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ON THE ZEROS OF EISENSTEIN SERIES FOR $\Gamma_0^*(p)$

Tesis

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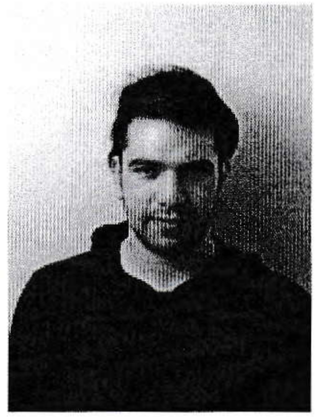

Nací en el año 1989 en Santiago de Chile. De niño me interesé por las matemáticas y la computación. De hecho para cuando había termiando la enseãnza media, ya sabía que debía estudiar matemáticas en la universidad. Entré a la Universida de Chile en el año 2008 a la carrera de licenciatura en matemáticas, y comprendí que debía dedicar mi vida a las matemáticas, fue por ello que luego de terminar la licenciatura ingresé al magister en matemáticas del cual estoy egresando.

Dedicado a Erika, a mis padres y a mis hermanos.

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Índice

Resumen

En 1970 Rankin y Swinnerton-Dyer probaron que los ceros de las series de Eisenstein asociadas a $SL_2(\mathbb{Z})$ se encuentran en el arco $\{z \in \mathbb{C} \mid |z| = 1, |\mathrm{Re}(z)| < \frac{1}{2}\}$. Desde entonces, el estudio de la ubicación de los ceros de diversas generalizaciones de las series de Eisenstein ha despertado gran interés

En la literatura actual existen trabajos respecto a la localización de los ceros de las series de Eisenstein asociadas a $\Gamma_0^*(p)$, para $p = 2, 3, 5, 7$. Esta tesis aborda el problema de la localización de los ceros de las series de Eisenstein asociadas a $\Gamma_0^*(p)$ para primos mayores que 7. Como una aplicación de los resultados generales, se probará que para $p = 13$, todos salvo quizás dos ceros, están en el borde inferior del dominio fundamental de $\Gamma_0^*(p)$.

Abstract

In 1970 Rankin and Swinnerton-Dyer proved that the zeros of Eisenstein series associated to $SL_2(\mathbb{Z})$ are located in the arc $\{z \in \mathbb{C} \mid |z| = 1, |\mathrm{Re}(z)| < \frac{1}{2}\}$. Since then, the study of the location of the zeros of diverse generalizations of Eisenstein series has caught people awareness.

Works about the zeros of Eisenstein series associated to $\Gamma_0^*(p)$ for $p = 2, 3, 5,$ and 7 can be founded in the current literature.

This thesis undertakes the problem of finding the location of the zeros of Eisenstein series associated to $\Gamma_0^*(p)$ for primes grater than 7.

As an application of the general results it will be shown that for $p = 13$, all but at most two zeros lie in the lower boundary of the fundamental domain of $\Gamma_0^*(p)$.

1. Introduction

Let $SL_2(\mathbb{Z})$ be the group of 2×2 matrices with integral coefficients and determinant equal to 1. Let k be an even integer greater than 2. The Eisenstein series of weight k associated to $SL_2(\mathbb{Z})$ is

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^k}.$$

This is the only modular form with value 1 at infinity which is orthogonal to all cuspidal modular forms of weight k over $SL_2(\mathbb{Z})$ under the Petersson inner product.

For any positive integer N define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}.$$

This is a subgroup of $SL_2(\mathbb{Z})$.

If $N = p$ prime then the involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$ normalizes $\Gamma_0(p)$, and $\Gamma_0^*(p) := \Gamma_0(p) \cup W_p\Gamma_0(p)$ is the full normalizer of $\Gamma_0(p)$ in $SL_2(\mathbb{R})$.

One can define modular forms over $\Gamma_0^*(p)$ and

$$E_{k,p}^*(z) = \frac{1}{2} \sum_{(c,d)=1,\ p|c} (cz+d)^{-k} + \frac{p^{k/2}}{2} \sum_{(c,d)=1,\ p|d} (cpz+d)^{-k}$$

is the corresponding Eisenstein series of weight k.

Let $\mathcal{F}$ be the standard fundamental domain of $SL_2(\mathbb{Z})$. Namely

$$\mathcal{F} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0, |\mathrm{Re}(z)| < 1/2 \text{ and } |z| > 1\} \cup B.$$

where $B = \left\{e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\right\} \cup \left\{-\frac{1}{2} + it \mid t \geq \frac{\sqrt{3}}{2}\right\}$.

Let f be a non-zero modular form of weight k over $SL_2(\mathbb{Z})$. It is known that the total number of its zeros in $\mathcal{F}$ is determined by the formula

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{\substack{z \in \mathcal{F} \\ z \neq i,\omega}} \nu_z(f) = \frac{k}{12} \tag{1}$$

where $\nu_p(f)$ is the order of the zero of f at p counting multiplicity, and $\omega := e^{2i\pi/3}$. This formula is called the "valence formula" for $SL_2(\mathbb{Z})$.

In [RSD] Rankin and Swinerton-Dyer proved that, for all k as above, every zero of E_k lies on the lower arc of $\mathcal{F}$. Indeed, they are in the set $\{z \in \mathcal{F} | \, |z| = 1 \text{ and } |Re(z)| < 1/2\}$.

The purpose of this work is to show that many zeros of $E^*_{k,p}$ are in the lower arcs of the fundamental domain $\mathcal{F}^*_p$ of $\Gamma^*_0(p)$.

$\mathcal{F}^*_p$ is a certain subset of $\mathbb{C}$ whose interior is the region

$$\left\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \, , \; |\mathrm{Re}(z)| < 1/2, |cz+d| > 1, \; \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^*_0(p) \text{ with } c \neq 0\right\}.$$

The boundary of $\mathcal{F}^*_p$ consists of two vertical lines and a finite set of arcs at the bottom of the region (see next chapter). They can be parametrized either by $n^{-1}(m + e^{i\theta}p^{-1/2})$ or by $p^{-1}(m + e^{i\theta})$ where n, m are fixed integers, and θ runs in an interval interval $[\beta, \pi/2 + \alpha] \subseteq (0, \pi)$. We call those arcs "lower arcs of $\mathcal{F}^*_p$". Among them we study a special class of arcs which we call "admissible". In those admissible arcs we are able to find many zeros of $E^*_{k,p}$.

There are some results about the location of the zeros of $E^*_{k,p}$ in the literature. For $p = 2$ and 3, all zeros of such Eisenstein series lie on the lower arcs of $\mathcal{F}^*_p$. This was proved by Miezaki, Nozaki and Shigezumi in [MNS]. For $p = 5$ and $p = 7$ Shigezumi in [Sh] proved that all but at most one zero of $E^*_{k,p}$, lie on the lower arcs of $\mathcal{F}^*_p$.

The argument given by Rankin and Swinerton-Dyer in [RSD] can be described as follows: The study of E_k restricted to the lower arc of $\mathcal{F}$ leads naturally to the study of the real-valued function $F_k(\theta) := e^{ik\theta/2}E_k(e^{i\theta})$ with $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. This is a continuous function, and has as many zeros as E_k in the arc $\{e^{i\theta} \mid \theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]\}$.

Rankin and Swinerton-Dyer proved that

$$F_k(\theta) = 2\cos(k\theta/2) + R_k(\theta)$$

where $|R_k(\theta)| < 2$ for all θ. This implies that the signs of $F_k(\theta)$ and $\cos(k\theta/2)$ coincide whenever $\cos(k\theta/2) = \pm 1$. Hence, there is a zero of F_k between two consecutive extreme values of $\cos(k\theta/2)$. In this way we get that there are at least as many zeros of F_k (and therefore of E_k) as extreme values of the function $\cos(k\theta/2)$ minus one. Next using the valence formula one checks that those are all the zeros of E_k.

The previous argument cannot be directly applied to the whole lower boundary of $\mathcal{F}_p^*$. This is why Shigezumi, in the cases $p = 5$ and 7, divides the lower boundary in two parts; one where the previous arguments can be extended, and another one where he uses some ad hoc methods to determine the number of zeros in that part of the boundary.

For primes greater than 7 there are many more obstacles, and the extra arguments used by Shigezumi are not enough. There are no results in the current literature about the location of zeros of $E_{k,p}^*$ when $p \geq 11$.

In this work we give a lower bound for the number of zeros of $E_{k,p}^*$ on the lower arcs in terms of the extreme values of $\cos(k\theta/2)$. This estimate is proved for the admissible arcs.

By an explicit calculation, one can check that for every $p \leq 41$ (except for $p = 23$) at least 55% of the arcs are admissible.

In this thesis we prove the following:

Theorem 1. *Let p be a prime. Let $\theta \mapsto \frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2} + \alpha]$ be a parametrization of an admissible arc, and let N_k be the number of extreme values of the function $\cos(k\theta/2)$ with $\theta \in [\beta, \pi/2 + \alpha]$.*

If $\alpha > 0$ and $\beta < \frac{\pi}{2}$ then the number of zeros of $E_{k,p}^*$ on such arc is greater than or equal to $N_k - 3$, for $k >> 0$ even.

Moreover if $\alpha \leq 0$ or $\beta \geq \pi/2$, then the number of zeros of $E_{k,p}^*$ on such arc is greater than or equal to $N_k - 2$, for $k >> 0$ even.

The theorem is a consequence of the following result.

Proposition 1. *Let p be a prime. Let $\theta \mapsto \frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2} + \alpha]$ be a parametrization of an admissible arc, suppose also $\alpha > 0$ and $\beta < \frac{\pi}{2}$. Then the function $F^*_{k,p}(\theta) := e^{ik\theta/2} E^*_{k,p}(\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}})$ is real-valued, and can be written as*

$$F^*_{k,p}(\theta) = 2\cos(k\theta/2) + R^*_{k,p}(\theta)$$

with $\left|R^*_{k,p}(\theta)\right| < 2$ if $\theta \in [\beta + \frac{\pi}{k}, \frac{\pi}{2} + \alpha - \frac{\pi}{k}]$ for $k >> 0$ even.

Moreover if $\alpha \leq 0$ (respectively $\beta \geq \pi/2$) then $\left|R^*_{k,p}(\theta)\right| < 2$ if $\theta \in [\beta + \frac{\pi}{k}, \frac{\pi}{2} + \alpha]$ (respectively $\theta \in [\beta, \frac{\pi}{2} + \alpha - \frac{\pi}{k}]$), for $k >> 0$ even.

As an application of our results, we prove the following enunciate for $p = 13$.

"If $k \equiv 0 \pmod{12}$ is large enough, then all but at most four zeros of $E^*_{k,13}$ lie in the lower arcs of $\mathcal{F}_{13}$".

CHAPTER 1

Basic concepts

1.1. Modular forms and Eisenstein series.

Let $\mathcal{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$ and k an even integer.

For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ define

$$f \mid_k [\gamma] := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

A modular form of weight k is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ which satisfy

$$f \mid_k [\gamma] = f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and is holomorphic at ∞. The last condition means that it has a Fourier expansion of type

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}. \tag{2}$$

The set of modular forms of weight k is a finite dimensional vector space denoted by $\mathcal{M}_k(SL_2(\mathbb{Z}))$. If the coefficient a_0 in (2) is equal to 0, we call f a *cuspidal* modular form. The set of cuspidal modular forms is a subspace of $\mathcal{M}_k(SL_2(\mathbb{Z}))$ and is denoted by $\mathcal{S}_k(SL_2(\mathbb{Z}))$.

For a couple of modular forms f, g of the same weight, at least one of them cuspidal, define

$$\langle f, g\rangle := \int_{\mathcal{F}} f(z)\overline{g(z)} y^{k-2} dx dy, \quad z = x + iy.$$

This is called the Petersson inner product of f and g.

An important example of modular form in $\mathcal{M}(SL_2(\mathbb{Z}))$ is given by the Eisenstein series

$$E_k(z) := e_k \sum_{\gamma \in SL_2(\mathbb{Z})/\Gamma_\infty} 1 \mid_k [\gamma] = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c,d\in\mathbb{Z}}} \frac{1}{(cz+d)^k},$$

where Γ_∞ is the stabilizer of the cusp at infinity, and e_k is a number chosen to ensure that E_k take the value 1 at infinity.

As stated above, this is the only modular form orthogonal to all cuspidal forms (with respect to the Petersson product) with value 1 at infinity. Much is known about $\mathcal{M}(SL_2(\mathbb{Z}))$ (see for example [Ko]).

The theory of modular forms can be extended to other first type Fuchsian groups such as $\Gamma_0(N)$ or

$$\Gamma_0^*(p) := \Gamma_0(p) \cup W_p\Gamma_0(p), \;\; W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$$

for a prime number p. (See for example [Miy]).

This work is about modular forms over the latter group. Throughout this thesis, p always represent a prime number.

The Eisenstein series for $\Gamma_0^*(p)$ is given by

$$E_{k,p}^*(z) = e_{k,p}^* \sum_{\gamma \in \Gamma_0^*(p)/\Gamma_\infty^*} 1 \mid_k [\gamma] = \frac{1}{2} \sum_{\substack{(c,d)=1,\ p\mid c \\ c,d\in\mathbb{Z}}} (cz+d)^{-k} + \frac{p^{k/2}}{2} \sum_{\substack{(c,d)=1,\ p\mid d \\ c,d\in\mathbb{Z}}} (cpz+d)^{-k}.$$

where $\Gamma_\infty^* \subseteq \Gamma_0^*(p)$ is the stabilizer of the cusp at infinity, and $e_{k,p}^*$ is a number chosen to ensure that $E_{k,p}^*$ take the value 1 at infinity.

Remark: The condition $k > 2$ even, ensures the absolute convergence of the Eisenstein series. This is why we can interchange the order of summation of the series when it is needed.

1.2. Fundamental domains

Let G be a group acting on a topological space X. A subset $F \subseteq X$ is a fundamental domain of such $G-$action if it is a connected set and satisfies the following properties

1. For all $x \in X$ there exists $g \in G$ such that $gx \in F$.

2. Any pair of distinct elements in F are not in the same orbit.

The group $SL_2(\mathbb{Z})$ acts on $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ via Mobius transformations. That is, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$\gamma z := \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathcal{H} \cup \mathbb{Q} - \{-\frac{d}{c}\}, \\ \frac{a}{c} & \text{if } z = \infty, \\ \infty & \text{if } z = -\frac{d}{c} \text{ and } c \neq 0. \end{cases}$$

The set $\mathcal{H}^*$ has a topology which extends the usual topology of $\mathcal{H}$ (see for example [Ko] p. 103-104).

The group $\Gamma_0^*(p)$ acts on $\mathcal{H}^*$ via Mobius transformation too, and we have the following proposition about its fundamental domain.

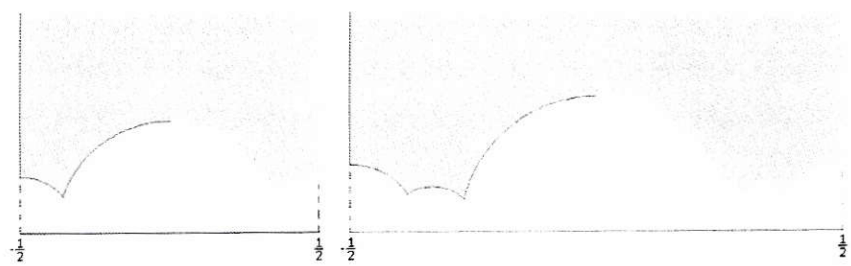
Proposition 2: There exists a fundamental domain for the action of $\Gamma_0^*(p)$ whose interior is

$$\left\{z \in \mathcal{H} \mid |Re(z)| < 1/2, |cz+d| > 1, \ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^*(p) \text{ with } c \neq 0\right\}.$$

(See for example [Miy] p. 22 and [A]).

Along this thesis we denote this fundamental domain by $\mathcal{F}_p^*$.

In general, there are many fundamental domains for each group. For example if $\gamma \in SL_2(\mathbb{Z})$ and F is a fundamental domain for $SL_2(Z)$, then γF is another fundamental domain for $SL_2(\mathbb{Z})$. However the results of this work are given in terms of the particular fundamental domain described in proposition 2. Here we exhibit a graphic presentation of such fundamental for primes $p = 7$ and $p = 13$.



Fundamental domains for $\Gamma_0^*(7)$ and $\Gamma_0^*(13)$

For more information of these fundamental domains see for example [Sh1] p. 33-36.

We finish this section with two technical concepts which are needed in this work.

1.3. Admissible arcs

As we mentioned in the introduction, in this thesis we work with the particular $\Gamma_0^*(p)$−fundamental domain $\mathcal{F}_p^*$ (see the previous section). It is known that its boundary consists of the two vertical lines $\{\pm 1/2 + it \mid t \in [(2\sqrt{p})^{-1}, \infty)\}$ and a set of parts of circumferences of certain circles. The elements in this set are called lower arcs of $\mathcal{F}_p^*$ by us, and for any given p there is a way to compute them explicitly.

Notice that all but two of such arcs have a left neighboring arc and a right neighboring arc in $\mathcal{F}_p^*$. The exceptions are the arcs in the extreme left and in the extreme right of $\mathcal{F}_p^*$.

The following two Lemmas give some properties of the lower arcs of $\mathcal{F}_p^*$. Afterwards we introduce the concept of admissible arc.

Lemma 1. *The lower arcs of the fundamental domain $\mathcal{F}_p^*$ are over the circumference of certain circles which belong to one of the following two finite families:*

1.) The circles of center $\frac{m}{n}$ and radius $\frac{1}{n\sqrt{p}}$, where m, n are integers such that $\gcd(m, n) = 1$, and $0 < |m| < n < 2\sqrt{p/3}$.

2.) The circles of center $\frac{n}{p}$ and radius $\frac{1}{p}$, where n is an integer such that $0 < |n| < p$.

Moreover, every point in $\mathcal{F}_p^*$ has imaginary part greater than or equal to $\frac{\sqrt{3}}{2p}$.

Lemma 2. *Let $\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}$ and $\frac{e^{i\theta}}{n_1\sqrt{p}} + \frac{m_1}{n_1}$ be parametrizations of two distinct neighboring arcs in the first family of the previous Lemma. Then $|mn_1 - m_1 n| \in \{1, 2\}$.*

For a proof of these Lemmas, see appendix B.

Notice that any lower arc of $\mathcal{F}_p^*$ on a circle of the first family of Lemma 1 can be parametrized as $\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2} + \alpha]$ for certain real numbers $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\beta \in [0, \pi]$, and certain integer numbers n, m with no common factors.

Definition 1. *Let $\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2} + \alpha]$ be a lower arc of $\mathcal{F}_p^*$ such that its left and right neighbors exist.*

Suppose that the left (respectively right) neighbor is defined by a circle with center $\frac{m_1}{n_1}$ (respectively $\frac{m_2}{n_2}$) and radius $\frac{1}{n_1\sqrt{p}}$ (respectively $\frac{1}{n_2\sqrt{p}}$).

If $\alpha, \beta \in (0, \frac{\pi}{2})$, we say that the initial arc is admissible if $m^2 p \equiv 1 \pmod{n}$ and the following two conditions hold.

1. $mn_1 - m_1 n = m_2 n - mn_2 = 1$.
2. $c_1 \exp\left(\frac{-\pi n_1 \sqrt{p}\cos(\alpha)}{n^2}\right) + c_2 \exp\left(\frac{-\pi n_2 \sqrt{p}\sin(\beta)}{n^2}\right) < 1$, *where c_i is defined by*

$$c_i := \begin{cases} 1 + \exp\left(-\frac{\pi\sqrt{3}}{2}\right) & \text{if } n_i^2 + nn_i + n^2 = p, \\ 1 + \exp\left(\frac{\pi\sqrt{3}}{2}\right) & \text{if } n_i > n \text{ and } n_i^2 - nn_i + n^2 = p, \\ 1 & \text{in any other case.} \end{cases} \quad (3)$$

If $\beta \geq \frac{\pi}{2}$, we say that initial arc is admissible if $n^2 < p+1$ and satisfies condition 1, and

$$2'. \quad c_1 \exp\left(\frac{-\pi n_1 \sqrt{p}\cos(\alpha)}{n^2}\right) < 1.$$

If $\alpha \leq 0$, we say that initial arc is admissible if $n^2 < p+1$ and satisfies condition 1, and

$$2''. \quad c_2 \exp\left(\frac{-\pi n_2 \sqrt{p}\sin(\beta)}{n^2}\right) < 1.$$

Remark: Clearly, the case $\alpha \leq 0$ and $b \geq \frac{\pi}{2}$ is meaningless. If one of the neighbors does not exist then either $\alpha \leq 0$ or $\beta \geq \pi/2$. In that case we say that the arc is admissible if it satisfies conditions 1. and condition 2'. or 2". according the case.

Notice that any admissible arc and its neighbors are always arcs of circles in the first family of Lemma 1 by definition.

1.4. Valence formula

Let $f : \mathcal{H} \to \mathbb{C}$ be a non-zero holomorphic function. We denote by $\nu_w(f)$ the order of zero of f at the complex number w. In other words, if the Taylor expansion of f at $z = w$ is

$$f(z) = \sum_{n=n_0}^{\infty} a_n (z-w)^n \,, \quad a_n \in \mathbb{C}, \;\; a_{n_0} \neq 0,$$

then $\nu_w(f) = n_0$.

As we stated above, if f in $\mathcal{M}_k(SL_2(\mathbb{Z}))$ is non-zero, the number of zeros of f in the fundamental domain is determined by formula (1).

For non-zero modular forms over $\Gamma_0^*(p)$ there is a similar relation. Namely

$$\nu_\infty(f) + \sum_{z \in \mathcal{F}_p^*} \frac{\nu_z(f)}{n_z^*} = \frac{k(p+1)}{24} \tag{4}$$

where $2n_z^*$ is the order of the stabilizer of z in $\Gamma_0^*(p)$, and the integer $\nu_\infty(f)$ is the order of zero in the Fourier expansion of f at infinity. A proof of (4) is given in appendix A.

CHAPTER 2

Main results

2.1. Main results.

In this section we prove first Proposition 1. Then we prove Theorem 1. In the proof of Proposition 1 we use three technical results (Lemma 3-5), which are proved in the next section for the sake of clarity.

Proof of Proposition 1: Assume that k is an even positive integer. Let $\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2} + \alpha]$, $\alpha, \beta \in (0, \frac{\pi}{2})$ be a parametrization of an admissible arc. As in Rankin and Swinnerton-Dyer's paper, we study the behavior of $E^*_{k,p}$ over this arc considering the function on the variable θ

$$F^*_{k,p}(\theta) := e^{ik\theta/2} E^*_{k,p}\left(\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}\right) \tag{5}$$

Note that the number of zeros of $F^*_{k,p}$ is the same as the number of zeros of $E^*_{k,p}$ on the admissible arc. Furthermore $F^*_{k,p}$ is a real-valued function on such an admissible arc as we see from the following.

Lemma 3. *For $c, d \in \mathbb{Z}$ let $\delta_{c,d} := \gcd(n, d - mc)^2$. If $m^2p \equiv 1 \pmod n$ and $p \nmid n$ then*

$$F^*_{k,p}(\theta) = \sum_{\substack{\gcd(c,d)=1 \\ p \nmid c}} \delta_{c,d}^{k/2} \mathrm{Re}\left\{\left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}\right\}. \tag{6}$$

Now we work with this expression.

Let $N = N(c,d) := c^2 + d^2$. The terms in (6) for which $N = 1$ are just the ones with $c = \pm 1$ and $d = 0$ (c can not be 0 because $p \nmid c$ in the sum). This implies

$$\sum_{\substack{\gcd(c,d)=1 \\ p \nmid c, N=1}} \delta_{c,d}^{k/2} \mathrm{Re}\left\{\left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}\right\} = 2\cos(k\theta/2)$$

since k is even. Hence we can write

$$F^*_{k,p}(\theta) = 2\cos(k\theta/2) + R^*_{k,p}(\theta) \tag{7}$$

where

$$R^*_{k,p}(\theta) := \sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N>1}} \delta_{c,d}^{k/2} \mathrm{Re}\left\{ \left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k} \right\}.$$

This identity gives the first part of the Proposition and it only remains to show that $|R^*_{k,p}(\theta)| < 2$ for $\theta \in [\beta + \pi/k, \pi/2 + \alpha - \pi/k]$ when $k >> 0$.

Let $v_k(c,d,\theta) := \left|\delta_{c,d}^{k/2}\left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}\right|$, i.e. $v_k(c,d,\theta) = \delta_{c,d}^{k/2}(c^2 + d^2 p + 2cd\sqrt{p}\cos(\theta))^{-k/2}$.

Notice that $v_k(c,d,\theta) = v_k(-c,-d,\theta)$ and that $v_k(c,d,\theta)$ is monotone as a function of θ (for k, c and d fixed, and $0 \le \theta \le \pi$). In fact $v_k(c,d,\theta)$ is increasing if $cd \ge 0$, decreasing otherwise.

Now we give an estimate for $R^*_{k,p}$

$$\begin{aligned} \left|R^*_{k,p}(\theta)\right| &\le \sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N>1}} \left|\delta_{c,d}^{k/2} Re\left\{\left(ce^{i\theta/2} + d\sqrt{p}e^{-i\theta/2}\right)^{-k}\right\}\right| \\ &\le 2 \sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N>1, d>0}} v_k(c,d,\theta) \end{aligned} \tag{8}$$

We can split the terms of the previous sum in two classes; those which converge to 0 as k goes to infinity uniformly in θ, and all the others. More precisely we have the following

Definition 2. *Let c, d be integers. We say that $v_k(c,d,\theta)$ is a "good term" if and only if*

$$\sup\left\{ v_k(c,d,\theta) \,\middle|\, \beta \le \theta \le \frac{\pi}{2} + \alpha \right\} \xrightarrow[k\to\infty]{} 0.$$

If $v_k(c,d,\theta)$ is not a good term, we say that it is a "bad term".

Notice that v_k reach its maximum at one extreme of the interval $[\beta, \frac{\pi}{2}+\alpha]$.

The following Lemma states that the sum of all good terms also converge to 0 when k goes to infinity, uniformly in θ, for $\beta \le \theta \le \frac{\pi}{2}+\alpha$.

Lemma 4. *There exists $\gamma > 0$ (which is independent of k, but depends on p and the arc), such that for any $M > 3$ we have*

$$\sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N\ge M, d>0}} v_k(c,d,\theta) \le 8(M-1)^{3/2} \cdot \left(\frac{\gamma(M-1)}{n^2}\right)^{-k/2}.$$

(Recall that $N = c^2+d^2$). On the other hand, for $n^2 < p+1$, if $\alpha \le 0$ no bad terms with $cd > 0$ exist, and if $\beta \ge \frac{\pi}{2}$ no bad term with $cd < 0$ exist.

Note that the right-hand side of the inequality in the Lemma converges to 0 when $k \to \infty$ if and only if $M > 1 + n^2/\gamma$. In particular for N large enough $v_k(c,d,\theta)$ goes to 0 as k goes to infinity uniformly in θ.

The following Lemma gives us information about bad terms:

Lemma 5. *Let $\frac{m}{n} + \frac{e^{i\theta}}{n\sqrt{p}}$ with $\theta \in [\beta, \frac{\pi}{2}+\alpha]$ and $\alpha, \beta \in (0, \frac{\pi}{2})$ be an admissible arc. Suppose that the left (respectively right) neighbor is defined by the circle with center $\frac{m_1}{n_1}$ (respectively $\frac{m_2}{n_2}$) and radius $\frac{1}{n_1\sqrt{p}}$ (respectively $\frac{1}{n_2\sqrt{p}}$). Then there are finitely many bad terms $v_k(c,d,\theta)$ with $d > 0$ (in fact, at most four), all of them with $d = 1$. Moreover the following hold*

a) $\displaystyle\lim_{k\to\infty} \sum_{\substack{\text{bad terms} \\ c>0}} v_k\left(c, 1, \tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right) = c_1 \exp\left(\frac{-\pi n_1\sqrt{p}\cos(\alpha)}{n^2}\right),$

b) $\displaystyle\lim_{k\to\infty} \sum_{\substack{\text{bad terms} \\ c<0}} v_k\left(c, 1, \beta+\tfrac{\pi}{k}\right) = c_2 \exp\left(\frac{-\pi n_2\sqrt{p}\sin(\beta)}{n^2}\right)$

where c_1, c_2 are the constants given in definition 1. (See equation (3)).

If $\alpha \le 0$ then b) holds, and if $\beta \ge \frac{\pi}{2}$ then a) holds.

Now we have enough information to complete the proof of Proposition 1. Recall we are assuming $\alpha > 0$ and $\beta < \frac{\pi}{2}$.

For every k, every $\theta \in [\beta + \frac{\pi}{k}, \frac{\pi}{2} + \alpha - \frac{\pi}{k}]$ and every $M > 1 + n^2/\gamma$ (with γ as in Lemma 4) consider the following expression

$$\begin{aligned} 2 \sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N>1, d>0}} v_k(c,d,\theta) &= 2 \sum_{\substack{\text{bad terms} \\ c<0}} v_k(c,d,\theta) + 2 \sum_{\substack{\text{bad terms} \\ c>0}} v_k(c,d,\theta) \\ &\quad + 2 \sum_{\substack{\text{good terms} \\ N<M}} v_k(c,d,\theta) + 2 \sum_{\substack{\text{good terms} \\ N\geq M}} v_k(c,d,\theta) \qquad (9) \end{aligned}$$

In the right-hand side all the sums are indexed by pairs of integers (c, d) which also satisfy the conditions $\gcd(c, d) = 1$, $p \nmid c$, $N > 1$ and $d > 0$.

By (8) the right-hand side of this equality is also an upper bound for $|R^*_{k,p}(\theta)|$.

Now we recall that for $d > 0$, $v_k(c, d, \theta)$ is a monotone decreasing (respectively, increasing) function of θ if $c < 0$ (respectively $c > 0$). This allows us to get a bound for the first and second sum in (9) by replacing their values at the corresponding extremes of the interval. Namely

$$\begin{aligned} |R^*_{k,p}(\theta)| &\leq 2 \sum_{\substack{\text{bad terms} \\ c<0}} v_k(c,1,\beta+\pi/k) + 2 \sum_{\substack{\text{bad terms} \\ c>0}} v_k(c,1,\pi/2+\alpha-\pi/k) \\ &\quad + 2 \sum_{\substack{\text{good terms} \\ N<M}} v_k(c,d,\theta) + 2 \sum_{\substack{\text{good terms} \\ N\geq M}} v_k(c,d,\theta) \end{aligned}$$

On the other hand, since the number of bad terms is finite, we can pick M large enough so that the last sum in the previous inequality coincide with the sum in Lemma 4. From this Lemma we get a bound for it with the extra condition $M > 1 + n^2/\gamma$.

In this way we get

$$\begin{aligned} |R^*_{k,p}(\theta)| &\leq 2 \sum_{\substack{\text{bad terms} \\ c<0}} v_k(c,1,\beta+\pi/k) + 2 \sum_{\substack{\text{bad terms} \\ c>0}} v_k(c,1,\pi/2+\alpha-\pi/k) \\ &\quad + 2 \sum_{\substack{\text{good terms} \\ N<M}} v_k(c,d,\theta) + 16(M-1)^{3/2} \cdot \left(\frac{\gamma(M-1)}{n^2}\right)^{-k/2}. \qquad (10) \end{aligned}$$

By Lemma 5 the sums over the bad terms converge to

$$2\left(c_1\exp\left(\frac{-\pi n_1\sqrt{p}\cos(\alpha)}{n^2}\right)+c_2\exp\left(\frac{-\pi n_2\sqrt{p}\sin(\beta)}{n^2}\right)\right)$$

when k goes to infinity. The other terms on the right-hand side of (10) are finitely many, and each of them converges to 0 uniformly in θ. (Here we use the definition of good term and the choice of $M > 1 + n^2/\gamma$).

This argument implies that the right-hand side of inequality (10) converges to

$$2\left(c_1\exp\left(\frac{-\pi n_1\sqrt{p}\cos(\alpha)}{n^2}\right)+c_2\exp\left(\frac{-\pi n_2\sqrt{p}\sin(\beta)}{n^2}\right)\right).$$

This number is strictly less than 2 because the arc is admissible (see Definition 1). This means that for any k large enough we have $\left|R^*_{k,p}(\theta)\right| < 2$ whenever $\theta \in [\beta + \pi/k, \pi/2 + \alpha - \pi/k]$. This complete the proof in the case $\alpha > 0$ and $\beta < \frac{\pi}{2}$.

Assume next $\alpha \leq 0$. By Lemma 4 we know that there are no bad terms with $cd > 0$. Then for $\theta \in [\beta + \frac{\pi}{k}, \frac{\pi}{2}]$ the expression (9) turns into

$$2\sum_{\substack{\gcd(c,d)=1\\ p\nmid c, N>1, d>0}} v_k(c,d,\theta) = 2\sum_{\substack{\text{bad terms}\\ c<0}} v_k\left(c,d,\theta\right)+2\sum_{\substack{\text{good terms}\\ N<M}} v_k(c,d,\theta)+2\sum_{\substack{\text{good terms}\\ N\geq M}} v_k(c,d,\theta) \tag{11}$$

As before, using Lemma 4 and bounding the bad terms by their value at the corresponding extreme of the interval we obtain

$$\begin{aligned}\left|R^*_{k,p}(\theta)\right| \;&\leq\; 2\sum_{\substack{\text{bad terms}\\ c<0}} v_k\left(c,1,\beta+\pi/k\right)\\ &\quad +2\sum_{\substack{\text{Good terms}\\ N<M}} v_k(c,d,\theta) + 16(M-1)^{3/2}\cdot\left(\frac{\gamma(M-1)}{n^2}\right)^{-k/2}.\end{aligned}$$

The right-hand side above converges to

$$2c_2\exp\left(\frac{-\pi n_2\sqrt{p}\sin(\beta)}{n^2}\right)$$

by Lemma 5, and this number is strictly less than 2 since the arc is admissible. This means that for k large enough we have $\left|R^*_{k,p}(\theta)\right| < 2$ whenever $\theta \in [\beta + \pi/k, \pi/2]$. This ends the proof in the case $\alpha \leq 0$.

The case $\beta \geq \frac{\pi}{2}$ is analogous. □

Proof of Theorem 1: As we noted in the previous proof, $F^*_{k,p}$ has the same number of zeros as $E^*_{k,p}$ over the admissible arc. Assume $\alpha > 0$ and $\beta < \frac{\pi}{2}$. By Proposition 1, for $k >> 0$ we can write

$$F^*_{k,p}(\theta) = 2\cos(k\theta/2) + R^*_{k,p}(\theta)$$

with $\left|R^*_{k,p}(\theta)\right| < 2$ if $\theta \in [\beta + \pi/k, \pi/2 + \alpha - \pi/k]$.

Hence $F^*_{k,p}$ has a zero between every pair of consecutive extreme values of $\cos(k\theta/2)$ because it is continuous (see (5)) and real-valued.

Let N_k denote the number of extreme values of $\cos(k\theta/2)$ in the interval $[\beta, \pi/2 + \alpha]$. Since there are at least $N_k - 2$ extreme values of $\cos(k\theta/2)$ in the interval $[\beta + \pi/k, \pi/2 + \alpha - \pi/k]$, we have at least $N_k - 3$ zeros of $F^*_{k,p}$ in $[\beta + \pi/k, \pi/2 + \alpha - \pi/k]$, and therefore $N_k - 3$ zeros of $E^*_{k,p}$ on the arc. This proves the first part of the theorem.

For the case $\alpha \leq 0$, from proposition 1 we have $\left|R^*_{k,p}(\theta)\right| < 2$ for all $\theta \in [\beta + \pi/k, \pi/2 + \alpha]$ if k is large enough. Note that at least $N_k - 1$ extreme values of $\cos(k\theta/2)$ are in $[\beta + \pi/k, \pi/2 + \alpha]$. Therefore, there are at least $N_k - 2$ zeros of $F^*_{k,p}$ on the interval $[\beta + \pi/k, \pi/2 + \alpha]$ and therefore $N_k - 2$ zeros of $E^*_{k,p}$ on the arc.

For the case $\beta \geq \frac{\pi}{2}$ the same argument works. □

2.2. Proof of the Lemmas.

In this section we prove Lemas 3,4 and 5.

Proof of Lemma 3: Recall that

$$E^*_{k,p}(z) = \frac{1}{2}\left(\sum_{(c,d)=1,p|c}(cz+d)^{-k} + p^{k/2}\sum_{(c,d)=1,p|d}(cpz+d)^{-k}\right)$$

Let $\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}$ with $\theta \in [\beta, \pi/2+\alpha]$ be an arc satisfying $m^2p \equiv 1 \pmod{n}$. From (5) we have

$$\begin{aligned}
F^*_{k,p}(\theta) &= e^{ik\theta/2}E^*_{k,p}\left(\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}\right) \\
&= \frac{e^{ik\theta/2}}{2}\sum_{(c,d)=1,p|c}\left(c\left(\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}\right)+d\right)^{-k} \\
&\quad + \frac{e^{ik\theta/2}}{2}\sum_{(c,d)=1,p|d}\left(cp^{1/2}\left(\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}\right)+dp^{-1/2}\right)^{-k} \\
&= \frac{e^{ik\theta/2}}{2}\sum_{(c,d)=1,p|c}\left(\frac{c}{n\sqrt{p}}e^{i\theta} + \frac{mc+nd}{n}\right)^{-k} \\
&\quad + \frac{e^{ik\theta/2}}{2}\sum_{(c,d)=1,p|d}\left(\frac{c}{n}e^{i\theta} + \frac{mcp+nd}{n\sqrt{p}}\right)^{-k} \\
&= \frac{1}{2}\sum_{(c,d)=1,p|c}\left(\frac{c}{n\sqrt{p}}e^{i\theta/2} + \frac{mc+nd}{n}e^{-i\theta/2}\right)^{-k} \\
&\quad + \frac{1}{2}\sum_{(c,d)=1,p|d}\left(\frac{mcp+nd}{n\sqrt{p}}e^{-i\theta/2} + \frac{c}{n}e^{i\theta/2}\right)^{-k} \qquad (12)
\end{aligned}$$

Let us assume that the following two set of ordered pairs are equal

$$\{(c, mc+nd) \mid \gcd(c,d)=1, p \mid c\} = \{(mcp+nd, c) \mid \gcd(c,d)=1, p \mid d\}$$

(this is something to be proved later). Then, the complex conjugate

$$\overline{\sum_{(c,d)=1,p|c}\left(\frac{c}{n\sqrt{p}}e^{i\theta/2} + \frac{mc+nd}{n}e^{-i\theta/2}\right)^{-k}} = \sum_{(c,d)=1,p|d}\left(\frac{mcp+nd}{n\sqrt{p}}e^{-i\theta/2} + \frac{c}{n}e^{i\theta/2}\right)^{-k}$$

This identity and (12) yield

$$F^*_{k,p}(\theta) = \sum_{(c,d)=1, p|d} Re\left\{ \left(\frac{mcp+nd}{n\sqrt{p}} e^{-i\theta/2} + \frac{c}{n} e^{i\theta/2} \right)^{-k} \right\}.$$

Notice that the index d in the last expression runs over the multiples of p, then we can change d by pd' taking some care with the indexes. Namely, we observe that

$$\{(c,d) \mid \gcd(c,d) = 1, p \mid d\} = \{(c, pd') \mid \gcd(c,d') = 1, p \nmid c\}$$

and therefore

$$\begin{aligned} F^*_{k,p}(\theta) &= \sum_{(c,d')=1, p\nmid c} Re\left\{ \left(\frac{mcp+npd'}{n\sqrt{p}} e^{-i\theta/2} + \frac{c}{n} e^{i\theta/2} \right)^{-k} \right\} \\ &= \sum_{(c,d')=1, p\nmid c} Re\left\{ \left(\frac{mc+nd'}{n} \sqrt{p} e^{-i\theta/2} + \frac{c}{n} e^{i\theta/2} \right)^{-k} \right\}. \end{aligned}$$

Now, collecting the terms with the same value of $\gcd(c,n)$ we obtain

$$\begin{aligned} F^*_{k,p}(\theta) &= \sum_{r|n} \sum_{\substack{(c,d')=1, p\nmid c \\ (c,n)=r}} Re\left\{ \left(\frac{mc+nd'}{n} \sqrt{p} e^{-i\theta/2} + \frac{c}{n} e^{i\theta/2} \right)^{-k} \right\} \\ &= \sum_{r|n} \sum_{\substack{(c,d')=1, p\nmid c \\ (c,n)=r}} \left(\frac{n}{r}\right)^k Re\left\{ \left((m\frac{c}{r} + \frac{n}{r} d') \sqrt{p} e^{-i\theta/2} + \frac{c}{r} e^{i\theta/2} \right)^{-k} \right\} \end{aligned}$$

Notice that c is a multiple of r, so we can change the variable c in the inner sum by $c'r$ using the following identity

$$\{(c,d') \mid \gcd(c,d') = 1, p \nmid c, \gcd(c,n) = r\}$$

$$= \{(c'r, d') \mid \gcd(c', d') = 1, p \nmid c', \gcd(c', n/r) = 1, \gcd(r, d') = 1\}.$$

(This will be proved later). Consequently

$$\begin{aligned} F^*_{k,p}(\theta) &= \sum_{r|n} \sum_{\substack{(c',d')=1, p\nmid c' \\ (c',n/r)=1, (r,d')=1}} \left(\frac{n}{r}\right)^k Re\left\{ \left((m\frac{c'r}{r} + \frac{n}{r} d') \sqrt{p} e^{-i\theta/2} + \frac{c'r}{r} e^{i\theta/2} \right)^{-k} \right\} \\ &= \sum_{r|n} \sum_{\substack{(c',d')=1, p\nmid c' \\ (c',n/r)=1, (r,d')=1}} \left(\frac{n}{r}\right)^k Re\left\{ \left((mc' + \frac{n}{r} d') \sqrt{p} e^{-i\theta/2} + c' e^{i\theta/2} \right)^{-k} \right\} \end{aligned}$$

Changing r by n/r and the equivalent conditions $(\frac{n}{r}, d') = 1$ by $(n, rd') = r$ we can write

$$F^*_{k,p}(\theta) = \sum_{r|n} \sum_{\substack{(c',d')=1, p\nmid c' \\ (c',r)=1, (n,rd')=r}} r^k Re\left\{\left((mc' + rd')\sqrt{p}e^{-i\theta/2} + c'e^{i\theta/2}\right)^{-k}\right\}$$

Finally we make the change of variables $c' = c$ and $rd' = d$ using the identity

$$\{(c', rd') \mid \gcd(c', d') = 1, p \nmid c', \gcd(c', r) = 1, \gcd(n, rd') = r\}$$

$$= \{(c, d) \mid \gcd(c, d) = 1, p \nmid c, \gcd(n, d) = r\}$$

(to be proved below), and get

$$\begin{aligned} F^*_{k,p}(\theta) &= \sum_{r|n} \sum_{\substack{(c,d)=1, p\nmid c \\ (n,d)=r}} r^k Re\left\{\left((mc + d)\sqrt{p}e^{-i\theta/2} + ce^{i\theta/2}\right)^{-k}\right\} \\ &= \sum_{r|n} \sum_{\substack{(c,d)=1, p\nmid c \\ (n,d-mc)=r}} r^k Re\left\{\left(d\sqrt{p}e^{-i\theta/2} + ce^{i\theta/2}\right)^{-k}\right\} \\ &= \sum_{(c,d)=1, p\nmid c} \delta_{c,d}^{k/2} Re\left\{\left(d\sqrt{p}e^{-i\theta/2} + ce^{i\theta/2}\right)^{-k}\right\}, \end{aligned}$$

where $\delta_{c,d} := \gcd(n, d - mc)^2$. This finishes the proof of Lemma 3, except for the three set identities which we now verify.

- $\{(c, mc + nd) \mid \gcd(c, d) = 1, p \mid c\} = \{(mup + nv, u) \mid \gcd(u, v) = 1, p \mid v\}$

  Take a pair $(c, mc + nd)$ in the first set and define $u := mc + nd$.

  Since the arc is admissible $n \mid (1 - m^2p)$, so we can define $v := c(1 - m^2p)/n - mpd \in \mathbb{Z}$. Note that

$$\begin{aligned} mup + nv &= m(mc + nd)p + n(c(1 - m^2p)/n - mpd) \\ &= m^2pc + mnpd + c - cm^2p - mnpd \\ &= c \end{aligned}$$

hence $(c, mc+nd) = (mup+nv, u)$. Observe that

$$\begin{pmatrix} m & n \\ (1-m^2p)/n & -mp \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and $\det \begin{pmatrix} m & n \\ (1-m^2p)/n & -mp \end{pmatrix} = -1$, imply $\gcd(u,v) = 1$. Moreover $p \mid c$ implies $p \mid v$. Therefore the pair $(c, mc+nd)$ taken in the first set is also in the second.

For the converse inclusion take $(mup+nv, u)$ in the second set and define $c := mup+nv$. As before $n \mid (1-m^2p)$, so $d := u(1-m^2p)/n - mv \in \mathbb{Z}$. It follows that

$$\begin{aligned} mc+nd &= m(mup+nv) + n(u(1-m^2p)/n - mv) \\ &= m^2up + mnv + u(1-m^2p) - nmv \\ &= u \end{aligned}$$

which implies $(mup+nv, u) = (c, mc+nd)$.

Note that

$$\begin{pmatrix} mp & n \\ (1-m^2p)/n & -m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

and $\det \begin{pmatrix} mp & n \\ (1-m^2p)/n & -m \end{pmatrix} = -1$. Thus $\gcd(c,d) = 1$. On the other hand, if $p \mid v$ then $p \mid c$. Therefore the second set is contained in the first.

- $\{(c,d) \mid \gcd(c,d) = 1, p \nmid c, \gcd(c,n) = r\}$
  $= \{(c'r, d) \mid \gcd(c',d) = 1, p \nmid c', \gcd(c', n/r) = 1, \gcd(r,d) = 1\}$

Let us start with the second set,

$$\begin{aligned}
&\{(c'r,d) \mid \gcd(c',d)=1, p\nmid c', \gcd(c',n/r)=1, \gcd(r,d)=1\}\\
= &\{(c'r,d) \mid \gcd(c'r,d)=1, p\nmid c', \gcd(c',n/r)=1\}\\
= &\{(c'r,d) \mid \gcd(c'r,d)=1, p\nmid c', \gcd(c'r,n)=r\}\\
= &\{(c,d) \mid \gcd(c,d)=1, p\nmid c, \gcd(c,n)=r\}
\end{aligned}$$

as desired

- $\{(c,rd) \mid \gcd(c,d)=1, p\nmid c, \gcd(c,r)=1, \gcd(n,rd)=r\} = \{(c,d') \mid \gcd(c,d')=1, p\nmid c, \gcd(n,d')=r\}$.

  Let us start with the first set

$$\begin{aligned}
&\{(c,rd) \mid \gcd(c,d)=1, p\nmid c, \gcd(c,r)=1, \gcd(n,rd)=r\}\\
= &\{(c,rd) \mid \gcd(c,rd)=1, p\nmid c, \gcd(n,rd)=r\}\\
= &\{(c,d') \mid \gcd(c,d')=1, p\nmid c, \gcd(n,d')=r\}
\end{aligned}$$

Concluding the proof of Lemma 3.

Proof of Lemma 4: Recall that

$$v_k(c,d,\theta) = \left(\frac{c^2+d^2p+2cd\sqrt{p}\cos(\theta)}{\delta_{c,d}}\right)^{-k/2}, \quad \theta\in[\beta,\pi/2+\alpha], \tag{13}$$

where $\delta_{c,d} = (\gcd(n,d-mc))^2$.

Suppose that $\alpha \le 0$ and $n^2 < p+1$. Let c,d be integers such that $cd>0$. Under these conditions, a term $v_k(c,d,\theta)$ is monotonously increasing in θ when $\theta\in[\beta,\pi/2+\alpha]\subseteq(0,\pi)$. This implies

$$v_k(c,d,\theta) \le v_k(c,d,\pi/2) = \left(\frac{c^2+d^2p}{n^2}\right)^{-k/2}.$$

Note that the right-hand side of the previous equation converges to 0 as k goes to infinity since $n^2 < p+1$, therefore $v_k(c,d,\theta)$ goes to 0 as k goes to infinity uniformly in θ. Hence any term $v_k(c,d,\theta)$ with $cd>0$ is not a bad term.

In the same way one checks for $\beta \ge \pi/2$ the nonexistence of bad terms with $cd<0$. This proves the second part of the statement in the Lemma.

Suppose next $\alpha > 0, \beta < \pi/2$ and that c, d are integers with $cd > 0$. In this case $v_k(c,d,\theta)$ is monotonously increasing in θ and we have $v_k(c,d,\theta) \leq v_k(c,d,\pi/2+\alpha)$. Hence

$$v_k(c,d,\theta) \leq \left(\frac{c^2 + d^2p - 2cd\sqrt{p}\sin(\alpha)}{\delta_{c,d}}\right)^{-k/2}. \tag{14}$$

First, we want to find γ_1 in the open interval $(0,1)$ such that

$$v_k(c,d,\theta) \leq \left(\frac{\gamma_1}{n^2}(c^2+d^2)\right)^{-k/2}. \tag{15}$$

In order to do so, by (14) it is enough to find γ_1 such that

$$\frac{c^2 - 2cd\sqrt{p}\sin(\alpha) + d^2p}{\delta_{c,d}} \geq \frac{\gamma_1}{n^2}(c^2+d^2).$$

Since $\delta_{c,d}$ is less than or equal to n^2, we only need γ_1 such that $c^2 - 2cd\sqrt{p}\sin(\alpha) + d^2p \geq \gamma_1(c^2+d^2)$. Notice that the last inequality is equivalent to

$$c^2(1-\gamma_1) - 2cd\sqrt{p}\sin(\alpha) + d^2(p-\gamma_1) \geq 0.$$

and that such a relation holds if $0 < \gamma_1 < 1$ satisfies

$$4d^2p\sin^2(\alpha) - 4d^2(1-\gamma_1)(p-\gamma_1) = 0, \tag{16}$$

(Because this is the discriminant of the polynomial $x^2(1-\gamma_1) - 2xd\sqrt{p}\sin(\alpha) + d^2(p-\gamma_1) \in \mathbb{R}[x]$). Therefore it is enough to find $\gamma_1 \in (0,1)$ such that (16) holds.

If $d \neq 0$, a solution of (16) is

$$\gamma_1 = \frac{1}{2}\left(p + 1 - \sqrt{(p+1)^2 - 4p\cos^2(\alpha)}\right) \tag{17}$$

Since the points on $\mathcal{F}_p^*$ have imaginary part greater than some positive constant, we have $\alpha < \pi/2$ and by hypothesis $\alpha > 0$. (Recall that $\frac{m}{n} + \frac{e^{i(\pi/2+\alpha)}}{n}$ is a point of the lower boundary of $\mathcal{F}_p^*$). Hence

$$0 < \frac{1}{2}\left(p + 1 - \sqrt{(p+1)^2 - 4p\cos^2(\alpha)}\right) < 1.$$

If $d = 0$, then equation (16) is vacuous and γ_1 in (17) satisfies (16). Consequently we have $\gamma_1 \in (0,1)$ such that (15) holds. Notice that (15) also holds for any γ such that $0 < \gamma \leq \gamma_1$.

On the other hand, if $cd < 0$, the same argument yields the bound

$$v_k(c,d,\theta) \leq \left(\frac{\gamma_2}{n^2}(c^2+d^2)\right)^{-k/2}, \tag{18}$$

for the number $\gamma_2 \in (0,1)$ given by

$$\gamma_2 = \frac{1}{2}\left(p+1-\sqrt{(p+1)^2 - 4p\sin^2(\beta)}\right).$$

In the extreme cases given either by $\alpha \leq 0$ and by $\beta \geq \pi/2$ we can deduce from (13) that

$$v_k(c,d,\theta) \leq \left(\frac{c^2+d^2}{n^2}\right)^{-k/2}. \tag{19}$$

Finally, taking $\gamma := \min\{\gamma_1, \gamma_2\}$ by (15), (18) and (19) we have

$$v_k(c,d,\theta) \leq \left(\frac{\gamma}{n^2}(c^2+d^2)\right)^{-k/2}. \tag{20}$$

for any pair of integers c, d.

Now we use this bound. Recall that $N = N(c,d) = c^2 + d^2$. Using (20) it is clear that

$$\sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N\geq M, d>0}} v_k(c,d,\theta) \leq \sum_{\substack{\gcd(c,d)=1 \\ p\nmid c, N\geq M, d>0}} \left(\frac{\gamma}{n^2}N\right)^{-k/2}.$$

Observe that the number of pairs c, d such that $c^2 + d^2 = N$ is not greater than $2\left(2\sqrt{N}+1\right) \le 8\sqrt{N}$. Then, assuming $k > 6$,

$$\begin{aligned}
\sum_{\substack{\gcd(c,d)=1\\ p\nmid c, N\ge M, d>0}} \left(\frac{\gamma}{n^2}N\right)^{-k/2} &\le \sum_{N\ge M} 8\sqrt{N}\left(\frac{\gamma}{n^2}N\right)^{-k/2} \\
&= 8\left(\frac{\gamma}{n^2}\right)^{-k/2}\sum_{N\ge M} N^{(1-k)/2} \\
&\le 8\left(\frac{\gamma}{n^2}\right)^{-k/2}\int_{x>M-1} x^{(1-k)/2}dx \\
&\le 8\left(\frac{\gamma}{n^2}\right)^{-k/2}\frac{2}{k-3}(M-1)^{(3-k)/2} \\
&< 8(M-1)^{3/2}\left(\frac{\gamma(M-1)}{n^2}\right)^{-k/2}.
\end{aligned}$$

This implies

$$\sum_{\substack{\gcd(c,d)=1\\ p\nmid c, N\ge M, d>0}} v_k(c,d,\theta) < 8(M-1)^{3/2}\left(\frac{\gamma(M-1)}{n^2}\right)^{-k/2},$$

as we wanted to prove. □

Proof of Lemma 5:

As we know from Lemma 1, the imaginary part of any point of $\mathcal{F}_p^*$ is greater than or equal to $\frac{\sqrt{3}}{2p}$. Then, for any $\theta \in [\beta, \pi/2+\alpha]$, the imaginary part of the point $\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}$ in the admissible arc is bounded below by $\frac{\sqrt{3}}{2p}$. In this way one gets

$$\sin(\theta) \ge \frac{n}{2}\sqrt{\frac{3}{p}}.$$

In turn, this inequality yields

$$|\cos(\theta)| \le \frac{\sqrt{4p-3n^2}}{2\sqrt{p}}. \tag{21}$$

Now we prove the Lemma in five steps.

Step 1: *If $v_k(c,d,\theta)$ with $d > 0$ is a bad term, then $d = 1$ and $mc \equiv 1 \pmod{n}$.*

Recall that

$$v_k(c,d,\theta) = \left(\frac{c^2 + d^2p + 2cd\sqrt{p}\cos(\theta)}{\delta_{c,d}}\right)^{-k/2}, \tag{22}$$

where $\delta_{c,d} = (\gcd(n, d - mc))^2$. Hence

$$v_k(c,d,\theta) \leq \left(\frac{c^2 + d^2p - 2\sqrt{p}\,|cd\cos(\theta)|}{\delta_{c,d}}\right)^{-k/2} \leq \left(\frac{c^2 + d^2p - |cd|\,\sqrt{4p - 3n^2}}{n^2}\right)^{-k/2}, \tag{23}$$

where in the last inequality we used (21) and $\delta_{c,d} \leq n^2$.

On the other hand, if $d > 1$ then the function $f(c) := c^2 + d^2p - |cd|\,\sqrt{4p - 3n^2} - n^2$ has no real roots and $\lim_{c\to\infty} f(c) = \infty$. Therefore

$$\frac{c^2 + d^2p - |cd|\,\sqrt{4p - 3n^2}}{n^2} > 1.$$

In this way, if $d > 1$ the right-hand side of (23) goes to zero as k goes to ∞, and $v_k(c,d,\theta)$ goes to zero as k goes to ∞ uniformly in θ. This shows that $v_k(c,d,\theta)$ is a good term whenever $d > 1$.

Suppose now that $\delta_{c,1} \neq n^2$. Since $\delta_{c,1}^{1/2} \mid n$, our assumption implies that $\delta_{c,1}^{1/2}$ is a proper divisor of n, hence $\delta_{c,1} \leq n^2/4$. Using this relation in the argument which yields (23) we obtain

$$v_k(c,1,\theta) \leq \left(\frac{c^2 + p - |c|\,\sqrt{4p - 3n^2}}{n^2/4}\right)^{-k/2}. \tag{24}$$

Since the function $f(c) := c^2 + p - |c|\,\sqrt{4p - 3n^2} - n^2/4$ has no real roots and $\lim_{c\to\infty} f(c) = \infty$, we deduce that

$$\frac{c^2 + p - |c|\,\sqrt{4p - 3n^2}}{n^2/4} > 1.$$

From the latter we conclude that the right-hand side of (24) converges to 0 when k goes to infinity, and therefore that $v_k(c,1,\theta)$ is a good term.

Consequently, if $v_k(c,1,\theta)$ is a bad term then $\delta_{c,1} = n^2$. This fact and the definition of $\delta_{c,1}$ yield $mc \equiv 1 \pmod{n}$.

This complete the proof of step 1.

Step 2: *The terms $v_k(n_1, 1, \theta)$ and $v_k(-n_2, 1, \theta)$ are always bad terms, where n_1 and n_2 are the parameters of the neighboring arcs.*

Since the arcs defined by the circles $\frac{e^{i\theta}}{n_1\sqrt{p}} + \frac{m_1}{n_1}$, $\frac{e^{i\theta}}{n\sqrt{p}} + \frac{m}{n}$ and $\frac{e^{i\theta}}{n_2\sqrt{p}} + \frac{m_2}{n_2}$ with $\theta \in \mathbb{R}$ are neighbors, we can compute their intersection points and get an expression for $\sin(\alpha)$ and $\cos(\beta)$. By elementary methods one finds that such expressions are given by

$$\sin(\alpha) = \frac{1}{2}\frac{n_1^2 + p(mn_1 - m_1n)^2 - n^2}{n_1\sqrt{p}(mn_1 - m_1n)}, \quad \text{and} \quad \cos(\beta) = \frac{1}{2}\frac{n_2^2 + p(m_2n - mn_2)^2 - n^2}{n_2\sqrt{p}(m_2n - mn_2)}.$$

Since the arc is admissible, we have $mn_1 - m_1n = m_2n - mn_2 = 1$. Then the expressions above turn into

$$\sin(\alpha) = \frac{n_1^2 + p - n^2}{2n_1\sqrt{p}}, \tag{25}$$

$$\text{and} \quad \cos(\beta) = \frac{n_2^2 + p - n^2}{2n_2\sqrt{p}}.$$

Manipulating the expression for $\sin(\alpha)$ one obtains

$$n_1^2 + p - 2n_1\sqrt{p}\sin(\alpha) = n^2.$$

Using this identity we get from (22)

$$v_k(n_1, 1, \pi/2 + \alpha) = \left(\frac{n_1^2 + p - 2n_1\sqrt{p}\sin(\alpha)}{n^2}\right)^{-k/2} = 1$$

This tells us that $v_k(n_1, 1, \theta)$ is a bad term (since $v_k(n_1, 1, \pi/2 + \alpha)$ does not converge to 0 when k goes to infinity). By the same argument one gets from the expression for $\cos(\beta)$ in (25) that $v_k(-n_2, 1, \theta)$ is a bad term.

Step 3: *There are at most two bad terms $v_k(c, 1, \theta)$ with $c > 0$ (respectively $c < 0$), and a necessary condition for the existence of two bad terms is that p can be written as*

$$n_1^2 + nn_1 + n^2 = p \quad \textit{or} \quad n_1^2 - nn_1 + n^2 = p$$

(respectively $n_2^2 + nn_2 + n^2 = p$ *or* $n_2^2 - nn_2 + n^2 = p$*).*

Moreover, if there are two bad terms then the numbers

$$\frac{\sqrt{4p - 3n^2} \pm n}{2}$$

(respectively $\dfrac{-\sqrt{4p-3n^2} \pm n}{2}$*) are the corresponding values of the parameter* c *of such terms.*

Take a bad term $v_k(c, 1, \theta)$ such that $c > 0$. By the monotonicity in θ of v_k (see equation (22)), we get

$$v_k(c, 1, \theta) \leq v_k(c, 1, \pi/2 + \alpha) = \left(\frac{c^2 + p - 2c\sqrt{p}\sin(\alpha)}{n^2} \right)^{-k/2}. \tag{26}$$

From this equation, it is clear that $v_k(c, 1, \theta)$ is a bad term if and only if

$$\frac{c^2 + p - 2c\sqrt{p}\sin(\alpha)}{n^2} \leq 1$$

(because the right-hand side of (26) converges to 0 when k goes to infinity if and only if $\frac{c^2+p-2c\sqrt{p}\sin(\alpha)}{n^2} > 1$).

The last inequality is equivalent to $c^2 + p - 2c\sqrt{p}\sin(\alpha) - n^2 \leq 0$. Therefore c must be an integer lying between the solutions of the equation $x^2 + p - 2x\sqrt{p}\sin(\alpha) - n^2 = 0$. In other words, c satisfies

$$\sqrt{p}\sin(\alpha) - \sqrt{p(\sin^2(\alpha) - 1) + n^2} \leq c \leq \sqrt{p}\sin(\alpha) + \sqrt{p(\sin^2(\alpha) - 1) + n^2}. \tag{27}$$

From (21) and the identity $\cos(\pi/2 + \alpha) = -\sin(\alpha)$ we have

$$|\sin(\alpha)| \leq \frac{\sqrt{4p - 3n^2}}{2\sqrt{p}}. \tag{28}$$

This upper bound can be used to simplify (27), i.e.

$$\sqrt{p}\sin(\alpha) - \frac{n}{2} \leq c \leq \sqrt{p}\sin(\alpha) + \frac{n}{2}. \tag{29}$$

This puts c in an interval of length n. But $mc \equiv 1 \pmod{n}$ as we showed in step 1. Hence there is just one such c in the case that one of the inequalities in (29) is strict. Otherwise there are at most two possibilities for c.

Clearly, the existence of two distinct such values of c requires equality in (28). Moreover, if we suppose the existence of two such c then

$$c = \sqrt{p}\sin(\alpha) \pm \frac{n}{2} = \frac{\sqrt{4p - 3n^2} \pm n}{2}. \tag{30}$$

This is because the only way in which we can have two integral solutions of the equation $mc \equiv 1 \pmod{n}$ in an interval of length n is that they are the extremes of the interval.

Now, by step 2, n_1 must be one of the values in (30). From this fact one gets directly that p has the form $p = n_1^2 + nn_1 + n^2$ or $p = n_1^2 - nn_1 + n^2$.

The same argument for $c < 0$ yields that if there are two bad terms $v_k(c, 1, \theta)$, the possible values of c are

$$c = \frac{-\sqrt{4p - 3n^2} \pm n}{2}, \tag{31}$$

and since $-n_2$ is one of them, one gets that p has the form $p = n_2^2 + nn_2 + n^2$ or $p = n_2^2 - nn_2 + n^2$.

This proves the claim in step 3 and the first statement of Lemma 5.

Step 4: *The bad terms $v_k(c, 1, \theta)$, other than $v_k(n_1, 1, \theta)$ and $v_k(-n_2, 1, \theta)$, are*

i) $v_k(n_1 - n, 1, \theta)$ if $n_1^2 - nn_1 + n^2 = p$ and $n_1 > n$.

ii) $v_k(n_1 + n, 1, \theta)$ if $n_1^2 + nn_1 + n^2 = p$.

iii) $v_k(-n_2 + n, 1, \theta)$ if $n_2^2 - nn_2 + n^2 = p$ and $n_2 > n$.

iv) $v_k(-n_2 - n, 1, \theta)$ if $n_2^2 + nn_2 + n^2 = p$.

Moreover, if i) and ii) (respectively iii) and iv)) do not occur, then $v_k(n_1, 1, \theta)$ (respectively $v_k(-n_2, 1, \theta)$) is the only bad term with $c > 0$ (respectively $c < 0$).

We prove only i) because the others cases are deduced similarly.

Suppose that $n_1 > n$ and $n_1^2 - n_1 n + n^2 = p$. Observe that under the condition $n_1 > n$, the equality $2n_1 - n = \sqrt{4p - 3n^2}$ is equivalent to $n_1^2 - n_1 n + n^2 = p$.

Using this identity in (25) we can compute $\sin(\alpha)$ in terms of n as

$$\sin(\alpha) = \frac{n_1^2 - n^2 + p}{2n_1\sqrt{p}} = \frac{2n_1 - n}{2\sqrt{p}} = \frac{\sqrt{4p - 3n^2}}{2\sqrt{p}}. \tag{32}$$

On the other hand, from (25) we know that n_1 is a solution of

$$x^2 - 2x\sqrt{p}\sin(\alpha) + p - n^2 = 0. \tag{33}$$

Using (32), one can check that the solutions of (33) are

$$x = \frac{\sqrt{4p - 3n^2} \pm n}{2}.$$

Therefore n_1 must be of this form. In fact, one can check that $n_1 = \dfrac{\sqrt{4p - 3n^2} + n}{2}$ (otherwise we would have $n_1^2 + nn_1 + n^2 = p$). Therefore the other solution is $n_1 - n$, i.e.

$$(n_1 - n)^2 - 2(n_1 - n)\sqrt{p}\sin(\alpha) + p - n^2 = 0.$$

Using this equation we can get from (22) the value of $v_k(n_1 - n, 1, \pi/2 + \alpha)$ as

$$v_k(n_1 - n, 1, \pi/2 + \alpha) = \left(\frac{(n_1 - n)^2 + p - 2(n_1 - n)\sqrt{p}\sin(\alpha)}{n^2}\right)^{-k/2} = 1.$$

Thus $v_k(n_1 - n, 1, \theta)$ is a bad term. This completes the proof of i).

Remark: In this proof we conclude that $n_1 - n$ and n_1 are solutions of (33). In the case ii) where the other bad term is $v_k(n_1 + n, 1, \theta)$, one gets that $n_1 + n$ is a solution of (33).

Similarly, in the cases iii) and iv) where the bad terms are $v_k(c, 1, \theta)$ with $c < 0$ one has that c satisfies

$$x^2 + p - 2x\sqrt{p}\cos(\beta) - n^2 = 0.$$

Now we prove the last statement in steep 4.

Suppose now that the conditions in i) and ii) do not hold. If we assume the existence of another bad term $v_k(c, 1, \theta)$ with $c > 0$ then we conclude that $p = n_1^2 - nn_1 + n^2$ and $n_1 < n$ from step 3 (because a necessary condition for the existence of two bad terms is $p = n_1^2 + nn_1 + n^2$ or $p = n_1^2 - nn_1 + n^2$).

Let $v_k(c', 1, \theta)$ be the other bad term with $c > 0$ (other than $v_k(n_1, 1, \theta)$). By step 3, the numbers

$$\frac{\sqrt{4p - 3n^2} \pm n}{2}$$

are the possible values of c'. Indeed we have

$$c' = \frac{\sqrt{4p - 3n^2} - n}{2}, \text{ and } n_1 = \frac{\sqrt{4p - 3n^2} + n}{2}$$

(otherwise $n_1 = \frac{\sqrt{4p-3n^2}-n}{2}$, and in such a case we would have $n_1^2 + nn_1 + n^2$ which is a contradiction). Hence $c' = n_1 - n$ and therefore $c' < 0$. This is a contradiction because we are assuming $c' > 0$. Consequently there is no other bad term.

The proof for bad terms with $c < 0$ is similar. This completes the proof of step 4.

Step 5: *The following equalities hold:*

a) $$\lim_{k\to\infty} \sum_{\substack{\text{bad terms} \\ c>0}} v_k\left(c, d, \tfrac{\pi}{2} + \alpha - \tfrac{\pi}{k}\right) = c_1 \exp\left(\frac{-\pi n_1 \sqrt{p}\cos(\alpha)}{n^2}\right)$$

b) $$\lim_{k\to\infty} \sum_{\substack{\text{bad terms} \\ c<0}} v_k\left(c, d, \beta + \tfrac{\pi}{k}\right) = c_2 \exp\left(\frac{-\pi n_2 \sqrt{p}\sin(\beta)}{n^2}\right)$$

where c_1, c_2 are the constants given in (3).

In the following we prove a). The proof of b) is analogous.

Let $v_k(c, 1, \theta)$ be a bad term with $c > 0$. From (22) we know that

$$v_k(c, 1, \pi/2 + \alpha - \pi/k) = \left(\frac{c^2 + p - 2c\sqrt{p}\sin(\alpha - \pi/k)}{n^2}\right)^{-k/2}.$$

Then

$$\begin{aligned}
\lim_{k\to\infty} v_k(c,1,\pi/2+\alpha-\pi/k) &= \lim_{k\to\infty}\left(\frac{c^2+p-2c\sqrt{p}\sin(\alpha-\pi/k)}{n^2}\right)^{-k/2} \\
&= \lim_{k\to\infty}\exp\left(-\frac{1}{2}\frac{\log\left(\frac{c^2+p-2c\sqrt{p}\sin(\alpha-\pi/k)}{n^2}\right)}{k^{-1}}\right) \\
&= \exp\left(-\frac{1}{2}\lim_{k\to\infty}\frac{\log\left(\frac{c^2+p-2c\sqrt{p}\sin(\alpha-\pi/k)}{n^2}\right)}{k^{-1}}\right) \quad (34)
\end{aligned}$$

By the remark in the proof of step 4, we have

$$\frac{c^2+p-2c\sqrt{p}\sin(\alpha)}{n^2}=1.$$

We can therefore use L'Hopital's rule to compute the limit in (34).

$$\begin{aligned}
\lim_{k\to\infty} v_k(c,1,\pi/2+\alpha-\pi/k) &= \exp\left(-\frac{1}{2}\lim_{k\to\infty}\frac{2c\sqrt{p}\pi\cos(\alpha-\pi/k)}{c^2+p-2c\sqrt{p}\sin(\alpha-\pi/k)}\right) \\
&= \exp\left(-\frac{c\sqrt{p}\pi\cos(\alpha)}{n^2}\right).
\end{aligned}$$

In the same way one proves the identity

$$\lim_{k\to\infty} v_k(c,1,\beta+\pi/k)=\exp\left(\frac{c\sqrt{p}\pi\sin(\beta)}{n^2}\right)$$

for a bad term with $c<0$.

At this point we recall step 4 which establish precisely which bad terms are relevant in the proof of a). We finish this proof by studying these three cases

Case 1: $p=n_1^2-nn_1+n^2$ and $n_1>n$.

By step 4, we have

$$\sum_{\substack{\text{bad terms}\\ c>0}} v_k\left(c,d,\tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right)=v_k(n_1,1,\pi/2+\alpha-\pi/k)+v_k(n_1-n,1,\pi/2+\alpha-\pi/k).$$

Then

$$\begin{aligned}
& \lim_{k\to\infty} \sum_{\substack{\text{bad terms}\\ c>0}} v_k\left(c,d,\tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right) \\
= \; & \lim_{k\to\infty} v_k(n_1,1,\pi/2+\alpha-\pi/k) + \lim_{k\to\infty} v_k(n_1-n,1,\pi/2+\alpha-\pi/k) \\
= \; & \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right) + \exp\left(\frac{-(n_1-n)\sqrt{p}\pi\cos(\alpha)}{n^2}\right) \\
= \; & \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right)\left(1+\exp\left(\frac{n\sqrt{p}\pi\cos(\alpha)}{n^2}\right)\right) \qquad (35)
\end{aligned}$$

As in the proof of step 4, from the equality $p = n_1^2 - nn_1 + n^2$ we deduce that

$$\sin(\alpha) = \frac{\sqrt{4p-3n^2}}{2\sqrt{p}},$$

and therefore $\cos(\alpha) = \frac{n}{2}\sqrt{\frac{3}{p}}$ (since $\alpha \in (0,\pi/2)$). Using the latter we can simplify (35) to get

$$\begin{aligned}
\lim_{k\to\infty} \sum_{\substack{\text{bad terms}\\ c>0}} v_k\left(c,d,\tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right) &= \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right)\left(1+\exp\left(\frac{\pi\sqrt{3}}{2}\right)\right) \\
&= c_1\exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right).
\end{aligned}$$

Case 2 $p = n_1^2 + nn_1 + n^2$.

The same argument of the previous case yields

$$\begin{aligned}
\lim_{k\to\infty} \sum_{\substack{\text{Bad terms}\\ c>0}} v_k\left(c,d,\tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right) &= \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right)\left(1+\exp\left(-\frac{\pi\sqrt{3}}{2}\right)\right) \\
&= c_1\exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right).
\end{aligned}$$

Case 3 The only bad term $v_k(c,1,\theta)$ with $c>0$ is $v_k(n_1,1,\theta)$.

In this case we have $c = n_1$ and

$$\lim_{k\to\infty} \sum_{\substack{\text{Bad terms}\\ c>0}} v_k\left(c,d,\tfrac{\pi}{2}+\alpha-\tfrac{\pi}{k}\right) = \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right) = c_1\exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right)$$

Consequently in all three cases we have obtained

$$\lim_{k\to\infty} \sum_{\substack{\text{bad terms} \\ c>0}} v_k\left(c, d, \tfrac{\pi}{2} + \alpha - \tfrac{\pi}{k}\right) = c_1 \exp\left(\frac{-n_1\sqrt{p}\pi\cos(\alpha)}{n^2}\right)$$

This completes the proof in the case $\alpha, \beta \in (0, \pi/2)$. If $\alpha \leq 0$ then $\beta < \pi/2$ (See remark in below Definition 1) and the previous proof for equation b) works. On the other hand, if $\beta \geq \pi/2$ then $\alpha > 0$ and the previous proof for equation a) works.

CHAPTER 3

Application

3.1. Application.

In this section we prove the following result.

Let $p = 13$, and $k \equiv 0 \pmod{12}$ large enough. Then all but at most four zeros of $E^*_{k,13}$ in $\mathcal{F}^*_{13}$ lie on the lower arcs of the fundamental domain $\mathcal{F}^*_{13}$.

From Lemma 1 one easily checks that the following is the fundamental domain $\mathcal{F}^*_{13}$ (for more details see for example [Sh1] p. 33-36).

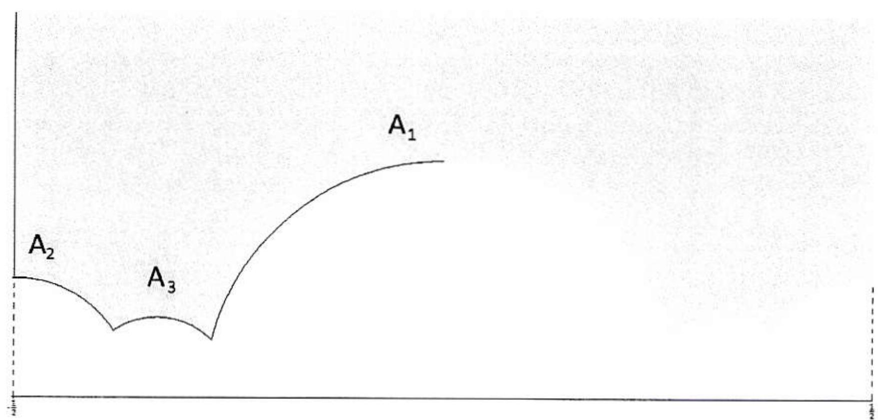


Figure 1: Fundamental domain for $\Gamma_0^*(13)$

The lower arcs A_2, A_3 and A_1 are parametrized as follows

- A_1: $\dfrac{e^{i\theta}}{\sqrt{13}}$ for $\theta \in [\pi/2 - \alpha_1, \pi/2 + \alpha_1]$, where $\alpha_1 = \arctan\left(7/\sqrt{3}\right)$.
- A_2: $\dfrac{e^{i\theta}}{2\sqrt{13}} + \dfrac{-1}{2}$ for $\theta \in [\beta_2, \pi/2]$, where $\beta_2 = \arctan(2/3)$.
- A_3: $\dfrac{e^{i\theta}}{3\sqrt{13}} + \dfrac{-1}{3}$ for $\theta \in [\beta_3, \pi/2 + \alpha_3]$, where $\beta_3 = \arctan\left(3\sqrt{3}/5\right)$ and $\alpha_3 = \arctan(2/3)$.

Observe that $\alpha_1 - \beta_3 = \frac{\pi}{6}$.

Now we check that the arcs A_1, A_2 and A_3 are admissible.

- The arc A_1.

  The neighbors of this arc have their centers at $\pm 1/3$, therefore $m = 0$, $n = 1$, $n_1 = n_2 = 3$, $m_1 = -1$ and $m_2 = 1$ in this case. Hence

$$m^2 13 \equiv 1 \pmod{n} \quad , \text{ and } \quad mn_1 - m_1 n = m_2 n - mn_2 = 1$$

  On the other hand $13 = n_i^2 + n_i \cdot n + n^2$ therefore

$$\begin{aligned}
& c_1 \exp\left(-3\sqrt{13}\pi \cos(\alpha_1)\right) + c_2 \exp\left(-3\sqrt{13}\pi \sin(\pi/2 - \alpha_1)\right) \\
= \; & 2\left(1 + \exp\left(\frac{-\pi\sqrt{3}}{2}\right)\right) \exp\left(-3\sqrt{13}\pi \cos(\alpha_1)\right) \\
= \; & 0.00060... < 1.
\end{aligned}$$

  This computation shows that the arc is admissible.

- For the arc A_3.

  The neighbors of this arc have their centers at $-1/2$ and 0 respectively, therefore $n = 3$, $m = -1$, $n_1 = 2$, $n_2 = 1$, $m_1 = -1$ and $m_2 = 0$ in this case. Hence

$$m^2 13 \equiv 1 \pmod{n} \quad , \text{ and } \quad mn_1 - m_1 n = m_2 n - mn_2 = 1$$

  On the other hand $13 \neq n_1^2 \pm n_1 \cdot n + n^2$ but $13 = n_2^2 + n_2 \cdot n + n^2$ therefore

$$\begin{aligned}
& c_1 \exp\left(\frac{-2\sqrt{13}\pi \cos(\alpha_3)}{9}\right) + c_2 \exp\left(\frac{-\sqrt{13}\pi \sin(\beta_3)}{9}\right) \\
= \; & \exp\left(\frac{-2\sqrt{13}\pi \cos(\alpha_3)}{9}\right) + \left(1 + \exp\left(\frac{-\pi\sqrt{3}}{2}\right)\right) \exp\left(\frac{-\sqrt{13}\pi \sin(\beta_3)}{9}\right) \\
= \; & 0.55349... < 1.
\end{aligned}$$

  This again shows that the arc is admissible.

- For the arc A_2.

  The only neighbor of this arc has its center at $-1/3$. Therefore in this case $n=2$, $m=-1$, $n_2=3$ and $m_2=-1$. Check that

$$m^2 13 \equiv 1 \pmod{n} \ , \ \text{and} \quad m_2 n - m n_2 = 1$$

  On the other hand $13 \neq n^2 \pm n \cdot n_2 + n_2^2$, so $c_2 = 1$. Hence

$$c_2 \exp\left(\frac{-3\sqrt{13}\pi \sin(\beta_2)}{4}\right) \quad = \quad 0.00898... < 1.$$

  Thus the arc is admissible.

Now we count the number of zeros of $E^*_{k,13}$, from now on assume that $12 \mid k$. Consider the following function (recall that $\alpha_1 - \beta_3 = \frac{\pi}{6}$ and $\alpha_3 = \beta_2$).

$$F_k(\theta) := \begin{cases} (-1)^{k/4} \exp\left(-\frac{ik}{2}\left(\theta + \frac{\pi}{2}\right)\right) E^*_{k,13}\left(\frac{\exp\left(i\left(\theta+\frac{\pi}{2}\right)\right)}{\sqrt{13}}\right) & \text{if } \theta \in [0, \alpha_1), \\ (-1)^{k/12} \exp\left(-\frac{ik}{2}\left(\theta - \frac{\pi}{6}\right)\right) E^*_{k,13}\left(-\frac{1}{3} + \frac{\exp\left(i\left(\theta-\frac{\pi}{6}\right)\right)}{3\sqrt{13}}\right) & \text{if } \theta \in [\alpha_1, \frac{2\pi}{3} + \alpha_3), \\ (-1)^{k/3} \exp\left(-\frac{ik}{2}\left(\theta - \frac{2\pi}{3}\right)\right) E^*_{k,13}\left(-\frac{1}{2} + \frac{\exp\left(i\left(\theta-\frac{2\pi}{3}\right)\right)}{2\sqrt{13}}\right) & \text{if } \theta \in [\frac{2\pi}{3} + \alpha_3, \frac{7\pi}{6}). \end{cases}$$

This is a continuous function (the continuity at the points α_1 and $\frac{2\pi}{3} + \alpha_3$ follows the hypothesis $12 \mid k$, and the equalities

$$\frac{\exp\left(i\left(\alpha_1 + \frac{\pi}{2}\right)\right)}{\sqrt{13}} = -\frac{1}{3} + \frac{\exp\left(i\left(\alpha_1 - \frac{\pi}{6}\right)\right)}{3\sqrt{13}}$$

$$\text{and} \quad -\frac{1}{3} + \frac{\exp\left(i\left(\frac{\pi}{2} + \alpha_3\right)\right)}{3\sqrt{13}} = -\frac{1}{2} + \frac{\exp\left(i\alpha_3\right)}{2\sqrt{13}},$$

which are expressions for the points where the arcs A_1, A_2 and A_3 intersect).

Notice that F_k has a zero at $\theta \in [0, \alpha_1)$ if and only if $E^*_{k,13}$ has a zero at $\frac{\exp\left(i\left(\theta+\frac{\pi}{2}\right)\right)}{\sqrt{13}} \in A_1$. Similarly, F_k has a zero at $\theta \in [\alpha_1, \frac{2\pi}{3} + \alpha_3)$ (respectively $[\frac{2\pi}{3} + \alpha_3, \frac{7\pi}{6}]$) if and only if $E^*_{k,13}$ has a zero at

$$-\frac{1}{3} + \frac{\exp(i(\theta - \frac{\pi}{6}))}{3\sqrt{13}} \in A_3 \text{ (respectively } -\frac{1}{2} + \frac{\exp(i(\theta - \frac{2\pi}{3}))}{2\sqrt{13}} \in A_2).$$

Therefore, the number of zeros of F_k in the interval $[0, \frac{7\pi}{6})$ is equal to the number of zeros of $E^*_{k,13}$ on the union of the arcs A_2, A_3 and the left half of A_1.

From Proposition 1 we have

$$F_k(\theta) := \begin{cases} 2(-1)^{k/4}\cos(\frac{k}{2}(\theta + \frac{\pi}{2})) + R_{1,k}(\theta) & \text{if } \theta \in [0, \alpha_1), \\ 2(-1)^{k/12}\cos(\frac{k}{2}(\theta - \frac{\pi}{6})) + R_{2,k}(\theta) & \text{if } \theta \in [\alpha_1, \frac{2\pi}{3} + \alpha_3), \\ 2(-1)^{k/3}\cos(\frac{k}{2}(\theta - \frac{2\pi}{3})) + R_{3,k}(\theta) & \text{if } \theta \in [\frac{2\pi}{3} + \alpha_3, \frac{7\pi}{6}), \end{cases}$$

where

$$\begin{aligned} |R_{1,k}(\theta)| < 2 \quad & \text{if} \quad \theta \in [0, \alpha_1 - \tfrac{\pi}{k}], \\ |R_{2,k}(\theta)| < 2 \quad & \text{if} \quad \theta \in [\alpha_1 + \tfrac{\pi}{k}, \tfrac{2\pi}{3} + \alpha_3 - \tfrac{\pi}{k}], \\ |R_{3,k}(\theta)| < 2 \quad & \text{if} \quad \theta \in [\tfrac{2\pi}{3} + \alpha_3 + \tfrac{\pi}{k}, \tfrac{7\pi}{6}], \end{aligned}$$

for k large enough.

Now, since $12 \mid k$, we can write

$$F_k(\theta) = 2\cos(k\theta/2) + R_k(\theta)$$

where $|R_k(\theta)| < 2$ for $\theta \in \big[0, \frac{7\pi}{6}\big] \backslash \big(\big[\alpha_1 - \frac{\pi}{k}, \alpha_1 + \frac{\pi}{k}\big] \cup \big[\frac{2\pi}{3} + \alpha_3 - \frac{\pi}{k}, \frac{2\pi}{3} + \alpha_3 + \frac{\pi}{k}\big]\big)$ for $k >> 0$.

Now, note that $\cos(k\theta/2)$ has $\frac{7k}{12} + 1$ extreme values (including 0 and $\frac{7\pi}{6}$) in the interval $[0, \frac{7\pi}{6}]$ and at most two of them are in $\big[\alpha_1 - \frac{\pi}{k}, \alpha_1 + \frac{\pi}{k}\big] \cup \big[\frac{2\pi}{3} + \alpha_3 - \frac{\pi}{k}, \frac{2\pi}{3} + \alpha_3 + \frac{\pi}{k}\big]$. Thus, since F_k is continuous and real-valued, at least $\frac{7k}{12} - 4$ zeros of F_k lie in the interval $[0, \frac{7\pi}{6}]$. Hence, $\frac{7k}{12} - 4$ zeros of $E^*_{k,13}$ lie in the union of the arcs A_2, A_3 and the left half of A_3.

On the other hand, the total numbers of zeros of $E^*_{k,13}$ is $\frac{7k}{12}$ according to the valence formula. This prove that all but at most 4 zeros of $E^*_{k,13}$ lie in the lower arcs of $\mathcal{F}^*_{13}$.

Notice that the previous result is a consequence of proposition 1.

Remark: Only the right boundary of $\mathcal{F}_{13}^*$ is considered in the fundamental domain, because there is an identification between the left and the right boundary $\mathcal{F}_{13}^*$ under $\Gamma_0^*(13)$.

Appendix

Appendix A

In this appendix we prove the following proposition

Proposition 2. *For any non-zero weight k modular form f over $\Gamma_0^*(p)$ we have*

$$\nu_\infty(f) + \sum_{z\in\mathcal{F}_p^*} \frac{\nu_z(f)}{n_z^*} = \frac{k(p+1)}{24}, \tag{36}$$

where n_z^* is the order of the stabilizer of z in $\Gamma_0^*(p)/\{\pm I\}$.

Proof: For any point $z \in \mathcal{H}$ let S_z denote the stabilizer of z in $\Gamma_0(p)$ and S_z^* the stabilizer of z in $\Gamma_0^*(p)$.

Let $n_z := \frac{1}{2}|S_z|$ and $n_z^* := \frac{1}{2}|S_z^*|$. Observe that n_z^* is the order of the stabilizer of z in $\Gamma_0^*(p)/\{\pm I\}$.

Since $S_z = \Gamma_0(p) \cap S_z^*$ we have that $n_z \mid n_z^*$. In particular $n_z \leq n_z^*$.

Moreover, $n_z < n_z^*$ if and only if there exists $\gamma \in \Gamma_0(p)$ such that $z = W_p\gamma z$ (because $\Gamma_0^*(p) = \Gamma_0(p) \cup W_p\Gamma_0(p)$). The latter is equivalent to $W_p z = \gamma z$ (because $W_p^{-1}z = W_p z$).

This means that $n_z < n_z^*$ if and only if z and $W_p z$ are in the same $\Gamma_0(p)$-orbit.

On the other hand, if there exists $\gamma \in \Gamma_0(p)$ such that $W_p\gamma \in S_z^*$ we have

$$\frac{\Gamma_0^*(p)}{\Gamma_0(p)} \cong \frac{S_z^*\Gamma_0(p)}{\Gamma_0(p)} \cong \frac{S_z^*}{S_z^* \cap \Gamma_0(p)} \cong \frac{S_z^*}{S_z},$$

therefore $2n_z = n_z^*$.

Since $\Gamma_0^*(p) = \Gamma_0(p) \cup W_p\Gamma_0(p)$ we know that $F := \mathcal{F}_p^* \cup W_p\mathcal{F}_p^*$ is a fundamental domain for $\Gamma_0(p)$. (see for example [Ko] p. 105).

For any non-zero modular form f over $\Gamma_0(p)$, the valence formula for $\Gamma_0(p)$ gives (see for example [BVHZ] pp. 11-12).

$$\nu_\infty(f) + \nu_0(f) + \sum_{z\in F} \frac{\nu_z(f)}{n_z} = \frac{k(p+1)}{12}. \tag{37}$$

Here $\nu_\infty(f)$ and $\nu_z(f)$ for $z \in \mathcal{H}$ are defined as in chapter 2. page 8. The integer $\nu_0(f)$ is the order of the zero at the cusp 0. The latter is defined as the index of the first non-zero term of the q-expansion of $f \mid_k \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, i.e.

$$f \mid_k \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = \sum_{n=\nu_0(f)}^{\infty} a_n q^n \quad \text{with} \quad a_{\nu_0(f)} \neq 0.$$

Now take a non-zero modular form $f \in \mathcal{M}_k(\Gamma_0^*(p)) \subseteq \mathcal{M}_k(\Gamma_0(p))$. Observe that

$$f(W_p z) = (\sqrt{p}z)^k f(z).$$

This implies $\nu_z(f) = \nu_{W_p z}(f)$ for $z \in \mathcal{H}$. One also has $\nu_0(f) = \nu_\infty(f)$ since $f \in \mathcal{M}_k(\Gamma_0^*(p))$ (see for example Lemma 2 in [Ko] p. 127).

Recall that 0 and infinity are in the same orbit under $\Gamma_0(p)$.

We now deduce (36) from (37).

Let $z \in \mathcal{F}_p^*$ be such that $\nu_z(f) > 0$, and let $z' := W_p z$.

Case 1: z and z' are not in the same $\Gamma_0(p)$-orbit.

In this case, z and z' are two distinct points in F. Therefore both terms $\frac{\nu_z(f)}{n_z}$ and $\frac{\nu_{z'}(f)}{n_{z'}}$ appear on the left-hand side of (37). Notice that

$$\frac{\nu_z(f)}{n_z} + \frac{\nu_{z'}(f)}{n_{z'}} = 2\frac{\nu_z(f)}{n_z} = 2\frac{\nu_z(f)}{n_z^*}$$

since $n_z = n_{z'}$ (because $S_{z'} = W_p S_z W_p^{-1}$). In the last equality we use that $n_z = n_z^*$ since z and z' are not in the same $\Gamma_0(p)-$orbit.

Case 2: z and z' are in the same $\Gamma_0(p)$-orbit.

In this case, only one element of $\{z, z'\}$ is in F, and therefore only one of the terms $\frac{\nu_z(f)}{n_z}$ and $\frac{\nu_{z'}(f)}{n_{z'}}$ appears in the sum (37). Notice that

$$\frac{\nu_z(f)}{n_z} = \frac{\nu_{z'}(f)}{n_{z'}}$$

as before. Moreover, since z and z' are in the same $\Gamma_0(p)-$orbit, $n_z = \frac{1}{2}n_z^*$ and so

$$\frac{\nu_z(f)}{n_z} = 2\frac{\nu_z(f)}{n_z^*}.$$

Using the two cases above, the remark about $\nu_0(f) = \nu_\infty(f)$, and (37) we deduce that

$$\begin{aligned}\frac{k(p+1)}{12} &= \nu_\infty(f) + \nu_0(f) + \sum_{z\in F}\frac{\nu_z(f)}{n_z} \\ &= 2\nu_\infty(f) + 2\sum_{z\in\mathcal{F}_p^*}\frac{\nu_z(f)}{n_z^*}.\end{aligned}$$

Therefore

$$\frac{k(p+1)}{24} = \nu_\infty(f) + \sum_{z\in\mathcal{F}_p^*}\frac{\nu_z(f)}{n_z^*},$$

as we wanted to prove.

Appendix B

In this appendix we prove Lemmas 1 and 2.

Proof of Lemma 1.

The boundary of the fundamental domain $\mathcal{F}_p^*$ consists of the two vertical lines $Re(z) = \pm 1/2$, and the maximal arcs defined by the circles $|az + b| = 1$ where a, b runs over the inferior row of the elements of $\Gamma_0^*(p)$. In this context maximal means that the arc is not in the interior of any disc $|az + b| \leq 1$, where a, b runs over the lower row of the elements of $\Gamma_0^*(p)$. (See for example [A]).

Since $\Gamma_0^*(p) = \Gamma_0(p) \cup W_p\Gamma_0(p)$, there are two types of matrices in this group, namely:

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \quad \text{and} \quad W_p \begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \begin{pmatrix} -\sqrt{p}c & -d/\sqrt{p} \\ a\sqrt{p} & b\sqrt{p} \end{pmatrix}, \quad \text{with} \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma_0(p)$$

In particular $\gcd(pc, d) = 1$ and $\gcd(a, b) = 1$.

Then an arc of the lower boundary of $\mathcal{F}_p^*$ is in one of the following two families

1. The arcs determined by the circles $\left|a\sqrt{p}z + b\sqrt{p}\right| = 1$ (this is the circle with center $\frac{-b}{a}$ and radius $\frac{1}{a\sqrt{p}}$), where a, b are integers such that $\gcd(a, b) = 1$

2. The arcs determined by the circles $|pcz + d| = 1$ (this is the circle with center $\frac{-d}{pc}$ and radius $\frac{1}{pc}$), where c, d are integers such that $\gcd(pc, d) = 1$

Notice that the arcs of the second family with $c = 1$ intersect each other at points with imaginary part $\frac{\sqrt{3}}{2p}$. This implies that the imaginary part of every point in the fundamental domain must be greater than or equal to $\frac{\sqrt{3}}{2p}$.

Therefore, if an arc appears in the boundary of the fundamental domain, the radius of the corresponding circle must be greater than $\frac{\sqrt{3}}{2p}$. This yields the restriction $|a| < 2\sqrt{\frac{p}{3}}$ for the first family and $c = 1$ for the second family.

On the other hand, if $|b| \geq a > 0$, one can easily check for $p \geq 5$ and any $\theta \in \mathbb{R}$ that

$$\left|\mathrm{Re}\left(\frac{e^{i\theta}}{a\sqrt{p}} - \frac{b}{a}\right)\right| > 1/2.$$

This means that the circle $\left|a\sqrt{p}z + b\sqrt{p}\right| = 1$ is outside of the vertical strip $|Re(z)| < \frac{1}{2}$. Hence it does not appear in the boundary of the fundamental domain. Therefore we can restrict the first family to pairs of integers a, b such that $\gcd(a, b) = 1$ and

$$0 < |a| < b < 2\sqrt{\frac{p}{3}}.$$

For the second family, note that if $|d| > p$ the arc does not appear in the fundamental domain since it would be contained in the region $|Re(z)| > 1/2$. Therefore, the second family can be restricted to the arcs in the circles $|pz + d| = 1$ where $|d| < p$. This proves Lemma 1.

Proof of Lemma 2.

Take two arcs as in the Lemma. They can be parametrized as

$$\left(x - \frac{m_1}{n_1}\right)^2 + y^2 = \frac{1}{n_1^2 p} \quad \text{and} \quad \left(x - \frac{m}{n}\right)^2 + y^2 = \frac{1}{n^2 p}$$

respectively. By elementary methods we find that any point (x, y) in their intersection satisfies

$$y^2 = \frac{-1}{4}\frac{\left(p(mn_1 - m_1 n)^2 - (n + n_1)^2\right)\left(p(mn_1 - m_1 n)^2 - (n - n_1)^2\right)}{p^2 n_1^2 n^2 (mn_1 - m_1 n)^2}.$$

Hence, a necessary condition for the existence of an intersection point is

$$\left(p(mn_1 - m_1 n)^2 - (n + n_1)^2\right)\left(p(mn_1 - m_1 n)^2 - (n - n_1)^2\right) < 0.$$

From Lemma 1 we know that

$$(n \pm n_1)^2 \leq \frac{16}{3}p,$$

thereby, if we assume $(mn_1 - m_1 n)^2 \geq 9$ we get

$$p(mn_1 - m_1 n)^2 - (n \pm n_1)^2 \geq 9p - \frac{16}{3}p = \frac{11}{3}p > 0.$$

This shows that there is no intersection point if $(mn_1 - m_1 n)^2 \geq 9$. Consequently $(mn_1 - m_1 n)^2 \leq 4$, i.e.

$$|mn_1 - m_1 n| \leq 2.$$

This proves the Lemma.

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