Electronic Journal of Statistics
Vol. 14 (2020) 2510–2528
ISSN: 1935-7524
https://doi.org/10.1214/20-EJS1719

Stein hypothesis and screening effect for covariances with compact support

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Abstract: In spatial statistics, the screening effect historically refers to the situation when the observations located far from the predictand receive a small (ideally, zero) kriging weight. Several factors play a crucial role in this phenomenon: among them, the spatial design, the dimension of the spatial domain where the observations are defined, the mean-square properties of the underlying random field and its covariance function or, equivalently, its spectral density.

The *tour de force* by Michael L. Stein provides a formal definition of the screening effect and puts emphasis on the Matérn covariance function, advocated as a good covariance function to yield such an effect. Yet, it is often recommended not to use covariance functions with a compact support. This paper shows that some classes of covariance functions being compactly supported allow for a screening effect according to Stein's definition, in both regular and irregular settings of the spatial design. Further, numerical experiments suggest that the screening effect under a class of compactly supported covariance functions is even stronger than the screening effect under a Matérn model.

Keywords and phrases: Compact Support, covariance function, generalized Wendland, Matérn, screening effect, spatial prediction.

Received November 2019.

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1. Introduction

Optimal unbiased linear prediction (kriging) is widely used in spatial statistics to interpolate point observations of a mean-square continuous random field. The notion of "screening effect" is used to describe a situation where the interpolant depends mostly on those observations that are located nearest to the predictand (Stein, 2002). The phenomenon has been of interest to geostatisticians for decades (Matheron, 1963, 1965, 1971; Chilès and Delfiner, 2012) since it allows reducing considerably the computational burden associated with the kriging predictor when handling large data sets.

1.1. Context and state of the art

The screening effect historically refers to the situation when the observations located far from the predictand receive a zero kriging weight. An early formalization is proposed by Matheron (Matheron, 1963, 1965), who examined the case when observations along a closed contour around the predictand perfectly screen out the influence of observations located outside this contour. Such a definition corresponds to the well-known Markov property in a one-dimensional space, which occurs when kriging a stationary random field with an exponential covariance or an intrinsic random field with a linear variogram (Matheron, 1971; Chilès and Delfiner, 2012). In multi-dimensional spaces, however, it requires defining a continuous version of kriging, with uncountably many observations located on a contour enclosing the predictand (Matheron, 1963, 1965). When dealing with finitely many observations, the evidence of screening effect has been mostly empirical, as well as a justification for the use of a moving neighborhood in the practice of kriging (David, 1976; Rivoirard, 1987; Isaaks and Srivastava, 1989). In the multivariate context, screening effects also arise with specific models of the joint correlation structure of the random fields being predicted (Rivoirard, 2004; Subramanyam and Pandalai, 2008).

Stein (1999, 2002, 2011, 2015) provides a slightly different formalization of the screening effect, under an infill asymptotics approximation. The motivation of this asymptotic approach to screening effect is the extreme difficulty in obtaining useful and general results for any fixed set of observations. Several examples in Stein (2011) demonstrate the complexity of the problem, which depends on the spatial design (how to locate the observation points), the dimension of the Euclidean space where the spatial domain is embedded, the covariance function attached to a Gaussian spatial random field (or, equivalently, its spectral density) and the mean-square differentiability in all directions of the spatial random field. A recent discussion about screening effect has been provided by Bao et al. (2020).

Specifically, let $\{Z(x): \; x \in D \subset \mathbb{R}^m\}$ be a mean-square continuous, zero mean and weakly stationary Gaussian random field with covariance function $K : \mathbb{R}^m \to \mathbb{R}$ having a spectral density

$$f(\omega) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}\omega\cdot x} K(x) \mathrm{d}x, \tag{1.1}$$

where $\cdot$ denotes the inner product between two commensurate vectors and $\mathfrak{i}^2 = -1$. Throughout, we are interested in predicting Z at $x = 0 \in \mathbb{R}^m$. Our notation largely follows Stein (2002): we let $Z(S)$ be a vector of observations at a nonempty set $S \subset \mathbb{R}^m$. Let F_ϵ, N_ϵ be two sets indexed by $\epsilon > 0$ such that N_ϵ contains the nearest observations to the predictand, and F_ϵ more distant observations. Let $e(S)$ be the error of the simple kriging interpolator to predict Z at $x = 0$ based on $Z(S)$. Stein (2002) says that N_ϵ *asymptotically screens out* F_ϵ when

$$\lim_{\epsilon \downarrow 0} \frac{\mathbb{E}e(N_\epsilon \cup F_\epsilon)^2}{\mathbb{E}e(N_\epsilon)^2} = 1. \tag{1.2}$$

Apparently, the configuration of points has a non-negligible impact on whether condition (1.2) happens. Stein (2002) shows that, for some $x_o \in \mathbb{R}^m$ not in the integer lattice, if $F_\epsilon = \{\epsilon(x_o + j)\}$, for $j \in \mathbb{Z}^m$ and if N_ϵ is the restriction of F_ϵ to some fixed region with 0 in its interior, then a sufficient condition for (1.2) to hold is that the spectrum f varies regularly at infinity (Bingham et al., 1987) in every direction with a common index of variation. This last aspect has been then constructively criticized in Stein (2011) when referring to space-time covariance models that exhibit different levels of differentiability in space and time. For the remainder of the paper, we call the above setting a *regular* asymptotics scheme, to distinguish formally from the *irregular* setting proposed by Stein (2011): for $x_1, \ldots, x_n$ being distinct nonzero elements of $\mathbb{R}^m$, $y_1, \ldots, y_N$ distinct elements of $\mathbb{R}^m$ and $y_0 \in \mathbb{R}^m$ being nonzero, we have $N_\epsilon = \epsilon x_1, \ldots, \epsilon x_n$ and $F_\epsilon = y_0 + \epsilon y_1, \ldots, y_0 + \epsilon y_N$. Stein (2011) starts from a reasonable condition on f: for every $R < \infty$,

$$\lim_{\|\omega\| \to \infty} \sup_{\|\tau\| < R} \left| \frac{f(\omega + \tau)}{f(\omega)} - 1 \right| = 0. \tag{1.3}$$

Throughout, we refer to Stein Hypothesis (`SH` throughout) as being verified when condition (1.3) is sufficient for (1.2) to happen under some mild additional conditions on f and N_ϵ. Indeed, Stein (2011) shows that `SH` is verified on $m = 1$ and $m = 2$ for mean-square continuous but non-differentiable random fields, under some specific design on N_ϵ.

Stein (2002, 2011) provides a wealth of examples showing when the screening effect is likely to happen under regular or irregular settings. Apparently, the Matérn model (Stein, 1999, see below for details) is a good candidate for both settings. Covariance models being compactly supported on the unit ball of $\mathbb{R}^m$ typically exhibit a lack of smoothness at ± 1 and are purported as negative examples for the screening effect to happen. The triangular and spherical models (Chilès and Delfiner, 2012) are prominent examples of covariance models with compact support in $\mathbb{R}^m$ for $m = 1$ and $m = 3$, respectively.

1.2. Our contribution

This paper addresses more comprehensively the problem of compact support within the context of both regular and irregular settings for screening effects.

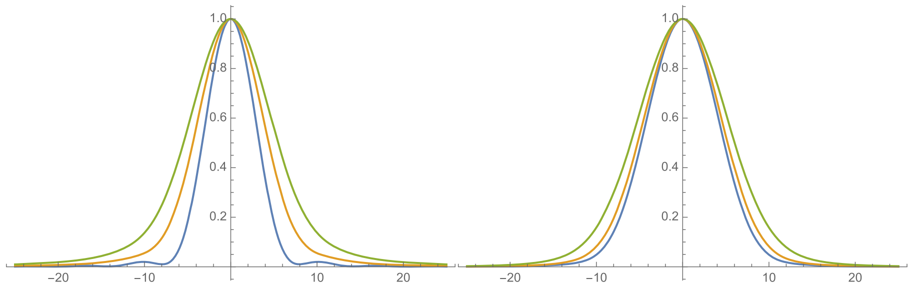

FIG 1. *Left: spectral density associated with the function $H_{\mu,\nu,2}$ in Equation (3.12) while fixing $\nu = 0$ and for increasing values of μ: $\mu = 1$ (blue line), $\mu = 2$ (orange) and $\mu = 3$ (green). Right: spectral density associated with the function $H_{\mu,\nu,2}$ in Equation (3.12) while fixing $\nu = 1$ and for increasing values of μ: $\mu = 2.5$ (blue line), $\mu = 3$ (orange) and $\mu = 4$ (green).*

We make special emphasis on the Generalized Wendland class (Bevilacqua et al., 2019) of compactly supported covariance functions, as well as on the Buhmann class (Buhmann, 2000; Zastavnyi, 2002). Just like the Matérn class of covariance functions, these classes allow for a continuous parameterization of smoothness of the underlying Gaussian random field, being additionally compactly supported.

Stein (1999) argues that the triangular model of covariances with compact support (being a special case of Generalized Wendland function) has poor performance in terms of best linear prediction, when $m = 1$. He shows that the poor behavior of this model is due to the tails of the spectral density, which oscillates away from the origin.

Figure 1 illustrates the situation. On the left-hand side we depict the spectral density of the Askey function (details are given in subsequent sections), with the triangular model (the blue line) being a special case of it. The orange and green lines illustrate how the behavior of the spectrum becomes more regular when a single parameter is changed. On the right-hand side, we depict the spectrum associated with the Generalized Wendland functions (details in subsequent section) when fixing the smoothing parameter so that the associated Gaussian random field is once mean-square differentiable. These examples motivate a deeper inspection of the properties of Generalized Wendland functions: given the very promising results obtained for the Generalized Wendland model under infill asymptotics (Bevilacqua et al., 2019), it makes sense to study conditions for regular and irregular screening effects to hold for the Generalized Wendland and Buhmann models. Specifically, we consider both classes as parametric families of compactly supported covariance functions. We then show under which restrictions on those parameters Stein's conditions (for regular or irregular settings) are met.

The plan of the paper is the following. Section 2 provides the necessary mathematical background and describes Stein's conditions under regular and irregular infill asymptotics schemes. Section 3 deals with theoretical results involving screening effect under Generalized Wendland and Buhmann classes of covariance functions. Section 4 provides an extensive simulation study that supports our theoretical findings. A short section concludes the paper. Mathematical proofs are deferred to the Appendix.

2. Mathematical background

2.1. Compactly supported covariance functions

For the remainder of the paper, m denotes a positive integer. A real-valued function $K : \mathbb{R}^m \to \mathbb{R}$ is positive semidefinite if, for any finite collection $\{x_k\}_{k=1}^N \subset \mathbb{R}^m$ and constants $\{c_k\}_{k=1}^N \subset \mathbb{R}$, we have $\sum_{k=1}^N \sum_{h=1}^N c_k K(x_k - x_h) c_h \geq 0$. There is a one-to-one correspondence between positive semidefinite functions and the covariance functions of Gaussian random fields in $\mathbb{R}^m$. The function K is called isotropic when $K(x) = \widetilde{K}(\|x\|)$ for some function $\widetilde{K}$ defined on $[0, \infty)$, and where $\|\cdot\|$ denotes the Euclidean norm. The function $\widetilde{K}$ is called the radial part of K, and we shall be ambiguous when calling $\widetilde{K}$ a covariance function. Being the Fourier pair of K, the spectral density f defined according to Equation (1.1) is also isotropic (Daley and Porcu, 2014) and is related to $\widetilde{K}$ through the following: $f(\omega) = \widetilde{f}_m(\|\omega\|)$, for $\omega \in \mathbb{R}^m$. Furthermore,

$$\widetilde{f}_m(\|\omega\|) = \frac{\|\omega\|^{1-m/2}}{(2\pi)^m} \int_0^\infty u^{m/2} J_{m/2-1}(u\|\omega\|) \widetilde{K}(u) \mathrm{d}u, \qquad \omega \in \mathbb{R}^m, \tag{2.4}$$

with J_m being a modified Bessel function (Abramowitz and Stegun, 1970). Throughout, we use r for $\|x\|$ and z for $\|\omega\|$. For details about isotropic covariance functions and spectral densities, the reader is referred to Daley and Porcu (2014), with the references therein.

The isotropic Matérn covariance model, $\mathcal{M}_\nu$, is defined as (Stein, 1999)

$$\mathcal{M}_\nu(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} r^\nu \mathcal{K}_\nu(r), \qquad r \geq 0. \tag{2.5}$$

$\mathcal{M}_\nu(r)$ is positive semidefinite on $\mathbb{R}^m$ (for all m) for any positive ν (Stein, 1999). Here, $\mathcal{K}_\nu$ is a modified Bessel function of the second kind of order ν. The parameter ν characterizes the differentiability at the origin and, as a consequence, the differentiability of the sample paths of a Gaussian random field with Matérn covariance function. In particular for a positive integer k, the sample paths are k times differentiable, in any direction, if and only if $\nu > k$.

When $\nu = 1/2 + k$ and k is a nonnegative integer, the Matérn function simplifies to the product of a negative exponential with a polynomial of degree k, and for ν tending to infinity, a rescaled version of the Matérn converges to a squared exponential model that is infinitely differentiable at the origin. Thus, the Matérn function allows for a continuous parameterization of its associated Gaussian random field in terms of smoothness.

The isotropic spectral density associated with the Matérn function, $\widehat{\mathcal{M}}_{\nu,m}$, has expression

$$\widehat{\mathcal{M}}_{\nu,m}(z) = \frac{\Gamma(\nu+m/2)}{\pi^{d/2}\Gamma(\nu)}(1+z^2)^{-\nu-m/2}, \qquad z \geq 0. \tag{2.6}$$

Covariance functions with compact support are identically zero outside a ball of $\mathbb{R}^m$ with a given radius. This paper works with functions being compactly supported on a ball of $\mathbb{R}^m$ with unit radius, without loss of generality. We now define the parametric classes that are used for the results coming subsequently. Let μ, ν be strictly positive parameters. We define the Generalized Wendland functions (Gneiting, 2002; Bevilacqua et al., 2019), denoted by $\psi_{\mu,\nu}$, through the identity

$$\psi_{\mu,\nu}(r) := \begin{cases} \frac{1}{B(2\nu,\mu+1)} \int_r^1 u(u^2-r^2)^{\nu-1}(1-u)^{\mu}\,\mathrm{d}u, & 0 \leq r < 1, \\ 0, & r \geq 1, \end{cases} \tag{2.7}$$

with B denoting the Beta function. We also define

$$\psi_{\mu,0}(r) := (1-r)_+^{\mu}, \qquad r \geq 0 \tag{2.8}$$

which is known as the Askey function (Askey, 1973), and where $(x)_+ = 0$ if $x > 0$, and 0 elsewhere. The special case $\psi_{1,0}$ is known as the triangular model (Stein, 1999; Chilès and Delfiner, 2012).

Let $G_{\mu,\nu,m} : [0,\infty) \to \mathbb{R}$ be the function defined through (Zastavnyi, 2006)

$$\begin{aligned} G_{\mu,\nu,m}(z) \quad = \quad & D(\mu,\nu,m) \times \\ & {}_1F_2\left(\frac{m-1}{2}+\nu; \frac{m-1}{2}+\nu+\frac{\mu}{2}, \frac{m-1}{2}+\nu+\frac{\mu+1}{2}; -\frac{z^2}{4}\right), \end{aligned} \tag{2.9}$$

$z \geq 0$, where $D(\mu,\nu,m)$ is a strictly positive constant and ${}_1F_2$ is a hypergeometric function. $G_{\mu,\nu,m}$ has to be nonnegative and integrable in $\mathbb{R}^m$ to ensure that the related Fourier pair, obtained through (2.4), is the radial part of a positive semidefinite function. (Zastavnyi, 2002, Theorem 11) shows that, for $m = 1$ and $\nu \geq 1$, this happens if and only if $\mu \geq \nu$. For $m \geq 2$ and if $\nu > 1/2$, it happens if and only if $\mu \geq (m-1)/2 + \nu$, and the reader is referred to Zastavnyi (2000) as well as to Remark 11 in Zastavnyi and Trigub (2002). In particular, Zastavnyi (2002) finds that $G_{\mu,\nu,m}$ is the radial Fourier transform $(2\pi)^m \widetilde{f}_m$ in (2.4) of the function

$$h_{\mu,\nu}(r) := \int_r^1 (2u-r) g_{\mu,\nu}(u) g_{\mu,\nu}(u-r)\,\mathrm{d}u,\ r < 1;\ \ h_{\mu,\nu}(r) := 0,\ r \geq 1,$$

where $g_{\mu,\nu}(u) := u^{\mu-1}(1-u^2)^{\nu-1}$, $u \in (0,1)$, for $\mu, \nu > 0$. The fact that (Zastavnyi, 2002)

$$\frac{h_{\mu,\nu+1}(r)}{h_{\mu,\nu+1}(0)} = \frac{\psi_{\mu,\nu}(r)}{\psi_{\mu,\nu}(0)}, \qquad r \geq 0,$$

Table 1

Special cases of Wendland functions $\psi_{\mu,k}$ and Matérn functions $\mathcal{M}_\nu(r)$. $SP(k)$ means that the sample paths of the associated Gaussian random field are k times differentiable. Reported from Bevilacqua et al. (2019)

k	$\psi_{\mu,k}(r)$	ν	$\mathcal{M}_\nu(r)$	$SP(k)$
0	$(1-r)_+^{\mu}$	0.5	e^{-r}	0
1	$(1-r)_+^{\mu+1}(1+r(\mu+1))$	1.5	$e^{-r}(1+r)$	1
2	$(1-r)_+^{\mu+2}(1+r(\mu+2)+r^2(\mu^2+4\mu+3)\frac{1}{3})$	2.5	$e^{-r}(1+r+\frac{r^2}{3})$	2
3	$(1-r)_+^{\mu+3}(1+r(\mu+3)+r^2(2\mu^2+12\mu+15)\frac{1}{5}$ $+r^3(\mu^3+9\mu^2+23\mu+15)\frac{1}{15})$	3.5	$e^{-r}(1+\frac{r}{2}+r^2\frac{6}{15}+\frac{r^3}{15})$	3

shows that the Fourier pair as in (2.4) associated with the Generalized Wendland class $\psi_{\mu,\nu}$ is positively proportional to $G_{\mu,\nu+1,m}$. This implies that $\psi_{\mu,\nu}$ is the radial part of a positive semidefinite function in $\mathbb{R}^m$, $m \geq 1$, if and only if $\mu \geq (m+1)/2+\nu$, for $\nu \geq 0$. For k a nonnegative integer, the functions $\psi_{\mu,k}$ are known as Wendland functions (Wendland, 1995). Some special cases are reported in Table 1, which also compares their behavior at the origin (for fixed values of k) in comparison with the Matérn class (for fixed values of μ).

We finish this exposition with a general class of compactly supported covariance functions. Let $\delta, \mu, \nu > 0$ and $\alpha \in \mathbb{R}$. We refer to Buhmann functions as the parametric class (Buhmann, 2000; Zastavnyi, 2006) defined by

$$\varphi_{\delta,\mu,\nu,\alpha}(r) := \begin{cases} \int_r^1 (s^2-r^2)^{\nu-1}(1-s^\delta)^{\mu-1}s^{\alpha-2\nu+1}\,\mathrm{d}s, & r<1 \\ 0, & r \geq 1. \end{cases} \tag{2.10}$$

An extensive inspection of the properties of this class can be found in Zastavnyi (2006) and we report here the essential facts. The class $\varphi_{\delta,\mu,\nu,\alpha}$ includes a wealth of interesting special cases. For instance, $\varphi_{\delta,\mu,1,\delta}(r)$ is proportional to $(1-r^\delta)_+^\mu$, which implies that $\varphi_{1,\mu,1,1}$ is proportional to the Askey functions $\psi_{\mu,0}$ (Askey, 1973). Thus, $\varphi_{1,1,1,1}(r) = \psi_{1,0}$ is the triangular model. Conditions for the radial Fourier transform of $\varphi_{\delta,\mu,\nu,\alpha}$ to be positive for general parameters δ, μ, ν and α are provided by Zastavnyi (2006) and are not reported here to favor a simplified exposition. Using again the arguments in Zastavnyi (2006) we have

$$\psi_{\mu,\nu}(r) = \frac{\varphi_{1,\mu+1,\nu,2\nu}(r)}{\varphi_{1,\mu+1,\nu,2\nu}(0)} = \frac{\varphi_{1,\mu,\nu+1,2\nu+1}(r)}{\varphi_{1,\mu,\nu+1,2\nu+1}(0)}, \qquad r \geq 0 \tag{2.11}$$

for $\mu, \nu > 0$. Arguments in Porcu et al. (2017) show that the Wu functions (Wu, 1995) and consequently the spherical model are special cases of the Buhmann class.

2.2. Stein's spectral conditions

This part of the presentation avoids mathematical obfuscation and will instead focus on intuitive interpretation of facts. For sharp statements, the reader is referred to Stein (2002, 2011).

We need some further notation to proceed. For nonnegative-valued functions a and b defined on a common domain D, write $a(x) \ll b(x)$ if there exists a finite $C > 0$ such that $a(x) \leq Cb(x)$ for all $x \in D$. Write $a(x) \asymp b(x)$ if $a(x) \ll b(x)$ and $b(x) \ll a(x)$.

We start with what has been termed regular setting for asymptotic screening. For this case, Stein (2002) elucidates the following assumptions. For a zero mean Gaussian random field in $\mathbb{R}^m$ with spectral density f,

(A.1) $f \asymp 1$ on bounded subsets of $\mathbb{R}^m$;

(A.2) $f(\omega) \asymp g(\omega)$ on $\mathbb{R}^m$, $f(\omega) \sim g(\omega)$ as $\|\omega\| \to \infty$ and $g(\omega) = \widetilde{g}(\|\omega\|)\theta(\omega/\|\omega\|)$, $\omega \in \mathbb{R}^m$, for some functions $\widetilde{g} : [0, \infty) \to \mathbb{R}$ and θ being defined on the unit ball of $\mathbb{R}^m$, such that $\theta \asymp 1$ and, for some $a > m$, $\widetilde{g}(z) = z^{-a}L(z)$ when $z > 1$ and $\widetilde{g}(z) = L(1)$ when $0 \leq z \leq 1$. Here, L is slowly varying at infinity, *i.e.*, L is positive on $[0, \infty)$ and for all $r > 0$, $L(rz)/L(z) \to 1$ as $z \to \infty$.

Stein (2002) shows that if f obeys both **(A.1)** and **(A.2)**, then N_ϵ asymptotically screens out F_ϵ, where both sets are specified in Theorem 1 of Stein (2002).

When working under the irregular asymptotics setting, it is extremely difficult to establish general results. Stein (2011) conjectures that, if f satisfies (1.3) and for $k = 1, ..., n$, if all mean-square derivatives of Z at the origin in the direction x_k can be predicted based on $Z(N_\epsilon)$ with mean-squared error tending to 0 as $\epsilon \downarrow 0$, then N_ϵ asymptotically screens out F_ϵ, where again we refer to Stein (2011) for an accurate assertion about the sets N_ϵ and F_ϵ.

Stein (2011) goes further and proves the conjecture in $\mathbb{R}$ and $\mathbb{R}^2$ for mean-square continuous but non-differentiable random fields, which simplifies things considerably. In $\mathbb{R}$, it is additionally required that $f(\omega) \asymp (1 + \|\omega\|)^{-a-1}$, for some $a \in (0, 2)$.

3. Results

We start with two theoretical results. The former is rather general and does not depend on any parametric form of the isotropic spectral density. The latter deals instead with spectral densities associated with Generalized Wendland functions.

Theorem 3.1. *Let m be a positive integer. Let the function $G : [0, \infty) \to \mathbb{R}$ be continuous and positive, with $G(z) \sim Cz^{-\gamma}$ as $z \to +\infty$, where $C > 0$, $\gamma > m$. Let $f(\omega) := G(\|\omega\|)$, $\omega \in \mathbb{R}^m$. Then, the following assertions are true:*

1. *f satisfies Stein's conditions **(A.1)** and **(A.2)**, with $L \equiv 1$, $\theta \equiv C$ and $a = \gamma$;*
2. *f satisfies Stein's condition in Equation (1.3).*

Theorem 3.2. *Let m be a positive integer. Let $G_{\mu,\nu,m}$ be the family of functions defined through Equation (2.9) and $f(\omega) := G_{\mu,\nu,m}(\|\omega\|)$, $\omega \in \mathbb{R}^m$. If, either:*

(i) *$m = 1$, $1/2 < \nu < 1$ and $\mu \geq 1$ or $\nu \geq 1$ and $\mu > \nu$;*
(ii) *$m \geq 2$, $\nu > 1/2$ and $\mu > (m-1)/2 + \nu$,*

then, the following assertions are true:

1. *f satisfies Stein's conditions **(A.1)** and **(A.2)**, with $L \equiv 1$ and $a = m + 2\nu - 1 > m$;*
2. *f satisfies Stein's condition in Equation (1.3).*

Some comments are in order. Call $H_{\mu,\nu,m}$ the radially symmetric spectral density associated with the Generalized Wendland functions as defined in Equation (2.7). Using the expression of the spectral density $G_{\mu,\nu,m}$ in Equation (2.9) in concert with arguments in Zastavnyi (2002), one can show that hypergeometric functions occur:

$$H_{\mu,\nu,m}(z) = C(\mu,\nu,m) \times {}_1F_2\left(\frac{m+1}{2}+\nu; \frac{m+1}{2}+\nu+\frac{\mu}{2}, \frac{m+1}{2}+\nu+\frac{\mu+1}{2}; -\frac{z^2}{4}\right), \tag{3.12}$$

$z \geq 0$, where $C(\mu,\nu,m)$ is a strictly positive constant. This implies the following:

(a) According to Conditions **(A.1)** and **(A.2)** in concert with Theorem 1 in Stein (2002), we deduce that the Generalized Wendland model allows for a regular asymptotic screening effect. The condition is that $\mu > (m+1)/2 + \nu$. Observe that this condition is not restrictive: $\mu \geq (m+1)/2 + \nu$ is already required for $\psi_{\mu,\nu}$ to be positive semidefinite in $\mathbb{R}^m$.

(b) The irregular setting for asymptotic screening effect is more intricate. When $m = 1$ and for non-differentiable fields, Theorem 1 in Stein (2011) in concert with our Theorem 3.2 explains that the Askey model $\psi_{\mu,0}(r)$ allows for irregular screening effect provided that $\mu > 1$. This finding does not contradict Stein's example (Stein, 2002) on the triangular model ($\mu = 1$). For $m = 2$, we can again make use of Theorem 2 in Stein (2011) to deduce that the Askey model allows for screening provided that $\mu > 3/2$.

(c) The Generalized Wendland model satisfies Stein's condition (1.3) that is the crux to verify `SH`.

(d) For the limit case $\mu = (m-1)/2 + \nu$ in Theorem 3.2, Condition (1.3) is not true. This can be verified by letting

$$t_n = \frac{\pi}{2}\left(\mu + \frac{m-1}{2} + \nu\right) + 2\pi n - \frac{\pi}{2}, \qquad n \in \mathbb{N}.$$

These facts, in concert with (A.15) below, allow deducing that, for $s > 0$, $\sin s \neq 0$,

$$\frac{G_{\mu,\nu,m}(t_n + s)}{G_{\mu,\nu,m}(t_n)} \to C \sin s + 1 \neq 1, \qquad \text{when} \quad n \to \infty,$$

with $C = 2^{-(m-3)/2-\nu}$.

A constructive proof of Theorem 3.2 is provided in the Appendix.

Our findings are now completed by reporting similar results for the Buhmann functions $\varphi_{\delta,\mu,\nu,\alpha}$ in Equation (2.10).

Theorem 3.3. *Let m be a positive integer. Let δ,μ,ν,α be strictly positive. Let $\mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})$ be the radial Fourier transform $(2\pi)^m\widetilde{f_m}$ in (2.4) of the Buhmann function $\varphi_{\delta,\mu,\nu,\alpha}$ as defined in Equation (2.10) and $f(\omega) := \mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(||\omega||)$, $\omega \in \mathbb{R}^m$. Let $\mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(z)$ be positive for all positive z. If, either,*

(i_1) *$\mu+\nu > \alpha + (m+1)/2$ and $\alpha < 2\nu$, or*
(i_2) *$\alpha = 2\nu$, $\mu > 1$, $\delta < 2$ and $\mu > (m+1)/2 + \delta + \nu$,*

then, the following assertions are true:

1. *f satisfies Stein's conditions **(A.1)** and **(A.2)**, with $L \equiv 1$ and $a = \gamma > m$, where $\gamma = m+\alpha$ for case (i_1), and $\gamma = m+\alpha+\delta$ for case (i_2);*
2. *f satisfies Stein's condition in Equation (1.3).*

4. Numerical examples

This section explores, through numerical examples, the strength of the screening effect when predicting a Gaussian random field using a Generalized Wendland covariance function $\psi_{\mu,\nu}(\cdot/\beta)$, where $\beta > 0$ is a compact support. A comparison with the screening effect under the Matérn model is also provided.

The simulation scenario we consider is similar to the one proposed in Stein (2002), with $m = 2$. Specifically, we first define $A_n = \{-n, -n+1, \dots, n-1, n\}^2$ and we compare the predictions of $Z(0.5\delta, 0.5\delta)$ based on observing Z at δA_n, with predictions based on observing Z at $\delta\mathbb{Z}^2$. Let $\mathbb{E}e(x,y)^2$ be the mean-squared error associated with the kriging predictor of $Z(x)$ using y. We consider how the mean-squared error of the simple kriging predictor of $Z(0.5\delta, 0.5\delta)$ behaves as a function of n, δ, μ and ν. Let us define

$$R_{\psi_{\mu,\nu,\beta}}(n,\delta) = \frac{\mathbb{E}e(\delta(0.5,0.5),\delta A_n)^2}{\mathbb{E}e(\delta(0.5,0.5),\delta\mathbb{Z}^2)^2} - 1.$$

Clearly, $R_{\psi_{\mu,\nu,\beta}}$ measures the strength of the screening effect under the Generalized Wendland model $\psi_{\mu,\nu}(\cdot/\beta)$. Specifically, when $R_{\psi_{\mu,\nu,\beta}}$ approaches zero, then the screening effect is stronger. Our numerical results suggest that replacing $\mathbb{E}e(\delta(0.5,0.5),\delta\mathbb{Z}^2)^2$ by $\mathbb{E}e(\delta(0.5,0.5),\delta A_{40})^2$ in $R_{\psi_{\mu,\nu,\beta}}(n,\delta)$ provides a good approximation to $R_{\psi_{\mu,\nu,\beta}}(n,\delta)$ for the values of n, ν and μ considered here. Figure 2 depicts our spatial setting: in particular δA_n for $\delta = 0.01$ and $n = 1, 4, 8, 12, 40$.

We consider a Generalized Wendland function setting $\sigma^2 = 1$, $\nu = 0, 0.5, 2, 15$, $\mu = (m+1)/2 + \nu + x$ with $x = 0, 0.5, 2, 20$ and $m = 2$. We note that the parameter ν characterizes the differentiability at the origin and, as a consequence, the mean-square differentiability of a Gaussian random field having Generalized Wendland covariance function. In particular, for a positive integer k, the random field is k times differentiable, if and only if $\nu > k - 0.5$.

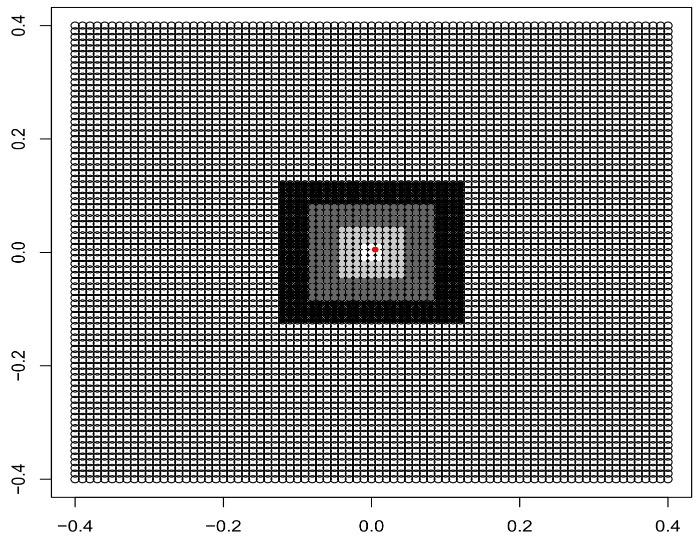

Fig 2. *Spatial setting for the numerical experiment. The outer to inner boxes are, respectively, δA_{40}, δA_{12}, δA_8, δA_4, δA_1 with $\delta = 0.01$. The red point ($\delta(0.5, 0.5)$) is the target point.*

As a comparison, we compute $R_{\mathcal{M}_{\nu,\alpha}}(n, \delta)$ for the Matérn covariance model $\mathcal{M}_\nu(\cdot/\alpha)$, $\alpha > 0$, setting $\nu = 0.5, 1, 2.5, 15.5$. The parameter ν characterizes the differentiability at the origin and, as a consequence, the mean-square differentiability of the random field. In particular for a positive integer k, the random field is k times differentiable, if and only if $\nu > k$.

Note that the parameters of the Generalized Wendland and Matérn models are chosen such that the associated random fields are $0, 2, 15$ times mean-square differentiable, respectively.

Finally, we fix the compact support of the Generalized Wendland function $\beta = 10\delta$ and the scale parameter α of the Matérn model is fixed such that the practical range is equal to 10δ. This choice has the effect of approximately fixing the spatial dependence proportional to the size of δA_{40}.

Figure 3 (second row) depicts Generalized Wendland correlation models with $\nu = 0, 0.5, 2$, $\mu = (m + 1)/2 + \nu + x$ with $x = 0, 0.5, 2, 20$ and compact support equal to 0.1. The third row of the Figure depicts a Matérn correlation model with $\kappa = 0.5, 1, 2.5$ and practical range equal to 0.1.

Table 2 shows $R_{\psi_{1.5+\nu+x,\nu,\beta}}(n, 0.01)$ for $x = 0, 0.5, 2, 20$, $\nu = 0, 0.5, 2, 15$ and $n = 1, \dots, 12$ with $\beta = 0.1$ and Table 3 shows $R_{\mathcal{M}_{\nu,\alpha}}(n, 0.01)$ for $\nu = 0.5, 1, 2.5, 15.5$ and $n = 1, \dots, 12$ and α such that the practical range is 0.1.

Some comments are in order. For a given level of differentiability, the screening effect is stronger under the Generalized Wendland than under the Matérn model. In particular, for the Generalized Wendland model, the screening effect increases when the μ parameter increases. We do not have any theoretical justification for this fact, which looks a bit surprising. The difference in the strength of the screening effect in-between the two models can be considerable, in par-

Table 2. $R_{\psi_{1.5+\nu+x,\nu,0.1}}(n, 0.01)$ *for the Generalized Wendland Model with different values of* x, ν *and* n.

	n	1	2	3	4	5	6	7	8	9	10	11	12
$\nu = 0$	$x = 0$	5.092991e-02	4.484664e-02	4.476267e-02	4.476104e-02	4.476066e-02	4.472874e-02	4.229398e-02	3.433339e-02	2.429962e-02	1.167579e-02	3.370656e-03	2.125518e-03
$\nu = 0$	$x = \frac{1}{2}$	7.291579e-03	1.926657e-03	1.845781e-03	1.842660e-03	1.840822e-03	1.831735e-03	1.752503e-03	1.461902e-03	1.033833e-03	4.773155e-04	9.049519e-05	6.540115e-06
$\nu = 0$	$x = 2$	4.174425e-03	8.287586e-05	7.529038e-06	4.160870e-06	3.168702e-06	2.829301e-06	2.386606e-06	1.682085e-06	9.402887e-07	3.078456e-07	1.539665e-08	4.961457e-09
$\nu = 0$	$x = 20$	1.417199e-05	3.256056e-07	5.262073e-10	2.932099e-12	1.776357e-14	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0
$\nu = \frac{1}{2}$	$x = 0$	1.238679e-01	2.518064e-02	2.186238e-02	2.170864e-02	2.170137e-02	2.165372e-02	2.038591e-02	1.704591e-02	1.136440e-02	5.399366e-03	1.854819e-03	7.696975e-04
$\nu = \frac{1}{2}$	$x = \frac{1}{2}$	9.580829e-02	4.375047e-03	1.264624e-03	1.118026e-03	1.111418e-03	1.110505e-03	1.025747e-03	7.978657e-04	5.080492e-04	2.103344e-04	4.473505e-05	6.540464e-06
$\nu = \frac{1}{2}$	$x = 2$	7.928570e-02	2.774658e-03	1.326394e-04	7.421395e-06	1.772594e-06	1.464125e-06	1.262922e-06	1.015172e-06	6.801762e-07	3.475151e-07	1.177511e-07	1.548480e-08
$\nu = \frac{1}{2}$	$x = 20$	2.817677e-03	3.156572e-05	3.839977e-07	4.898443e-09	6.456302e-11	8.653078e-13	1.199041e-14	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0	0.000000e+0
$\nu = 2$	$x = 0$	1.906047e+0	2.298318e-01	4.936823e-02	1.376110e-02	6.211718e-03	4.427011e-03	3.911247e-03	3.374716e-03	2.740038e-03	2.027527e-03	1.289726e-03	6.921458e-04
$\nu = 2$	$x = \frac{1}{2}$	1.748272e+0	2.140837e-01	4.340609e-02	9.815036e-03	2.654434e-03	1.048069e-03	6.605041e-04	5.272317e-04	4.262353e-04	3.183656e-04	2.038321e-04	1.078267e-04
$\nu = 2$	$x = 2$	1.367888e+0	1.789414e-01	3.558611e-02	7.572603e-03	1.638356e-03	3.571073e-04	7.710742e-05	1.591468e-05	3.079431e-06	6.402223e-07	1.813578e-07	6.325228e-08
$\nu = 2$	$x = 20$	7.379256e-02	5.772543e-03	5.081329e-04	4.596307e-05	4.214214e-06	3.899937e-07	3.634979e-08	3.407452e-09	3.209182e-10	3.034262e-11	2.880141e-12	2.744471e-13
$\nu = 15$	$x = 0$	1.284557e+01	2.725173e+0	9.531585e-01	4.084754e-01	1.919252e-01	9.425634e-02	4.731806e-02	2.401927e-02	1.226161e-02	6.277599e-03	3.218749e-03	1.651642e-03
$\nu = 15$	$x = \frac{1}{2}$	1.117221e+01	2.408608e+0	8.438059e-01	3.590051e-01	1.665825e-01	8.053529e-02	3.972315e-02	1.978863e-02	9.907069e-03	4.972322e-03	2.498739e-03	1.256494e-03
$\nu = 15$	$x = 2$	7.646261e+0	1.712488e+0	5.981033e-01	2.476250e-01	1.102478e-01	5.071996e-02	2.369082e-02	1.114485e-02	5.260525e-03	2.487002e-03	1.176665e-03	5.569126e-04
$\nu = 15$	$x = 20$	5.530243e-01	1.001050e-01	2.107670e-02	4.583492e-03	1.004122e-03	2.203619e-04	4.838269e-05	1.062475e-05	2.333417e-06	5.125118e-07	1.125772e-07	2.473042e-08

Table 3. $R_{\mathcal{M}_{\nu,\alpha}}(n, 0.01)$ *for the Matérn Model with different values of* ν *and* n *with* α *such that the practical range is* 0.1.

	n	1	2	3	4	5	6	7	8	9	10	11	12
$\nu = 0.5$		3.993871e-03	5.592063e-05	6.017734e-07	1.108406e-07	1.976435e-08	4.135736e-09	1.014062e-09	2.771223e-10	8.221135e-11	2.602474e-11	8.683720e-12	3.026690e-12
$\nu = 1.0$		7.656900e-02	2.606023e-03	1.149040e-04	5.281975e-06	2.461492e-07	1.157238e-08	5.479242e-10	2.623302e-11	1.402434e-12	2.373657e-13	1.982858e-13	1.167955e-13
$\nu = 2.5$		1.390118e+0	1.707814e-01	3.333572e-02	6.943729e-03	1.475033e-03	3.159579e-04	6.803041e-05	1.470515e-05	3.188743e-06	6.933220e-07	1.511040e-07	3.300342e-08
$\nu = 15.5$		3.173124e+07	2.041359e+05	5.300267e+03	3.841679e+02	5.980949e+01	1.590339e+01	6.004259e+0	2.825557e+0	1.522168e+0	8.964182e-01	5.599226e-01	3.657309e-01

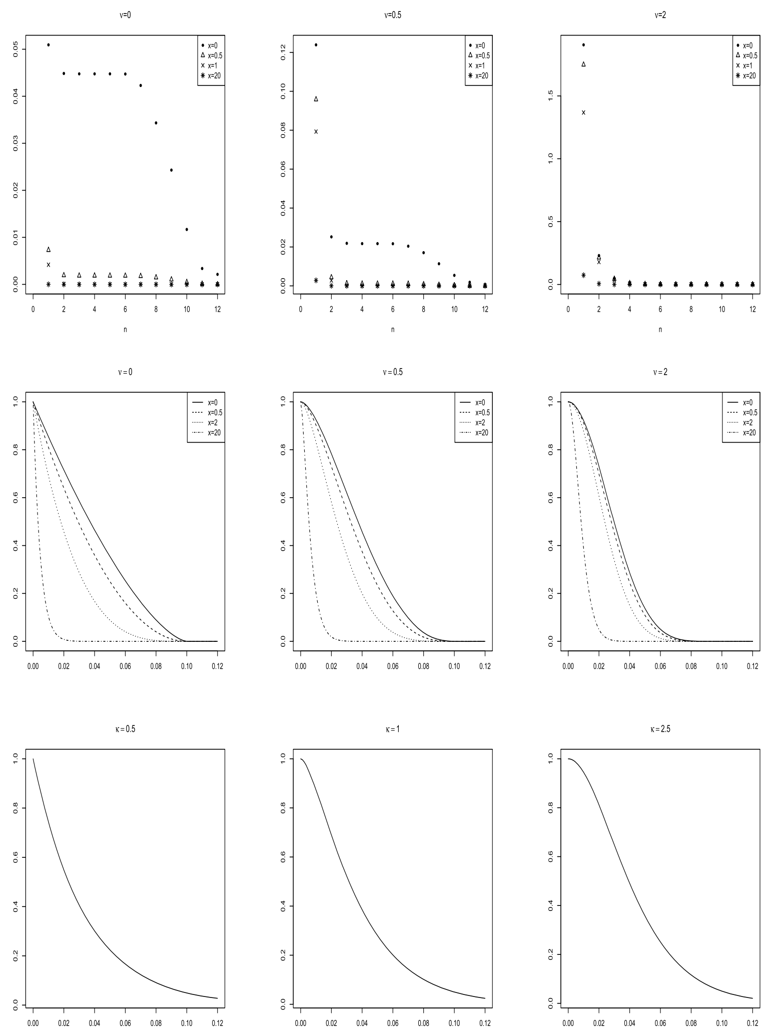

FIG 3. Top*: plots of* $R_{\psi_{\mu,\nu,\beta}}(n, \delta, \mu, \nu)$ *versus* δn *for* $n = 1, \ldots, 12$*, with* $\delta = 0.01$*,* $\nu = 0, 0.5, 2$ *(from left to right) and* $\mu = (m+1)/2 + \nu + x$ *with* $x = 0, 0.5, 2, 20$. Center*: Generalized Wendland correlation models with* $\nu = 0, 0.5, 2$ *(from left to right),* $\mu = (m+1)/2 + \nu + x$ *with* $x = 0, 0.5, 2, 20$ *and compact support equal to* 0.1. Bottom*: Matérn correlation models with* $\kappa = 0.5, 1, 2.5$ *(from left to right) and practical range equal to* 0.1.

ticular for high levels of differentiability (see the case 15 and 15.5. and $n = 1$). On the other hand, as expected, when n increases, $R_{\psi_{\mu,\nu,\beta}}(n, \delta)$ decreases for the Generalized Wendland model, regardless of μ and ν. This is also apparent for $R_{\mathcal{M}_{\nu,\alpha}}(n, \delta)$.

Figure 3 (first row) shows the plot of $R_{\psi_{\mu,\nu,10\delta}}(n, \delta)$ versus δn for $n = 1, \ldots, 12$, with $\delta = 0.01$, $\nu = 0, 0.5, 2$ (from left to right) and $\mu = (m+1)/2+\nu+x$

with $x = 0, 0.5, 2, 20$. Finally, we repeat our numerical experiments by changing the value of δ but, as in Stein (2002), the conclusions are not significantly different.

5. Conclusion

The results provided in this paper, supported by numerical experiments, rehabilitate some compactly supported covariance models as being good candidates to provide a screening effect in spatial prediction. As emphasized by Stein (2002, 2011), the theoretical analysis of the screening effect is extremely complex and results can be stated only with respect to specific classes of covariance functions. A future direction of research is to consider space-time Gaussian random fields having different levels of regularity along different directions. In particular, a comparison between some classes of space-time covariance functions might provide useful indications about the screening effect for space-time prediction.

Acknowledgments

The authors are grateful to the Associate Editor and to an anonymous reviewer for their constructive comments and acknowledge the support of the Millennium Science Initiative of the Chilean Ministry of Economy, Development, and Tourism, through grant "Millenium Nucleus Center for the Discovery of Structures in Complex Data" (Emilio Porcu and Moreno Bevilacqua), of the National Agency for Research and Development of Chile, through grant CONICYT PIA AFB180004 (Xavier Emery) and grant Fondecyt number 1200068 from Chilean Ministry of Education (Moreno Bevilacqua).

Appendix A: Mathematical proofs

A.1. *Proof of Theorem 3.1*

Proof. The assumptions on the function G imply

$$G(z) \asymp (1+z)^{-\gamma}, \quad z \geq 0. \tag{A.13}$$

Let us prove Assertion 1. Equation (A.13) implies that $G(||\omega||) \asymp 1$ on bounded subsets of $\mathbb{R}^m$. Thus, condition **(A.1)** holds. To show that **(A.2)** holds also, we set $L(r) := 1$ for all $r \geq 0$. Then,

$$\widetilde{g}(z) = \begin{cases} z^{-\gamma}, & z \geq 1 \\ \\ 1, & 0 \leq z \leq 1. \end{cases}$$

Thus, we have that the function g in **(A.2)** can be written as $g(\omega) = \widetilde{g}(\|\omega\|)\theta(\omega/\|\omega\|)$, $\omega \in \mathbb{R}^m$, where $\theta(u) \equiv C$. The function $\widetilde{g}$ is continuous and positive on $[0,\infty)$ and $\widetilde{g}(z) \sim z^{-\gamma}$ as $z \to +\infty$. It follows that $\widetilde{g}(z) \asymp (1+z)^{-\gamma}$, $z \geq 0$ and therefore $f(\omega) = G(\|\omega\|) \sim g(\omega)$ as $\|\omega\| \to \infty$ and $f(\omega) \asymp g(\omega)$ on $\mathbb{R}^m$ (see Equation (A.13)).

To prove Assertion 2., we make use of the assumption $G(z) \sim Cz^{-\gamma}$ as $z \to \infty$, to imply that

$$G(z) = \frac{C}{z^{\gamma}} + \frac{B(z)}{z^{\gamma}}, \quad z > 0; \quad \lim_{z\to\infty} B(z) = 0; \quad B^*(z) := \sup_{u \geq z} \Big|B(u)\Big| \to 0 \tag{A.14}$$

as $z \to \infty$. Let $R > 0$, $\omega, \tau \in \mathbb{R}^m$, $z := \|\omega\| \geq R+1$, $\|\tau\| \leq R$ and $s := \|\omega+\tau\| - \|\omega\|$. Then, $|s| \leq \|\tau\| \leq R$. From (A.14) we have

$$\Big|G(\|\omega+\tau\|) - G(\|\omega\|)\Big| = \Big|G(z+s) - G(z)\Big| \leq C\left|\frac{1}{(z+s)^{\gamma}} - \frac{1}{z^{\gamma}}\right| + \frac{2B^*(z-R)}{(z-R)^{\gamma}}.$$

By the mean-value theorem, there exists a point, ξ, between z and $z+s$ such that

$$\left|\frac{1}{(z+s)^{\gamma}} - \frac{1}{z^{\gamma}}\right| = \frac{\gamma\,|s|}{\xi^{\gamma+1}} \leq \frac{\gamma\, R}{(z-R)^{\gamma+1}},$$

where the inequality on the right-hand side follows from the fact that $\xi \geq \min\{z, z+s\} \geq z-R$. Thus, for $R>0$ and $z = \|\omega\| \geq R+1$, we conclude that

$$\sup_{\|\tau\|\leq R} \Big|G(\|\omega+\tau\|) - G(\|\omega\|)\Big| \leq \frac{C\gamma\, R}{(z-R)^{\gamma+1}} + \frac{2B^*(z-R)}{(z-R)^{\gamma}}.$$

The proof is completed by noting that

$$\sup_{\|\tau\|\leq R} \left|\frac{G(\|\omega+\tau\|)}{G(\|\omega\|)} - 1\right| \leq \left(\frac{C\gamma\, R}{(z-R)^{\gamma+1}} + \frac{2B^*(z-R)}{(z-R)^{\gamma}}\right) \times (1+z)^{\gamma} \sup_{u\geq 0} \frac{(1+u)^{-\gamma}}{G(u)} \longrightarrow 0, z \to \infty. \qquad \square$$

A.2. Proof of Theorem 3.2

Proof. We start by noting that, if $m=1$, $\nu > 1/2$, $\mu \geq \max\{\nu, 1\}$ and $(\mu,\nu) \neq (1,1)$, or if $m \geq 2$, $\nu > 1/2$ and $\mu \geq (m-1)/2 + \nu$, then $G_{\mu,\nu,m}(z) > 0$ for all $z \geq 0$. These are well-known results (Fields and Ismail, 1975; Gasper, 1975; Moak, 1987). Therefore, in both cases **(i)** and **(ii)** $G_{\mu,\nu,m}(z) > 0$ for all $z \geq 0$.

We first note that, for μ, ν and z strictly positive, we have (Zastavnyi, 2006, Proposition 6)

$$G_{\mu,\nu,m}(z) = \frac{\Gamma(\mu)\Gamma(\nu)2^{\nu-1/2}}{\sqrt{\pi} z^{\mu+(m-1)/2+\nu}} \left(\cos\left(z - \pi/2\,(\mu + (m-1)/2 + \nu)\right) + \mathcal{O}(1/z)\right) + \frac{C_{\nu,m}}{z^{m-1+2\nu}} + o\left(\frac{1}{z^{m-1+2\nu}}\right), \quad z \to \infty, \tag{A.15}$$

with

$$C_{\nu,m} = \frac{\Gamma((m-1)/2+\nu)\Gamma(\nu)}{\sqrt{\pi}} 2^{m/2+2\nu-2}.$$

From (A.15) it follows that if $\mu > (m-1)/2+\nu$ (this condition is satisfied in both cases **(i)** and **(ii)**), then $G_{\mu,\nu,m}(z) \sim C_{\nu,m}\, z^{-\gamma}$ as $z \to \infty$, where $\gamma = m + 2\nu - 1 > m$. We can thus use Theorem 3.1 to conclude the proof. □

A.3. Proof of Theorem 3.3

Before providing a proof, we start with some preliminary remarks. If δ, μ, ν and α are positive, then the function $\mathbb{R} \ni t \mapsto \varphi_{\delta,\mu,\nu,\alpha}(t)$ is continuous on $\mathbb{R}$ if and only if $\mu+\nu-1>0$ (see Zastavnyi, 2006, Theorem 1). Also, Theorem 3 in Zastavnyi (2006), if $\delta,\mu,\nu,\alpha > 0$, then

$$\mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(z) = 2^{\nu-1}\Gamma(\nu)\mathrm{I}_{\delta,\mu,(m-1)/2+\nu,m-1+\alpha}(z), \qquad z \geq 0, \tag{A.16}$$

where

$$\mathrm{I}_{\delta,\mu,\nu,\alpha}(z) := \int_0^1 \left(1-x^\delta\right)^{\mu-1} x^\alpha j_{\nu-1/2}(zx)\mathrm{d}x, \qquad z>0,$$

and $j_\nu(z) = J_\nu(z)/z^\nu$ and J_ν as defined in (2.4). The well-known cases for positiveness of the function $\mathrm{I}_{\delta,\mu,\nu,\alpha}$ were already given before Theorem 4 in Zastavnyi (2006). Theorems 4, 5 and 6 in Zastavnyi (2006) assess conditions for positiveness of the function $\mathrm{I}_{\delta,\mu,\nu,\alpha}$. In particular, we quote a necessary condition from Theorem 5 of Zastavnyi (2006): if $\delta,\mu,\nu,\alpha>0$, $\mathrm{I}_{\delta,\mu,(m-1)/2+\nu,m-1+\alpha}(z) \geq 0$ for all $z>0$, then necessary conditions are $\mu+\nu \geq \alpha + (m-1)/2$ and, either,
(a) $\alpha < 2\nu$, or
(b) $\alpha = 2\nu$, $\mu > 1$, $\delta < 2$ and $\mu \geq (m+1)/2 + \nu + \delta$.

Proof. Using Equation (A.16) in concert with Proposition 6 in Zastavnyi (2006), we have that

$$\begin{aligned}
&\mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(z) = \\
&\frac{\Gamma(\nu)\Gamma(\mu)\delta^{\mu-1}}{\sqrt{\pi}} \cdot \frac{2^{\nu-1/2}}{z^{\mu+(m-1)/2+\nu}} \cdot \left(\cos\left(z - \frac{\pi}{2}\left(\mu + \frac{m-1}{2} + \nu\right)\right) + \mathcal{O}\left(\frac{1}{z}\right)\right) \\
&+ \frac{\Gamma((\alpha+m)/2)\Gamma(\nu)}{\Gamma(\nu-\alpha/2)} \cdot \frac{2^{\alpha+m/2-1}}{z^{m+\alpha}} - \frac{\Gamma((\alpha+m+\delta)/2)\Gamma(\nu)(\mu-1)}{\Gamma(\nu-(\alpha+\delta)/2)} \cdot \frac{2^{\alpha+m/2+\delta-1}}{z^{m+\alpha+\delta}} \\
&+ o\left(\frac{1}{z^{m+\alpha+\delta}}\right), \qquad z \to \infty.
\end{aligned} \tag{A.17}$$

Equation (A.17) implies that in cases (i_1) and (i_2) from Theorem 3.3, as $z \to \infty$,

$$\mathcal{F}_m\left(\varphi_{\delta,\mu,\nu,\alpha}\right)(z) \sim \begin{cases} \frac{\Gamma((\alpha+m)/2)\Gamma(\nu)}{\Gamma(\nu-\alpha/2)}\, 2^{\alpha+m/2-1}\, z^{-m-\alpha}, & (i_1) \\[2ex] \frac{\Gamma((2\nu+m+\delta)/2)\Gamma(\nu)\delta(\mu-1)}{\Gamma(1-\delta/2)}\, 2^{2\nu+m/2+\delta-2}\, z^{-m-\alpha-\delta}, & (i_2). \end{cases}$$

In both cases $\mathcal{F}_m\left(\varphi_{\delta,\mu,\nu,\alpha}\right)(z) \sim Cz^{-\gamma}$ as $z \to \infty$, where $C > 0$ and $\gamma > m$. If, in addition, $\mathcal{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(z) > 0$ for all $z \geq 0$, then we can apply the Theorem 3.1. $\square$

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