# On the extension complexity of scheduling * 

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#### Abstract

Linear programming is a powerful method in combinatorial optimization with many applications in theory and practice. For solving a linear program quickly it is desirable to have a formulation of small size for the given problem. A useful approach for this is the construction of an extended formulation, which is a linear program in a higher dimensional space whose projection yields the original linear program. For many problems it is known that a small extended formulation cannot exist. However, most of these problems are either NP-hard (like TSP), or only quite complicated polynomial time algorithms are known for them (like for the matching problem). In this work we study the minimum makespan problem on identical machines in which we want to assign a set of $n$ given jobs to $m$ machines in order to minimize the maximum load over the machines. We prove that the canonical formulation for this problem has extension complexity $2^{\Omega(n / \log n)}$, even if each job has size 1 or 2 and the optimal makespan is 2 . This is a case that a trivial greedy algorithm can solve optimally! For the more powerful configuration integer program we even prove a lower bound of $2^{\Omega(n)}$. On the other hand, we show that there is an abstraction of the configuration integer program admitting an extended formulation of size $f($ opt $) \cdot$ poly $(n, m)$. In addition, we give an $O(\log n)$-approximate integral formulation of polynomial size, even for arbitrary processing times and for the far more general setting of unrelated machines.


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## 1 Introduction

In order to solve a linear program quickly one is interested in a formulation with as few variables and constraints as possible. A useful technique for this are extended formulations. A polytope $Q$ is said to be an extended formulation or extension of a polytope $P$ if $P$ is a linear projection of $Q$. There are many examples of polytopes $P$ that require many constraints to be described but that admit extended formulations that are much smaller. For instance, the convex hull of all characteristic vectors of spanning trees in a graph with $n$ vertex needs $2^{\Omega(n)}$ inequalities to be described [4], but it admits an extended formulation of size $O\left(n^{3}\right)$ [13]. The extension complexity xc $(P)$ of a polytope $P$ is the minimum number of inequalities needed to describe an extended formulation of it, see [2, 11, 20, 21] for surveys on the topic. We study the classical scheduling problem of assigning jobs on identical machines to minimize the makespan, also known as $P \| C_{\max }$ in the scheduling literature. We are given a set $J$ of $n$ jobs and a set $M$ of $m$ identical machines and every job $j \in J$ has a processing time $p_{j} \in \mathbb{N}$. The goal is to assign each job on a machine in order to minimize the maximum load over the machines, where the load of a machine is the sum of the processing times of the jobs assigned to it.

### 1.1 Our Contribution

A natural formulation for $P \| C_{\text {max }}$, known as the assignment integer program, uses a variable $x_{i j}$ for each combination of a machine $i$ and a job $j$, modelling whether $j$ is assigned to $i$. The makespan is then modeled by an additional variable $T$. Then, its linear relaxation is given by

$$
\begin{array}{lll}
\min & T & \\
\text { s.t. } & \sum_{i \in M} x_{i j}=1 & \text { for all } j \in J, \\
\sum_{j \in J} x_{i j} p_{j} \leq T & \text { for all } i \in M, \\
x_{i j} \geq 0 & \text { for all } i \in M, \text { for all } j \in J . \tag{4}
\end{array}
$$

Lower bounds on the extension complexity. We prove that there are instances with $O(n)$ jobs and machines such that the convex hull $P_{I}$ of all integral solutions to the above linear program has an extension complexity of $2^{\Omega(n / \log n)}$. The optimal solutions form a face of $P_{I}$ and our bound also holds for this face and hence for all polytopes containing it as a face. Our instances satisfy that $p_{j} \in\{1,2\}$ for each job $j \in J$ and the optimal makespan is 2 . Such instances can be solved optimally by a simple greedy algorithm in time $O(n+m)$. Our key insight is that there are faces of $P_{I}$ in which some jobs cannot be assigned to certain machines, e.g., defined via equalities of the form $x_{i j}=0$. Hence, an extended formulation of $P_{I}$ also yields such a formulation for the polytope of any instance of the restricted assignment problem in which we have the same input as in $P \| C_{\max }$ and additionally for each job $j$ there is a set of machines $M_{j}$ such that $j$ must be assigned to a machine in $M_{j}$. Using a result in [6] we show that there are 3-Bounded-2-SAT instances such that the polytope describing all feasible solutions for them has extension complexity $2^{\Omega(n / \log n)}$. This might be of independent interest, in particular since 2-SAT can be solved easily in polynomial time. We reduce these instances to the restricted assignment problem using a reduction from 3-Bounded-2-SAT to the restricted assignment problem such that $p_{j} \in\{1,2\}$ for each job $j \in J$ and the optimal makespan is 2 [3]. Hence, the polytope of all optimal solutions has extension complexity $2^{\Omega(n / \log n)}$.

By the above this also holds for the polytope of the corresponding instance of $P \| C_{\max }$ where we ignore the sets $M_{j}$. Moreover, we show that this holds also for each subpolytope containing all optimal solutions.

Then we consider the variation of the above formulation in which we omit the variable $T$ and the second constraint. The set of integral points is simply the set of all schedules and the resulting (polynomial size) LP is integral. Note that hence in this space there exists an integral linear program of small complexity containing all optimal schedules. However, we show that if we restrict this polytope to the convex hull of all optimal schedules then resulting polytope again has extension complexity $2^{\Omega(n / \log n)}$. We show that our bounds are almost tight by giving an extended formulation of size $2^{O(n)} m$ for the convex hull of all optimal solutions, also for arbitary processing times and for the formulation in which the makespan $T$ is a variable.

Approximate schedules: Lower bound. For $P \| C_{\text {max }}$ there is a polynomial time $(1+\epsilon)$-approximation algorithm known and even an EPTAS, see e.g., [8, 9]. Therefore, one might wonder whether we can obtain a polynomial size extended formulations that contains all optimal schedules and possibly also some $\alpha$-approximate schedules, e.g., for $\alpha=1+\epsilon$. We show that this is not possible for any $\alpha<\frac{3}{2}$. Even more, if we ask for a small polytope that contains all $\alpha$-approximate schedules for some value $\alpha$ we show that this is does not exist for any $\alpha \leq m^{1-\epsilon}$ with $\epsilon>0$. Moreover, this is tight in the sense that the set of all $m$-approximate schedules is simply the set of all schedules which admits the polynomial size formulation mentioned above.

Approximate schedules: Upper bound. Despite this negative result, we show that there is a polynomial size formulation for a polytope that contains all optimal schedules and some $O(\log n)$-approximate schedules if the target makespan $T$ is fixed. Our construction works for arbitrary job processing times and even in the more complex setting of unrelated machines, i.e., $R \| C_{\max }$, where for each combination of a machine $i$ and a job $j$ there is a value $p_{i j} \in \mathbb{N} \cup\{\infty\}$ denoting the processing time of $j$ when it is assigned to machine $i$. Key to our extended formulation is to construct an instance of bipartite matching in which there is a vertex for each job which can be matched to vertices representing slots on the machines. Then we prove that in the space with the makespan $T$ as a variable such a formulation cannot exist which yields a separation between the two spaces even though they might appear very similar at first glance.

Configuration integer program. Finally, we study the configuration integer program which is a popular approach in scheduling, e.g., $[19,10]$, with connections to bin packing, see e.g., $[7,15]$. There is a variable $y_{i C}$ for each combination of a machine $i$ and a configuration $C \in \mathcal{C}(T)$ where $\mathcal{C}(T)$, contains all sets of jobs whose total processing size does not exceed an upper bound $T$ on the optimal makespan. While for large $T$ already the number of variables can be exponential, for constant $T$ this number is only polynomial and could potentially admit a small extended formulation. However, we prove that already for the case that $p_{j}=1$ for each job $j$ and $T=2$ there are instances with $O(n)$ jobs and machines such that this linear program has extension complexity $2^{\Omega(n)}$. To show this, we establish the maybe surprising connection that there are such instances for which the corresponding polytope is an extended formulation of the perfect matching polytope in a graph with $n$ vertices and the latter has extension complexity $2^{\Omega(n)}$ [16]. On the other hand, there is an abstraction of the configuration integer program which instead of assigning a configuration $C \in \mathcal{C}(T)$ to each machine $i$ only assigns a pattern that describes how many jobs of each size are assigned
to $i$ but without specifying the actual jobs. We prove that in contrast to the configuration integer program this abstraction admits an extended formulation of size $O(f(T) \cdot \operatorname{poly}(n, m))$ for some function $f$.

### 1.2 Related work

Lower bounds. There are many examples known of polytopes that do not admit small extended formulations, i.e., formulations of polynomial size. For instance, Yanakakis [22] proved that for TSP there can be no such formulation that is symmetric, i.e., stays invariant under permutation of cities. Recently, it became an active field of research to prove such lower bounds. For instance, Fiorini et al. [5] extended the above result for TSP to arbitrary (possibly non-symmetric) formulations and Avis et al. [1] showed that neither for 3-SAT, subset sum, 3D-matching, nor for MaxCut for suspensions of cubic planar graphs, there can be small extended formulations. Note that all these problems are NP-hard and hence a polynomial size extended formulation for any of them would be very surprising.

Easy problems with large extension complexity. There are only few problems in P for which the corresponding polytope is known to have large extension complexity. The most famous example is probably the perfect matching polytope for which Rothvoss showed in his celebrated result that it has exponential extension complexity [16]. While the matching problem is in $P$, the polynomial time algorithm for it is highly complicated. Also, Rothvoss showed that there exists a family of matroids whose associated polytopes have exponential extension complexity [14]. This contrasts with the fact that we can optimize over any matroid in polynomial time using the greedy algorithm [17].

## 2 Extension complexity: Lower bound

Suppose that we are given an instance $(J, M)$ of $P \| C_{\max }$. We consider the linear program defined by (1)-(4) in Section 1.1. Denote by $P(J, M)$ the convex hull of all its integral solutions. In the remainder of this section we prove the following theorem.

Theorem 1. For every $n$ there exists an instance $(J, M)$ of $P\left|p_{j} \in\{1,2\}\right| C_{\text {max }}$ with $O(n)$ jobs, $O(n)$ machines, and $\operatorname{opt}(J, M)=2$ such that $\mathrm{xc}(P(J, M)) \geq 2^{\Omega(n / \log n)}$.

Let $n \in \mathbb{N}$. For any given SAT formula $\Phi$ with $n$ variables we define the polytope $\operatorname{SAT}(\Phi)$ as the convex hull of all satisfying assignments, i.e., $\operatorname{SAT}(\Phi):=\operatorname{conv}\left(\left\{y \in\{0,1\}^{n}: \Phi(y)=1\right\}\right)$. We use the following theorem that follows easily from [6].

Theorem 2. For every $n \in \mathbb{N}$ there exists a 2-SAT formula $\Phi$ with $O(n)$ variables and $O(n)$ clauses such that $\operatorname{xc}(\operatorname{SAT}(\Phi)) \geq 2^{\Omega(n / \log n)}$. Each clause of $\Phi$ contains exactly two literals.

Proof of Theorem 2. Let $n \in \mathbb{N}$. In [6] it is shown that there exists a graph $G=(V, E)$ with $n$ vertices such that for its independent set polytope $P_{G}$ it holds that $\operatorname{xc}\left(P_{G}\right) \geq 2^{\Omega(n / \log n)}$, i.e., $P_{G} \subseteq[0,1]^{n}$ is the convex hull of all incidence vectors of independent sets of $G$. Moreover, the degree of $G$ is bounded by a global constant that is independent of $n$. Based on $G$ we construct a 2-SAT formula $\Phi$. For each node $v \in V$ we introduce a variable $x_{v}$, the intuition being that $x_{v}$ is true if $v$ is in the independent set. For each edge $\{u, v\} \in E$ we introduce a clause $\left(\neg x_{v} \vee \neg x_{u}\right)$, modelling that not both $u$ and $v$ can be in the independent set. The number of variables is $n$ and since $G$ has bounded
degree each variable appears in at most $O(1)$ clauses. Hence, the number of clauses is also $O(n)$. Each satisfying assignment of $\Phi$ corresponds to an independent set of $G$ and vice versa. Therefore, $\mathrm{xc}(\operatorname{SAT}(\Phi)) \geq 2^{\Omega(n / \log n)}$.

Let $\Phi$ denote the formula due to Theorem 2. We transform $\Phi$ into an equivalent 3-Bounded-2-SAT formula $\Phi^{\prime}$ using a standard reduction i.e., $\Phi^{\prime}$ is a 2-SAT formula in which each variable appears at most three times. Let $x_{i}$ be a variable in $\Phi$ and assume that $x_{i}$ occurs $k$ times. We introduce $k$ new variables $x_{i}^{(1)}, \ldots, x_{i}^{(k)}$ and for each $\ell \in[k]$ we replace the $\ell$-th occurrence of $x_{i}$ with $x_{i}^{(\ell)}$. Additionally, we add the clauses $\left(\neg x_{i}^{(\ell)} \vee x_{i}^{(\ell+1)}\right)$ for each $\ell \in\{1, \ldots, k-1\}$ and the clause $\left(\neg x_{i}^{(k)} \vee x_{i}^{(1)}\right)$. Hence, in any satisfying assignment of the resulting formula, either $x_{i}^{(\ell)}=1$ for each $\ell \in[k]$ or $x_{i}^{(\ell)}=0$ for each $\ell \in[k]$. We do this transformation with each variable in $\Phi$. Let $\Phi^{\prime}$ denote the resulting formula. By construction, we have that $\Phi^{\prime}$ has $O(n)$ variables and $O(n)$ clauses, using that both quantities are linear in the number of literals in $\Phi$ and the latter is bounded by $O(n)$. By construction, each variable appears exactly three times and at most two times positively and at most two times negatively. Also, each clause contains exactly two literals.

### 2.1 Reduction to the restricted assignment problem

Construction. Next, based on $\Phi^{\prime}$ we construct an instance of the restricted assignment problem. We invoke the reduction from [3]. For each variable $x$ in $\Phi^{\prime}$ we introduce a machine $i(x)$, a machine $i(\neg x)$, and a job $j(x)$. The job $j(x)$ has processing time $p_{j(x)}=2$ and it can be assigned only on $i(x)$ and $i(\neg x)$, i.e., $M_{j(x)}=\{i(x), i(\neg x)\}$. The intuition behind is that if $j(x)$ is scheduled on $i(x)$ then $x$ is true and if $j(x)$ is scheduled on $i(\neg x)$ then $x$ is false. For each clause $c$ we introduce one machine $i(c)$. For each variable $x$ that occurs in $c$ we introduce a job $j(c, x)$ with $p_{j(c, x)}=1$. If $x$ occurs positively in $c$ then we define $M_{j(c, x)}=\{i(c), i(\neg x)\}$, otherwise we define $M_{j(c, x)}=\{i(c), i(x)\}$. Finally, for each clause $c$ we introduce a job $j(c)$ with $p_{j(c)}=1$ and $M_{j(c)}=\{i(c)\}$.

Since the total number of variables and clauses is $O(n)$, we introduced $O(n)$ jobs and machines. Let $J^{\prime}$ denote the set of jobs and let $\bar{m}$ denote the number of machines defined so far. We want that in solutions with makespan 2 each machine has a load of exactly 2 . To this end, we introduce a set $D$ of $2 \bar{m}-\sum_{j \in J^{\prime}} p_{j}$ dummy jobs of length 1 each. For each dummy job $j \in D$, it can be assigned to any machine, i.e., $M_{j}=M$.

Correctness. For each satisfying assignment of $\Phi^{\prime}$ there is a schedule of makespan 2: If a variable $x$ is assigned to be $x=1$ in the satisfying assignment then we schedule $j(x)$ on machine $i(x)$, otherwise we schedule $j(x)$ on machine $i(\neg x)$. Consider a clause $c$. There must be at least one variable that satisfies $c$. For each such variable $x$, if $x$ occurs positively in $c$ then we assign $j(c, x)$ on $i(\neg x)$, otherwise we assign $j(c, x)$ on $i(x)$. For each variable $y$ that does not satisfy $c$ we assign $j(c, y)$ to $i(c)$. Also, we assign $j(c)$ to $i(c)$. Using that each variable appears at most twice positively and at most twice negatively, one can check that the resulting makespan is 2. Finally, we assign the dummy jobs to the machines such that each machine still has a makespan of at most 2. This is the optimal solution since the largest processing time is 2 . One can also easily show that if there is a solution of makespan 2 then there exists a satisfying assignment for $\Phi^{\prime}$, see [3] for details.

Faces of scheduling polytope. Let $J$ and $M$ denote the set of jobs and machines in the construction above, respectively. Also, let $M_{j}$ denote the set of allowed machines for each job $j \in J$. We
consider the polyhedron $P(J, M)$. Note that $P(J, M)$ ignores the sets $M_{j}$ of allowed machines for each job $j$. We argue that there is a face $P^{\prime}(J, M)$ of $P(J, M)$ such that each vertex of $P^{\prime}(J, M)$ corresponds to a schedule in which each job $j \in J$ is assigned on a machine in $M_{j}$ and the makespan is 2. Observe that the inequalities $\sum_{j \in J} \sum_{i \in M \backslash M_{j}} x_{i j} \geq 0$ and $T \geq 2$ are valid inequalities for $P(J, M)$. Hence, the set

$$
P^{\prime}(J, M)=P(J, M) \cap\left\{(x, T): \sum_{j \in J} \sum_{i \in M \backslash M_{j}} x_{i j}=0 \text { and } T=2\right\}
$$

is a face of $P(J, M)$. Also, $\mathrm{xc}\left(P^{\prime}(J, M)\right) \leq \mathrm{xc}(P(J, M))$.
Proof of Theorem 1. We describe now a linear projection $f: P^{\prime}(J, M) \rightarrow \operatorname{SAT}\left(\Phi^{\prime}\right)$ of $P^{\prime}(J, M)$ to $\operatorname{SAT}\left(\Phi^{\prime}\right)$. Given a point $x \in P^{\prime}(J, M)$ we define that the component of $f(x)$ corresponding to the variable $y_{\ell}$ equals to $x_{i\left(y_{\ell}\right), j\left(y_{\ell}\right)}$. By the construction above and the proof of correctness of the reduction, for each integral point $y \in \operatorname{SAT}\left(\Phi^{\prime}\right)$ there exists a feasible schedule for ( $J, M$ ) with makespan 2 in which each job $j$ is assigned to a machine in $M_{j}$. Additionally, for each variable $y$ in $\Phi^{\prime}$ it holds that if $y=1$ then $j(y)$ is assigned on machine $i(y)$ in this schedule. Thus, there exists an integral point $x \in P^{\prime}(J, M)$ such that $f(x)=y$. Similarly, for each integral point $x \in P^{\prime}(J, M)$ we have that $f(x) \in \operatorname{SAT}\left(\Phi^{\prime}\right)$. Thus, $f\left(P^{\prime}(J, M)\right)=\operatorname{SAT}\left(\Phi^{\prime}\right)$ and therefore, $\mathrm{xc}\left(\operatorname{SAT}\left(\Phi^{\prime}\right)\right) \leq \mathrm{xc}\left(P^{\prime}(J, M)\right)$. Finally we give a linear projection $g: \operatorname{SAT}\left(\Phi^{\prime}\right) \rightarrow \operatorname{SAT}(\Phi)$. Recall that for each variable $x_{i}$ in $\Phi$ we introduced a set of new variables $x_{i}^{(1)}, \ldots, x_{i}^{(k)}$ in $\Phi^{\prime}$. By construction, in each satisfying assignment for $\Phi^{\prime}$ all these variables $x_{i}^{(1)}, \ldots, x_{i}^{(k)}$ have the same value. Therefore, we define that the component of $g(x)$ corresponding to $x_{i}$ equals $x_{i}^{(1)}$ for each variable $x_{i}$ in $\Phi$. We obtain $g\left(\operatorname{SAT}\left(\Phi^{\prime}\right)\right)=\operatorname{SAT}(\Phi)$. Therefore, $\operatorname{xc}(\operatorname{SAT}(\Phi)) \leq \operatorname{xc}\left(\operatorname{SAT}\left(\Phi^{\prime}\right)\right)$. Hence, $\mathrm{xc}(P(J, M)) \geq \mathrm{xc}\left(P^{\prime}(J, M)\right) \geq \mathrm{xc}\left(\operatorname{SAT}\left(\Phi^{\prime}\right)\right) \geq \mathrm{xc}(\operatorname{SAT}(\Phi)) \geq 2^{\Omega(n / \log n)}$. This completes the proof of Theorem 1.

Since already the face $P^{\prime}(J, M)$ of $P(J, M)$ containing the optimal solutions to $(J, M)$ has extension complexity $2^{\Omega(n / \log n)}$ we obtain the following corollary.
Corollary 1. For every $n$ there exists an instance $(J, M)$ of $P\left|p_{j} \in\{1,2\}\right| C_{\max }$ with $O(n)$ jobs, $O(n)$ machines, and $\operatorname{opt}(J, M)=2$ such that for any integral polyhedron $\bar{P}(J, M) \subseteq P(J, M)$ that contains all optimal and possibly also some other solutions to $(J, M)$ it holds that $\mathrm{xc}(\bar{P}(J, M)) \geq 2^{\Omega(n / \log n)}$.

### 2.2 Extensions

Above we proved that $P(J, M)$ has large extension complexity. The polyhedron $P(J, M)$ is defined using the variable $T$ which represents an upper bound on the makespan of the respective solution. This raises the question whether there exist compact extended formulations in the space defined only via the variables $x$, without the variable $T$. Formally, we consider the polyhedron

$$
Q(J, M)=\operatorname{conv}\left(\left\{x \in\{0,1\}^{M \times J}: \sum_{i \in M} x_{i j}=1 \text { for all } j \in J\right\}\right)
$$

describing the set of all schedules, including all optimal ones. One can easily show that its extension complexity is $O(n m)$ by simply taking its linear relaxation. Observe that Corollary 1 rules out such a polyhedron in the space lifted with the variable $T$.

Proposition 1. The polyhedron $Q(J, M)$ has extension complexity $O(n m)$.
This follows by observing that the extreme points of the linear relaxation are integral. Let $x$ be a feasible point in the relaxation such that $x_{i j} \in(0,1)$ for some machine $i \in M$ and some job $j \in J$. Due to Equality (2) there must be another machine $\ell \neq i$ such that $x_{\ell j} \in(0,1)$. For sufficiently small $\epsilon>0$ we define a new solution $\tilde{x}=x+\varepsilon\left(\mathbb{1}_{i j}-\mathbb{1}_{\ell j}\right)$, where $\mathbb{1}_{a}$ is the canonical vector with value 1 at entry $a$ and zero otherwise. Similarly, we define a new solution $\hat{x}=x+\varepsilon\left(\mathbb{1}_{\ell j}-\mathbb{1}_{i j}\right)$. We can find $\varepsilon>0$ such that $\tilde{x}, \hat{x} \geq 0$ and $x=\frac{1}{2}(\tilde{x}+\hat{x})$, and hence $x$ is not an extreme point. Therefore, the extension complexity of $Q(J, M)$ is $O(n m)$.

However, when we minimize the makespan, we are interested only in the set of all integral points in $Q(J, M)$ that correspond to optimal solutions. Note that their convex hull corresponds to the face of $P(J, M)$ containing all points $(x, T)$ with $T=$ opt. Due to Corollary 1 this polytope has extension complexity $2^{\Omega(n / \log n)}$. We can strengthen this statement. In the instance $(J, M)$ constructed above the optimal makespan is 2 and any non-optimal solution has a makespan of at least 3. Thus, all integral polytopes $\bar{Q}(J, M) \subseteq Q(J, M)$ that contain all optimal solutions to ( $J, M$ ) and possibly some $(3 / 2-\epsilon)$-approximate solutions have large extension complexity. Note that this contrasts the fact that $P \| C_{\max }$ admits an EPTAS [9].

Corollary 2. Let $\epsilon>0$ with $\epsilon \leq 1 / 2$. For every $n$ there exists an instance $(J, M)$ of $P\left|p_{j} \in\{1,2\}\right| C_{\max }$ with $O(n)$ jobs, $O(n)$ machines, and $\operatorname{opt}(J, M)=2$ such that for any integral polytope $\bar{Q}(J, M) \subseteq Q(J, M)$ whose vertices consist of all optimal solutions and possibly some $\left(\frac{3}{2}-\epsilon\right)$-approximate solutions to $(J, M)$ it holds that $\operatorname{xc}(\bar{Q}(J, M)) \geq 2^{\Omega(n / \log n)}$.

Next, we show that there cannot be an integral polytope of polynomial extension complexity containing all $\alpha$-approximate solutions of $Q(J, M)$ for any $\alpha \leq n^{1-\epsilon}$.

Corollary 3. For every $n \in \mathbb{N}$ and every $\alpha \geq 1$ there exists an instance $\left(J^{\prime}, M^{\prime}\right)$ of $P\left|p_{j} \in\{1,2\}\right| C_{\max }$ with $O(\alpha n)$ jobs and $O(\alpha n)$ machines such that for any integral polytope $\tilde{Q}(J, M) \subseteq Q(J, M)$ whose vertices are all $\alpha$-approximate solutions to $\left(J^{\prime}, M^{\prime}\right)$ it holds that $\mathrm{xc}(\tilde{Q}(J, M)) \geq 2^{\Omega(n / \log n)}$. In particular, if $\alpha=n^{1 / \epsilon}$ for some $\epsilon>0$ then the instance has $\bar{n}=O\left(n^{1+1 / \epsilon}\right)$ jobs and machines and $\operatorname{xc}(\tilde{Q}(J, M)) \geq 2^{\bar{n}^{\Omega(\epsilon)}}$.

Proof. Assume by contradiction that such a polytope exists and suppose that $\alpha \in \mathbb{N}$. Take the instance $(J, M)$ defined above. We add $(2 \alpha-2)|M|$ dummy jobs of length 1 each, denote them by $J_{\text {dum }}$. Also, we add $(\alpha-1)|M|$ dummy machines, denote them by $M_{\text {dum }}$. Note that the optimal makespan is still 2 and thus any $\alpha$-approximate solution has makespan $2 \alpha$. Let ( $J^{\prime}, M_{\tilde{\prime}}^{\prime}$ ) denote the resulting instance and observe that it has $O(\alpha n)$ jobs and machines. We consider $\tilde{Q}\left(J^{\prime}, M^{\prime}\right)$. Then for each dummy machine $i \in M_{\text {dum }}$ the inequality $\sum_{j \in J^{\prime}} x_{i j} \geq 0$ is a valid inequality. Also, consider an assignment of the jobs in $J_{\text {dum }}$ to the machines $M^{\prime} \backslash M_{\text {dum }}$ such that each machine $i \in M^{\prime} \backslash M_{\text {dum }}$ gets a load of $2 \alpha-2$ in this way. Formally, we define a map $g: J_{\text {dum }} \rightarrow M^{\prime} \backslash M_{\text {dum }}$ such that $\left|g^{-1}(i)\right|=2 \alpha-2$ for each $i \in M^{\prime} \backslash M_{\text {dum }}$. Then for each job $j \in J_{\text {dum }}$ the inequality $x_{g(j) j} \leq 1$ is a valid inequality. Therefore, the set $\tilde{Q}^{\prime}\left(J^{\prime}, M^{\prime}\right)$ defined as

$$
\tilde{Q}(J, M) \cap\left\{x: \sum_{j \in J^{\prime}} x_{i j}=0 \text { for all } i \in M_{\mathrm{dum}} \text { and } x_{g(j) j}=1 \text { for all } j \in J_{\mathrm{dum}}\right\}
$$

is a face of $\tilde{Q}(J, M)$. Therefore, $\tilde{Q}^{\prime}\left(J^{\prime}, M^{\prime}\right)$ contains exactly the solutions in which the dummy jobs are assigned as described by the map $g$ and the non-dummy jobs $J=J^{\prime} \backslash J_{\text {dum }}$ are assigned such
that they give a load of 2 on each machine. Thus, there is a linear projection of $\tilde{Q}^{\prime}\left(J^{\prime}, M^{\prime}\right)$ to the polytope $\bar{Q}(J, M)$ as defined in Theorem 2. This implies that $\operatorname{xc}\left(\tilde{Q}\left(J^{\prime}, M^{\prime}\right)\right) \geq \operatorname{xc}\left(\tilde{Q}^{\prime}\left(J^{\prime}, M^{\prime}\right)\right) \geq$ $\mathrm{xc}(\bar{Q}(J, M)) \geq 2^{\Omega(n / \log n)}$.

Finally, we can show that our lower bounds from Theorem 1 and Corollary 2 are almost tight by giving an upper bound of $2^{O(n)} m$.

Theorem 3. Let $(J, M)$ be an instance of $P \| C_{\max }$ with $n$ jobs and $m$ machines. Let $\bar{Q}(J, M)$ denote the convex hull of the vertices corresponding to all optimal solutions to $(J, M)$ in $Q(J, M)$. It holds that $\operatorname{xc}(\bar{Q}(J, M)) \leq 2^{O(n)} m$ and $\mathrm{xc}(P(J, M)) \leq 2^{O(n)} m$.

In the following we prove Theorem 3. Given an instance ( $J, M$ ), we first describe a dynamic program that computes a solution to $(J, M)$ in time $2^{O(n)} m$ for some given target makespan $T$, assuming that such a solution exists. Then based on it we define an extended formulation of $\bar{Q}(J, M)$ of size $2^{O(n)} m$.

Dynamic program. We introduce a cell $\left(J^{\prime}, m^{\prime}, T\right)$ for each subset $J^{\prime} \subseteq J$ of jobs and each integer $m^{\prime} \in\{1, \ldots, m\}$. In $\left(J^{\prime}, m^{\prime}, T\right)$ we want to store a schedule for the jobs $J^{\prime}$ on the machines $\left\{m^{\prime}, \ldots, m\right\}$ such that each machine has a makespan of at most $T$, assuming that such a schedule exists. Consider the case $m^{\prime}=m$, that is, we look for a schedule in the single machine $m$. For each set of jobs $J^{\prime}$ with $p\left(J^{\prime}\right) \leq T$ we store in the cell $\left(J^{\prime}, m, T\right)$ the schedule that assigns all jobs in $J^{\prime}$ to machine $m$. For each set of jobs $J^{\prime}$ with $p\left(J^{\prime}\right)>T$ we store $\perp$ in the cell $\left(J^{\prime}, m, T\right)$, indicating that no feasible schedule exists for $\left(J^{\prime}, m, T\right)$. Now suppose we are given a cell $\left(J^{\prime}, m^{\prime}, T\right)$ with $m^{\prime}<m$. If there is a subset $J^{\prime \prime} \subseteq J^{\prime}$ with $p\left(J^{\prime \prime}\right) \leq T$ such that in the cell $\left(J^{\prime} \backslash J^{\prime \prime}, m^{\prime}+1, T\right)$ we stored a schedule (and not $\perp$ ) then in ( $\left.J^{\prime}, m^{\prime}, T\right)$ we store the schedule that assigns $J^{\prime \prime}$ on machine $m^{\prime}$ and the schedule in the cell $\left(J^{\prime} \backslash J^{\prime \prime}, m^{\prime}+1, T\right)$ on the machines $\left\{m^{\prime}+1, \ldots, m\right\}$. If no such set $J^{\prime \prime} \subseteq J^{\prime}$ exists then we store $\perp$ in $\left(J^{\prime}, m^{\prime}, T\right)$. Finally, if we stored a schedule $(J, 1, T)$ then we output this schedule, otherwise $(J, 1, T)$ contains $\perp$ and we output that there is no schedule for $J$ on $m$ machines with makespan at most $T$. This dynamic program table has $2^{n} m$ cells and evaluating one cell takes $2^{n}$ time, which yields a total running time of $2^{O(n)} m$.

In the transition above, for each cell $\left(J^{\prime}, m^{\prime}, T\right)$ we took an arbitrary subset $J^{\prime \prime}$ such that $p\left(J^{\prime \prime}\right) \leq$ $T$ and $\left(J^{\prime} \backslash J^{\prime \prime}, m^{\prime}+1, T\right)$ does not contain $\perp$. In fact, we can construct any schedule of makespan at most $T$ in this way if for each cell $\left(J^{\prime}, m^{\prime}, T\right)$ we choose for $J^{\prime \prime}$ the set of jobs that the respective schedule assigns on machine $m^{\prime}$. In the sequel, we define a graph $G$ with $2^{O(n)} m$ vertices including two special vertices $s$ and $t$ such that any path from $s$ to $t$ corresponds to a solution that the above dynamic program might compute for suitable choices for $J^{\prime \prime}$. Then we define a linear program whose vertices are exactly these paths which then yield an extended formulation to $\bar{Q}(J, M)$ if we choose $T=$ opt.

Construction of the graph. Let $T \geq$ opt. Let $\mathcal{V}_{T}$ be the set of cells $\left(J^{\prime}, m^{\prime}, T\right)$ of the dynamic program table such that there exists a schedule for the jobs $J^{\prime}$ on machines $\left\{m^{\prime}, \ldots, m\right\}$ of makespan at most $T$. For convenience we add a dummy element $(\emptyset, m+1, T)$ to $\mathcal{V}_{T}$. Then, in our graph $G$ the set of vertices corresponds to $\mathcal{V}_{T}$. For each $J^{\prime}, \bar{J}^{\prime} \subseteq J, m^{\prime} \in\{1, \ldots, m\}$ we introduce an arc $\left(\left(J^{\prime}, m^{\prime}, T\right),\left(\bar{J}^{\prime}, m^{\prime}+1, T\right)\right)$ if and only if $\left(J^{\prime}, m^{\prime}, T\right),\left(\bar{J}^{\prime}, m^{\prime}+1, T\right) \in \mathcal{V}_{T}$ and there is a set $J^{\prime \prime} \subseteq J^{\prime}$ with $p\left(J^{\prime \prime}\right) \leq T$ such that $\bar{J}^{\prime}=J^{\prime} \backslash J^{\prime \prime}$. Let $s=(J, 1, T)$ be the source and $t=(\emptyset, m+1, T)$ the sink. We call $\mathcal{A}_{T}$ the set of arcs, and then $G=\left(\mathcal{V}_{T}, \mathcal{A}_{T}\right)$.

Lemma 1. Every s-t path in the graph $G$ corresponds to a schedule $S$ of makespan at most $T$. Furthermore, every schedule $S$ of makespan at most $T$ induces an s-t path in $G$.

Proof. Let $P$ be an $s$-t path with vertices $\left(J_{1}^{\prime}, 1, T\right),\left(J_{2}^{\prime}, 2, T\right), \ldots,\left(J_{m}^{\prime}, m, T\right),\left(J_{m+1}^{\prime}, m+1, T\right)$, where $J_{1}^{\prime}=J$ and $J_{m+1}^{\prime}=\emptyset$. Then, our construction of $G$ guarantees that $p\left(J_{\ell+1}^{\prime} \backslash J_{\ell}^{\prime}\right) \leq T$ for each $\ell \in\{1, \ldots, m\}$. Thus, for each arc $\left(\left(J_{\ell}^{\prime}, \ell, T\right),\left(J_{\ell+1}^{\prime}, \ell+1, T\right)\right)$ we assign the jobs in $J_{\ell}^{\prime} \backslash J_{\ell+1}^{\prime}$ to machine $\ell$ which yields a schedule of makespan at most $T$. Conversely, consider a schedule $S$ of makespan at most $T$ and for each $i \in\{1, \ldots, m\}$ let $J_{i} \subseteq J$ be the jobs that are assigned to machine $i$ in $S$. Then, there is a path that for each $i \in\{1, \ldots, m+1\}$ visiting the vertex $\left(\bigcup_{\ell=i}^{m} J_{\ell}, i, T\right)$. Observe that for each $i \in\{1, \ldots, m\}$ we have that $\left(\left(\bigcup_{\ell=i}^{m} J_{\ell}, i, T\right),\left(\bigcup_{\ell=i+1}^{m} J_{\ell}, i+1, T\right)\right) \in \mathcal{A}_{T}$ since $\bigcup_{\ell=i}^{m} J_{\ell} \backslash \bigcup_{\ell=i+1}^{m} J_{\ell}=J_{i}$ and $p\left(J_{i}\right) \leq T$.

Extended formulation. We define a linear program with a variable $y_{a}$ for each arc $a \in \mathcal{A}_{T}$. We add constraints that describe a flow in $G$ such that we send exactly one unit of flow from $s$ to $t$. In particular, we require that one unit of flow leaves $s$, one unit of flow enters $t$, and on all other vertices there is flow conservation. For every job $j \in J$, let $\mathcal{J}_{i j}^{T}=\left\{\left(\left(J^{\prime}, i, T\right),\left(\bar{J}^{\prime}, i+1, T\right)\right) \in \mathcal{A}_{T}\right.$ : $\left.j \in J^{\prime} \backslash \bar{J}^{\prime}\right\}$ which are the arcs such that if one of them is contained in an s-tpath $P$ then in the schedule corresponding to $P$ job $j \in J$ is assigned on machine $i \in M$. Then, consider the following linear program

$$
\begin{align*}
& \sum_{a \in \delta^{+}(v)} y_{a}-\sum_{a \in \delta^{-}(v)} y_{a}= \begin{cases}0 & \text { for all } v \in \mathcal{V}_{T} \backslash\{s, t\}, \\
1 & \text { if } v=s, \\
-1 & \text { if } v=t .\end{cases}  \tag{5}\\
& \sum_{a \in \mathcal{J}_{i j}^{T}} y_{a}=x_{i j} \text { for all } i \in M, \text { for all } j \in J,  \tag{6}\\
& x_{i j} \geq 0 \quad \text { for all } i \in M, \text { for all } j \in J,  \tag{7}\\
& y_{a} \geq 0 \quad \text { for all } a \in \mathcal{A}_{T}, \tag{8}
\end{align*}
$$

where for each vertex $v$, we have that $\delta^{+}(v), \delta^{-}(v)$ denote the out-going and in-going edges at $v \in \mathcal{V}_{T}$, respectively.

Proof of Theorem 3. For $T=$ opt, we project the polytope of the linear program above to $\bar{Q}(J, M)$ by defining $x_{i j}$ according to (6) for each job $j \in J$ and each machine $i \in M$. By Lemma 1 this yields an extended formulation of size $2^{O(n)} \mathrm{m}$. We describe now how to extend the above to an extended formulation for $P(J, M)$ of size $2^{O(n)} m$. For that, we show how to modify the dynamic program used for constructing the extended formulation of $\bar{Q}(J, M)$.

Note that for the makespan of any feasible solution there are only $2^{n}$ options since the makespan equals the sum of the processing times of the jobs on some machine and there are only $2^{n}$ options for this quantity. Let $\mathcal{T}$ be a set containing all these options. We define a cell $\left(J^{\prime}, m^{\prime}, T\right)$ for each subset $J^{\prime} \subseteq J$ of jobs, each integer $m^{\prime}$ with $m^{\prime} \in\{1, \ldots, m\}$ and for each $T \in \mathcal{T}$. For each such cell in which the previous dynamic program above does not store $\perp$ we define a vertex like above and also arcs as before. Note that there are no arcs between two vertices $\left(J^{\prime}, m^{\prime}, T\right),\left(\bar{J}^{\prime}, \bar{m}^{\prime}, \bar{T}\right)$ with $T \neq \bar{T}$. We define a new source vertex $s^{\prime}$ and introduce an $\operatorname{arc}\left(s^{\prime},(J, 1, T)\right)$ for each $T \in \mathcal{T}$, assuming that the vertex $(J, 1, T)$ exists. Similarly, we define a new sink vertex $t^{\prime}$ and an arc $\left((\emptyset, m+1, T), t^{\prime}\right)$ for each $T \in \mathcal{T}$. A path from $s^{\prime}$ to $t^{\prime}$ represents the choice of a makespan $T \in \mathcal{T}$ and defining a
schedule with makespan at most $T$. As before, consider a linear program computing $s$ - $t$ paths in this new graph, using a flow formulation and $y$ variables as before, that is, constraints (5) and (8). We lift this linear program by adding a variable $\mathbf{T}$ and introducing a new constraint

$$
\begin{equation*}
\mathbf{T} \geq \sum_{T \in \mathcal{T}} T \cdot y_{\left(s^{\prime},(J, 1, T)\right)} . \tag{9}
\end{equation*}
$$

Then we project each point $(y, \mathbf{T})$ in the resulting polyhedron to the point $(x, \mathbf{T}) \in P(J, M)$ by setting $x_{i j}=\sum_{T \in \mathcal{T}} \sum_{a \in \mathcal{J}_{i j}^{T}} y_{a}$ for each $i \in M$ and each $j \in J$. That concludes the proof of the theorem.

## 3 Approximate polynomial size extended formulation

In the previous section we showed in Corollary 3 that there can be no polynomial size extended formulation of $Q(J, M)$ whose vertices are exactly all $\alpha$-approximate solutions, for essentially any $\alpha$. However, in this section we show that there is a small extended formulation that contains all optimal schedules and some approximate schedules. Our formulation works even in the more general setting of unrelated machines, i.e., $R \| C_{\max }$, where the processing time of a job $j$ can depend on the machine $i$ that it is assigned to and for each such combination the input contains a value $p_{i j} \in \mathbb{N} \cup\{\infty\}$. In this setting the polytope $Q(J, M)$ defined exactly in the same as way as before, however, now a solution $x$ is optimal if $\sum_{j \in J} x_{i j} p_{i j} \leq$ opt for each machine $i \in M$.

Construction of the extended formulation. In the sequel let $T \geq$ opt. The intuition behind our construction is the following. We define a bipartite graph $G$ in which we have one vertex $v_{j}$ for each job $j \in J$ and $n$ vertices $w_{(i, 1)}, \ldots, w_{(i, n)}$ for each machine $i \in M$. Each vertex $w_{(i, \ell)}$ corresponds to a slot for machine $i$. We search for a matching that assigns each job vertex $v_{j}$ to some slot vertex $w_{(i, \ell)}$ and we allow this assignment if and only if $p_{i j} \leq T / \ell$, i.e., we introduce an edge $\left\{v_{j}, w_{(i, \ell)}\right\}$ if and only if $p_{i j} \leq T / \ell$. The intuition is that in opt for each machine $i$ there can be at most one job $j$ assigned to $i$ with $p_{i j} \in(T / 2, T]$, at most two jobs $j$ with $p_{i j} \in(T / 3, T]$ and more general at most $\ell$ jobs $j$ with $p_{i j} \in(T /(\ell+1), T]$. Hence, there is a matching in $G$ that corresponds to opt. On the other hand, in any matching the total processing time of the jobs on any machine $i$ is bounded by $T+T / 2+T / 3+\ldots+T / n=T \cdot H_{n}=O(T \log (n))$ which is at most opt $\cdot O(\log (n))$ if $T=O($ opt $)$ (e.g., set $T=$ opt or set $T$ to be the makespan found by a 2-approximation algorithm for $\left.R \| C_{\max }[12,18]\right)$.

We define an integral polytope that models this bipartite matching. For each $(i, \ell)$ let $J_{i \ell}$ be the subset of jobs $j$ whose vertex $v_{j}$ has an edge incident to $w_{(i, \ell)}$. For each edge $\left\{v_{j}, w_{(i, \ell)}\right\}$ we have a variable $y_{j i \ell}$ that intuitively indicates whether job $j$ is assigned to the slot $(i, \ell)$. On top of this, we
project the resulting polytope to the space of the variables $x$ of $Q(J, M)$ which yields $\bar{Q}(J, M)$.

$$
\begin{align*}
\sum_{j \in J_{i}} y_{j i \ell} & \leq 1 & & \text { for all } i \in M, \text { for all } \ell \in\{1, \ldots, n\},  \tag{10}\\
\sum_{i \in M} \sum_{\ell=1}^{n} y_{j i \ell} & =1 & & \text { for all } j \in J,  \tag{11}\\
\sum_{\ell: j \in J_{i \ell}} y_{j i \ell} & =x_{i j} & & \text { for all } i \in M, \text { for all } j \in J,  \tag{12}\\
x_{i j} & \geq 0 & & \text { for all } i \in M, \text { for all } j \in J,  \tag{13}\\
y_{i j \ell} & \geq 0 & & \text { for all } i \in M, \text { for all } j \in J, \text { for all } \ell \in\{1, \ldots, n\} . \tag{14}
\end{align*}
$$

Theorem 4. Given an instance $(J, M)$ of $R \| C_{\text {max }}$ there exists an extended formulation of size $O\left(n^{2} m\right)$ for an integral polytope $\bar{Q}(J, M) \subseteq Q(J, M)$ whose vertices correspond to all optimal solutions and some $O(\log n)$-approximate solutions to $(J, M)$.

Proof of Theorem 4. Consider an optimal schedule and let $x$ be its corresponding solution in $Q(J, M)$. For each machine $i \in M$, let $J_{i}(x)$ be the subset of job vertices $v_{j}$ in $G$ such that $x_{i j}=1$, that is, $J_{i}(x)=\left\{v_{j}: x_{i j}=1\right\}$, and consider $G_{i}$ the bipartite subgraph of $G$ that is induced by the vertices $J_{i}(x) \cup\left\{w_{(i, 1)}, \ldots, w_{(i, n)}\right\}$. We check that there exists a matching in $G_{i}$ that covers $J_{i}(x)$. Consider $W \subseteq J_{i}(x)$. There exists at least one job in $t \in W$ with processing time at most $T /|W|$, otherwise $x$ would exceed the makespan $T$ in machine $i$. Therefore, $\mathrm{job} t$ is connected to $w_{(i, \ell)}$ for every $\ell \leq|W|$, and its degree in $G_{i}$ is at least $|W|$. In particular, $W$ is connected to at least $|W|$ slots in the bipartite subgraph $G_{i}$. By Hall's theorem we conclude that exists a matching in $G_{i}$ covering every job in $J_{i}(x)$. For every job in $J_{i}(x)$ we define $y_{j i \ell}=1$ if $v_{j}$ is connected to $w_{(i, \ell)}$ in the matching, and $y_{j i \ell}=0$ otherwise. By construction the solution $(x, y)$ satisfies the constraints in the program above.

On the other hand, consider a vertex solution $(x, y)$. The integrality of $(x, y)$ comes from the fact that the linear program restricted to $y$ variables is a bipartite matching formulation. We bound the makespan of the schedule obtained from $x$. For each machine $i \in M$, we have

$$
\sum_{j \in J} p_{i j} x_{i j}=\sum_{j \in J} p_{i j} \sum_{\ell: j \in J_{i \ell}} y_{j i \ell}=\sum_{\ell=1}^{n} \sum_{j \in J_{i \ell}} p_{i j} y_{j i \ell} .
$$

For every $j \in J_{i \ell}$, we have that $p_{i j} \leq T / \ell$. This, together with constraint (10) allows us to upper bound the last summation above by

$$
\sum_{\ell=1}^{n} \frac{T}{\ell} \sum_{j \in J_{i \ell}} y_{j i \ell} \leq T \sum_{\ell=1}^{n} \frac{1}{\ell}=T \cdot H_{n}=O(T \log n) .
$$

## 4 Extension complexity of configuration-LP

An alternative way to formulate $P \| C_{\max }$ as an integer linear program is via a configuration integer program. For a given target makespan $T \geq 0$ we define the set of all configurations $\mathcal{C}(T)$ to be all subsets of jobs whose total processing time is at most $T$, i.e., $\mathcal{C}(T):=\left\{L \subseteq J: \sum_{j \in L} p_{j} \leq T\right\}$.

For each machine $i \in M$ and each configuration $C \in \mathcal{C}(T)$ we introduce a variable $y_{i C} \in\{0,1\}$ that models whether machine $i$ gets exactly the jobs in configuration $C$ assigned to it. Denote by $P_{\text {config }}(J, M, T)$ the convex hull of all solutions to the integer program below.

$$
\begin{array}{rll}
\sum_{i \in M} \sum_{C \in \mathcal{C}(T): j \in C} y_{i C} & =1 & \text { for all } j \in J, \\
\sum_{C \in \mathcal{C}(T)} y_{i C} & =1 & \text { for all } i \in M, \\
y_{i C} & \in\{0,1\} & \text { for all } i \in M, \text { for all } C \in \mathcal{C}(T) . \tag{17}
\end{array}
$$

For arbitrary values of $T$ the number of variables of $P_{\text {config }}(J, M, T)$ can be exponential in the input length. However, for constant $T$ there are only a polynomial number of variables (and constraints) since the number of possible configurations is bounded by $\binom{n}{T}=n^{O(T)}$. This raises the questions whether $P_{\text {config }}(J, M, T)$ admits a small extended formulation for such values of $T$. If $T=2$ and $p_{j} \in\{1,2\}$ for each $j \in J$ Theorem 1 implies a lower bound of $2^{\Omega(n / \log n)}$ since there is an easy projection of the configuration-LP to $P(J, M)$. We strengthen this to the case where $p_{j}=1$ for each $j \in J$ and to a lower bound of $2^{\Omega(n)}$.

Theorem 5. For every $n \in \mathbb{N}$ there is an instance $(J, M, T)$ of $P \| C_{\max }$ with $n$ machines and $O(n)$ jobs such that the polytope $P_{\text {config }}(J, M, T)$ has an extension complexity of $2^{\Omega(n)}$. It holds that $T=2$ and $p_{j}=1$ for each $j \in J$.

Proof of Theorem 5. For every $n \in \mathbb{N}$ we construct an instance of $(J, M, T)$ by $T=2, M$ contains $n$ machines, and $J$ contains $2 n$ jobs with $p_{j}=1$ for each $j \in J$. The set $\mathcal{C}(T)$ is thus the set of all pairs of jobs. We show that there is a linear map that projects each solution to $P_{\text {config }}(J, M, T)$ to a matching in a complete graph on $2 n$ vertices. Let $G=(V, E)$ be the complete graph on $2 n$ vertices, i.e., $G=K_{2 n}$. Consider the perfect matching polytope of $G$, given by

$$
\operatorname{PM}(G)=\operatorname{conv}\left(\left\{\chi_{M} \in \mathbb{R}^{E}: M \subseteq E \text { is a perfect matching }\right\}\right) .
$$

We define a linear map $f: P_{\text {config }}(J, M, T) \rightarrow \operatorname{PM}(G)$ as follows: for each edge $e=\{u, v\} \in E$ we define $f_{e}(y):=\sum_{i \in M} y_{i\{u, v\}}$. For each vertex $y$ of $P_{\text {config }}(J, M, T)$ it holds that $f(y) \in \operatorname{PM}(G)$ and therefore $f\left(P_{\text {config }}(J, M, T)\right) \subseteq \operatorname{PM}(G)$. On the other hand, let $x$ be a vertex of $\operatorname{PM}(G)$, i.e., $x$ represents a perfect matching $\left\{\left\{u_{1}, v_{1}\right\},\left\{u_{2}, v_{2}\right\}, \ldots,\left\{u_{n}, v_{n}\right\}\right\}$ in $G$. Then we can construct a vertex $y$ of $P_{\text {config }}(J, M, T)$ such that $f(y)=x$ as follows: assume that the machines are numbered $\{1, \ldots, n\}$. Then for each machine $i$ we define $y_{i\left\{u_{i}, v_{i}\right\}}=1$ and $y_{i C}=0$ for each $C \in \mathcal{C}(T) \backslash\left\{\left\{u_{i}, v_{i}\right\}\right\}$. Then $f(y)=x$. Rothvoss showed that the extension complexity of $\operatorname{PM}(G)$ is $2^{\Omega(n)}$ [16]. Our construction implies that $P_{\text {config }}(J, M, T)$ is an extended formulation for $\operatorname{PM}(G)$. Therefore, the extension complexity of $P_{\text {config }}(J, M, T)$ is also $2^{\Omega(n)}$.

## 5 An extended formulation with small parameterized size

In this section we consider an abstraction of the configuration integer program that exhibits low extension complexity parameterized by $T$. Instead of encoding configurations as subsets of jobs, we consider how many jobs in the configuration have a certain processing time $p$. That is, in this context a configuration is a multiset $C$ of $\left\{p_{j}: j \in J\right\}$, and let $m(p, C)$ be the multiplicity of $p$ in
$C$, namely, the number of times that $p$ appears in $C$. For every $p \in\left\{p_{j}: j \in J\right\}$, let $n_{p}$ be the number of jobs in $J$ with processing time equal to $p$. We denote by $\mathcal{C}_{*}(T)$ the set of configurations having total processing time at most $T$. The key fact about this encoding is that two or more machines can be scheduled in the same configuration and there are fewer solutions overall than for the configuration-LP. As before, we propose a formulation where for every combination of machine $i \in M$ and configuration $C \in \mathcal{C}_{*}(T)$ we have variable $y_{i C}$ indicating whether $i$ is scheduled according to $C$. That is,

$$
\begin{array}{rll}
\sum_{C \in \mathcal{C}_{*}(T)} y_{i C} & =1 & \text { for all } i \in M, \\
\sum_{i \in M} \sum_{C \in \mathcal{C}_{*}(T)} m(p, C) y_{i C} & =n_{p} & \text { for all } p \in\left\{p_{j}: j \in J\right\}, \\
y_{i C} \in\{0,1\} & \text { for all } i \in M, \text { for all } C \in \mathcal{C}_{*}(T) . \tag{20}
\end{array}
$$

Theorem 6. The extension complexity of the integer hull for the above formulation is $O(f(T) \cdot p o l y(n, m))$ for some function $f$.

We define a program deciding for each configuration $C \in \mathcal{C}_{*}(T)$ how many machines get configuration $C$ such that we have enough slots for all jobs, that is,

$$
\begin{align*}
& \sum_{C \in \mathcal{C}_{*}(T)} z_{C}=m,  \tag{21}\\
& \sum_{C \in \mathcal{C}_{*}(T)} m(p, C) z_{C}=n_{p}  \tag{22}\\
& z_{C} \in \mathbb{N} \quad \text { for all } p \in\left\{p_{j}: j \in J\right\},  \tag{23}\\
& \text { for all } C \in \mathcal{C}_{*}(T) .
\end{align*}
$$

We denote $P_{*}(J, M, T)$ the convex hull of the solutions to the integer program above. Since the number of processing times ranges in $[1, T]$, the total number of configurations in $\mathcal{C}_{*}(T)$ is upper bounded by $T^{T}$.

Lemma 2. There exists a function $h$ such that the number of vertices of the polytope $P_{*}(J, M, T)$ is upper bounded by $h(T)$.

Observe that the lemma above holds even when number of machines and jobs on the right hand side of the integer program above are not necessarily bounded by a function of $T$, which is in general the case. Now we can prove Theorem 6 via a Dantzig-Wolfe reformulation with a variable for each of the $h(T)$ vertices of $P_{*}(J, M, T)$. Since the size of $P_{*}(J, M, T)$ is bounded by $h(T)$, we have at most that many variables of this type. Then, we model the values of the variables $y_{i C}$ as the solution to a suitable transshipment problem.

Proof of Lemma 2. Let $w$ be a vertex of the polytope $P_{*}(J, M, T)$. We argue that there are at most $h(T)$ possibilities for $w$ for some suitable function $h$. For a value $g(T)$ to be defined later define a new point $w^{1}$ by setting $w_{C}^{1}:=w_{C}$ if $w_{C} \leq g(T)$ and $w_{C}^{1}:=0$ otherwise for each configuration
$C \in \mathcal{C}_{*}(T)$. Define $w^{2}=w-w^{1} \geq 0$, which is feasible for the following integer linear program

$$
\begin{array}{rlrl}
\sum_{C \in \mathcal{C}_{*}(T)} z_{C} & =m-\sum_{C \in \mathcal{C}_{*}(T)} w_{C}^{1}, & \\
\sum_{C \in \mathcal{C}_{*}(T)} m(p, C) z_{C} & =n_{p}-m(p, C) w_{C}^{1} & & \text { for all } p \in\left\{p_{j}: j \in J\right\}, \\
z_{C} & \in \mathbb{N} & & \text { for all } C \in \mathcal{C}_{*}(T) .
\end{array}
$$

One can easily show that $w^{2}$ is a vertex of the integer hull of the solutions to the program above, since otherwise $w$ is non-trivial convex combination of feasible solutions of $P_{*}(J, M, T)$. Let $\mathcal{C}_{*}^{\prime}(T) \subseteq$ $\mathcal{C}_{*}(T)$ denote the configurations in the support of $w^{2}$. We identify each configuration $C \in \mathcal{C}_{*}^{\prime}(T)$ with a vector which is the column in the matrix of the program above corresponding to $C$, i.e., the first entry of each such vector is a 1 and the other entries are given by the values $m(p, C)_{p \in\left\{p_{j}: j \in J\right\}}$. If the vectors in $\mathcal{C}_{*}^{\prime}(T)$ are linearly independent then the above IP has a unique solution. Hence, there is at most one vertex $w$ of $P_{*}(J, M, T)$ of the form $w=w^{1}+\hat{w}$ where $\hat{w}$ is a solution to the above IP. In the sequel we argue that it cannot be that the vectors in $\mathcal{C}_{*}^{\prime}(T)$ are linearly dependent if $g(T)$ is sufficiently large. Then the claim of the lemma follows since the vector $w^{1}$ satisfies that $w_{C}^{1} \in\{0, \ldots, g(T)\}$ for each $C \in \mathcal{C}_{*}(T)$ and the number of such vectors can be bounded by a value $h(T)$.

If the vectors in $\mathcal{C}_{*}^{\prime}(T)$ are not linearly independent then there exists a configuration $C \in \mathcal{C}_{*}^{\prime}(T)$ and a set of linearly independent vectors $C_{1}, \ldots, C_{k} \in \mathcal{C}_{*}^{\prime}(T)$ and values $\lambda_{1}, \ldots, \lambda_{k}$ such that $C=$ $\sum_{i=1}^{k} \lambda_{i} C_{i}$. Let $\Lambda$ denote the smallest value $\Lambda^{\prime} \in \mathbb{N}$ such that $\Lambda^{\prime} C=\sum_{i=1}^{k} \Lambda^{\prime} \lambda_{i} C_{i}$ such that $\Lambda^{\prime} \lambda_{i} \in \mathbb{Z}$ for each $i$. If $g(T) \geq \Lambda$ and $g(T) \geq \Lambda \lambda_{i}$ for each $i$ then we can write $w^{2}$ as the convex combination of two integral vectors and hence $w^{2}$ is not a vertex. To ensure the former we define $g(T)$ to be the maximum over all values $\Lambda$ and $\Lambda \lambda_{i}$ that we can obtain in this way, i.e., by selecting one $C \in \mathcal{C}_{*}(T)$, expressing it as a linear combination $C=\sum_{i=1}^{k} \lambda_{i} C_{i}$ of a set of linearly independent configurations $C_{1}, \ldots, C_{k} \in \mathcal{C}_{*}(T)$, and finding the smallest value $\Lambda^{\prime} \in \mathbb{N}$ such that $\Lambda^{\prime} \lambda_{i} \in \mathbb{Z}$. Note that the number of values $\Lambda^{\prime}$ obtained in this way is finite and hence $g(T)$ is well-defined (and finite).

Proof of Theorem 6. Let $V(J, M, T)$ be the set of vertices in the polytope $P_{*}(J, M, T)$. By Lemma 2, there exists a function $h$ such that the size $V(J, M, T)$ is bounded by $h(T)$. Since we can restrict the optimization problem to its set of vertices, we consider the linear program obtained by lifting the integer program above by the Dantzig-Wolfe reformulation using the set of vertices $V(J, M, T)$. For each vertex $v \in V(J, M, T)$, consider a variable $\lambda_{v}$ indicating whether we pick or not the vertex solution $v$. In addition, for each configuration we consider as before a variables $z_{C}$ indicating how many times the configuration $C$ is used. Finally, for each combination of machine $i \in M$ and configuration $C \in \mathcal{C}_{*}(T)$ we have a variable $y_{i C}$ indicating whether machine $i$ is scheduled according to $C$.

The idea behind the extended formulation is to first select a vertex $v \in V(J, M, T)$ by using the Dantzig-Wolfe reformulation in the variables $(\lambda, z)$, which provides for each configuration $C \in$ $\mathcal{C}_{*}(T)$ the number of times, $z_{C}$, that is used. Then we formulate a transportation problem between machines and configurations satisfying the offer $z_{C}$ for each configuration.

More specifically, consider a complete bipartite graph $G$ where we have one vertex $v_{i}$ for each machine $i \in M$, and we have a vertex $w_{C}$ for each configuration $C \in \mathcal{C}_{*}(T)$. For each vertex $w_{C}$ we have an offer $z_{C}$, and every vertex $v_{i}$ has a demand of 1 . The variable $y_{i C}$ indicates whether
the demand of the machine vertex $v_{i}$ is satisfied by the configuration vertex $w_{C}$. More specifically, consider

$$
\begin{align*}
& \sum_{v \in V(J, M, T)} \lambda_{v}=1,  \tag{24}\\
& \sum_{v \in V(J, M, T)} v \lambda_{v}=z,  \tag{25}\\
& \sum_{C \in \mathcal{C}_{*}(T)} z_{C}=m,  \tag{26}\\
& \sum_{C \in \mathcal{C}_{*}(T)} m(p, C) z_{C}=n_{p} \quad \text { for all } p \in\left\{p_{j}: j \in J\right\},  \tag{27}\\
& \sum_{i \in M} y_{i C}=z_{C} \quad \text { for all } C \in \mathcal{C}_{*}(T),  \tag{28}\\
& \sum_{C \in \mathcal{C}_{*}(T)} y_{i C}=1 \quad \text { for all } i \in M,  \tag{29}\\
& \lambda_{v} \geq 0  \tag{30}\\
& z_{C} \geq 0  \tag{31}\\
& y_{i C} \text { for all } v \in V \quad \text { for all } C \in \mathcal{C}_{*}(T),  \tag{32}\\
& \text { for all } i \in M, \text { for all } C \in \mathcal{C}_{*}(T) .
\end{align*}
$$

Observe that constraints (26) and (29) guarantees that the total demand equals the total offer in the transportation problem over $G$. It holds that a vertex $(\lambda, z, y)$ of the linear program above is integral. If not, suppose that $\lambda$ is fractional, otherwise the integrality of $\lambda$ implies the integrality of $z$ and in turns the integrality of $y$ since the transportation program in the graph $G$ given by constraints (28)-(29) is integral. If $\lambda$ is fractional, $z$ is a non-trivial convex combination of the vertices $\left\{v \in V(J, M, T): \lambda_{v}>0\right\}$, and each of these vertices is feasible for the constraints (26)-(27), which implies that $(\lambda, z)$ is a convex combination of the vectors $\left\{\left(e_{v}, v\right): \lambda_{v}>0\right\}$, where $e_{v} \in$ $\{0,1\}^{V(J, M, T)}$ is the canonical vector that is 1 for entry $v$ and zero otherwise. For each $v$ such that $\lambda_{v}>0$, the constraints (28)-(29) solve a transportation problem between the machine vertices $\left\{v_{i}: i \in M\right\}$ and the configurations vertices $\left\{w_{C}: C \in \mathcal{C}_{*}(T)\right\}$, where the offer for $w_{C}$ is equal to $z_{C}$. The vertices with positive offer are given by $\mathcal{C}_{v}=\left\{w_{C}: v_{C}>0\right\}$, and then any solution to this problem is a convex combination of the integral solutions to the transportation problem over $G$. Since every solution is supported over $M \times \mathcal{C}_{v}$, and therefore we contradicted the fact that $(\lambda, z, y)$ is a vertex of the polytope.

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