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# KINKS OF THE SINE-GORDON EQUATION ON A WORMHOLE

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**KINKS DE LA ECUACIÓN DE SINE-GORDON EN UN AGUJERO DE GUSANO**

En esta tesis estudiamos la ecuación de Sine-Gordon estacionaria en un agujero de gusano (SGWH) con parámetro $a$. Específicamente, establecemos algunos resultados para la ecuación de Sine-Gordon (SG) estacionaria en un espacio-tiempo plano y su solución *kink* $H_{SG}$. Encontramos la solución 1-*kink* $H_a(r)$ de la ecuación SGWH, estudiamos su comportamiento asintótico cuando $|r| \to +\infty$ y probamos que converge cuadráticamente a $H_{SG}$ si $a \to +\infty$. Adicionalmente, el espectro del operador de SGWH linealizado es analizado, y mostramos que el primer valor propio $\lambda_a$ converge al valor propio de SG $\lambda_{SG}$ con tasa cuadrática en $a$. Finalmente, discutimos la existencia de soluciones $n$-*kink* para la ecuación SGWH.

**KINKS OF THE SINE-GORDON EQUATION ON A WORMHOLE**

In this thesis we study the stationary Sine-Gordon equation on a wormhole (SGWH) with parameter $a$. Specifically, we establish some results for the stationary Sine-Gordon (SG) equation in flat spacetime and its kink solution $H_{SG}$. We find the 1-kink solution $H_a(r)$ for the SGWH equation, study its asymptotic behavior as $|r| \to +\infty$ and prove that it converges quadratically to $H_{SG}$ as $a \to +\infty$. In addition, the spectrum of the linearized SGWH operator is analyzed, and we show that the first eigenvalue $\lambda_a$ converges to the SG eigenvalue $\lambda_{SG}$ at a quadratic rate on $a$. Finally, we discuss the existence of $n$-kink solutions for the SGWH equation.

*Sic transit gloria mundi*

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# Introduction

Dispersive partial differential equations (PDEs) are used to model many important physical phenomena, particularly those with wave-like behavior. Their defining feature is that the phase velocities of a solution depend on their wavelength, a property that reflects how appropiate they are to describe waves. Among these equations, the non-linear kind express a variety of interesting structures and results that arise thanks to their complexity. Solitons are a classical example, drawing the attention of many mathematicians since their discovery in the nineteenth century [1].

Solitons have been a subject of study in non-linear PDEs for a long time [1]. A fully formal definition is not widely agreed upon, but following [2] there are three key properties that a solution must satisfy in some form:

1. They are localized in space.
2. Their shape does not vary with time.
3. Interacting with other solitons only results in phase shifts.

In this sense, solitons are 'simple' solutions to complex problems that owe their existence to the delicate balance between dispersive and non-linear effects [3].

In addition to solitons, there is another type of solution that arises in dispersive equations. In the Schrödinger equation one encounters functions whose mass disperses through space in a way that any compact set fails to capture it. These radiative objects appear in many similar equations [4], a fact that hints they may be more closely related to the nature of these problems.

Indeed, both these functions and solitons play an important role in the soliton resolution conjecture. This conjecture stems from the common drive in mathematics to describe objects in terms of simpler constructs, and PDEs are no exception. A natural question in physics inquires about the long-time behavior of systems, and since solitons remain unchanged with time this question morphs into the following: can solutions of non-linear dispersive equations in the asymptotic regime be written as the superposition of solitons plus radiative components? This is the *soliton resolution conjecture* [4] , and it is a very difficult question indeed. This is an active line of research and many results exist in the case of the non-linear Schrödinger equation [5], the Korteweg-deVries equation [6] and the Sine-Gordon equation [7].

While it is true that the conjecture in its full generality has yet to be proven, it points towards a promising direction. It also encourages us to study solitons in greater detail: though interesting in their own right, analyzing their behavior could lead us to a better understanding of other solutions in the long-time regime. While radiative components will eventually 'evaporate', solitons conserve their shape, even after interacting with other solitons. It follows that their superposition could reflect an approximation of the behavior of the general solution. Considering it is no easy task to find a general solution for non-linear PDEs, this would be quite helpful.

Solitons for the Sine-Gordon (SG) equation, known as *kinks* and *multi-kinks*, are well understood. Since modified SG equations have been proposed [8], it is reasonable to study how their respective kinks change compared to the original. While they may be similar or have new kinks, they may have none. The modification we are interested in relies on the geometric description of our spacetime. To understand it, we will present some of the background needed in differential geometry in Section 1.1, then discuss the classical SG equation and its kink solutions in Section 1.2. Finally, we describe the modified spacetime and the new SG equation in Section 1.3.

## 1.1. Preliminaries

This section will focus on the theory of pseudo-Riemannian manifolds to establish the context of the problem studied. We elude complicated concepts and intrincate definitions for the sake of brevity, as this is intended to give the minimum knowledge required to understand the setting. No proofs are included because the results presented here are either standard and readily available in books such as [9] and [10], or easily adapted from their Riemannian counterparts in the pseudo-Riemannian case.

**Remark** We shall employ Einstein's summation convention to avoid unnecessary notation. Index variables that appear twice in a term will be summed over their range if they appear both as a subscript and a superscript. For example, if $a = (a_0, a_1, a_2, a_3)$ and $b = (b^0, b^1, b^2, b^3)$ are vectors in $\mathbb{R}^4$, then their inner product $a \cdot b$ can be written using the summation convention like this:

$$a \cdot b = \sum_{i=0}^{3} a_i b^i = a_i b^i.$$

**Definition 1.1** *An n-*dimensional topological manifold *is a Hausdorff, second-countable topological space $M$ such that every point $p$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Intuitively, this means that a manifold resembles regular euclidean space, at least locally. For the rest of this section, $n$ shall refer to the dimension of the manifold in question.

**Definition 1.2** *A* chart *on a manifold $M$ is a pair $(U, x)$, where $U \subset M$ is open and $x\colon U \to x(U) \subset \mathbb{R}^n$ is an homeomorphism.*

**Remark** The term chart can also refer to the function $x$ or the set $U$, and we use it interchangeably. It is also common to refer to $x$ as *coordinates*, and its inverse $x^{-1}$ as a *parametrization*.

**Definition 1.3** *An* $n$-dimensional smooth manifold *is a topological manifold and a collection of charts* $\{(U_\alpha, x_\alpha)\}$ *that satisfy:*

1. $\bigcup_\alpha U_\alpha = M$.

2. *If* $(U_\alpha, x_\alpha)$ *and* $(U_\beta, x_\beta)$ *intersect, then the* transition maps

$$x_\beta \circ x_\alpha^{-1} \colon x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta),$$
$$x_\alpha \circ x_\beta^{-1} \colon x_\beta(U_\alpha \cap U_\beta) \to x_\alpha(U_\alpha \cap U_\beta)$$

   *are smooth as* $\mathbb{R}^n \to \mathbb{R}^n$ *functions.*

3. *The collection* $\{(U_\alpha, x_\alpha)\}$ *is maximal with respect to these two properties.*

It is not useful for our purposes to dwell on the technicalities associated to $\{(U_\alpha, x_\alpha)\}$ (known as an atlas). The important property here is the compatibility condition, which ensures it is possible to change from one set of coordinates to another in a smooth way.

**Definition 1.4** *Let* $M$ *and* $N$ *be two smooth manifolds of dimension* $m$ *and* $n$ *respectively. A map* $f \colon M \to N$ *is* smooth at a point $p \in M$ *if there is a chart* $(U, x)$ *in* $M$ *that contains* $p$ *and a chart* $(V, y)$ *in* $N$ *that contains* $f(p)$, *such that* $f(U) \subset V$ *and the map*

$$y \circ f \circ x^{-1} \colon x(U) \to y(V)$$

*is smooth at* $x(p)$. *We say that it is smooth on* $M$ *(or simply smooth) if it is smooth at every* $p \in M$.

**Remark** The sets $x(U)$ and $y(V)$ are subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively; therefore the differentiability of the map $y \circ f \circ x^{-1}$ is well understood.

From now on we will use $M$ to refer to an $n$-dimensional smooth manifold, unless stated otherwise. A natural construction is what is known as a *tangent space*. This is a generalization of the tangent plane to a regular surface, though slightly more complex due to the absence of an ambient space.

**Definition 1.5** *Given* $p \in M$, *the* tangent space at $p$ *is the vector space* $T_pM$ *of all (real) linear maps* $v \colon C^\infty \to \mathbb{R}$ *that satisfy the Leibniz rule:*

$$v(fg) = f(p)v(g) + v(f)g(p) \qquad \forall f, g \in C^\infty(M).$$

*We call elements of* $T_pM$ *tangent vectors at* $p$.

There are other equivalent definitions that provide different insights on the nature of tangent vectors; for reasons of space, however, we do not present them here. Note that given a coordinate chart $(U, x)$, there are $n$ tangent vectors induced by $x$.

**Proposition 1.1** *Let* $p \in M$ *be a point and* $(U, x)$ *a chart containing* $p$. *Let* $x^i$ *be the component functions of* $x$, *that is,* $x(q) = (x^1(q), \dots, x^n(q)) \in \mathbb{R}^n$ *for* $q \in U$. *For* $1 \le i \le n$,

*$x^i$ defines a tangent vector at $p$:*

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R}$$
$$f \mapsto \partial_i(f \circ x^{-1})(x(p)),$$

*where $\partial_i$ denotes the usual partial derivative on $\mathbb{R}^n$.*

**Remark** We shall dispense with the evaluation at $p$ if it is clear that the vector belongs to $T_pM$ for a certain $p$. Moreover, if the chart $x$ is understood, then we will simply write $\partial_i$. Since the functions that $\partial_i$ acts on are defined on $M$ instead of $\mathbb{R}^n$, there is no confusion with the euclidean partial derivative.

**Proposition 1.2** *Given $p$ and $(U, x)$ as above, the set $\{ \partial_i : 1 \leq i \leq n \}$ is a basis for $T_pM$. If $v$ is an element of $T_pM$, its components in this basis are $v^i = v(x^i)$.*

**Definition 1.6** *The* cotangent space*, denoted $T_p^*M$, is the dual of $T_pM$. The dual basis to $\{\partial_i\}_{i=1}^n$ is $\{ \mathrm{d}x^i : 1 \leq i \leq n \}$.*

As each tangent space is a vector space, it is natural to ask if there is a way to measure angles an distances in these spaces in a manner that is consistent with the manifold and its differential structure. Such an object is called a pseudo-Riemannian metric.

**Definition 1.7** *A* pseudo-Riemannian metric *$g$ is a function that assigns to each point $p \in M$ a symmetric, non-degenerate bilinear form $g_p$ on $T_pM$ that is smooth in the following sense: for any chart $(U, x)$ around $p$ and indices $1 \leq i, j \leq n$, the functions*

$$g_{ij} : U \to \mathbb{R}$$
$$q \mapsto g_{ij}(q) = g_q(\partial_i, \partial_j)$$

*are smooth. If $g_p$ is positive definite for all $p \in M$ then we call it a* Riemannian metric*. The pair $(M, g)$ is known as a (pseudo) Riemannian manifold.*

Thanks to bilinearity, the functions $g_{ij}$ determine $g$ in $U$. If $v = a^i\partial_i$ and $w = b^j\partial_j$ are tangent vectors at $p \in M$, we see that

$$g_p(v, w) = g_p(a^i\partial_i, b^j\partial_j) = a^i b^j g_{ij}(p).$$

Thus, inside a chart one can think of $g$ as a function that maps a point $p$ to a symmetric, non-singular matrix $(g_{ij})$. We can also write $g$ using covectors: by definition the dual basis of the cotangent space satisfies $\mathrm{d}x^i(\partial_j) = \delta^i_j$, with the Kronecker delta on the right side, it follows that

$$g_{kl}\mathrm{d}x^k(v)\mathrm{d}x^l(w) = g_{kl}a^i b^j \mathrm{d}x^k(\partial_i)\mathrm{d}x^l(\partial_j) = g_{kl}a^i b^j \delta^k_i \delta^l_j = g_{ij}a^i b^j = g_p(v, w).$$

Defining the action of $g_{ij}\mathrm{d}x^i\mathrm{d}^j$ on a pair of vectors $v, w$ using the previous expression (evaluating the covectors and multiplicating the result) we can employ the notation $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, which is often used in physics. This discussion can be made rigorous using the language of tensors and fields, but this escapes the scope of this section.

At any point $p \in M$ we can find an orthonormal basis of $T_pM$ with respect to $g$. In this basis, the representation of $g$ is given by a diagonal matrix such that $g_{ii} = \pm 1$ (none of the entries are zero because $g$ is non-degenerate). The number of positive and negative entries is known as the *signature of* $g$; it is denoted either by an ordered pair of natural numbers $(a, b)$ where $a + b = n$ or a $n$-tuple of signs $(-, \ldots, -, +, \ldots, +)$, and it can be shown that it does not depend on the point $p$.

The usefulness of this concept lies in the ability to classify manifolds. Of particular interest in physics are *Lorentzian manifolds*, where the metric has signature $(1, n-1)$ (or $(n-1, 1)$ depending on the convention chosen). These metrics model spacetime in special and general relativity.

**Example** Euclidean $n$-dimensional space can be realized as a Riemannian manifold by taking $M = \mathbb{R}^n$ with the standard topology, a single chart $(\mathbb{R}^n, \mathrm{Id})$ and $g$ as the identity matrix.

**Example** Consider $M = \mathbb{R}^4$, with the usual topology and a single chart $(\mathbb{R}^4, \mathrm{Id})$. *Minkowski spacetime* is the 4-dimensional Lorentzian manifold $(M, \eta)$, where

$$\eta = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

This manifold models flat spacetime in special relativity.

**Example** The sphere $S^2 \subset \mathbb{R}^3$ has a standard Riemannian structure inherited from its ambient space. Given the polar angle $\phi \in (0, \pi)$ and azimuthal angle $\theta \in (0, 2\pi)$, one defines the usual spherical coordinates like this: define $V \coloneqq (0, 2\pi) \times (0, \pi)$ and the map $f(\theta, \phi) = (\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), \cos(\phi) \in S^2$. Since this is an homeomorphism onto its image $U \coloneqq f(V)$, it defines a chart $(U, x)$ through its inverse $x \coloneqq f^{-1}$. This chart alone does not cover $S^2$, but the complement $S^2 \backslash U$ is negligible and it is easily covered by additional charts derived from this one.

Given the parametrization $f$, its partial derivatives $\partial_\theta f$ and $\partial_\phi f$ define geometric vectors:

$$v = \partial_\theta f = (\cos(\theta)\sin(\phi), -\sin(\theta)\sin(\phi), \cos(\phi)),$$
$$w = \partial_\phi f = (\sin(\theta)\cos(phi), \cos(\theta)\cos(\phi), -\sin(\phi)).$$

These are clearly tangent to the sphere because $f \cdot v = f \cdot w = 0$, and they correspond to the usual directional derivatives $\nabla_v$ and $\nabla_w$, which are the basis vectors $\partial_\theta$ and $\partial_\phi$ at $p = f(\theta, \phi)$. With this identification the inner product in $\mathbb{R}^3$ induces a Riemannian structure on $S^2$: define $h_{11}(p)$ as the inner product $v \cdot v$, $h_{12}(p) = h_{21}(p) = v \cdot w$ and $h_{22}(p) = w \cdot w$. The full metric is

$$h = \mathrm{d}\phi^2 + \sin^2(\phi)\mathrm{d}\theta^2.$$

**Example** A simple scaling argument reveals that the metric for a sphere of radius $r > 0$ is $r^2 h$. With this we can write the flat metric in $\mathbb{R}^3$ in spherical coordinates by taking advantage of the independence between the radius $r$ and $\phi, \theta$:

$$g = \mathrm{d}r^2 + r^2 h.$$

A similar reasoning results in the expression for flat spacetime in spherical coordinates:

$$\eta = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 h.$$

It can be shown that a pseudo-Riemannian metric defines a unique object on the manifold, called the Levi-Civita connection. This is a differential operator that generalizes the directional derivative of a vector field in $\mathbb{R}^n$, and it characterizes geometric concepts like curvature which *a priori* are not defined on a pseudo-Riemannian manifold alone. With this machinery in place, it is possible to define classical differential operators such as the divergence, gradient and consequently the Laplacian.

**Definition 1.8** (Laplace-Beltrami operator) *The* Laplace-Beltrami *operator is the operator* $\Delta\colon C^\infty(M) \to C^\infty(M)$ *defined in local coordinates by the expression*

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j f \right),$$

*where* $|g| = |\det(g_{ij})|$ *and* $g^{ij}$ *refers to the components of the inverse of the matrix* $(g_{ij})$*.*

It is important to mention that this object does not depend on the chart used to calculate $\Delta f$. More sophisticated definitions avoid this issue entirely and coincide with the one given here (see Annex A for an alternative presentation).

**Example** The Laplace-Beltrami operator in euclidean space is the standard laplacian $\Delta$:

$$\Delta_g f = \Delta f = \sum_{i=1}^{n} \partial_{ii}^2 f.$$

**Example** We can compute the previous operator in spherical coordinates. The matrix $(g_{ij})$, its inverse $(g^{ij})$ and determinant $|g|$ are:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\phi) \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2}(\phi) \end{pmatrix}, \quad |g| = r^4 \sin^2(\phi).$$

Note that in the domain $\phi \in (0, \pi)$, $\sin(\phi)$ is non-negative and $\sqrt{|g|} = r^2 |\sin(\phi)| = r^2 \sin(\phi)$. Expanding the implicit sum:

$$\sqrt{|g|} \Delta_g f = \partial_r \left( \sqrt{|g|} g^{rj} \partial_j f \right) + \partial_\phi \left( \sqrt{|g|} g^{\phi j} \partial_j f \right) + \partial_\theta \left( \sqrt{|g|} g^{\theta j} \partial_j f \right).$$

Because $(g^{ij})$ is diagonal, the sum $g^{ij} \partial_j f$ reduces to $g^{ii} \partial_i f$ which translates to

$$\sqrt{|g|} \Delta_g f = \partial_r \left( \sqrt{|g|} g^{rr} \partial_j f \right) + \partial_\phi \left( \sqrt{|g|} g^{\phi\phi} \partial_\phi f \right) + \partial_\theta \left( \sqrt{|g|} g^{\theta\theta} \partial_\theta f \right).$$

Now we can replace $g^{rr} = 1, g^{\phi\phi} = r^{-2}, g^{\theta\theta} = r^{-2}\sin^{-2}(\phi)$ and expand the derivatives:

$$r^2 \sin(\phi)\Delta_g f = 2r\sin(\phi)\partial_r f + r^2 \sin(\phi)\partial_{rr} f + \cos(\phi)\partial_\phi f + \sin(\phi)\partial_{\phi\phi} f + \frac{1}{\sin(\phi)}\partial_{\theta\theta} f,$$

thus we arrive at the classical expression for the standard laplacian in spherical coordinates:

$$\Delta_g f = \partial_{rr} f + \frac{2}{r}\partial_r f + \frac{\cos(\phi)}{r^2\sin(\phi)}\partial_\phi f + \frac{1}{r^2}\partial_{\phi\phi} f + \frac{1}{r^2\sin^2(\phi)}\partial_{\theta\theta} f.$$

**Example** In Minkowski spacetime, the Laplace-Beltrami operator acting on $f \in C^\infty(M)$ is

$$\Delta_\eta f = -\frac{\partial^2 f}{\partial t^2} + \Delta f.$$

Under the appropiate sign convention, $\Delta_\eta$ is the wave operator $\Box$ (also known as the d'Alembert operator).

## 1.2. The Sine-Gordon equation

The Klein-Gordon equation is a linear PDE that arises in the study of relativistic waves, and it has been the subject of much interest thanks to the discovery of soliton solutions. Writing $\varphi'$ and $\dot{\varphi}$ for derivatives in space and time respectively, the $1+1$ dimensional Klein-Gordon equation is

$$\varphi'' - \varphi = \ddot{\varphi},$$

where the physical constants have been set to 1 for simplicity.

The Sine-Gordon equation (shortened to SG from now on) results from replacing $\varphi$ with a non-linear term $\sin(\varphi)$:

$$\varphi'' - \sin(\varphi) = \ddot{\varphi}.$$

The change of variables $2u = x + t$, $2v = x - t$ gives the equivalent formulation:

$$\partial_{uv}\varphi - \sin(\varphi) = 0.$$

The latter is the original form of the SG equation, introduced in the study of surfaces in $\mathbb{R}^3$ with constant Gaussian curvature $K = -1$ [11]. This is not its only application, however, as it is also used to describe varied physical phenomena: the dislocation of certain crystals, elementary particles and a series of rigid pendulums attached to a rubber band [12].

There are some identities in the equation that are important. The first is that for some solution $\varphi$, the function $\varphi + 2\pi$ will be a solution too thanks to the periodicity of $\sin(\,\cdot\,)$. The second is similar: the change $\varphi \mapsto \varphi + \pi$ leads to the equivalent equation

$$\varphi'' + \sin(\varphi) = \ddot{\varphi},$$

where the sign accompanying $\sin(\varphi)$ changed. Finally, rescaling $\varphi \mapsto 2\varphi$, $x \mapsto x/\sqrt{2}$, $t \mapsto t\sqrt{2}$ produces another formulation

$$\varphi'' + \sin(2\varphi) = \ddot{\varphi},$$

and this is the one we shall study.

In 1+3 dimensions, the same equation can be written using the standard Laplace operator:

$$\Delta\varphi - \ddot{\varphi} + \sin(2\varphi) = 0.$$

Using the wave operator with the Minkowski metric $\eta$ allows us to write

$$\Box_\eta \varphi + \sin(2\varphi) = 0,$$

which gives an immediate generalization to other geometries: one simply replaces $\Box_\eta$ with the wave operator $\Box_g$ corresponding to the given Lorentzian metric.

In this coordinate-free form, one deduces from the properties of $\Box_\eta$ that the possible solutions are symmetric under the action of the Poincaré group, the group of symmetries of Minkowski spacetime. This group is comprised of spacetime translations, rotations in space, reflections and another type of transformation called Lorentz boost. Translation invariance is obvious, as are space rotations due to rotational symmetry of the standard Laplace operator $\Delta$. Lorentz boosts originate from the special relativity postulate that the speed of light should not depend on the reference frame, and it is a well known fact that the wave operator is invariant under these transformations.

Among the solutions of this equation, one can find particular functions that exhibit soliton-like behavior. Consider the ansatz

$$\varphi(r,t) = 2\arctan\Big(e^{\gamma(r-vt)+\delta}\Big) - \frac{\pi}{2} \tag{1.1}$$

with velocity $v$, phase $\delta$ and a parameter $\gamma$ to be determined. The equation implies that $\gamma^2 = 2/(1-v^2)$: choosing the positive root leads to the *kink solution*, and the negative root to the *anti-kink solution*; these are the simplest among the Sine-Gordon solitons.

While it is true that $\varphi$, as presented in (1.1), is not localized in space, the quantity $\sin(2\varphi)$ does satisfy this condition. Going back to the pendulum model, $\varphi$ describes the angle of a pendulum at position $r$ and time $t$ (up to scaling and translation). Therefore, the periodicity of such a quantity results in a localized "twist" or angle fluctuation that asymptotically approaches a given "rest angle" from opposite sides for any given time.

Figure 1.1 shows this behavior more clearly in the stationary case $v = 0$. The curve describes the head of each pendulum attached to a rubber band lying in the $r$ axis, and we can see that the twist is localized. In addition to this, both the kink and anti-kink have a constant shape and it can be proven that their interactions only result in phase shifts, which makes them qualify as solitons in the sense discussed at the start of this chapter. Specific combinations of these solutions lead to *multi-solitons*, and while we shall not focus on them, they are nonetheless interesting in their own right.

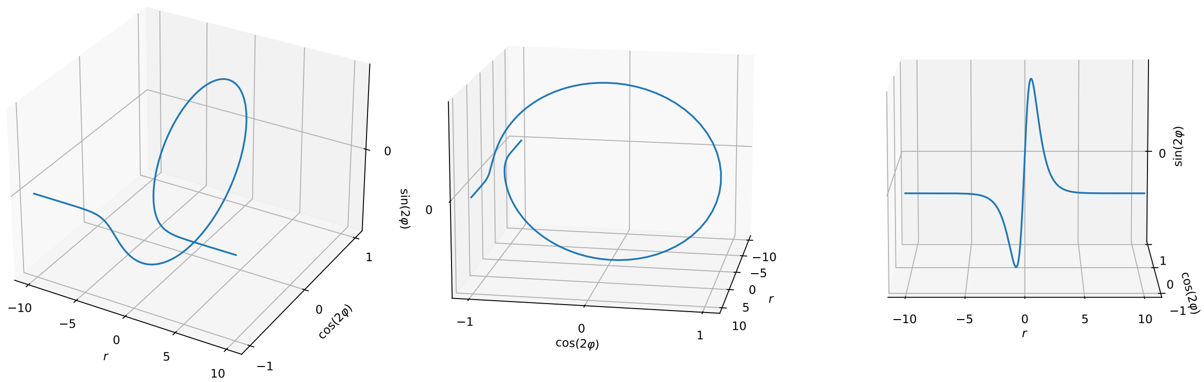


Figure 1.1: Visual representation of the stationary kink in the pendulum model.

## 1.3. Wormhole spacetime

Remember that Minkowski spacetime in spherical coordinates is described by the metric

$$\eta = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\phi^2 + \sin^2(\phi)\mathrm{d}\theta^2).$$

From here, introducing a wormhole to this spacetime becomes a two-step procedure. First, we extend the domain of $r$ to the real line $\mathbb{R}$: this gives us two copies of flat spacetime connected through an artificial singularity at $r = 0$, one for $r > 0$ and another for $r < 0$. The next step is replacing the factor $r^2$ accompanying the spherical metric $\mathrm{d}\omega^2$ with $r^2 + a^2$, where $a \neq 0$ is a parameter. This serves two purposes: it eliminates the singularity at the origin and replaces it with a spherical "neck" or "throat" of radius $a$. Explicitly, the metric in question is

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + (r^2 + a^2)(\mathrm{d}\phi^2 + \sin^2(\phi)\mathrm{d}\theta^2).$$

This model was introduced independently by Ellis [13] and Bronnikov [14], back in 1973. A complete justification beyond the intuition presented here escapes the scope of this thesis, but a detailed treatment can be found in [15].

We use Definition 1.8 to compute the Laplace-Beltrami operator of the wormhole metric. Note that this manifold is Lorentzian; thus, the result will be a hyperbolic differential operator. The procedure is straightforward: first, we write the matrix representation of $g$,

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 + a^2 & 0 \\ 0 & 0 & 0 & (r^2 + a^2)\sin^2(\phi) \end{pmatrix},$$

its inverse,

$$(g^{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (r^2+a^2)^{-1} & 0 \\ 0 & 0 & 0 & (r^2+a^2)^{-1}\sin^{-2}(\phi) \end{pmatrix},$$

and the modulus of its determinant $|g|$. As $\sin(\phi)$ is non-negative in $\phi \in (0, \pi)$, it follows that $\sqrt{|g|} = (r^2 + a^2)\sin(\phi)$.

The matrices are diagonal, simplifying the sums of the form $g^{ij}\partial_j f$. So far we have the expression

$$\sqrt{|g|}\Delta_g f = \partial_t\Big(\sqrt{|g|}g^{tt}\partial_t f\Big) + \partial_r\Big(\sqrt{|g|}g^{rr}\partial_r f\Big) + \partial_\phi\Big(\sqrt{|g|}g^{\phi\phi}\partial_\phi f\Big) + \partial_\theta\Big(\sqrt{|g|}g^{\theta\theta}\partial_\theta f\Big).$$

Using the values for $g^{ii}$ and $\sqrt{|g|}$ we get

$$\begin{aligned}(r^2+a^2)\sin(\phi)\Delta_g f = & -\partial_t\Big((r^2+a^2)\sin(\phi)\partial_t f\Big) + \partial_r\Big((r^2+a^2)\sin(\phi)\partial_r f\Big) \\ & + \partial_\phi(\sin(\phi)\partial_\phi f) + \partial_\theta\left(\frac{1}{\sin(\phi)}\partial_\theta f\right),\end{aligned}$$

or equivalently

$$\Delta_g f = -\partial_{tt} f + \frac{2r}{r^2+a^2}\partial_r f + \partial_{rr} f + \frac{\cos(\phi)}{(r^2+a^2)\sin(\phi)}\partial_\phi f + \frac{1}{r^2+a^2}\partial_{\phi\phi} f + \frac{1}{(r^2+a^2)\sin^2(\phi)}\partial_{\theta\theta} f.$$

This operator resembles the usual wave operator in spherical coordinates, but with terms involving $r^2$ replaced with $r^2 + a^2$: this is not a surprise, as this reproduces the modification introduced in the metric $g$. A notable difference is that, other than time translations, the new operator does not exhibit Poincaré symmetry. Although the time component of the operator is left unchanged, the parameter $a$ present in the spatial terms breaks the symmetry for transformations involving space components (translations, rotations and Lorentz boosts).

Because we are interested in the spherically symmetric problem, we impose the additional assumption that $f$ does not depend on the angles $\phi$ and $\theta$. In other words, $f = f(t, r)$ and $\partial_\phi f = \partial_\theta f = 0$. Finally, we switch notations to $\Box_g$ to remind us that the operator is hyperbolic:

$$\Box_g f = -\partial_{tt} f + \partial_{rr} f + \frac{2r}{r^2+a^2}\partial_r f.$$

Due to the symmetries imposed, the operator does not depend on the coordinates $\phi$ and $\theta$. We will dispense with them from now on and use $\phi, \varphi$ to refer to radial functions. The *Sine-Gordon equation on a wormhole* (SGWH equation) is simply $\Box_g \phi + \sin(2\phi) = 0$:

$$\ddot{\phi} = \phi'' + \frac{2r}{r^2+a^2}\phi' + \sin(2\phi), \tag{1.2}$$

where we used dots for differentiation in $t$ and primes for differentiation in $r$. Two important remarks: first, we have reduced our problem to 1+1 dimensions thanks to the symmetry hypothesis. Second, the term $r(r^2 + a^2)^{-1}$ decays as $a \to +\infty$ and the equation starts to resemble the classical SG equation in the limit. It is therefore a natural course of action to consider this problem and its solutions as perturbations of the simpler one, and we gather insights on these objects from this point of view.

Particularly, we are interested in the analogue of the kink solutions described in the preceding section and their properties for the stationary regime, as it is simpler. If $\phi$ does not depend on time, then $\dot{\phi} = \ddot{\phi} = 0$ and the equation reduces to

$$\phi'' + \frac{2r}{r^2 + a^2}\phi' + \sin(2\phi) = 0 \qquad \forall r \in \mathbb{R}; \tag{1.3}$$

we will see in this thesis that this additional term can be controlled in a way that makes solutions resemble those of the SG equation.

This task has been carried out in detail in [8]. The authors did an extensive study of the kinks of the SGWH equation, their stability and the linearized operator around them. In the case of a particular kink they also derive a decay rate. In addition, they analyze the soliton resolution conjecture using numerical simulations and provide numerical solutions to the SGWH equation.

Because the scope of [8] is vast, some (comparatively easier) results are mentioned in passing or the proofs are sketched. The aim of this work is to complement this article: we will provide rigorous demonstrations where applicable, focusing on the 1-kink. This study includes decay rates, convergence rates as a function of the parameter $a$, and the spectrum of the linearized operator.

# Chapter 2

# Results

This chapter can be divided into four parts that provide rigorous proofs to many of the results in [8]. First, we focus on the existence of solutions with asymptotic conditions in Section 2.1, then we move on to examine their convergence rates in Section 2.2. Section 2.3 discusses the linearized Sine-Gordon operator and its spectrum. Section 2.4 studies the dependence on the parameter $a$ for the kink solution and shows that it is unique, and later we turn our attention to the linearized operator in Section 2.5. Finally, in Section 2.6 we talk about the $n$-kink family of solutions.

## 2.1. Existence

Consider the stationary SG equation in flat spacetime:

$$\phi'' + \sin(2\phi) = 0 \qquad \forall r \in \mathbb{R}. \tag{2.1}$$

This is an ordinary differential equation, which is easier to solve than the dynamic problem. One can interpret $r \in \mathbb{R}$ as a new time variable, where the equation describes the motion of a particle. Multiplying equation (2.1) by $\phi'$ and integrating the result, one sees that the quantity $(\phi')^2/2 + \sin^2(\phi)$ is conserved; this can be viewed as the sum of the particle's kinetic and potential energies.

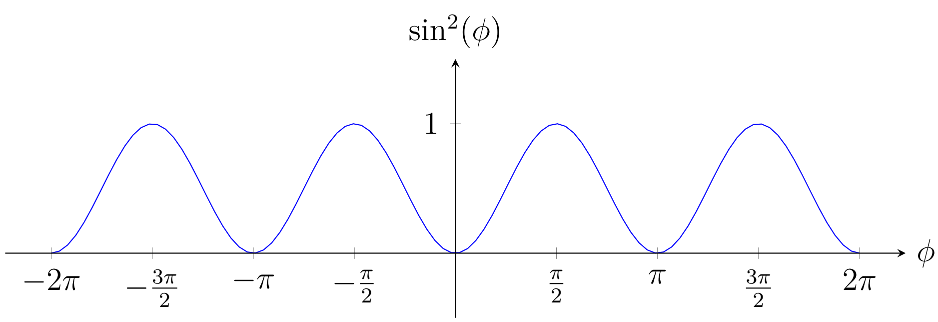


Figure 2.1: Potential energy as a function of the particle's position.

Now remember that the stationary SGWH equation (1.3) is

$$\phi'' + \frac{2r}{r^2 + a^2}\phi' + \sin(2\phi) = 0 \qquad \forall r \in \mathbb{R}.$$

The same procedure allows us to find the kinetic and potential energies, but this time something has changed: the term $2r/(r^2+a^2)\phi'$ introduced by the wormhole results in a loss of total energy in the system. This is not a surprise, because this expression is similar to those who model friction and other damping forces.

Our goal in this section is to find an analogue of the kink solution (1.1) for the SGWH equation: a solution of (1.3) that has a localized twist, that is, a function $H_a$ that satisfies the asymptotic conditions

$$\lim_{r\to\pm\infty} H_a(r) = \pm\frac{\pi}{2}.$$

We need some preliminary results, which we state now.

**Lemma 2.1** *For each $b \in (0,+\infty)$, the initial value problem*

$$(\tilde{P}) \begin{cases} \phi'' + \dfrac{2r}{r^2+a^2}\phi' + \sin(2\phi) = 0, \quad r \in (0,+\infty), \\ \phi(0) = 0, \\ \phi'(0) = b, \end{cases}$$

*has a unique solution that depends continuously on $b$.*

Proof. Consider the function

$$f\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$$
$$(r,\phi,\psi) \mapsto \left(\psi,\ -\frac{2r}{r^2+a^2}\psi - \sin(2\phi)\right)^\top,$$

which allows us to write $(\tilde{P})$ as

$$(\tilde{P}) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = f(r,\phi,\psi), \quad r \in (0,+\infty), \\ \begin{pmatrix}\phi(0)\\ \psi(0)\end{pmatrix} = \begin{pmatrix}0\\ b\end{pmatrix}. \end{cases}$$

The function $f$ is smooth, and its Jacobian is

$$J_{\phi,\psi} f(r,\phi,\psi) = \begin{pmatrix} 0 & 1 \\ -2\cos(2\phi) & -\frac{2r}{r^2+a^2} \end{pmatrix}.$$

The matrix components are uniformly bounded in $(r,\phi,\psi) \in \mathbb{R}\times\mathbb{R}^2$ by $K := \max\left\{2, \frac{1}{a}\right\}$; therefore, $f$ is globally Lipschitz in $(\phi,\psi)$ with constant $K$.

It follows that $f$ is continuous in the first variable and Lipschitz-continuous in the second (viewed as a variable in $\mathbb{R}^2$). Using the global existence and uniqueness theorem for ordinary differential equations, we deduce that problem $(\tilde{P})$ has a unique solution in $(0,+\infty)$. This solution depends continuously on the initial conditions, in the following sense: if $X = (\phi,\psi)^\top$ and $\tilde{X} = (\tilde{\phi},\tilde{\psi})^\top$ are solutions of $(\tilde{P})$ with Cauchy data $X(0) = X_0$ and $\tilde{X}(0) = \tilde{X}_0$ respectively, then for a certain constant $L > 0$:

$$\left|X(r) - \tilde{X}(r)\right| \leq \left|X_0 - \tilde{X}_0\right| e^{L|r|}.$$

This proves the lemma. □

**Remark** Solutions of $(\tilde{P})$ cannot have zero derivative at potential peaks. More specifically: if $\phi$ is a solution to $(\tilde{P})$, then for all $r > 0$ and $n \in \mathbb{N}$ (not necessarily odd), $\phi'(r) = 0$ implies $\phi(r) \neq n\pi/2$. Otherwise, uniqueness affirms that $\phi$ equals the constant solution $\phi \equiv n\pi/2$, which has different initial conditions.

In addition to the previous lemma, we derive an energy-like formula for the differential equation.

**Proposition 2.1** *The solution to* $(\tilde{P})$ *satisfies the identity*

$$\frac{1}{2}(\phi'(r))^2 + \sin^2(\phi(r)) + \int_0^r \frac{2s}{s^2 + a^2}(\phi'(s))^2 \mathrm{d}s = \frac{1}{2}b^2 \qquad \forall r \geq 0. \tag{2.2}$$

Proof. First we multiply the equation by $\phi$:

$$\phi'\phi'' + \frac{2r}{r^2 + a^2}(\phi')^2 + \sin(2\phi)\phi' = 0.$$

Integrate this expression over $(0, r)$ to see that

$$\frac{1}{2}(\phi'(r))^2 - \frac{1}{2}b^2 - \frac{1}{2}\cos(2\phi(r)) + \frac{1}{2} + \int_0^r \frac{2s}{s^2 + a^2}(\phi'(s))^2 \mathrm{d}s = 0 \qquad \forall r \geq 0.$$

Rearranging terms we get:

$$\frac{1}{2}(\phi'(r))^2 + \frac{1 - \cos(2\phi(r))}{2} + \int_0^r \frac{2s}{s^2 + a^2}(\phi'(s))^2 \mathrm{d}s = \frac{1}{2}b^2 \qquad \forall r \geq 0.$$

Using trigonometric identities we deduce that $\frac{1}{2}(1 - \cos(2\phi)) = \sin^2(\phi)$, concluding the proof. □

For each $b > 0$, let $\phi$ be the unique solution to $(\tilde{P})$ given previously.

**Lemma 2.2** *Suppose* $|\phi|$ *is bounded by a constant* $C$ *for all* $b > 0$*. Then*

$$\phi'(r) \geq b - \frac{4C}{a} - r \qquad \forall r > 0.$$

Proof. We derive a lower bound on $r_0$ that depends on $b$. Indeed, thanks to the fundamental theorem of calculus we have

$$\phi'(r) = b + \int_0^r \phi''(s)\mathrm{d}s = b - \int_0^r \frac{2s}{s^2 + a^s}\phi'(s)\mathrm{d}s - \int_0^r \sin(2\phi(s))\mathrm{d}s \qquad \forall r > 0.$$

Integration by parts shows that:

$$\int_0^r \frac{2s}{s^2+a^s}\phi'(s)\mathrm{d}s = \frac{2r}{r^2+a^2}\phi(r) + 2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s.$$

Replacing in the previous expression:

$$\phi'(r) = b - \frac{2r}{r^2+a^2}\phi(r) - 2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s - \int_0^r \sin(2\phi(s))\mathrm{d}s \qquad \forall r > 0.$$

Let $C > 0$ be the constant that satisfies $|\phi(r)| < C$ for all $r > 0$. Then

$$-\frac{2r}{r^2+a^2}\phi(r) \geq -\frac{2Cr}{r^2+a^2}.$$

Clearly, $-\sin(2\phi(s)) \geq -1$. This gives the inequality

$$\phi'(r) \geq b - \frac{2Cr}{r^2+a^2} - 2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s - r \qquad r > 0.$$

What follows is an analysis of the integral that is left. If $r < a$, then $(s^2-a^2)/(s^2+a^2)^2$ is negative in $(0,r)$, implying that

$$-2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s \geq 2C\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\mathrm{d}s = -2C\left.\frac{s}{s^2+a^2}\right|_0^r = -\frac{2Cr}{r^2+a^2}.$$

If $r > a$, then we split the integral:

$$-2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s = -2\int_0^a \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s - 2\int_a^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s.$$

First, we have that

$$-2\int_0^a \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s \geq 2C\int_0^a \frac{s^2-a^2}{(s^2+a^2)^2}\mathrm{d}s = -2C\left.\frac{s}{s^2+a^2}\right|_0^a = -\frac{C}{a}.$$

In the interval $(a,r)$, the quantity $(s^2-a^2)/(s^2+a^2)^2$ is positive:

$$-2\int_a^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s \geq -2C\int_a^r \frac{s^2-a^2}{(s^2+a^2)^2}\mathrm{d}s = 2C\left.\frac{s}{s^2+a^2}\right|_a^r = \frac{2Cr}{r^2+a^2} - \frac{C}{a}.$$

Combine these inequalities to see that

$$-2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s \geq \frac{2Cr}{r^2+a^2} - \frac{2C}{a}$$

No matter how $r$ compares to $a$, this lower bound holds:

$$-2\int_0^r \frac{s^2-a^2}{(s^2+a^2)^2}\phi(s)\mathrm{d}s \geq -\frac{2Cr}{r^2+a^2} - \frac{2C}{a} \qquad \forall r > 0.$$

These bounds imply that

$$\phi'(r) \geq b - \frac{2Cr}{r^2+a^2} - \frac{2Cr}{r^2+a^2} - \frac{2C}{a} - r = b - \frac{2C}{a} - \frac{4Cr}{r^2+a^2} - r \qquad \forall r > 0.$$

The expression $r/(r^2+a^2)$ is bounded above by $1/2a$ for $r > 0$, so we can derive a simpler bound:

$$\phi'(r) \geq b - \frac{2C}{a} - 4C\frac{1}{2a} - r = b - \frac{4C}{a} - r \qquad \forall r > 0.$$

As a remark, $C$ may depend on $b$. Where this inequality becomes useful is in arguments by contradiction, where we assume $|\phi|$ is bounded for all $b > 0$. □

**Lemma 2.3** *Let $n > 0$ be an odd natural number, and $b > 0$ an initial velocity. If there is a $r_0 > 0$ such that $\phi(r_0) \in \left((n-2)\frac{\pi}{2}, \frac{n\pi}{2}\right)$ and $\phi'(r_0) = 0$, then*

$$\left|\phi(r) - (n-1)\frac{\pi}{2}\right| < \left|\phi(r_0) - (n-1)\frac{\pi}{2}\right| \qquad \forall r > r_0.$$

*In other words, if $\phi$ stops in a potential well, it will remain in said potential well.*

Proof. To start, we provide some key remarks. The first is that $\phi(r_0) \neq (n-1)\pi/2$, because uniqueness would imply $\phi \equiv (n-1)\pi/2$. Moreover, we know that

$$\phi''(r_0) = -\frac{2r_0}{r_0^2+a^2}\phi'(r_0) - \sin(2\phi(r_0)) = -\sin(2\phi(r_0)).$$

There are two options: if $\phi(r_0) \in ((n-1)\pi/2, n\pi/2)$ (the right half of the potential well), then $\sin(2\phi(r_0)) > 0$ and $\phi''(r_0) < 0$. If $\phi(r_0) \in ((n-2)\pi/2, (n-1)\pi/2)$, then $\sin(2\phi(r_0)) < 0$ and $\phi''(r_0) > 0$. In either case, $\phi$ is pulled towards the potential's local minimum $(n-1)\pi/2$ and $|\phi(r) - (n-1)\pi/2|$ decreases immediately after $r_0$.

Because $\phi'$ is non-zero immediately after $r_0$, the following inequality is strict:

$$\frac{1}{2}b^2 - \int_0^{r_0} \frac{2s}{s^2+a^2}(\phi'(s))^2 \mathrm{d}s > \frac{1}{2}b^2 - \int_0^{r} \frac{2s}{s^2+a^2}(\phi'(s))^2 \mathrm{d}s \qquad \forall r > r_0.$$

Thanks to the energy identity (2.2), we have the inequality

$$\sin^2(\phi(r_0)) = \frac{1}{2}(\phi'(r_0))^2 + \sin^2(\phi(r_0)) > \frac{1}{2}(\phi'(r))^2 + \sin^2(\phi(r)) \geq \sin^2(\phi(r)) \qquad \forall r > r_0.$$

Meanwhile, a simple trigonometric identity states that

$$2\sin^2(\phi(r)) = 1 - \cos(2\phi(r)) = 1 - \cos\left(2\left(\phi(r) - (n-1)\frac{\pi}{2}\right) + (n-1)\pi\right).$$

Should there be a $r_1 > r_0$ such that $|\phi(r_1) - (n-1)\pi/2| = |\phi(r_0) - (n-1)\pi/2|$, since $n$ is odd and $\cos(\,\cdot\,)$ is an even function, it follows that

$$2\sin^2(\phi(r_0)) = 1 - \cos\left(2\left|\phi(r_0) - (n-1)\frac{\pi}{2}\right|\right) = 1 - \cos\left(2\left|\phi(r_1) - (n-1)\frac{\pi}{2}\right|\right) = 2\sin^2(\phi(r_1)),$$

and the contradiction is evident. From this, we deduce that there cannot be a $r_1 > r_0$ where $|\phi(r_1) - (n-1)\pi/2| = |\phi(r_0) - (n-1)\pi/2|$; continuity of $\phi$ gives the desired inequality:

$$\left|\phi(r) - (n-1)\frac{\pi}{2}\right| < \left|\phi(r_0) - (n-1)\frac{\pi}{2}\right| \qquad \forall r > r_0.$$

□

This lemma follows from the observation that a particle modeled by the SGWH equation loses energy. For example, in Figure 2.2, after the particle $\phi$ stops in the instant $r_0$, its potential energy will never ascend beyond $\sin^2(\phi(r_0)) = 2/3$. This barrier is represented by the dashed line.

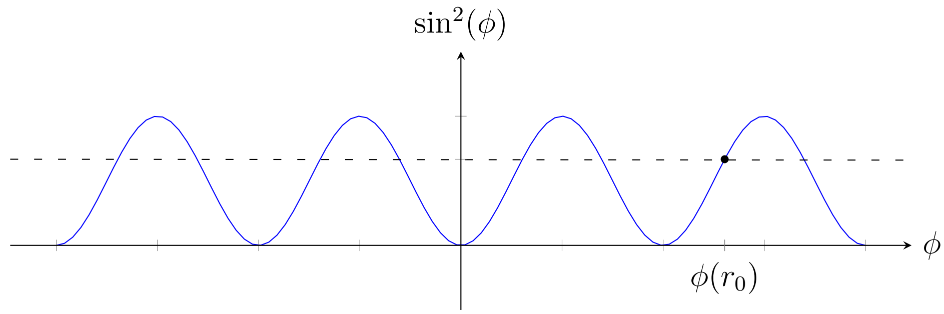


Figure 2.2: Particle stopping at $r_0$.

**Corollary 2.1** *If $b > 0$, then $\phi$ is bounded below by $-\pi/2$.*

Proof. For $\phi$ to become negative, its derivative must vanish at some point. Uniqueness states that $\phi$ cannot stop at potential peaks, and so there must be a $r_0 > 0$ and an odd natural $n \in \mathbb{N}$ where $\phi(r_0) \in ((n-2)\pi/2, n\pi))$ and $\phi'(r_0) = 0$. Thanks to the preceding lemma, we know that $\phi(r)$ will remain in $((n-2)\pi/2, n\pi)$ for all $r > r_0$; but $(n-2)\pi/2 \geq -\pi/2$, and since $\phi(r) > 0$ for $r \in (0, r_0)$ we conclude that $-\pi/2 < \phi(r)$ for all $r > 0$. □

Existence is proven using a shooting argument. Going back to the particle analogy, for a given initial velocity $b$, the corresponding trajectory $\phi$ will fall into one of three possible categories:

1. If the particle does not have enough energy to surmount the local maximum at $\pi/2$, it will stay in the potential well, oscillating and losing energy.

2. In contrast, if $b$ is large, the particle will surmount the hill, never to return to $\pi/2$.

3. For a certain velocity, will approach $\pi/2$ in infinite time.

The third outcome is what we are looking for. We define the sets $A$ and $B$, that capture the first and second cases, and show that there must be a $b$ in the complement of their union: this will be the initial velocity of the desired solution.

**Lemma 2.4** *The set defined as*

$$A \coloneqq \{b > 0 \colon \exists r > 0, \phi(r) > \pi/2\}$$

*is both open and non-empty.*

Proof. Continuous dependence on initial conditions shows that $A$ is open. Now suppose that $A = \varnothing$; this means that for all $b > 0$ and $r > 0$, $\phi(r) \leq \pi/2$, and the lower bound from the corollary gives us the inequality $|\phi| < \pi/2$. This allows us to invoke Lemma 2.2 and state that

$$\phi'(r) \geq b - \frac{2\pi}{a} - r \qquad \forall r > 0.$$

Integrate both sides from 0 to $r$. Since $\phi(0) = 0$, it is true that for any $b > 0$,

$$\phi(r) \geq \left(b - \frac{2\pi}{a}\right) r - \frac{1}{2} r^2 \qquad \forall r > 0.$$

The right side in this inequality is a parabola, with a maximum attained at $r = b - 2\pi/a$. Evaluate at this point to see that

$$\phi\left(b - \frac{2\pi}{a}\right) \geq \frac{1}{2}\left(b - \frac{2\pi}{a}\right)^2.$$

Choose $b > \sqrt{\pi} + 2\pi/a$ to deduce that

$$\phi\left(b - \frac{2\pi}{a}\right) \geq \frac{\pi}{2},$$

which contradicts our assumption that $A$ is empty. This demonstrates that $A \neq \varnothing$. □

**Lemma 2.5** *The set*

$$B := \{b > 0 \colon \exists r > 0, \phi(r) < \pi/2 \wedge \phi'(r) < 0\}$$

*is open and non-empty.*

Proof. Once again, $B$ is open thanks to continuous dependence on initial conditions. Assume that, for all $b > 0$ and $r > 0$, $\phi'(r) \geq 0$ or $\phi(r) \geq \pi/2$. Fix $b \ll 1$, and suppose there exists $\tilde{r}_1 > 0$ such that $\phi(\tilde{r}_1) \geq \pi/2$. Because $\phi$ is continuous, the intermediate value theorem guarantees the existence of a $r_1 > 0$ that satisfies $\phi(r_1) = \pi/2$. We take said $r_1$ to be minimal so that $\phi(r) < \pi/2$, and by hypothesis, $\phi'(r) \geq 0$ in $(0, r_1)$. Evaluate the energy identity (2.2) at $r_1$; the integral is non-negative, which gives us the inequality

$$0 \leq \frac{1}{2}(\phi'(r_1))^2 \leq \frac{1}{2} b^2 - 1.$$

If $b$ is sufficiently small this does not make sense, because the right side would be negative. Then, for $b$ close to 0, it is true that $\phi < \pi/2$ and the claim $B = \varnothing$ reduces to the following: for all $r > 0$, $\phi' \geq 0$. This also implies that

$$\phi'' = -\sin(2\phi) - \frac{2r}{r^2 + a^2}\phi' \leq 0.$$

Particularly, $\phi'$ is decreasing and bounded below, so it has a limit at infinity. Said limit must be 0: if that were not the case, $\phi$ would grow to infinity as $r \to +\infty$, contradicting the fact that $\phi < \pi/2$.

The solution $\phi$ is increasing and bounded above by $\pi/2$, so it has a limit at infinity:

$$L := \lim_{r\to+\infty} \phi(r) \in \left(0, \frac{\pi}{2}\right].$$

Suppose $L < \pi/2$: then $\sin(2L) > 0$, and there is some $\varepsilon > 0$ and $R > 0$ such that, for all $r > R$, $\sin(2\phi(r)) > \varepsilon$. We can use the equation satisfied by $\phi$ to see that

$$0 = \phi'' + \frac{2r}{r^2+a^2}\phi' + \sin(2\phi) > \phi'' + \sin(2\phi) > \phi'' + \varepsilon \qquad \forall r > R.$$

A simple application of the fundamental theorem of calculus shows that integrating this inequality in the interval $(R, r)$ leads to

$$\phi'(R) - \phi'(r) > \varepsilon(r - R) \qquad \forall r > R.$$

As mentioned earlier, $\phi'$ tends to 0 at infinity. Then, the left side remains bounded, while the right side grows linearly in $r$. This is not possible, so $L$ must equal $\pi/2$.

These results are valid for any sufficiently small $b$, under the assumption that $B = \varnothing$. Take the limit in equality (2.2) to see that

$$0 < \int_0^{+\infty} \frac{2s}{s^2+a^2}(\phi'(s))^2 \mathrm{d}s = \frac{1}{2}b^2 - \sin^2(L) = \frac{1}{2}b^2 - 1.$$

But we have chosen $b \ll 1$, contradicting the previous inequality. It follows that $B$ must be non-empty, concluding the proof. □

The set $A$ describes the trajectories where $\phi$ surpasses our target $\pi/2$. Likewise, the set $B$ captures the trajectories where $\phi$ stops before reaching $\pi/2$. If we can find an initial velocity that belongs to neither set, we will have our candidate for the desired solution.

**Lemma 2.6** *The sets $A$ and $B$ are disjoint.*

Proof. Let $b \in B$. By definition its associated trajectory $\phi$ has a point $r_0$ where $\phi(r_0) < \pi/2$ and $\phi'(r_0) = 0$. Lemma 2.3 says that for $n = 1$:

$$|\phi(r)| < |\phi(r_0)| < \frac{\pi}{2} \qquad \forall r > r_0.$$

From this we deduce that, if $\phi$ were to surpass $\pi/2$, it would have to do so before $r_0$. But in that case, since $\phi(r_0) < \pi/2$, there must be an $r_1 < r_0$ such that $\phi(r_1) > \pi/2$ and $\phi'(r_1) = 0$: to come back to $\phi(r_0)$, $\phi'$ must change sign. The same lemma states that $\phi$ would forever remain in the potential well corresponding to $\pi$, which is clearly not the case because $b \in B$. In other words, $\phi$ cannot surpass $\pi/2$ before nor after $r_0$, that is to say $\phi \le \pi/2$. Thus, $b \notin A$ and the result is proven. □

We are now ready to prove the existence of a solution for the limit problem.

**Theorem 2.1** *The following problem*

$$(P)\begin{cases}\phi'' + \dfrac{2r}{r^2+a^2}\phi' + \sin(2\phi) = 0, \quad r \in (0,+\infty),\\ \phi(0) = 0,\\ \lim\limits_{r\to+\infty}\phi(r) = \dfrac{\pi}{2},\end{cases}$$

*has a solution in* $[0,+\infty)$.

Proof. Remember that the sets $A$ and $B$ are defined as:

$$A \coloneqq \{b > 0\colon \exists r > 0, \phi(r) > \pi/2\},$$
$$B \coloneqq \{b > 0\colon \exists r > 0, \phi'(r) < 0 \wedge \phi(r) < \pi/2\}.$$

The preceding lemmas state that they are non-empty, disjoint, open subsets of $(0,+\infty)$. Because $(0,+\infty)$ is connected, there exists at least one positive real number that does not belong to $A \cup B$: we call this element $b$.

On one hand, we know that the associated solution satisfies $\phi(r) \leq \pi/2$ for all $r \geq 0$, because $b \notin A$. On the other, if $\phi(r) < \pi/2$ then $\phi'(r) \geq 0$, because $b \notin B$. It is obvious that $\phi'(r)$ cannot be negative; that implies $\phi(r) = \pi/2$, and immediately after $\phi$ would return to $(0,\pi/2)$, where the derivative is non-negative: continuity assures that this cannot happen. It follows that $\phi'(r) \geq 0$ for all $r > 0$. In addition to this, if $\phi(r) = \pi/2$ for some $r$, then either $\phi'(r) > 0$, meaning $\phi$ would surpass $\pi/2$ immediately after, or $\phi'(r) = 0$, implying that $b = 0$ because of uniqueness of trajectories. Either case is a contradiction, and we deduce that $\phi < \pi/2$.

We know $\phi < \pi/2$ and $\phi' \geq 0$ for all $r > 0$, which are the same hypothesis we used in the proof of Lemma 2.5 to prove that $\phi$ has a limit

$$L \coloneqq \lim_{r\to+\infty}\phi(r) \in \left(0, \frac{\pi}{2}\right].$$

We repeat the argument used to show $L = \pi/2$: we know $\phi(0) = 0$, and since $\phi$ is non-decreasing, we deduce that $\phi(r) \in (0, L]$ for positive $r$. If $L < \pi/2$, then there exists an $R > 0$ and $\varepsilon > 0$ such that $\sin(2\phi) > \varepsilon$ for all $r > R$. The equation for $\phi$ implies that

$$\varepsilon < -\phi'' \qquad \forall r > R;$$

Integrating this inequality from $R$ to $r$ gives the expression

$$\phi'(R) - \phi'(r) > \varepsilon(r - r_0).$$

We can take $r$ large enough so that the right side grows to infinity, but the left side is bounded: this contradiction shows that $L = \pi/2$.

We have found a solution of $(\tilde{P})$ whose limit at infinity is $\pi/2$, which makes it a solution of $(P)$. □

**Corollary 2.2** *The problem*

$$(\overline{P})\begin{cases}\phi'' + \dfrac{2r}{r^2+a^2}\phi' + \sin(2\phi) = 0, & r \in \mathbb{R},\\ \displaystyle\lim_{r\to\pm\infty}\phi(r) = \pm\frac{\pi}{2}, & \end{cases}$$

*has a solution.*

Proof. Let $\phi$ be the solution found in the previous lemma and consider its odd extension, which we call $H_a$. Explicitly:

$$H_a(r) = -\phi(-r) \qquad \forall r \in (-\infty, 0].$$

Using the chain rule, we see that in this interval:

$$H_a''(r) + \frac{2r}{r^2+a^2}H_a'(r) + \sin(2H_a(r)) = -H_a''(-r) + \frac{2r}{r^2+a^2}H_a'(-r) + \sin(-2H_a(-r)).$$

Because $H_a$ is a solution in $-r \in [0, +\infty)$, it follows that

$$-\left(H_a''(-r) + \frac{2(-r)}{r^2+a^2}H_a'(-r) + \sin(2H_a(-r))\right) = 0$$

and we deduce that $H_a$ satisfies the equation for all $r \in \mathbb{R}$. Also, since $\phi(r) \to \pi/2$ as $r \to +\infty$, it follows from the definition that $H_a(r) \to -\pi/2$ as $r \to -\infty$. □

We call the function $H_a$ the 1-*kink*, since it travels a total distance of $n\pi$ with $n = 1$. Now that we know that a solution exists, it is natural to ask if it is unique: this question will be answered in Section 2.4.

## 2.2. Growth

In this section we study the behavior of the 1-kink as $|r|$ grows to infinity. It suffices to do this for $r \to +\infty$, due to the anti-symmetric nature of the solution. With this in mind, we turn our attention to the difference $\pi/2 - H_a$, where $H_a$ is the solution of $(\overline{P})$. Due to the fact that $H_a$ tends to $\pi/2$ at positive infinity, we know that this quantity decays to 0.

**Definition 2.1** *Define the function $\varphi$ as*

$$\varphi := \pi/2 - H_a.$$

*It satisfies the conditions*

$$\lim_{r\to-\infty}\varphi(r) = \pi, \qquad \lim_{r\to+\infty}\varphi(r) = 0.$$

This is merely for convenience. Convergence of $H_a$ to $\pi/2$ is equivalent to $\varphi$ converging to 0, so we search for convergence rates for $\varphi$ instead. Any bounds of the form $|\varphi(r)| < O(g(r))$ as $r \to +\infty$ for some function $f$ will translate to $|\pi/2 - H_a(r)| < O(g(r))$.

**Theorem 2.2** *For all sufficienlty small $\delta > 0$, there is a $R = R(\delta) > 0$ such that*

$$|\varphi(r)| \le Me^{-\sqrt{2-\delta}r} \qquad \forall r > R,$$

*where $M := \pi e^{\sqrt{2-\delta}R}$. In other words, $\varphi(r) = O(e^{-\sqrt{2-\delta}r})$ as $r \to +\infty$ for all sufficiently small $\delta > 0$.*

Proof. The differential equation satisfied by the new function is

$$\varphi'' + \frac{2r}{r^2+a^2}\varphi' - \sin(2\varphi) = -H_a'' - \frac{2r}{r^2+a^2}H_a' - \sin(2\phi) = 0.$$

In other words, if we define the differential operator

$$L_\varphi := -\frac{\partial^2}{\partial r^2} - \frac{2r}{r^2+a^2}\frac{\partial}{\partial r} + \frac{\sin(2\varphi(r))}{\varphi(r)}$$

then $L\varphi = 0$. It is obvious that $L_\varphi$ is elliptic. By definition, $\varphi$ is positive and tends to 0, and so the term $\sin(2\varphi(r))/\varphi(r)$ approaches 2 from below. Fix $\delta > 0$ close to 0: there exists a $R(\delta) > 0$ such that for all $r > R$, $\sin(2\varphi(r))/\varphi(r) > 2 - \delta/2$.

Define the auxiliary function

$$\psi_{M,\varepsilon}(r) := Me^{-\sigma(r-R)} + \varepsilon e^{\sigma(r-R)},$$

where $\sigma$, $M$ are positive constants to be determined and $\varepsilon$ is a positive parameter. We will use this function to obtain a bound for $\varphi$ from $L_\varphi$, given appropriate choices of constants. The derivatives of $\psi_{M,\varepsilon}$ are

$$\psi_{M,\varepsilon}'(r) = -\sigma Me^{-\sigma(r-R)} + \sigma\varepsilon e^{\sigma(r-R)}, \quad \psi_{M,\varepsilon}''(r) = \sigma^2 Me^{-\sigma(r-R)} + \sigma^2\varepsilon e^{\sigma(r-R)}.$$

Then

$$\begin{aligned} L_\varphi\psi_{M,\varepsilon} =& -\sigma^2 Me^{-\sigma(r-R)} - \sigma^2\varepsilon e^{\sigma(r-R)} - \frac{2r}{r^2+a^2}\left(-\sigma Me^{-\sigma(r-R)} + \sigma\varepsilon e^{\sigma(r-R)}\right) \\ &+ \frac{\sin(2\varphi)}{\varphi}\left(Me^{-\sigma(r-R)} + \varepsilon e^{\sigma(r-R)}\right), \end{aligned}$$

and it follows that

$$L_\varphi\psi_{M,\varepsilon} = \left(\frac{\sin(2\varphi)}{\varphi} - \sigma^2\right)\left(Me^{-\sigma(r-R)} + \varepsilon e^{\sigma(r-R)}\right) + \frac{2\sigma r}{r^2+a^2}\left(Me^{-\sigma(r-R)} - \varepsilon e^{\sigma(r-R)}\right),$$

which is equivalent to

$$L_\varphi\psi_{M,\varepsilon} = \left(\frac{\sin(2\varphi)}{\varphi} - \sigma^2 + \frac{2\sigma r}{r^2+a^2}\right)Me^{-\sigma(r-R)} + \left(\frac{\sin(2\varphi)}{\varphi} - \sigma^2 - \frac{2\sigma r}{r^2+a^2}\right)\varepsilon e^{\sigma(r-R)}.$$

For reasons that will be clear soon, we want this quantity to be non-negative on the set $I := (R, \overline{r})$ for arbitrarily large $\overline{r} > R$. If we choose $\sigma = \sqrt{2-\delta}$, then we have the following lower bound:

$$\frac{\sin(2\varphi)}{\varphi} - \sigma^2 > 2 - \frac{\delta}{2} - \sigma^2 > 2 - \frac{\delta}{2} - 2 + \delta > \frac{\delta}{2} > 0 \qquad \forall r > R.$$

Also, if $R$ is sufficiently large (redefining it if needed, but keeping $\delta$ fixed), we have that

$$\frac{2\sigma r}{r^2 + a^2} < \frac{\delta}{2} \qquad \forall r > R$$

and it follows that $L_\varphi \psi_{M,\varepsilon} \geq 0$ in $I$. This choice of $R$ can be made independently of $a$.

From the previous analysis we see that

$$L_\varphi(-\varphi - \psi_{M,\varepsilon}) = L_\varphi(\varphi - \psi_{M,\varepsilon}) = -L_\varphi \psi_{M,\varepsilon} \leq 0$$

holds in $I$. The weak maximum principle for elliptical operators asserts that

$$\max_{r \in \overline{I}} (\varphi(r) - \psi_{M,\varepsilon}(r)) \leq \max_{r \in \partial I} (\varphi(r) - \psi_{M,\varepsilon}(r))^+,$$

where the superscript + denotes the positive part of a function. The boundary of $I$ consists of two points, $R$ and $\overline{r}$: the values are

$$\varphi(R) - \psi_{M,\varepsilon}(R) = \varphi(R) - M - \varepsilon, \qquad \varphi(\overline{r}) - \psi_{M,\varepsilon}(\overline{r}) = \varphi(\overline{r}) - Me^{-\sigma(\overline{r}-R)} - \varepsilon e^{\sigma(\overline{r}-R)}.$$

The auxiliary function satisfies $\|\varphi\|_\infty = \pi$, and the supremum (over $\mathbb{R}$) is not attained. The choice $M = \pi$ translates to $\varphi(R) - \psi_{M,\varepsilon}(R) < 0$ and the maximum over $\partial I$ reduces to

$$\max_{r \in \partial I} (\varphi(r) - \psi_{M,\varepsilon}(r))^+ = (\varphi(\overline{r}) - \psi_{M,\varepsilon}(\overline{r}))^+.$$

So far we have the inequality

$$\max_{r \in [R, \overline{r}]} (\varphi(r) - \psi_{M,\varepsilon}(r)) \leq \left(\varphi(\overline{r}) - \pi e^{-\sigma(\overline{r}-R)} - \varepsilon e^{\sigma(\overline{r}-R)}\right)^+,$$

which is true for arbitrary $\overline{r} > R$. In particular, if $\overline{r}$ is large compared to $R$ (which is fixed), then the expression inside the parenthesis on the right becomes negative. Taking the limit $\overline{r} \to +\infty$, this means that its positive part is equal to zero, and the maximum is non-positive for all $\varepsilon > 0$:

$$\max_{r \geq R} (\varphi(r) - \psi_{M,\varepsilon}(r)) \leq 0.$$

The last inequality is interchangeable with the statement

$$\varphi(r) \leq \psi_{M,\varepsilon}(r) = \pi e^{-\sigma(r-R)} + \varepsilon e^{\sigma(r-R)} \qquad \forall r \geq R.$$

This is true for all positive $\varepsilon$. The fact that $M, \sigma$ and $R$ only depend on $\delta$ justifies taking the limit $\varepsilon \to 0$, and we deduce that

$$\varphi(r) \leq \pi e^{\sigma R} e^{-\sigma r} \qquad \forall r \geq R.$$

Remember that $\phi$ is positive by definition, so the previous inequality is also true for $|\varphi|$. We have shown $\varphi(r) = O(e^{-\sigma r})$ as $r \to +\infty$ if $\sigma = \sqrt{2-\delta}$, where $\delta > 0$ is small; redefine $M = \pi e^{\sigma R}$ to conclude. Note that we cannot take the limit as $\delta$ tends to 0 in the previous inequality, because $R$ depends on it. □

**Corollary 2.3** *$\varphi(r) = O(e^{-(\sqrt{2}-\delta)r})$ as $r \to +\infty$ for all sufficiently small $\delta > 0$.*

Proof. If $\delta < 2\sqrt{2} - 1$, the following inequality is elementary:

$$(\sqrt{2} - \delta)^2 = 2 - 2\sqrt{2}\delta + \delta^2 < 2 - \delta$$

Furthermore, if $\delta < \sqrt{2}$, we can take the square root $\sqrt{2} - \delta < \sqrt{2-\delta}$. Since the exponential is monotone increasing we get

$$e^{-\sqrt{2-\delta}r} < e^{-(\sqrt{2}-\delta)r} \qquad \forall r > 0.$$

If $\delta$ is small, Theorem 2.2 states that there is a $R > 0$ such that

$$|\varphi(r)| \leq Me^{-\sqrt{2-\delta}r} < Me^{-(\sqrt{2}-\delta)r} \qquad \forall r > R,$$

that is, $\varphi = O(e^{-(\sqrt{2}-\delta)r})$. □

One would like to refine this rate of convergence and assert that $\sigma = \sqrt{2}$. Remember that $\varphi$ as defined previously satisfies the equation

$$\varphi'' + \frac{2r}{r^2 + a^2}\varphi' - \sin(2\varphi) = 0,$$

and $\varphi$ decreases to 0 at positive infinity. We study a slightly different function given by the expression

$$\varphi = \frac{\psi}{(r^2 + a^2)^{\frac{1}{2}}}.$$

The motive behind this change is that $\psi$ satisfies a new equation that does not depend on $\psi'$, unlike $\varphi$. Rates of convergence for $\psi$ of the form $O(g(r))$ will correspond to rates for $\varphi$ of the form $O(g(r)/\sqrt{r^2 + a^2})$, and these will be valid for our original function $H_a$.

**Definition 2.2** *We define the auxiliary function $\psi$:*

$$\varphi = \frac{\psi}{\sqrt{r^2 + a^2}}$$

**Definition 2.3** *We introduce the positive potential*

$$U_a(r) := \frac{a^2}{(r^2 + a^2)^2} \qquad \forall r \in \mathbb{R}.$$

*It is even, integrable, bounded by $\|U_a\|_\infty = 1/a^2$ and the maximum is attained at $r = 0$.*

Next, some preliminary results about $\psi$.

**Proposition 2.2** *There exists a $R > 0$, independent of $a$, that satisfies:*

$$\|\psi\|_{L^\infty(\mathbb{R}_+)} \leq 2\sqrt{R^2 + a^2}\|\varphi\|_\infty = 2\pi\sqrt{R^2 + a^2}.$$

PROOF. Fix $\delta > 0$ and $R(\delta) > 0$ so that

$$|\varphi(r)| \leq M e^{-(\sqrt{2}-\delta)r} \qquad \forall r > R;$$

The constant $M$ depends on $\delta$ and $R(\delta)$; it is given by $M = \pi e^{\sqrt{2-\delta}R}$. Choose $R$ in a way that $Me^{-(\sqrt{2}-\delta)r} < \psi(0) = a\pi/2$ when $r > 2R$. Then

$$\sup_{r \in \mathbb{R}_+} |\psi(r)| = \sup_{r \in [0,2R]} |\psi(r)|,$$

and from the definition $\psi(r) = \varphi(r)\sqrt{r^2 + a^2}$, we get

$$\|\psi\|_{L^\infty(\mathbb{R}_+)} \leq \sqrt{4R^2 + a^2}\|\varphi\|_\infty \leq 2\pi\sqrt{R^2 + a^2}.$$

□

**Proposition 2.3** *$\psi$ satisfies the differential equation*

$$\psi'' - U_a\psi = \sqrt{r^2 + a^2}\sin\left(\frac{2\psi}{\sqrt{r^2 + a^2}}\right),$$

*and tends to 0 as $r \to +\infty$.*

PROOF. A simple calculation shows that the derivatives of $\psi$ satisfy:

$$\varphi' = \frac{\psi'}{(r^2 + a^2)^{\frac{1}{2}}} - \frac{r\psi}{(r^2 + a^2)^{\frac{3}{2}}}, \quad \varphi'' = \frac{\psi''}{(r^2 + a^2)^{\frac{1}{2}}} - \frac{2r\psi'}{(r^2 + a^2)^{\frac{3}{2}}} - \psi\frac{a^2 - 2r^2}{(r^2 + a^2)^{\frac{5}{2}}}.$$

Replace these expressions in the equation for $\varphi$ and multiply by $\sqrt{r^2 + a^2}$. This leads to the following equation for $\psi$:

$$\psi'' - \left(\frac{a^2 - 2r^2}{(r^2 + a^2)^2} + \frac{2r^2}{(r^2 + a^2)^2}\right)\psi - (r^2 + a^2)^{\frac{1}{2}}\sin\left(\frac{2\psi}{(r^2 + a^2)^{\frac{1}{2}}}\right) = 0,$$

and cancelling the terms accompanying $\psi$ gives the desired equality. The limit is a consequence of the exponential behavior of $\varphi$. □

The term inside the sine tends to zero, which hints at a Taylor expansion near $r \to +\infty$ to deduce the convergence rate of $\psi$. To better reflect this, consider the equivalent equation

$$\psi'' - 2\psi = -U_a\psi + \sqrt{r^2 + a^2}\sin\left(\frac{2\psi}{\sqrt{r^2 + a^2}}\right) - 2\psi;$$

we need an estimate on the right side, which we call $f$.

**Lemma 2.7** *Define*

$$f(r) := -2\psi(r) + U_a(r)\psi(r) + \sqrt{r^2+a^2}\sin(2\varphi(r)).$$

*Then there is a* $R > 0$*, that does not depend on* $a$*, such that*

$$\|f\|_{L^\infty(\mathbb{R}_+)} \leq 2\Big(2\pi + 1 + \frac{\pi}{a^2}\Big)\sqrt{R^2+a^2}.$$

*Also,* $f = U_a\psi + \sqrt{r^2+a^2}\,o(|\varphi|^3)$ *and* $f = O((r^2+a^2)^{-\frac{3}{2}}e^{-\sqrt{2-\delta}r})$ *as* $r \to +\infty$*.*

Proof. Choose $R > 0$ so that $\varphi$ is close to zero. It is important to mention that this choice can be made independently of $a$, as the proof of Theorem 2.2 shows. A first order Taylor expansion of $\sin(\,\cdot\,)$ around 0 says that

$$\sin(2\varphi) = 2\varphi + o(|\varphi|^3) \qquad \forall r > R,$$

Remember that $\varphi = \psi/\sqrt{r^2+a^2}$:

$$\sqrt{r^2+a^2}\sin(2\varphi) = 2\psi + \sqrt{r^2+a^2}\,o(|\varphi|^3) \qquad \forall r > R,$$

implying that

$$f(r) = U_a(r)\psi(r) + \sqrt{r^2+a^2}\,o(|\varphi|^3) \qquad \forall r > R.$$

The asymptotic behavior of $\psi$ is determined by $\varphi$. From the definition of $U_a$ we get

$$f(r) = \frac{a^2}{(r^2+a^2)^{\frac{3}{2}}}\varphi(r) + \sqrt{r^2+a^2}\,o(|\varphi|^3) \qquad \forall r > R.$$

As shown in Theorem 2.2, $\varphi = O(e^{-\sqrt{2-\delta}r})$ for sufficiently small $\delta > 0$;

$$|f(r)| \leq O\bigg(\frac{e^{-\sqrt{2-\delta}r}}{(r^2+a^2)^{\frac{3}{2}}}\bigg) + O\Big(\sqrt{r^2+a^2}e^{-3\sqrt{2-\delta}r}\Big) = O\bigg(\frac{e^{-\sqrt{2-\delta}r}}{(r^2+a^2)^{\frac{3}{2}}}\bigg).$$

Finding a bound for $|f|$ is now trivial. Due to the fact that it converges to zero, to estimate its supremum over $\mathbb{R}_+$ it is sufficient to consider a sufficiently large interval $[0, 2R]$. If $R$ is chosen appropriately, we can utilize the inequality in Proposition 2.2:

$$|f(r)| \leq (2 + \|U_a\|_\infty)\|\psi\|_{L^\infty(\mathbb{R}_+)} + \sqrt{4R^2+a^2} \leq 2\Big(2\pi + 1 + \frac{\pi}{a^2}\Big)\sqrt{R^2+a^2} \qquad \forall r \in [0, 2R].$$

Although this quantity depends on $a$ and $R(\delta)$, since $\delta$ is fixed there are no issues. □

**Lemma 2.8** *In the interval* $[0, +\infty)$*, the function* $\psi$ *has the form*

$$\psi(r) = -\frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s - \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\Big(-a\pi\sqrt{2} - \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^r e^{\sqrt{2}s}f(s)\mathrm{d}s\Big).$$

Proof. Remember that $\psi$ satisfies the equation

$$\psi'' - U_a\psi = \sqrt{r^2+a^2}\sin(2\varphi).$$

Add and substract $-2\psi$ to see that

$$\psi'' - 2\psi = -2\psi + U_a\psi + \sqrt{r^2+a^2}\sin(2\varphi) = f,$$

which is the ODE we will solve.

The homogeneous equation has a solution $\psi_H = c_1 e^{\sqrt{2}r} + c_2 e^{-\sqrt{2}r}$ for constants $c_1, c_2$, and its wronskian is $-2\sqrt{2}$. Variation of parameters affirms

$$\psi(r) = A(r)e^{\sqrt{2}r} + B(r)e^{-\sqrt{2}r},$$

with the functions $A$ and $B$ given by

$$A(r) = \frac{1}{2\sqrt{2}}\int_R^r e^{-\sqrt{2}s} f(s)\mathrm{d}s + c_1, \qquad B(r) = -\frac{1}{2\sqrt{2}}\int_R^r e^{\sqrt{2}s} f(s)\mathrm{d}s + c_2.$$

for some $R > 0$. Explicitly:

$$\psi(r) = \frac{e^{\sqrt{2}r}}{2\sqrt{2}}\Big(c_1 + \int_R^r e^{-\sqrt{2}s} f(s)\mathrm{d}s\Big) - \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\Big(c_2 + \int_R^r e^{\sqrt{2}s} f(s)\mathrm{d}s\Big). \tag{2.3}$$

We must take advantage of the fact that $\psi \to 0$ as $r \to +\infty$. In particular, the limit

$$c_1 + \lim_{r\to+\infty}\int_R^r e^{-\sqrt{2}s} f(s)\mathrm{d}s = 0.$$

must hold, to cancel the exponential $e^{\sqrt{2}r}$. The only way this can be true is when

$$c_1 = -\int_R^{+\infty} e^{-\sqrt{2}s} f(s)\mathrm{d}s;$$

Using l'Hopital's rule:

$$\lim_{r\to+\infty}\frac{1}{2\sqrt{2}e^{-\sqrt{2}r}}\int_r^{+\infty} e^{-\sqrt{2}s} f(s)\mathrm{d}s = \lim_{r\to+\infty}\frac{e^{-\sqrt{2}r} f(r)}{4e^{-\sqrt{2}r}} = \frac{1}{4}\lim_{r\to+\infty} f(r) = 0,$$

and the choice of $c_1$ is justified.

The procedure to determine $c_2$ is similar. According to the definition of $\psi$, evaluating at $r = 0$ shows that $\psi(0) = a\varphi(0) = a\pi/2$. This gives us the equation

$$\psi(0) = \frac{a\pi}{2} = -\frac{1}{2\sqrt{2}}\int_0^{+\infty} e^{-\sqrt{2}s} f(s)\mathrm{d}s - \frac{1}{2\sqrt{2}}\Big(c_2 + \int_R^0 e^{\sqrt{2}s} f(s)\mathrm{d}s\Big),$$

allowing us to pick the correct value of $c_2$:

$$c_2 = -a\pi\sqrt{2} - \int_0^{+\infty} e^{-\sqrt{2}s} f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s} f(s)\mathrm{d}s.$$

Replace $c_1$ and $c_2$ in (2.3):

$$\psi(r) = -\frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s - \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\Big(-a\pi\sqrt{2} - \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^r e^{\sqrt{2}s}f(s)\mathrm{d}s\Big),$$

valid for $r \geq 0$.

□

This will be useful when proving the next theorem

**Theorem 2.3** *For fixed $a > 0$, there is a $R > 0$ and a constant $M > 0$, both independent of $a$, that ensure the inequality*

$$|\psi(r)| \leq K_a e^{-\sqrt{2}r} \qquad \forall r > R$$

*holds, for the constants*

$$M_a = \frac{M^3}{4-6\sqrt{2}\delta}\left(a + \frac{1}{\sqrt{2}-3\delta}\right), \quad N_a = \frac{a\pi}{2} + \frac{e^{\sqrt{2}R}}{2}\left(2\pi + 1 + \frac{\pi}{a^2}\right)\sqrt{R^2+a^2},$$
$$K_a = 2e^{-\sqrt{2}R}M_a + N_a.$$

Proof. For large $r > R$, the function $f$ has the form

$$f = U_a\psi + \sqrt{r^2+a^2}\,o(|\varphi|^3),$$

as seen in Lemma 2.7; we can take advantage of this approximation by replacing it in the expression for $\psi$ obtained in Lemma 2.8. To save space, define the auxiliary function

$$g(r) := \psi(r) + \frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}U_a(s)\psi(s)\mathrm{d}s + \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\int_R^r e^{\sqrt{2}s}U_a(s)\psi(s)\mathrm{d}s,$$

so

$$\begin{aligned} g(r) = &-\frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}\,o(|\varphi|^3)\mathrm{d}s - \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\int_R^r e^{\sqrt{2}s}\sqrt{s^2+a^2}\,o(|\varphi|^3)\mathrm{d}s \quad \forall r > R \\ &- \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\left(-a\pi\sqrt{2} - \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s\right). \end{aligned} \tag{2.4}$$

Note that $\psi > 0$, therefore all the terms in the definition of $g$ are positive. In particular, $\psi < g$.

We analyze the asymptotic behavior of each term on the right separately. First, there is a constant $A > 0$ and $R > 0$ (we can assume it is larger than the previous $R$ by taking the maximum between the two) such that:

$$\frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}\,o(|\varphi|^3)\mathrm{d}s \leq \frac{e^{\sqrt{2}r}}{2\sqrt{2}}\int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}A|\varphi(s)|^3\mathrm{d}s$$

and because $|\varphi| = O(e^{(\sqrt{2}-\delta)r})$ for small $\delta > 0$, we deduce that

$$\frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \le \frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}M^3 e^{-3(\sqrt{2}-\delta)r}\mathrm{d}s.$$

Here, we absorbed $A$ into the constant $M$ given by Theorem 2.2. This $M$ depends on $\delta$ and $R(\delta)$, and although it is exponential in the latter, this is not a concern as $\delta$ is fixed. The last inequality is equivalent to

$$\frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \le M^3\frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-4\sqrt{2}s}e^{3\delta s}\sqrt{s^2+a^2}\mathrm{d}s;$$

since $e^{-3\sqrt{2}s} \le e^{-3\sqrt{2}r}$ for all $s > r > 0$, we can take some exponentials out of the integral. This leads to

$$\frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \le M^3\frac{e^{-2\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s,$$

and if $\delta < \sqrt{2}/3$ the integral can be bounded like this: the integrand is positive, thus

$$\int_r^{+\infty} e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s \le \int_0^{+\infty} e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s.$$

Then integrate by parts:

$$\int_0^{+\infty} e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s = \frac{a}{\sqrt{2}-3\delta} + \frac{1}{\sqrt{2}-3\delta}\int_0^{+\infty} \frac{s}{\sqrt{s^2+a^2}} e^{-(\sqrt{2}-3\delta)s}\mathrm{d}s.$$

The quantity $s/\sqrt{s^2+a^2}$ is bounded above by 1, meaning that

$$\int_0^{+\infty} \frac{s}{\sqrt{s^2+a^2}} e^{-(\sqrt{2}-3\delta)s}\mathrm{d}s \le \int_0^{+\infty} e^{-(\sqrt{2}-3\delta)s}\mathrm{d}s = \frac{1}{\sqrt{2}-3\delta},$$

With

$$M_a := \frac{M^3}{4-6\sqrt{2}\delta}\left(a + \frac{1}{\sqrt{2}-3\delta}\right),$$

the bound we are looking for is

$$\frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \le M_a e^{-2\sqrt{2}r} \qquad \forall r > R. \tag{2.5}$$

The second term is bounded in a similar way to the first. Thanks to the known bound on $|\varphi|$, it is true that

$$\frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^{r} e^{\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \le M^3\frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^{r} e^{\sqrt{2}s}\sqrt{s^2+a^2}e^{-3(\sqrt{2}-\delta)s}\mathrm{d}s.$$

It follows from $R < r$ that $e^{-\sqrt{2}r} < e^{-\sqrt{2}R}$, thus

$$\frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^r e^{\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \leq M^3 e^{-\sqrt{2}R}\frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^r e^{-(\sqrt{2}+3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s.$$

Because $\delta < \sqrt{2}/3$, we can repeat a previous argument to obtain an upper bound for the integral

$$\int_R^r e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s \leq \int_0^{+\infty} e^{-(\sqrt{2}-3\delta)s}\sqrt{s^2+a^2}\mathrm{d}s \leq \frac{1}{\sqrt{2}-3\delta}\left(a+\frac{1}{\sqrt{2}-3\delta}\right),$$

and deduce that

$$\frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^r e^{\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \leq e^{-\sqrt{2}R}M_a e^{-\sqrt{2}r} \qquad \forall r > R. \tag{2.6}$$

Finally, we study the constants:

$$\int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s \leq \frac{1}{\sqrt{2}}\|f\|_{L^\infty(\mathbb{R}_+)}, \qquad \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s \leq \frac{1}{\sqrt{2}}(e^{\sqrt{2}R}-1)\|f\|_{L^\infty(\mathbb{R}_+)},$$

implying that

$$a\pi\sqrt{2} + \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s \leq a\pi\sqrt{2} + \frac{e^{\sqrt{2}R}}{\sqrt{2}}\|f\|_{L^\infty(\mathbb{R}_+)}.$$

We take advantage of the bound for $f$ from Lemma 2.7:

$$a\pi\sqrt{2} + \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s \leq a\pi\sqrt{2} + \frac{2}{\sqrt{2}}e^{\sqrt{2}R}\left(2\pi+1+\frac{\pi}{a^2}\right)\sqrt{R^2+a^2},$$

and define

$$N_a := \frac{a\pi}{2} + \frac{e^{\sqrt{2}R}}{2}\left(2\pi+1+\frac{\pi}{a^2}\right)\sqrt{R^2+a^2}$$

to get the inequality

$$\frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\left(a\pi\sqrt{2} + \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s\right) \leq N_a e^{-\sqrt{2}r} \qquad \forall r > R. \tag{2.7}$$

As $g$ is positive, $|g| = g$. Take the absolute value in equation (2.4):

$$\begin{aligned} |g(r)| \leq{}& \frac{e^{\sqrt{2}r}}{2\sqrt{2}} \int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s + \frac{e^{-\sqrt{2}r}}{2\sqrt{2}} \int_R^r e^{\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \\ & \frac{e^{-\sqrt{2}r}}{2\sqrt{2}}\left(a\pi\sqrt{2} + \int_0^{+\infty} e^{-\sqrt{2}s}|f(s)|\mathrm{d}s + \int_0^R e^{\sqrt{2}s}|f(s)|\mathrm{d}s\right) \end{aligned}$$

Because $|\psi| = \psi < g = |g|$, inequalities (2.5), (2.6) and (2.7) assert that

$$|\psi(r)| \leq M_a e^{-2\sqrt{2}r} + e^{-\sqrt{2}R}M_a e^{-\sqrt{2}r} + N_a e^{-\sqrt{2}r} \qquad \forall r > R.$$

Also, for $R < r$, the inequality $e^{2\sqrt{2}r} < e^{-\sqrt{2}R}e^{-\sqrt{2}r}$. This means that

$$|\psi(r)| \leq \left(2e^{-\sqrt{2}R}M_a + N_a\right)e^{-\sqrt{2}r} \qquad \forall r > R;$$

define $K_a \coloneqq 2e^{-\sqrt{2}R}M_a + N_a$ to recover the result. □

**Corollary 2.4** *For fixed $a > 0$, the function $\varphi$ satisfies*

$$\varphi(r) = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2 + a^2}}\right)$$

*as $r \to +\infty$.*

Proof. The previous theorem states that $\psi = O(e^{-\sqrt{2}r})$ as $r \to +\infty$. The corollary follows directly from the definition of $\psi$. □

It is convenient to have a bound that does not depend on $a$.

**Theorem 2.4** *For $a > 1$, there are positive constants $C$ and $R$, both independent of $a$, that satisfy*

$$|\varphi(r)| \leq Ce^{-\sqrt{2}r} \qquad r > R$$

Proof. As Theorem 2.3 affirms, there is a $R > 0$ such that

$$|\varphi(r)| \leq \frac{1}{\sqrt{r^2 + a^2}}\left(2e^{-\sqrt{2}R}M_a + N_a\right)e^{-\sqrt{2}r} \qquad \forall r > R.$$

Notice that $1/\sqrt{r^2 + a^2} < 1/\sqrt{R^2 + a^2}$ for all $r > R$. With this, we can bound each constant as follows:

$$\frac{M_a}{\sqrt{r^2 + a^2}} \leq \frac{M^3}{4 - 6\sqrt{2}\delta}\left(\frac{a}{\sqrt{R^2 + a^2}} + \frac{1}{\sqrt{2} - 3\delta}\frac{1}{\sqrt{R^2 + a^2}}\right) \qquad \forall r > R.$$

As a consequence of the hypothesis that $a > 1$, the bound

$$\frac{M_a}{\sqrt{r^2 + a^2}} \leq \frac{M^3}{4 - 6\sqrt{2}\delta}\left(1 + \frac{1}{\sqrt{2} - 3\delta}\right) \qquad \forall r > R.$$

is true.

The other bound is analogous to the first:

$$\frac{N_a}{\sqrt{r^2 + a^2}} \leq \frac{a}{\sqrt{R^2 + a^2}}\frac{\pi}{2} + \frac{e^{\sqrt{2}R}}{2}\left(2\pi + 1 + \frac{\pi}{a^2}\right) \qquad \forall r > R.$$

Again, $a > 1$ provides the desired inequality

$$\frac{N_a}{\sqrt{r^2 + a^2}} \leq \frac{\pi}{2} + \frac{e^{\sqrt{2}R}}{2}(3\pi + 1) \qquad \forall r > R.$$

Combine these with the inequality for $\varphi$ to obtain

$$|\varphi(r)| \leq \left(\frac{2M^3e^{-\sqrt{2}R}}{4-6\sqrt{2}\delta}\left(1+\frac{1}{\sqrt{2}-3\delta}\right)+\frac{\pi}{2}+\frac{e^{\sqrt{2}R}}{2}(3\pi+1)\right)e^{-\sqrt{2}r} \qquad \forall r > R.$$

The constant we are looking for is

$$C := \left(\frac{M^3e^{-\sqrt{2}R}}{2-3\sqrt{2}\delta}\left(1+\frac{1}{\sqrt{2}-3\delta}\right)+\frac{\pi}{2}+\frac{e^{\sqrt{2}R}}{2}(3\pi+1)\right).$$

This expression only depends on $\delta$ and $R = R(\delta)$, which are fixed. This is exactly what we wanted:

$$|\varphi(r)| \leq Ce^{-\sqrt{2}r} \qquad \forall r > R.$$

□

Next, we search for exponential bounds on the derivatives of $\varphi$.

**Theorem 2.5** *With the constants $K_a$, $M_a$ and $N_a$ defined as in Theorem 2.3, there is a $R > 0$, independent of $a$, such that*

$$|\psi'(r)| \leq \tilde{K}_a e^{-\sqrt{2}r} \qquad \forall r > R,$$

*where the constant $\tilde{K}_a$ is*

$$\tilde{K}_a = \frac{\pi K_a}{4a} + 2\sqrt{2}M_a e^{-\sqrt{2}R} + \sqrt{2}N_a.$$

Proof. Differentiate the expression for $\psi$ found in Lemma 2.8:

$$\begin{aligned}\psi'(r) = &-\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \frac{f(r)}{2\sqrt{2}} + \frac{e^{-\sqrt{2}r}}{2}\int_R^r e^{\sqrt{2}s}f(s)\mathrm{d}s - \frac{f(r)}{2\sqrt{2}} \qquad \forall r > R\\ &+\frac{e^{-\sqrt{2}r}}{2}\left(-a\pi\sqrt{2} - \int_0^{+\infty} e^{-\sqrt{2}s}f(s)\mathrm{d}s + \int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s\right).\end{aligned}$$

As usual, the way forward is to study each exponential separately. Since $f = U_a\psi + \sqrt{r^2+a^2}o(|\varphi|^3)$ if $R$ is large enough, we have the expression:

$$\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}|f(s)|\mathrm{d}s = \frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}\left(U_a(s)\psi(s) + \sqrt{s^2+a^2}o(|\varphi|^3)\right)\mathrm{d}s,$$

where we used the fact that $U_a$ and $\psi$ are positive to eliminate the absolute value. We already know from inequality (2.5) in the proof of Theorem 2.3 that

$$\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \leq \sqrt{2}M_a e^{-2\sqrt{2}r} \qquad \forall r > R;$$

the very same theorem states that

$$\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}U_a(s)\psi(s)\mathrm{d}s \leq \frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}U_a(s)K_a e^{-\sqrt{2}s}\mathrm{d}s \qquad \forall r > R,$$

and

$$\frac{K_a}{2}e^{\sqrt{2}r}\int_r^{+\infty} e^{-2\sqrt{2}s}U_a(s)\mathrm{d}s \leq \frac{K_a}{2}e^{-\sqrt{2}r}\int_r^{+\infty} U_a(s)\mathrm{d}s \leq \frac{K_a}{2}e^{-\sqrt{2}r}\int_0^{+\infty} U_a(s)\mathrm{d}s \qquad \forall r > R.$$

The integral of $U_a$ is

$$\int_0^{+\infty} \frac{a^2}{(s^2+a^2)^2}\mathrm{d}s = \frac{1}{2}\left(\frac{1}{a}\arctan\left(\frac{s}{a}\right) + \frac{s}{s^2+a^2}\right)\Big|_0^{+\infty} = \frac{\pi}{4a},$$

and in turn we get the bound for the original expression:

$$\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}|f(s)|\mathrm{d}s \leq \frac{\pi K_a}{8a}e^{-\sqrt{2}r} + \sqrt{2}M_a e^{-2\sqrt{2}r} \qquad \forall r > R.$$

As we have done many times before, we take advantage of the inequality $e^{-2\sqrt{2}r} \leq e^{-\sqrt{2}R}e^{-\sqrt{2}r}$ for $R < r$:

$$\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty} e^{-\sqrt{2}s}|f(s)|\mathrm{d}s \leq \left(\frac{\pi K_a}{8a} + \sqrt{2}M_a e^{-\sqrt{2}R}\right)e^{-\sqrt{2}r} \qquad \forall r > R.$$

The next exponential is analogous to the previous case. Replace $|f|$ by its estimate:

$$\frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}|f(s)|\mathrm{d}s \leq \frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}\Big(U_a(s)\psi(s) + \sqrt{s^2+a^2}o(|\varphi|^3)\Big)\mathrm{d}s.$$

One expression was already bounded in inequality (2.6):

$$\frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}\sqrt{s^2+a^2}o(|\varphi|^3)\mathrm{d}s \leq \sqrt{2}e^{-\sqrt{2}R}M_a e^{-\sqrt{2}r} \qquad \forall r > R.$$

The other is bounded by

$$\frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}U_a(s)\psi(s)\mathrm{d}s \leq \frac{K_a}{2}e^{-\sqrt{2}r}\int_R^{r} e^{\sqrt{2}s}U_a(s)e^{-\sqrt{2}s}\mathrm{d}s \qquad \forall r > R,$$

where the integral of $U_a$ can be extended to $\mathbb{R}_+$ to obtain the bound

$$\frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}U_a(s)\psi(s)\mathrm{d}s \leq \frac{\pi K_a}{8a}e^{-\sqrt{2}r} \qquad \forall r > R.$$

These two inequalities show that

$$\frac{e^{-\sqrt{2}r}}{2}\int_R^{r} e^{\sqrt{2}s}|f(s)|\mathrm{d}s \leq \left(\frac{\pi K_a}{8a} + \sqrt{2}M_a e^{-\sqrt{2}R}\right)e^{-\sqrt{2}r} \qquad \forall r > R.$$

Finally, inequality (2.7) states that

$$\frac{e^{-\sqrt{2}r}}{2}\left(a\pi\sqrt{2}+\int_0^{+\infty}e^{-\sqrt{2}s}f(s)\mathrm{d}s+\int_0^R e^{\sqrt{2}s}f(s)\mathrm{d}s\right)\leq\sqrt{2}N_a e^{-\sqrt{2}r}\qquad\forall r>R.$$

Take the absolute value of $\psi'$:

$$\begin{aligned}|\psi'(r)|\leq\ &\frac{e^{\sqrt{2}r}}{2}\int_r^{+\infty}e^{-\sqrt{2}s}|f(s)|\mathrm{d}s+\frac{e^{-\sqrt{2}r}}{2}\int_R^r e^{\sqrt{2}s}|f(s)|\mathrm{d}s \qquad \forall r>R\\ &+\frac{e^{-\sqrt{2}r}}{2}\left(a\pi\sqrt{2}+\int_0^{+\infty}e^{-\sqrt{2}s}|f(s)|\mathrm{d}s+\int_0^R e^{\sqrt{2}s}|f(s)|\mathrm{d}s\right).\end{aligned}$$

It follows from the preceding discussion that

$$|\psi'(r)|\leq\left(\frac{\pi K_a}{4a}+2\sqrt{2}M_a e^{-\sqrt{2}R}+\sqrt{2}N_a\right)e^{-\sqrt{2}r}\qquad\forall r>R..$$

□

**Corollary 2.5** *For fixed $a>0$, the derivative $\varphi'$ satisfies*

$$|\varphi'(r)|=O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right)$$

*as $r\to+\infty$.*

Proof. The derivative of $\varphi'$ is

$$\varphi'=\frac{\psi'}{\sqrt{r^2+a^2}}-\frac{r\psi}{(r^2+a^2)^{\frac{3}{2}}}.$$

The absolute value satisfies the inequality

$$|\varphi'(r)|\leq\frac{|\psi'(r)|}{\sqrt{r^2+a^2}}+\frac{r|\psi(r)|}{(r^2+a^2)^{\frac{3}{2}}}\qquad\forall r>0.$$

For $r>1$, it is easy to see that $r/(r^2+a^2)^{\frac{3}{2}}\leq 1/\sqrt{r^2+a^2}$:

$$|\varphi'(r)|\leq\frac{1}{\sqrt{r^2+a^2}}(|\psi(r)|+|\psi'(r)|)\qquad\forall r>1,$$

If $R$ is chosen to be sufficiently large, both Theorem 2.3 and 2.5 are applicable and we deduce the result

$$|\varphi'(r)|\leq\left(K_a+\tilde{K}_a\right)\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\qquad\forall r>R.$$

□

**Theorem 2.6** *If $a > 1$, then the inequality*

$$|\varphi'(r)| \le \tilde{C} e^{-\sqrt{2}r} \qquad \forall r > R$$

*holds for some positive constants $\tilde{C}$ and $R$ that do not depend on $a$.*

Proof. In Theorem 2.4, we demonstrated that both $M_a/\sqrt{r^2+a^2}$ and $N_a/\sqrt{r^2+a^2}$ are bounded above by a constant that does not depend on $a$, given that $a > 1$. Immediately we deduce that this is also true for both $K_a/\sqrt{r^2+a^2}$ and $\tilde{K}_a/\sqrt{r^2+a^2}$, because they are linear combinations of $M_a$ and $N_a$: in other words, there is a constant $\tilde{C}$, independent of $a$, that satisfies

$$\frac{K_a}{\sqrt{r^2+a^2}} + \frac{\tilde{K}_a}{\sqrt{r^2+a^2}} \le \tilde{C} \qquad \forall r > R.$$

The proof is then complete:

$$|\varphi'(r)| \le \tilde{C} e^{-\sqrt{2}r} \qquad \forall r > R.$$

□

**Theorem 2.7** *The second derivative $\varphi''$ is of order*

$$|\varphi''(r)| = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right)$$

*as $r \to +\infty$, for fixed $a > 0$. In addition, if $a > 1$ then*

$$\varphi'' = O\left(e^{-\sqrt{2}r}\right)$$

*independently of $a$.*

Proof. The differential equation solved by $\varphi$ is

$$\varphi'' = -\frac{2r}{r^2+a^2}\varphi' + \sin(2\varphi).$$

In a neighborhood of $\varphi = 0$, expand $\sin(\varphi)$ up to first order terms:

$$\varphi'' = -\frac{2r}{r^2+a^2}\varphi' + 2\varphi + o(|\varphi|^3) \qquad \forall r > R,$$

where we pick a sufficiently large $R > 0$, so that $\varphi$ is small and $r/(r^2+a^2) < 1$ when $r > R$. Now we can apply the absolute value in this equality to obtain

$$|\varphi''| \le 2|\varphi'| + 2|\varphi| + o(|\varphi|^3) \qquad \forall r > R.$$

The term $o(|\varphi|^3)$ can be ignored, since it is of higher order. For an appropriate $R$, we use corollaries 2.4 and 2.5 to see that

$$|\varphi''| = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right) + O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right) = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right).$$

as $r \to +\infty$. Likewise, if $a > 1$ then theorems 2.4 and 2.6 imply that

$$|\varphi''| = O\left(e^{-\sqrt{2}r}\right),$$

independently of $a$. □

We extend these results to derivatives of arbitrary order, but first we enunciate a couple of lemmas.

**Lemma 2.9** *For all non-negative integers $n \geq 0$, there is a constant $C_n > 0$ such that*

$$\left|\left(\frac{1}{r^2 + a^2}\right)^{(n)}\right| \leq \frac{C_n}{r^n} \qquad \forall r > 1.$$

Proof. Write $f(x) = 1/x$ and $g(r) = r^2 + a^2$. Introduce the set

$$S_n := \left\{ m \in \mathbb{Z}_+^n : \sum_{k=1}^{n} k m_k = n \right\};$$

Faà di Bruno's formula for the derivative of the composition $f(g(r)) = 1/(r^2 + a^2)$ says

$$\left(\frac{1}{r^2 + a^2}\right)^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}r^n} f(g(r)) = \sum_{m \in S_n} n! f^{(m_1 + \cdots + m_n)}(g(r)) \prod_{k=1}^{n} \frac{1}{m_k!} \left(\frac{g^{(k)}(r)}{k!}\right)^{m_k}.$$

The derivatives of $g$ are $g'(r) = 2r$, $g''(r) = 2$ and $g^{(k)}(r) = 0$ for $k > 2$. Therefore, the only terms of the sum that are not zero are those where $m_k = 0$ for $k > 2$:

$$\left(\frac{1}{r^2 + a^2}\right)^{(n)} = \sum_{\substack{m_1 + 2m_2 = n \\ m_1, m_2 \geq 0}} n! f^{(m_1 + m_2)}(g(r)) \frac{(2r)^{m_1}}{m_1!} \frac{2^{m_2}}{m_2! 2^{m_2}}.$$

The derivative $f^{(m_1 + m_2)}$ has an explicit formula:

$$f^{(m_1 + m_2)}(x) = (-1)^{m_1 + m_2} (m_1 + m_2)! \frac{1}{x^{m_1 + m_2}} \qquad \forall x > 0,$$

and it follows that

$$\left(\frac{1}{r^2 + a^2}\right)^{(n)} = \sum_{\substack{m_1 + 2m_2 = n \\ m_1, m_2 \geq 0}} n! (-1)^{m_1 + m_2} 2^{m_1} \frac{(m_1 + m_2)!}{m_1! m_2!} \frac{r^{m_1}}{(r^2 + a^2)^{m_1 + m_2}}.$$

The rational function can be bounded above in this way:

$$\frac{r^{m_1}}{(r^2 + a^2)^{m_1 + m_2}} \leq \frac{r^{m_1}}{r^{2(m_1 + m_2)}} = \frac{r^{m_1}}{r^{m_1}} \frac{1}{r^{m_1 + 2m_2}} = \frac{1}{r^n} \qquad \forall r > 0,$$

which provides the inequality for the derivative of order $n$. Indeed,

$$\left|\left(\frac{1}{r^2+a^2}\right)^{(n)}\right| \leq \sum_{\substack{m_1+2m_2=n\\ m_1,m_2\geq 0}} n!2^{m_1}\binom{m_1+m_2}{m_1}\frac{1}{r^n},$$

and if we define

$$C_n \coloneqq n! \sum_{\substack{m_1+2m_2=n\\ m_1,m_2\geq 0}} 2^{m_1}\binom{m_1+m_2}{m_1},$$

the inequality stated in the lemma is true:

$$\left|\left(\frac{1}{r^2+a^2}\right)^{(n)}\right| \leq \frac{C_n}{r^n}.$$

□

**Lemma 2.10** *There is a constant $C_n > 0$, which depends only on $n$, such that*

$$\left|\left(\frac{r}{r^2+a^2}\right)^{(n)}\right| \leq C_n \qquad \forall r > 1.$$

*for all non-negative integers $n \geq 0$.*

Proof. For $n = 0$, we have

$$\frac{r}{r^2+a^2} \leq \frac{r}{r^2} = \frac{1}{r} \qquad \forall r > 0.$$

Meanwhile, for $n = 1$:

$$\left|\left(\frac{r}{r^2+a^2}\right)'\right| = \left|\frac{a^2-r^2}{(r^2+a^2)^2}\right| \leq \frac{r^2+a^2}{(r^2+a^2)^2} = \frac{1}{r^2+a^2} \leq 1 \qquad \forall r > 0$$

In the case where $n > 1$, use the general Leibniz rule:

$$\left(\frac{r}{r^2+a^2}\right)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} r^{(k)}\left(\frac{1}{r^2+a^2}\right)^{(n-k)} \qquad \forall r > 0.$$

Clearly, $r^{(k)} = 0$ for $k > 1$, so

$$\left(\frac{r}{r^2+a^2}\right)^{(n)} = r\left(\frac{1}{r^2+a^2}\right)^{(n)} + n\left(\frac{1}{r^2+a^2}\right)^{(n-1)} \qquad \forall r > 0.$$

and the absolute value satisfies

$$\left|\left(\frac{r}{r^2+a^2}\right)^{(n)}\right| \leq r\left|\left(\frac{1}{r^2+a^2}\right)^{(n)}\right| + n\left|\left(\frac{1}{r^2+a^2}\right)^{(n-1)}\right| \qquad \forall r > 0.$$

The previous lemma allows us to get an estimate on the derivatives with constants that do not depend on $a$:

$$\left|\left(\frac{r}{r^2+a^2}\right)^{(n)}\right| \leq r\frac{C_n}{r^n} + n\frac{C_{n-1}}{r^{n-1}} = (C_n + nC_{n-1})\frac{1}{r^{n-1}},$$

implying that, for $r > 1$, the bound

$$\left|\left(\frac{r}{r^2+a^2}\right)^{(n)}\right| \leq C_n + nC_{n-1} \qquad \forall r > 1,$$

holds. The conclusion follows from absorbing constants. □

**Theorem 2.8** *The $n$-th derivative $\varphi^{(n)}$ is of order*

$$\left|\varphi^{(n)}\right| = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right)$$

*as $r \to +\infty$, for fixed $a > 0$. In addition, if $a > 1$ then*

$$\left|\varphi^{(n)}\right| = O\left(e^{-\sqrt{2}r}\right)$$

*independently of $a$.*

Proof. So far we have shown that this is true for $n = 0, 1, 2$. Induction is the argument of choice: let $m > 1$ be an integer and write $n = m + 2$. Assume that the result holds for all non-negative integers less than $n$. From the equation satisfied by $\varphi$ we see that

$$\frac{\mathrm{d}^m \varphi''}{\mathrm{d}r^m} = -2\frac{\mathrm{d}^m}{\mathrm{d}r^m}\left(\frac{r}{r^2+a^2}\varphi'\right) + \frac{\mathrm{d}^m}{\mathrm{d}r^m}\sin(2\varphi).$$

Expand the first term on the right using the general Leibniz rule:

$$\frac{\mathrm{d}^m}{\mathrm{d}r^m}\left(\frac{r}{r^2+a^2}\varphi'\right) = \sum_{k=0}^{m}\binom{m}{k}\left(\frac{r}{r^2+a^2}\right)^{(m-k)}(\varphi')^{(k)} = \sum_{k=0}^{m}\binom{m}{k}\left(\frac{r}{r^2+a^2}\right)^{(m-k)}\varphi^{(k+1)}$$

The $(m-k)$-th derivative is bounded by a constant independent of $a$ and $r$ for all large $r$, as shown in the preceding lemma. This implies that

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\left(\frac{r}{r^2+a^2}\varphi'\right)\right| \leq \sum_{k=0}^{m} C_{m,k}\left|\varphi^{(k+1)}\right|;$$

the highest order derivative is of order $m + 1 < n$, therefore we can use the induction hypothesis to deduce that

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\left(\frac{r}{r^2+a^2}\varphi'\right)\right| \leq \sum_{k=0}^{m} C_{m,k} O\left(e^{-\sqrt{2}r}\right) = O\left(e^{-\sqrt{2}r}\right).$$

The derivatives of $\sin(2\varphi)$ are more difficult to handle. An equivalent formulation of Faà di Bruno's identity gives an expression for the $m$-th derivative of a composition of functions in terms of the Bell polynomials:

$$\frac{\mathrm{d}^m}{\mathrm{d}r^m}\sin(2\varphi) = \sum_{k=1}^{m} \sin^{(k)}(2\varphi) B_{m,k}\Big(2\varphi', 2\varphi'', \ldots, 2\varphi^{(m-k+1)}\Big).$$

Here, $B_{m,k}$ denotes the partial Bell polynomial

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \sum \frac{m!}{j_1! j_2! \cdots j_{m-k+1}!} \prod_{\alpha=1}^{m-k+1} \left(\frac{x_\alpha}{\alpha!}\right)^{j_\alpha},$$

where the sum ranges over all sequences $j_1, j_2, \ldots, j_{m-k+1}$ of non-negative integers that satisfy the conditions

$$\sum_{\alpha=1}^{m-k+1} j_\alpha = k, \qquad \sum_{\alpha=1}^{m-k+1} \alpha j_\alpha = m.$$

We will call the set containing these sequences $S_{m,k} \subseteq \mathbb{Z}_+^{m-k+1}$.

The absolute value plus the triangle inequality show that

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\sin(2\varphi)\right| \leq \sum_{k=1}^{m} \left|\sin^{(k)}(2\varphi)\right| \left|B_{m,k}\Big(2\varphi', 2\varphi'', \ldots, 2\varphi^{(m-k+1)}\Big)\right|.$$

The inequality $\left|\sin^{(k)}(2\varphi)\right| \leq 1$ is trivial. Moreover, the Bell polynomials are

$$\left|B_{m,k}\Big(2\varphi', 2\varphi'', \ldots, 2\varphi^{(m-k+1)}\Big)\right| \leq \sum_{(j_1,\ldots,j_{m-k+1})\in S_{m_k}} \frac{m!}{j_1! j_2! \cdots j_{m-k+1}!} \prod_{\alpha=1}^{m-k+1} 2^{j_\alpha} \left|\frac{\varphi^{(\alpha)}}{\alpha!}\right|^{j_\alpha}.$$

As before, if $k \leq m$, then the highest order derivatives that appear in the polynomials $B_{m,k}$ are of order $m+1 < n$. Their multiplication will be small compared to $O(e^{-\sqrt{2}r})$, thus

$$\left|B_{m,k}\Big(2\varphi', 2\varphi'', \ldots, 2\varphi^{(m-k+1)}\Big)\right| \leq \sum_{(j_1,\ldots,j_{m-k+1})\in S_{m_k}} \frac{2^k m!}{j_1! j_2! \cdots j_{m-k+1}!} O\Big(e^{-\sqrt{2}r}\Big) = O\Big(e^{-\sqrt{2}r}\Big).$$

Going back to the derivative of $\sin(2\varphi)$, we have shown that

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\sin(2\varphi)\right| \leq \sum_{k=1}^{m} O\Big(e^{-\sqrt{2}r}\Big) = O\Big(e^{-\sqrt{2}r}\Big).$$

The result follows from

$$\left|\varphi^{(n)}\right| = \left|\varphi^{(m+2)}\right| \leq 2\left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\left(\frac{r}{r^2+a^2}\varphi'\right)\right| + \left|\frac{\mathrm{d}^m}{\mathrm{d}r^m}\sin(2\varphi)\right| = O\Big(e^{-\sqrt{2}r}\Big).$$

Note that none of the constants involved depend on $a$, either by the induction hypothesis (in the case of higher order derivatives), or by direct analysis (the derivatives of the rational function $r/(r^2+a^2)$). Because $n$ is arbitrary, by the principle of induction we conclude that the previous convergence rate is true for all non-negative integers $n$. The exact same argument works if we replace $O(e^{-\sqrt{2}r})$ with $O((r^2+a^2)^{-\frac{1}{2}}e^{-\sqrt{2}r})$, and the theorem is proven. □

Finally, we present the results of this section in terms of the original 1-kink $H_a$ . They follow immediately from the definition $\varphi = \pi/2 - H_a$ and the parity of $H_a$.

**Corollary 2.6** *The convergence rates*

$$\left|H_a(r) \mp \frac{\pi}{2}\right| = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right), \qquad \left|H_a^{(n)}(r)\right| = O\left(\frac{e^{-\sqrt{2}r}}{\sqrt{r^2+a^2}}\right),$$

*hold as $r \to \pm\infty$ for all $n \in \mathbb{N}$. Moreover, if $a > 1$ then*

$$\left|H_a(r) \mp \frac{\pi}{2}\right| = O\left(e^{-\sqrt{2}r}\right), \qquad \left|H_a^{(n)}(r)\right| = O\left(e^{-\sqrt{2}r}\right),$$

*as $r \to \pm\infty$ and the constants involved do not depend on $a$.*

## 2.3. Linearized SG operator

Compared to the SG equation, the SGWH equation has an extra term that tends to zero as $a$ grows infinitely large. It is therefore natural to expect that quantities and objects related to the SGWH model would converge (in an appropriate sense) to their analogues in flat spacetime. In other words, for large values of $a$, objects arising from the SGWH model could potentially be described as *perturbations*. To do this, however, we need to understand the base objects themselves, and in this section we study the SG model to achieve this task.

**Definition 2.4** *The function $H_{SG}$ is defined as the unique solution to the problem*

$$(SG)\begin{cases} \phi'' + \sin(2\phi) = 0, & r \in \mathbb{R}, \\ \phi(0) = 0, & \\ \lim\limits_{r\to\pm\infty} \phi(r) = \pm\dfrac{\pi}{2}. & \end{cases}$$

**Theorem 2.9** *$H_{SG}$ is given by the expression:*

$$H_{SG}(r) = 2\arctan\left(e^{\sqrt{2}r}\right) - \frac{\pi}{2}.$$

Proof. The derivatives are:

$$H'_{SG} = \frac{2\sqrt{2}e^{\sqrt{2}r}}{1+e^{2\sqrt{2}r}} = \sqrt{2}\,\mathrm{sech}\left(\sqrt{2}r\right), \quad H''_{SG} = 4e^{\sqrt{2}r}\frac{1-e^{2\sqrt{2}r}}{(1+e^{2\sqrt{2}r})^2} = -2\tanh\left(\sqrt{2}r\right)\mathrm{sech}\left(\sqrt{2}r\right).$$

Meanwhile, the other term is:

$$\sin(2H_{SG}) = \sin\left(4\arctan\left(e^{\sqrt{2}r}\right) - \pi\right) = -\sin\left(4\arctan\left(e^{\sqrt{2}r}\right)\right),$$

and trigonometric identities imply that

$$\begin{aligned}\sin\left(4\arctan\left(e^{\sqrt{2}r}\right)\right) = \;& 4\sin\left(\arctan\left(e^{\sqrt{2}r}\right)\right)\cos\left(\arctan\left(e^{\sqrt{2}r}\right)\right)\cos^2\left(\arctan\left(e^{\sqrt{2}r}\right)\right) \\ & - 4\sin\left(\arctan\left(e^{\sqrt{2}r}\right)\right)\cos\left(\arctan\left(e^{\sqrt{2}r}\right)\right)\sin^2\left(\arctan\left(e^{\sqrt{2}r}\right)\right).\end{aligned}$$

It is known that $\sin(\arctan(x)) = x/\sqrt{1+x^2}$ and $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, so

$$\sin\Big(4\arctan\Big(e^{\sqrt{2}r}\Big)\Big) = 4e^{\sqrt{2}r}\frac{1-e^{2\sqrt{2}r}}{(1+e^{2\sqrt{2}r})^2}.$$

It follows that

$$H''_{SG} + \sin(2H_{SG}) = 4e^{\sqrt{2}r}\frac{1-e^{2\sqrt{2}r}}{(1+e^{2\sqrt{2}r})^2} - 4e^{\sqrt{2}r}\frac{1-e^{2\sqrt{2}r}}{(1+e^{2\sqrt{2}r})^2} = 0.$$

The initial condition is

$$H_{SG}(0) = 2\arctan(1) - \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

The limits at infinity are

$$\lim_{r\to-\infty} H_{SG}(r) = -\frac{\pi}{2}, \qquad \lim_{r\to+\infty} H_{SG}(r) = \pi - \frac{\pi}{2} = \frac{\pi}{2},$$

and the theorem is proven. □

Before we continue, we state some useful remarks regarding $H_{SG}$ and its derivatives.

**Remark** $H_{SG}$ and its derivatives are

$$H_{SG}(r) = 2\arctan\Big(e^{\sqrt{2}r}\Big) - \frac{\pi}{2}, \qquad H'_{SG}(r) = \sqrt{2}\,\mathrm{sech}\Big(\sqrt{2}r\Big),$$
$$H''_{SG}(r) = -2\tanh\Big(\sqrt{2}r\Big)\mathrm{sech}\Big(\sqrt{2}r\Big), \quad H'''_{SG}(r) = 2\sqrt{2}\,\mathrm{sech}\Big(\sqrt{2}r\Big)\Big(1 - 2\,\mathrm{sech}^2\Big(\sqrt{2}r\Big)\Big).$$

From the equations $H_{SG}$ and $H'_{SG}$ satisfy we deduce

$$\sin(2H_{SG}(r)) = 2\tanh\Big(\sqrt{2}r\Big)\mathrm{sech}\Big(\sqrt{2}r\Big), \qquad \cos(2H_{SG}(r)) = 2\,\mathrm{sech}^2\Big(\sqrt{2}r\Big) - 1.$$

A crucial component of this study is the *linearized Sine-Gordon operator* $L_{SG}$. The rest of this section is devoted to this object.

**Definition 2.5** *The* Sine-Gordon differential operator *is defined as*

$$D_{SG} \coloneqq -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \sin(2\,\cdot\,).$$

*Note that we used the opposite sign to maintain positivity.*

**Proposition 2.4** *The linearization of $D_{SG}$ at $H_{SG}$ in $H^2(\mathbb{R})$ is given by the densely defined operator*

$$\begin{aligned} L_{SG}\colon H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ \phi &\mapsto -\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} - 2\cos(2H_{SG})\phi. \end{aligned}$$

Proof. The Fréchet derivative of $D_{SG}$ is $-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\cos(2\,\cdot\,)$. Thus, near $H_{SG}$ it is true that

$$D_{SG}(\phi) = D_{SG}(H_{SG}) - \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} - 2\cos(2H_{SG})\phi + o(\phi) \qquad \forall\phi \in H^2(\mathbb{R}).$$

By definition $D_{SG}(H_{SG}) = 0$, and we deduce the proposition. □

**Definition 2.6** *We define the quadratic form $Q_{SG}$ corresponding to $L_{SG}$ as*

$$Q_{SG}(\phi) \coloneqq \langle L_{SG}\phi, \phi\rangle \qquad \forall\phi \in H^2(\mathbb{R}).$$

*Alternatively, thanks to integration by parts we have:*

$$Q_{SG}(\phi) = \int (\phi')^2 \mathrm{d}r - 2\int \cos(2H_{SG})\phi \mathrm{d}r \qquad \forall\phi \in H^1(\mathbb{R}).$$

**Definition 2.7** *Define*

$$L_{PT} \coloneqq -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 4\operatorname{sech}^2\left(\sqrt{2}r\right) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_{PT}.$$

*This is a special case of a* Pöschl-Teller operator*. Earlier remarks allow us to write $L_{SG}$ in this way:*

$$L_{SG} = L_{PT} + 2.$$

$L_{SG}$ being a translation of a Pöschl-Teller operator is convenient, because the latter's eigenvalues are well known.

**Lemma 2.11** *The operator $A \coloneqq -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ defined on $H^2(\mathbb{R})$ is self-adjoint, and its spectrum is*

$$\sigma(A) = \sigma_{ess}(A) = [0, +\infty).$$

Proof. For $\phi, \psi \in H^2(\mathbb{R})$, a formal calculation provides the equality

$$-\int \phi''\psi \mathrm{d}r = -\left.\phi'\psi\right|_{-\infty}^{+\infty} + \int \phi'\psi' \mathrm{d}r = -\left.\phi'\psi\right|_{-\infty}^{+\infty} + \left.\phi\psi'\right|_{-\infty}^{+\infty} - \int \phi\psi'' \mathrm{d}r.$$

Because $\phi, \psi \in H^2(\mathbb{R})$, the asymptotic terms vanish and we have the equality

$$\langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle \qquad \forall\phi, \psi \in H^2(\mathbb{R}),$$

implying symmetry.

The essential spectrum of $A$ is $[0, +\infty)$. This is a standard result in spectral theory, but for the sake of completeness we sketch a proof here. Let $\lambda \in \mathbb{C}$, $\phi \in H^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$, so the Fourier transform of $A\phi \in L^2(\mathbb{R})$ is well defined. Consider the identity $(\lambda - A)\phi = \psi$ and take the Fourier transform:

$$(\lambda - \xi^2)\widehat{\phi} = \widehat{\psi}.$$

If $\lambda \in \mathbb{C}\backslash[0, +\infty)$, then $\lambda - \xi^2 \neq 0$ and we can obtain an inverse for $\lambda - A$ thanks to the inverse Fourier transform $\mathcal{F}^{-1}$:

$$\psi = \mathcal{F}^{-1}((\lambda - \xi^2)^{-1}\widehat{\psi}).$$

As a result, $\lambda \in \rho(A)$, i.e. $\sigma(A) \subseteq [0, +\infty)$.

The other inclusion can be obtained constructing a Weyl sequence. Let $\lambda \in [0, +\infty)$, and define $\phi(r) := e^{i\sqrt{\lambda}r}$. For $n \in \mathbb{N}$, write $I_n := (-n, n)$ and choose a family $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$ of cutoff functions such that $0 \leq f_n \leq 1$, $\operatorname{supp} f_n \subseteq I_{n+1}$, $f_n|_{I_n} = 1$ and the first two derivatives are bounded. Note that the derivatives $f_n'$ and $f_n''$ are only non-zero in the intervals $(n, n+1)$ and $(-n-1, -n)$, which have constant length. Therefore, the bound on the derivatives can be chosen to be uniform in $n$.

We define the new sequence $\phi_n = c_n f_n \phi$, where $c_n > 0$ is a normalization constant that ensures $\|\phi_n\|_{L^2} = 1$. If we want this to be a Weyl sequence, we need to study $\|(\lambda - A)\phi_n\|_{L^2}$. Use the fact that $f_n|_{I_n} = 1$:

$$1 = \|\phi_n\|_{L^2}^2 \geq c_n^2 \int_{-n}^{n} |f_n\phi|^2 \mathrm{d}r = c_n^2 \int_{-n}^{n} \left|e^{i\sqrt{\lambda}r}\right|^2 \mathrm{d}r = c_n^2 \int_{-n}^{n} \mathrm{d}r = 2nc_n^2,$$

thus $c_n^2 \leq 1/2n$. Now, the operator $(\lambda - A)$ acts on $\phi_n$ in the following way:

- In $I_n$, we have that $f_n = 0$ and $f_n' = f_n'' = 0$, so

$$(\lambda - A)\phi_n = \lambda c_n \phi + c_n \phi'' = c_n(\lambda - \lambda) = 0.$$

- In $I_{n+1}^{\mathsf{c}}$, $f_n = 0$ and $\phi_n = (\lambda - A)\phi_n = 0$.
- In $I_{n+1} \backslash I_n$:

$$(\lambda - A)\phi_n = \lambda c_n f_n \phi + c_n(f_n''\phi + f_n'\phi' + f_n\phi'') = c_n\phi(\lambda f_n + f_n'' + i\sqrt{\lambda} f_n' - \lambda f_n),$$

  and the bound

$$\|(\lambda - A)\phi_n\|_\infty \leq c_n(\|f_n''\|_\infty + \sqrt{\lambda}\|f_n'\|_\infty) =: c_n\alpha_\lambda$$

  holds, where $\alpha_\lambda$ is a positive constant that does not depend on $n$.

With this knowledge we can estimate the $L^2(\mathbb{R})$ norm:

$$\|(\lambda - A)\phi_n\|_{L^2}^2 = \int |(\lambda - A)\phi_n|^2 \mathrm{d}r = \int_{I_{n+1}\backslash I_n} |(\lambda - A)\phi_n|^2 \mathrm{d}r \leq \|(\lambda - A)\phi_n\|_\infty^2 \int_{I_{n+1}\backslash I_n} \mathrm{d}r$$

Replace $\|(\lambda - A)\phi_n\|_\infty^2$ with $c_n^2\alpha_\lambda^2$, and remember that $c_n^2 \leq 1/2n$ to see that

$$\|(\lambda - A)\phi_n\|_{L^2}^2 \leq c_n^2\alpha_\lambda^2 \int_{I_{n+1}\backslash I_n} \mathrm{d}r = 2c_n^2\alpha_\lambda^2 \leq \frac{\alpha_\lambda^2}{n}.$$

We have demonstrated that there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ such that $\|\phi_n\|_{L^2} = 1$ and $\|(\lambda - A)\phi_n\|_{L^2} \to 0$ as $n \to +\infty$, that is, we have constructed a Weyl sequence for $\lambda$. This implies that $\lambda \in \sigma(A)$, and because $\lambda \in [0, +\infty)$ is arbitrary, together with the other inclusion we conclude that $\sigma(A) = [0, +\infty)$.

The operator is also closed. To see this, take a sequence $(\phi_n)_{n\in\mathbb{N}} \subseteq H^2(\mathbb{R})$ and functions $\phi$, $\psi$ in $L^2(\mathbb{R})$ such that $\phi_n \to \phi$ and $A\phi_n \to \psi$ (in $L^2(\mathbb{R})$). First, we show that $\phi \in H^2(\mathbb{R})$; we need to find its weak derivatives. Let $n,\ m \in \mathbb{N}$:

$$\|\phi_n' - \phi_m'\|_{L^2}^2 = \int (\phi_n')^2 \mathrm{d}r + \int (\phi_m')^2 \mathrm{d}r - 2\int \phi_n' \phi_m \mathrm{d}r.$$

Because $\phi_n$ and $\phi_m$ belong to $H^2(\mathbb{R})$, both functions and their derivatives vanish at infinity. Next, integrate by parts:

$$\|\phi_n' - \phi_m'\|_{L^2}^2 = -\int \phi_n \phi_n'' \mathrm{d}r - \int \phi_m \phi_m'' \mathrm{d}r + 2\int \phi_n \phi_m'' \mathrm{d}r,$$

and rearrange terms:

$$\|\phi_n' - \phi_m'\|_{L^2}^2 = \int \phi_n(\phi_n'' - \phi_m'') \mathrm{d}r + \int \phi_m''(\phi_n - \phi_m) \mathrm{d}r.$$

The Cauchy-Schwarz inequality allows us to assert that

$$\|\phi_n' - \phi_m'\|_{L^2}^2 \leq \|\phi_n\|_{L^2} \|\phi_n'' - \phi_m''\|_{L^2} + \|\phi_m''\|_{L^2} \|\phi_n - \phi_m\|_{L^2};$$

both $\|\phi_n\|_{L^2}$ and $\|\phi_m''\|_{L^2} = \|A\phi_m\|_{L^2}$ are bounded, because they converge. They are also Cauchy sequences, implying that $(\phi_n')_{n\in\mathbb{N}} \subseteq L^2(\mathbb{R})$ is a Cauchy sequence too. Completeness says that there is a function $f \in L^2(\mathbb{R})$ that satisfies $\phi_n' \to f$. It is clear that this is the weak derivative of $\phi$; simply take the limit $n \to +\infty$ in both sides of the equality

$$\int \phi_n' \eta \mathrm{d}r = -\int \phi_n \eta' \mathrm{d}r \qquad \forall \eta \in C_c^\infty(\mathbb{R}).$$

Then, $f = \phi'$ and $\phi \in H^1(\mathbb{R})$.

Now we prove that $A\phi = \psi$ and $\phi \in H^2(\mathbb{R})$. Because $\phi_n \in H^2(\mathbb{R})$ for all $n \in \mathbb{N}$, we know that

$$\int A\phi_n \eta \mathrm{d}r = -\int \phi_n'' \eta \mathrm{d}r = \int \phi_n' \eta' \mathrm{d}r \qquad \forall \eta \in C_c^\infty(\mathbb{R}).$$

Once again take the limit to see that

$$\int \psi \eta = \int \phi' \eta' \mathrm{d}r \qquad \forall \eta \in C_c^\infty(\mathbb{R}).$$

This is the definition of $\phi'' = -\psi$, which belongs to $L^2(\mathbb{R})$ by hypothesis. In other words, $A\phi = \psi$ and $\phi \in H^2(\mathbb{R})$, meaning that $A$ is closed.

The operator $A$ is symmetric, closed and its spectrum $\sigma(A) = [0, +\infty)$ is a subset of $\mathbb{R}$. It follows from Theorem 2.18 in [16] that $A$ is self-adjoint. This implies that $\sigma(A) = \sigma_{\text{disc}}(A) \cup \sigma_{\text{ess}}(A)$; since $\sigma(A)$ has no isolated points, necessarily

$$\sigma(A) = \sigma_{\text{ess}}(A) = [0, +\infty).$$

□

**Theorem 2.10** *The operator $L_{PT}$ is self-adjoint.*

Proof. Start by defining the differential operator $A := -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ in $H^2(\mathbb{R})$ as before, and the multiplication operator $V_{PT} = -4\,\mathrm{sech}^2\big(\sqrt{2}r\big)$ in $L^2(\mathbb{R}) \supseteq H^2(\mathbb{R})$.

Clearly, $V_{PT}$ is symmetric. Moreover, for $\phi \in H^2(\mathbb{R})$ we have that

$$\|V_{PT}\phi\|_{L^2} \leq \left\|4\,\mathrm{sech}^2\big(\sqrt{2}\,\cdot\,\big)\right\|_\infty \|\phi\|_{L^2} = 4\|\phi\|_{L^2} \leq 4\|\phi\|_{H^2}.$$

This implies that $V_{PT}$ is bounded relative to $A$, with $A$-bound equal to 0:

$$\|V_{PT}\phi\|_{L^2} \leq 0 \cdot \|A\phi\|_{L^2} + 4\|\phi\|_{H^2}$$

So far, from Lemma 2.11 we know $A$ is self-adjoint in $H^2(\mathbb{R})$, $V_{PT}$ is symmetric in $L^2(\mathbb{R})$ and its $A$-bound is less than one. The Kato-Rellich theorem (see Theorem 6.4 in [16]) states that $A + V_{PT} = L_{PT}$ is self-adjoint in $H^2(\mathbb{R})$, completing the proof. □

**Corollary 2.7** *The operator $L_{SG}$ is self-adjoint.*

Proof. The sum $L_{SG} = L_{PT} + 2$ of a self-adjoint and a bounded operator is self-adjoint. Alternatively, because $2\cos(2H_{SG})$ is bounded one can apply the Kato-Rellich theorem directly to $L_{SG}$. □

**Theorem 2.11** *The only eigenvalue of the operator $L_{PT}$ is $\lambda_{SG} = 0$.*

Proof. The eigenvalues of $L_{PT}$ are studied in [17], specifically in Problem 12 from Section 1. To paraphrase the result, for a Pöschl-Teller operator of the form

$$-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - V_0\,\mathrm{sech}^2\left(\frac{x}{a}\right)$$

with positive constants $\hbar, \mu, V_0$ and $a$, the eigenvalues are

$$E_n = -\frac{\hbar^2}{2\mu a^2}\left(\frac{1}{2}\sqrt{\frac{8\mu V_0 a^2}{\hbar^2} + 1} - \left(n + \frac{1}{2}\right)\right)^2,$$

with $n = 0, 1, \ldots, N$ and $N$ the largest non-negative integer such that

$$2N < \sqrt{\frac{8\mu V_0 a^2}{\hbar^2} + 1} - 1.$$

The change of variables $x = \hbar r/\sqrt{2\mu}$, $\psi(x) = \phi(r)$ takes $L_{PT}$ to the form described in [17]:

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} - 4\,\mathrm{sech}^2\big(\sqrt{2}r\big)\phi(r) = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - 4\,\mathrm{sech}^2\left(\frac{x}{\frac{\hbar}{2\sqrt{\mu}}}\right)\psi(x).$$

In our case, we see that $V_0 = 4$ and $a = \hbar/2\sqrt{\mu}$. This means that

$$2N = 2 < \sqrt{\frac{8\mu 4\hbar^2}{4\hbar^2\mu} + 1} - 1 = \sqrt{9} - 1 = 2;$$

therefore, $N = 0$ and the only eigenvalue $\lambda_{PT}$ of $L_{PT}$ is given by the expression

$$\lambda_{PT} = E_0 = -\frac{\hbar^2}{2\mu}\frac{4\mu}{\hbar^2}\left(\frac{3}{2} - \frac{1}{2}\right)^2 = -2.$$

□

**Theorem 2.12** *The spectrum of $L_{PT}$ is*

$$\sigma(L_{PT}) = \{-2\} \cup [0, +\infty).$$

Proof. We invoke Theorem 2.15 from [18], which we paraphrase now. Define $A$ as the unique self-adjoint extension of $-\frac{\mathrm{d}^2}{\mathrm{d}r^2} \colon C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$: clearly, $A$ is just the second derivative in $H^2(\mathbb{R})$. Let $V \colon \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ be a bounded, (piecewise) continuous potential such that $V(r) \to 0$ as $|r| \to +\infty$. Assume that $V$ is $A$-bounded with relative bound strictly less than 1. Then $A + V$ is self-adjoint and

$$\sigma_{\text{ess}}(A + V) = \sigma_{\text{ess}}(A) = [0, +\infty).$$

Our potential $V_{PT}(r) = -4\,\mathrm{sech}^2\left(\sqrt{2}r\right)$ is obviously bounded, continuous and tends to 0 as $|r|$ grows. Moreover, we demonstrated in the proof of Theorem 2.10 that it is $A$-bounded by 0. The theorem then states that

$$\sigma_{\text{ess}}(L_{PT}) = \sigma_{\text{ess}}(A + V_{PT}) = [0, +\infty).$$

Additionally, because $L_{PT}$ is self-adjoint, we can characterize its discrete spectrum $\sigma_{\text{disc}} = \sigma(L_{PT}) \backslash \sigma_{\text{ess}}(L_{PT})$ as the set of isolated eigenvalues with finite multiplicity. Theorem 2.11 shows that $\sigma_{\text{disc}}(L_{PT}) \subseteq \{-2\}$, as $-2$ is an eigenvalue that is isolated from $[0, +\infty)$. Because $L_{PT}$ is a second order differential operator, the dimension of the eigenspace of $\lambda_{PT}$ can be 2 at most (and we shall see that it is actually 1). This implies that $\sigma_{\text{disc}}(L_{PT}) = \{-2\}$, and the result

$$\sigma(L_{PT}) = \sigma_{\text{disc}}(L_{PT}) \cup \sigma_{\text{ess}}(L_{PT}) = \{-2\} \cup [0, +\infty)$$

follows. □

**Corollary 2.8** *The spectrum of $L_{SG}$ is*

$$\sigma(L_{SG}) = \{0\} \cup [2, +\infty).$$

A visual representation of $\sigma(L_{SG})$ is shown in Figure 2.3.

We also find the eigenfunction associated to $\lambda_{SG} = 0$.

**Lemma 2.12** *The general solution to the equation*

$$-L_{SG}v = v'' + 2\cos(2H_{SG})v = 0$$

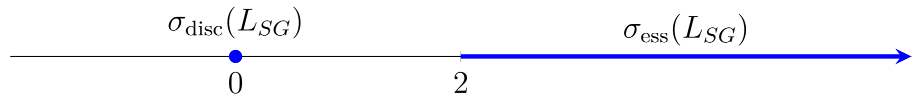


Figure 2.3: Spectrum of $L_{SG}$.

*is of the form*

$$v(r) = c_1 \operatorname{sech}\left(\sqrt{2}r\right) + c_2 \operatorname{sech}\left(\sqrt{2}r\right)\left(2\sqrt{2}r + \sinh\left(2\sqrt{2}r\right)\right)$$

*where $c_1$ and $c_2$ are real constants.*

Proof. We check that $v_1 = H'_{SG}$ is indeed a solution:

$$(H''_{SG})' + 2\cos(2H_{SG})H'_{SG} = -(\sin(2H_{SG}))' + (\sin(2H_{SG}))' = 0.$$

Now we can apply the method of reduction of order. Let $v_2 = uv_1$ be the other solution. The function $u$ must satisfy the identity

$$u''H'_{SG} + 2u'H''_{SG} = 0,$$

and using the fact that $H'_{SG}$ is positive everywhere, we solve by inspection to obtain

$$u(r) = c_2 + c_3 \int_{r_0}^{r} \frac{1}{(H'_{SG})^2}\mathrm{d}s.$$

The expression for $v_2$ that we have found allows us to write the general solution as

$$v(r) = c_1 \operatorname{sech}\left(\sqrt{2}r\right) + c_2 \operatorname{sech}\left(\sqrt{2}r\right) \int_{r_0}^{r} \cosh^2\left(\sqrt{2}s\right)\mathrm{d}s.$$

The integral can be solved explictly to obtain the desired result:

$$v(r) = c_1 \operatorname{sech}\left(\sqrt{2}r\right) + c_2 \operatorname{sech}\left(\sqrt{2}r\right)\left(2\sqrt{2}r + \sinh\left(2\sqrt{2}r\right)\right).$$

□

**Remark** After absorbing constants, the general solution can be written as

$$v(r) = c_1 H'_{SG} + c_2\left(rH'_{SG} - \frac{H''_{SG}}{(H'_{SG})^2}\right).$$

**Corollary 2.9** *The eigenspace of $\lambda_{SG}$ is spanned by $H'_{SG}$. In other words, $H'_{SG}$ is the only eigenfunction of $\lambda_{SG}$, up to a constant.*

Proof. It is true that the equation $L_{SG}v = 0$ has two linearly independent solutions $v_1 = H'_{SG}$ and $v_2$. However, $v_2$ is unbounded; only $H'_{SG}$ belongs to $L^2(\mathbb{R})$, making it the eigenfunction that corresponds to the eigenvalue $\lambda_{SG} = 0$. □

The final theorem of this section will ve vital for future proofs.

**Remark** Any mention of orthogonality refers to the usual $L^2(\mathbb{R})$ inner product, unless stated otherwise.

**Theorem 2.13** *The operator $L_{SG}$ is non-negative. It is also coercive in* $\text{span}\{H'_{SG}\}^\perp \cap H^2(\mathbb{R}) \subseteq L^2(\mathbb{R})$ *for both the $L^2$ and $H^1$-norms:*

$$Q_{SG}(\phi) \geq 2\|\phi\|^2_{L^2}, \qquad Q_{SG}(\phi) \geq \frac{2}{5}\|\phi\|^2_{H^1} \qquad \forall\phi \in \text{span}\{H'_{SG}\}^\perp \cap H^2(\mathbb{R}).$$

Proof. As $L_{SG}$ is self-adjoint, from Theorem 2.19 in [16] we see that

$$0 = \inf \sigma(L_{SG}) = \inf_{\substack{\psi\in H^2(\mathbb{R})\\ \|\psi\|_{L^2}=1}} \langle L_{SG}\psi, \psi\rangle \leq \langle L_{SG}\phi, \phi\rangle \qquad \forall\phi \in H^2(\mathbb{R}).$$

Coercivity in the $L^2$-norm follows from the min-max theorem for self-adjoint operators (Theorem 4.10 in [16]). For $\eta \in L^2(\mathbb{R})$, define the set $S_\eta := \text{span}\{\eta\}^\perp \cap H^2(\mathbb{R})$; the intersection with $H^2(\mathbb{R})$ ensures that we stay in the domain of $L_{SG}$. The only eigenvalue is 0, and the infimum of $\sigma_{\text{ess}}(L_{SG})$ is 2. Suppose that

$$\inf_{\substack{\psi\in S_{H'_{SG}}\\ \|\psi\|_{L^2}=1}} \langle L_{SG}\psi, \psi\rangle =: \beta < 2;$$

then $\beta$ would be an eigenvalue, with a corresponding eigenfunction $\psi_\beta$. But the only eigenvalue of $L_{SG}$ is 0, and it has one eigenfunction $H'_{SG}$ that is orthogonal to $\psi_\beta$. This is clearly a contradiction, so

$$2 \leq \inf_{\substack{\psi\in S_{H'_{SG}}\\ \|\psi\|_{L^2}=1}} \langle L_{SG}\psi, \psi\rangle = \inf_{\psi\in S_{H'_{SG}}} \frac{\langle L_{SG}\psi, \psi\rangle}{\|\psi\|^2_{L^2}}.$$

Equivalently:

$$\langle L_{SG}\psi, \psi\rangle \geq 2\|\psi\|^2_{L^2}$$

for all $\psi \in H^2(\mathbb{R})$ that are orthogonal to $H'_{SG}$ with the $L^2(\mathbb{R})$ inner product.

Coercivity in $H^1(\mathbb{R})$ is deduced from the previous result. For any $\phi \in \text{span}\{H'_{SG}\}^\perp$ whose derivative vanishes at infinity, integration by parts applies to the definition for $Q_{SG}$ says that

$$\langle L_{SG}\phi, \phi\rangle = Q_{SG}(\phi) = -\int \phi''\phi \mathrm{d}r - \int 2\cos(2H_{SG})\phi^2 \mathrm{d}r = \int (\phi')^2 \mathrm{d}r - \int 2\cos(2H_{SG})\phi^2 \mathrm{d}r.$$

Let $\delta > 0$, and substract $\delta\|\phi'\|^2_{L^2}$ on both sides:

$$Q_{SG}(\phi) - \delta\|\phi'\|^2_{L^2} = (1-\delta)\|\phi'\|^2_{L^2} - \int 2\cos(2H_{SG})\phi^2 \mathrm{d}r.$$

Add and substract terms to obtain

$$Q_{SG}(\phi) - \delta\|\phi'\|^2_{L^2} = (1-\delta)\Big(\|\phi'\|^2_{L^2} - \int 2\cos(2H_{SG})\phi^2 \mathrm{d}r\Big) - \delta\int 2\cos(2H_{SG})\phi^2 \mathrm{d}r.$$

It is easy to see that $\|\cos(2H_{SG})\|_\infty = 1$, which gives us the lower bound

$$Q_{SG}(\phi) - \delta\|\phi'\|_{L^2}^2 \geq (1-\delta)Q_{SG}(\phi) - 2\delta\int \phi^2 \mathrm{d}r = (1-\delta)Q_{SG}(\phi) - 2\delta\|\phi\|_{L^2}^2.$$

Thanks to coercivity in $L^2(\mathbb{R})$, the next inequality holds for $\delta < 1$:

$$Q_{SG}(\phi) - \delta\|\phi'\|_{L^2}^2 \geq 2(1-\delta)\|\phi\|_{L^2}^2 - 2\delta\|\phi\|_{L^2}^2 = 2(1-2\delta)\|\phi\|_{L^2}^2.$$

Choose $\delta = 2/5$: then $2 - 4\delta = 2/5$, and we have

$$Q_{SG} \geq \frac{2}{5}\|\phi'\|_{L^2}^2 + \frac{2}{5}\|\phi\|_{L^2}^2 = \frac{2}{5}\|\phi\|_{H^1}^2.$$

This shows that $L_{SG}$ is coercive for the $H^1$-norm in $\operatorname{span}\{H'_{SG}\}^\perp$, thus the proof is complete. □

## 2.4. Dependence on the parameter

To simplify notation, we replace the parameter $a > 0$ with $\varepsilon \coloneqq 1/a$ so that the 1-kink $H_\varepsilon$ solves the problem:

$$(\overline{P})\begin{cases} \phi'' + \dfrac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\phi' + \sin(2\phi) = 0, & r \in \mathbb{R}, \\ \phi(0) = 0, \\ \lim\limits_{r\to\pm\infty} \phi(r) = \pm\dfrac{\pi}{2}. \end{cases}$$

The term accompanying $H'_\varepsilon$ in $(\overline{P})$ tends to 0 as $\varepsilon$ decreases, therefore the expectation is that $H_\varepsilon$ will converge to $H_{SG}$.

**Theorem 2.14** *For any sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers that converges to zero, there is a subsequence $(\varepsilon_n)$ such that $H_{\varepsilon_n}$ and its derivatives converge uniformly on compact sets to those of $H_{SG}$.*

Proof. Thanks to the symmetry present in the definition of $H_\varepsilon$, it is enough to prove this result in $\mathbb{R}_+$. For small $\varepsilon$, we know from Section 2.1 that $H_\varepsilon$, $H'_\varepsilon$ and $H''_\varepsilon$ are bounded uniformly in $\varepsilon$ on every interval. Explicitly, if we have a compact interval $I = [a, b] \subset \mathbb{R}_+$, then there are constants $C_i, i = 0, 1, 2$, independent of $\varepsilon$ such that

$$\left\|H_\varepsilon^{(i)}\right\|_{L^\infty(I)} < C_i, \quad i = 0, 1, 2.$$

To check equicontinuity we use the fundamental theorem of calculus. Take $I$ as before and $x, y \in I$.

$$|H_\varepsilon(x) - H_\varepsilon(y)| = \left|\int_x^y H'_\varepsilon(s)\mathrm{d}s\right| \leq \|H'_\varepsilon\|_{L^\infty(I)}|y - x| < C_1|x - y|.$$

We obtain Lipschitz-continuity and in consequence, uniform equicontinuity. Thanks to the Arzelà-Ascoli theorem we deduce the existence of a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $H_{\varepsilon_k}$ converges uniformly on $I$ to $H_{SG}$. The same argument holds for $H'_\varepsilon$, and we find another subsequence (which we also call $(\varepsilon_k)$) that guarantees $H'_{\varepsilon_k}$ will converge uniformly on $I$ to $H'_{SG}$.

Equicontinuity for $H''_\varepsilon$ follows from the differential equation

$$|H''_\varepsilon(x) - H''_\varepsilon(y)| = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}|H'_\varepsilon(x) - H'_\varepsilon(y)| + |\sin(2H_\varepsilon(x)) - \sin(2H_\varepsilon(y))|.$$

Since $\sin(\,\cdot\,)$ is uniformly continuous on compact intervals and both $H_\varepsilon$ and $H'_\varepsilon$ are equicontinuous, it follows that $H''_\varepsilon$ has the same property. Applying the Arzelà-Ascoli theorem again gives us a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $H_{\varepsilon_k}$, $H'_{\varepsilon_k}$ and $H''_{\varepsilon_k}$ converge uniformly in $I$.

So far we have proven that for any compact interval $I$, there is a subsequence that makes $H_{\varepsilon_k}$ and its derivatives converge uniformly in that interval. This is not exactly what we want, because a single subsequence should guarantee uniform convergence for all compact sets. We employ a diagonal argument to remedy this problem: define $I_1 := [0, 1]$, and let $(\varepsilon_{k_1})$ be the subsequence that ensures uniform convergence in $I_1$ for our functions. Now define $I_2 := [0, 2]$ and $(\varepsilon_{k_2})$ to be the subsequence of $(\varepsilon_{k_1})$ that gives convergence in $I_2$. Inductively, one defines $I_n := [0, n]$ and takes $(\varepsilon_{k_n}) \subseteq (\varepsilon_{k_{n-1}})$ to be the subsequence that makes the functions converge uniformly in $I_n$. Define the subsequence $(\varepsilon_n) \subseteq (\varepsilon_k)$ to be the $n$-th term of $(\varepsilon_{k_n})$; we claim that this is the desired subsequence.

Checking that this is true is simple. If $A$ is any compact subset of $[0, +\infty)$, then it is contained in $[0, n]$ for all naturals $n$ greater than some $N > 0$. By construction $H_{\varepsilon_n}$ and its derivatives will converge uniformly in $A$, and the theorem is proven. □

**Corollary 2.10** *The subsequence in Theorem 2.14 converges uniformly over* $\mathbb{R}$*.*

Proof. Through an abuse of notation let $(\varepsilon_k)_{k\in\mathbb{N}}$ be the sequence given by Theorem 2.14. Because $H_{SG}$ and $H_\varepsilon$ tend to the same limit as $r \to +\infty$, for any given $\delta > 0$ there is a $R > 0$ such that $|H_{SG} - H_\varepsilon| < \delta$ for all $r > R$, or equivalently

$$\sup_{r>R} |H_{SG}(r) - H_{\varepsilon_k}(r)| < \delta.$$

It is important to mention that $R$ does not depend on $\varepsilon_k$, because in Section 2.2 we derived an exponential rate of convergence that is independent of $a = 1/\varepsilon$. Because $[0, R]$ is a compact set, for large $k$ we have that

$$\sup_{r\in[0,R]} |H_{SG}(r) - H_{\varepsilon_k}(r)| < \delta,$$

and it follows that

$$\sup_{r\geq 0} |H_{SG}(r) - H_{\varepsilon_k}(r)| < \delta.$$

Parity allows us to conclude that $\|H_{SG} - H_{\varepsilon_k}\|_\infty < \delta$, deducing uniform convergence. For the derivatives $H'_{SG} - H'_{\varepsilon_k}$ and $H''_{SG} - H''_{\varepsilon_k}$ one employs an identical argument. □

Remember that $H_\varepsilon$ and $H_{SG}$ satisfy the non-linear ODEs:

$$\begin{aligned} H_\varepsilon'' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_\varepsilon' + \sin(2H_\varepsilon) = 0, \\ H_{SG}'' + \sin(2H_{SG}) = 0. \end{aligned}$$

Taking the difference and adding extra terms gives us an equation for the difference.

**Definition 2.8** *We define the function $h_\varepsilon := H_\varepsilon - H_{SG}$. It solves the ODE:*

$$h_\varepsilon'' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' + \sin(2H_\varepsilon) - \sin(2H_{SG}) = 0. \tag{2.8}$$

Now we can prove the main results of this subsection.

**Theorem 2.15** *If the difference $h_\varepsilon = H_\varepsilon - H_{SG}$ tends to zero uniformly, then its $H^1(\mathbb{R})$ norm decreases to zero quadratically in $\varepsilon$. In other words, $\|h_\varepsilon\|_{H^1} = O(\varepsilon^2)$.*

Proof. Recall that $h_\varepsilon$ satisfies equation (2.8):

$$h_\varepsilon'' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' + \sin(2H_\varepsilon) - \sin(2H_{SG}) = 0.$$

As usual, a first order approximation says that

$$\sin(2H_\varepsilon) = \sin(2H_{SG} + 2h_\varepsilon) = \sin(2H_{SG}) + 2\cos(2H_{SG})h_\varepsilon + O(|h_\varepsilon|^2),$$

and this translates into a linearized equation for $h_\varepsilon$:

$$h_\varepsilon'' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' + 2\cos(2H_{SG})h_\varepsilon + O(|h_\varepsilon|^2) = -\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}'. \tag{2.9}$$

Multiply by $h_\varepsilon$ to obtain

$$h_\varepsilon'' h_\varepsilon + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' h_\varepsilon + 2\cos(2H_{SG})h_\varepsilon^2 + O(|h_\varepsilon|^3) = -\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' h_\varepsilon.$$

Next we integrate over $\mathbb{R}$, decay properties and integration by parts say that

$$\begin{aligned} \int h_\varepsilon'' h_\varepsilon \mathrm{d}r = -\int (h_\varepsilon')^2 \mathrm{d}r, \\ \int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' h_\varepsilon \mathrm{d}r = -\int \varepsilon^2 \frac{1 - \varepsilon^2 r^2}{(1 + \varepsilon^2 r^2)^2} h_\varepsilon^2 \mathrm{d}r. \end{aligned}$$

The equation becomes

$$\int (h_\varepsilon')^2 \mathrm{d}r + \int \left( \varepsilon^2 \frac{1 - \varepsilon^2 r^2}{(1 + \varepsilon^2 r^2)^2} - 2\cos(2H_{SG}) \right) h_\varepsilon^2 \mathrm{d}r = \int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' h_\varepsilon \mathrm{d}r + \int O(|h_\varepsilon|^3) \mathrm{d}r.$$

We call the quadratic form on the left side $Q$, and remark that it is identical to $Q_{SG}$ except for a correction term in the potential:

$$Q(h_\varepsilon) = Q_{SG}(h_\varepsilon) + \varepsilon^2 \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2} h_\varepsilon^2 \mathrm{d}r = \int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H'_{SG} h_\varepsilon \mathrm{d}r + \int O(|h_\varepsilon|^3) \mathrm{d}r.$$

The function $h_\varepsilon = H_\varepsilon - H_{SG}$ is odd, and $H'_{SG}$ is even. Their product is odd and its integral over $\mathbb{R}$ vanishes:

$$\langle h_\varepsilon, H'_{SG} \rangle = \int h_\varepsilon H'_{SG} \mathrm{d}r = 0.$$

In other words, $h_\varepsilon$ is orthogonal to $H_{SG}$. Theorem 2.13 says that

$$Q_{SG}(h_\varepsilon) \geq 2\|h_\varepsilon\|_{L^2}^2,$$

and we recover an inequality for $Q$ by adding the leftover term:

$$Q(h_\varepsilon) \geq \int \left(2 + \varepsilon^2 \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}\right) |h_\varepsilon|^2 \mathrm{d}r.$$

Notice that

$$\frac{|1-\varepsilon^2 r^2|}{(1+\varepsilon^2 r^2)^2} \leq 1 \qquad \forall r \in \mathbb{R},$$

and with this we can get a lower bound for $Q$ in terms of the $L^2$-norm:

$$Q(h_\varepsilon) \geq 2\|h_\varepsilon\|_{L^2}^2 + \varepsilon \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2} |h_\varepsilon|^2 \mathrm{d}r \geq 2\|h_\varepsilon\|_{L^2}^2 - \varepsilon^2 \int |h_\varepsilon|^2 \mathrm{d}r = (2-\varepsilon^2)\|h_\varepsilon\|_{L^2}^2 \quad (2.10)$$

We also search for an upper bound on $Q$. We can take the absolute value to get

$$Q(h_\varepsilon) \leq \int \frac{2\varepsilon^2}{\varepsilon^2 r^2 + 1} |r H'_{SG}| |h_\varepsilon| \mathrm{d}r + \int \left| O(|h_\varepsilon|^3) \right| \mathrm{d}r.$$

By definition, there exists a constant $C > 0$ such that $\left| O(|h_\varepsilon|^3) \right| < C|h_\varepsilon|^3$:

$$Q(h_\varepsilon) \leq \int \frac{2\varepsilon^2}{\varepsilon^2 r^2 + 1} |r H'_{SG}| |h_\varepsilon| \mathrm{d}r + C \int |h_\varepsilon|^3 \mathrm{d}r. \quad (2.11)$$

On one hand, we use the Cauchy-Schwartz inequality on the first integral:

$$\int \frac{2\varepsilon^2}{\varepsilon^2 r^2 + 1} |r H'_{SG}| |h_\varepsilon| \mathrm{d}r \leq \left(4\varepsilon^4 \int \frac{r^2 (H'_{SG})^2}{(\varepsilon^2 r^2 + 1)^2} \mathrm{d}r\right)^{\frac{1}{2}} \left(\int |h_\varepsilon|^2 \mathrm{d}r\right)^{\frac{1}{2}},$$

and noting that $ab \leq a^2/2 + b^2/2$ for any real numbers $a, b$ we get:

$$\int \frac{2\varepsilon^2}{\varepsilon^2 r^2 + 1} |r H'_{SG}| |h_\varepsilon| \mathrm{d}r \leq 2\varepsilon^4 \int \frac{r^2 (H'_{SG})^2}{(\varepsilon^2 r^2 + 1)^2} \mathrm{d}r + \frac{1}{2} \int |h_\varepsilon|^2 \mathrm{d}r. \quad (2.12)$$

On the other hand:

$$\int_R |h_\varepsilon|^3 \mathrm{d}r < \|h_\varepsilon\|_\infty \int |h_\varepsilon|^2 \mathrm{d}r. \quad (2.13)$$

These two inequalities help us find a bound for $\|h_\varepsilon\|_{L^2}$, because inequality (2.11) expands to

$$Q(h_\varepsilon) \leq 2\varepsilon^4 \int \frac{r^2(H'_{SG})^2}{(\varepsilon^2 r^2+1)^2}\mathrm{d}r + \frac{1}{2}\int |h_\varepsilon|^2 \mathrm{d}r + C\|h_\varepsilon\|_\infty \int |h_\varepsilon|^2\mathrm{d}r.$$

The lower bound (2.10) states that

$$(2-\varepsilon^2)\|h_\varepsilon\|_{L^2}^2 \leq 2\varepsilon^4 \int \frac{r^2(H'_{SG})^2}{(\varepsilon^2 r^2+1)^2}\mathrm{d}r + \frac{1}{2}\|h_\varepsilon\|_{L^2}^2 + C\|h_\varepsilon\|_\infty \|h_\varepsilon\|_{L^2}^2,$$

or alternatively

$$\left(\frac{3}{2} - \varepsilon^2 - C\|h_\varepsilon\|_\infty\right)\|h_\varepsilon\|_{L^2}^2 \leq 2\varepsilon^4 \int \frac{r^2(H'_{SG})^2}{(\varepsilon^2 r^2+1)^2}\mathrm{d}r \leq 2\varepsilon^4 \int r^2(H'_{SG})^2\mathrm{d}r.$$

The integral on the right side is positive, finite and constant for $\varepsilon$. The key remark here is that $\|h_\varepsilon\|_\infty$ converges to zero as $\varepsilon \to 0^+$ by hypothesis. This means that the coefficient $3/2 - \varepsilon^2 - C\|h_\varepsilon\|_\infty$ can be bounded below by a positive constant that is also independent of $\varepsilon$. Abusing notation we absorb all these constants into another, called $C > 0$, to assert that, for sufficiently small $\varepsilon$:

$$\|h_\varepsilon\|_{L^2}^2 \leq C\varepsilon^4.$$

Equivalently

$$\|h_\varepsilon\|_{L^2} = O(\varepsilon^2). \tag{2.14}$$

The next step is studying $\|h'_\varepsilon\|_{L^2}$. We know that

$$\int (h'_\varepsilon)^2\mathrm{d}r = -\int \left(\varepsilon^2 \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2} - 2\cos(2H_{SG})\right) h_\varepsilon^2 \mathrm{d}r + \int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1} H'_{SG} h_\varepsilon \mathrm{d}r + \int O(|h_\varepsilon|^3)\mathrm{d}r,$$

and taking the modulus shows that

$$\int (h'_\varepsilon)^2\mathrm{d}r \leq \int \left|\varepsilon^2 \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2} - 2\cos(2H_{SG})\right| |h_\varepsilon|^2 \mathrm{d}r + \int \frac{2\varepsilon^2}{\varepsilon^2 r^2+1} |rH'_{SG}||h_\varepsilon|\mathrm{d}r + \int \left|O(|h_\varepsilon|^3)\right|\mathrm{d}r.$$

The expression

$$\left|\varepsilon^2 \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2} - 2\cos(2H_{SG})\right|$$

is bounded in $\mathbb{R}$ by a constant independent of $\varepsilon$, for all $\varepsilon$ sufficiently small. For the other integrals we use inequalities (2.12) and (2.13) to see that (after absorbing constants)

$$\int (h'_\varepsilon)^2\mathrm{d}r \leq C\|h_\varepsilon\|_{L^2}^2 + C\varepsilon^4 + \frac{1}{2}\|h_\varepsilon\|_{L^2}^2 + C\|h_\varepsilon\|_\infty \|h_\varepsilon\|_{L^2}^2.$$

But we know that $\|h_\varepsilon\|_{L^2}^2$ is of order $O(\varepsilon^4)$. The hypothesis that $\|h_\varepsilon\|_\infty$ vanishes as $\varepsilon \to 0^+$ ensures that it is not an obstruction for the estimates we desire. Thus, we deduce that

$$\|h'_\varepsilon\|_{L^2} = O(\varepsilon^2). \tag{2.15}$$

Inequalities (2.14) and (2.15) imply that

$$\|h_\varepsilon\|_{H^1} = \left(\|h_\varepsilon\|_{L^2}^2 + \|h'_\varepsilon\|_{L^2}^2\right)^{\frac{1}{2}} = O(\varepsilon^2),$$

proving the theorem.

□

**Corollary 2.11** *If the difference* $h_\varepsilon = H_\varepsilon - H_{SG}$ *tends to zero uniformly, then* $\|h_\varepsilon\|_\infty = O(\varepsilon^2)$.

Proof. A standard embedding result (see Theorem 8.8 in [19]) asserts that there is a constant $k$ such that $\|h_\varepsilon\|_\infty \leq k\|h_\varepsilon\|_{H^1}$. This is sufficient to conclude that

$$\|h_\varepsilon\|_\infty = O(\varepsilon^2).$$

□

**Theorem 2.16** *The subsequence* $(h_{\varepsilon_n})_n$ *given by Theorem 2.14 converges uniformly and in* $H^1$*, and the rate of convergence is quadratic for both. The same is true for* $(h'_{\varepsilon_n})_n$.

Proof. A direct application of Corollary 2.10, Theorem 2.15 and Corollary 2.11 proves the first part. To alleviate notation we obviate the subscript $n$ and write $h_\varepsilon$.

As seen in the proof of Theorem 2.15, $h_\varepsilon$ satisfies the linearized equation (2.9):

$$h''_\varepsilon + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}h'_\varepsilon + 2\cos(2H_{SG})h_\varepsilon + O(|h_\varepsilon|^2) = -\frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}H'_{SG}.$$

Multiply both sides by $h''_\varepsilon$ and integrate to arrive at a new equation:

$$\|h''_\varepsilon\|^2_{L^2} + \int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}h'_\varepsilon h''_\varepsilon \mathrm{d}r + \int 2\cos(2H_{SG})h_\varepsilon h''_\varepsilon \mathrm{d}r + \int h''_\varepsilon O(|h_\varepsilon|^2)\mathrm{d}r = -\int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}H'_{SG}h''_\varepsilon \mathrm{d}r.$$

$h_\varepsilon$ and its derivatives converge to zero as $|r| \to +\infty$; thus, integration by parts shows that

$$\begin{aligned}
\int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}h'_\varepsilon h''_\varepsilon \mathrm{d}r &= -\varepsilon^2 \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}(h'_\varepsilon)^2 \mathrm{d}r,\\
\int 2\cos(2H_{SG})h_\varepsilon h''_\varepsilon \mathrm{d}r &= 4\int \sin(2H_{SG})H'_{SG}h_\varepsilon h'_\varepsilon \mathrm{d}r - 2\int \cos(2H_{SG})(h'_\varepsilon)^2 \mathrm{d}r,\\
\int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2+1}H'_{SG}h''_\varepsilon \mathrm{d}r &= -2\varepsilon^2 \int \frac{r}{\varepsilon^2 r^2+1}H''_{SG}h'_\varepsilon \mathrm{d}r - 2\varepsilon^2 \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}H'_{SG}h'_\varepsilon \mathrm{d}r.
\end{aligned}$$

This implies the equality

$$\begin{aligned}
\|h''_\varepsilon\|^2_{L^2} = \;& 2\varepsilon^2 \int \frac{r}{\varepsilon^2 r^2+1}H''_{SG}h'_\varepsilon \mathrm{d}r + 2\varepsilon^2 \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}H'_{SG}h'_\varepsilon \mathrm{d}r + \varepsilon^2 \int \frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}(h'_\varepsilon)^2 \mathrm{d}r\\
&+ 2\int \cos(2H_{SG})(h'_\varepsilon)^2 \mathrm{d}r - 4\int \sin(2H_{SG})H'_{SG}h_\varepsilon h'_\varepsilon \mathrm{d}r - \int h''_\varepsilon O(|h_\varepsilon|^2)\mathrm{d}r.
\end{aligned}$$

Now we take the modulus and bound some expressions, listed here for clarity:

$$\left|\frac{r}{\varepsilon^2 r^2+1}\right| \leq |r|, \quad \left|\frac{1-\varepsilon^2 r^2}{(1+\varepsilon^2 r^2)^2}\right| \leq 1, \quad |\cos(2H_{SG})| \leq 1, \quad |\sin(2H_{SG})H'_{SG}| \leq 1.$$

With this we can bound the $L^2$-norm of $h''_\varepsilon$:

$$\begin{aligned}\|h''_\varepsilon\|^2_{L^2} \leq\ & 2\varepsilon^2 \int |rH''_{SG}||h'_\varepsilon|\mathrm{d}r + 2\varepsilon^2 \int |H'_{SG}||h'_\varepsilon|\mathrm{d}r + \varepsilon^2 \int |h'_\varepsilon|^2\mathrm{d}r \\ & + 2\int |h'_\varepsilon|^2\mathrm{d}r + 4\int |h_\varepsilon||h'_\varepsilon|\mathrm{d}r + \int |h''_\varepsilon|\Big|O(|h_\varepsilon|^2)\Big|\mathrm{d}r.\end{aligned}$$

An application of Cauchy-Schwarz's inequality on the first, second and fifth integrals result in

$$\begin{aligned}\|h''_\varepsilon\|^2_{L^2} \leq\ & 2\varepsilon^2\|rH''_{SG}\|_{L^2}\|h'_\varepsilon\|_{L^2} + 2\varepsilon^2\|H'_{SG}\|_{L^2}\|h'_\varepsilon\|_{L^2} + \varepsilon^2\|h'_\varepsilon\|^2_{L^2} \\ & + 2\|h'_\varepsilon\|^2_{L^2} + 4\|h_\varepsilon\|_{L^2}\|h'_\varepsilon\|_{L^2} + \int |h''_\varepsilon|\Big|O(|h_\varepsilon|^2)\Big|\mathrm{d}r.\end{aligned}$$

We know that $H''_{SG}(r) = -2\tanh\Big(\sqrt{2}r\Big)\,\mathrm{sech}\Big(\sqrt{2}r\Big)$, which means $\|rH''_{SG}\|_{L^2}$ is finite. In addition, $|h''_\varepsilon|$ is bounded uniformly and indepently of $\varepsilon$ thanks to Corollary 2.10: simply take $\varepsilon$ small enough to guarantee that $\|h''_\varepsilon\|_\infty < 1$. Also, $4\|h_\varepsilon\|_{L^2}\|h'_\varepsilon\|_{L^2} \leq 2\|h_\varepsilon\|^2_{L^2} + 2\|h'_\varepsilon\|^2_{L^2}$. Using that $\Big|O(|h_\varepsilon|^2)\Big| < C|h_\varepsilon|^2$ for some positive constant $C$ we have the equality:

$$\|h''_\varepsilon\|^2_{L^2} = 2\varepsilon^2(\|rH''_{SG}\|_{L^2} + \|H'_{SG}\|_{L^2})\|h'_\varepsilon\|_{L^2} + \Big(2+\varepsilon^2\Big)\|h'_\varepsilon\|^2_{L^2} + 2\|h_\varepsilon\|^2_{H^1} + C\|h_\varepsilon\|^2_{L^2}.$$

As demonstrated earlier, in equations (2.14) and (2.15) we see that both $\|h_\varepsilon\|_{L^2}$ and $\|h'_\varepsilon\|_{L^2}$ are quadratic in $\varepsilon$. Thanks to this we deduce that

$$\|h''_\varepsilon\|^2_{L^2} = 2(\|rH''_{SG}\|_{L^2} + \|H'_{SG}\|_{L^2})\varepsilon^2 O(\varepsilon^2) + \Big(2+\varepsilon^2\Big)O(\varepsilon^4) + 2O(\varepsilon^4) + CO(\varepsilon^4).$$

This gives us the rate of convergence we need:

$$\|h''_\varepsilon\|_{L^2} = O(\varepsilon^2).$$

Like we mentioned previously, $\|h'_\varepsilon\|_{L^2}$ is also of order $O(\varepsilon^2)$. This implies that

$$\|h'_\varepsilon\|_{H^1} = O(\varepsilon^2).$$

Finally, we use the same Sobolev embedding invoked in the proof of Corollary 2.11 to conclude that

$$\|h'_\varepsilon\|_\infty = O(\varepsilon^2).$$

□

From now on, when we take limits of the form $\varepsilon \to 0^+$ that involve $H_\varepsilon$, they will be understood to mean the subsequence $H_{\varepsilon_k}$ from Theorem 2.14. As an application of this result, we show that the solution $H_\varepsilon$ for problem $(\overline{P}_1)$ in Section 2.1 is unique.

**Theorem 2.17** *The function $H_\varepsilon$ is the only odd solution to problem*

$$(\overline{P})\begin{cases}\phi'' + \dfrac{2r}{r^2+a^2}\phi' + \sin(2\phi) = 0, & r \in \mathbb{R}, \\ \displaystyle\lim_{r\to\pm\infty}\phi(r) = \pm\frac{\pi}{2}, & \end{cases}$$

*for all sufficiently small $\varepsilon$.*

Proof. Suppose there is another odd solution $\overline{H}_\varepsilon$. Define $h_\varepsilon = H_\varepsilon - H_{SG}$ as usual, and $\overline{h}_\varepsilon = \overline{H}_\varepsilon - H_{SG}$. These functions are odd, and hence orthogonal to $H'_{SG}$ as explained in the proof of Theorem 2.15. The results so far apply to both, since they do not rely on uniqueness: this means both are quadratic in $\varepsilon$.

The non-linear equation for $h_\varepsilon$ is

$$h''_\varepsilon + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}(h'_\varepsilon + H'_{SG}) + \sin(2H_\varepsilon) - \sin(2H_{SG}) = 0.$$

The Taylor series for $f := \sin(2\,\cdot\,)$ centered at $H_{SG}$ is

$$f(H_\varepsilon) = \sin(2H_\varepsilon) = \sum_{n=0}^{\infty} a_n (H_\varepsilon - H_{SG})^n = \sum_{n=0}^{\infty} a_n h_\varepsilon^n, \qquad a_n = \frac{f^{(n)}(H_{SG})}{n!} \quad \forall n \in \mathbb{N}_0,$$

which can be replaced in the equation for $h_\varepsilon$ to obtain

$$h''_\varepsilon + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}(h'_\varepsilon + H'_{SG}) + \sin(2H_{SG}) + 2\cos(2H_{SG})h_\varepsilon + \sum_{n=2}^{\infty} a_n h_\varepsilon^n - \sin(2H_{SG}) = 0.$$

This is equivalent to

$$L_{SG} h_\varepsilon = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}(h'_\varepsilon + H'_{SG}) + \sum_{n=2}^{\infty} a_n h_\varepsilon^n,$$

and the very same argument shows that

$$L_{SG} \overline{h}_\varepsilon = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\left(\overline{h}'_\varepsilon + H'_{SG}\right) + \sum_{n=2}^{\infty} a_n \overline{h}_\varepsilon^n.$$

Define $g_\varepsilon := h_\varepsilon - \overline{h}_\varepsilon = H_\varepsilon - \overline{H}_\varepsilon$, which is also orthogonal to $H_{SG}$ and of order $O(\varepsilon^2)$. The difference between the previous equalities shows that $g_\varepsilon$ solves

$$L_{SG} g_\varepsilon = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} g'_\varepsilon + \sum_{n=2}^{\infty} a_n \left(h_\varepsilon^n - \overline{h}_\varepsilon^n\right).$$

Use the elemental identity

$$h_\varepsilon^n - \overline{h}_\varepsilon^n = (h_\varepsilon - \overline{h}_\varepsilon) \sum_{j=0}^{n-1} h_\varepsilon^{n-j-1} \overline{h}_\varepsilon^j$$

to see that

$$L_{SG} g_\varepsilon = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} g'_\varepsilon + g_\varepsilon \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} a_n h_\varepsilon^{n-j-1} \overline{h}_\varepsilon^j.$$

Multiply by $g_\varepsilon$ and integrate to obtain an expression for $\langle L_{SG} g_\varepsilon, g_\varepsilon \rangle = Q_{SG}(g_\varepsilon)$. Integration by parts shows that

$$\int \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} g_\varepsilon g'_\varepsilon \mathrm{d}r = \varepsilon^2 \int \frac{\varepsilon^2 r^2 - 1}{(\varepsilon^2 r^2 + 1)^2} g_\varepsilon^2 \mathrm{d}r,$$

thus

$$Q_{SG}(g_\varepsilon) = \varepsilon^2 \int \frac{\varepsilon^2 r^2 - 1}{(\varepsilon^2 r^2 + 1)^2} g_\varepsilon^2 \mathrm{d}r + \int g_\varepsilon^2 \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} a_n h_\varepsilon^{n-j-1} \overline{h}_\varepsilon^j \mathrm{d}r.$$

Next we take the absolute value of the right side:

$$Q_{SG}(g_\varepsilon) \le \varepsilon^2 \int \frac{|\varepsilon^2 r^2 - 1|}{(\varepsilon^2 r^2 + 1)^2} g_\varepsilon^2 \mathrm{d}r + \int g_\varepsilon^2 \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} |a_n| |h_\varepsilon|^{n-j-1} \left|\overline{h}_\varepsilon\right|^j \mathrm{d}r.$$

On one hand, it is easy to see that

$$\frac{|\varepsilon^2 r^2 - 1|}{(\varepsilon^2 r^2 + 1)^2} \le 1;$$

on the other, we have that

$$|a_n| = \begin{cases} \frac{2^n}{n!} |\sin(2H_{SG})| \text{ if } n \text{ is even,} \\ \frac{2^n}{n!} |\cos(2H_{SG})| \text{ if } n \text{ is odd.} \end{cases}$$

In either case, $|a_n| \le 2^n/n!$ and we deduce that

$$Q_{SG}(g_\varepsilon) \le \varepsilon^2 \|g_\varepsilon\|_{L^2}^2 + \int g_\varepsilon^2 \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} \frac{2^n}{n!} |h_\varepsilon|^{n-j-1} \left|\overline{h}_\varepsilon\right|^j \mathrm{d}r.$$

To proceed, bound $|h_\varepsilon|$ and $\left|\overline{h}_\varepsilon\right|$ by $\|h_\varepsilon\|_\infty$ and $\left\|\overline{h}_\varepsilon\right\|_\infty$ respectively. Earlier results indicate that these norms are of order $O(\varepsilon^2)$, that is, there is a constant $C > 0$ (we can choose the same constant for both without loss of generality) such that $\|h_\varepsilon\|_\infty \le C\varepsilon^2$ and $\left\|\overline{h}_\varepsilon\right\|_\infty \le C\varepsilon^2$ if $\varepsilon^2$ is sufficiently small. It follows that

$$\sum_{j=0}^{n-1} \frac{2^n}{n!} \|h_\varepsilon\|_\infty^{n-j-1} \left\|\overline{h}_\varepsilon\right\|_\infty^j \le \sum_{j=0}^{n-1} \frac{2^n}{n!} (C\varepsilon^2)^{n-j-1} (C\varepsilon^2)^j = \sum_{j=0}^{n-1} \frac{2^n}{n!} (C\varepsilon^2)^{n-1} = \frac{2^n}{n!} (C\varepsilon^2)^{n-1} n.$$

The series can then be bounded by

$$\sum_{n=2}^{\infty} \sum_{j=0}^{n-1} \frac{2^n}{n!} |h_\varepsilon|^{n-j-1} \left|\overline{h}_\varepsilon\right|^j \le 2 \sum_{n=2}^{\infty} \frac{2^{n-1}}{(n-1)!} (C\varepsilon^2)^{n-1} = 2 \sum_{n=1}^{\infty} \frac{1}{n!} (2C\varepsilon^2)^n = 2(e^{2C\varepsilon^2} - 1),$$

and the integral obeys the inequality

$$\int g_\varepsilon^2 \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} \frac{2^n}{n!} |h_\varepsilon|^{n-j-1} \left|\overline{h}_\varepsilon\right|^j \mathrm{d}r \le 2(e^{2C\varepsilon^2} - 1) \int g_\varepsilon^2 \mathrm{d}r = 2(e^{2C\varepsilon^2} - 1) \|g_\varepsilon\|_{L^2}^2.$$

So far we have shown that

$$Q_{SG}(g_\varepsilon) \le \varepsilon^2 \|g_\varepsilon\| + 2(e^{2C\varepsilon^2} - 1) \|g_\varepsilon\|_{L^2}^2.$$

Remember from Theorem 2.13 that $L_{SG}$ is coercive with constant 2 for functions orthogonal to $H'_{SG}$:

$$2\|g_\varepsilon\|^2_{L^2} \leq (\varepsilon^2 + 2e^{2C\varepsilon^2} - 2)\|g_\varepsilon\|^2_{L^2}. \tag{2.16}$$

Both $\varepsilon^2$ and $e^{2C\varepsilon^2} - 1$ tend to zero; therefore, if $\varepsilon$ is sufficiently small, it is true that

$$\varepsilon^2 + 2e^{2C\varepsilon^2} - 2 < 2.$$

The only way inequality (2.16) can hold in this case is if $\|g_\varepsilon\|_{L^2} = 0$, which implies $g_\varepsilon = 0$ almost everywhere. Since $g_\varepsilon$ is continuous, this means that $g_\varepsilon = 0$ everywhere and we conclude that $h_\varepsilon = \overline{h}_\varepsilon$. □

We believe that the solution should be unique for all $\varepsilon > 0$, that is, prove Theorem 2.17 without the hypothesis on $\varepsilon$. A proof of this statement has eluded us.

## 2.5. Linearized SGWH operator

In this section, we study the spectrum of the linearized SGWH operator in accordance with [8] and expand the arguments therein to provide more detail. Remember that the stationary equation is

$$H''_\varepsilon + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H'_\varepsilon + \sin(2H_\varepsilon) = 0.$$

To simplify future analysis we consider the non-linear differential operator with opposite sign.

**Definition 2.9** *The* Sine-Gordon differential operator on a wormhole*, or* SGWH operator*, is*

$$D_\varepsilon \coloneqq -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\frac{\mathrm{d}}{\mathrm{d}r} - \sin(2\,\cdot\,).$$

**Proposition 2.5** *The linearization of $D_{SG}$ at $H_\varepsilon$ is the operator*

$$\begin{aligned} L_\varepsilon \colon H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ \phi &\mapsto -\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} - \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\frac{\mathrm{d}\phi}{\mathrm{d}r} - 2\cos(2H_\varepsilon)\phi \end{aligned}$$

Proof. Its Fréchet derivative at $H_\varepsilon$ is

$$L_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\frac{\mathrm{d}}{\mathrm{d}r} - 2\cos(2H_\varepsilon),$$

hence the linearization of $D_\varepsilon$ in a neighborhood of $H_\varepsilon$ can be written as

$$D_\varepsilon(H_\varepsilon + \phi) = D_\varepsilon(H_\varepsilon) + L_\varepsilon\phi + o(\phi) = L_\varepsilon\phi + o(\phi) \qquad \forall \phi \in H^2(\mathbb{R}).$$

□

We study the operator $L_\varepsilon$ in terms of the auxiliary function $\psi = \sqrt{r^2 + 1/\varepsilon^2}\phi$, like we did in Section 2.2. The derivatives of $\phi$ expressed in terms of $\psi$ are

$$\phi' = \frac{\psi'}{\sqrt{r^2+\frac{1}{\varepsilon^2}}} - \frac{r\psi}{\left(r^2+\frac{1}{\varepsilon^2}\right)^{\frac{3}{2}}}, \qquad \phi'' = \frac{\psi''}{\sqrt{r^2+\frac{1}{\varepsilon^2}}} - \frac{2r\psi'}{\left(r^2+\frac{1}{\varepsilon^2}\right)^{\frac{3}{2}}} + \left(\frac{3r^2}{\left(r^2+\frac{1}{\varepsilon^2}\right)^{\frac{5}{2}}} - \frac{1}{\left(r^2+\frac{1}{\varepsilon^2}\right)^{\frac{3}{2}}}\right)\psi.$$

With this change the operator $L_\varepsilon$ takes the form

$$\sqrt{r^2+\frac{1}{\varepsilon^2}}L_\varepsilon\psi = -\psi'' + \frac{\varepsilon^2}{(\varepsilon^2r^2+1)^2}\psi - 2\cos(2H_\varepsilon)\psi.$$

**Definition 2.10** *We define the operator* $\mathcal{L}_\varepsilon := \sqrt{r^2+1/\varepsilon^2}L_\varepsilon$ *on* $H^2(\mathbb{R})$. *With the potential*

$$\mathcal{V}_\varepsilon(r) = \frac{\varepsilon^2}{(\varepsilon^2r^2+1)^2} - 2\cos(2H_\varepsilon)$$

*it becomes*

$$\mathcal{L}_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\varepsilon^2}{(\varepsilon^2r^2+1)^2} - 2\cos(2H_\varepsilon) = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \mathcal{V}_\varepsilon.$$

**Remark** The term $\varepsilon^2/(\varepsilon^2r^2+1)$ is the potential $U_a$ defined in Section 2.2, expressed in terms of $\varepsilon$:

$$U_a(r) = \frac{a^2}{(r^2+a^2)^2} = \frac{\varepsilon^4}{\varepsilon^2(\varepsilon^2r^2+1)^2} = \frac{\varepsilon^2}{(\varepsilon^2r^2+1)^2}.$$

We refer to it as $U_\varepsilon$ for convenience.

**Proposition 2.6** *Let* $L_{SG}$ *be the linearization of the Sine-Gordon operator near* $H_{SG}$. *Consider also the functions*

$$W := \cos(2H_{SG}) - \cos(2H_\varepsilon), \qquad V_\varepsilon := U_\varepsilon + 2W.$$

*Then* $\mathcal{L}_\varepsilon$ *is of the form*

$$\mathcal{L}_\varepsilon = L_{SG} + V_\varepsilon$$

Proof. Remember that Proposition 2.4 says $L_{SG} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\cos(2H_{SG})$. The result is an immediate consequence of adding and substracting $2\cos(2H_{SG})$ in the definition of $\mathcal{L}_\varepsilon$. □

**Definition 2.11** *The quadratic form* $\mathcal{Q}_\varepsilon$ *is defined as*

$$\mathcal{Q}_\varepsilon(\psi) := \langle\mathcal{L}_\varepsilon\psi,\psi\rangle = Q_{SG}(\psi) + \int V_\varepsilon\psi^2\mathrm{d}r \qquad \forall\psi\in H^1(\mathbb{R}).$$

**Proposition 2.7** *The potential* $V_\varepsilon$ *is of order* $O(\varepsilon^2)$ *as* $\varepsilon\to 0^+$.

Proof. The term $U_\varepsilon$ is clearly quadratic in $\varepsilon^2$. As for $W$, a first order expansion shows that

$$W = 2\sin(2H_{SG})(H_\varepsilon - H_{SG}) - o(|H_\varepsilon - H_{SG}|^2).$$

Theorem 2.16 states that $h_\varepsilon = H_\varepsilon - H_{SG}$ is of order $O(\varepsilon^2)$, and the result follows. □

We mimic the procedure used in Section 2.3 and utilize an auxiliary operator to understand the spectrum of $\mathcal{L}_\varepsilon$.

**Definition 2.12** *Define the potential*

$$\tilde{\mathcal{V}}_\varepsilon(r) = -4\operatorname{sech}^2\left(\sqrt{2}r\right) + V_\varepsilon(r)$$

*and the differential operator*

$$\tilde{\mathcal{L}}_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 4\operatorname{sech}^2\left(\sqrt{2}r\right) + V_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \tilde{\mathcal{V}}_\varepsilon.$$

**Remark** The definition implies $\mathcal{L}_\varepsilon = \tilde{\mathcal{L}}_\varepsilon + 2$. This is the analogue of the Pöschl-Teller operator utilized previously.

**Theorem 2.18** *The operator $\tilde{\mathcal{L}}_\varepsilon$ is self-adjoint for all small $\varepsilon$.*

Proof. The proof resembles that of Theorem 2.10. We write $A := -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ defined in $H^2(\mathbb{R})$ and the multiplication operator $\tilde{\mathcal{V}}_\varepsilon$. We already know $A$ is self-adjoint; the only obstacle is to show $\tilde{\mathcal{V}}_\varepsilon$ is $A$-bounded with constant less than one.

The potential $\tilde{\mathcal{V}}_\varepsilon$ is bounded:

$$\left|\tilde{\mathcal{V}}_\varepsilon\right| \leq 4\left\|\operatorname{sech}^2\left(\sqrt{2}\,\cdot\right)\right\|_\infty + \|V_\varepsilon\|_\infty \leq 4 + O(\varepsilon^2) < 5.$$

With this, demonstrating $\tilde{\mathcal{V}}_\varepsilon$ is $A$-bounded is trivial:

$$\left\|\tilde{\mathcal{V}}_\varepsilon\phi\right\|_{L^2} \leq 5\|\phi\|_{L^2} \leq 5\|\phi\|_{H^2},$$

and the $A$-bound is 0. The result is obtained by applying the Kato-Rellich theorem to $\tilde{\mathcal{L}}_\varepsilon = A + \tilde{\mathcal{V}}_\varepsilon$. □

**Corollary 2.12** *The operator $\tilde{\mathcal{L}}_\varepsilon$ is self-adjoint.*

Proof. The sum of a self-adjoint and a bounded operator is self-adjoint. □

**Theorem 2.19** *The essential spectrum of $\tilde{\mathcal{L}}_\varepsilon$ is*

$$\sigma_{ess}(\tilde{\mathcal{L}}_\varepsilon) = [0, +\infty).$$

Proof. Once again, Theorem 2.15 from [18] is the key, just like in the proof of theorem 2.12. The potential $V_\varepsilon$ is continuous and vanishes at infinity: checking this for the term $\varepsilon^2/(\varepsilon^2 r^2 + 1)^2$ is trivial. Remember that both $H_{SG}$ and $H_\varepsilon$ have the same limit conditions $\pm\pi/2$, so

$$\lim_{r\to+\infty} \cos(2H_{SG}(r)) = \cos(\pi) = \lim_{r\to+\infty} \cos(2H_\varepsilon(r))$$

and the same holds for $r \to -\infty$. We deduce that

$$\lim_{|r|\to+\infty} W(r) = 0,$$

giving us the limit $V_\varepsilon(r) \to 0$ as $|r| \to +\infty$.

As a consequence, $\tilde{\mathcal{V}}_\varepsilon$ is continuous, bounded and goes to zero as $|r| \to +\infty$. From the proof of Theorem 2.19 we know that it is also $A$-bounded by 0, where $A$ is defined as before. These are the hypothesis of the theorem, and we obtain the result

$$\sigma_{\text{ess}}(\tilde{\mathcal{L}}_\varepsilon) = \sigma_{\text{ess}}(A) = [0, +\infty).$$

□

**Corollary 2.13** *The essential spectrum of $\mathcal{L}_\varepsilon$ is*

$$\sigma_{ess}(\mathcal{L}_\varepsilon) = [2, +\infty).$$

After describing the essential spectrum, the eigenvalues are next.

**Theorem 2.20** *The operator $\tilde{\mathcal{L}}_\varepsilon$ has at least one eigenvalue, if $\varepsilon$ is sufficiently small.*

Proof. To determine the existence of an eigenvalue, we use Proposition 2.17 in [18], which states the following: take $A$ as the unique self-adjoint extension of $-\frac{\mathrm{d}^2}{\mathrm{d}r^2} \colon C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$. Let $V \colon \mathbb{R}^d \to \mathbb{R}$ be a bounded, (piecewise) continuous potential that converges to 0 as $|r| \to +\infty$. Suppose that $V$ is $A$-bounded, with the constant being strictly less than 1. If there is a function $\psi \in H^2(\mathbb{R})$ with $\langle (A + V)\psi, \psi\rangle < 0$, then $A + V$ has at least one negative eigenvalue.

Most of the hypothesis were checked in the proof of Theorem 2.19, the only one left is to find a function $\psi \in H^2(\mathbb{R})$ that satisfies $\langle \tilde{\mathcal{L}}_\varepsilon \psi, \psi\rangle < 0$. We claim that $\psi = H'_{SG}$ is the correct choice. By definition, we know that

$$\tilde{\mathcal{L}}_\varepsilon H'_{SG} = L_{PT} H'_{SG} + V_\varepsilon H'_{SG}.$$

Use the fact that $L_{SG} H'_{SG} = L_{PT} H'_{SG} + 2H'_{SG} = 0$ to see that

$$\tilde{\mathcal{L}}_\varepsilon H'_{SG} = (-2 + V_\varepsilon) H'_{SG},$$

and take the inner product with $H'_{SG}$:

$$\langle \tilde{\mathcal{L}}_\varepsilon H'_{SG}, H'_{SG}\rangle = -2\|H'_{SG}\|^2_{L^2} + \langle V_\varepsilon H'_{SG}, H'_{SG}\rangle.$$

Now we take advantage of Proposition 2.7. Write the inner product explicitly and bound $V_\varepsilon$:

$$|\langle V_\varepsilon H'_{SG}, H'_{SG}\rangle| = \left| \int V_\varepsilon (H'_{SG})^2 \mathrm{d}r \right| \leq \int |V_\varepsilon| (H'_{SG})^2 \mathrm{d}r = \|V_\varepsilon\|_\infty \|H'_{SG}\|^2_{L^2}.$$

This can be combined with the previous inequality to obtain

$$\langle \tilde{\mathcal{L}}_\varepsilon H'_{SG}, H'_{SG}\rangle \leq -2\|H'_{SG}\|^2_{L^2} + |\langle V_\varepsilon H'_{SG}, H'_{SG}\rangle| \leq (-2 + \|V_\varepsilon\|_\infty)\|H'_{SG}\|^2_{L^2}.$$

Since $\|V_\varepsilon\|_\infty$ tends to zero as $\varepsilon \to 0^+$, there is a $\varepsilon_0$ such that, for all positive $\varepsilon < \varepsilon_0$, the quantity $-2+\|V_\varepsilon\|_\infty$ is negative. This is to say, for all $\varepsilon < \varepsilon_0$, the inner product $\langle \tilde{\mathcal{L}}_\varepsilon H'_{SG}, H'_{SG}\rangle$ is negative. The hypothesis are satisfied, meaning there is at least one negative eigenvalue of

$\tilde{\mathcal{L}}_\varepsilon$. □

**Corollary 2.14** *The operator $\mathcal{L}_\varepsilon$ has at least one eigenvalue.*

Our next task is to corroborate our intuition that the first eigenvalue $\lambda_\varepsilon$ of $\mathcal{L}_\varepsilon$ tends to $\lambda_{SG} = 0$, and we do so through its Rayleigh quotient. To properly estimate this quantity we need intermediate results.

**Lemma 2.13** *The solution to*

$$\begin{cases} w'' + 2\cos(2H_{SG})w = -2rH'_{SG}, \quad r \in \mathbb{R}, \\ w(0) = 0, \\ \lim_{|r|\to+\infty} w(r) = 0 \end{cases}$$

*is of the form*

$$w = A(r)\,\mathrm{sech}\big(\sqrt{2}r\big) + B(r)\,\mathrm{sech}\big(\sqrt{2}r\big)\Big(2\sqrt{2}r + \sinh\big(2\sqrt{2}r\big)\Big),$$

*where*

$$A(r) = r^2\tanh\big(\sqrt{2}r\big) - \int_0^r s\tanh\big(\sqrt{2}s\big)\mathrm{d}s,$$
$$B(r) = -\frac{1}{2}\int_{-\infty}^r s\,\mathrm{sech}^2\big(\sqrt{2}s\big)\mathrm{d}s.$$

Proof. We know from Lemma 2.12 that the general solution to the homogeneous equation is $c_1\,\mathrm{sech}\big(\sqrt{2}r\big) + c_2\,\mathrm{sech}\big(\sqrt{2}r\big)\Big(2\sqrt{2}r + \sinh\big(2\sqrt{2}r\big)\Big)$, and that $H'_{SG}(r) = \sqrt{2}\,\mathrm{sech}\big(\sqrt{2}r\big)$. The wronskian of these functions is constant and equal to $4\sqrt{2}$; therefore, variation of parameters states that the particular solution is

$$w = A(r)\,\mathrm{sech}\big(\sqrt{2}r\big) + B(r)\,\mathrm{sech}\big(\sqrt{2}r\big)\Big(2\sqrt{2}r + \sinh\big(2\sqrt{2}r\big)\Big).$$

The functions $A$ and $B$ are given by the formulas

$$A(r) = \frac{1}{2}\int_0^r 2\sqrt{2}s^2\,\mathrm{sech}^2\big(\sqrt{2}s\big)\mathrm{d}s + \frac{1}{2}\int_0^r s\,\mathrm{sech}^2\big(\sqrt{2}s\big)2\sinh\big(\sqrt{2}s\big)\cosh\big(\sqrt{2}s\big)\mathrm{d}s + c_1,$$
$$B(r) = -\frac{1}{2}\int_0^r s\,\mathrm{sech}^2\big(\sqrt{2}s\big)\mathrm{d}s + c_2,$$

with real constants $c_1$ and $c_2$. Algebraic manipulations plus integration by parts in the first integral reduce $A$ to

$$A(r) = r^2\tanh\big(\sqrt{2}r\big) - \int_0^r s\tanh\big(\sqrt{2}s\big)\mathrm{d}s + c_1.$$

The next step is to incorporate the conditions into this solution. At the origin we have

$$w(0) = A(0) = c_1 = 0.$$

The first term of $w$ tends to zero at infinity: tanh is bounded, thus $A$ grows polynomially and the exponential decay of sech eliminates it. The term accompanying $B$ is unbounded, but setting $c_2 = -\frac{1}{2}\int_{-\infty}^{0} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s$ ensures that $w$ vanishes. Indeed, with this choice we have that

$$B(r) = -\frac{1}{2}\int_{-\infty}^{r} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s,$$

which approaches 0 as $r \to \pm\infty$ because it is the integral of an odd function. It follows that the expression

$$\sqrt{2}r\,\mathrm{sech}\left(\sqrt{2}r\right)\int_{-\infty}^{r} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s$$

also tends to zero in the limit.

The more interesting term is

$$\sinh\left(\sqrt{2}r\right)\int_{-\infty}^{r} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s = \frac{1}{\mathrm{csch}\left(\sqrt{2}r\right)}\int_{-\infty}^{r} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s,$$

where both the numerator and denominator tend to zero. This suggests the use of l'Hopital's rule, which grants us the equality

$$\lim_{r\to+\infty}\frac{\int_{-\infty}^{r} s\,\mathrm{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s}{\mathrm{csch}\left(\sqrt{2}r\right)} = \lim_{r\to+\infty}\frac{r\,\mathrm{sech}^2\left(\sqrt{2}r\right)}{-\sqrt{2}\cosh\left(\sqrt{2}r\right)\mathrm{csch}^2\left(\sqrt{2}r\right)} = -\frac{1}{\sqrt{2}}\lim_{r\to+\infty} r\frac{\sinh^2\left(\sqrt{2}r\right)}{\cosh^3\left(\sqrt{2}r\right)} = 0.$$

Therefore,

$$\lim_{|r|\to+\infty} B(r)\,\mathrm{sech}\left(\sqrt{2}r\right)\left(2\sqrt{2}r + \sin\left(2\sqrt{2}r\right)\right) = 0$$

and the limit condition for $w$ is verified. Since all constants are known, the proof is complete. □

**Lemma 2.14** *For sufficiently small $\varepsilon$, the integral*

$$\int (H'_{SG})^2 W\mathrm{d}r = \int (\cos(2H_{SG}) - \cos(2H_\varepsilon))(H'_{SG})^2\mathrm{d}r$$

*is positive.*

Proof. For small $\varepsilon$, $H_\varepsilon$ is a perturbation of $H_{SG}$ as shown in Theorem 2.16. A first order expansion shows that

$$\cos(2H_{SG}) - \cos(2H_\varepsilon) = 2\sin(2H_{SG})h_\varepsilon + o(|h_\varepsilon|^2),$$

and this implies

$$\int (H'_{SG})^2 W\mathrm{d}r = 2\int \sin(2H_{SG})h_\varepsilon (H'_{SG})^2\mathrm{d}r + \int o(|h_\varepsilon|^2)(H'_{SG})^2\mathrm{d}r.$$

Theorem 2.16 says that both $h_\varepsilon$ and $h'_\varepsilon$ are (uniformly) quadratic in $\varepsilon$; therefore, $O(|h_\varepsilon|^2) = O(\varepsilon^4)$. Meanwhile, the quantity $\varepsilon^2 r/(\varepsilon^2 r^2 + 1)$ is of order $O(\varepsilon)$:

$$\max_{r\in\mathbb{R}}\left|\frac{\varepsilon^2 r}{\varepsilon^2 r^2 + 1}\right| = \varepsilon^2\max_{r>0}\frac{r}{\varepsilon^2 r^2 + 1} = \varepsilon^2\frac{\varepsilon^{-1}}{\varepsilon^2\varepsilon^{-1} + 1} = \frac{\varepsilon}{\varepsilon + 1} = O(\varepsilon).$$

It follows that

$$\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' = O(\varepsilon^3).$$

We are interested in the sign of the integral, where $h_\varepsilon$ plays a crucial role. We derived the linearized equation (2.9) for $h_\varepsilon$ in the proof of Theorem 2.15:

$$h_\varepsilon'' + \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} h_\varepsilon' + 2\cos(2H_{SG})h_\varepsilon + O(|h_\varepsilon|^2) = -\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}'.$$

This can be simplified by focusing on terms of order $O(\varepsilon^2)$, and leaving the expressions studied earlier as a function $O(\varepsilon^3)$:

$$h_\varepsilon'' + 2\cos(2H_{SG})h_\varepsilon = -\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' - O(\varepsilon^3). \tag{2.17}$$

Note that

$$r - \frac{r}{\varepsilon^2 r^2 + 1} = \frac{\varepsilon^2 r^3}{\varepsilon^2 r^2 + 1}$$

This implies that

$$\left| \varepsilon^2 r H_{SG}' - \frac{\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' \right| = \varepsilon^4 \left| \frac{H_{SG}' r^3}{\varepsilon^2 r^2 + 1} \right| \leq C\varepsilon^4,$$

for some finite constant $C > 0$, because $H_{SG}' r^3$ remains bounded. From this we deduce the approximation given by

$$\frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' = \frac{2\varepsilon^2 r}{\varepsilon^2 r^2 + 1} H_{SG}' + 2\varepsilon^2 r H_{SG}' - 2\varepsilon^2 r H_{SG}' = 2\varepsilon^2 r H_{SG}' + O(\varepsilon^4),$$

and combined with equation (2.17) we get a new approximation for $h_\varepsilon$:

$$h_\varepsilon'' + 2\cos(2H_{SG})h_\varepsilon = -2\varepsilon^2 r H_{SG}' - O(\varepsilon^3).$$

The objective here is to determine the sign of the integral, and to do so we have to know $h_\varepsilon$ explicitly. The advantage of the previous approximation is that perturbations of higher order in $\varepsilon$ will not affect the sign of $h_\varepsilon$, if $\varepsilon$ is small. Thus, the sign of

$$\int (\cos(2H_{SG}) - \cos(2H_\varepsilon))(H_{SG}')^2 \mathrm{d}r$$

equals the sign of

$$2\varepsilon^2 I := 2\varepsilon^2 \int \sin(2H_{SG})(H_{SG}')^2 w \ \mathrm{d}r$$

where $w$ solves the linear, inhomogeneous equation

$$w'' + 2\cos(2H_{SG})w = -2rH_{SG}'.$$

The previous lemma gives an explicit formula for $w$, which lets us write $I$ as

$$\begin{aligned} I = \; & 4\int \tanh\left(\sqrt{2}r\right)\operatorname{sech}^4\left(\sqrt{2}r\right)\left(r^2\tanh\left(\sqrt{2}r\right) - \int_0^r s\tanh\left(\sqrt{2}s\right)\mathrm{d}s\right)\mathrm{d}r \\ & - 2\int \tanh\left(\sqrt{2}r\right)\operatorname{sech}^4\left(\sqrt{2}r\right)\left(2\sqrt{2}r + \sinh\left(2\sqrt{2}r\right)\right)\left(\int_{-\infty}^r s\operatorname{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s\right)\mathrm{d}r. \end{aligned}$$

The integrand is even, so:

$$\begin{aligned} I = \; & 8\int_0^{+\infty} \tanh\left(\sqrt{2}r\right)\operatorname{sech}^4\left(\sqrt{2}r\right)\left(r^2\tanh\left(\sqrt{2}r\right) - \int_0^r s\tanh\left(\sqrt{2}s\right)\mathrm{d}s\right)\mathrm{d}r \\ & - 4\int_0^{+\infty} \tanh\left(\sqrt{2}r\right)\operatorname{sech}^4\left(\sqrt{2}r\right)\left(2\sqrt{2}r + \sinh\left(2\sqrt{2}r\right)\right)\left(\int_{-\infty}^r s\operatorname{sech}^2\left(\sqrt{2}s\right)\mathrm{d}s\right)\mathrm{d}r. \end{aligned}$$

The integral is difficult to compute explicitly, and we opt to use numerical methods to determine its value and, more importantly, its sign. We use the module `scipy` in Python 3.6.13 to integrate from $r = 0$ to $r = 250$; this value is chosen because of numerical instability as the interval grows larger. Note that the integrand decreases exponentially, thus the biggest contribution to the value of the integral lies near the origin. Numerical integration gives a value of 0.707106781186547, which is positive. Annex B contains the Python code written for this purpose. □

**Corollary 2.15** *For sufficiently small $\varepsilon$,*

$$\int (H_{SG}')^2 V_\varepsilon \mathrm{d}r > 0.$$

Proof. From the definition of $V_\varepsilon$:

$$\int (H_{SG}')^2 V_\varepsilon \mathrm{d}r = \int (H_{SG}')^2 U_\varepsilon \mathrm{d}r + \int (H_{SG}')^2 W \mathrm{d}r.$$

The first integral on the right side is positive because $U_\varepsilon > 0$, and the second is also positive thanks to Lemma 2.14. □

We are now ready to study the first eigenvalue of $\mathcal{L}_\varepsilon$. The order of this quantity is shown to be quadratic, as proven in [8].

**Theorem 2.21** *The first eigenvalue $\lambda_\varepsilon$ of $\mathcal{L}_\varepsilon$ is of order $O(\varepsilon^2)$ as $\varepsilon \to 0^+$.*

Proof. Let $\psi \in H^2(\mathbb{R})$ be a function, and decompose it into a co-linear and an orthogonal part:

$$\psi = \alpha H_{SG}' + \beta\eta,$$

where $\alpha, \beta \in \mathbb{R}$ are constants and $\eta \perp H_{SG}'$, all of them depending on $\psi$. The norm of $\psi$ equals

$$\|\psi\|_{L^2}^2 = \alpha^2\|H_{SG}'\|_{L^2} + \beta^2\|\eta\|_{L^2},$$

due to orthogonality.

Apply the quadratic form $\mathcal{Q}_\varepsilon$ to $\psi$:

$$\mathcal{Q}_\varepsilon(\psi) = \alpha^2\langle \mathcal{L}_\varepsilon H_{SG}', H_{SG}'\rangle + \alpha\beta\langle \mathcal{L}_\varepsilon H_{SG}', \eta\rangle + \alpha\beta\langle H_{SG}', \mathcal{L}_\varepsilon\eta\rangle + \beta^2\langle \mathcal{L}_\varepsilon\eta, \eta\rangle.$$

Because $L_{SG}H'_{SG} = 0$, the action of $\mathcal{L}_\varepsilon$ on $H'_{SG}$ reduces to $\mathcal{L}_\varepsilon H'_{SG} = V_\varepsilon H'_{SG}$. Moreover, $\mathcal{L}_\varepsilon$ being self-adjoint implies that $\langle H'_{SG}, \mathcal{L}_\varepsilon \eta\rangle = \langle V_\varepsilon H'_{SG}, \eta\rangle$. In summary:

$$\mathcal{Q}_\varepsilon(\psi) = \alpha^2\langle V_\varepsilon H'_{SG}, H'_{SG}\rangle + 2\alpha\beta\langle V_\varepsilon H'_{SG}, \eta\rangle + \beta^2\langle \mathcal{L}_\varepsilon \eta, \eta\rangle.$$

Corollary 2.15 implies that $\alpha^2\langle V_\varepsilon H'_{SG}, H'_{SG}\rangle \geq 0$. Similarly, coercivity for $L_{SG}$ from Theorem 2.13 shows that

$$\mathcal{Q}_\varepsilon(\eta) \geq 2\|\eta\|^2_{L^2} + \langle V_\varepsilon \eta, \eta\rangle.$$

We can bound $V_\varepsilon$ below by $-\|V_\varepsilon\|_\infty$ to see that $\langle V_\varepsilon \eta, \eta\rangle \geq -\|V_\varepsilon\|_\infty \|\eta\|^2_{L^2}$:

$$\langle V_\varepsilon \eta, \eta\rangle = \int V_\varepsilon \eta^2 \mathrm{d}r \geq -\|V_\varepsilon\|_\infty \int \eta^2 \mathrm{d}r = -\|V_\varepsilon\|_\infty \|\eta\|^2_{L^2}.$$

Therefore,

$$\mathcal{Q}_\varepsilon(\eta) \geq (2 - \|V_\varepsilon\|_\infty)\|\eta\|^2_{L^2}.$$

Thanks to Proposition 2.7, there is a constant $k > 0$ such that $\|V_\varepsilon\|_\infty < k\varepsilon^2$ for all $\varepsilon$ close to zero. It follows that

$$\mathcal{Q}_\varepsilon(\eta) \geq (2 - k\varepsilon^2)\|\eta\|^2_{L^2}.$$

The term in the middle can be bounded using the Cauchy-Schwarz inequality:

$$2|\alpha\beta||\langle V_\varepsilon H'_{SG}, \eta\rangle| \leq 2|\alpha\beta|\|V_\varepsilon H'_{SG}\|_{L^2}\|\eta\|_{L^2}.$$

The potential $V_\varepsilon$ can be taken out of the norm as $\|V_\varepsilon\|_\infty$:

$$\|V_\varepsilon H'_{SG}\|_{L^2} = \left(\int V_\varepsilon^2 (H'_{SG})^2 \mathrm{d}r\right)^{\frac{1}{2}} \leq \left(\|V_\varepsilon\|^2_\infty \int (H'_{SG})^2 \mathrm{d}r\right)^{\frac{1}{2}} = \|V_\varepsilon\|_\infty \|H'_{SG}\|_{L^2}.$$

The well-known inequality $2ab \leq a^2 + b^2$ for real numbers $a, b$ implies that

$$2|\alpha\beta|\|V_\varepsilon H'_{SG}\|_{L^2}\|\eta\|_{L^2} \leq \|V_\varepsilon\|_\infty\Big(\alpha^2\|H'_{SG}\|^2_{L^2} + \beta^2\|\eta\|^2_{L^2}\Big),$$

where we chose $a = |\alpha|\|H'_{SG}\|_{L^2}$ and $b = |\beta|\|\eta\|_{L^2}$. As $\Big(\alpha^2\|H'_{SG}\|^2_{L^2} + \beta^2\|\eta\|^2_{L^2}\Big)$ equals $\|\psi\|^2_{L^2}$, we obtain:

$$2|\alpha\beta\langle V_\varepsilon H'_{SG}, \eta\rangle| \leq k\varepsilon^2\|\psi\|^2_{L^2}.$$

These bounds illustrate that

$$\mathcal{Q}_\varepsilon(\psi) \geq -k\varepsilon^2\|\psi\|^2_{L^2} + \beta^2(2 - k\varepsilon^2)\|\eta\|^2_{L^2}$$

The term $2\beta^2\|\eta\|^2_{L^2}$ is non-negative. In addition, $\beta^2\|\eta\|^2_{L^2} \leq \alpha^2\|H'_{SG}\|^2_{L^2} + \beta^2\|\eta\|^2_{L^2}$, thus $\beta^2\|\eta\|^2_{L^2} \leq \|\psi\|^2_{L^2}$. From this, it follows that

$$\mathcal{Q}_\varepsilon(\psi) \geq -2k\varepsilon^2\|\psi\|^2_{L^2} \qquad \forall \psi \in H^2(\mathbb{R}).$$

Divide by $\|\psi\|^2_{L^2}$ and take the infimum over $\psi \in H^2(\mathbb{R})$ to deduce a lower bound for the eigenvalue:

$$\lambda_\varepsilon = \inf_{\psi \in H^2(\mathbb{R})} \frac{\mathcal{Q}_\varepsilon(\psi)}{\|\psi\|^2_{L^2}} \geq -2k\varepsilon^2.$$

An upper bound is easier to find. We can evaluate the Rayleigh quotient at $H'_{SG}$:

$$\lambda_\varepsilon = \inf_{\psi \in H^2(\mathbb{R})} \frac{\langle \mathcal{L}_\varepsilon \psi, \psi \rangle}{\|\psi\|_{L^2}^2} \leq \frac{\langle \mathcal{L}_\varepsilon H'_{SG}, H'_{SG} \rangle}{\|H'_{SG}\|_{L^2}^2}.$$

Remember that $\mathcal{L}_\varepsilon H'_{SG} = V_\varepsilon H'_{SG}$. Like we did for $\eta$, we see that

$$\langle V_\varepsilon H'_{SG}, H'_{SG} \rangle \leq \|V_\varepsilon\|_\infty \|H'_{SG}\|_{L^2}^2 \leq k\varepsilon^2 \|H'_{SG}\|_{L^2}^2.$$

Combining these inequalities we get

$$\lambda_\varepsilon \leq \frac{\langle \mathcal{L}_\varepsilon H'_{SG}, H'_{SG} \rangle}{\|H'_{SG}\|_{L^2}^2} \leq \frac{k\varepsilon^2 \|H'_{SG}\|_{L^2}^2}{\|H'_{SG}\|_{L^2}^2} = k\varepsilon^2 < 2k\varepsilon^2.$$

The lower and upper bounds together imply that

$$|\lambda_\varepsilon| \leq 2k\varepsilon^2 = O(\varepsilon^2),$$

for all sufficiently small $\varepsilon$. □

**Theorem 2.22** *The first eigenvalue $\lambda_\varepsilon$ of $\mathcal{L}_\varepsilon$ is strictly positive for all sufficiently small $\varepsilon$.*

Proof. Let $\psi_\varepsilon$ be the normalized eigenfunction corresponding to the first eigenvalue $\lambda_\varepsilon$. Since $\mathcal{L}_\varepsilon$ is a perturbation of $L_{SG}$, we expect that $\psi_\varepsilon$ to be a perturbation of $H'_{SG}$, the eigenfunction for $L_{SG}$. Introducing an auxiliary function $\eta_\varepsilon \in L^2(\mathbb{R})$, this translates to $\psi_\varepsilon = \alpha_\varepsilon(H'_{SG} + \varepsilon^2 \eta_\varepsilon)$, where $\alpha_\varepsilon > 0$ is a normalization constant. Note that we can take $\eta_\varepsilon$ orthogonal to $H_{SG}$, because the co-linear component can be incorporated into the constant $\alpha_\varepsilon$. The eigenvalue equation becomes

$$\varepsilon^2 L_{SG} \eta_\varepsilon + V_\varepsilon H'_{SG} + \varepsilon^2 V_\varepsilon \eta_\varepsilon = \lambda_\varepsilon (H'_{SG} + \varepsilon^2 \eta_\varepsilon),$$

or equivalently

$$L_{SG} \eta_\varepsilon + (V_\varepsilon - \lambda_\varepsilon)\eta_\varepsilon = \frac{1}{\varepsilon^2}(\lambda_\varepsilon - V_\varepsilon) H'_{SG}.$$

Before we verify that $\lambda_\varepsilon$ is positive, we need a bound for $\|\eta_\varepsilon\|_{L^2}$. Take the preceding equation, multiply by $\eta_\varepsilon$ and integrate:

$$\langle L_{SG} \eta_\varepsilon, \eta_\varepsilon \rangle = \lambda_\varepsilon \frac{1}{\varepsilon^2} \langle H'_{SG}, \eta_\varepsilon \rangle - \frac{1}{\varepsilon^2} \langle V_\varepsilon H'_{SG}, \eta_\varepsilon \rangle + \lambda_\varepsilon \|\eta_\varepsilon\|_{L^2}^2 - \langle V_\varepsilon \eta_\varepsilon, \eta_\varepsilon \rangle.$$

Orthogonality eliminates the first inner product on the right side, and thanks to Theorem 2.13 we can use coercivity:

$$2\|\eta_\varepsilon\|_{L^2}^2 \leq \langle L_{SG} \eta_\varepsilon, \eta_\varepsilon \rangle = -\frac{1}{\varepsilon^2} \langle V_\varepsilon H'_{SG}, \eta_\varepsilon \rangle + \lambda_\varepsilon \|\eta_\varepsilon\|_{L^2}^2 - \langle V_\varepsilon \eta_\varepsilon, \eta_\varepsilon \rangle. \tag{2.18}$$

The norm $\|H'_{SG}\|_{L^2}$ is a finite constant, which we call $C > 0$. We can take the absolute value on the right side and use the Cauchy-Schwarz inequality:

$$2\|\eta_\varepsilon\|_{L^2}^2 \leq \frac{1}{\varepsilon^2} Ck\varepsilon^2 \|\eta_\varepsilon\|_{L^2} + |\lambda_\varepsilon| \|\eta_\varepsilon\|_{L^2}^2 + k\varepsilon^2 \|\eta_\varepsilon\|_{L^2}^2.$$

Divide by $\|\eta_\varepsilon\|_{L^2}$ and rearrange the resulting expression:

$$(2 - |\lambda_\varepsilon| - k\varepsilon^2)\|\eta_\varepsilon\|_{L^2} \leq Ck.$$

Both $|\lambda_\varepsilon|$ and $k\varepsilon^2$ approach zero, as seen in Theorem 2.21. We take advantage of this by choosing $\varepsilon$ in a way that ensures $1 < 2 - |\lambda_\varepsilon| - k\varepsilon^2$, as this implies

$$\|\eta_\varepsilon\|_{L^2} \leq Ck;$$

in other words, $\|\eta_\varepsilon\|_{L^2}$ is bounded independently of $\varepsilon$. Additionally, the normalization constant tends to $1/C$ as $\varepsilon$ decreases:

$$1 = \|\psi_\varepsilon\|_{L^2}^2 = \alpha_\varepsilon^2 \left\| H'_{SG} + \varepsilon^2 \eta_\varepsilon \right\|_{L^2}^2 = \alpha_\varepsilon^2 \Big( \|H'_{SG}\|_{L^2}^2 + \varepsilon^2 \|\eta_\varepsilon\|_{L^2}^2 \Big) = \alpha_\varepsilon^2 C^2 (1 + O(\varepsilon^2)).$$

Now we can prove that $\lambda_\varepsilon$ is positive. The quadratic form applied to $\psi_\varepsilon$ is

$$\frac{1}{\alpha_\varepsilon^2} \mathcal{Q}_\varepsilon(\psi_\varepsilon) = \langle \mathcal{L}_\varepsilon H'_{SG}, H'_{SG} \rangle + 2\varepsilon^2 \langle V_\varepsilon H'_{SG}, \eta_\varepsilon \rangle + \varepsilon^4 \|\eta_\varepsilon\|_{L^2}^2.$$

Bounding by $\|V_\varepsilon\|_\infty$ shows that

$$|\langle V_\varepsilon H'_{SG}, \eta_\varepsilon \rangle| \leq C \|V_\varepsilon\|_\infty \|\eta_\varepsilon\|_{L^2} = O(\varepsilon^2) O(1) = O(\varepsilon^2).$$

So far, we have the estimate

$$\frac{1}{\alpha_\varepsilon^2} \mathcal{Q}_\varepsilon(\psi_\varepsilon) = \langle \mathcal{L}_\varepsilon H'_{SG}, H'_{SG} \rangle + O(\varepsilon^4) = \langle V_\varepsilon H'_{SG}, H'_{SG} \rangle + O(\varepsilon^4).$$

We know from Corollary 2.15 that $\langle V_\varepsilon H'_{SG}, H'_{SG} \rangle > 0$, and we proved earlier that it is quadratic in $\varepsilon$. The normalization constant $\alpha_\varepsilon$ does not interfere in our analysis, because it approaches $1/C$ as $\varepsilon \to 0^+$. The conclusion comes from the realization that the sign of $\mathcal{Q}_\varepsilon(\psi_\varepsilon)$ only depends on lower order terms, if $\varepsilon$ is sufficiently small:

$$\lambda_\varepsilon = \mathcal{Q}_\varepsilon(\psi_\varepsilon) = \alpha_\varepsilon^2 \langle V_\varepsilon H'_{SG}, H'_{SG} \rangle + O(\varepsilon^4) > 0.$$

□

It is also known from [8] that the eigenvalue is unique, shown directly by establishing the inequality $\mathcal{L}_\varepsilon \geq L_{SG}$ . This completes the description of $\sigma(\mathcal{L}_\varepsilon)$, illustrated in Figure 2.4. We believe an alternative proof of this fact could use the technique known as Darboux factorization, used to demonstrate a similar conjecture for a different model in [20].

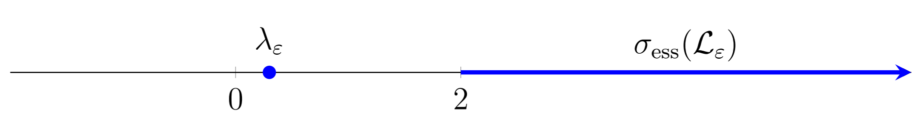


Figure 2.4: Essential spectrum of $\mathcal{L}_\varepsilon$ and its first eigenvalue $\lambda_\varepsilon$.

## 2.6. Brief discussion about the $n$-kinks

The stationary SG equation in flat spacetime is:

$$\phi'' + \sin(2\phi) = 0 \qquad \forall r \in \mathbb{R}.$$

The potential energy is $\sin^2(\phi)$, shown in Figure 2.1. The sinusoidal shape plus conservation of energy implies there are three types of solutions that depend only on the starting velocity:

1. If not given enough kinetic energy, the particle $\phi$ will not be able to traverse the first hill and will instead oscillate forever in the interval $(-\pi/2, \pi/2)$.

2. For a unique velocity, $\phi$ will approach the top of the hill in the limit $r \to +\infty$.

3. If given more energy than the previous case, the particle will surmount the hill and, thanks to conservation of energy, will traverse the following ones as well. This trajectory is unbounded and escapes to infinity as $r \to +\infty$.

As discussed in Section 2.1, the new term in the SGWH equation introduces an analogue for friction, meaning that the potential and kinetic energies of the particle $\phi$ are no longer conserved. This changes the possible outcomes of the trajectories:

1. If the particle does not have enough kinetic energy to surmount a hill. it will stay in the potential well, oscillating and losing energy.

2. Given the exact amount of kinetic energy required, $\phi$ will approach the top in infinite time.

3. Larger amounts of kinetic energy will allow the particle to continue travelling past the hill. However, because of friction, it will keep losing energy until it can no longer move past a crest. At this point, we return to the previous two cases: the particle will either stay in whatever potential well it is in, or it will approach the next local maximum as $r \to +\infty$.

The difference is clear: solutions of equation (2.1) can only stop at the first potential peak (in infinite time), but if it surpasses $\pi/2$, then it will never stop. In contrast, solutions of equation (1.3) can stop at any peak

Given $n \in \mathbb{N}$, the aim is to find a solution $H_a$ of equation (1.3) that travels between peaks in the potential, for a total distance of $n\pi$, as $r$ varies from $-\infty$ to $+\infty$. These multiples of $\pi$ are known as topological sectors; the solutions, known as $n$-kinks, are said to connect the 0 and $n$ topological sectors. In the model described, this condition can be established in this way: for odd $n$, the solutions we are interested in satisfy

$$H_a(0) = 0, \quad \lim_{r \to \pm\infty} H_a(r) = \pm\frac{n\pi}{2}.$$

Figure 2.5 shows the potential for the equation. The function $H_a$ travels from the peak at $-n\pi/2$, which corresponds to $H_a(-\infty)$, to the opposite peak at $n\pi/2$, corresponding to $H_a(+\infty)$.

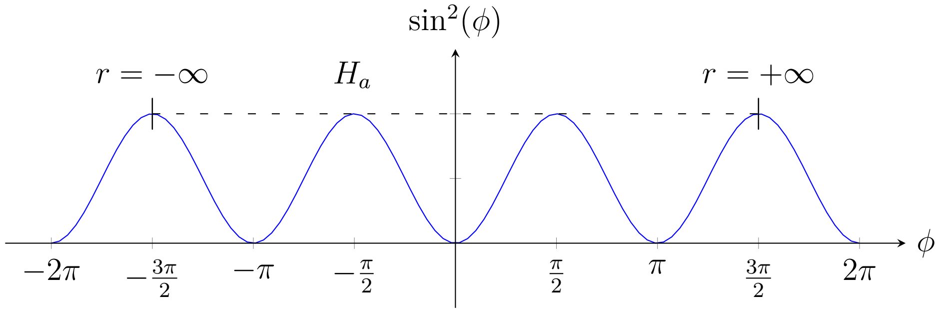


Figure 2.5: Distance covered by the kink for $n = 3$.

Figure 2.6 shows a rough sketch (*not* a numerical simulation) of the kink that covers a distance of $3\pi$, or 3-kink. It is anti-symmetrical and tends to $\pm 3\pi/2$ as $r \to \pm\infty$, never reaching it in finite time.

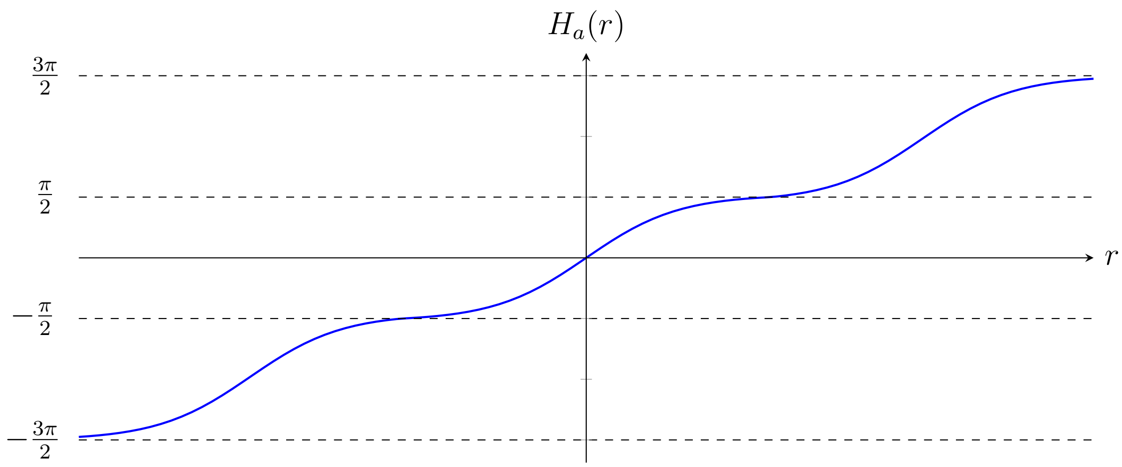


Figure 2.6: Sketch of the 3-kink.

The case for even $n$ is slightly different. The previous limit condition would place $H_a$ at potential wells at infinity, which is the opposite of what we want. To remedy this problem, we have to sacrifice symmetry and impose the asymptotic conditions

$$\lim_{r \to -\infty} H_a(r) = -(n-1)\frac{\pi}{2}, \quad \lim_{r \to +\infty} H_a(r) = (n+1)\frac{\pi}{2}.$$

Figure 2.7 illustrates this change. To travel a total distance of $n\pi$ for even $n$, and also land in the potential peaks at both infinities, it is necessary to replace symmetry at $r = 0$ with respect to the origin with symmetry relative to $\pi/2$. It is worth mentioning that one can recover the original symmetry with a translation; this leads to a sign change in the potential, and the problem is equivalent to the one we study.

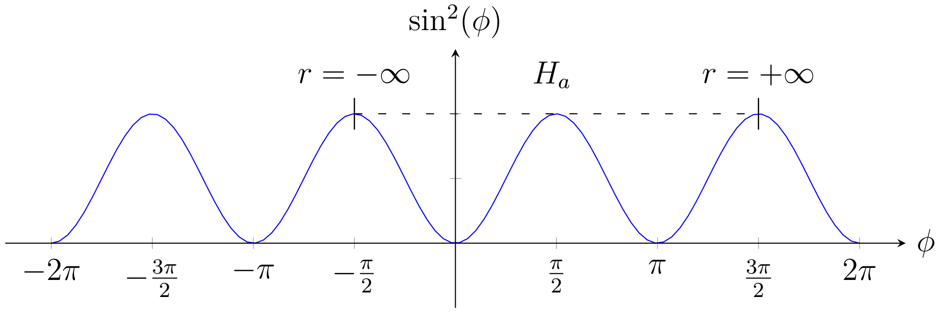


Figure 2.7: Distance covered by the kink for $n = 2$.

A sketch of the 2-kink can be found in Figure 2.8, where the symmetry wih respect to $\pi/2$ is more apparent.

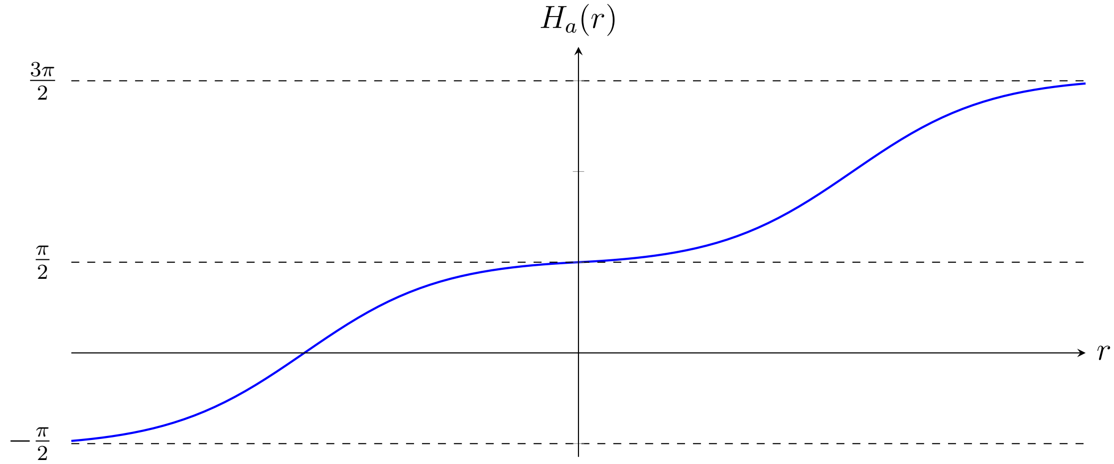


Figure 2.8: Sketch of the 2-kink.

An idea for a proof would be to use the shooting method in a similar way as the 1-kink, though care must be taken in the case for even $n$ because of the different formulation. Let $b_k$ refer to the initial velocity of the $k$-kink. For odd $n > 1$, one could employ an inductive argument by defining

$$A_n := \left\{ b > b_{n-2} : \exists r > 0,\ \phi(r) > \frac{n\pi}{2} \right\};$$

in fact, the same proof that shows $A$ is open and non-empty works for $A_n$ with minor modifications.

The difficulties arise when studying the set

$$B_n := \left\{ b > b_{n-2} : \exists r > 0,\ \phi(r) < \frac{n\pi}{2} \wedge \phi'(r) < 0 \right\}.$$

It is trivial to see that it is open, but finding an element in it is not so simple. The crux of the matter lies in the relationship between $b$ and $\phi$: one would expect that if $b > b_{n-2}$, then the trajectory would surpass the $(n-2)$-kink. However, a proper justification for this intuition has eluded us.

Should one prove $B_n$ to be non-empty, the rest of the demonstration closely resembles that of the 1-kink. The set $A_n$ excludes trajectories that surpass $n\pi/2$, while $B_n$ excludes those that stop before. It is important to mention that $k$-kinks for odd $k < n$ are bounded above by $n\pi/2$ and never stop before reaching their targets; this is why we take $b > b_{n-2}$ in the definition for $A_n$ and $B_n$, to make sure no other velocities $b_k$ can be found in the complement $(b_{n-2}, +\infty)\backslash A \cup B$.

The case where $n$ is even is different, because the potential is inverted. This is the price to pay for symmetry at $r = 0$, the translation changes the sign of $\sin^2(\phi)$ (the added constant is irrelevant) and forces $\phi$ to start at a local potential maximum instead of a minimum. However, we do not think this is a major complication, and the underlying ideas should still ring true.

# Chapter 3

# Conclusion

The aim of this work was to study the stationary Sine-Gordon equation on a wormhole, and a particular solution known $H_a$ as the 1-kink. In this context, we have described the Sine-Gordon equation, the wormhole geometry and how it changes the problem of finding appropiate solutions, providing a brief background on the differential geometry concepts needed to understand the setting. We have proven the existence and uniqueness of the 1-kink, showcasing many of the intricacies and complexities underlying the equation and its solutions. Many of the intermediate results and arguments used in the proofs are interesting in their own right, and help pant a more detailed picture of the stationary model.

In this vein, we delved further into the nature of the 1-kink, in an attempt to establish properties that mimic those of the kink of the SG equation. This lead us to derive asymptotic convergence rates for both $H_a$ and its derivatives as $r$ grows to infinity; we found that one can obtain a purely exponential rate independent of the parameter $a$. Related to this quantity is the convergence rate of $H_a$ to the flat SG kink $H_{SG}$, which is shown to be quadratic for both the supremum norm and the Sobolev norm in the space $H^1$. This is expected, as the term in the SGWH equation that corresponds to the parameter tends to zero as $a$ goes to infinity.

In addition, our study of a slightly-modified linearized SGWH operator $\mathcal{L}_\varepsilon$ points in the same direction: the first eigenvalue of $\mathcal{L}_\varepsilon$ displays quadratic convergence to zero, the eigenvalue of the linearized SG operator $L_{SG}$. These results support and extend the work realized in [8], fulfilling the goal of this thesis.

There are multiple ways to extend this research. For example, one could focus on the detailed study of the numerical aspects of the wormhole equation, and provide simulated solutions through careful implementation of the shooting method. Another path would be to follow the discussion in section 2.6, and show the existence of $n$-kinks that travel a total distance of $n\pi$. This family of solutions is particularly interesting, because it is absent in the flat Sine-Gordon model. We suspect that many of the lemmas and theorems presented here generalize to these functions. Finally, while the stationary model has value in its own right, it is still limited compared to the dynamic case. Results that translate some of the ideas shown in this thesis to time-dependent solutions would be very valuable, especially as a way to approach the soliton resolution conjecture.

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# Annexes

## Annex A. Laplace-Beltrami operator

The following is an alternative derivation of the Laplace-Beltrami operator on a wormhole using the Hessian tensor, for those with a background in differential geometry. The Koszul formula allows us to compute the Christoffel symbols of the Levi-Civita connection associated to this metric. The only nonzero symbols are:

$$\Gamma^r_{\phi\phi} = -r, \qquad \Gamma^r_{\theta\theta} = -r\sin^2(\phi), \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{r}{r^2+a^2},$$
$$\Gamma^\phi_{\theta\theta} = \sin(\phi)\cos(\phi), \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{r}{r^2+a^2}, \qquad \Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} = \frac{\cos(\phi)}{\sin(\phi)}.$$

With this information we can compute the hessian tensor of a given function $f \in C^\infty(M)$.

By definition
$$\operatorname{Hess} f := \nabla^2 f = \nabla\nabla f = \nabla \mathrm{d}f.$$

Let $p \in M$ be an event in spacetime, and $X, Y \in T_pM$ tangent vectors at $p$. It follows from the definition of covariant derivatives for differential forms that

$$\nabla \mathrm{d}f(X,Y) = (\nabla_X \mathrm{d}f)Y = X(\mathrm{d}f(Y)) - \mathrm{d}f(\nabla_X Y).$$

Label the coordinates $x^0 = t, x^1 = r, x^2 = \phi, x^3 = \theta$. Introducing the coordinate frame $\{\, \partial_\mu : \mu = 0,1,2,3 \,\}$ and its coframe $\{\, \mathrm{d}x^\nu : \nu = 0,1,2,3 \,\}$, and expressing $X = X^\mu\partial_\mu$, $Y = Y^\nu\partial_\nu$ in this frame we see that

$$\nabla(\mathrm{d}f)(X,Y) = X(Y^\nu\partial_\nu f) - \partial_\nu f \mathrm{d}x^\nu\Big(X^\mu\partial_\mu Y^\alpha\partial_\alpha + X^\mu Y^\nu\Gamma^\alpha_{\mu\nu}\partial_\alpha\Big).$$

Thanks to the Leibniz rule,

$$\nabla(\mathrm{d}f)(X,Y) = X^\mu Y^\nu\partial_{\mu\nu}f + X^\mu\partial_\mu Y^\nu\partial_\nu f - X^\mu\partial_\mu Y^\nu\partial_\nu f - X^\mu Y^\nu\Gamma^\alpha_{\mu\nu}\partial_\alpha f.$$

Thus, in local coordinates the components of $\operatorname{Hess} f$ are given by

$$(\operatorname{Hess} f)_{\mu\nu} = \partial_{\mu\nu}f - \Gamma^\alpha_{\mu\nu}\partial_\alpha f.$$

The metric $g$ is Lorentzian, therefore its Laplace-Beltrami operator is hyperbolic and we refer to it as the wave operator $\Box_g$:

$$\Box_g f = \operatorname{tr} \operatorname{Hess} f = g^{\mu\nu} (\operatorname{Hess} f)_{\mu\nu}.$$

Since $g$ is diagonal, this reduces to

$$\Box_g f = \sum_{\mu=0}^{3} g^{\mu\mu} (\operatorname{Hess} f)_{\mu\mu} = \sum_{\mu=0}^{3} g^{\mu\mu} \Big( \partial_{\mu\mu} f - \Gamma^{\alpha}_{\mu\mu} \partial_\alpha f \Big).$$

Under the additional assumption that $f = f(t, r)$ is radial, we have that $\partial_\phi f = \partial_\theta f = 0$, so

$$\Box_g f = \sum_{\mu=0}^{3} g^{\mu\mu} \Big( \partial_{\mu\mu} f - \Gamma^{t}_{\mu\mu} \partial_t f - \Gamma^{r}_{\mu\mu} \partial_r f \Big).$$

Note that $\Gamma^t_{\mu\mu}$ is always zero. Now we are ready to expand the sum using the expressions for the Christoffel symbols:

$$\Box_g f = -\partial_{tt} f + \partial_{rr} f + \frac{1}{r^2 + a^2} (-(-r)\partial_r f) + \frac{1}{\sin^2(\phi)(r^2 + a^2)} (-(-r \sin^2(\phi))\partial_r f).$$

In conclusion, the wave operator for a radial function $f$ on a wormhole is

$$\Box_g f = -\partial_{tt} f + \partial_{rr} f + \frac{2r}{r^2 + a^2} \partial_r f.$$

# Annex B. Code

This is the code used to determine the sign of the integral $I$ in the proof for Lemma 2.14 in Chapter 2, Section 2.5. Script run on Python 3.6.13, with the modules NumPy 1.19.2 and SciPy 1.5.2.

Code B.1: Code used to calculate the integral.

```
import numpy as np
from scipy import integrate

R = 250     # Interval [0,R] used for integration
rt2 = np.sqrt(2)
k2 = 1/2 * integrate.quad(lambda s: s * np.cosh(rt2*s)**-2, 0, R)[0]

# Integrands
h1 = lambda r: np.tanh(rt2*r) * np.cosh(rt2*r)**-4

g1 = lambda r: r**2 * np.tanh(rt2*r)
g2 = lambda r: integrate.quad(lambda s: s * np.tanh(rt2*s), 0, r)[0]

f1 = lambda r: integrate.quad(lambda s: s * np.cosh(rt2*s)**-2, -R, r)[0]
f2 = lambda r: 2 * rt2 * r + np.sinh(2*rt2*r)

g3 = lambda r: f1(r) * f2(r)

h2 = lambda r: g1(r) - g2(r) - 1/2 * g3(r)

H = lambda r:  h1(r) * h2(r)

# Integrate functions
result = integrate.quad(H, 0, R)
I = 8*result[0]

print('I = ', I)
```