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On large prime actions on Riemann surfaces

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Abstract. In this article, we study compact Riemann surfaces of genus g with an automorphism of prime order $g+1$. The main result provides a classification of such surfaces. In addition, we give a description of them as algebraic curves, determine and realise their full automorphism groups and compute their fields of moduli. We also study some aspects of their Jacobian varieties such as isogeny decompositions and complex multiplication. Finally, we determine the period matrix of the Accola–Maclachlan curve of genus four.

1 Introduction and statement of the results

Let $\mathscr{M}_g$ denote the moduli space of compact Riemann surfaces (smooth irreducible complex algebraic curves) of genus $g \geqslant 2$. It is classically known that $\mathscr{M}_g$ is endowed with the structure of a complex analytic space of dimension $3g-3$, and that, for $g \geqslant 4$, its singular locus agrees with the branch locus of the canonical projection

$$T_g \to \mathscr{M}_g,$$

where T_g stands for the Teichmüller space of genus g. In other words, if $g \geqslant 4$, then

$$\mathrm{Sing}(\mathscr{M}_g) = \{[S] \in \mathscr{M}_g : \mathrm{Aut}(S) \neq 1\},$$

where $\mathrm{Aut}(S)$ denotes the full automorphism group of S.

The classification of groups of automorphisms of compact Riemann surfaces is a classical problem which has attracted broad interest ever since it was proved that the full automorphism group of a compact Riemann surface S of genus $g \geqslant 2$ is finite, and that

$$|\mathrm{Aut}(S)| \leqslant 84(g-1).$$

It is well known that there are infinitely many values of g for which there is no compact Riemann surfaces of genus g possessing $84(g-1)$ automorphisms. Regarding this matter, Accola [1] and Maclachlan [46] proved that, for fixed g,

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the largest order $n_0(g)$ of the full automorphism group of a compact Riemann surface of genus g satisfies

$$n_0(g) \geqslant 8(g+1) \tag{1.1}$$

and that, for infinitely many values of g, the inequality (1.1) turns into an equality.

We denote by X_8 the so-called Accola–Maclachlan curve, namely the compact Riemann surface of genus g with $8(g+1)$ automorphisms given by the algebraic curve

$$y^2 = x^{2(g+1)} - 1.$$

The Accola–Maclachlan curve is a remarkable example of a compact Riemann surface determined by the order of its full automorphism group. More precisely, Kulkarni in [43] succeeded in proving that, up to finitely many values of the genus, if $g \not\equiv 3 \bmod 4$, then X_8 is the unique compact Riemann surface of genus g with exactly $8(g+1)$ automorphisms.

The analogous problem of finding $n_0(g)$ but for uniparametric families of compact Riemann surfaces was studied in [22]. Concretely, the existence was proved of a closed equisymmetric complex one-dimensional family, henceforth denoted by $\bar{\mathscr{C}}_g$, of hyperelliptic compact Riemann surfaces of genus g with a group of automorphisms isomorphic to

$$\mathbf{D}_{g+1} \times C_2 \quad \text{acting with signature } (0; 2, 2, 2, g+1)$$

(we shall recall the precise definition of *signature* in § 2.1 and § 2.2). It was then shown that $4(g+1)$ is the largest order of the full automorphism group of complex one-dimensional families of compact Riemann surfaces of genus g appearing for all g. These results were recently extended to the three and four-dimensional cases in [40] while the two-dimensional case is derived from the results of [56].

It is a well-known fact that if a compact Riemann surface of genus $g \geqslant 2$ has an automorphism of prime order q such that $q > g$, then either $q = 2g + 1$ or $q = g + 1$. The former case corresponds to the so-called Lefschetz surfaces. This paper deals with the latter case.

Let $q \geqslant 5$ be a prime number. Consider the singular sublocus

$$\mathscr{M}_{q-1}^{q} \subset \operatorname{Sing}(\mathscr{M}_{q-1})$$

consisting of all those compact Riemann surfaces of genus $q-1$ endowed with an automorphism of order q. This sublocus was studied by Urzúa in [68] from a hyperbolic geometry point of view, and later by Costa and Izquierdo in [21] when the existence of complex one-dimensional isolated strata of the singular locus of the moduli space was proved.

This paper is devoted to classifying and describing the surfaces lying in $\mathscr{M}_{q-1}^{q}$ and to study some aspects of the corresponding Jacobians in the singular locus of the moduli space of principally polarised abelian varieties of dimension $q-1$. In other words, we shall consider all those compact Riemann surfaces of genus $g \geqslant 4$ (and their Jacobian varieties) with a group of automorphisms of order

$$\lambda(g+1), \quad \text{where } \lambda \geqslant 1 \text{ is an integer,}$$

under the assumption that $q := g+1$ is a prime number.

The classification

The first result of the paper provides a classification of these surfaces.

Theorem 1. *Let $q \geqslant 7$ be a prime number. If S is a compact Riemann surface of genus $g = q-1$ endowed with a group of automorphisms of order λq for some integer $\lambda \geqslant 1$, then $\lambda \in \{1, 2, 3, 4, 8\}$.*

Assume $\lambda = 8$. Then S is isomorphic to the Accola–Maclachlan curve X_8.

Assume $\lambda = 4$.

(1) *If $q \equiv 3 \bmod 4$, then S belongs to the closed family $\bar{\mathscr{C}}_g$.*

(2) *If $q \equiv 1 \bmod 4$, then S belongs to the closed family $\bar{\mathscr{C}}_g$ or S is isomorphic to the unique compact Riemann surface X_4 with full automorphism group isomorphic to*

$$C_q \rtimes_4 C_4 \quad \textit{acting with signature } (0; 4, 4, q).$$

Moreover, if $\mathscr{C}_g$ stands for the interior of $\bar{\mathscr{C}}_g$, then

$$\bar{\mathscr{C}}_g - \mathscr{C}_g = \{X_8\}.$$

Assume $\lambda = 3$. Then S is isomorphic to the unique compact Riemann surface X_3 with full automorphism group isomorphic to

$$C_q \times C_3 \quad \textit{acting with signature } (0; 3, q, 3q).$$

Assume $\lambda = 2$. Then one of the following statements holds.

(1) *S is isomorphic to one of the $\frac{q-3}{2}$ pairwise non-isomorphic compact Riemann surfaces $X_{2,k}$ for $k \in \{1, \ldots, \frac{q-3}{2}\}$ with full automorphism group isomorphic to*

$$C_q \times C_2 \quad \textit{acting with signature } (0; q, 2q, 2q).$$

(2) *S belongs to the closed family* $\bar{\mathscr{K}_g}$ *of compact Riemann surfaces with a group of automorphisms isomorphic to*

$$\mathbf{D}_q \quad \textit{acting with signature } (0; 2, 2, q, q).$$

Moreover, the closed family $\bar{\mathscr{K}_g}$ *consists of at most*

$$\begin{cases} \frac{q+3}{4} & \textit{if } q \equiv 1 \bmod 4, \\ \frac{q+1}{4} & \textit{if } q \equiv 3 \bmod 4, \end{cases}$$

equisymmetric strata; one of them being $\mathscr{C}_g$. *Furthermore, if* $\mathscr{K}_g$ *stands for the interior of* $\bar{\mathscr{K}_g}$, *then the full automorphism group of* $S \in \mathscr{K}_g - \mathscr{C}_g$ *is isomorphic to* $\mathbf{D}_q$ *and*

$$\bar{\mathscr{K}_g} - \mathscr{K}_g = \begin{cases} \{X_4, X_8\} & \textit{if } q \equiv 1 \bmod 4, \\ \{X_8\} & \textit{if } q \equiv 3 \bmod 4. \end{cases}$$

Remark 1. We point out some observations concerning Theorem 1.

(1) If $S \in \mathscr{M}_{q-1}^q$, then either $\mathrm{Aut}(S) \cong C_q$ or S lies in one of the cases described in the theorem. These two possible situations were considered in [21] where the focus was put on finding isolated equisymmetric strata of $\mathrm{Sing}(\mathscr{M}_g)$. We shall discuss later the results of [21] in terms of our terminology (see Remark 2 in § 3).

(2) The case $q = 5$ is slightly different. As a matter of fact, if S has genus 4 and is endowed with a group of automorphisms of order 5λ for some $\lambda \geqslant 1$, then in addition to the case $\mathrm{Aut}(S) \cong C_5$ and the possibilities given in the theorem, λ can equal 12 and 24. In the last two cases, S is isomorphic to the classical Bring curve; see [19, 20, 44].

(3) We conjecture that the upper bound given in the theorem for the number of equisymmetric strata of the closed family $\bar{\mathscr{K}_g}$ is sharp. By means of computer routines developed in [7], one can see the sharpness of the bound for small primes ($q \leqslant 23$).

(4) We emphasise that the equisymmetric family $\bar{\mathscr{C}_g}$ is contained in the family $\bar{\mathscr{K}_g}$.

(5) An analogous classification as in the theorem but for the compact Riemann surfaces lying in $\mathscr{M}_{q+1}^q$ was obtained in a series of articles due to Belolipetsky, Izquierdo, Jones and the first author; see [8, 38, 39, 56].

Algebraic description

Although the literature still shows few general results in this direction, there is a great interest in providing descriptions of compact Riemann surfaces as algebraic curves in an explicit manner. The following result gives such a description for the surfaces appearing in Theorem 1, as well as a realisation of their full automorphism groups.

Theorem 2. *Let $q \geqslant 5$ be a prime number, and let $g = q - 1$. Set $\omega_l = \exp\left(\frac{2\pi i}{l}\right)$.*

If S belongs to the closed family $\bar{\mathscr{C}}_g$, then S is isomorphic to the normalisation of the singular affine algebraic curve

$$\mathscr{X}_t : y^2 = (x^q - 1)(x^q - t) \quad \textit{for some } t \in \mathbb{C} - \{0, 1\}.$$

In addition, if $S \in \mathscr{C}_g$, then the full automorphism group of $S \cong \mathscr{X}_t$ is generated by

$$(x, y) \mapsto (\omega_q x, -y) \quad \textit{and} \quad (x, y) \mapsto \left(\sqrt[q]{t}\frac{1}{x}, \sqrt{t}\frac{y}{x^q} \right).$$

Assume $q \equiv 1 \bmod 4$, and choose $\rho \in \{2, \ldots, q-2\}$ such that $\rho^4 \equiv 1 \bmod q$. Then X_4 is isomorphic to the normalisation of the singular affine algebraic curve

$$y^q = (x-1)(x-i)^\rho (x+1)^{q-1}(x+i)^{q-\rho},$$

where $i^2 = -1$. In the previous model, the full automorphism group of X_4 is generated by

$$(x, y) \mapsto (x, \omega_q y) \quad \textit{and} \quad (x, y) \mapsto (ix, \varphi(x) y^\rho),$$

where

$$\varphi(x) = \frac{-(x+i)^{e-\rho}}{(x-i)^{e-1}(x+1)^{\rho-1}} \quad \textit{and} \quad e = \frac{\rho^2+1}{q}.$$

The surface X_3 is isomorphic to the normalisation of the singular affine algebraic curve

$$y^3 = x^q - 1,$$

and in this model, its full automorphism group is generated by

$$(x, y) \mapsto (\omega_q x, \omega_3 y).$$

For each $k \in \{1, \ldots, \frac{q-3}{2}\}$, there exists $n_k \in \{1, \ldots, q-1\}$ different from $q-2$ such that $X_{2,k}$ is isomorphic to the normalisation of the singular affine algebraic curve

$$y^q = x^{n_k}(x^2 - 1)$$

and, in this model, its full automorphism group is generated by

$$(x, y) \mapsto (x, \omega_q y) \quad \textit{and} \quad (x, y) \mapsto (-x, (-1)^{n_k} y).$$

If S belongs to the closed family $\bar{\mathscr{K}_g}$, then S is isomorphic to the normalisation of the singular affine algebraic curve

$$\mathscr{Z}_t : y^q = (x-1)(x+1)^{q-1}(x-t)(x+t)^{q-1} \quad \textit{for some } t \in \mathbb{C} - \{0, \pm 1\}$$

and, if $S \neq X_4$ and $S \notin \bar{\mathscr{C}_g}$, then the full automorphism group of $S \cong \mathscr{Z}_t$ is generated by

$$(x, y) \mapsto (x, \omega_q y) \quad \textit{and} \quad (x, y) \mapsto (-x, \phi_t(x) y^{-1}),$$

where $\phi_t(x) = (x^2 - 1)(x^2 - t^2)$.

The theorem above overlaps results obtained in [68, § 11].

Hyperelliptic surfaces

Arakelian and Speziali in [3] studied groups of automorphisms of large prime order of (non-necessarily smooth) projective absolutely irreducible algebraic curves over algebraically closed fields of any characteristic. In terms of our terminology, in [3, Theorem 4.7], they proved that if $q \geqslant 7$ is a prime number and S is a compact Riemann surface of genus $q-1$ with a group of automorphisms of order λq, then

$$S \text{ is non-hyperelliptic implies } \lambda \in \{1, 2, 3, 4\}.$$

The following result lengthens the implication above; it follows from Theorem 1.

Proposition 1. *Let $q \geqslant 7$ be a prime number. The compact Riemann surfaces lying in $\mathscr{M}_{q-1}^q$ that are non-hyperelliptic are $X_{2,k}, X_3, X_4$, the surfaces which belong to $\mathscr{K}_g - \mathscr{C}_g$ and those for which* $\mathrm{Aut}(S) \cong C_q$.

Jacobian variety

Let S be a compact Riemann surface of genus $g \geqslant 2$. We denote by JS the Jacobian variety of S, that is, the quotient

$$JS = \mathscr{H}^1(S, \mathbb{C})^* / H_1(S, \mathbb{Z}),$$

where $\mathscr{H}^1(S, \mathbb{C})^*$ stands for the dual of the g-dimensional complex vector space of holomorphic forms of S and $H_1(S, \mathbb{Z})$ stands for the first integral homology group of S.

We emphasise the following two classical facts (see, for example, [10]):

(1) JS is an irreducible principally polarised abelian variety of dimension g, and

(2) up to isomorphism, the surface is determined by its Jacobian (Torelli's theorem).

If G is a group acting on S, then G also acts on JS and this action, in turn, induces the so-called group algebra decomposition of JS. Concretely,

$$JS \sim A_1 \times \cdots \times A_r \sim B_1^{n_1} \times \cdots \times B_r^{n_r},$$

where the factors A_j are pairwise non-G-isogenous abelian subvarieties of JS uniquely determined, and in correspondence with central idempotents generating the simple algebras decomposing the rational group algebra of G. Each A_j decomposes further as $B_j^{n_j}$, where the abelian subvarieties B_j are no longer unique and are related to the decomposition of each simple algebra as a product of minimal left ideals. The numbers r and n_j depend only on the algebraic structure of G. See [16, 45].

The following result provides the group algebra decomposition of the Jacobian varieties of the surfaces of Theorem 1, with the exception of X_3 and $X_{2,k}$. In fact, the group algebra decomposition of JX_3 is trivial whilst that of $JX_{2,k}$ agrees with the classical decomposition

$$JX_{2,k} \sim J(X_{2,k}/H) \times \mathrm{Prym}(X_{2,k} \to X_{2,k}/H),$$

where Prym stands for the Prym variety and $H \leqslant \mathrm{Aut}(X_{2,k})$ is isomorphic to C_2.

Theorem 3. *Let $q \geqslant 5$ be a prime number, and let $g = q - 1$.*

The Jacobian variety JX_8 decomposes, up to isogeny, as the square power $JX_8 \sim JY_8^2$, where Y_8 is quotient compact Riemann surface given by the action of $\langle z \rangle$ on X_8, where

$$\mathrm{Aut}(X_8) \cong \langle x, y, z : x^{2q} = y^2 = z^2 = 1,\ [x, y] = [z, y] = 1,\ zxz = x^{-1}y \rangle.$$

The Jacobian variety JX_4 of X_4 decomposes, up to isogeny, as the fourth power $JX_4 \sim JY_4^4$, where Y_4 is quotient compact Riemann surface given by the action of $\langle B \rangle$ on X_4, where

$$\mathrm{Aut}(X_4) \cong \langle A, B : A^q = B^4 = 1,\ BAB^{-1} = A^\rho \rangle$$

and ρ is a primitive fourth root of unity in $\mathbb{Z}_q$.

The Jacobian variety JS of $S \in \mathscr{K}_g$ decomposes, up to isogeny, as the square power $JS \sim JX^2$, where X is the quotient compact Riemann surface given by the action of $\langle s \rangle$ on S, where

$$\mathrm{Aut}(S) \cong \begin{cases} \mathbf{D}_q & \text{if } S \in \mathscr{K}_g - \mathscr{C}_g, \\ \mathbf{D}_q \times C_2 & \text{if } S \in \mathscr{C}_g, \end{cases}$$

and $\mathbf{D}_q = \langle r, s : r^q = s^2 = (sr)^2 = 1 \rangle$.

Field of moduli and fields of definition

Let $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ denote the group of field automorphisms of $\mathbb{C}$. The correspondence

$$\mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \times \mathscr{M}_g \to \mathscr{M}_g \quad \text{given by } (\sigma, [S]) \mapsto [S^\sigma],$$

where S^σ is the Galois σ-transformed of S (considered as algebraic curve), defines an action.

The field of moduli of a compact Riemann surface S is the fixed field $\mathcal{M}(S)$ of the isotropy group of S under the aforementioned action, namely

$$\mathcal{M}(S) = \mathrm{fix}\{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) : S^\sigma \cong S\}.$$

The field of moduli of S agrees with the intersection of all its fields of definition and, as proved by Koizumi in [41], S can be defined over a finite degree extension of $\mathcal{M}(S)$.

Necessary and sufficient conditions under which S can be defined over its field of moduli were provided by Weil in [69] (see also [34] for a constructive proof of Weil's theorem); these conditions are trivially satisfied if S has no non-trivial automorphisms. Besides, as proved by Wolfart in [71], if S is quasiplatonic, then S can be defined over its field of moduli.

The general question of deciding whether or not the field of moduli is a field of definition is a challenging problem; see, for example, [4, 27, 30, 33, 35, 42, 54]. In this direction, it is a known fact that if the genus of $S/\mathrm{Aut}(S)$ is zero, then either S can be defined over $\mathcal{M}(S)$ or over a quadratic extension of it; see [23] and also [31] for recent results.

We now study the aforementioned problem for the compact Riemann surfaces of Theorem 1. First, note that, for the quasiplatonic ones, the problem is trivial. Indeed,

(1) as proved in Theorem 2, the surfaces X_3, $X_{2,k}$ and X_8 are defined over $\mathbb{Q}$, and therefore their fields of moduli are $\mathbb{Q}$.

(2) As mentioned above, the fact that X_4 is quasiplatonic implies that it can be defined over its field of moduli. Moreover, the uniqueness of X_4 implies that its field of moduli is $\mathbb{Q}$. In fact, we shall see later (Remark 3 in § 4) that X_4 is isomorphic to

$$y^q = x(x+1)^\rho (x-1)^{q-\rho}.$$

The remaining cases (that is, the surfaces lying in the family $\mathscr{K}_g$ since it contains $\mathscr{C}_g$) are given in the following proposition.

Proposition 2. *Let $q \geqslant 5$ be prime, and let $g = q-1$. If S belongs to the family $\mathscr{K}_g$ and*

$$S \cong \mathscr{Z}_t = \{(x,y) : y^q = (x-1)(x+1)^{q-1}(x-t)(x+t)^{q-1}\}$$

for $t \in \mathbb{C} - \{0, \pm 1\}$, then the field of moduli of S is $\mathbb{Q}(t)$.

It it worth mentioning that a compact Riemann surface and its Jacobian variety can be defined over the same fields and that their fields of moduli agree; see [63] and also [47].

The following result is a direct consequence of the above.

Corollary 1. *The compact Riemann surfaces of Theorem 1 and their Jacobian varieties can be defined over their fields of moduli.*

The sublocus of $\mathcal{A}_g$ with G-action

It is well known that the moduli space $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g is isomorphic to the quotient

$$\pi \colon \mathscr{H}_g \to \mathcal{A}_g \cong \mathscr{H}_g / \operatorname{Sp}(2g, \mathbb{Z})$$

of the Siegel upper half-space $\mathscr{H}_g$ by the action of the symplectic group $\operatorname{Sp}(2g, \mathbb{Z})$. If the isomorphism class of JS is represented by $Z_S \in \mathscr{H}_g$, then there is an isomorphism of groups

$$\operatorname{Aut}(JS) \cong \Sigma_S := \{R \in \operatorname{Sp}(2g, \mathbb{Z}) : R \cdot Z_S = Z_S\},$$

where Σ_S is well-defined up to conjugation in $\operatorname{Sp}(2g, \mathbb{Z})$. The subset of $\mathscr{H}_g$ given by

$$\mathscr{S}_S := \{Z \in \mathscr{H}_g : R \cdot Z = Z \text{ for all } R \in \Sigma_S\}$$

consists of those matrices representing principally polarised abelian varieties of dimension g admitting an action which is equivalent to that of $\operatorname{Aut}(JS)$. This subset is, indeed, an analytic submanifold of $\mathscr{H}_g$ closely related with some special subvarieties and Shimura families of $\mathcal{A}_g$.

Observe that if $\bar{\mathscr{U}}_g$ is an equisymmetric family of compact Riemann surfaces of genus g and if S is any surface lying in the interior $\mathscr{U}_g$ of $\bar{\mathscr{U}}_g$, then

$$\{JX : X \in \mathscr{U}_g\} \subseteq \pi(\mathscr{S}_S).$$

In general, those loci of $\mathcal{A}_g$ do not agree. Nonetheless, the uncommon cases in which these dimensions do agree have been useful in finding Jacobians with complex multiplication.

Although a satisfactory description of the matrices in $\mathscr{S}_S$ seems to be a difficult problem, as we shall see in § 2.7, there is a simple representation-theoretic way to compute the dimension of the (component which contains JS of) $\mathscr{S}_S$. We shall denote the aforementioned dimension by N_S.

Theorem 4. *Let $q \geqslant 5$ be prime, let $g = q - 1$, and let $S \in \mathscr{K}_g$. Then*

$$N_{X_8} = N_{X_3} = N_{X_{2,k}} = 0, \quad N_{X_4} = \frac{q-1}{4} \quad \textit{and} \quad N_S = \frac{q-1}{2}.$$

According to results due to Streit in [67] (and later generalised in [26] for higher dimension), if N_S is zero, then the full automorphism group of S determines the period matrix for JS and JS admits complex multiplication. We refer to [49, 50] for recent applications of this result for quasiplatonic curves that are hyperelliptic and superelliptic.

As a direct consequence of the previous theorem, we recover the following known result.

Corollary 2. *The Jacobian varieties of X_3, $X_{2,k}$ and X_8 admit complex multiplication.*

In spite of the fact that the problem of determining the period matrix of a given Jacobian variety is, in general, intractable, interesting results have been obtained for some famous Riemann surfaces. For instance, the period matrices of Macbeath's curve of genus seven and of Bring's curve were determined in [9] and [60] respectively. A method to find the period matrices of the Accola–Maclachlan and Kulkarni surfaces was given in [13]. In addition, in [13, Example 3.7], the authors went even further and employed their method to provide the period matrix of the Accola–Maclachlan curve of genus two in an explicit way.

At the end of the paper, we determine explicitly the period matrix of the Accola–Maclachlan curve of genus four.

This article is organised as follows. In § 2, we succinctly review the basic preliminaries: Fuchsian groups and group action on Riemann surfaces and abelian varieties. The proof of Theorem 1 is given in § 3, and the proofs of Theorem 2 and Proposition 2 are given in § 4. In § 5, we prove some basic algebraic lemmata needed to prove, in § 6, Theorems 3 and 4. Finally, we include an addendum in which the period matrix of the Accola–Maclachlan curve of genus four is computed.

2 Preliminaries

2.1 Fuchsian groups

A *Fuchsian group* is a discrete group of automorphisms of the upper half-plane $\mathbb{H}$. If Δ is a Fuchsian group and the orbit space $\mathbb{H}/\Delta$ given by the action of Δ on $\mathbb{H}$ is compact, then the algebraic structure of Δ is determined by its *signature*

$$\sigma(\Delta) = (\gamma; k_1, \ldots, k_s), \tag{2.1}$$

where γ is the genus of $\mathbb{H}/\Delta$ and $k_1, \ldots, k_s$ are the branch indices in the universal canonical projection $\mathbb{H} \to \mathbb{H}/\Delta$. In this case, Δ has a canonical presentation in terms of *canonical generators* $\alpha_1, \ldots, \alpha_\gamma$, $\beta_1, \ldots, \beta_\gamma$, $x_1, \ldots, x_s$ and relations

$$x_1^{k_1} = \cdots = x_s^{k_s} = \prod_{i=1}^{\gamma} [\alpha_i, \beta_i] \prod_{i=1}^{s} x_i = 1, \tag{2.2}$$

where the brackets stand for the commutator. The Teichmüller space of Δ is a complex analytic manifold homeomorphic to the complex ball of dimension $3\gamma - 3 + s$.

Let Δ' be a group of automorphisms of $\mathbb{H}$ such that $\Delta \leqslant \Delta'$ of finite index. Then Δ' is also Fuchsian, and they are related by the so-called Riemann–Hurwitz formula

$$2\gamma - 2 + \sum_{i=1}^{s}\Big(1 - \frac{1}{k_i}\Big) = [\Delta' : \Delta] \cdot \Big[2\gamma' - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{k_i'}\Big)\Big],$$

where $\sigma(\Delta') = (\gamma'; k_1', \ldots, k_r')$.

2.2 Group action on Riemann surfaces

Let S be a compact Riemann surface of genus $g \geqslant 2$. A finite group G acts on S if there is a group monomorphism $\epsilon \colon G \to \mathrm{Aut}(S)$. The orbit space S/G given by the action of $G \cong \epsilon(G)$ on S inherits naturally a Riemann surface structure such that the canonical projection $S \to S/G$ is holomorphic.

By the classical uniformisation theorem, there is a unique, up to conjugation, Fuchsian group Γ of signature $(g; -)$ such that $S \cong \mathbb{H}/\Gamma$. Moreover, G acts on S if and only if there is a Fuchsian group Δ containing Γ together with a group epimorphism

$$\theta\colon \Delta \to G \quad \text{such that} \quad \ker(\theta) = \Gamma. \tag{2.3}$$

It is said that G acts on S with signature $\sigma(\Delta)$ and that the action is represented by the *surface-kernel epimorphism* (2.3); henceforth, we write *ske* for short. Abusing notation, we shall also identify θ with the tuple of the images of the canonical generators of Δ.

2.3 Extending actions

Assume that G' is a finite group such that $G \leqslant G'$. The action of G on S represented by the ske (2.3) is said to *extend* to an action of G' on S if

(1) there is a Fuchsian group Δ' containing Δ,

(2) the Teichmüller spaces of Δ and Δ' have the same dimension, and

(3) there exists an ske

$$\Theta\colon \Delta' \to G' \quad \text{in such a way that} \quad \Theta|_{\Delta} = \theta \text{ and } \ker(\theta) = \ker(\Theta).$$

An action is called *maximal* if it cannot be extended in the previous sense. Singerman in [66] determined the complete list of pairs of signatures of Fuchsian groups Δ and Δ' for which it may be possible to have an extension as before. See also [62, 65].

2.4 Equivalence of actions

Two actions $\epsilon_i\colon G \to \mathrm{Aut}(S)$ are *topologically equivalent* if there exist $\omega \in \mathrm{Aut}(G)$ and an orientation preserving self-homeomorphism f of S such that

$$\epsilon_2(g) = f\epsilon_1(\omega(g))f^{-1} \quad \text{for all } g \in G. \tag{2.4}$$

Each f satisfying (2.4) yields an automorphism f^* of Δ, where $\mathbb{H}/\Delta \cong S/G$. If $\mathscr{B}$ is the subgroup of $\mathrm{Aut}(\Delta)$ consisting of them, then $\mathrm{Aut}(G) \times \mathscr{B}$ acts on the set of skes defining actions of G on S with signature $\sigma(\Delta)$ by

$$((\omega, f^*), \theta) \mapsto \omega \circ \theta \circ (f^*)^{-1}.$$

Two skes $\theta_1, \theta_2\colon \Delta \to G$ define topologically equivalent actions if and only if they belong to the same $(\mathrm{Aut}(G) \times \mathscr{B})$-orbit; see, for example, [12]. If the genus of S/G is zero, then $\mathscr{B}$ is generated by the so-called *braid transformations* Φ_i, for $1 \leqslant i < l$, defined by

$$x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \quad \text{and} \quad x_j \mapsto x_j \quad \text{when } j \neq i, i+1.$$

2.5 Equisymmetric stratification of $\mathscr{M}_g$

Following [11], the singular locus of $\mathscr{M}_g$ admits an equisymmetric stratification where each equisymmetric stratum, if nonempty, corresponds to one topological class of maximal actions (see also [29]). More precisely,

$$\mathrm{Sing}(\mathscr{M}_g) = \bigcup_{G,\theta} \bar{\mathscr{M}}_g^{G,\theta},$$

where the *equisymmetric stratum* $\mathscr{M}_g^{G,\theta}$ consists of surfaces of genus g with full automorphism group isomorphic to G such that the action is topologically equivalent to θ. In addition, the *closure* $\bar{\mathscr{M}}_g^{G,\theta}$ of $\mathscr{M}_g^{G,\theta}$ is a closed irreducible algebraic subvariety of $\mathscr{M}_g$ and consists of surfaces of genus g with a group of automorphisms isomorphic to G such that the action is topologically equivalent to θ.

The subset $\bar{\mathscr{F}}_g(G,\sigma) = \bar{\mathscr{F}}_g$ of $\mathscr{M}_g$ of all those compact Riemann S surfaces of genus g with a group of automorphisms isomorphic to a given group G acting with a given signature σ will be called a *closed family*. Observe that if the signature of the action of G on S is (2.1), then

$$\dim(\bar{\mathscr{F}}_g) = 3\gamma - 3 + s.$$

Assume that the action of G is maximal. Then

(1) the *interior* $\mathscr{F}_g$ of $\bar{\mathscr{F}}_g$ consists of those surfaces S such that $G = \mathrm{Aut}(S)$,

(2) $\mathscr{F}_g$ is formed by finitely many equisymmetric strata that are in correspondence with the pairwise non-equivalent topological actions of G, and

(3) the set $\bar{\mathscr{F}}_g - \mathscr{F}_g$ is formed by those surfaces S such that $G < \mathrm{Aut}(S)$ properly.

2.6 Abelian varieties

A complex *abelian variety* is a complex torus which is also a complex projective algebraic variety. Each abelian variety $X = V/\Lambda$ admits a polarisation, that is, a non-degenerate real alternating form Θ on V such that, for all $v, w \in V$,

$$\Theta(iv, iw) = \Theta(v, w) \quad \text{and} \quad \Theta(\Lambda \times \Lambda) \subset \mathbb{Z}.$$

If each elementary divisor of $\Theta|_{\Lambda\times\Lambda}$ is equal to 1, then Θ is called *principal* and X is called a *principally polarised abelian variety*; from now on, we write *ppav* for short. In this case, there exists a basis for Λ such that the matrix for $\Theta_{\Lambda\times\Lambda}$ with respect to it is given by

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \quad \text{where } g = \dim(X); \tag{2.5}$$

such a basis is called *symplectic*. In addition, there exists a basis for V with respect to which the period matrix for X is

$$\Pi = (I_g Z), \quad \text{where } Z \in \mathscr{H}_g = \{Z \in \mathrm{M}(g, \mathbb{C}) : Z = Z^t, \mathrm{Im}(Z) > 0\},$$

with Z^t denoting the transpose matrix of Z. The space $\mathscr{H}_g$ is called the Siegel upper half-space.

By an isomorphism of ppavs, we mean an isomorphism of the underlying complex tori preserving the involved polarisations. In other words, if $(I_g Z_i)$ is the period matrix of X_i, then an isomorphism $X_1 \to X_2$ is given by invertible matrices

$$M \in \mathrm{GL}(g, \mathbb{C}) \text{ and } R \in \mathrm{GL}(2g, \mathbb{Z}) \quad \text{such that} \quad M(I_g Z_1) = (I_g Z_2)R. \tag{2.6}$$

Since R preserves the polarisation (2.5), it belongs to the *symplectic group*

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{R \in \mathrm{M}(2g, \mathbb{Z}) : R^t J R = J\}.$$

It follows from (2.6) that the correspondence $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathscr{H}_g \to \mathscr{H}_g$ given by

$$\left(R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \mapsto R \cdot Z := (A + ZC)^{-1}(B + ZD)$$

defines an action that identifies period matrices representing isomorphic ppavs. Hence

$$\mathscr{H}_g \to \mathcal{A}_g := \mathscr{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$$

is the moduli space of isomorphism classes of ppavs of dimension g. See [51].

2.7 Abelian varieties with G-action

Let S be a compact Riemann surface of genus $g \geqslant 2$. Consider the Jacobian variety JS and its full (polarisation-preserving) automorphism group $\mathrm{Aut}(JS)$. Every automorphism of S induces a unique automorphism of JS. In fact,

$$[\mathrm{Aut}(JS) : \mathrm{Aut}(S)] \in \{1, 2\}$$

according to whether or not S is hyperelliptic; moreover, in the latter case,

$$\mathrm{Aut}(JS)/\,\mathrm{Aut}(S) = \{\pm 1\}.$$

As mentioned in the introduction, once a symplectic basis of $\Lambda = H_1(S, \mathbb{Z})$ is fixed, there is an isomorphism

$$\mathrm{Aut}(JS) \cong \Sigma_S := \{R \in \mathrm{Sp}(2g, \mathbb{Z}) : R \cdot Z_S = Z_S\},$$

where $(I_g\, Z_S)$ is the period matrix of JS. A change of basis induces a different but equivalent choice of Z_S and a conjugate subgroup Σ_S. One obtains a well-defined analytic submanifold

$$\mathscr{S}_S := \{Z \in \mathscr{H}_g : R \cdot Z = Z \text{ for all } R \in \Sigma_S\}$$

of $\mathscr{H}_g$ whose points represent ppavs admitting an action equivalent to that of $\mathrm{Aut}(JS)$ in the symplectic group. Equivalently, as $-1 \in \Sigma_S$, the previous submanifold represents ppavs admitting an action equivalent to that of $\mathrm{Aut}(S)$. Clearly, $Z_S \in \mathscr{S}_S$.

According to [67] (see also [26, Lemma 3.8]), the dimension N_S of (the component which contains JS of) $\mathscr{S}_S$ agrees with

$$\dim(\mathrm{Sym}^2 \mathscr{H}^{1,0}(S, \mathbb{C}))^{\mathrm{Aut}(S)},$$

where $\mathrm{Sym}^2 \mathscr{H}^{1,0}(S, \mathbb{C})$ stands for the symmetric square of $\mathscr{H}^{1,0}(S, \mathbb{C})$. It follows that

$$N_S = \langle \chi_{\rho_a}^{\mathrm{sym}} \mid 1 \rangle_G, \quad \text{where } G = \mathrm{Aut}(S)$$

and $\chi_{\rho_a}^{\mathrm{sym}}$ denotes the character of the symmetric square of the analytic representation ρ_a of G and the brackets denote the usual inner product of characters of G.

It is worth mentioning that $\mathscr{S}_S$ is related to some special subvarieties of $\mathcal{A}_g$. Indeed, as $\mathrm{Aut}(JS)$ can be considered as a subgroup of

$$L_S := \mathrm{End}_{\mathbb{Q}}(JS) = \mathrm{End}(JS) \otimes_{\mathbb{Z}} \mathbb{Q},$$

one sees that $\mathscr{S}_S$ contains a complex submanifold of $\mathscr{H}_g$ of matrices representing ppavs containing L_S in their endomorphism algebras. This submanifold is called a Shimura domain for S, and the corresponding ppavs form a so-called Shimura family for S; this a *special* subvariety of $\mathcal{A}_g$ (see [48, § 3] for a precise definition). We refer to [70, § 3] for more details.

2.8 The group algebra decomposition

The action of a group G on a compact Riemann surface S induces a $\mathbb{Q}$-algebra homomorphism from the rational group algebra of G to L_S,

$$\Xi \colon \mathbb{Q}[G] \to L_S.$$

Let $W_1, \ldots, W_r$ be the rational irreducible representations of G, and for each W_l, let V_l be a complex irreducible representation of G associated to it. Following [45], the equality

$$1 = e_1 + \cdots + e_r \quad \text{in } \mathbb{Q}[G], \tag{2.7}$$

where e_l is a uniquely determined central idempotent associated to W_l, yields an isogeny

$$JS \sim A_1 \times \cdots \times A_r, \quad \text{where } A_l := \Xi(\alpha_l e_l)(JS)$$

which is G-equivariant, with $\alpha_l \geqslant 1$ chosen to satisfy $\alpha_l e_l \in \mathbb{Z}[G]$. Additionally, there are idempotents $f_{l1}, \ldots, f_{ln_l}$ such that

$$e_l = f_{l1} + \cdots + f_{ln_l}, \tag{2.8}$$

where $n_l = d_l/s_l$ is the quotient of the degree d_l and the Schur index s_l of V_l. These idempotents provide n_l pairwise isogenous subvarieties of JS. If we denote by B_l one of them for each l, then (2.7) and (2.8) provide the isogeny

$$JS \sim B_1^{n_1} \times \cdots \times B_r^{n_r} \tag{2.9}$$

known as the *group algebra decomposition* of JS with respect to G. See [16].

Let H be a subgroup of G. We denote by d_l^H the dimension of the vector subspace of V_l of those elements which are fixed under H. Following [16, Proposition 5.2], the group algebra decomposition (2.9) induces the following isogeny of the Jacobian $J(S/H)$ of the quotient S/H:

$$J(S/H) \sim B_1^{n_1^H} \times \cdots \times B_r^{n_r^H}, \quad \text{where } n_l^H = d_l^H/s_l. \tag{2.10}$$

The previous isogeny has proved to be fruitful in finding Jacobians JS isogenous to a product of Jacobians of quotients of S. See, for example, [58] and also [59].

Assume that $(\gamma; k_1, \ldots, k_s)$ is the signature of the action of G on S and that this action is represented by the ske $\theta\colon \Delta \to G$, with Δ as in (2.2). Observe that if $V_1 = W_1$ denotes the trivial representation of G, then $B_1 \sim J(S/G)$, and therefore $\dim B_1 = \gamma$. If $l \geqslant 2$, then according to [62, Theorem 5.12], we have that

$$\dim B_l = m_l \left[d_l(\gamma - 1) + \frac{1}{2} \sum_{j=1}^{s} (d_l - d_l^{\langle \theta(x_j) \rangle}) \right], \tag{2.11}$$

where m_l is the degree of $\mathbb{Q} \leq L_l$ with L_l denoting a minimal field of definition for V_l.

For decompositions of Jacobians and families of Jacobians with respect to special groups, we refer to the articles [5, 14, 15, 24, 25, 32, 52, 53, 55, 57, 61].

3 Proof of Theorem 1

The proof of Theorem 1 is presented as a consequence of a series of propositions proved in this section. Hereafter, we assume $q \geqslant 7$ to be prime and S to be a Riemann surface of genus $g := q - 1$ with a group of automorphisms G of order λq, where $\lambda \geqslant 1$ is an integer.

Proposition 3.1. *If $\lambda = 3$, then G is cyclic and acts with signature $(0; 3, q, 3q)$. Moreover, S is unique up to isomorphism, and G is its full automorphism group.*

Proof. Let $(\gamma; k_1, \ldots, k_l)$ be the signature of the action of G on S. The Riemann–Hurwitz formula implies that

$$2(q-2) \geqslant 3q\Big(2\gamma - 2 + \frac{2}{3}l\Big). \tag{3.1}$$

Observe that if $\gamma \geqslant 1$, then $l = 0$, and therefore $q = 2$, contradicting the assumption $q \geqslant 7$. We then assume $\gamma = 0$, and therefore (3.1) shows that $l = 3$. It follows that the signature of the action of G is

$$(0; k_1, k_2, k_3), \quad \text{where } \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{3} + \frac{4}{3q} \text{ and } k_j \in \{3, q, 3q\}.$$

After a routine computation, one sees that the unique solution of the previous equation is, up to permutation, $k_1 = 3$, $k_2 = q$ and $k_3 = 3q$. The last equality implies that

$$G \cong C_q \times C_3 = \langle \alpha, \beta : \alpha^q = \beta^3 = 1,\ [\alpha, \beta] = 1 \rangle.$$

Consider the Fuchsian group Δ of signature $(0; 3, q, 3q)$ canonically presented

$$\Delta = \langle w_1, w_2, w_3 : w_1^3 = w_2^q = w_3^{3q} = w_1 w_2 w_3 = 1 \rangle,$$

and let $\theta \colon \Delta \to G$ be an ske representing an action of G on S. It is not difficult to see that, up to an automorphism of G, the ske θ is given by

$$\theta(w_1) = \beta, \quad \theta(w_2) = \alpha \quad \text{and} \quad \theta(w_3) = \alpha^{-1}\beta^2;$$

this proves the uniqueness of S. By the results of [66], if G is strictly contained in the full automorphism group $\mathrm{Aut}(S)$ of S, then $\mathrm{Aut}(S)$ has order $12q$, acts on S with signature $(0; 2, 3, 3q)$ and G is a non-normal subgroup of it. By the classical Sylow theorem, if a group of order $12q$ with $q > 11$ has a non-normal subgroup isomorphic to G, then it is isomorphic to $C_q \times A_4$, where A_4 stands for the alternating group of order 12. However, the product of an element of order two and an element of order three of $C_q \times A_4$ cannot have order $3q$. The cases $q \leqslant 11$ are not realised either; see [18]. The proof is complete. □

Proposition 3.2. *λ is different from 5, 6 and 7.*

Proof. If λ is equal to 5, 6 or 7, then G is a large group of automorphisms (that is, $|G| > 4(g-1)$), and therefore (see, for example, [43, § 2.3]), the signature of the action is either

(1) $(0; k_1, k_2, k_3)$ for some $2 \leqslant k_1 \leqslant k_2 \leqslant k_3$,

(2) $(0; 2, 2, 2, k)$ for some $k \geqslant 3$, or

(3) $(0; 2, 2, 3, k)$ for some $3 \leqslant k \leqslant 5$.

If λ is equal to 5 or 7, then G has no involutions; then the signature of the action G is

$$(0; k_1, k_2, k_3) \quad \text{for some } k_j \in \{5, q, 5q\} \text{ or } k_j \in \{7, q, 7q\}$$

respectively. The Riemann–Hurwitz formula implies that

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{3}{5} + \frac{4}{5q} \quad \text{and} \quad \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{5}{7} + \frac{4}{7q}$$

respectively, and this, in turn, implies that q is negative, a contradiction.

We now assume that G has order $6q$. If the signature of the action of G is $(0; 2, 2, 2, k)$, then by the Riemann–Hurwitz formula, $q + 4$ divides $6q$, and therefore $q = 2$, contradicting the assumption $q \geqslant 7$. The signatures $(0; 2, 2, 3, 4)$ and $(0; 2, 2, 3, 5)$ cannot be realised either since a group of order $6q$ does not have elements of order 4 nor 5. Besides, a direct computation shows that the signature $(0; 2, 2, 3, 3)$ contradicts the Riemann–Hurwitz formula.

It follows that the signature of the action is

$$(0; k_1, k_2, k_3), \quad \text{where } k_j \in \{2, 3, 6, q, 2q, 3q, 6q\}$$

satisfy, by the Riemann–Hurwitz formula, the equality

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{3} + \frac{2}{3q}.$$

Set $v = \#\{k_j : k_j = 3\}$. It is clear that $v \neq 2, 3$. Assume $v = 1$ and say $k_1 = 3$. If $k_2, k_3 \geqslant 6$, then $q \leqslant 0$. Then we can assume $k_2 = 2$, and therefore

$$\frac{1}{k_3} = -\frac{1}{6} + \frac{2}{3q}, \quad \text{showing that} \quad q \leqslant 3.$$

Thus $v = 0$. Now, let $u = \#\{k_j : k_j = 2\}$, and observe that $u \neq 2, 3$. If $u = 1$, then

$$\frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{3q} - \frac{1}{6}, \quad \text{and therefore} \quad q \leqslant 3.$$

All the above ensures that each $k_j \geqslant 6$. It follows that

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{3} + \frac{2}{3q} \leqslant \frac{1}{2},$$

and therefore $q < 0$, contradicting the assumption $q \geqslant 7$. □

Proposition 3.3. *If $\lambda \geqslant 8$, then $\lambda = 8$ and $S \cong X_8$.*

Proof. If the order of G is at least $8(g+1)$, then following [2, p. 77], the signature of the action of G is either

(1) $(0; 2, 2, 2, 3)$,

(2) $(0; 2, 3, k)$, where $k \geqslant 7$,

(3) $(0; 2, 4, k)$, where $k \geqslant 5$,

(4) $(0; 2, 5, k)$, where $5 \leqslant k \leqslant 19$,

(5) $(0; 2, 6, k)$, where $6 \leqslant k \leqslant 11$,

(6) $(0; 2, 7, k)$, where $7 \leqslant k \leqslant 9$,

(7) $(0; 3, 3, k)$, where $4 \leqslant k \leqslant 11$, or

(8) $(0; 3, 4, k)$, where $4 \leqslant k \leqslant 5$.

We observe that cases (1), (5) and (8) are not realised. Indeed, this fact follows from the contradiction between the fourth and fifth columns in the following table.

Case	Signature	$\lvert G\rvert$	Condition	Riemann–Hurwitz formula
(1)	$(0; 2, 2, 2, 3)$	$6\lambda' q$	$\lambda' \geqslant 2$	$\lambda' = 1 - \frac{2}{q}$
(5)	$(0; 2, 6, k)$	$6\lambda' q$	$\lambda' \geqslant 2$	$\lambda' = \frac{k}{k-3}(1 - \frac{2}{q})$
(8.1)	$(0; 3, 4, 4)$	$12\lambda' q$	$\lambda' \geqslant 1$	$\lambda' = (1 - \frac{2}{q})$
(8.2)	$(0; 3, 4, 5)$	$60\lambda' q$	$\lambda' \geqslant 1$	$\lambda' = \frac{2}{13}(1 - \frac{2}{q})$

We also note that cases (4), (6) and (7) are not realised.

Case	Signature	$\lvert G\rvert$	Condition	Riemann–Hurwitz formula
(4)	$(0; 2, 5, k)$	$10\lambda' q$	$\lambda' \geqslant 1$	$\lambda' = \frac{2k}{3k-10}(1 - \frac{2}{q})$
(6)	$(0; 2, 7, k)$	$2\lambda' q$	$\lambda' \geqslant 4$	$\lambda' = \frac{14k}{5k-14}(1 - \frac{2}{q})$
(7)	$(0; 3, 3, k)$	$3\lambda' q$	$\lambda' \geqslant 3$	$\lambda' = \frac{2k}{k-3}(1 - \frac{2}{q})$

Note that,

(a) in case (4), we have $\lambda' = 1$, and therefore $q = 4k/(10 - k)$ and $5 \leqslant k \leqslant 9$. However, for each k as before, we obtain that q is not prime;

(b) in case (6), we have $\lambda' = 4$, and therefore q is equal to 14, 28 and 126 for $k = 7, 8$ and 9 respectively;

(c) in case (7), the facts that $\lambda \geqslant 3$ and $q \geqslant 7$ imply that $k \leqslant 8$. If $k = 4$, then q is not prime, if $k = 5$, then $\lambda' = 3$ or 4 and $q = 5$ or 10, and if $k = 6, 7, 8$, then $\lambda' = 3$ and q is not prime.

We claim that case (2) is not realised either. Indeed, note that otherwise the order of G is equal to $6\lambda' q$ for some $\lambda' \geqslant 2$ and the Riemann–Hurwitz formula reads

$$k = \frac{6q\lambda'}{q(\lambda' - 2) + 4}, \quad \text{and therefore} \quad k' := \frac{6\lambda'}{q(\lambda' - 2) + 4} \in \mathbb{Z}^+.$$

(1) If $k' = 1$, then $\lambda' = 2 + 8/(q - 6)$, showing that $q = 7$ and $\lambda' = 10$. Consequently, the order of G is 420 and acts on S of genus six with signature $(0; 2, 3, 7)$. However, such a Riemann surface does not exist because the maximal number of automorphisms that a Riemann surface of genus six can admit is 150 (see, for instance, [18]).

(2) If $k' \geqslant 2$, then $\lambda' \leqslant 2 + 2/(q - 3)$, and therefore $\lambda' = 2$. It follows that G has order $12q$ and acts on S with signature $(0; 2, 3, 3q)$. Observe that the signature of the action shows, in particular, that S has a cyclic subgroup $H < G$ of automorphisms of order $3q$. However, as proved in Proposition 3.1, if S has a group of automorphisms of order $3q$, then S does not have more automorphisms, a contradiction.

This proves the claim.

All the above ensures that the signature of the action is $(0; 2, 4, k)$ for some $k \geqslant 5$. Observe that the order of G is $4\lambda' q$ for some $\lambda' \geqslant 2$ and the Riemann–Hurwitz formula says

$$\lambda' = \frac{2k}{k - 4}\left(1 - \frac{2}{q}\right) < \frac{2k}{k - 4}.$$

It follows that one of the following statements holds.

(1) $k = 5$ and $\lambda' \in \{3, \ldots, 9\}$, and therefore $q = 20/(10 - \lambda')$.

(2) $k = 6$ and $\lambda' \in \{3, 4, 5\}$, and therefore $q = 12/(6 - \lambda')$.

(3) $k \geqslant 5$ and $\lambda' = 2$, and therefore $q = k/2$.

The first two cases must be disregarded because q is not prime; then G has order $8q$ and acts with signature $(0; 2, 4, 2q)$. By [43, § 5], we obtain that $S \cong X_8$, as desired. □

We recall that, following [22], the closed family $\bar{\mathscr{C}}_g$ consists of all those compact Riemann surfaces of genus g endowed with a group of automorphisms G isomorphic to

$$\mathbf{D}_q \times C_2 \cong \mathbf{D}_{2q}$$

acting with signature $(0; 2, 2, 2, q)$. Moreover, if S belongs to the interior of $\bar{\mathscr{C}}_g$, then G agrees with the full automorphism group of S. It was also observed in [22] that $X_8 \in \bar{\mathscr{C}}_g - \mathscr{C}_g$.

Proposition 3.4. $\bar{\mathscr{C}}_g - \mathscr{C}_g = \{X_8\}$.

Proof. Observe that if X belongs to $\bar{\mathscr{C}}_g - \mathscr{C}_g$, then its automorphism group has order $4tq$ for some $t \geqslant 2$. It follows from Proposition 3.3 that $t = 2$ and that $X \cong X_8$. □

For later and repeated use, we recall here that

$$\mathrm{Aut}(X_8) \cong \langle x, y, z : x^{2q} = y^2 = z^2 = 1,\ [x, y] = [z, y] = 1,\ zxz = x^{-1}y\rangle, \quad (3.2)$$

and its action on X_8 is represented by the ske

$$\Theta\colon \Delta_8 \to \mathrm{Aut}(X_8) \quad \text{given by } (z_1, z_2, z_3) \to (z, zx, x^{-1}), \qquad (3.3)$$

where Δ_8 is a Fuchsian group of signature $(0; 2, 4, 2q)$ presented as

$$\Delta_8 = \langle z_1, z_2, z_3 : z_1^2 = z_2^4 = z_3^{2q} = z_1z_2z_3 = 1\rangle. \qquad (3.4)$$

Proposition 3.5. *If X is a compact Riemann surface of genus g with a group of automorphisms isomorphic to $C_q \times C_2^2$ acting with signature $(0; 2, 2q, 2q)$, then $X \cong X_8$.*

Proof. Let Δ_2 be a Fuchsian group of signature $(0; 2, 2q, 2q)$ presented as

$$\Delta_2 = \langle y_1, y_2, y_3 : y_1^2 = y_2^{2q} = y_3^{2q} = y_1y_2y_3 = 1\rangle, \qquad (3.5)$$

and consider the group $G \cong C_q \times C_2^2$ presented as

$$\langle A, B, C : A^q = B^2 = C^2 = (BC)^2 = [A, B] = [A, C] = 1\rangle.$$

Let $\theta\colon \Delta_2 \to G$ be an ske representing the action of G on X. Up to an automorphism of G, we can assume $\theta(y_1) = B$. Moreover, after considering the automorphism of G given by

$$A \mapsto A, \quad B \mapsto B, \quad C \mapsto BC,$$

we can assume that $\theta(y_2)$ is equal to either $A^i B$ or $A^i C$ for some $i \in \mathbb{Z}_q^*$. Note that the former case is impossible since $\theta(y_1 y_2)$ would not have order $2q$. Thus, after sending A to an appropriate power of it, we obtain that θ is equivalent to

$$\Delta_2 \to C_q \times C_2^2 \quad \text{given by } (y_1, y_2, y_3) \mapsto (B, AC, (ABC)^{-1}). \tag{3.6}$$

Observe that, with the notation of (3.4), the elements

$$\hat{y}_1 := z_2^{-2}, \quad \hat{y}_2 := z_3^{-1} \quad \text{and} \quad \hat{y}_3 = z_2^{-1} z_3^{-1} z_2$$

generate a subgroup of Δ_8 isomorphic to Δ_2, and the restriction of (3.3) to it

$$\Delta_2 \to C_q \times C_2^2 \quad \text{is given by } (\hat{y}_1, \hat{y}_2, \hat{y}_3) \mapsto (y, x, x^{-1} y). \tag{3.7}$$

By letting $x = AC$ and $y = B$, we see that (3.6) and (3.7) agree; consequently, $X \cong X_8$. □

Proposition 3.6. *If Y is a compact Riemann surface of genus g with a group of automorphisms isomorphic to $C_q \rtimes_2 C_4$ acting with signature $(0; 4, 4, q)$, then $Y \cong X_8$.*

Proof. Let Δ_3 be a Fuchsian group of signature $(0; 4, 4, q)$ presented as

$$\Delta_3 = \langle y_1, y_2, y_3 : y_1^4 = y_2^4 = y_3^q = y_1 y_2 y_3 = 1 \rangle, \tag{3.8}$$

and consider the group $G \cong C_q \rtimes_2 C_4$ presented as

$$\langle A, B : A^q = B^4 = 1,\ BAB^{-1} = A^{-1} \rangle.$$

Let $\theta\colon \Delta_3 \to G$ be an ske representing the action of G on Y. Then, after sending B to B^{-1} if necessary, θ is given by

$$(y_1, y_2, y_3) \mapsto (A^i B, A^j B^{-1}, A^k) \quad \text{for some } i, j \in \mathbb{Z}_q \text{ and } k \in \mathbb{Z}_q^*.$$

Up to conjugation, we can assume $i = 0$, and after sending A to an appropriate power of it, we can assume $k = 1$. It follows that $j = 1$, and therefore θ is equivalent to

$$\Delta_3 \to C_q \rtimes_2 C_4 \quad \text{given by } (y_1, y_2, y_3) \mapsto (B, AB^{-1}, A). \tag{3.9}$$

Now, as in the previous proposition, with the notation of (3.4), we see that

$$\tilde{y}_1 := z_2, \quad \tilde{y}_2 := z_3 z_2 z_3^{-1} \quad \text{and} \quad \tilde{y}_3 = z_3^2$$

generate a subgroup of Δ_8 isomorphic to Δ_3, and the restriction of (3.3) to it

$$\Delta_3 \to C_q \rtimes_2 C_4 \quad \text{is given by} \quad (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \mapsto (zx, x^{-3}z, x^{-2}). \tag{3.10}$$

Write

$$A = x^{-2} \quad \text{and} \quad B = zx$$

to see that (3.9) and (3.10) agree; consequently $Y \cong X_8$. □

Proposition 3.7. *Assume $q \equiv 1 \bmod 4$. There exists a unique, up to isomorphism, compact Riemann surface X_4 of genus g with full automorphism group isomorphic to $C_q \rtimes_4 C_4$ acting on it with signature* $(0; 4, 4, q)$.

Proof. Consider the Fuchsian group Δ_3 as in (3.8), and the group

$$G \cong C_q \rtimes_4 C_4 = \langle A, B : A^q = B^4 = 1,\ BAB^{-1} = A^{\rho} \rangle, \tag{3.11}$$

where ρ is a primitive fourth root of unity in $\mathbb{Z}_q$. If $\theta\colon \Delta_3 \to C_q \rtimes_4 C_4$ is an ske representing the action of G on a compact Riemann surface Z of genus g, then by proceeding similarly to the previous proposition, one sees that θ is equivalent to

$$\theta_1(y_1, y_2, y_3) = (A^{-1}B, B^{-1}, A) \quad \text{or} \quad \theta_2(y_1, y_2, y_3) = (A^{-1}B^{-1}, B, A).$$

It follows that, up to isomorphism, there are at most two surfaces Z as before, namely

$$Z_j := \mathbb{H}/K_j, \quad \text{where } K_j = \ker(\theta_j) \quad \text{for } j = 1, 2.$$

Observe that if the full automorphism group of Z_j is different from G, then by Proposition 3.3, necessarily $Z_j \cong X_8$ and, in particular, $\mathrm{Aut}(X_8)$ contains a subgroup isomorphic to $C_q \rtimes_4 C_4$. However, this is not possible. Indeed, with the notation of (3.2), if $\iota \in \mathrm{Aut}(X_8)$ has order 4, then ι^2 is equal to the central element y. It follows that $\mathrm{Aut}(Z_j) \cong G$ for $j = 1, 2$.

We record here that Z_1 and Z_2 are isomorphic if and only if K_1 and K_2 are conjugate in $\mathrm{Aut}(\mathbb{H})$. As the normaliser of each K_j is Δ_3, it can be seen that K_1 and K_2 are conjugate if and only if they are conjugate in the normaliser $N(\Delta_3)$ of Δ_3. Now, the action by conjugation of $N(\Delta_3)$ on $\{K_1, K_2\}$ has orbits of length $[N(\Delta_3) : \Delta_3]$ which is, by [66, Theorem 1], equal to 2. Hence K_1 and K_2 are conjugate, and thus $X_4 := Z_1 \cong Z_2$, as desired. □

Proposition 3.8. *If $\lambda = 4$ and $S \notin \bar{\mathscr{C}}_g$, then $q \equiv 1 \bmod 4$ and $S \cong X_4$.*

Proof. As in the proof of Proposition 3.2, the signature of the action of G on S is either

(1) $(0; k_1, k_2, k_3)$ for some $2 \leqslant k_1 \leqslant k_2 \leqslant k_3$,

(2) $(0; 2, 2, 2, k)$ for some $k \geqslant 3$, or

(3) $(0; 2, 2, 3, k)$ for some $3 \leqslant k \leqslant 5$.

The third case must be disregarded since there is no group of order $4q$ with an element of order three. If the signature is as in the second case, then the Riemann–Hurwitz formula implies that $k = q$. We now assume the signature to be as in the first case. The Riemann–Hurwitz formula says

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{2} + \frac{1}{q}. \tag{3.12}$$

Note that, among k_1, k_2, k_3, not two or three of them can be equal to 2. It follows that, up to permutation, there are two cases to consider.

(1) Assume $k_1 = 2$ and $k_2, k_3 \geqslant 4$. Then (3.12) turns into $\frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{q}$. Note that if $k_2 = 4$, then $k_3 \leqslant 0$. It follows that

$$k_2, k_3 \geqslant q, \quad \text{and therefore} \quad k_2 = k_3 = 2q.$$

(2) Assume $k_1, k_2, k_3 \geqslant 4$. If the number of periods k_j that are equal to 4 is 2, then (3.12) implies that the signature is $(0; 4, 4, q)$; otherwise $q \leqslant 4$.

Thereby, the signature of the action of G on S is either

$$(0; 2, 2, 2, q), \quad (0; 2, 2q, 2q) \quad \text{or} \quad (0; 4, 4, q).$$

We recall that if $q \equiv 3 \bmod 4$, then G is isomorphic to either C_{4q}, $C_q \times C_2^2$, $\mathbf{D}_{2q}$ or $C_q \rtimes_2 C_4$, and if $q \equiv 1 \bmod 4$, then in addition, G can be isomorphic to $C_q \rtimes_4 C_4$.

(1) If G acts with signature $(0; 2, 2, 2, q)$, then G is generated by three involutions, and therefore $G \cong \mathbf{D}_{2q}$, showing that $S \in \bar{\mathscr{C}}_g$.

(2) If G acts with signature $(0; 2, 2q, 2q)$, then G is generated by two elements of order $2q$ whose product is an involution; thus $G \cong C_q \times C_2^2$. By Proposition 3.5, we see that $S \cong X_8$, and therefore $S \in \bar{\mathscr{C}}_g$.

(3) If G acts with signature $(0; 4, 4, q)$, then G is generated by two elements of order 4, and therefore G is isomorphic to $C_q \rtimes_2 C_4$ or $C_q \rtimes_4 C_4$. By Proposition 3.6, the former case implies $S \cong X_8$, whilst by Proposition 3.7, the latter case implies $S \cong X_4$.

This finishes the proof. $\square$

Proposition 3.9. *If* $\lambda = 2$ *and* G *is cyclic acting with signature* $(0; 2, 2, q, q)$*, then* $S \in \bar{\mathscr{C}}_g$.

Proof. Let Δ_4 be a Fuchsian group of signature $(0; 2, 2, q, q)$ presented as

$$\Delta_4 = \langle x_1, x_2, x_3, x_4 : x_1^2 = x_2^2 = x_3^q = x_4^q = x_1x_2x_3x_4 = 1\rangle, \tag{3.13}$$

and consider the cyclic group of order $2q$ generated by a of order q and b of order two. As G has only one involution, it is clear that, after sending a to a suitable power of it, each ske representing an action of G on S is equivalent to

$$\theta_0 \colon \Delta_4 \to C_q \times C_2 \quad \text{such that} \quad \theta_0(x_1, x_2, x_3, x_4) = (b, b, a, a^{-1}). \tag{3.14}$$

Then such surfaces S form an equisymmetric complex one-dimensional family. Let

$$\Delta_1 = \langle y_1, y_2, y_3, y_4 : y_1^2 = y_2^2 = y_3^2 = y_4^q = y_1y_2y_3y_4 = 1\rangle \tag{3.15}$$

be a Fuchsian group of signature $(0; 2, 2, 2, q)$, and consider the group

$$\mathbf{D}_{2q} = \langle R, T : R^{2q} = T^2 = (TR)^2 = 1\rangle. \tag{3.16}$$

We recall that, following [22], the action of $\mathbf{D}_{2q}$ on $S' \in \bar{\mathscr{C}}_g$ is represented by the ske

$$\theta \colon \Delta_1 \to \mathbf{D}_{2q} \quad \text{given by} \quad (y_1, y_2, y_3, y_4) \mapsto (R^q, T, TR, R^{q-1}) \tag{3.17}$$

Now, the elements of Δ_1,

$$\hat{x}_1 := y_1, \quad \hat{x}_2 := y_2y_1y_2, \quad \hat{x}_3 := y_4 \quad \text{and} \quad \hat{x}_4 := y_1y_2y_4y_2y_1,$$

generate a Fuchsian group isomorphic to Δ_4. The restriction of (3.17) to it,

$$\Delta_4 \to \langle R\rangle \cong G, \quad \text{is given by} \quad (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \mapsto (R^q, R^q, R^{q-1}, R^{1-q}). \tag{3.18}$$

Set $a := R^{q-1}$ and $b := R^q$ to see that (3.18) agrees with (3.14), and the result follows. □

Proposition 3.10. *Let* $\lambda = 2$*, and let* G *be a cyclic group acting with signature* $(0; q, 2q, 2q)$*. Then either* $S \cong X_8$ *or the full automorphism group of* S *agrees with* G*. In the latter case, there are exactly* $\frac{q-3}{2}$ *pairwise non-isomorphic compact Riemann surfaces.*

Proof. Let Δ_5 be a Fuchsian group of signature $(0; q, 2q, 2q)$ presented as

$$\Delta_5 = \langle x_1, x_2, x_3 : x_1^q = x_2^{2q} = x_3^{2q} = x_1x_2x_3 = 1\rangle,$$

and consider the cyclic group of order $2q$ generated by a of order q and b of order two.

If $\theta\colon \Delta_5 \to G$ is an ske representing the action of G on S, then after sending a to an appropriate power of it, we see that θ is equivalent to

$$\theta_j = (a, a^j b, a^{-j-1}b) \quad \text{for some } j \neq -1, 0.$$

Let S_j be the compact Riemann surface defined by θ_j, and write $j^* = \frac{q-1}{2}$. We claim that $S_{j^*} \cong X_8$. To prove that, we note that, by Proposition 3.5, it suffices to verify that θ_{j^*} extends to the action of $C_q \times C_2^2$ with signature $(0; 2, 2q, 2q)$. With the notation of the proof of Proposition 3.5, the elements

$$\hat{x}_1 := y_3^2, \quad \hat{x}_2 := y_1y_2y_1 \quad \text{and} \quad \hat{x}_3 := y_2$$

generate a subgroup of (3.5) isomorphic to Δ_5, and the restriction of (3.6) to it,

$$\Delta_5 \to \langle A, C\rangle \cong G, \quad \text{is given by} \quad (\hat{x}_1, \hat{x}_2, \hat{x}_3) \mapsto (A^{-2}, AC, AC). \tag{3.19}$$

By setting $a := A^{-2}$, $b := C$, we see that (3.19) is equivalent to θ_{j^*}, as desired.

Let $j \neq j^*$. If $\mathrm{Aut}(S_j) \neq G$, then by [66] and Proposition 3.3, the action θ_j must extend to the action (3.6) of $C_q \times C_2^2$ with signature $(0; 2, 2q, 2q)$. Observe that an element y of (3.5) has order $2q$ if and only if it is conjugate to y_2^k or to y_3^k for some $k \in \{1, \dots, 2q-1\}$ odd and different from q. As the target group is abelian, the image of y under (3.6) is either AC or $(ABC)^{-1}$. Now, if Δ' is a subgroup of (3.5) isomorphic to Δ_5, then the restriction of (3.6) to the canonical generators of Δ' must be

$$(A^{-2}, AC, AC), \quad (B, AC, (ABC)^{-1}) \quad \text{or} \quad (A^2, (ABC)^{-1}, (ABC)^{-1}).$$

The second case is impossible since it does not have the required signature; the other two cases are equivalent to θ_{j^*}. We conclude that $\mathrm{Aut}(S_j) = G$ if $j \neq j^*$, and in particular, S_j is not isomorphic to S_{j^*}.

Write $K_j = \ker(\theta_j)$ for each $j \in \mathbb{Z}_q - \{-1, 0, j^*\}$. As argued in the proof of Proposition 3.7, we have that S_{j_1} and S_{j_2} are isomorphic if and only if K_{j_1} and K_{j_2} are conjugate in the normaliser $N(\Delta_5)$ of Δ_5. The action by conjugation of $N(\Delta_5)$ on

$$\{K_j : j \in \mathbb{Z}_q - \{-1, 0, j^*\}\}$$

has orbits of length $[N(\Delta_5) : \Delta_5]$ which is, by [66, Theorem 1], equal to 2. Hence

$$\{S_j : j \in \mathbb{Z}_q - \{-1, 0, j^*\}\}$$

splits into $\frac{q-3}{2}$ isomorphism classes. Finally, observe that the elements x_2 and x_3 of Δ_5 are conjugate in $N(\Delta_5)$; thus S_j and S_{-j-1} are isomorphic, and therefore the isomorphism classes are represented by S_j, where $1 \leqslant j \leqslant \frac{q-3}{2}$. □

Proposition 3.11. *There exists a closed family $\bar{\mathscr{K}_g}$ of compact Riemann surfaces with a group of automorphisms isomorphic to the dihedral group of order $2q$ acting with signature $(0; 2, 2, q, q)$. The number of equisymmetric strata of $\bar{\mathscr{K}_g}$ is at most*

$$\begin{cases} \frac{q+3}{4} & \text{if } q \equiv 1 \bmod 4, \\ \frac{q+1}{4} & \text{if } q \equiv 3 \bmod 4, \end{cases}$$

and, independently of q, one of them is equal to $\mathscr{C}_g$.

Proof. Let Δ_4 be a Fuchsian group of signature $(0; 2, 2, q, q)$ presented as in (3.13), and consider the dihedral group of order $2q$,

$$G \cong \mathbf{D}_q = \langle r, s : r^q = s^2 = (sr)^2 = 1\rangle.$$

The existence of the family $\bar{\mathscr{K}_g}$ follows after considering the ske

$$\Phi\colon \Delta_4 \to G \quad \text{given by} \quad (x_1, x_2, x_3, x_4) \mapsto (s, s, r^{-1}, r).$$

Let us now assume that $\theta\colon \Delta_4 \to G$ is an ske representing the action of G on $S \in \bar{\mathscr{K}_g}$. If $\theta(x_1) = \theta(x_2)$, then after a conjugation and after sending r to an appropriate power of it, we see that θ is equivalent to Φ. On the other hand, if $\theta(x_1) \neq \theta(x_2)$, then after considering a suitable automorphism of G, we see that θ is equivalent to the ske

$$\theta_i := (s, sr, r^i, r^{-i-1}) \quad \text{for some } i \in \mathbb{Z}_q - \{-1, 0\}.$$

The braid transformation Φ_3 (see § 2.4) shows that $\theta_i \cong \theta_{-i-1}$. The rule

$$i \mapsto -i - 1$$

has order two, restricts to a bijection of $\mathbb{Z}_q - \{-1, 0\}$ and has exactly one fixed point, namely $i^* = \frac{q-1}{2}$. Observe that if φ_u is the automorphism of G given by $r \mapsto r^u$, then

$$\Phi = \Phi_2^2 \circ \varphi_{(i^*)^{-1}}(\theta_{i^*}).$$

All the above ensures that θ is equivalent to either

$$\Phi \text{ or } \theta_i \quad \text{for some } i \in \left\{1, \ldots, \frac{q-3}{2}\right\}.$$

Now, for each $i \in \{1, \ldots, \frac{q-3}{2}\}$, the transformation $\varphi_{i^{-1}} \circ \Phi_2^2$ provides an equivalence

$$\theta_i \cong \theta_{-i(2i+1)^{-1}}.$$

The rule $i \mapsto -i(2i+1)^{-1}$ has order two and (up to the identification $i \sim -i-1$) restricts to a bijection of $\{1, \ldots, \frac{q-3}{2}\}$; it has a fixed point if and only if

$$-i-1 = -i(2i+1)^{-1} \iff 2i^2 + 2i + 1 = 0, \tag{3.20}$$

and the quadratic equation above has solution in $\mathbb{Z}_q$ if and only if $q \equiv 1 \bmod 4$. It follows that the number of pairwise non-equivalent skes θ is at most

$$1 + \frac{1}{2}\left(\frac{q-3}{2}\right) = \frac{q+1}{4} \quad \text{and} \quad 2 + \frac{1}{2}\left(\frac{q-3}{2} - 1\right) = \frac{q+3}{4}$$

if $q \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 4$ respectively. Finally, with the notation of (3.15), define

$$\hat{x}_1 := y_1 y_2 y_1, \quad \hat{x}_2 := y_2, \quad \hat{x}_3 := y_2 y_1 y_4 y_1 y_2 \quad \text{and} \quad \hat{x}_4 := y_4,$$

and note that they generate a Fuchsian group isomorphic to Δ_4. The restriction of (3.17) to it is given by

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \mapsto (T, T, R^{1-q}, R^{q-1}). \tag{3.21}$$

If we write $s := T, r := R^{q-1}$, we see that (3.21) agrees with Φ. Hence the action of Φ extends to (3.17), and therefore the stratum defined by Φ agrees with $\mathscr{C}_g$. $\square$

Proposition 3.12. *If $\mathscr{K}_g$ stands for the interior of the closed family $\bar{\mathscr{K}}_g$, then the full automorphism group of $S \in \mathscr{K}_g$ is isomorphic to either $\mathbf{D}_q$ or $\mathbf{D}_{2q}$. In addition,*

$$\bar{\mathscr{K}}_g - \mathscr{K}_g = \begin{cases} \{X_8, X_4\} & \text{if } q \equiv 1 \bmod 4, \\ \{X_8\} & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

Proof. We keep the notation of the proof of Proposition 3.11. The first statement is clear since the full automorphism group of S is isomorphic to $\mathbf{D}_{2q}$ or $\mathbf{D}_q$ according to whether or not θ_i is equivalent to Φ.

Let $\mathscr{K}_g^i$ denote the equisymmetric stratum defined by θ_i.

(1) If θ_i is equivalent to Φ, then Proposition 3.11 and Proposition 3.4 imply that $\bar{\mathscr{K}}_g^i - \mathscr{K}_g^i = \{X_8\}$.

(2) If θ_i is non-equivalent to Φ, then by Proposition 3.8, we see that,

   (a) if $q \equiv 3 \bmod 4$, then $\bar{\mathscr{K}}_g^i - \mathscr{K}_g^i$ is empty, and

   (b) if $q \equiv 1 \bmod 4$, then $\bar{\mathscr{K}}_g^i - \mathscr{K}_g^i$ is either empty or $\{X_4\}$.

Assume $q \equiv 1 \bmod 4$. We recall that the full automorphism group of X_4 is isomorphic to (3.11), and the corresponding action is given by the ske

$$\Delta_3 \to \mathrm{Aut}(X_4) \quad \text{such that} \quad (y_1, y_2, y_3) \mapsto (A^{-1}B, B^{-1}, A), \tag{3.22}$$

where Δ_3 is as in (3.8). The elements

$$\hat{x}_1 := y_1 y_2^2 y_1^{-1}, \quad \hat{x}_2 := y_1^2, \quad \hat{x}_3 := y_1^{-1} y_3 y_1 \quad \text{and} \quad \hat{x}_4 := y_3$$

generate a Fuchsian group isomorphic to Δ_4. The restriction of (3.22) to it is equivalent to

$$(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \to (B^2, B^2A, A^e, A^{-e-1}), \quad \text{where } e := -\rho(\rho-1)^{-1}. \tag{3.23}$$

By letting $r := A$ and $s := B^2$, we see that (3.23) agrees with θ_e. As e solves equation (3.20), we conclude that $\bar{\mathscr{K}}_g^i - \mathscr{K}_g^i$ is equal to $\{X_4\}$ if i solves (3.20) and is empty otherwise. □

Proposition 3.13. *If $\lambda = 2$, then S belongs to $\bar{\mathscr{K}}_g$ or S is isomorphic to one of the $\frac{q-3}{2}$ pairwise non-isomorphic surfaces of Proposition* 3.10.

Proof. Assume that the signature of the action of G on S is $(\gamma; k_1, \ldots, k_l)$. Observe that $\gamma = 0$. Indeed, otherwise the Riemann–Hurwitz formula implies that

$$l \leqslant 2\Big(1 - \frac{2}{q}\Big), \quad \text{showing that} \quad l = 0 \text{ or } l = 1.$$

In both cases, we see that $\gamma = 1$, and therefore $q = 2$ and 4 respectively, a contradiction.

As each $k_j \geqslant 2$, we see that

$$l \leqslant 6 - \frac{4}{q}, \quad \text{and therefore} \quad l \in \{3, 4, 5\}.$$

Let $v = \#\{k_j : k_j = 2\}$.

(1) If $l = 5$, then

$$\sum_{j=1}^{5} \frac{1}{k_j} = 2 + \frac{2}{q} \leqslant \frac{v}{2} + \frac{5-v}{q},$$

showing that $v = 5$. But, in this case, $q = 4$, contradicting the fact that q is prime.

(2) If $l = 4$, then

$$\sum_{j=1}^{4} \frac{1}{k_j} = 1 + \frac{2}{q} \leqslant \frac{v}{2} + \frac{4-v}{q},$$

showing that $v \in \{2, 3, 4\}$. Note that if $v = 4$, then $q = 2$, and if $v = 3$, then $k_4 < 0$. It follows that $v = 2$, and therefore $k_3 = k_4 = q$.

(3) If $l = 3$, then clearly $v \neq 3$. If $v = 2$, then $k_3 < 0$, and if $v = 1$, then $q < 4$. Thus

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{q}, \quad \text{where } k_j \in \{q, 2q\}. \tag{3.24}$$

It is easy to verify that the unique solution of (3.24) is $k_1 = q, k_2 = k_3 = 2q$.

Thereby, the signature of the action of G on S is either

$$(0; 2, 2, q, q) \quad \text{or} \quad (0; q, 2q, 2q).$$

We record here the simple fact that a group of order $2q$ is either cyclic or dihedral.

(1) If the signature is $(0; q, 2q, 2q)$, then G is cyclic, and therefore, by Proposition 3.10, we obtain that $S \cong X_8 \in \bar{\mathscr{K}}_g$ or S is isomorphic to one of the $\frac{q-3}{2}$ pairwise non-isomorphic surfaces with full automorphism group isomorphic to $C_q \times C_2$.

(2) If the signature is $(0; 2, 2, q, q)$, then the following holds.
   (a) If G is cyclic, then by Proposition 3.9, we have that $S \in \bar{\mathscr{C}}_g \subset \bar{\mathscr{K}}_g$.
   (b) If G is dihedral, then by Proposition 3.11, we have that $S \in \bar{\mathscr{K}}_g$.

This proves the proposition. □

Remark 2. Let $S \in \mathscr{M}_{q-1}^{q}$, and denote by t the automorphism of S of order q. According to [21], the action of $G = \langle t \rangle$ on S is equivalent to the action represented by one of the following skes:

$$\theta_1 = (t, t, t, t^{-3}), \qquad \theta_2 = (t, t, t^{-1}, t^{-1}),$$

$$\theta_{3,i} = (t, t^{-1}, t^i, t^{-i}), \quad \theta_{4,i} = (t, t, t^i, t^{q-2-i}), \quad \theta_{5,ij} = (t, t^i, t^j, t^{q-1-i-j}),$$

where $2 \leqslant i \leqslant \frac{q-1}{2}$, $j \notin \{1, q-1, i, q-i\}$ and $q - 1 - i - j \notin \{1, i, j\}$.

In terms of our terminology, the results of [21] allow us to claim that the stratum determined by θ_1 contains X_3, the stratum determined by θ_2 agrees with the family $\bar{\mathscr{C}}_g$, the strata determined by θ_{3i} agree with the equisymmetric strata of the family $\bar{\mathscr{K}}_g$ that are different from $\bar{\mathscr{C}}_g$, and the strata determined by θ_{4i} contain the surfaces $X_{2,k}$. We also mention that the isolated strata of dimension one of $\mathrm{Sing}(\mathscr{M}_{q-1})$ are those determined by the skes θ_{5ij}.

4 Proof of Theorem 2 and Proposition 2

Let $q \geqslant 5$ be a prime number, and set $g = q - 1$. We write $\omega_l = \exp\left(\frac{2\pi i}{l}\right)$.

The family $\bar{\mathscr{C}}_g$

We recall that if $S \in \bar{\mathscr{C}}_g$, then the action of

$$G = \mathbf{D}_{2q} = \langle R, T : R^{2q} = T^2 = (TR)^2 = 1 \rangle$$

on S has signature $(0; 2, 2, 2, q)$ and is represented by the ske

$$\theta = (R^q, T, TR, R^{q-1}).$$

Let $H = \langle R^q \rangle \cong C_2$. We consider the associated two-fold regular covering map

$$\pi \colon S \to \Sigma := S/H$$

and observe that Σ has genus zero and π ramifies over $2q$ values. If we denote them by

$$u_1, \ldots, u_q \quad \text{and} \quad v_1 \ldots, v_q, \tag{4.1}$$

then it is classically known that S is isomorphic to the normalisation of

$$y^2 = \prod_{i=1}^{q} (x - u_i)(x - v_i)$$

Note that $\Sigma \cong \mathbb{P}^1$ admits an action of $K = G/H \cong \mathbf{D}_q$ in such a way that $\Sigma/K \cong S/G$. It follows that the values in (4.1) form two orbits of length q under the action of the cyclic subgroup of order q of K. Without loss of generality, we can assume that

$$u_i = \omega_q^i \quad \text{and} \quad v_i = \lambda\omega_q^i, \quad \text{where } 1 \leqslant i \leqslant q$$

and λ is a nonzero complex number such that $t := \lambda^q \neq 1$. Hence S is isomorphic to the normalisation of the singular affine algebraic curve

$$\mathscr{X}_t := \{(x, y) \in \mathbb{C}^2 : y^2 = (x^q - 1)(x^q - t)\}$$

for some $t \neq 0, 1$. It is a direct computation to verify that the transformations

$$\mathfrak{r}(x, y) = (\omega_q x, -y) \quad \text{and} \quad \mathfrak{t}(x, y) = \left(\sqrt[q]{t}\frac{1}{x}, \sqrt{t}\frac{y}{x^q} \right)$$

restrict to automorphisms of $\mathscr{X}_t$ and that $\langle \mathfrak{r}, \mathfrak{t} \rangle \cong \mathbf{D}_{2q}$.

Note that $\mathfrak{r}^q(x, y) = (x, -y)$ is the hyperelliptic involution and that $X_8 \cong \mathscr{X}_{-1}$.

The surface X_4

Following the proof of Proposition 3.7, the action of

$$G = C_q \rtimes_4 C_4 = \langle A, B : A^q = B^4 = 1, BAB^{-1} = A^\rho \rangle$$

on X_4 has signature $(0; 4, 4, q)$ and is represented by the ske $\theta = (A^{-1}B, B^{-1}, A)$. Let $Q = \langle a \rangle$, and observe that the associated q-fold regular covering map

$$\pi \colon X_4 \to \Sigma := X_4/Q \cong \mathbb{P}^1$$

ramifies over four values marked with q. As Σ admits the action of

$$K = G/Q \cong C_4 \quad \text{with signature } (0; 4, 4)$$

and $\Sigma/K \cong S/G$, the branch values of π form one orbit under the action of K. Without loss of generality, we can assume these branch values to be 1, i, -1 and $-i$ (where $i^2 = -1$) and that their corresponding rotation numbers (modulo q) are 1, ρ, ρ^2 and ρ^3 respectively. Then, following [28] (see also [29, 72]), X_4 is isomorphic to the normalisation of

$$y^q = (x-1)(x-i)^\rho (x+1)^{q-1}(x+i)^{q-\rho}. \tag{4.2}$$

Since ρ has order 4 in $\mathbb{Z}_q$, there exists $e \in \mathbb{Z}$ such that $\rho^2 + 1 = eq$. Set

$$\mathfrak{a}(x, y) = (x, \omega_q y) \quad \text{and} \quad \mathfrak{b}(x, y) = \left(ix, \frac{-(x+i)^{e-\rho}}{(x-i)^{e-1}(x+1)^{\rho-1}} y^\rho \right),$$

and note that they restrict to automorphisms of (4.2). If φ is as in the statement of the theorem, then routine computations show that

$$\varphi(-ix)\varphi(-x)^\rho \varphi(ix)^{\rho^2} \varphi(x)^{\rho^3} = y^{1-\rho^4}.$$

This implies that $\mathfrak{b}$ has order four. Now, it is direct to see that $\langle \mathfrak{a}, \mathfrak{b} \rangle \cong C_q \rtimes_4 C_4$.

Remark 3. The Möbius transformation $\iota \colon \mathbb{P}^1 \cong \Sigma \to \mathbb{P}^1$ given by

$$\iota(z) = i\frac{z-1}{z+1} \quad \text{satisfies} \quad \iota(1, i, -1, -i) = (0, -1, \infty, 1)$$

and lifts to obtain an isomorphism between X_4 and the Riemann surface given by

$$y^q = x(x+1)^\rho (x-1)^{q-\rho}$$

This provides the explicit model for X_4 defined over its field of moduli $\mathbb{Q}$.

The surface X_3

Similarly to before, the normalisation of $y^3 = x^q - 1$ defines a Riemann surface of genus g and $(x, y) \mapsto (\omega_q x, \omega_3 y)$ restricts to an automorphism of it of order $3q$.

The surfaces $X_{2,k}$

Following the proof of Proposition 3.10, the action of

$$G = C_q \times C_2 = \langle a, b : a^q = b^2 = [a, b] = 1\rangle$$

on $X_{2,k}$ has signature $(0; q, 2q, 2q)$ and is determined by the ske

$$\theta_k = (a, a^k b, a^{-k-1} b), \quad \text{where } 1 \leqslant k \leqslant \frac{q-3}{2}.$$

If $Q = \langle a\rangle$, then the associated regular covering map

$$\pi_k \colon X_{2,k} \to \Sigma_k := X_{2,k}/Q \cong \mathbb{P}^1$$

ramifies over four values marked with q. As Σ_k admits the action of

$$K = G/Q \cong C_2 \quad \text{with signature } (0; 2, 2)$$

and $\Sigma/K \cong S/G$, two branch values of π_k form an orbit, and the remaining ones are fixed under the action of K. Thus we can assume that the branch values of π_k are 1, -1, ∞ and 0, where the first two form an orbit. It follows that $X_{2,k}$ is isomorphic to

$$y^q = n^{n_k}(x-1)(x+1) \tag{4.3}$$

for some $1 \leqslant n_k \leqslant q-1$ such that $n_k \neq q-2$. It is easy to see that

$$\mathfrak{a}(x, y) = (x, \omega_q y) \quad \text{and} \quad \mathfrak{b}_k(x, y) = (-x, (-1)^{n_k} y)$$

are automorphisms of (4.3) and that $\langle \mathfrak{a}, \mathfrak{b}_k\rangle \cong C_q \times C_2$.

The family $\bar{\mathscr{K}}_g$

Following the proof of Proposition 3.11, the action of

$$G = \mathbf{D}_q = \langle r, s : r^q = s^2 = (sr)^2 = 1\rangle \tag{4.4}$$

on $S \in \bar{\mathscr{K}}_g$ has signature $(0; 2, 2, q, q)$ and is determined by the ske

$$\theta_i = (s, sr, r^i, r^{-i-1}) \quad \text{for some } 1 \leqslant i \leqslant \frac{q-1}{2}.$$

If $Q = \langle r \rangle$, then the associated regular covering map

$$\pi \colon S \to \Sigma := S/Q \cong \mathbb{P}^1$$

ramifies over four values marked with q. As Σ admits the action of

$$K = G/Q \cong C_2 \quad \text{with signature } (0; 2, 2)$$

and $\Sigma/K \cong S/G$, the branch values of π form two orbits under the action of K. We can assume these values to be $1, -1$ and $t, -t$ for some $t \neq 0, \pm 1$. In addition, the rotation numbers are 1, $q-1$, 1 and $q-1$ respectively. Thus S is isomorphic to

$$\mathscr{Z}_t := \{(x, y) \in \mathbb{C}^2 : y^q = (x-1)(x+1)^{q-1}(x-t)(x+t)^{q-1}\}$$

It is straightforward to verify that the transformations

$$\mathrm{r}(x, y) = (x, \omega_q y) \quad \text{and} \quad \mathfrak{s}(x, y) \mapsto (-x, (x^2-1)(x^2-t^2)y^{-1})$$

restrict to automorphisms of $\mathscr{Z}_t$ and $\langle \mathrm{r}, \mathfrak{s} \rangle \cong \mathbf{D}_q$.

Proof of Proposition 2

Assume that $S \in \mathscr{K}_q$, and let G be as in (4.4). The covering

$$\Sigma \to \Sigma/K \cong S/G \quad \text{can be chosen as} \quad z \mapsto z^2,$$

and therefore, if $S \cong \mathscr{Z}_t$, then the branch values of $S \to S/G$ are ∞ and 0 marked with 2, and 1 and t^2 marked with q.

Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, and assume S_t and $(S_t)^\sigma = S_{\sigma(t)}$ to be isomorphic. The facts that

(1) $G \cong \mathrm{Aut}(S)$ provided that $S \in \mathscr{K}_g - \mathscr{C}_g$, and

(2) G is the unique group of automorphisms of S isomorphic to $\mathbf{D}_q$ provided that $S \in \mathscr{C}_g$

imply that there is a Möbius transformation $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\varphi(\{\infty, 0\}) = \{\infty, 0\} \quad \text{and} \quad \varphi(\{1, t^2\}) = \{1, \sigma(t)^2\}.$$

We have two possible cases.

(1) If $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, then either $\varphi(z) = z$ or $\varphi(z) = \sigma(t)^2 z$.

(2) If $\varphi(0) = \infty$ and $\varphi(\infty) = 0$, then either $\varphi(z) = \frac{1}{z}$ or $\varphi(z) = \frac{\sigma(t)^2}{z}$.

Observe that the latter case in (1) and the former case in (2) imply $t\sigma(t) = \pm 1$, a contradiction. It follows that $\sigma(t)^2 = t^2$ showing that $\sigma(t) = t$. Conversely, if $\sigma(t) = t$, then it is clear that $S = (S_t)^\sigma$. Hence the field of moduli of $S \cong \mathscr{X}_t$ is

$$\mathrm{fix}\{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}) : \sigma(t) = t\} = \mathbb{Q}(t),$$

as desired.

5 Some algebraic lemmata

In this section, we collect some facts related to the representations of the groups appearing in Theorem 1; these results will be needed to prove Theorems 3 and 4. Set $\omega_l = \exp(\frac{2\pi i}{l})$.

Rational and complex irreducible representations

Lemma 1. *Let $q \geqslant 5$ be a prime number. The group*

$$G_8 = \langle x, y, z : x^{2q} = y^2 = z^2 = 1,\ [x, y] = [z, y] = 1,\ zxz = x^{-1}y\rangle$$

has four complex irreducible representations of degree one given by

$$\chi_0^{1+}\colon \begin{cases} x \mapsto 1, \\ y \mapsto 1, \\ z \mapsto 1, \end{cases} \quad \chi_0^{2+}\colon \begin{cases} x \mapsto 1, \\ y \mapsto 1, \\ z \mapsto -1, \end{cases} \quad \chi_q^{1+}\colon \begin{cases} x \mapsto -1, \\ y \mapsto 1, \\ z \mapsto 1, \end{cases} \quad \chi_q^{2+}\colon \begin{cases} x \mapsto -1, \\ y \mapsto 1, \\ z \mapsto -1, \end{cases}$$

and $2q-1$ complex irreducible representations of degree two given by the following table.

Representation	x	y	z
χ_j^{+}, $1 \leqslant j \leqslant q-1$	$\mathrm{diag}(\omega_{2q}^{j}, -\omega_{2q}^{q-j})$	I_2	J_2
χ_j^{1-}, $0 \leqslant j \leqslant \frac{q-1}{2}$	$\mathrm{diag}(\omega_{2q}^{j}, \omega_{2q}^{q-j})$	$-I_2$	J_2
χ_j^{2-}, $1 \leqslant j \leqslant \frac{q-1}{2}$	$\mathrm{diag}(-\omega_{2q}^{j}, -\omega_{2q}^{q-j})$	$-I_2$	J_2

Here I_2 stands for the 2×2 identity matrix and $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof. The proof is an application of the method of Wigner and Mackey to build the complex irreducible representations of certain semidirect products. See, for example, [64, § 8.2]. □

Lemma 2. *Let $q \geqslant 5$ be a prime number such that $q \equiv 1 \bmod 4$. Let ρ be a primitive fourth root of unity in $\mathbb{Z}_q$, and choose a maximal subset $\mathcal{P} \subset \mathbb{Z}_q^*$ of representatives of the relation $k \sim k\rho \sim -k \sim -k\rho$ over $\mathbb{Z}_q^*$. The group*

$$G_4 = \langle A, B : A^q = B^4 = 1,\ BAB^{-1} = A^\rho \rangle$$

has four complex irreducible representations of degree one given by

$$\chi_1\colon \begin{cases} A \mapsto 1 \\ B \mapsto 1 \end{cases}, \quad \chi_i\colon \begin{cases} A \mapsto 1, \\ B \mapsto i, \end{cases} \quad \chi_{-1}\colon \begin{cases} A \mapsto 1, \\ B \mapsto -1, \end{cases} \quad \chi_{-i}\colon \begin{cases} A \mapsto 1, \\ B \mapsto -i, \end{cases}$$

and $\frac{q-1}{4}$ complex irreducible representations of degree four given by

$$\phi_j\colon A \mapsto \operatorname{diag}(\omega_q^j, \omega_q^{\rho j}, \omega_q^{-j}, \omega_q^{-\rho j}), \quad B \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for $j \in \mathcal{P}$. In addition, the rational irreducible representations of G_4 are

$$\chi_1,\ \chi_{-1},\ \chi_i \oplus \chi_{-i} \text{ and } \varrho = \bigoplus_{j \in \mathcal{P}} \phi_j$$

Proof. The construction of the representations follows from [64, § 8.2], and we only need to prove the last statement. As G_4 has four conjugacy classes of cyclic subgroups, it has four pairwise non-equivalent rational irreducible representations: three of them are χ_1, χ_{-1} and $\chi_i \oplus \chi_{-i}$. It follows that ϕ_j are Galois conjugate and added up together produce the remaining rational irreducible representation. □

Symmetric square character formula

Let H be a finite group, and let $\rho\colon H \to \mathrm{GL}(V)$ be a complex representation of H. Consider the associated representation of H on the symmetric square vector space of V,

$$\mathrm{Sym}^2(\rho)\colon H \to \mathrm{GL}(\mathrm{Sym}^2(V)).$$

According to [64, Proposition 2.3], the character χ_ρ^{sym} of $\mathrm{Sym}^2(\rho)$ is given by

$$\chi_\rho^{\mathrm{sym}}(h) = \frac{1}{2}[\chi_\rho(h)^2 + \chi_\rho(h^2)] \quad \text{for } h \in H, \tag{5.1}$$

where χ_ρ denotes the character of ρ.

Lemma 3. *Let χ be a character of H, and let $\bar{\chi}$ denote its complex-conjugate. Then*

$$\sum_{h\in H} \chi^{\mathrm{sym}} = \frac{1}{2}\Big[\sum_{h\in H}(\chi+\bar{\chi})^{\mathrm{sym}}(h) - |H|\langle \chi \mid \chi\rangle_H\Big].$$

Proof. According to [64, Exercise 2.1], for any pair of characters χ_1 and χ_2, we have

$$(\chi_1+\chi_2)^{\mathrm{sym}} = \chi_1^{\mathrm{sym}} + \chi_2^{\mathrm{sym}} + \chi_1\chi_2.$$

If we write $\chi_1 = \chi$ and $\chi_2 = \bar{\chi}$, then the previous equality implies

$$\sum_{h\in H} \chi^{\mathrm{sym}}(h) = \sum_{h\in H} [(\chi+\bar{\chi})^{\mathrm{sym}}(h) - (\bar{\chi})^{\mathrm{sym}}(h) - \chi(h)\bar{\chi}(h)].$$

Since

$$\sum_{h\in H} (\bar{\chi})^{\mathrm{sym}}(h) = \sum_{h\in H} \chi^{\mathrm{sym}}(h^{-1}) = \sum_{h\in H} \chi^{\mathrm{sym}}(h),$$

the conclusion follows after noting that $\sum_{h\in H}\chi(h)\bar{\chi}(h) = |H|\langle \chi \mid \chi\rangle_H$. □

The analytic representation

Let S be a compact Riemann surface of genus g, and let H be a group of automorphisms of S. The action of H induces a complex representation

$$\rho_a \colon H \to \mathrm{GL}(\mathscr{H}^{1,0}(S,\mathbb{C})) \cong \mathrm{GL}(g,\mathbb{C}),$$

called the analytic representation of H. Let $\mathrm{Irr}(H)$ denote the set of complex irreducible representations of H, up to equivalence. If we write

$$\rho_a \cong \bigoplus_{\rho\in\mathrm{Irr}(H)} \mu_\rho\rho, \quad \text{where } \mu_\rho \in \mathbb{N}\cup\{0\},$$

then μ_ρ can be computed using the classically known Chevalley–Weil formula; see [17].

Lemma 4. *Assume that the action of H on S has signature $\sigma = (0; k_1, \dots, k_s)$ and is represented by the ske $\theta\colon \Delta \to H$, where Δ is a Fuchsian group of signature σ canonically presented as in (2.2). Then $\mu_\rho = 0$ if ρ is the trivial representation; otherwise*

$$\mu_\rho = -d_\rho + \sum_{l=1}^{s}\sum_{j=1}^{k_l} N_{l,j}^{\rho}\Big(1-\frac{j}{k_l}\Big), \tag{5.2}$$

where d_ρ is the degree of ρ and $N_{l,j}^{\rho}$ is the number of eigenvalues of $\rho(\theta(x_l))$ that are equal to $\omega_{k_l}^{j}$.

Computation of the dimension $N_{S,G}$

Let S be a compact Riemann surface of genus g. As mentioned in § 2.7, following [67] and [26, Lemma 3.8] together with formula (5.1), the dimension of (the component which contains JS of) the submanifold $\mathscr{S}_S$ of $\mathscr{H}_g$ of matrices representing ppavs admitting an action equivalent to that of $\mathrm{Aut}(S)$ is

$$N_{S,G} = \langle \chi_{\rho_a}^{\mathrm{sym}} \mid 1 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_a}^{\mathrm{sym}}(g)$$
$$= \frac{1}{2|G|} \sum_{g \in G} [\chi_{\rho_a}(g)^2 + \chi_{\rho_a}(g^2)], \tag{5.3}$$

where ρ_a is the analytic representation of $G = \mathrm{Aut}(S)$.

As a direct consequence of Lemma 3, the previous equality can be rewritten as follows.

Lemma 5. *Assume that χ_{ρ_a} denotes the character of the analytic representation of $G = \mathrm{Aut}(S)$. Then*

$$N_{S,G} = \frac{1}{2|G|} \Big[\sum_{g \in G} (\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g) - |G| \langle \chi_{\rho_a} \mid \chi_{\rho_a} \rangle_G \Big].$$

Remark 4. Note the following.

(1) The computation of $N_{S,G}$ depends both on the group G and on its action on the Riemann surface S. It then makes sense to compute $N_{S,H}$ for a subgroup H of $G = \mathrm{Aut}(S)$. When considering $G = \mathrm{Aut}(S)$, we write N_S instead of $N_{S,G}$.

(2) Observe that if $H_1 \leqslant H_2$ are two groups of automorphisms of S, then we have $N_{S,H_2} \leqslant N_{S,H_1}$. In particular, if $N_{S,H} = 0$ for some group of automorphisms H of S, then $N_S = 0$.

(3) According to (5.1), the summand $(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g)$ in Lemma 5 is equal to

$$(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g) = \frac{1}{2}[(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^2(g) + (\chi_{\rho_a} + \bar{\chi}_{\rho_a})(g^2)] \tag{5.4}$$

and is, indeed, the sum of two rational numbers.

6 Proof of Theorems 3 and 4

The surface X_8

With the notation of Lemma 1, the representations

$$\chi_j^{1-} \text{ and } \chi_j^{2-} \quad \text{for } j = 1, \ldots, \frac{q-1}{2}$$

of G_8 are pairwise Galois conjugate, showing that their direct sum yields a rational irreducible representation of degree $2(q-1)$. We denote this last representation by ϱ, namely

$$\varrho \cong \left(\bigoplus_{j=1}^{\frac{q-1}{2}} \chi_j^{1-} \right) \oplus \left(\bigoplus_{j=1}^{\frac{q-1}{2}} \chi_j^{2-} \right).$$

As explained in § 2.8, the group algebra decomposition of JX_8 with respect to $G_8 \cong \mathrm{Aut}(X_8)$ has the form $JX_8 \sim B^n \times P$, where B is the abelian subvariety of JX_8 associated to ϱ and P is the product of the factors associated to the remaining rational irreducible representations of G_8. Following [36, Proposition (10.8)], the Schur index of χ_1^{1-} is one, and thus $n = 2$.

We recall that the action of G_8 on X_8 is represented by the ske $\theta = (z, zx, x^{-1})$. The dimension of the fixed subspaces of χ_1^{1-} under the action of the subgroups $\langle z \rangle$, $\langle zx \rangle$ and $\langle x^{-1} \rangle$ is 1, 0 and 0 respectively; this is clear by noting that $\chi_1^{1-}(z) = J_2$ and that

$$\chi_1^{1-}(zx) = \begin{pmatrix} 0 & \omega_{2q}^{q-1} \\ \omega_{2q} & 0 \end{pmatrix} \quad \text{and} \quad \chi_1^{1-}(x^{-1}) = \mathrm{diag}(\omega_{2q}^{-1}, \omega_{2q}^{1-q})$$

do not have 1 as an eigenvalue. In addition, it is easy to see that the character field of χ_1^{1-} has degree $q-1$ over the rationals. We then apply equation (2.11) to conclude that

$$\dim B = (q-1)\Big[-2 + \frac{1}{2}\big((2-1) + (2-0) + (2-0)\big)\Big] = \frac{q-1}{2}.$$

Since the dimension of JX_8 is $q-1$, it follows that $P = 0$, and therefore

$$JX_8 \sim B^2. \tag{6.1}$$

Finally, we consider the subgroup $\langle z \rangle$ of G_8 and write $Y_8 = X_8/\langle z \rangle$. The induced isogeny (2.10) applied to (6.1) implies that

$$JY_8 = J(X_8/\langle z \rangle) \sim B, \quad \text{and therefore} \quad JX_8 \sim JY_8^2,$$

as claimed in Theorem 3.

We now proceed to prove that $N_{X_8} = 0$. Let $H = \langle x \rangle \cong C_{2q}$, and consider the maps

$$\rho_k \colon H \to \mathbb{C} \quad \text{given by} \quad x \mapsto \omega_{2q}^k \quad \text{for } k = 0, \ldots, 2q-1.$$

We claim that the analytic representation ρ_a of H decomposes as the direct sum

$$\rho_a \cong \bigoplus_{j=q+1}^{2q-1} \rho_j.$$

To prove that, observe that $\{\rho_0, \ldots, \rho_{2q-1}\}$ is a full set of pairwise non-equivalent complex irreducible representations of H. Besides, as noted in the proof of Proposition 3.10, the induced action of H on X_8 has signature $(0; q, 2q, 2q)$ and is represented by the ske (x^2, x^{-1}, x^{-1}) (see also the algorithm in [7] based on [66]). If we write

$$\rho_a \cong \bigoplus_{k=0}^{2q-1} \mu_k \rho_k \quad \text{for some } \mu_k \in \mathbb{N} \cup \{0\},$$

then according to Lemma 4, we have $\mu_0 = 0$ and

$$\mu_k = -1 + 2\sum_{j=1}^{2q-1} N_{1,j}^k \Big(1 - \frac{j}{2q}\Big) + \sum_{j=1}^{q-1} N_{2,j}^k \Big(1 - \frac{j}{q}\Big)$$

for each $k \in \{1, \ldots, 2q-1\}$, where $N_{1,j}^k = 1$ if and only if $j = 2q - k$ and

$$N_{2,j}^k = \begin{cases} 1 & \text{if and only if } j = k \text{ for } k \leqslant q, \\ 1 & \text{if and only if } j = k - q \text{ for } k \geqslant q + 1. \end{cases}$$

In this way, we obtain if $k \leqslant q$, then $\mu_k = 0$, and if $k \geqslant q + 1$, then $\mu_k = 1$. The claim follows.

We then can construct the following table.

h	Order	$(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^2(h)$	$(\chi_{\rho_a} + \bar{\chi}_{\rho_a})(h^2)$
1	1	$(2(q-1))^2$	$2(q-1)$
x^q	2	0	$2(q-1)$
$x^{2k}, 1 \leqslant k \leqslant q-1$	q	$(-2)^2$	-2
$x^{2k-1}, 1 \leqslant k \leqslant q, k \neq \frac{q+1}{2}$	$2q$	0	-2

Then we obtain, by (5.4), that $\sum_{h \in H} (\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(h)$ is equal to

$$\frac{1}{2}\big(4(q-1)^2 + 2(q-1) + 2(q-1) + 2(q-1) - 2(q-1)\big) = 2(q^2 - q).$$

Now, the fact that $|H| \langle \chi_{\rho_a} \mid \chi_{\rho_a} \rangle_H = 2q(q-1)$ together with Lemma 3 imply that

$$\sum_{h \in H} \chi_{\rho_a}^{\mathrm{sym}}(h) = \frac{1}{2}[2(q^2 - q) - 2q(q-1)] = 0,$$

and therefore we obtain $N_{X_8, H} = 0$. Hence $N_{X_8} = 0$, as claimed in Theorem 4.

The surface X_4

With the notation of Lemma 2, the group algebra decomposition of JX_4 with respect to $G_4 \cong \mathrm{Aut}(X_4)$ is $JX_4 \sim D_1 \times D_i \times D_{-1} \times D^4$, where the factor D_k is associated to the representation χ_k and D is associated to ϱ. Observe that the character field of each ϕ_j has degree $\frac{q-1}{4}$ over the rationals.

We recall that the action of G_4 on X_4 is represented by the ske

$$\Theta = (A^{-1}B, B^{-1}, A).$$

The dimension of the fixed subspace of ϕ_j under the action of $\langle A^{-1}B\rangle$, $\langle B^{-1}\rangle$ and $\langle A\rangle$ is 1, 1 and 0 respectively. Consequently, equation (2.11) implies

$$\dim D = \frac{q-1}{4}\Big[-4 + \frac{1}{2}\big((4-1) + (4-1) + (4-0)\big)\Big] = \frac{q-1}{4}.$$

The previous equality shows, in addition, that

$$D_1 = D_i = D_{-1} = 0, \quad \text{and therefore} \quad JX_4 \sim D^4.$$

Finally, we consider the subgroup $\langle B\rangle$ of G_4 and write $Y_4 = X_4/\langle B\rangle$. The induced isogeny (2.10) applied to the previous isogeny implies that

$$JY_4 = J(X_4/\langle B\rangle) \sim D, \quad \text{and therefore} \quad JX_4 \sim JY_4^4,$$

as claimed in Theorem 3.

We now proceed to prove that $N_{X_4} = \frac{q-1}{4}$. A routine application of Lemma 4 shows that the analytic representation ρ_a of G_4 is

$$\rho_a \cong \bigoplus_{j\in\mathscr{P}} \phi_j, \quad \text{and therefore} \quad |G_4|\langle \chi_{\rho_a} \mid \chi_{\rho_a}\rangle_{G_4} = q(q-1),$$

where $\mathscr{P}$ is as in Lemma 2. It is not difficult to see that

$$(\chi_{\rho_a} + \bar{\chi}_{\rho_a})(g) = \begin{cases} 2(q-1) & \text{if } g = 1, \\ 0 & \text{if } |g| = 2, 4, \\ -2 & \text{if } |g| = q, \end{cases}$$

and therefore (5.4) allows us to write that $(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(1) = (q-1)(2q-1)$ and that

$$\sum_{g\in G_4,\, |g|=\epsilon} (\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g) = \begin{cases} q(q-1) & \text{if } \epsilon = 2, \\ 0 & \text{if } \epsilon = 4, \\ q-1 & \text{if } \epsilon = q. \end{cases}$$

Thus $\sum_{g\in G_4}(\chi_{\rho_a}+\bar{\chi}_{\rho_a})^{\mathrm{sym}}(g)=3q(q-1)$. Now, it follows from Lemma 3 that

$$\sum_{g\in G_4}\chi_{\rho_a}^{\mathrm{sym}}(g)=\frac{1}{2}\big(3q(q-1)-q(q-1)\big)=q(q-1),$$

and consequently, Lemma 5 says that $N_{X_4}=\frac{q-1}{4}$, as desired.

The surface X_3

The complex irreducible representations of $G_3=\langle x : x^{3q}=1\rangle$ are

$$\chi_k\colon G_3\to\mathbb{C},\quad x\mapsto\omega_{3q}^k\quad\text{for } k\in\{0,\dots,3q-1\}.$$

We denote by ρ_r the complexification of the rational representation corresponding to the action of G_3 on X_3. We claim that

$$\rho_r\cong\bigoplus_{\sigma\in\mathscr{G}}\chi_1^{\sigma},\quad\text{where } \mathscr{G}=\mathrm{Gal}(\mathbb{Q}(\omega_{3q})/\mathbb{Q})$$

In fact, according to [62, Theorem 5.10], the multiplicity of χ_1 in the decomposition of ρ_r as a sum of irreducible representations is one. In addition, since ρ_r is indeed defined over the rationals, we can deduce that all the orbit of χ_1 under $\mathscr{G}$ appears in the decomposition of ρ_r. The claim follows after noting that the aforementioned orbit has length $2(q-1)$, and this number agrees with the degree of ρ_r.

Since $\rho_r\cong\rho_a\oplus\bar{\rho}_a$, the previous claim says that ρ_a decomposes into $q-1$ pairwise non-equivalent complex irreducible representations of degree one of G_3, and thereby

$$\langle\chi_{\rho_a}\mid\chi_{\rho_a}\rangle_{G_3}=q-1. \tag{6.2}$$

In order to determine the character χ_{ρ_r} of ρ_r, it is convenient to decompose ρ_r in the following different but equivalent way. Let

$$\Lambda(q)=\{t\in\{1,\dots,3q-1\}:\gcd(t,3q)=1\},$$

and consider the subset $\Lambda'(q)$ of $\Lambda(q)$ of cardinality $q-1$ obtained by removing the additive inverses modulo $3q$ (that is, $k\in\Lambda'(q)$, then $-k \bmod 3q\notin\Lambda'$). It is not difficult to see that

$$\rho_r\cong\bigoplus_{k\in\Lambda'(q)}(\chi_k\oplus\bar{\chi}_k),\quad\text{and therefore}\quad\chi_{\rho_r}=\sum_{k\in\Lambda'(q)}(\chi_k+\bar{\chi}_k).$$

(1) Clearly, $\chi_{\rho_r}(1)=2(q-1)$, the degree of ρ_r.

(2) For the elements of order 3 (that is, x^q and x^{2q}), we have that

$$(\chi_k + \bar{\chi}_k)(x^q) = \omega_{3q}^{kq} + \omega_{3q}^{-kq}$$
$$= -1 \text{ for each } k \in \Lambda' \implies \chi_{\rho_r}(x^q) = -(q-1).$$

Analogously, one sees that $\chi_{\rho_r}(x^{2q}) = -(q-1)$.

(3) For the elements of order q (that is, x^{3j} with $j \in \{1, \dots, q-1\}$), we have that

$$(\chi_k + \bar{\chi}_k)(x^{3j}) = \omega_{3q}^{3kj} + \omega_{3q}^{-3kj} = \omega_q^{kj} + \omega_q^{-kj}$$

for every $k \in \Lambda'$. Then

$$\chi_{\rho_r}(x^{3j}) = \sum_{k \in \Lambda'(q)} (\chi_k + \bar{\chi}_k)(x^{3j}) = \sum_{k \in \Lambda'(q)} (\omega_q^{kj} + \omega_q^{-kj}),$$

showing that

$$\chi_{\rho_r}(x^{3j}) = -2 \quad \text{for all } j \in \{1, \dots, q-1\}.$$

(4) For the elements of order $3q$ (that is, x^t with $t \in \Lambda(q)$), we have that

$$\chi_{\rho_r}(x^t) = \sum_{k \in \Lambda'(q)} (\chi_k + \bar{\chi}_k)(x^t) = \sum_{k \in \Lambda'(q)} (\omega_{3q}^{kt} + \omega_{3q}^{-kt}),$$

and this corresponds to the sum of all primitive $3q$-th roots of unity. It is a known fact that this sum corresponds to the Möbius function $\mu(3q)$, which is 1; thus

$$\chi_{\rho_r}(x^t) = 1 \quad \text{for every } t \in \Lambda(q).$$

We summarise all the above in the third column of the following table; the fourth column follows from all the above and (5.4).

Order	g	$\chi_{\rho_r}(g) = (\chi_{\rho_a} + \bar{\chi}_{\rho_a})(g)$	$2(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g)$
1	1	$2(q-1)$	$2(q-1) + (2(q-1))^2$
3	x^q, x^{2q}	$-(q-1)$	$-(q-1) + (q-1)^2$
q	x^{3j}, $j = 1, \dots, q-1$	-2	$-2 + 4 = 2$
$3q$	x^t, $t \in \Lambda(q)$	1	$1 + 1 = 2$

It follows that

$$\sum_{g \in G_3} (\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g) = 3q(q-1),$$

and the desired result $N_{X_3} = 0$ follows from Lemma 3 together with equality (6.2).

The surfaces $X_{2,k}$

Let $1 \leqslant k \leqslant \frac{q-3}{2}$. Consider the complex irreducible representations of

$$\mathrm{Aut}(X_{2,k}) \cong G_2 = \langle a, b : a^q = b^2 = [a, b] = 1 \rangle$$

given by $\chi_1 \colon x \mapsto \omega_{2q}$ and $\chi_2 \colon x \mapsto \omega_q$, where $x := ab$. If $\mathscr{G}_1$ and $\mathscr{G}_2$ are the Galois groups of the extensions of $\mathbb{Q}$ by $\mathbb{Q}(\omega_{2q})$ and $\mathbb{Q}(\omega_q)$ respectively, then

$$\bigoplus_{\sigma \in \mathscr{G}_1} \chi_1^\sigma \quad \text{and} \quad \bigoplus_{\sigma \in \mathscr{G}_2} \chi_2^\sigma$$

are rational irreducible representations of G_2 of degree $q-1$.

Let ρ_r denote the complexification of the rational representation corresponding to the action of G_2 on $X_{2,k}$. By arguing as in the case of X_3, one obtains that

$$\rho_r \cong \Big(\bigoplus_{\sigma \in \mathscr{G}_1} \chi_1^\sigma \Big) \oplus \Big(\bigoplus_{\sigma \in \mathscr{G}_2} \chi_2^\sigma \Big),$$

and if ρ_a is the analytic representation of the involved action, then

$$\langle \chi_{\rho_a} \mid \chi_{\rho_a} \rangle_{G_2} = q-1, \tag{6.3}$$

where χ_{ρ_a} is the character of ρ_a. Moreover, the character χ_{ρ_r} of ρ_r is given by

$$\chi_{\rho_r}(g) = \begin{cases} 2(q-1) & \text{if } g = 1, \\ 0 & \text{if } |g| = 2, \\ -2 & \text{if } |g| = q, \\ 0 & \text{if } |g| = 2q, \end{cases}$$

and therefore we can construct the following table.

Order	g	$\chi_{\rho_r}(g) =$ $(\chi_{\rho_a} + \bar{\chi}_{\rho_a})(g)$	$2(\chi_{\rho_a} + \bar{\chi}_{\rho_a})^{\mathrm{sym}}(g)$
1	1	$2(q-1)$	$2(q-1) + (2(q-1))^2$
2	b	0	$2(q-1) + 0^2$
q	a^j, $j = 1, \ldots, q-1$	-2	$-2 + (-2)^2 = 2$
$2q$	$a^j b$, $\gcd(j, 2q) = 1$	0	$-2 + 0^2$

The desired conclusion $N_{X_{2,k}} = 0$ follows from the previous table, equation (6.3) and Lemma 5.

The family $\mathscr{C}_g$

The complex representation $\psi\colon \mathbf{D}_{2q} \to \mathrm{GL}(2,\mathbb{C})$ (with the dihedral group presented as in (3.16)) given by

$$\psi(R) = \mathrm{diag}(\omega_{2q}, \omega_{2q}^{-1}) \quad \text{and} \quad \psi(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has Schur index 1 and field of characters of degree $\frac{q-1}{2}$ over the rationals. It follows that the group algebra decomposition of JS for each $S \in \mathscr{C}_g$ with respect to $\mathbf{D}_{2q}$ has the form

$$JS \sim B^2 \times P, \tag{6.4}$$

where B is the abelian subvariety of JS associated to ψ. The dimension of the fixed subspaces of ψ under the action of $\langle R^q \rangle$, $\langle T \rangle$, $\langle TR \rangle$ and $\langle R^{q-1} \rangle$ equal 0, 1, 1 and 0 respectively. Thereby, equation (2.11) together with the fact that the action is represented by the ske $\theta = (R^q, T, TR, R^{q-1})$ imply that

$$\dim B = \frac{q-1}{2}\Big(-2 + \frac{1}{2}\big((2-0) + (2-1) + (2-1) + (2-0)\big)\Big) = \frac{q-1}{2},$$

and therefore $P = 0$. Now, we consider the subgroup $\langle T \rangle \cong C_2$ of $\mathbf{D}_{2q}$ and write $X = S/\langle T \rangle$. The induced isogeny (2.10) applied to (6.4) implies that

$$JX = J(S/\langle T \rangle) \sim B, \quad \text{and therefore} \quad JS \sim JX^2.$$

A routine application of (5.2) permits us to see that the analytic representation ρ_a of the action of $\mathbf{D}_{2q}$ on S is equivalent to the Galois orbit of ψ, namely

$$\rho_a \cong \bigoplus_{\sigma} \psi^{\sigma},$$

where σ runs over the Galois group associated to character field of ψ.

The following table (taken from [37, Proposition 6.1]) collects the character of ρ_a and of its symmetric square for representatives of the conjugacy classes of the group.

g	χ_{ρ_a}	$\chi_{\rho_a}^{\mathrm{sym}}$
1	$q-1$	$(q-1)^2 + (q-1)$
R^{2j-1}, $j = 1, \dots, q$, $j \neq \frac{q+1}{2}$	1	$1 + (-1)$
R^{2j}, $j = 1, \dots, q-1$	-1	$1 + (-1)$
R^q	$-(q-1)$	$(q-1)^2 + (q-1)$
T	0	$0 + (q-1)$
TR	0	$0 + (q-1)$

Thus, by equation (5.3), we obtain that

$$N_S = \frac{1}{8q}\big((q-1)^2 + (q-1) + (q-1)^2 + (q-1) + 2q(q-1)\big) = \frac{q-1}{2}.$$

The family $\mathscr{K}_g$

It is well known that $\mathbf{D}_q$ has two complex irreducible representations of degree one, and $\frac{q-1}{2}$ of degree two given by

$$\psi_j : r \mapsto \operatorname{diag}(\omega_q^j, \omega_q^{-j}) \quad \text{and} \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{where } j \in \left\{1, \ldots, \frac{q-1}{2}\right\};$$

all of them are Galois conjugate. Clearly, their direct sum

$$W \cong \bigoplus_{j=1}^{\frac{q-1}{2}} \psi_j$$

is a rational irreducible representation of $\mathbf{D}_q$ of degree $q-1$. It follows that the group algebra decomposition of JS for each $S \in \mathscr{K}_g - \mathscr{C}_g$ with respect to $\mathbf{D}_q$ has the form

$$JS \sim B^2 \times P, \tag{6.5}$$

where B is the abelian subvariety of JS associated to W. Observe that

(1) the dimension of the fixed subspace of ψ_1 under $\langle r^i \rangle$ is equal to 0 for each $1 \leqslant i \leqslant q-1$;

(2) the dimension of the fixed subspace of ψ_1 under $\langle sr^i \rangle$ is equal to 1 for each $0 \leqslant i \leqslant q-1$.

It follows that, independently of the equisymmetric stratum to which S belongs (see the ske θ_i given in the proof of Proposition 3.11), equation (2.11) implies

$$\dim B = \frac{q-1}{2}\Big[-2 + \frac{1}{2}\big((2-1) + (2-1) + (2-0) + (2-0)\big)\Big] = \frac{q-1}{2},$$

and therefore $P = 0$. Now, if $X = S/\langle s \rangle$, then (2.10) applied to (6.5) implies that $JS \sim JX^2$.

By Lemma 4, one sees that the analytic representation ρ_a of the action of $\mathbf{D}_q$ on S is equivalent to W. The character of ρ_a and of ρ_a^{sym} is summarised in the following table.

g	χ_{ρ_a}	$\chi_{\rho_a}^{\mathrm{sym}}$
1	$q-1$	$(q-1)^2+(q-1)$
$sr^j,\ j=0,\dots,q-1$	0	$q-1$
$r^j,\ j=1,\dots,q-1$	-1	$-1+(-1)^2$

Finally, by equation (5.3), we obtain that

$$N_S = \frac{1}{4q}\big((q-1)^2+(q-1)+q(q-1)\big) = \frac{q-1}{2}.$$

A Addendum

We recall here the fact that the full automorphism group of the Accola–Maclachlan curve X_8 determines its Jacobian variety JX_8 (that is, $N_{X_8}=0$), and therefore it allows us to determine its period matrix. In this addendum, we determine explicitly the period matrix $(I_4\ Z)$, where $Z \in \mathscr{H}_4$, of JX_8 provided that the genus of X_8 is four (that is, for $q=5$). To accomplish this task, we apply the results on adapted hyperbolic polygons and algorithms programed in [6] to realise the action of the full automorphism group of X_8 in the symplectic group. Explicitly, the rational representation $\rho_r\colon \mathrm{Aut}(X_8) \to \mathrm{Sp}(8,\mathbb{Z})$ is given by

$$\rho_r(x^{-1}) = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_r(zx) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, if we write

$$Z = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & j \\ d & g & j & k \end{pmatrix},$$

then with the notation of § 2.7, the fact that

$$R \cdot Z = Z \quad \text{for each } R \in \langle \rho_r(x^{-1}), \rho_r(zx) \rangle$$

implies that the coefficients of Z satisfy the relations

(1) $a = -100/11k^3 - 140/11k$,

(2) $b = -5k^2 - 31/4$,

(3) $c = d = 5k^2 + 29/4$,

(4) $e = 20/11k^3 + 39/11k$,

(5) $f = -10/11k^3 - 39/22k$,

(6) $g = -50/11k^3 - 151/22k$,

(7) $h = k$,

(8) $j = 60/11k^3 + 84/11k$,

where the parameter k satisfies the following equation:

$$k^4 + 5/2k^2 + 121/80 = 0. \tag{A.1}$$

The solutions of (A.1) are

$$k_1 = -\frac{1}{2} i \sqrt{\frac{2}{5}\sqrt{5} + 5}, \quad k_2 = \frac{1}{2} i \sqrt{\frac{2}{5}\sqrt{5} + 5},$$
$$k_3 = -\frac{1}{2} i \sqrt{-\frac{2}{5}\sqrt{5} + 5}, \quad k_4 = \frac{1}{2} i \sqrt{-\frac{2}{5}\sqrt{5} + 5}.$$

The values k_1 and k_4 must be disregarded; indeed,

$$a(k_1) := -100/11k_1^3 - 140/11k_1 \quad \text{and} \quad a(k_4) := -100/11k_4^3 - 140/11k_4$$

do not have imaginary part positive, and therefore the corresponding matrices do not belong to $\mathscr{H}_4$. Now, the fact that $Z \in \mathscr{H}_4$ also implies that

$$\delta := \det \operatorname{Im} \begin{pmatrix} a & b \\ b & e \end{pmatrix} \text{ must be positive.}$$

With the help of numerical approximations of [73] one sees that δ is positive only for $k = k_2$.

We replace the value of $k = k_2$ in equalities (1), ..., (8) to finally obtain that the period matrix of JX_8 is $(I_4\ Z)$, where Z is given below:

$$\begin{pmatrix} \frac{25}{22}iz^{\frac{3}{2}} - \frac{70}{11}i\sqrt{z} & \frac{1}{2}\sqrt{5} - \frac{3}{2} & -\frac{1}{2}\sqrt{5} + 1 & -\frac{1}{2}\sqrt{5} + 1 \\ \frac{1}{2}\sqrt{5} - \frac{3}{2} & -\frac{5}{22}iz^{\frac{3}{2}} + \frac{39}{22}i\sqrt{z} & \frac{5}{44}iz^{\frac{3}{2}} - \frac{39}{44}i\sqrt{z} & \frac{25}{44}iz^{\frac{3}{2}} - \frac{151}{44}i\sqrt{z} \\ -\frac{1}{2}\sqrt{5} + 1 & \frac{5}{44}iz^{\frac{3}{2}} - \frac{39}{44}i\sqrt{z} & \frac{1}{2}i\sqrt{z} & -\frac{15}{22}iz^{\frac{3}{2}} + \frac{42}{11}i\sqrt{z} \\ -\frac{1}{2}\sqrt{5} + 1 & \frac{25}{44}iz^{\frac{3}{2}} - \frac{151}{44}i\sqrt{z} & -\frac{15}{22}iz^{\frac{3}{2}} + \frac{42}{11}i\sqrt{z} & \frac{1}{2}i\sqrt{z} \end{pmatrix}$$

with $z = \frac{2}{5}\sqrt{5} + 5$.

Remark 5. The results of [6] extend those of [13]; thus, if we repeat this procedure for the Accola–Maclachlan curve of genus two, then we recover the period matrix given in [13].

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