# Subdifferential of the supremum function <br> Moving back and forth between continuous and non-continuous settings 

R. Correa • A. Hantoute • M. A. López

Received: date / Accepted: date


#### Abstract

In this paper we develop general formulas for the subdifferential of the pointwise supremum of convex functions, which cover and unify both the compact continuous and the non-compact non-continuous settings. From the noncontinuous to the continuous setting, we proceed by a compactification-based approach which leads us to problems having compact index sets and upper semicontinuously indexed mappings, giving rise to new characterizations of the subdifferential of the supremum by means of upper semicontinuous regularized functions and an enlarged compact index set. In the opposite sense, we rewrite the subdifferential of these new regularized functions by using the original data, also leading us to new results on the subdifferential of the supremum. We give two applications in the last section, the first one concerning the nonconvex Fenchel duality, and the second one establishing Fritz-John and KKT conditions in convex semi-infinite programming.


Keywords Supremum of convex functions • subdifferentials • Stone-Čech compactification • convex semi-infinite programming • Fritz-John and KKT optimality conditions

Mathematics Subject Classification (2010) 46N10 • 52A41 • 90C25
Research supported by CONICYT (Fondecyt 1190012 and 1190110), Proyecto/Grant PIA AFB-170001, MICIU of Spain and Universidad de Alicante (Grant Beatriz Galindo BEAGAL 18/00205), and Research Project PGC2018-097960-B-C21 from MICINN, Spain. The research of the third author is also supported by the Australian ARC - Discovery Projects DP 180100602.
R. Correa

Universidad de O'Higgins, Chile, and DIM-CMM of Universidad de Chile
E-mail: rcorrea@dim.uchile.cl
A. Hantoute

Center for Mathematical Modeling (CMM), Universidad de Chile, and Universidad de Alicante, Spain
E-mail: ahantoute@dim.uchile.cl (corresponding author)
M. A. López

Universidad de Alicante, Spain, and CIAO, Federation University, Ballarat, Australia
E-mail: marco.antonio@ua.es

## 1 Introduction

In this paper we deal with the characterization of the subdifferential of the pointwise supremum $f:=\sup _{t \in T} f_{t}$ of a family of convex functions $f_{t}: X \rightarrow \mathbb{R} \cup\{ \pm \infty\}$, $t \in T$, with $T$ being an arbitrary nonempty set, defined on a separated locally convex space $X$. We obtain new characterizations which allow us to unify both the compact continuous and the non-compact non-continuous setting ([8], 9], [27, 30], etc.). The first setting relies on the following standard conditions in the literature of convex analysis and non-differentiable semi-infinite programming:
$T$ is compact and the mappings $f_{(\cdot)}(z), z \in X$, are upper semi-continuous.
In the other framework, called the non-compact non-continuous setting, we do not assume the above conditions. In other words (see, i.e., [14, [15], 21], 18, [29], [30], 31, etc.):
$T$ is an arbitrary set, possibly infinite and without any prescribed topology, and no requirement is imposed on the mappings $f_{(\cdot)}(z)$.

Going from the non-continuous to the continuous setting, we follow an approach based on the Stone-Čech compactification of the index set $T$. At the same time, we build an appropriate enlargement of the original family $f_{t}, t \in T$, which ensures the fulfillment of the upper semi-continuity property required in the compact setting. Since the new setting is naturally compact, by applying the results in [8, 9 , we obtain new characterizations given in terms of the exact subdifferential at the reference point of the new functions and the extended active set. In this way, we succeed in unifying both settings. In [10, we gave the first steps in this direction, using compactification arguments, but in the current paper we go further into the subject with some enhanced formulas.

To move in the other direction, we rewrite the subdifferential of these new regularizing functions in terms of the original data, and this also leads us to new results on the subdifferential of the supremum. In this last case, the characterizations are given upon limit processes on the $\varepsilon$-subdifferentials at the reference point of the almost-active original functions. These limit processes also involve approximations by finite-dimensional sections of the domain of the supremum function.

The main results of this paper are applied to derive formulas for the subdifferential of the conjugate function ([3, [4], [5]). Our approach permits simple proofs of these results, with the aim of relating the solution set of a nonconvex optimization problem and its convexified relaxation. Additionally, our results give rise to new Fritz-John and KKT conditions in convex semi-infinite programming.

The paper is organized as follows. After a short section introducing the notation, in section 3 we present some preliminary results in the continuous setting. In section 4 we apply our compactification approach to obtain, in Theorem 4 a first characterization of the subdifferential of the supremum. Such a theorem constitutes an improved version of the main result in [10, as the requirement of equipping $T$ with a completely regular topology is eliminated. Theorem 4 is enhanced in Section 5 allowing for a more natural interpretation of the regularized functions. The main result in section 6 is Theorem 11] involving only the $\varepsilon$-subdifferentials of the original data functions. This theorem, whose proof is based on Lemmas 9
and 10, is crucial in the proposed approach to move from the continuous to the non-continuous setting. Finally, in section 7 we give two applications. The first one addresses the extension of the classical Fenchel duality to nonconvex functions, and the second one establishes Fritz-John and KKT optimality conditions for convex semi-infinite optimization.

## 2 Notation

Let $X$ be a (real) separated locally convex space, with topological dual $X^{*}$ endowed with the $w^{*}$-topology. By $\mathcal{N}_{X}\left(\mathcal{N}_{X^{*}}\right)$ we denote the family of closed, convex, and balanced neighborhoods of the origin in $X\left(X^{*}\right)$, also called $\theta$-neighborhoods. The spaces $X$ and $X^{*}$ are paired in duality by the bilinear form $\left(x^{*}, x\right) \in X^{*} \times X \mapsto$ $\left\langle x^{*}, x\right\rangle:=\left\langle x, x^{*}\right\rangle:=x^{*}(x)$. The zero vectors in $X$ and $X^{*}$ are both denoted by $\theta$. We use the notation $\overline{\mathbb{R}}:=\mathbb{R} \cup\{-\infty,+\infty\}$ and $\mathbb{R}_{\infty}:=\mathbb{R} \cup\{+\infty\}$, and adopt the convention $(+\infty)+(-\infty)=(-\infty)+(+\infty)=+\infty$.

Given two nonempty sets $A$ and $B$ in $X$ (or in $X^{*}$ ), we define the algebraic (or Minkowski) sum by

$$
\begin{equation*}
A+B:=\{a+b: a \in A, b \in B\}, \quad A+\emptyset=\emptyset+A=\emptyset . \tag{1}
\end{equation*}
$$

By $\operatorname{co}(A)$, cone $(A)$, and $\operatorname{aff}(A)$, we denote the convex, the conical convex, and the affine hulls of the set $A$, respectively. Moreover, $\operatorname{int}(A)$ is the interior of $A$, and $\operatorname{cl} A$ and $\bar{A}$ are indistinctly used for denoting the closure of $A$. We use $\operatorname{ri}(A)$ to denote the (topological) relative interior of $A$ (i.e., the interior of $A$ in the topology relative to $\operatorname{aff}(A)$ if aff $(A)$ is closed, and the empty set otherwise).

Associated with $A \neq \emptyset$ we consider the polar set and the orthogonal subspace given respectively by

$$
A^{\circ}:=\left\{x^{*} \in X^{*}:\left\langle x^{*}, x\right\rangle \leq 1 \text { for all } x \in A\right\},
$$

and

$$
A^{\perp}:=\left\{x^{*} \in X^{*}:\left\langle x^{*}, x\right\rangle=0 \text { for all } x \in A\right\} .
$$

The following relation holds

$$
\begin{equation*}
\bigcap_{L \in \mathcal{F}}\left(A+L^{\perp}\right) \subset \operatorname{cl} A \tag{2}
\end{equation*}
$$

where $\mathcal{F}$ is the family of finite-dimensional linear subspaces in $X$.
If $A \subset X$ is convex and $x \in X$, we define the normal cone to $A$ at $x$ as

$$
\mathrm{N}_{A}(x):=\left\{x^{*} \in X^{*}:\left\langle x^{*}, z-x\right\rangle \leq 0 \text { for all } z \in A\right\}
$$

if $x \in A$, and the empty set otherwise.
The basic concepts in this paper are traced from [23,28. Given a function $f: X \longrightarrow \overline{\mathbb{R}}$, its (effective) domain and epigraph are, respectively,

$$
\operatorname{dom} f:=\{x \in X: f(x)<+\infty\} \text { and epi } f:=\{(x, \lambda) \in X \times \mathbb{R}: f(x) \leq \lambda\}
$$

We say that $f$ is proper when $\operatorname{dom} f \neq \emptyset$ and $f(x)>-\infty$ for all $x \in X$. By cl $f$ and $\overline{\mathrm{co}} f$ we respectively denote the closed and the closed convex hulls of $f$, which are
the functions such that $\operatorname{epi}(\operatorname{cl} f)=\operatorname{cl}(\operatorname{epi} f)$ and $\operatorname{epi}(\overline{\operatorname{co} f})=\overline{\operatorname{co}}(\operatorname{epi} f)$. We say that $f$ is lower semicontinuous (lsc, for short) at $x$ if $(\mathrm{cl} f)(x)=f(x)$, and lsc if $\mathrm{cl} f=f$.

Given $x \in X$ and $\varepsilon \geq 0$, the $\varepsilon$-subdifferential of $f$ at $x$ is

$$
\partial_{\varepsilon} f(x)=\left\{x^{*} \in X^{*}: f(y) \geq f(x)+\left\langle x^{*}, y-x\right\rangle-\varepsilon \text { for all } y \in X\right\},
$$

when $x \in \operatorname{dom} f$, and $\partial_{\varepsilon} f(x):=\emptyset$ when $f(x) \notin \mathbb{R}$. The elements of $\partial_{\varepsilon} f(x)$ are called $\varepsilon$-subgradients of $f$ at $x$. The subdifferential of $f$ at $x$ is $\partial f(x):=\partial_{0} f(x)$, and its elements are called subgradients of $f$ at $x$. If $f$ and $g$ are convex functions such that one of them is finite and continuous at a point of the domain of the other one, then Moreau-Rockafellar's theorem says that

$$
\begin{equation*}
\partial(f+g)=\partial f+\partial g \tag{3}
\end{equation*}
$$

Given a function $f: X \rightarrow \overline{\mathbb{R}}$, the (Fenchel) conjugate of $f$ is the function $f^{*}: X^{*} \rightarrow$ $\overline{\mathbb{R}}$ defined as

$$
f^{*}\left(x^{*}\right):=\sup _{x \in X}\left\{\left\langle x^{*}, x\right\rangle-f(x)\right\} .
$$

The indicator and the support functions of $A \subset X$ are respectively defined as

$$
\mathrm{I}_{A}(x):= \begin{cases}0, & \text { if } x \in A \\ +\infty, & \text { if } x \in X \backslash A\end{cases}
$$

and

$$
\sigma_{A}:=\mathrm{I}_{A}^{*}
$$

Provided that $f^{*}$ is proper, by Moreau's theorem we have

$$
\begin{equation*}
f^{* *}=\overline{\operatorname{co}} f, \tag{4}
\end{equation*}
$$

where $f^{* *}:=\left(f^{*}\right)^{*}$. For example, if $\left\{f_{i}, i \in I\right\}$ is a nonempty family of proper lsc convex functions, then

$$
\begin{equation*}
\left(\sup _{i \in I} f_{i}\right)^{*}=\overline{\operatorname{co}}\left(\inf _{i \in I} f_{i}^{*}\right), \tag{5}
\end{equation*}
$$

provided that the supremum function $\sup _{i \in I} f_{i}$ is proper. Thus, given a nonempty family of closed convex sets $A_{i} \subset X, i \in I$, such that $\cap_{i \in I} A_{i} \neq \emptyset$, we have $\mathrm{I}_{\cap_{i \in I} A_{i}}(x)=\sup _{i \in I} \mathrm{I}_{A_{i}}(x)$ and, so, by taking the conjugate in the equalities $\mathrm{I}_{\cap_{i \in I} A_{i}}(x)=\sup _{i \in I} \mathrm{I}_{A_{i}}(x)=\sup _{i \in I} \sigma_{A_{i}}^{*}(x)$, we obtain

$$
\sigma_{\cap_{i \in I} A_{i}}=\left(\mathrm{I}_{\cap_{i \in I} A_{i}}\right)^{*}=\left(\sup _{i \in I} \mathrm{I}_{A_{i}}\right)^{*}=\overline{\operatorname{co}}\left(\inf _{i \in I} \sigma_{A_{i}}\right)
$$

## 3 Preliminary results in the continuous framework

In Section 4 we develop a compactification process addressed to give new characterizations of the subdifferential of the pointwise supremum, with the aim of unifying both the compact and non-compact settings. In this section we gather some preliminary results in the continuous setting.

Given the family of convex functions $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, and the supremum function $f:=\sup _{t \in T} f_{t}$, we start from the following characterization of $\partial f(x)$ in the continuous setting, given in [8, Proposition 2], where the following notation is
used:

$$
\begin{equation*}
\mathcal{F}(x):=\{L \subset X: L \text { is a finite-dimensional linear subspace containing } x\} \tag{6}
\end{equation*}
$$

and

$$
T_{\varepsilon}(x):=\left\{t \in T: f_{t}(x) \geq f(x)-\varepsilon\right\},
$$

for $\varepsilon \geq 0$; we set $T(x):=T_{0}(x)$.

Proposition 1 [8, Proposition 2] Fix $x \in X$ and $\varepsilon>0$ such that $T_{\varepsilon}(x)$ is compact Hausdorff and, for each net $\left(t_{i}\right)_{i} \subset T_{\varepsilon_{0}}(x)$ converging to $t$,

$$
\begin{equation*}
\limsup _{i} f_{t_{i}}(z) \leq f_{t}(z) \quad \text { for all } z \in \operatorname{dom} f \tag{7}
\end{equation*}
$$

that is, the functions $f_{(\cdot)}(z)$ are upper semi-continuos (usc, in brief) relatively to $T_{\varepsilon_{0}}(x)$. Then we have

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{t \in T(x)} \partial\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} \tag{8}
\end{equation*}
$$

It is worth recalling that the intersection over the $L$ 's in (8) is removed in finite dimensions ([8, Theorem 3]) and, more generally, if ri(dom $f) \neq \emptyset$ and $f_{\mid \operatorname{aff}(\operatorname{dom} f)}$ is continuous on $\operatorname{ri}(\operatorname{dom} f)$, then we have (see [9, Corollary 3.9])

$$
\partial f(x)=\overline{\mathrm{co}}\left\{\bigcup_{t \in T(x)} \partial\left(f_{t}+\mathrm{I}_{\mathrm{dom} f}\right)(x)\right\}
$$

Consequently, if $f$ is continuous somewhere in its domain, then ( 9 , Theorem 3.12])

$$
\partial f(x)=\overline{\operatorname{co}}\left\{\bigcup_{t \in T(x)} \partial f_{t}(x)\right\}+\mathrm{N}_{\operatorname{dom} f}(x)
$$

and the closure is removed in finite dimensions. In particular, when $f$ is continuous at the reference point $x$, the normal cone above collapses to $\theta$ and we recover Valadier's formula in [30].

On the other hand, in the general setting, when either $T$ is not compact and/or some of the mappings $t \rightarrow f_{t}(z), z \in \operatorname{dom} f$, fail to be usc, the active index set $T(x)$ as well as the subdifferential sets $\partial f_{t}(x)$ may be empty. To overcome this situation, the following result given in [15, Theorem 4] (see, also, 14 for finite dimensions) appeals to the $\varepsilon$-active set $T_{\varepsilon}(x)$ and the $\varepsilon$-subdifferentials.

## Proposition 2 If

$$
\begin{equation*}
\operatorname{cl} f=\sup _{t \in T}\left(\operatorname{cl} f_{t}\right) \tag{9}
\end{equation*}
$$

then for every $x \in X$

$$
\begin{equation*}
\partial f(x)=\bigcap_{\varepsilon>0, L \in \mathcal{F}(x)} \overline{\bar{c}}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right\} \tag{10}
\end{equation*}
$$

Also here, the intersection over the $L$ 's is dropped out if $\operatorname{ri}(\operatorname{dom} f) \neq \emptyset(15$, Corollary 8]). Moreover, if $f$ is continuous somewhere, so that (9) holds automat-
ically ([15, Corollary 9]), then the last formula reduces to

$$
\partial f(x)=\mathrm{N}_{\mathrm{dom} f}(x)+\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right\} .
$$

Hence, provided that $f$ is continuous at $x$, we obtain the formula in [31 (where the underlying space $X$ is additionally assumed to be normed).

Condition (9) guarantees the possibility of characterizing $\partial f(x)$ by means of the $f_{t}$ 's, and not via the augmented functions $f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}$ as in Proposition 1 Thus, to complete the analysis, we give next a consequence of (10), which avoids to appeal to condition (9).

Proposition 3 For every $x \in X$,

$$
\begin{equation*}
\partial f(x)=\bigcap_{\varepsilon>0, L \in \mathcal{F}(x)} \overline{\operatorname{co}}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} . \tag{11}
\end{equation*}
$$

Proof Fix $x \in \operatorname{dom} f$ and $L \in \mathcal{F}(x)$, and denote

$$
g_{t}:=f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}, t \in T ; \quad g:=\sup _{t \in T} g_{t} .
$$

We have $\operatorname{dom} g_{t}=L \cap \operatorname{dom} f$ and

$$
\operatorname{dom} g \cap\left(\cap_{t \in T} \operatorname{ri}\left(\operatorname{dom} g_{t}\right)\right)=(L \cap \operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} f \cap L)=\operatorname{ri}(\operatorname{dom} f \cap L) \neq \emptyset
$$

so that, by [15, Corollary 9 (iv)], the family $\left\{g_{t}, t \in T\right\}$ satisfies condition (9). At the same time we have, for all $\varepsilon \geq 0$,

$$
\left\{t \in T: g_{t}(x) \geq g(x)-\varepsilon\right\}=T_{\varepsilon}(x) .
$$

Then, since that $\partial f(x) \subset \partial\left(f+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)=\partial g(x)$, by Proposition 2 we obtain that

$$
\begin{aligned}
\partial f(x) & \subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} g_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} g}(x)\right\} \\
& \subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\},
\end{aligned}
$$

and the inclusion " $\subset$ " in (11) follows as $L$ was arbitrarily chosen. The opposite inclusion is straightforward, and we are done.

## 4 Compactification approach to the subdifferential

Our main objective in this section is to give a new characterization for $\partial f(x)$, which covers both formula (8) in the compact-continuous setting, using the active set and the exact subdifferential, and formula (11) in the non-compact non-continuous framework, given in terms of $\varepsilon$-active indices and $\varepsilon$-subdifferentials. To this aim, we develop a compactification approach which works by extending the original index set $T$ to a compact set $\widehat{T}$, and building new appropriate functions $f_{\gamma}, \gamma \in \widehat{T}$, that satisfy property (7) of Proposition 1 To make the paper self-contained, we resume here the main features of the compactification process, which can be also found in [10].

We start by assuming that $T$ is endowed with some topology $\tau$, for instance the discrete topology. If

$$
\begin{equation*}
\mathcal{C}(T,[0,1]):=\{\varphi: T \rightarrow[0,1]: \varphi \text { is } \tau \text {-continuous }\} \tag{12}
\end{equation*}
$$

we consider the product space $[0,1]^{C(T,[0,1])}$, which is compact for the product topology (by Tychonoff theorem). We regard the index set $T$ as a subset of $[0,1]^{C(T,[0,1])}$. For this purpose we consider the continuous embedding $\mathfrak{w}: T \rightarrow$ $[0,1]^{C(T,[0,1])}$ which assigns to each $t \in T$ the evaluation function $\mathfrak{w}(t)=\gamma_{t}$, defined as

$$
\begin{equation*}
\gamma_{t}(\varphi):=\varphi(t), \quad \varphi \in \mathcal{C}(T,[0,1]) \tag{13}
\end{equation*}
$$

The closure of $\mathfrak{w}(T)$ in $[0,1]^{C(T,[0,1])}$ for the product topology is the compact set

$$
\begin{equation*}
\widehat{T}:=\operatorname{cl}(\mathfrak{m}(T)), \tag{14}
\end{equation*}
$$

which is the so-called Stone-Čech compactification of $T$, also denoted by $\beta T$. The convergence in $\widehat{T}$ is the pointwise convergence; i.e., for $\gamma \in \widehat{T}$ and a net $\left(\gamma_{i}\right)_{i} \subset \widehat{T}$ we have $\gamma_{i} \rightarrow \gamma$ if and only if

$$
\begin{equation*}
\gamma_{i}(\varphi) \rightarrow \gamma(\varphi) \text { for all } \varphi \in \mathcal{C}(T,[0,1]) \tag{15}
\end{equation*}
$$

Hence, provided that $T$ is completely regular (when endowed with the discrete topology, for isntance), the mapping $\mathfrak{w}$ is an homeomorphism between $T$ and $\mathfrak{w}(T)$, and if $\gamma_{i}=\gamma_{t_{i}}$ and $\gamma=\gamma_{t}$ for some $t, t_{i} \in T$, then $\gamma_{i} \rightarrow \gamma$ if and only if $t_{i} \rightarrow t$ in $T$.

Next, we enlarge the original family $\left\{f_{t}, t \in T\right\}$ by introducing the functions $f_{\gamma}: X \rightarrow \overline{\mathbb{R}}, \gamma \in \widehat{T}$, defined by

$$
\begin{equation*}
f_{\gamma}(z):=\lim \sup _{\gamma_{t} \rightarrow \gamma, t \in T} f_{t}(z) \tag{16}
\end{equation*}
$$

It can be easily verified that the functions $f_{\gamma}, \gamma \in \widehat{T}$, are all convex and satisfy $\sup _{\gamma \in \widehat{T}} f_{\gamma} \leq f$. Moreover, if $\left(t_{n}\right)_{n} \subset T$ verifies $f(z)=\lim _{n} f_{t_{n}}(z)$, with $z \in X$, then there exist a subnet $\left(t_{i}\right)_{i}$ of $\left(t_{n}\right)_{n}$ and $\gamma \in \widehat{T}$ such that $\gamma_{t_{i}} \rightarrow \gamma$. Hence,

$$
f_{\gamma}(z) \geq \lim \sup _{i} f_{t_{i}}(z)=\lim _{i} f_{t_{i}}(z)=\lim _{n} f_{t_{n}}(z)=f(z)
$$

and so $\sup _{\gamma \in \widehat{T}} f_{\gamma} \geq f$. In other words, the functions $f_{\gamma}$ provide the same supremum $f$ as the original $f_{t}$ 's,

$$
\sup _{\gamma \in \widehat{T}} f_{\gamma}=\sup _{t \in T} f_{t}=f .
$$

If $f(x) \in \mathbb{R}$ and $\varepsilon \geq 0$, then the extended $\varepsilon$-active index set of $f$ at $x$ is

$$
\begin{equation*}
\widehat{T}_{\varepsilon}(x):=\left\{\gamma \in \widehat{T}: f_{\gamma}(x) \geq f(x)-\varepsilon\right\} \tag{17}
\end{equation*}
$$

with $\widehat{T}(x):=\widehat{T}_{0}(x)$; when $f(x) \notin \mathbb{R}$ we set $\widehat{T}_{\varepsilon}(x):=\emptyset$ for all $\varepsilon \geq 0$. By the compactness of $\widehat{T}$ and the simple fact that, for each $t \in T$,

$$
\begin{aligned}
f_{\gamma_{t}}(x)=\limsup _{\gamma_{s} \rightarrow \gamma_{t}} f_{s}(x) & =\sup \left\{\lim _{i} f_{t_{i}}(x), \gamma_{t_{i}} \rightarrow \gamma_{t}\right\} \\
& \geq \sup \left\{\lim _{i} f_{t_{i}}(x), t_{i} \rightarrow t\right\} \geq f_{t}(x)
\end{aligned}
$$

we verify that $\widehat{T}_{\varepsilon}(x) \neq \emptyset$. Also, the closedness of $\widehat{T}_{\varepsilon}(x)$ comes by using a diagonal process.

The way that the functions $f_{\gamma}, \gamma \in \widehat{T}$, are constructed ensures the fulfillment of the upper semi-continuity property required in Proposition 1 . More precisely, assuming that $f(x) \in \mathbb{R}$ and $\varepsilon \geq 0$, for every net $\left(\gamma_{i}\right)_{i} \subset \widehat{T}_{\varepsilon}(x)$ with an accumulation point $\gamma \in \widehat{T}_{\varepsilon}(x)$, and every $z \in \operatorname{dom} f$, we verify that

$$
\begin{equation*}
\limsup _{i} f_{\gamma_{i}}(z) \leq f_{\gamma}(z) \tag{18}
\end{equation*}
$$

Indeed, we may assume without loss of generality that $\gamma_{i} \rightarrow \gamma$ and $\lim \sup _{i} f_{\gamma_{i}}(z)=$ $\lim _{i} f_{\gamma_{i}}(z)=\alpha \in \mathbb{R}$. Next, for each $i$ there exists a net $\left(t_{i j}\right)_{j} \subset T$ such that

$$
\gamma_{t_{i j}} \rightarrow_{j} \gamma_{i}, \quad f_{\gamma_{i}}(z)=\lim _{j} f_{t_{i j}}(z) ;
$$

that is, $\left(\gamma_{t_{i j}}, f_{t_{i j}}(z)\right) \rightarrow_{j}\left(\gamma_{i}, f_{\gamma_{i}}(z)\right)$ and $\left(\gamma_{i}, f_{\gamma_{i}}(z)\right) \rightarrow_{i}(\gamma, \alpha)$. Then we can find a diagonal net $\left(t_{i j_{i}}\right)_{i} \subset T$ such that $\left(\gamma_{t_{i j_{i}}}, f_{t_{i j_{i}}}(z)\right) \rightarrow_{i}(\gamma, \alpha)$, and we obtain

$$
f_{\gamma}(z) \geq \lim \sup _{i} f_{t_{i j_{i}}}(z)=\alpha=\limsup { }_{i} f_{\gamma_{i}}(z) .
$$

The compactification process above covers in a natural way the compact framework. Namely, if $T$ is compact Hausdorff (hence, complete regular), then the family $\left\{f_{\gamma}, \gamma \in \widehat{T}\right\}$ above turns out to be the family of the usc regularization of the functions $f_{(\cdot)}(z)$, given by

$$
\bar{f}_{t}(z):=\limsup _{s \rightarrow t} f_{s}(z) .
$$

In this case, the indexed set $T$ does not change; i.e., $\widehat{T}=T$. Consequently, if additionally the functions $f_{(\cdot)}(z), z \in \operatorname{dom} f$, are already usc, then we recover the classical compact and continuous setting, originally proposed in 30.

The following theorem characterizes $\partial f(x)$ in terms of the functions $f_{\gamma}$ (see (16)) and the compact set $\widehat{T}(x)$, when $\tau$ is any topology on $T$. This result is crucial in the subsequent sections.

Theorem 4 Let $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, be convex functions and $f=\sup _{t \in T} f_{t}$. Then, for every $x \in X$,

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(f_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} . \tag{19}
\end{equation*}
$$

Proof First, we consider that the topology $\tau$ in $T$ is the discrete topology $\tau_{d}$, so that $C(T,[0,1]):=[0,1]^{T}$ and $\widehat{T}$ is compact. Moreover, since $\left(T, \tau_{d}\right)$ is completely regular, $\widehat{T}$ is Hausdorff (see, i.e., [24, §38]). Since $f=\sup _{\gamma \in \widehat{T}} f_{\gamma}$ and (18) holds, Proposition 1 applies and yields

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}^{d}(x)} \partial\left(f_{\gamma}^{d}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} \tag{20}
\end{equation*}
$$

where $f_{\gamma}^{d}$ and $\widehat{T}^{d}(x)$ are defined as in (16) and (17), respectively, but with respect to the topology $\tau_{d}$.

Now, let $\tau$ be any topology, so that $\tau \subset \tau_{d}$ and, for any $\left(\gamma_{t_{i}}\right)_{i} \subset \widehat{T}$,

$$
\begin{aligned}
\gamma_{t_{i}} \rightarrow \tau_{d} \gamma & \Longleftrightarrow \varphi\left(t_{i}\right) \rightarrow \gamma(\varphi) \text { for all } \varphi \in[0,1]^{T} \\
& \Longleftrightarrow \varphi\left(t_{i}\right) \rightarrow \gamma(\varphi) \text { for all } \varphi \in C(T,[0,1]) \\
& \Longleftrightarrow \gamma_{t_{i}} \rightarrow \tau \gamma
\end{aligned}
$$

hence, for every $z \in X$,

$$
f_{\gamma}^{d}(z)=\limsup _{\gamma_{t} \rightarrow \tau_{d} \gamma, t \in T} f_{t}(z) \leq \limsup _{\gamma_{t} \rightarrow \tau \gamma, t \in T} f_{t}(z)=f_{\gamma}(z) .
$$

Moreover, since for all $\gamma \in \widehat{T}^{d}(x)$ we have

$$
f(x)=f_{\gamma}^{d}(x) \leq f_{\gamma}(x) \leq f(x)
$$

we deduce that

$$
\begin{equation*}
\widehat{T}^{d}(x) \subset \widehat{T}(x) \text { and } \partial\left(f_{\gamma}^{d}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x) \subset \partial\left(f_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x) . \tag{21}
\end{equation*}
$$

Thus, by (20),

$$
\partial f(x) \subset \bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(f_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\}
$$

and (19) follows as the opposite inclusion is straightforward.
It is worth observing, from the inclusions in (21), that the discrete topology provides the simplest characterization of $\partial f(x)$, since it possibly involves less and smaller sets. Also observe that the intersection over finite-dimensional $L$ in (19) is superfluous in finite dimensions.

Theorem 4 covers the classical Valadier's setting where $T$ is compact Hausdorff and the mappings $f_{(\cdot)}(z), z \in \operatorname{dom} f$, are usc. In this case, formula (19) reduces to (see Proposition (1)

$$
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in T(x)} \partial\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\}
$$

Let us also observe that when $T$ admits a one-point compactification $T_{\Omega}:=$ $T \cup\{\Omega\}(\Omega \notin T)$, which occurs if and only if $T$ is locally compact Hausdorff (hence, complete regular), instead of $\left\{f_{\gamma}, \gamma \in \widehat{T}\right\}$ we can use the family $\left\{f_{\gamma_{t}}, t \in T ; f_{\Omega}\right\}$, where

$$
\begin{equation*}
f_{\Omega}(z):=\limsup _{t \rightarrow \Omega} f_{t}(z), \quad z \in X \tag{22}
\end{equation*}
$$

Indeed, in this case the Stone-Čech compactification of $T$ is

$$
\widehat{T}:=\left\{\gamma_{t}, t \in T\right\} \cup\left\{\lim _{i} \gamma_{t_{i}}:\left(t_{i}\right)_{i} \subset T, t_{i} \rightarrow \Omega\right\}
$$

where the limits $\lim _{i} \gamma_{t_{i}}$ and $t_{i} \rightarrow \Omega$ are in $[0,1]^{C(T,[0,1])}$ and $T_{\Omega}$, respectively. In this way we obtain, for all $t \in T$,

$$
\begin{equation*}
f_{\gamma_{t}}=\limsup _{\gamma_{s} \rightarrow \gamma_{t}, s \in T} f_{s}=\limsup _{s \rightarrow t, s \in T} f_{s}, \text { for } t \in T \text {, } \tag{23}
\end{equation*}
$$

due to the topological identification of $T$ with $\mathfrak{w}(T)$, and

$$
f_{\gamma}=\limsup _{\gamma_{t} \rightarrow \gamma, t \in T} f_{t}=\limsup _{\gamma_{t} \rightarrow \gamma, t \rightarrow \Omega, t \in T} f_{t}, \text { for } \gamma \in \widehat{T} \backslash T \text {. }
$$

Now, we observe that

$$
\left.\sup _{\gamma \in \widehat{T} \backslash T} f_{\gamma}=\sup _{\gamma \in \widehat{T} \backslash T} \limsup _{t} \rightarrow \gamma, t \rightarrow \Omega, t \in T\right) ~ f_{t}=\limsup _{t \rightarrow \Omega} f_{t}=f_{\Omega} .
$$

It is clear that the family $\left\{f_{\gamma_{t}}, t \in T ; f_{\Omega}\right\}$ and the (one-point compactification) index set $T \cup\{\Omega\}$ satisfy the assumption of Proposition [1 together with $f=$ $\sup \left\{f_{\gamma_{t}}, t \in T ; f_{\Omega}\right\}$. Thus, it suffices to consider Theorem With this new family $^{\text {w }}$ $\left\{f_{\gamma_{t}}, t \in T ; f_{\Omega}\right\}$ instead of the one of the original $f_{\gamma}$ 's.

In the particular case when $T=\mathbb{N}$, endowed with the discrete topology, for each $n \in \mathbb{N}$ we obtain

$$
f_{\gamma_{n}}=\limsup _{\gamma_{k} \rightarrow \gamma_{n}, k \in \mathbb{N}} f_{k}=\limsup _{k \rightarrow n, k \in \mathbb{N}} f_{k}=f_{n},
$$

so that the family to consider in Theorem 4 is

$$
\left\{f_{n}, n \in \mathbb{N} ; f_{\infty}\right\}
$$

where

$$
f_{\infty}=\limsup _{n \rightarrow \infty} f_{n}
$$

Corollary 5 Assume that $T$ is locally compact Hausdorff. Then for every $x \in X$ formula (19) holds with

$$
\widehat{T}(x)= \begin{cases}\left\{\gamma_{t}, t \in T, f_{\gamma_{t}}(x)=f(x)\right\}, & \text { if } f_{\Omega}(x)<f(x), \\ \left\{\gamma_{t}, t \in T, f_{\gamma_{t}}(x)=f(x), \Omega\right\}, & \text { if } f_{\Omega}(x)=f(x),\end{cases}
$$

and, when $T=\mathbb{N}$,

$$
\widehat{T}(x)= \begin{cases}\left\{n \in \mathbb{N}, f_{n}(x)=f(x)\right\}, & \text { if } f_{\infty}(x)<f(x), \\ \left\{n \in \mathbb{N}, f_{n}(x)=f(x), \infty\right\}, & \text { if } f_{\infty}(x)=f(x) .\end{cases}
$$

## 5 From non-continuous to continuous. Enhanced formulas

We give in this section some new characterizations of $\partial f(x)$, which provide additional insight to Theorem 4 and that are applied in Section 6 .

According to Theorem $4 \partial f(x)$ only involves the active functions $f_{\gamma}$, i.e., when $\gamma \in \widehat{T}(x)$. The idea behind the following result is to replace these $f_{\gamma}$ 's by the new functions $\tilde{f}_{\gamma}: X \rightarrow \mathbb{R}_{\infty}, \gamma \in \widehat{T}$, defined as

$$
\begin{equation*}
\tilde{f}_{\gamma}(z):=\limsup _{\gamma_{t} \rightarrow \gamma, f_{t}(x) \rightarrow f(x), t \in T} f_{t}(z), \tag{24}
\end{equation*}
$$

considering only those nets $\left(t_{i}\right)_{i} \subset T$ associated with functions $f_{t_{i}}$ approaching the supremum function $f$ at the nominal point $x$. Observe that if $\gamma \in \widehat{T} \backslash \widehat{T}(x)$, then $\tilde{f}_{\gamma} \equiv-\infty$ by the convention $\sup \emptyset=-\infty$, and this function is ignored when taking the supremum.

Remember that $T$ is endowed with any topology.
Theorem 6 For every $x \in X$ we have

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\tilde{f}_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} \tag{25}
\end{equation*}
$$

where $\tilde{f}_{\gamma}$ and $\widehat{T}(x)$ are defined in (24) and (17), respectively.

Proof We only need to check the inclusion " $\subset$ " when $\tau$ is the discrete topology $\tau_{d}$, and $\partial f(x) \neq \emptyset$; hence, $f$ is lsc at $x$ and proper, and we may suppose, without loss of generality, that $x=\theta$ and $f(\theta)=0$. Let us fix a closed convex neighborhood $U$ of $\theta$ such that $f(z) \geq-1$, for all $z \in U$, and denote by $g_{t}: X \rightarrow \mathbb{R}_{\infty}, t \in T$, the functions given by

$$
\begin{equation*}
g_{t}(z):=\max \left\{f_{t}(z),-1\right\} . \tag{26}
\end{equation*}
$$

Thus, for all $z \in U$,

$$
f(z)=\max \{f(z),-1\}=\sup _{t \in T} \max \left\{f_{t}(z),-1\right\}=\sup _{t \in T} g_{t}(z),
$$

and so, applying (19) with the discrete topology $\tau_{d}$ on $T$ to the family $\left\{g_{t}, t \in T\right\}$,

$$
\begin{equation*}
\partial f(\theta)=\partial\left(\sup _{t \in T} g_{t}\right)(\theta)=\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcup_{\gamma \in \widetilde{T}(\theta)} \partial\left(g_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)\right\} \tag{27}
\end{equation*}
$$

where $g_{\gamma}:=\limsup _{\gamma_{t} \rightarrow \gamma, t \in T} g_{t}$ and $\widetilde{T}(\theta):=\left\{\gamma \in \widehat{T}: g_{\gamma}(\theta)=0\right\}$.
Let us first verify that

$$
\begin{equation*}
\widetilde{T}(\theta)=\widehat{T}(\theta) \tag{28}
\end{equation*}
$$

Indeed, if $\gamma \in \widetilde{T}(\theta)$ so that

$$
0=g_{\gamma}(\theta)=\limsup _{\gamma_{t} \rightarrow \gamma, t \in T} g_{t}(\theta) \leq \max \left\{f_{\gamma}(\theta),-1\right\} \leq \max \{f(\theta),-1\}=0,
$$

then $f_{\gamma}(\theta)=0$ and, so, $\gamma \in \widehat{T}(\theta)$. Conversely, if $\gamma \in \widehat{T}(\theta)$, then

$$
0=f_{\gamma}(\theta) \leq g_{\gamma}(\theta) \leq \sup _{\gamma \in \widehat{T}} g_{\gamma}(\theta)=\sup _{t \in T} g_{t}(\theta)=f(\theta)=0
$$

and so $\gamma \in \widetilde{T}(\theta)$.
Next, we fix $\gamma \in \widetilde{T}(\theta)$ and, by the definition of this set, let $\left(\bar{t}_{i}\right)_{i} \subset T$ be a net such that $\gamma_{\bar{t}_{i}} \rightarrow \gamma$ and $\lim _{i} g_{\bar{t}_{i}}(\theta)=0$; hence,

$$
\begin{equation*}
\lim _{i} f_{\bar{t}_{i}}(\theta)=\lim _{i} g_{\bar{t}_{i}}(\theta)=0 . \tag{29}
\end{equation*}
$$

We also introduce the functions $\varphi_{z}, z \in \operatorname{dom} f$, defined on $T$ as follows

$$
\varphi_{z}(t):=(\max \{f(z)+1,1\})^{-1}\left(g_{t}(z)+1\right)
$$

which are $\left(\tau_{d^{-}}\right)$continuous functions such that $\varphi_{z}(t) \in[0,1]$ for all $t \in T$, because

$$
-1 \leq g_{t}(z) \leq \max \{f(z),-1\}<+\infty \text { for all } t \in T \text { and } z \in \operatorname{dom} f
$$

Hence, for every $\gamma_{t_{i}} \rightarrow \gamma$ we have $\varphi_{z}\left(t_{i}\right) \rightarrow_{i} \gamma\left(\varphi_{z}\right)$, and this entails

$$
\begin{equation*}
g_{t_{i}}(z) \rightarrow_{i}-1+(\max \{f(z)+1,1\}) \gamma\left(\varphi_{z}\right) \in \mathbb{R} \tag{30}
\end{equation*}
$$

Consequently, by taking into account that $\gamma_{\bar{t}_{i}} \rightarrow \gamma$ and $\lim _{i} f_{\bar{t}_{i}}(\theta)=0$ (see (29)) we obtain

$$
\begin{equation*}
g_{\gamma}=\limsup _{\gamma_{t} \rightarrow \gamma, t \in T} g_{t}=\lim _{\gamma_{t} \rightarrow \gamma} g_{t}=\lim _{\gamma_{t} \rightarrow \gamma, f_{t}(\theta) \rightarrow 0} g_{t}, \tag{31}
\end{equation*}
$$

which leads us to

$$
\begin{align*}
g_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f} & =\lim _{\gamma_{t} \rightarrow \gamma, f_{t}(\theta) \rightarrow 0}\left(g_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)  \tag{32}\\
& \leq \max \left\{\limsup _{\gamma_{t} \rightarrow \gamma, f_{t}(\theta) \rightarrow 0}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right),-1\right\} . \tag{33}
\end{align*}
$$

But the two functions on the left and the right have the same value 0 at $\theta$, and so

$$
\begin{aligned}
\partial\left(g_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta) & \subset \partial\left(\max \left\{\limsup _{\gamma_{t} \rightarrow \gamma, f_{t}(\theta) \rightarrow 0} f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f,-1}\right\}\right)(\theta) \\
& =\partial\left(\limsup _{\gamma_{t} \rightarrow \gamma, f_{t}(\theta) \rightarrow 0} f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)=\partial\left(\tilde{f}_{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)
\end{aligned}
$$

where the first equality comes from Proposition 1 applied to the finite family $\left\{\tilde{f}_{\gamma},-1\right\}$. Finally, the desired inclusion follows thanks to (27) and (28).

Let us introduce a function which asigns to each given $\gamma \in \widehat{T}(x)$ a net $\left(t_{i}^{\gamma}\right)_{i} \subset T$ such that

$$
\begin{equation*}
\gamma_{t_{i}^{\gamma}} \rightarrow \gamma, f_{t_{i}^{\gamma}}(x) \rightarrow f(x) . \tag{34}
\end{equation*}
$$

Then, according to (33),

$$
\lim _{\gamma_{t} \rightarrow \gamma}\left(g_{t}+\mathrm{I}_{\operatorname{dom} f}\right)=\lim _{i}\left(g_{t_{i}^{\gamma}}+\mathrm{I}_{L \cap \operatorname{dom} f}\right) \leq \max \left\{\limsup _{i}\left(f_{t_{i}^{\gamma}}+\mathrm{I}_{L \cap \operatorname{dom} f}\right),-1\right\}
$$

and we obtain, reasoning as above,

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\limsup f_{i}^{\gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} . \tag{35}
\end{equation*}
$$

The use of the functions $g_{t}$ allows us to formulate $\partial f(x)$ involving only limits instead of upper limits. In fact, from (27), (28) and (31) we get

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\lim _{\gamma_{t} \rightarrow \gamma, f_{t}(x) \rightarrow f(x)}\left(g_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)\right)(x)\right\} \tag{36}
\end{equation*}
$$

Corollary 7 Suppose that the function $f$ is finite and continuous somewhere. Then, for every $x \in X$,

$$
\begin{align*}
\partial f(x) & =\overline{\operatorname{co}}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\lim \sup _{i} f_{t_{i}^{\gamma}}\right)(x)\right\}+\mathrm{N}_{\operatorname{dom} f}(x)  \tag{37}\\
& =\operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\lim \sup _{i} f_{t_{i}^{\gamma}}\right)(x)\right\}+\mathrm{N}_{\operatorname{dom} f}(x) \quad\left(\text { if } X=\mathbb{R}^{n}\right) \tag{38}
\end{align*}
$$

where $\left(t_{i}^{\gamma}\right)$ is defined in (34).

Proof Suppose, without loss of generality, that $x=\theta$ and $f(\theta)=0$. According to (35), and using (3),

$$
\begin{aligned}
\partial f(\theta) & =\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(\theta)} \partial\left(\lim \sup _{i} f_{t_{i}^{\gamma}}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)\right\} \\
& =\bigcap_{L \in \mathcal{F}(\theta)}\left(\operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(\theta)} \partial\left(\lim \sup _{i} f_{t_{i}^{\gamma}}\right)(\theta)\right\}+\mathrm{N}_{\operatorname{dom} f}(\theta)+L^{\perp}\right),
\end{aligned}
$$

and (38) follows. To prove (37) we first obtain, due to the last relation and (2),

$$
\begin{equation*}
\partial f(\theta) \subset \operatorname{cl}(A+B)=\partial \sigma_{A+B}(\theta)=\partial\left(\sigma_{A}+\sigma_{B}\right)(\theta) \tag{39}
\end{equation*}
$$

where $A:=\operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(\theta)} \partial\left(\limsup _{i} f_{t_{i}^{\gamma}}\right)(\theta)\right\}$ and $B:=\mathrm{N}_{\operatorname{dom} f}(\theta)$.
Since $\limsup \sup _{i} f_{t_{i}^{\gamma}} \leq f$ and both functions coincide at $\theta$, we have $A \subset \partial f(\theta)$. There also exist $m \geq 0, x_{0} \in \operatorname{dom} f$ and $\theta$-neighborhood $U \subset X$ such that $f\left(x_{0}+\right.$ $y) \leq m$, for all $y \in U$. Then

$$
\begin{equation*}
\sigma_{A}\left(x_{0}+y\right) \leq \sigma_{\partial f(\theta)}\left(x_{0}+y\right) \leq f\left(x_{0}+y\right) \leq m \text { for all } y \in U \tag{40}
\end{equation*}
$$

that is, $\sigma_{A}$ is continuous at $x_{0}$. Consequently, since $\sigma_{B}\left(x_{0}\right) \leq 0$, (39) and (3) entail

$$
\partial f(\theta) \subset \partial \sigma_{A}(\theta)+\partial \sigma_{B}(\theta)=\operatorname{cl}(A)+B
$$

and the inclusion " $\subset$ " in (37) follows. The opposite inclusion is straightforward.
The following corollary provides a characterization of $\partial f(x)$ in terms only of the active original functions $f_{t}$ 's.

Corollary 8 Fix $x \in X$. If for each net $\left(t_{i}\right)_{i} \subset T$ satisfying $f_{t_{i}}(x) \rightarrow f(x)$, there exist a subnet $\left(t_{i_{j}}\right)_{j} \subset T$ of $\left(t_{i}\right)_{i}$ and an index $t \in T$ such that

$$
\begin{equation*}
\lim \sup _{j} f_{t_{i_{j}}}(z) \leq f_{t}(z) \quad \text { for all } z \in \operatorname{dom} f \tag{41}
\end{equation*}
$$

then we have

$$
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{t \in T(x)} \partial\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\}
$$

Proof Given any $\gamma \in \widehat{T}(x)$ such that $\gamma_{t_{i}} \rightarrow \gamma$ and $f_{t_{i}}(x) \rightarrow f(x)$, for some net $\left(t_{i}\right)_{i} \subset T$, we choose a subnet $\left(t_{i_{j}}^{\gamma}\right)_{j}$ in (34) satisfying (41) for a certain $t^{\gamma} \in T$.

Then $t^{\gamma} \in T(x)$, taking into account (41) with $z=x$, and by (35)

$$
\begin{aligned}
\partial f(x) & =\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(\lim \sup _{j} f_{t_{i_{j}}^{\gamma}}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\} \\
& \subset \bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{T}(x)} \partial\left(f_{t \gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\}
\end{aligned}
$$

where the last inclusion holds as $\lim \sup _{j} f_{t_{i_{j}}}+\mathrm{I}_{L \cap \operatorname{dom} f} \leq f_{t \gamma}+\mathrm{I}_{L \cap \operatorname{dom} f}$, by (41), and these two functions take the same value at $x$. The inclusion " $\subset$ " follows as we have shown that $t^{\gamma} \in T(x)$. The opposite inclusion is immediate.

## 6 From continuous to non-continuous

In this section, we consider again a family $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, of convex functions defined on $X$, and the supremum function $f:=\sup _{t \in T} f_{t}$. Based on the results of the previous section we provide characterizations of $\partial f(x)$ involving only the $f_{t}$ 's and not the regularized ones, i.e, the $f_{\gamma}$ 's. We shall need the following technical lemmas. In what follows, $\mathrm{cl}^{s}$ stands for the strong topology on $X^{*}$ (usually denoted by $\beta\left(X^{*}, X\right)$ ).

Lemma 9 Assume that the convex functions $f_{t}, t \in T$, are proper, lsc, and such that $f_{\mid \operatorname{aff}(\operatorname{dom} f)}$ is continuous on $\operatorname{ri}(\operatorname{dom} f)$, assumed nonempty. Let $x \in \operatorname{dom} f$ and the net $\left(z_{i}^{*}\right)_{i \in I} \subset X^{*}$ such that

$$
\begin{equation*}
\lim _{i}\left(\left\langle z_{i}^{*}, x\right\rangle-\inf _{t \in T} f_{t}^{*}\left(z_{i}^{*}\right)\right)=f(x) \tag{42}
\end{equation*}
$$

and for all $z \in \operatorname{dom} f$

$$
\begin{equation*}
\limsup _{i}\left(\left\langle z_{i}^{*}, z\right\rangle-\inf _{t \in T} f_{t}^{*}\left(z_{i}^{*}\right)\right)>-\infty \tag{43}
\end{equation*}
$$

Then, there exist a subnet $\left(z_{i_{j}}^{*}\right)_{j}$ of $\left(z_{i}^{*}\right)_{i}$ and $z^{*} \in X^{*}$ such that

$$
\begin{equation*}
z^{*} \in \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+(\operatorname{aff}(\operatorname{dom} f))^{\perp}\right), \text { for all } \varepsilon>0 \tag{44}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle z_{i_{j}}^{*}-z^{*}, z\right\rangle \rightarrow_{j} 0, \text { for all } z \in \operatorname{aff}(\operatorname{dom} f) . \tag{45}
\end{equation*}
$$

In particular, if $\operatorname{dom} f$ is finite-dimensional, then 44) also holds with $\mathrm{cl}^{s}$ instead of cl.

Proof We may assume that $x=\theta$ and $f(\theta)=0$, and denote $E:=\operatorname{aff}(\operatorname{dom} f)$ which is a closed subspace with dual $E^{*}$. We also denote $h:=\inf _{t \in T} f_{t}^{*}$, so that (see (44))

$$
\begin{equation*}
h^{*}=\left(\inf _{t \in T} f_{t}^{*}\right)^{*}=\sup _{t \in T} f_{t}^{* *}=\sup _{t \in T} f_{t}=f \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
h^{*}(\theta)+h\left(z_{i}^{*}\right)=f(\theta)+h\left(z_{i}^{*}\right)=h\left(z_{i}^{*}\right) \rightarrow 0 . \tag{47}
\end{equation*}
$$

Hence, for every fixed $\varepsilon>0$, there is some $i_{0} \in I$ such that for all $i \succeq i_{0}$

$$
\begin{equation*}
h^{*}(\theta)+h\left(z_{i}^{*}\right)=\sup _{t \in T} f_{t}(\theta)+\inf _{t \in T} f_{t}^{*}\left(z_{i}^{*}\right)=h\left(z_{i}^{*}\right)<\varepsilon \tag{48}
\end{equation*}
$$

and so

$$
\begin{equation*}
\left(z_{i}^{*}\right)_{i \succeq i_{0}} \subset \partial_{\varepsilon} h^{*}(\theta)=\partial_{\varepsilon} f(\theta) \tag{49}
\end{equation*}
$$

Now, using the continuity assumption, we choose $x_{0} \in \operatorname{dom} f$, a $\theta$-neighborhood $U \subset X$ and $r \geq 0$ such that

$$
\begin{equation*}
f\left(x_{0}+y\right) \leq r \text { for all } y \in U \cap E \tag{50}
\end{equation*}
$$

and, by (43) with $z=x_{0}$ and (47),

$$
\limsup _{i}\left\langle z_{i}^{*}, x_{0}\right\rangle>-\infty
$$

Therefore we may assume, up to some subnet, that $\inf _{i}\left\langle z_{i}^{*}, x_{0}\right\rangle>-\infty$ and, so, by (49) and (50), there is some $m>0$ such that

$$
\begin{equation*}
\left\langle z_{i}^{*}, y\right\rangle \leq f\left(x_{0}+y\right)+\varepsilon-\inf _{i}\left\langle z_{i}^{*}, x_{0}\right\rangle \leq m, \text { for all } y \in U \cap E \text { and for all } i \tag{51}
\end{equation*}
$$

that is $\left(z_{i}^{*}\right)_{i} \subset(U \cap E)^{\circ}$. Since the last set is weak*-compact in $E^{*}$, by the Alaoglu-Banach-Bourbaki theorem, there exists a subnet $\left(z_{i_{j} \mid E}^{*}\right)_{j}$ and $\tilde{z}^{*} \in E^{*}$ such that

$$
\begin{equation*}
\left\langle z_{i_{j} \mid E}^{*}-\tilde{z}^{*}, u\right\rangle \rightarrow_{j} 0 \text { for all } u \in E . \tag{52}
\end{equation*}
$$

Moreover, by the Hahn-Banach theorem, $\tilde{z}^{*} \in E^{*}$ is extended to some $z^{*} \in X^{*}$, which satisfies

$$
\begin{equation*}
\left\langle z_{i_{j}}^{*}-z^{*}, u\right\rangle=\left\langle z_{i_{j} \mid E}^{*}-\tilde{z}^{*}, u\right\rangle \rightarrow_{j} 0 \text { for all } u \in E . \tag{53}
\end{equation*}
$$

Now, using (48), we see that for each $i$ there exists $t_{i} \in T$ such that

$$
f_{t_{i}}(\theta)+f_{t_{i}}^{*}\left(z_{i}^{*}\right) \leq f_{t_{i}}^{*}\left(z_{i}^{*}\right)<\varepsilon,
$$

entailing that $z_{i}^{*} \in \partial_{\varepsilon} f_{t_{i}}(\theta)$ and

$$
-f_{t_{i}}(\theta)=\left\langle z_{i}^{*}, \theta\right\rangle-f_{t_{i}}(\theta) \leq f_{t_{i}}^{*}\left(z_{i}^{*}\right)<\varepsilon ;
$$

that is, $t_{i} \in T_{\varepsilon}(\theta)$ and so,

$$
z_{i}^{*} \in \bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta) .
$$

We fix a weak* (strong, when $\operatorname{dom} f$ is finite-dimensional) $\theta$-neighborhood $V \subset X^{*}$. Since $E^{*}$ is isomorphic to the quotient space $X_{/ E \pm}^{*}$, then $V_{\mid E}:=\left\{u_{\mid E}^{*}: u^{*} \in V\right\} \in$ $\mathcal{N}_{E^{*}}([12])$, where $u_{\mid E}^{*}$ denotes the restriction of $u^{*}$ to $E^{*}$. Consequently, writing

$$
z_{i_{j} \mid E}^{*} \in A:=\left\{u_{\mid E}^{*} \in E^{*}: u^{*} \in \bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)\right\}
$$

and passing to the limit on $j$, (53) leads us to

$$
\begin{equation*}
z_{\mid E}^{*} \in A+V_{\mid E} . \tag{54}
\end{equation*}
$$

In other words, there are $u^{*} \in \bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)$ and $v^{*} \in V$ such that $z_{\mid E}^{*}=$ $u_{\mid E}^{*}+v_{\mid E}^{*}$; that is,

$$
\left\langle z^{*}, u\right\rangle=\left\langle u^{*}+v^{*}, u\right\rangle \quad \text { for all } u \in E,
$$

implying that

$$
z^{*} \in u^{*}+v^{*}+E^{\perp} \subset \bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+E^{\perp}+V .
$$

The conclusion follows then by intersecting over $V$ and, after, over $\varepsilon>0$.

In the currrent framework, $\widehat{X^{*}}$ is the Stone-Čech compactification of $X^{*}$, with respect to the discrete topology, and the mappings $\gamma_{z^{*}}:[0,1]^{X^{*}} \rightarrow[0,1], z^{*} \in X^{*}$, are defined as in (13), so that the convergence $\gamma_{z_{i}^{*}} \rightarrow \gamma$ for for a net $\left(z_{i}^{*}\right)_{i} \subset X^{*}$ and $\gamma \in \widehat{X^{*}}$ means

$$
\varphi\left(z_{i}^{*}\right) \rightarrow \gamma(\varphi) \quad \text { for all } \varphi \in[0,1]^{X^{*}}
$$

Lemma 10 Assume in Lemma 9 that the net $\left(\gamma_{z_{i}^{*}}\right)_{i}$ converges in $\widehat{X^{*}}$. Then for the function

$$
\psi(z):=\limsup _{i}\left(\left\langle z_{i}^{*}, z\right\rangle-\inf _{t \in T} f_{t}^{*}\left(z_{i}^{*}\right)+\mathrm{I}_{\operatorname{dom} f}(z)\right), z \in X
$$

we have

$$
\begin{aligned}
\partial \psi(x) & \subset \mathrm{N}_{\operatorname{dom} f}(x)+\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+(\operatorname{aff}(\operatorname{dom} f))^{\perp}\right) \\
& \subset \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+\mathrm{N}_{\operatorname{dom} f}(\theta)\right)
\end{aligned}
$$

with $\mathrm{cl}^{s}$ instead of cl when $\operatorname{dom} f$ is finite-dimensional.

Proof We may suppose that $x=\theta$ and $f(\theta)=0$. By Lemma 9 there exist a subnet $\left(z_{i_{j}}^{*}\right)_{j}$ of $\left(z_{i}^{*}\right)_{i}$ and

$$
z^{*} \in \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+(\operatorname{aff}(\operatorname{dom} f))^{\perp}\right)
$$

such that $\left(z_{i_{j}}^{*}\right)_{j}$ weak*-converges to $z^{*}$ in $E^{*}($ where $E=\operatorname{aff}(\operatorname{dom} f))$.
We introduce the functions $g_{u^{*}}: X \rightarrow \mathbb{R}_{\infty}, u^{*} \in X^{*}$, defined as

$$
g_{u^{*}}:=\max \left\{u^{*}-h\left(u^{*}\right),-1\right\},
$$

where $h=\inf _{t \in T} f_{t}^{*}$ (already used in the proof of Lemma 9). Observe that (recall (46))

$$
-1 \leq g_{u^{*}} \leq \max \left\{h^{*},-1\right\}=\max \{f,-1\},
$$

and

$$
\varphi_{z}\left(u^{*}\right):=\frac{g_{u^{*}}(z)+1}{\max \{f(z)+1,1\}} \in[0,1], \text { for all } z \in \operatorname{dom} f .
$$

Hence, since $\varphi_{z}$ is obviously continuous on $X^{*}$ endowed with the discrete topology, the convergence assumption of $\left(\gamma_{z_{i}^{*}}\right)_{i}$ ensures that, for each $z \in \operatorname{dom} f$, the net

$$
\gamma_{z_{i}^{*}}\left(\varphi_{z}\right)=\frac{g_{z_{i}^{*}}(z)+1}{\max \{f(z)+1,1\}}
$$

also converges, as well as the net $\left(g_{z_{i}^{*}}(z)\right)_{i}$. Then, taking into account (42) and (45), we obtain

$$
\begin{aligned}
\lim _{i} g_{z_{i}^{*}}(z) & =\lim _{i} \max \left\{\left\langle z_{i}^{*}, z\right\rangle-h\left(z_{i}^{*}\right),-1\right\} \\
& =\lim _{j} \max \left\{\left\langle z_{i_{j}}^{*}, z\right\rangle,-1\right\}=\max \left\{\left\langle z^{*}, z\right\rangle,-1\right\},
\end{aligned}
$$

which gives

$$
\limsup _{i}\left\langle z_{i}^{*}, z\right\rangle \leq \limsup _{i}\left(\max \left\{\left\langle z_{i}^{*}, z\right\rangle,-1\right\}\right)=\max \left\{\left\langle z^{*}, z\right\rangle,-1\right\}
$$

But both functions $\lim \sup _{i} z_{i}^{*}+\mathrm{I}_{\mathrm{dom} f}$ and $\max \left\{z^{*},-1\right\}+\mathrm{I}_{\mathrm{dom} f}$ coincide at $\theta$, and so

$$
\partial\left(\limsup _{i}\left(z_{i}^{*}+\mathrm{I}_{\operatorname{dom} f}\right)\right)(\theta) \subset \partial\left(\max \left\{z^{*}+\mathrm{I}_{\operatorname{dom} f},-1\right\}\right)(\theta)
$$

and (19) applied to the (finite) family $\left\{z^{*}+\mathrm{I}_{\operatorname{dom} f},-1\right\}$ yields (recall (42))

$$
\begin{aligned}
\partial \psi(\theta) & =\partial\left(\limsup _{i}\left(z_{i}^{*}+\mathrm{I}_{\mathrm{dom} f}\right)\right)(\theta) \\
& \subset z^{*}+\mathrm{N}_{\operatorname{dom} f}(\theta) \\
& \subset \mathrm{N}_{\operatorname{dom} f}(\theta)+\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+(\operatorname{aff}(\operatorname{dom} f))^{\perp}\right) \\
& \subset \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+\mathrm{N}_{\operatorname{dom} f}(\theta)\right)
\end{aligned}
$$

Theorem 11 Let $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, be convex functions and $f=\sup _{t \in T} f_{t}$. Then, for every $x \in X$,

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right)\right\} \tag{55}
\end{equation*}
$$

If, in addition,

$$
\begin{equation*}
\operatorname{cl} f=\sup _{t \in T}\left(\operatorname{cl} f_{t}\right) \tag{56}
\end{equation*}
$$

then

$$
\begin{equation*}
\partial f(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)+\mathrm{N}_{L \cap \operatorname{dom} f}(x)\right)\right\} . \tag{57}
\end{equation*}
$$

Remark 1 (before the proof) Formula (55) leads straightforwardly to the following characterization of $\partial f(x)$, using the strong closure

$$
\partial f(x)=\bigcap_{L \in \mathcal{F}(x), \varepsilon>0} \overline{\cos }^{s}\left\{\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(x)\right\}
$$

improving the one of Proposition 3 which is given in terms of the weak*-closure. However, on despite that both formulas involve similar elements, the order in taking the intersection over $\varepsilon$ leads to different interpretations of $\partial f(x)$. For instance, if $T$ is finite, $T=T(x)$ and $f$ is continuous, then (55) reads

$$
\partial f(x)=\operatorname{co}\left\{\bigcup_{t \in T(x)} \partial f_{t}(x)\right\},
$$

giving Valadier's formula (see, e.g., 30), while Proposition 3 yields

$$
\partial f(x)=\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left\{\bigcup_{t \in T(x)} \partial_{\varepsilon} f_{t}(x)\right\},
$$

which turns out to be the Brøndsted formula (1] see, also, [15, Corollary 12]).
Proof The inclusions " $\supset$ " in both formulas are straightforward. We may suppose, without loss of generality, that $x=\theta, f(\theta)=0$ and $\partial f(\theta) \neq \emptyset$; hence,

$$
\begin{equation*}
\partial(\mathrm{cl} f)(\theta)=\partial f(\theta) \text { and } f(\theta)=(\mathrm{cl} f)(\theta)=0 \tag{58}
\end{equation*}
$$

We proceed in three steps:
Step 1. We assume that all the $f_{t}$ 's are proper and lsc; hence, (56) obviously holds. We fix $L \in \mathcal{F}(\theta)$, and define the functions

$$
\begin{equation*}
\tilde{f}_{t}:=f_{t}+\mathrm{I}_{L}, t \in T, \quad \text { and } \quad h:=\inf _{t \in T} \tilde{f}_{t}^{*} . \tag{59}
\end{equation*}
$$

The $\tilde{f_{t}}$ 's are proper and lsc, and we have (see (4))

$$
\begin{equation*}
\left(f+\mathrm{I}_{L}\right)(z)=\sup _{t \in T} \tilde{f}_{t}(z)=\sup _{t \in T} \tilde{f}_{t}^{* *}(z)=\left(\inf _{t \in T} \tilde{f}_{t}^{*}\right)^{*}(z)=h^{*}(z) \tag{60}
\end{equation*}
$$

that is,

$$
\left(f+\mathrm{I}_{L}\right)(z)=\sup \left\{\left\langle z, z^{*}\right\rangle-h\left(z^{*}\right), z^{*} \in X^{*}\right\},
$$

and (35) applied with $T=X^{*}$ (endowed with the discrete topology) yields

$$
\begin{equation*}
\partial\left(f+\mathrm{I}_{L}\right)(\theta) \subset \operatorname{co}\left\{\bigcup_{\gamma \in \widehat{X^{*}}(\theta)} \partial\left(\limsup _{i}\left(z_{i}^{* \gamma}-h\left(z_{i}^{*^{\gamma}}\right)+\mathrm{I}_{L \cap \operatorname{dom} f}\right)\right)(\theta)\right\} \tag{61}
\end{equation*}
$$

where $\widehat{X^{*}}(\theta)$ repesents the set $\widehat{T}(\theta)$ given in (17); that is,

$$
\widehat{X^{*}}(\theta)=\left\{\gamma \in \widehat{X^{*}}: \limsup _{\gamma_{z^{*}} \rightarrow \gamma}\left(-h\left(z^{*}\right)\right)=0\right\}
$$

and $\left(z_{i}^{* \gamma}\right)_{i} \subset X^{*}$ is a fixed net such that $\gamma_{z_{i}^{* \gamma}} \rightarrow \gamma$ and $h\left(z_{i}^{* \gamma}\right) \rightarrow 0$ (by (34)). Consequenlty, for every $\gamma \in \widehat{X^{*}}(\theta)$, Lemma 10 applies and yields

$$
\begin{align*}
& \partial\left(\underset{i}{\left.\limsup \left(z_{i}^{*^{\gamma}}-h\left(z_{i}^{*^{\gamma}}\right)+\mathrm{I}_{L \cap \operatorname{dom} f}\right)\right)(\theta)}\right. \\
& \quad \subset \bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T_{\varepsilon}^{1}(\theta)} \partial_{\varepsilon} \tilde{f}_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right), \tag{62}
\end{align*}
$$

where

$$
\begin{equation*}
T_{\varepsilon}^{1}(\theta):=\left\{t \in T: \tilde{f}_{t}(\theta) \geq-\varepsilon\right\}=T_{\varepsilon}(\theta) . \tag{63}
\end{equation*}
$$

Indeed, condition (43) is satisfied when the left-hand side in (62) is nonempty, and thus the function $\lim \sup _{i}\left(z_{i}^{*^{\gamma}}-h\left(z_{i}^{*^{\gamma}}\right)+\mathrm{I}_{L \cap \operatorname{dom} f}\right)$ is proper. Consequently, combining (61), (62) and (63),

$$
\begin{equation*}
\partial\left(f+\mathrm{I}_{L}\right)(\theta) \subset \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} \tilde{f}_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right\} \tag{64}
\end{equation*}
$$

and the inclusion " $\subset$ " in (55) follows since $\partial f(\theta) \subset \partial\left(f+\mathrm{I}_{L}\right)(\theta)$ and

$$
\partial_{\varepsilon} \tilde{f}_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta) \subset \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta) .
$$

Moreover, due to the fact that $\partial_{\varepsilon} \tilde{f}_{t}(\theta) \subset \operatorname{cl}\left(\partial_{\varepsilon} f_{t}(\theta)+L^{\perp}\right)$ (see, e.g., [17]), (64) implies that

$$
\begin{align*}
\partial f(\theta) & \subset \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \operatorname{cl}\left(\partial_{\varepsilon} f_{t}(\theta)+L^{\perp}\right)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right\} \\
& \subset \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right)\right\} \\
& =\operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{\varepsilon} f_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right\}, \tag{65}
\end{align*}
$$

which yields the inclusion " $\subset$ " in (57).
Step 2. We suppose that (56) holds and we fix $L \in \mathcal{F}(\theta)$. By (58) we choose a $\theta$-neighborhood $U \subset X$ such that

$$
\begin{equation*}
f(z) \geq(\operatorname{cl} f)(z) \geq-1, \text { for all } z \in U, \tag{66}
\end{equation*}
$$

and denote $S:=\left\{t \in T:\right.$ cl $f_{t}$ is proper $\}$. We define the functions

$$
g_{t}:=\operatorname{cl} f_{t}, \text { if } t \in S, \text { and } g_{t}:=\max \left\{\mathrm{cl} f_{t},-1\right\}, \text { otherwise. }
$$

Then (see the proof of [15, Theorem 4], page 871) $g_{t}$ is proper, lsc and convex,

$$
g(z):=\sup _{t \in T} g_{t}(z)=(\operatorname{cl} f)(z), \text { for all } z \in U
$$

hence, $g(\theta)=0$,

$$
\begin{gathered}
\left.\left\{t \in T: g_{t}(\theta) \geq-\varepsilon\right\} \subset T_{\varepsilon}(\theta) \cap S, \forall \varepsilon \in\right] 0,1[, \\
\left.\partial_{\varepsilon} g_{t}(\theta) \subset \partial_{2 \varepsilon} f_{t}(\theta), \partial_{\varepsilon}\left(g_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta) \subset \partial_{2 \varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta), \forall \varepsilon \in\right] 0,1[,
\end{gathered}
$$

and

$$
\begin{equation*}
\partial f(\theta)=\partial(\operatorname{cl} f)(\theta)=\partial g(\theta) \tag{67}
\end{equation*}
$$

Consequently, by Step 1,

$$
\begin{aligned}
\partial f(\theta)=\partial g(\theta) & =\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T, g_{t}(\theta) \geq-\varepsilon} \partial_{\varepsilon}\left(g_{t}+\mathrm{I}_{L \cap \operatorname{dom} g}\right)(\theta)\right)\right\} \\
& \subset \bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{2 \varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)\right)\right\}
\end{aligned}
$$

entailing the desired inclusion " $\subset$ " in (55).

Similarly, (65) yields

$$
\begin{align*}
\partial f(\theta) & =\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{0<\varepsilon<1} \operatorname{cl}\left(\bigcup_{t \in T, g_{t}(\theta) \geq-\varepsilon} \partial_{\varepsilon} g_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} g}(\theta)\right)\right\} \\
& \subset \bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{0<\varepsilon<1} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\theta)} \partial_{2 \varepsilon} f_{t}(\theta)+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right\} \tag{68}
\end{align*}
$$

which easily leads to the inclusion " $\subset$ " in (57).
Step 3. We prove (55) in the general case, without assuming (56). We fix $L \in \mathcal{F}(\theta)$ and define

$$
\hat{f_{t}}:=f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}
$$

so that

$$
\begin{gathered}
f_{L}:=\sup _{t \in T} \hat{f_{t}}=f+\mathrm{I}_{L \cap \operatorname{dom} f}=f+\mathrm{I}_{L}, \\
\hat{f}_{t}(\theta)=f_{t}(\theta), f_{L}(\theta)=0, \text { and } \operatorname{dom} f_{L}=L \cap \operatorname{dom} f .
\end{gathered}
$$

Moreover, the family $\left\{\hat{f}_{t}, t \in T\right\}$ satisfies condition (56) (see the proof of Proposition (3). Since (see [8, Lemma 3.1])

$$
\partial f(\theta)=\bigcap_{L \in \mathcal{F}(\theta)} \partial\left(f+\mathrm{I}_{L}\right)(\theta)=\bigcap_{L \in \mathcal{F}(\theta)} \partial f_{L}(\theta)
$$

applying Step 2 to the family $\left\{\hat{f}_{t}, t \in T\right\}$ we get

$$
\begin{aligned}
\partial f(\theta) & =\bigcap_{L \in \mathcal{F}(\theta)} \partial f_{L}(\theta) \\
& \subset \bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{t \in T, \hat{f}_{t}(\theta) \geq-\varepsilon} \partial_{\varepsilon}\left(\hat{f}_{t}+\mathrm{I}_{L \cap \operatorname{dom} f_{L}}\right)(\theta)\right)\right\} \\
& =\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{s}\left(\bigcup_{T_{\varepsilon}(\theta)} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{L \cap \operatorname{dom} f}\right)(\theta)\right)\right\}
\end{aligned}
$$

and the inclusion " $\subset$ " in (55) follows.
The following corollary closing this section considers a frequent hypothesis in the literature.
Corollary 12 Let $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, be convex functions. If $f=\sup _{t \in T} f_{t}$ is finite and continuous at some point, then for every $x \in X$

$$
\begin{aligned}
\partial f(x) & =\mathrm{N}_{\operatorname{dom} f}(x)+\overline{\operatorname{co}}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)\right\} \\
& =\mathrm{N}_{\operatorname{dom} f}(x)+\operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)\right\} \quad\left(\text { if } X=\mathbb{R}^{n}\right)
\end{aligned}
$$

Proof The proof is similar to the one of Theorem 11, but with the use of the formulas in Corollary 7 instead of formula (35).

We close this section with an extension of Theorem 11 to nonconvex functions. We also refer to [22, and references therein, for other studies on the subdifferential of the supremum of nonconvex functions.

Corollary 13 Let $f_{t}: X \rightarrow \overline{\mathbb{R}}, t \in T$, be a family of non-necessarily convex functions and $f:=\sup _{t \in T} f_{t}$. Assume that

$$
f^{* *}=\sup _{t \in T} f_{t}^{* *} .
$$

Then 57) holds.
Proof It suffices to prove the inclusion " $\subset$ " in (57) for $x$ such that $\partial f(x) \neq \emptyset$; hence, $f^{*}$ is proper, $f(x)=f^{* *}(x)$ and $\partial f(x)=\partial(\overline{\operatorname{co}} f)(x)=\partial f^{* *}(x)$. Thus, applying the second statement in Theorem 11 to the family $\left\{f_{t}^{* *}, t \in T\right\}$,
$\partial f(x)=\partial f^{* *}(x)=\bigcap_{L \in \mathcal{F}(x)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}^{1}(x)} \partial_{\varepsilon} f_{t}^{* *}(x)+\mathrm{N}_{L \cap \operatorname{dom} f^{* *}}(x)\right)\right\}$,
where $T_{\varepsilon}^{1}(x):=\left\{t \in T: f_{t}^{* *}(x) \geq f(x)-\varepsilon\right\}$. Observe that every $t \in T_{\varepsilon}^{1}(x)$ satisfies

$$
f_{t}(x) \geq f_{t}^{* *}(x) \geq f(x)-\varepsilon \geq f_{t}(x)-\varepsilon
$$

hence, $t \in T_{\varepsilon}(x)$ and $\partial_{\varepsilon} f_{t}^{* *}(x) \subset \partial_{2 \varepsilon} f_{t}(x)$. Additionally, the inequality $f^{* *} \leq f$ implies that $\mathrm{N}_{L \cap \operatorname{dom} f^{* *}}(x) \subset \mathrm{N}_{L \cap \operatorname{dom} f}(x)$, and the desired inclusion follows.

## 7 Two applications in optimization

First, in this section, we apply the previous results to extend the classical Fenchel duality to the nonconvex framework. This will lead us to recover some of the results in [3 4,5] (see, also, [26]), relating the solution set of a nonconvex optimization problem and its convexified relaxation. Second, we establish Fritz-John and KKT optimality conditions for convex semi-infinite optimization problems, improving similar results in [8].

Given a function $g: X \rightarrow \mathbb{R}_{\infty}$, we recall that the Fenchel conjugate of $g$ is the function $f: X^{*} \rightarrow \overline{\mathbb{R}}$, given by

$$
\begin{equation*}
f\left(x^{*}\right):=\sup _{x \in X}\left(\left\langle x, x^{*}\right\rangle-g(x)\right) . \tag{69}
\end{equation*}
$$

When $g$ is proper, lsc and convex, the classical Fenchel duality, together with (4), yields

$$
\begin{equation*}
\partial f=(\partial g)^{-1} \tag{70}
\end{equation*}
$$

We extend this relation to non-necessarily convex functions. We denote below the closure with respect to the weak topology in $X$ by cl ${ }^{w}$.

Proposition 14 Assume that the function $f$ is proper. Then, for every $x^{*} \in X^{*}$,

$$
\partial f\left(x^{*}\right)=\bigcap_{L \in \mathcal{F}\left(x^{*}\right)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{w}\left(\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)+\mathrm{N}_{L \cap \operatorname{dom} f}\left(x^{*}\right)\right)\right\}
$$

If, in addition, $f$ is finite and (weak*-) continuous somewhere, then

$$
\begin{aligned}
\partial f\left(x^{*}\right) & =\overline{\operatorname{co}}\left\{\left(\left(\partial\left(\operatorname{cl}^{w} g\right)\right)^{-1}\left(x^{*}\right)\right)\right\}+\mathrm{N}_{\operatorname{dom} f}\left(x^{*}\right) \\
& =\operatorname{co}\left\{\left((\partial(\operatorname{cl} g))^{-1}\left(x^{*}\right)\right)\right\}+\mathrm{N}_{\operatorname{dom} f}\left(x^{*}\right) \quad\left(\text { if } X=\mathbb{R}^{n}\right),
\end{aligned}
$$

where $\mathrm{cl}^{w} g$ is the weak-lsc hull of $g$.

Proof We define the convex functions $f_{x}: X^{*} \rightarrow \overline{\mathbb{R}}, x \in X$, as

$$
f_{x}\left(x^{*}\right):=\left\langle x, x^{*}\right\rangle-g(x), x \in \operatorname{dom} g,
$$

so that $f_{x}$ are weak ${ }^{*}$-continuous and $f=\sup _{x \in \operatorname{dom} g} f_{x}$. Then, according to formula (57), for every $x^{*} \in X^{*}$ we have

$$
\partial f\left(x^{*}\right)=\bigcap_{L \in \mathcal{F}\left(x^{*}\right)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{w}\left(\bigcup_{x \in T_{\varepsilon}\left(x^{*}\right)} \partial_{\varepsilon} f_{x}\left(x^{*}\right)+\mathrm{N}_{L \cap \operatorname{dom} f}\left(x^{*}\right)\right)\right\}
$$

where

$$
T_{\varepsilon}\left(x^{*}\right):=\left\{x \in \operatorname{dom} g: f_{x}\left(x^{*}\right) \geq f\left(x^{*}\right)-\varepsilon\right\}=\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)
$$

Consequently, the first formula comes from the fact that $\partial_{\varepsilon} f_{x}\left(x^{*}\right)=\{x\}$.
Assume now that $f$ is finite and weak*-continuous somewhere. Then, arguing in a similar way, but using Corollary 12 instead of (57),

$$
\begin{aligned}
\partial f\left(x^{*}\right) & =\overline{\operatorname{co}}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{w}\left(\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)\right)\right\}+\mathrm{N}_{\operatorname{dom} f}\left(x^{*}\right) \\
& =\operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)\right)\right\}+\mathrm{N}_{\operatorname{dom} f}\left(x^{*}\right) \quad\left(\text { if } X=\mathbb{R}^{n}\right) .
\end{aligned}
$$

The desired formulas follow as

$$
\begin{equation*}
\bigcap_{\varepsilon>0} \mathrm{cl}^{w}\left(\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)\right)=\left(\partial\left(\mathrm{cl}^{w} g\right)\right)^{-1}\left(x^{*}\right) \tag{71}
\end{equation*}
$$

according to [6, Lemma 2.3].

Observing that $\operatorname{Argmin}(\overline{\operatorname{co} g})=\partial f(\theta)$, the previous proposition gives:
Corollary 15 Assume that the function $f$ is proper. Then we have

$$
\operatorname{Argmin}(\overline{\operatorname{cog}} g)=\bigcap_{L \in \mathcal{F}(\theta)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}^{w}\left(\varepsilon-\operatorname{Argmin} g+\mathrm{N}_{L \cap \operatorname{dom} f}(\theta)\right)\right\}
$$

If, in addition, $f$ is finite and continuous at some point, then

$$
\begin{aligned}
\operatorname{Argmin}(\overline{\operatorname{co}} g) & =\overline{\operatorname{co}}\left(\operatorname{Argmin}\left(\mathrm{cl}^{w} g\right)\right)+\mathrm{N}_{\mathrm{dom} f}(\theta) \\
& =\operatorname{co}(\operatorname{Argmin}(\operatorname{cl} g))+\mathrm{N}_{\operatorname{dom} f}(\theta) \quad\left(\text { if } X=\mathbb{R}^{n}\right) .
\end{aligned}
$$

When $X$ is a normed space, the set $\partial f\left(x^{*}\right)$ is also seen as a subset of the bidual space, whereas Proposition 14 characterizes only the part of $\partial f\left(x^{*}\right)$ in the subspace $X$ of $X^{* *}$. A light adaptation of Proposition 14 allows us to have a complete picture of $\partial f\left(x^{*}\right)$, as a proper set of the bidual space $X^{* *}$. In such a setting, we denote the weak*-topology $\sigma\left(X^{* *}, X^{*}\right)$ in $X^{* *}$ by $w^{* *}$, and introduce the function $\bar{g}^{w^{* *}}: X^{* *} \rightarrow \overline{\mathbb{R}}$ defined by

$$
\bar{g}^{w^{* *}}(y)=\liminf _{x \rightarrow w^{* *}} y(x), y \in X^{* *}
$$

We refer, e.g., to [2, Chapter 1] for these concepts.

Proposition 16 Assume that $X$ is a normed space and $X^{*}$ is endowed with the dual norm topology. If the function $f$ is proper, then for every $x^{*} \in X^{*}$

$$
\partial f\left(x^{*}\right)=\bigcap_{L \in \mathcal{F}\left(x^{*}\right)} \operatorname{co}\left\{\bigcap_{\varepsilon>0} \mathrm{cl}^{w^{* *}}\left(\left(\partial_{\varepsilon} g\right)^{-1}\left(x^{*}\right)+\mathrm{N}_{L \cap \operatorname{dom} f}\left(x^{*}\right)\right)\right\} .
$$

If, in addition, $f$ is finite and (norm-) continuous somewhere, then

$$
\partial f\left(x^{*}\right)=\overline{\operatorname{co}}\left\{\left(\partial \bar{g}^{w^{* *}}\right)^{-1}\left(x^{*}\right)\right\}+\mathrm{N}_{\operatorname{dom} f}\left(x^{*}\right) .
$$

Proof Following similar arguments as those used in [4, we apply Proposition 14 in the duality pair $\left(\left(X^{* *}, w^{* *}\right),\left(X^{*},\| \|_{*}\right)\right)$, replacing the function $g$ in (69) by the function $\hat{g}$ defined on $X^{* *}$ as

$$
\hat{g}(y)=g(y), \quad \text { if } y \in X^{* *} ;+\infty, \text { otherwise } .
$$

Observe that the $w^{* *}$-lsc hull of $\hat{g}$ is precisely the function $\bar{g}^{w^{* *}}$.

Now, as in [8], we consider the following convex semi-infinite optimization problem

$$
(\mathcal{P}): \quad \operatorname{Inf} f_{0}(x), \text { subject to } f_{t}(x) \leq 0, t \in T,
$$

where $T$ is a given set, and $f_{0}, f_{t}: \mathbb{R}^{n} \rightarrow \mathbb{R}_{\infty}, t \in T$, are proper and convex. We assume, without loss of generality, that $0 \notin T$, and denote

$$
f:=\sup _{t \in T} f_{t}
$$

The following result establishes new Fritz-John and KKT optimality conditions for problem $(\mathcal{P})$, improving similar results in [8, 10]. Here we adopt the convention $\mathbb{R}_{+} \emptyset=\left\{0_{n}\right\}$.

Proposition 17 Let $\bar{x}$ be an optimal solution of $(\mathcal{P})$ such that $f(\bar{x})=0$. Then we have

$$
\begin{equation*}
0_{n} \in \operatorname{co}\left\{\partial\left(f_{0}+\mathrm{I}_{\mathrm{dom} f}\right)(\bar{x}) \cup \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\bar{x})} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{\mathrm{dom} f \cap \operatorname{dom} f_{0}}\right)(\bar{x})\right)\right\} \tag{72}
\end{equation*}
$$

Moreover, if the Slater condition holds; that is, $f\left(x_{0}\right)<0$ for some $x_{0} \in \operatorname{dom} f_{0}$, then

$$
\begin{equation*}
0_{n} \in \partial\left(f_{0}+\mathrm{I}_{\operatorname{dom} f}\right)(\bar{x})+\operatorname{cone} \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\bar{x})} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{\mathrm{dom} f \cap \operatorname{dom} f_{0}}\right)(\bar{x})\right) \tag{73}
\end{equation*}
$$

and, provided in addition that $f$ is continuous at some point in $\operatorname{dom} f_{0} \cap \operatorname{dom} f$,

$$
\begin{equation*}
0_{n} \in \partial f_{0}(\bar{x})+\operatorname{cone} \bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\bar{x})} \partial_{\varepsilon} f_{t}(\bar{x})\right)+\mathrm{N}_{\mathrm{dom} f}(\bar{x}) \tag{74}
\end{equation*}
$$

Proof We consider the function $g: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$, defined as

$$
g(x):=\sup \left\{f_{0}(x)-f_{0}(\bar{x}), f_{t}(x), t \in T\right\}=\max \left\{f_{0}(x)-f_{0}(\bar{x}), f(x)\right\}
$$

so that $\operatorname{dom} g=\operatorname{dom} f_{0} \cap \operatorname{dom} f$. Then $\bar{x}$ is a global minimum of $g$; that is, $0_{n} \in$ $\partial g(\bar{x})$. To proceed, we first apply Proposition to the (finite) family $\left\{f_{0}-f_{0}(\bar{x}), f\right\}$
and obtain

$$
\begin{equation*}
0_{n} \in \operatorname{co}\left\{\partial\left(f_{0}+\mathrm{I}_{\mathrm{dom} f}\right)(\bar{x}) \cup \partial\left(f+\mathrm{I}_{\mathrm{dom} f_{0}}\right)(\bar{x})\right\} \tag{75}
\end{equation*}
$$

But Theorem 11] applied to the family $\left\{f_{t}+\mathrm{I}_{\text {dom } f_{0}}, t \in T\right\}$, yields

$$
\begin{equation*}
\partial\left(f+\mathrm{I}_{\mathrm{dom} f_{0}}\right)(\bar{x})=\operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(\bar{x})} \partial_{\varepsilon}\left(f_{t}+\mathrm{I}_{\mathrm{dom} f \cap \operatorname{dom} f_{0}}\right)(\bar{x})\right)\right\} \tag{76}
\end{equation*}
$$

and (72) follows from (75).
Finally, it can be easily seen from (75) that the Slater condition precludes that $0_{n} \in \partial\left(f+\mathrm{I}_{\mathrm{dom} f_{0}}\right)(\bar{x})$. So, (73) follows from (72). Under the supplementary continuity condition, Corollary 12 ensures that

$$
\begin{aligned}
\partial\left(f+\mathrm{I}_{\mathrm{dom} f_{0}}\right)(\bar{x}) & =\mathrm{N}_{\mathrm{dom} f_{0}}(\bar{x})+\partial f(\bar{x}) \\
& =\mathrm{N}_{\mathrm{dom} f_{0}}(\bar{x})+\mathrm{N}_{\mathrm{dom} f}(x)+\operatorname{co}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)\right\},
\end{aligned}
$$

and (74) follows, taking into account (3) and

$$
\begin{aligned}
0_{n} & \in \partial\left(f_{0}+\mathrm{I}_{\text {dom } f}\right)(\bar{x})+\mathbb{R}_{+} \partial\left(f+\mathrm{I}_{\operatorname{dom} f_{0}}\right)(\bar{x}) \\
& =\partial f_{0}(\bar{x})+\mathrm{N}_{\operatorname{dom} f} f(\bar{x})+\mathbb{R}_{+} \partial\left(f+\mathrm{I}_{\mathrm{dom}} f_{0}\right)(\bar{x}) \\
& =\partial f_{0}(\bar{x})+\operatorname{cone}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)\right\}+\mathrm{N}_{\operatorname{dom} f_{0}}(\bar{x})+\mathrm{N}_{\operatorname{dom} f}(x) \\
& \subset \partial f_{0}(\bar{x})+\operatorname{cone}\left\{\bigcap_{\varepsilon>0} \operatorname{cl}\left(\bigcup_{t \in T_{\varepsilon}(x)} \partial_{\varepsilon} f_{t}(x)\right)\right\}+\mathrm{N}_{\operatorname{dom} f}(x) .
\end{aligned}
$$

## 8 Conclusions

The main conclusion of this work is that the compactification method proposed in the paper allows us to move from the non-continuous setting to the continuous one and the other way around, as well as to develop a unifying theory which inspires new results and applications. The main results in relation to the subdifferential of the supremum are stated in Theorems 4 6 and 11, which are established in the most general framework, free of assumptions on the index set and the data functions. Our results cover most of the existing formulas such as those obtained in $[7,8,9,10,11,14,15,16,18,19,20,25,26,27,29,30,31$. The Fritz-John and KKT conditions for convex semi-infinite optimization are expressed in the most general scenario and, consequently, extend some previous results which can be found in [11, 13, 16, 20 .

## References

1. A. Brøndsted, On the subdifferential of the supremum of two convex functions. Math. Scand., 31 (1972), 225-230.
2. J. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems. Springer Series in Operations Research. Springer-Verlag, New York, 2000.
3. R. Correa, A. Hantoute, New formulas for the Fenchel subdifferential of the conjugate function. Set-Valued Var. Anal. 18 (2010), 405-422.
4. R. Correa, A. Hantoute, Subdifferential of the conjugate function in general Banach spaces. TOP 20 (2012), 328-346.
5. R. Correa, A. Hantoute, Lower semicontinuous convex relaxation in optimization. SIAM J. Optim. 23 (2013), 54-73.
6. R. Correa, A. Hantoute, Y. García, Nonconvex integration using $\varepsilon$-subdifferentials. Optimization 67 (2018), 2205-2227.
7. R. Correa, A. Hantoute, M. A. López, Weaker conditions for subdifferential calculus of convex functions. J. Funct. Anal. 271 (2016), 1177-1212.
8. R. Correa, A. Hantoute, M. A. López, Moreau-Rockafellar type formulas for the subdifferential of the supremum function. SIAM J. Optim. 29 (2019), 1106-1130.
9. R. Correa, A. Hantoute, M. A. López, Valadier-like formulas for the supremum function II: the compactly indexed case. J. Convex Anal. 26 (2019), 299-324
10. R. Correa, A. Hantoute, M. A. López, Subdifferential of the supremum via compactification of the index set. To appear in Vietnam J. Math. (2020).
11. N. Dinh, M. A. Goberna and M. A. López, From linear to convex systems: consistency, Farkas' lemma and applications. J. Convex Anal. 13 (2006), 113-133.
12. M. Fabian, P. Habala, P. Háyek, V. Montesinos, J. Pelant, and V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, Springer, New York (2001).
13. M. A. Goberna and M. A. López, Linear Semi-infinite Optimization. J. Wiley, Chichester, 1998.
14. A. Hantoute, M. A. López, A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions, J. Convex Anal. 15 (2008), 831-858.
15. A. Hantoute, M. A. López, C. Zălinescu, Subdifferential calculus rules in convex analysis: A unifying approach via pointwise supremum functions, SIAM J. Optim. 19 (2008), 863-882.
16. J.-B. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms. I. Fundamentals. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 305. Springer-Verlag, Berlin, 1993.
17. J.-B. Hiriart-Urruty, R.R. Phelps, Subdifferential calculus using $\varepsilon$-subdifferentials, J. Funct. Anal., 118 (1993), 154-166.
18. A. D. Ioffe, A note on subdifferentials of pointwise suprema. Top 20 (2012), 456-466.
19. A. D. Ioffe, U. L. Levin, Subdifferentials of convex functions, Trudy Moskov Mat. Obshch, 26 (1972) 3-73 (Russian).
20. A. D. Ioffe, V. H. Tikhomirov, Theory of Extremal Problems, Studies in Mathematics and its Applications, Vol. 6, North-Holland, Amsterdam, 1979.
21. C. Li, K. F. Ng, Subdifferential calculus rules for supremum functions in convex analysis, SIAM J. Optim. 21 (2011), 782-797.
22. B. S. Mordukhovich, T. T. A. Nghia, Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with Lipschitzian data. SIAM J. Optim. 23 (2013), 406-431.
23. J.-J. Moreau, Fonctionnelles Convexes. In: Leray, J. (ed.) Séminaire sur les équations aux dérivées partielles, vol. 2, pp. 1-108, 1965 (French).
24. J. Munkres, Topology, 2nd Edition, Prentice Hall, Upper Saddle River, 2000.
25. O. Lopez, L. Thibault, Sequential formula for subdifferential of upper envelope of convex functions. J. Nonlinear Convex Anal. 14 (2013), 377-388.
26. M. A. López, M. Volle, A formula for the set of optimal solutions of a relaxed minimization problem. Applications to subdifferential calculus. J. Convex Anal. 17 (2010), 1057-1075.
27. B. N. Pschenichnyi, Convex programming in a normalized space, Kibernetika, 5 (1965), 46-54 (Russian); translated as Cybernetics 1 (1965) no. 5, 46-57 (1966).
28. R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N. J., 1970.
29. V. N. Solov'ev, The subdifferential and the directional derivatives of the maximum of a family of convex functions, Izvestiya RAN: Ser. Mat., 65 (2001), 107-132.
30. M. Valadier, Sous-différentiel d'une borne supérieure d'une somme continue de fonctions convexes, C. R. Acad. Sci. Paris Sér. A-B, 268 (1969), A39-A42.
31. M. Volle, Sous-différentiel d'une enveloppe supérieure de fonctions convexes, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), 845-849.
