag{27}$$

In this case the intertwining relation $\widetilde{P}^n \Lambda = \Lambda \overleftarrow{P}^n$ implies $\overleftarrow{\pi}'_n = \widetilde{\pi}'_n \Lambda$ for $n \geq 0$, where $\overleftarrow{\pi}_n$ and $\widetilde{\pi}_n$ are the distributions of $\overleftarrow{X}_n$ and $\widetilde{X}_n$ respectively.

Since Λ is stochastic it has a left probability eigenvector π'_Λ, so $\pi'_\Lambda = \pi'_\Lambda \Lambda$ and π_Λ is linked with itself. If Λ is non irreducible then π_Λ could fail to be strictly positive, which is the case for the Siegmund kernel where Λ is given by (5) and one can check that $\mathbf{e}_1$ is the unique left eigenvector satisfying $\mathbf{e}'_1 = \mathbf{e}'_1 \Lambda$. So, the initial conditions $\widetilde{X}_1 \sim \delta_1$ and $\overleftarrow{X}_0 \sim \delta_1$ are linked. Assume P is monotone. From relation (5) one gets that (27) is equivalent to $\overleftarrow{\pi}_0(x)/\pi(x) = \sum_{y \geq x} \widetilde{\pi}(y)/\pi^c(y)$ for all $x \in I$. (See relation (4.7) and (4.10) in [5]). In the finite case $I = I_N$, Corollary 3 (iii) states that if P is monotone then $\widetilde{\partial} = N$ is the unique absorbing state for $\widetilde{X}$. Let us now introduce the sharp dual.

Definition 3. *The process $\widetilde{X}$ is a sharp dual to $\overleftarrow{X}$ if it has an absorbing state $\widetilde{\partial}$, and when $\overleftarrow{X}$ and $\widetilde{X}$ start from linked initial conditions $\widetilde{X}_0 \sim \widetilde{\pi}_0$, $\overleftarrow{X}_0 \sim \overleftarrow{\pi}_0$ with $\overleftarrow{\pi}'_0 = \widetilde{\pi}'_0 \Lambda$, then it holds*

$$sep(\overleftarrow{\pi}_n, \pi) = \mathbb{P}_{\widetilde{\pi}_0}(\widetilde{\tau}_{\widetilde{\partial}} > n), \; \forall n \geq 0.$$

□

We recall that if $\widetilde{\partial}$ is an absorbing state for $\widetilde{P}$, then $\pi' = \mathbf{e}'_{\widetilde{\partial}} \Lambda$ (this is (2)).

We now state the sharpness result alluded to in Remark 2.39 of [5] and in Theorem 2.1 in [7]. The hypotheses stated in Remark 2.39 are understood as the condition (28) below. The results of this section were proven in [10].

We recall the definition made in Section 4 in [2]: A state $d \in I$ is called separable for X when

$$\mathrm{sep}(\pi_n, \pi) = 1 - \frac{\pi_n(d)}{\pi(d)}, \; n \geq 1.$$

On the other hand, $d \in I$ was called a witness state if it satisfies

$$\Lambda \mathbf{e}_d = \pi(d) \mathbf{e}_{\widetilde{\partial}}. \tag{28}$$

We note that in the monotone finite case in I_N, the state $d = N$ is a witness state, this is exactly (7) in Corollary 3.

Proposition 10. *Let X be an irreducible positive recurrent Markov chain, $\widetilde{X}$ be a $\Lambda-$intertwining of $\overleftarrow{X}$ having $\widetilde{\partial}$ as an absorbing state. If $d \in I$ is a witness state then d is a separable state, $\widetilde{X}$ is a sharp dual to $\overleftarrow{X}$ and it is satisfied,*

$$sep(\overleftarrow{\pi}_n, \pi) = 1 - \frac{\overleftarrow{\pi}_n(d)}{\pi(d)} = \mathbb{P}_{\widetilde{\pi}_0}(\widetilde{\tau}_{\widetilde{\partial}} > n), \; n \geq 1.$$

□

Remark 5. *Since we have shown that being witness implies being separable, Corollary* 4.1 *of* [2] *stated for separable states applies for a witness state, this is* $\overleftarrow{T}^{\pi_0}_{d} = \overleftarrow{T}^{\pi_0}_{\pi} + Z$ *is an independent sum and* $Z \sim \overleftarrow{T}^{\pi}_{d}$. $\square$

Remark 6. *The condition of being witness can be stated in terms of the dual function* H. *In fact, in* [11] *it was shown that if there exists* $\widehat{a} \in \widehat{I}$ *and* $d \in I$ *such that for some constants* $c_1 > 0$, $c > 0$ *one has*

$$H\mathbf{e}_{\widehat{a}} = c_1 \mathbf{1} \;\text{ and } \mathbf{e}'_d H = c\mathbf{e}'_{\widehat{a}}.$$

Then d *is a witness state and* $\widetilde{X}$ *is a sharp dual to* $\overleftarrow{X}$. $\square$

Corollary 11. *For a monotone irreducible stochastic kernel* P, *the* $\Lambda-$*intertwining Markov chain* $\widetilde{X}$ *has* N *as an absorbing state and it is a sharp dual of* $\overleftarrow{X}$. *Also,* N *is a separable state and the initial conditions* $\overleftarrow{X}_0 = \delta_1$ *and* $\widetilde{X} = \delta_1$ *are linked.* $\square$

7. The Diaconis-Fill Coupling

Let $\widetilde{P}$ be a $\Lambda-$intertwining of P. Consider the following stochastic kernel $\underline{P}$ defined on $I \times \widetilde{I}$, which was introduced in [5],

$$\underline{P}\left((x,\widetilde{x}),(y,\widetilde{y})\right) = \frac{P(x,y)\,\widetilde{P}(\widetilde{x},\widetilde{y})\,\Lambda(\widetilde{y},y)}{(\Lambda P)(\widetilde{x},y)} \mathbf{1}\left((\Lambda P)(\widetilde{x},y) > 0\right). \tag{29}$$

Let $\underline{X} = (\underline{X}_n : n \geq 0)$ be the chain taking values in $I \times \widetilde{I}$, evolving with the kernel $\underline{P}$ and with initial distribution

$$\mathbb{P}\left(X_0 = x_0, \widetilde{X}_0 = \widetilde{x}_0\right) = \underline{\pi}_0(x_0,\widetilde{x}_0) = \pi'_0(\widetilde{x}_0)\Lambda(\widetilde{x_0},x_0),\; x_0 \in I, \widetilde{x}_0 \in \widetilde{I},$$

where $\widetilde{\pi}'_0$ is an initial distribution of $\widetilde{X}$. In [5] it was proven that $\underline{X}$ starting from $\underline{\pi}'_0$ is a coupling of the chains X and $\widetilde{X}$ starting from $\pi'_0 = \widetilde{\pi}'_0\Lambda$ and $\widetilde{\pi}'_0$, respectively. Since X and $\widetilde{X}$ are the components of $\underline{X}$ one puts $\underline{X}_n = (X_n, \widetilde{X}_n)$.

Let $\mathbb{P}$ be the probability measure on $(I \times \widetilde{I})^{\mathbb{N}}$ induced by the coupling transition kernel $\underline{P}$ and assume $\underline{X}$ starts from linked initial conditions. In [5] (also [4]) it was shown,

$$\forall\, n \geq 0 : \quad \Lambda(\widetilde{x}_n, x_n) = \mathbb{P}\left(X_n = x_n \mid \widetilde{X}_n = \widetilde{x}_n\right) = \mathbb{P}\left(X_n = x_n \mid \widetilde{X}_0 = \widetilde{x}_0 \cdots \widetilde{X}_n = \widetilde{x}_n\right). \tag{30}$$

Hence, the equality $\pi' = \mathbf{e}'_{\widetilde{\partial}}\Lambda$ in (2) together with relation (30) give

$$\pi(x) = \Lambda(\widetilde{\partial}, x) = \mathbb{P}(X_n = x \mid \widetilde{X}_0 = \widetilde{x}_0 \cdots \widetilde{X}_n = \widetilde{\partial}).$$

Then, the following result shown in [5] holds.

Theorem 12. *Let* X *be an irreducible positive recurrent Markov chain with stationary distribution* π' *and let* $\widetilde{X}$ *be a* $\Lambda-$*intertwining of* X. *Assume the initial conditions are linked, meaning* $\pi'_0 = \widetilde{\pi}'_0\Lambda$. *Then,* $\widetilde{X}$ *is called a strong stationary dual of* X, *which means that the following equality is satisfied,*

$$\pi(x) = \mathbb{P}\left(X_n = x \mid \widetilde{X}_0 = \widetilde{x}_0 \cdots \widetilde{X}_{n-1} = \widetilde{x}_{n-1}, \widetilde{X}_n = \widetilde{\partial}\right), \forall x \in I,\, n \geq 0,$$

where $\widetilde{x}_0 \cdots \widetilde{x}_{n-1} \in \widetilde{I}$ *satisfy* $\mathbb{P}\left(\widetilde{X}_0 = \widetilde{x}_0 \cdots \widetilde{X}_{n-1} = \widetilde{x}_{n-1}, \widetilde{X}_n = \widetilde{\partial}\right) > 0$. $\square$

7.1. **Quasi-stationarity and coupling.** Below we state a property on quasi-stationarity of the coupling. Let us recall some elements on quasi-stationarity (see for instance [3]). Let $Y = (Y_n : n \geq 0)$ be a Markov chain with values on a countable set J and transition stochastic kernel Q. Let K be a nonempty strictly subset of J and let τ_K be the hitting time of K. A probability measure μ on $J \setminus K$ is a quasi-stationary distribution (q.s.d.) for Y and the forbidden set K if

$$\mathbb{P}_\mu(Y_n = j \,|\, \tau_K > n) = \mu(j), \; j \in J \setminus K. \tag{31}$$

Note that the q.s.d. does not depend on the behavior of the chain Y on K, so we assume Y is absorbed at K. In order that μ is a q.s.d. it suffices to satisfy (31) for $n = 1$, which is equivalent for μ being a left eigenvector of Q, so

$$\mu' Q = \gamma \mu'.$$

In the finite case, μ is the normalized left Perron-Frobenius eigenvector with Perron-Frobenius eigenvalue γ. It is easily checked that

$$\gamma = \mathbb{P}_\mu(\tau_K > 1) = \sum_{j \in J \setminus K} \mu_j \sum_{j' \in J \setminus K} P(j, j').$$

The hitting time τ_K starting from μ is geometrically distributed: $\mathbb{P}_\mu(\tau_K > n) = \gamma^n$, this is why γ is called the survival decay rate. If the chain is irreducible in $J \setminus K$, then every q.s.d. is strictly positive and for all $j \in J \setminus K$ one has $\mathbb{P}_j(\tau_K > n) \leq C_j \gamma^n$ with $C_j = \mu_j^{-1}$.

In the next result we put in relation the q.s.d. of the process $\widetilde{X}$ with forbidden state $\widetilde{\partial}$ and the q.s.d. of the process $(X, \widetilde{X})$ with forbidden set $\underline{\partial} = I \times \{\widetilde{\partial}\}$. Since $\widetilde{X} = \widetilde{\partial}$ is equivalent to $(X, \widetilde{X}) \in \underline{\partial}$, then it is straightforward that the survival decay rates for $\widetilde{X}$ and $(X, \widetilde{X})$ with respect to $\widetilde{\partial}$ and $\underline{\partial}$ respectively, are the same (that is the Perron-Frobenius eigenvalue is common for both processes).

Proposition 13. *Assume $\widetilde{\mu}'$ is a q.s.d. for the process $\widetilde{X}$ with the forbidden state $\widetilde{\partial}$. Then, the probability measure*

$$\underline{\mu}(x_0, \widetilde{x}_0) = \widetilde{\mu}(\widetilde{x}_0) \Lambda(\widetilde{x}_0, x_0), \; (x_0, \widetilde{x}_0) \in I \times (\widetilde{I} \setminus \{\widetilde{\partial}\}),$$

is a q.s.d. for $(X, \widetilde{X})$ with forbidden set $\underline{\partial} = I \times \{\widetilde{\partial}\}$.

Proof. From the hypothesis we have

$$\sum_{\widetilde{x} \in \widetilde{I}, \widetilde{x} \neq \widetilde{\partial}} \widetilde{\mu}(\widetilde{x}) \widetilde{P}(\widetilde{x}, \widetilde{y}) = \gamma \widetilde{\mu}(\widetilde{y}), \; \widetilde{y} \in \widetilde{I} \setminus \{\widetilde{\partial}\}, \text{ with } \gamma = 1 - \sum_{\widetilde{x} \in \widetilde{I}, \widetilde{x} \neq \widetilde{\partial}} \mu(\widetilde{x}) \widetilde{P}(\widetilde{x}, \widetilde{\partial}).$$

Now

$$\begin{aligned}
&\sum_{x \in I} \sum_{\widetilde{x} \in \widetilde{I} \setminus \{\widetilde{\partial}\}} \widetilde{\mu}(\widetilde{x}) \Lambda(\widetilde{x}, x) \underline{P}((x, \widetilde{x}), (y, \widetilde{y})) \\
&= \sum_{x \in I} \sum_{\widetilde{x} \in \widetilde{I} \setminus \{\widetilde{\partial}\}} \widetilde{\mu}(\widetilde{x}) \Lambda(\widetilde{x}, x) P(x, y) \widetilde{P}(\widetilde{x}, \widetilde{y}) \Lambda(\widetilde{y}, y) \mathbf{1}((\Lambda P)(\widetilde{x}, y) > 0) \Lambda(\widetilde{x}, y)^{-1} \\
&= \sum_{\widetilde{x} \in \widetilde{I} \setminus \{\widetilde{\partial}\}} \widetilde{\mu}(\widetilde{x}) \widetilde{P}(\widetilde{x}, \widetilde{y}) \Lambda(\widetilde{y}, y) \mathbf{1}((\Lambda P)(\widetilde{x}, y) > 0) (\Lambda P)(\widetilde{x}, y)^{-1} (\sum_{x \in I} \Lambda(\widetilde{x}, x) P(x, y)) \\
&= \sum_{\widetilde{x} \in \widetilde{I} \setminus \{\widetilde{\partial}\}} \widetilde{\mu}(\widetilde{x}) \widetilde{P}(\widetilde{x}, \widetilde{y}) \Lambda(\widetilde{y}, y) \mathbf{1}((\Lambda P)(\widetilde{x}, y) > 0).
\end{aligned}$$

Now, if $P(\widetilde{x},\widetilde{y}) > 0$ and $\Lambda(\widetilde{y},y) > 0$, then we have $(\Lambda P)(\widetilde{x},y) = (\widetilde{P}\Lambda)(\widetilde{x},y) > 0$ and so we get

$$\sum_{x\in I}\sum_{\widetilde{x}\in\widetilde{I}\setminus\{\widetilde{\partial}\}} \widetilde{\mu}'(\widetilde{x})\underline{P}((x,\widetilde{x}),(y,\widetilde{y})) = \sum_{\widetilde{x}\in\widetilde{I}\setminus\{\widetilde{\partial}\}} \widetilde{\mu}(\widetilde{x})\widetilde{P}(\widetilde{x},\widetilde{y})\Lambda(\widetilde{y},y)$$
$$= \gamma\widetilde{\mu}(\widetilde{y})\Lambda(\widetilde{y},y) = \gamma\underline{\mu}(y,\widetilde{y}).$$

Hence, $\underline{\mu}$ is a q.s.d. for $\underline{X}$ with forbidden set $\underline{\partial}$. □

Remark 7. *Based upon basic relations on quasi-stationarity (see Theorem* 2.6 *in* [3]*), it can be shown that starting from* $\underline{\mu}$ *the random variables* $\underline{X}_{\tau_{\underline{\partial}}} = (X_{\tau_{\underline{\partial}}},\underline{\partial})$ *and* $\tau_{\underline{\partial}}$ *are independent, so* $X_{\tau_{\underline{\partial}}}$ *and* $\tau_{\underline{\partial}}$ *are independent. But in the setting of the Diaconis-Fill coupling this property is contained in Theorem 12. In fact, the latter result ensures a much stronger result which is that starting from any linked initial condition* $\underline{\pi}_0$ *one has that* $X_{\tau_{\underline{\partial}}}$ *and* $\tau_{\underline{\partial}}$ *are independent and* $X_{\tau_{\underline{\partial}}} \sim \pi'$. □

Let P be monotone on I_N. From Corollary 3 (i), $\Lambda(\widetilde{x},y) = \mathbf{1}(\widetilde{x} \ge y)\pi(y)/\pi^c(\widetilde{x})$ and $\widetilde{P}(\widetilde{x},\widetilde{y}) = \widehat{P}(\widetilde{x},\widetilde{y})\pi^c(\widetilde{y})/\pi^c(\widetilde{x})$. The coupling (29) for $\overleftarrow{P}$ satisfies,

$$\Lambda\overleftarrow{P}(\widetilde{x},y) = \sum_{z\le\widetilde{x}} \frac{\pi(z)}{\pi^c(\widetilde{x})}P(y,z)\frac{\pi(y)}{\pi(z)} = \frac{\pi(y)}{\pi^c(\widetilde{x})}(\sum_{z\le\widetilde{x}} P(y,z)).$$

Then, $\Lambda\overleftarrow{P}(\widetilde{x},y) > 0$ is equivalent to $\sum_{z\le\widetilde{x}} P(y,z) > 0$, for $\widetilde{x},y\in I$, so

$$\overleftarrow{\underline{P}}((x,\widetilde{x}),(y,\widetilde{y})) = \frac{\overleftarrow{P}(x,y)\,\widetilde{P}(\widetilde{x},\widetilde{y})\,\Lambda(\widetilde{y},y)}{(\Lambda\overleftarrow{P})(\widetilde{x},y)}\mathbf{1}\left((\Lambda P)(\widetilde{x},y) > 0\right)$$
$$= \frac{\pi(y)}{\pi(x)}\mathbf{1}(\widetilde{y}\ge y)P(y,x)\widehat{P}(\widetilde{x},\widetilde{y})\mathbf{1}(\sum_{z\le\widetilde{x}} P(y,z) > 0)(\sum_{z\le\widetilde{x}} P(y,z))^{-1}.$$

Now, in this coupling we can set the truncations kernel P_N of the mean expected value, whose reversed time kernel satisfies $\overleftarrow{P}_N(x,y) = \pi_N(x)^{-1}P_N(y,x)\pi_N(y)$. From Proposition 9 one gets that the time for $\widetilde{P}_N$ to attain the absorbing state N is stochastically smaller that the time for $\widetilde{P}_{N+1}$ to attain $N+1$. Then, the time for $\overleftarrow{P}_N$ to attain the stationary distribution π_N is stochastically smaller than the time for $\overleftarrow{P}_{N+1}$ to attain π_{N+1}.

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