UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

MINIMUM DEGREE CONDITIONS FOR MONOCHROMATIC CYCLE PARTITIONING IN BIPARTITE GRAPHS

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN CIENCIAS DE LA INGENIERÍA,
MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

MATÍAS ANDRÉS AZÓCAR CARVAJAL

PROFESORA GUÍA:
MAYA STEIN

MIEMBROS DE LA COMISIÓN:
MARTÍN MATAMALA VÁSQUEZ
HIỆP HÀN

Este trabajo ha sido parcialmente financiado por FONDECYT REGULAR 1180830,
CMM ANID PIA AFB170001, CMM ANID BASAL ACE210010 y
CMM ANID BASAL FB210005.

SANTIAGO DE CHILE
2023

RESUMEN TESIS PARA OPTAR AL
GRADO DE MAGÍSTER EN CIENCIAS DE LA
INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO
CIVIL MATEMÁTICO
POR: MATÍAS ANDRÉS AZÓCAR CARVAJAL
FECHA: 2023
PROFESORA GUÍA: MAYA STEIN

CONDICIONES DE GRADO MÍNIMO PARA PARTICIONES DE CICLOS MONOCROMÁTICOS EN GRAFOS BIPARTITOS

Si coloreamos con r colores las aristas de un grafo con grado mínimo $n/2 + 1200r\log(n)$ es posible construir una partición del conjunto de vértices, compuesta únicamente ciclos monocromáticos, de tamaño $O(r^2)$. Este resultado, probado por Korándi, Lang, Letzter y Pokrovskiy en [26], es el que motiva el estudio de esta tesis.

El resultado de que se presenta aquí es una adaptación de la condición de grado mínimo, condicionado a que ahora el grafo estudiado sea bipartito balanceado. Más precisamente, para todo $\eta > 0$, para todo grafo bipartito balanceado r-arista-coloreado suficientemente grande con grado mínimo $(1/4 + \eta)n$, es posible asegurar la existencia de un *vertex cover* de tamaño $O(r^2)$ compuesto únicamente por ciclos monocromáticos vértice disjuntos.

Para la demostración del resultado, se presenta el concepto de grafos *birobustamente emparejables* y usamos el lema de regularidad, en su versión de r colores. Posterior a esto, utilizamos un método propuesto por Łuczak para cubrir casi todo el grafo. Finalizamos utilizando el *"blow-up lemma"* para cubrir los vértices faltantes.

RESUMEN TESIS PARA OPTAR AL
GRADO DE MAGÍSTER EN CIENCIAS DE LA
INGENIERÍA, MENCIÓN MATEMÁTICAS APLICADAS
MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO
CIVIL MATEMÁTICO
POR: MATÍAS ANDRÉS AZÓCAR CARVAJAL
FECHA: 2023
PROFESORA GUÍA: MAYA STEIN

MINIMUM DEGREE CONDITIONS FOR MONOCHROMATIC CYCLE PARTITIONING IN BIPARTITE GRAPHS

If we colour with r colours the edges of a graph with minimum degree $n/2 + 1200r\log(n)$ it is possible to construct a partition of the vertex set, which only contains monochromatic cycles, of size $O(r^2)$. This result, proved by Korándi, Lang, Letzter, and Pokrovskiy in [26], is the motivation for the study of this thesis.

The result presented here is an adaptation of the minimum degree condition, conditional on the fact that now the studied graph is balanced bipartite. More precisely, for every $\eta > 0$ and for any sufficiently large balanced bipartite r-edge-coloured graph with minimum degree $(1/4 + \eta)n$, it is possible to ensure the existence of a vertex cover of size $O(r^2)$ composed only of vertex-disjoint monochromatic cycles.

For the proof of the result, we present the concept of *birobustly matchable* graphs and use the regularity lemma, in its r-colour version. Subsequently, we use a method proposed by Łuczak to cover almost the whole graph. We finish by using the "*blow-up lemma*" to cover the remaining vertices.

Education never ends, Watson. It is a series of lessons, with the greatest for the last.

Sherlock Holmes

Agradecimientos

Me gustaría comenzar esta sección agradeciendo a mi familia. A mi madre, por su infinita entrega desde el comienzo y a lo largo de toda mi vida. A mi padre, por su compañía, guía y admiración. A mi hermana, Cami, por ser desde siempre una persona que ha estado a mi lado, en las buenas, malas y peores. A todos ellos, infinitas gracias desde el fondo de mi corazón. Quiero que sepan que sin ustedes sé que no estaría donde estoy.

Quiero también agradecer especialmente a mi profesora guía, Maya Stein. Estoy sumamente agradecido desde el momento en que aceptó tomarme como tesista. Gracias por introducirme en la teoría extremal de grafos, por su entrega, por cada comentario, cada idea, cada corrección, cada palabra de aliento, cada reunión, cada deseo de que siguiera apuntando más alto y cada paso dado para poder completar este proceso de la mejor manera posible. Gracias por ser una gran mentora para mí y por ayudarme a crecer como estudiante y matemático.

A *los Chacritas*, Barri, Benji, Choco, Gitano, Javo, Madariaga, Majo, Nico, Pavlito, Pele, Vicho y Vivi, quienes me acompañaron a lo largo de (casi) todo mi paso por el DIM. Gracias por todos esos momentos de amistad, de trabajo conjunto, de sacadas de vuelta, sufrimiento y compañerismo, de grandes aventuras, de rabias, penas y otros que no podrían haber sido mejores. Hicieron de mi carrera una etapa muy acompañada, fructífera, entretenida y llena de grandes momentos que recordaré siempre.

A Jorge, por su incansable y eterna amistad desde que nos conocimos, hace ya más de 10 años. Por acompañarme en la facultad que nos vio crecer en una nueva etapa después del liceo.

A María José, por ser mi primera gran amiga en la universidad y ser una compañía presente durante toda mi etapa en ella, como compañera y colega. Por ser un apoyo e impulso enorme, incluso en mis peores momentos. Por compartir penas y alegrías a lo largo de los años. Gracias por todo lo que me brindaste y ser una persona tan valiosa para mí.

A Cami y Javier, con quienes estoy muy feliz de haber coincidido en momentos que fueron complejos. Fueron un soporte emocional importante y una compañía reconfortante, personal y académica estando, probablemente, más cerca que con ningún, por nuestra afinidad de áreas y gustos. Muchísimas gracias por su amistad y lo compartido.

A Javiera Herrera y Valentina Aguilar, grandes amigas que estuvieron en momentos clave de mi vida y me permitieron estar en las suyas. Por su incondicional amistad y paciencia.

A Álvaro, un amigo que llegó hace poco, pero se siente como de toda la vida.

A Lucho, por ser un fichaje mucho mejor de lo que nadie hubiera imaginado y convertirse en un gran sucesor, amigo, compañero y jefe. Espero siga creciendo y siendo una gran persona.

A Anto, Carlos y Diego por su alegre compañía, por recibirme con agrado durante este último semestre y por su inesperada e impagable entrega, apoyo y amistad sincera.

A *Dragon Loli*, *Demonios Koreanos* y *Bruno Express FC*, grupos de amigos que han estado desde hace tiempo o llegaron de forma repentina. Tuvieron un rol sumamente relevante en mantener mi cordura durante el período de pandemia (y después de ella, hasta el día de hoy).

A las y los funcionarios del DIM, Eterin, Karen, Silvia, Gladys, María Cecilia, Luis, Oscar, Christian, Carlos, Kuki, Linda y Juanito. Gracias por cada vez que me ayudaron a sacar adelante distintas actividades y por ser el mecanismo que mantiene al DIM funcionando.

A las generaciones anteriores del DIM, que me hicieron sentir bienvenido desde el primer día y a quienes siempre recuerdo con cariño, por los consejos y ánimo entregados. Me gustaría destacar a Rolando, Rodrigo, Felipe, Ian y Arturo, por acompañarme desde el principio. Y a las generaciones siguientes (sobre todo al grupo *Pichangas DIM*), quienes me han mostrado un afecto, respeto y compañerismo tremendo.

A los profesores que tuve durante la carrera, quienes aportaron con un poquito de su esencia para formarme durante estos años. Quiero agradecer, especialmente, a Sebastián Donoso por ayudarme -muchas veces contra el tiempo- en numerosas ocasiones.

Agradezco especialmente también a los profesores Martín Matamala y a Hiệp Hàn por acceder a ser parte de mi comisión y por sus valiosos comentarios respecto de esta tesis.

Agradezco a Richard Lang, por tener la amabilidad de comentar conmigo dudas que tuve respecto de su artículo con lujo de detalle y ayudarme con problemas que tuve durante la tesis.

A mis amigos y amigas de plan común, quienes fueron las personas que me abrieron las puertas para pertenecer a un grupo y me acompañaron durante (al menos) dos largos años. Me entregaron recuerdos invaluables que siempre llevaré conmigo.

A mi gatita Coco, y al resto de las criaturitas de la casa, Kuru, Pandora y Pucho, por acompañarme y sacarme más de una sonrisa desde su llegada a mi vida.

A las personas que me permitieron enseñarles un poquito. Gracias por permitirme desarrollar una veta que siempre me ha parecido demasiado fascinante. Quisiera hacer mención especial a Antonia Suazo, una persona muy real, gran alumna y amiga.

Finalmente, me gustaría hacer mención especial a Natacha, quien fue una persona muy importante en mi formación. Gracias por cada oportunidad, por las distintas instancias y conversaciones, por cada favor hecho a última hora, por cada chistecito, por los recreos para acompañarte un cigarro, por esos sábados interminables, por ser una jefa dura, pero gentil y por motivarme, al igual que cada persona en estos agradecimientos, a terminar este trabajo. Aprecio mucho todo lo que pudimos realizar durante estos años.

Table of Content

1. Introduction **1**
1.1. Vertex covers in monochromatic pieces 1
1.2. Overview of the proof 5

2. Notation and previous results **7**
2.1. Initial definitions 7
2.2. Matching theorem 7
2.3. Regularity 7
2.4. Robustly matchable graphs and matching lemmas 9
2.5. Auxiliary lemmas for paths and cycles 9
2.6. Graphs and probabilities 10
2.7. Main covering theorems 11
2.8. Similarities and differences 11

3. Proof of the principal theorem **15**
3.1. Birobustly matchable graphs 15
3.2. Monochromatic cycle partition of birobustly matchable graphs 19
3.3. Turning balanced bipartite graphs into birobustly matchable graphs 27
3.4. Proof of Theorem 1.3 28

Bibliography **29**

Annex **32**

List of Figures

1.1. Diagram of graph with $\delta(G) = (1/4 - \eta)n$ which cannot be covered with less than $O(n)$ monochromatic cycles. 5

3.1. Diagram of the example of a birobustly matchable graph. 16

3.2. Example of H' with $V(H) = \{x, y, z\}$, $E(H) = \{(x, y), (x, z)\}$ and $b_1(x) = 5$, $b_1(y) = 2$, $b_1(z) = 3$ 18

Chapter 1

Introduction

This chapter will introduce, in section 1.1, the previous studies done on vertex covers in monochromatic pieces as a historical context for this thesis, going through different approaches that have led to different results and conjectures. Finally, in section 1.2 we conclude with an overview of the proof of the main theorem. For the sake of simplicity, every time we mention a 2-edge-colouring without any further details, we are going to assume the two colours are red and blue.

1.1. Vertex covers in monochromatic pieces

The study of vertex covers in monochromatic pieces has been a very popular topic in extremal graph theory through the years. Going back to 1967, Gerencsér and Gyárfás [13] proved that the vertex set of any 2-edge-coloured complete graph K_n can be partitioned into a red and a blue path. This arises as the first approach to the study of path covering in r-edge-coloured graphs.

Roughly ten years later, in 1979, Lehel conjectured that the vertex set of any 2-edge-coloured complete graph K_n can be partitioned into a red and a blue cycle; this conjecture was first cited in [4]. Here we admit an isolated vertex and K_2 as cycles, allowing some particular graphs to have this decomposition. These cycles will be called "*degenerated cycles*". Gyárfás [21] proved in 1983 a slightly weaker statement which was that the vertex set of any 2-edge-coloured complete graph can be covered by a red and a blue cycle such that they intersect in at most one vertex. Please note that covering the vertex set means to cover the vertices, but not necessarily the edges.

It was not until several years after Gyárfás had presented his proof that Lehel's conjecture could be fully demonstrated. First, in 1998, Łuczak, Rödl, and Szemerédi [40] proved Lehel's conjecture for large enough graphs using the regularity lemma. Allen [1], in 2008, improved the result of Łuczak, Rödl, and Szemerédi by reducing the order required in [40] for the graphs. It is noteworthy that he did not use the regularity lemma. Finally in 2010, Bessy and Thomassé [6] fully proved Lehel's conjecture using an inductive proof.

Lehel's conjecture can be understood as a particular case of a more general conjecture proposed by Erdős, Gyárfás, and Pyber [11] in 1991. They conjectured that any r-edge-coloured complete graph can be partitioned into at most r monochromatic cycles. They also

showed there exist r-edge-coloured complete graphs that can only be covered by at least r monochromatic paths (therefore, the number of cycles needed is also r). An example provided by them is the following:
Consider a complete graph whose vertices are partitioned into sets A_i where $|A_i| = 2i-1$ for $i = 1, 2, \ldots, r$. The edge xy gets colour $\min\{i : A_i \cap \{x, y\} \neq \emptyset\}$. Then colour 1 must be used to cover A_1 and if colour i is not used for some $1 < i \leq r$ then paths of colour $1, 2, \ldots, i-1$ can cover at most $\sum_{i=1}^{i-1} 2^j = 2^i - 2 < |A_i|$ vertices of A_i. Since edges of colour $j > i$ cannot cover any vertex of A_i, some vertex of A_i remains uncovered. Thus all the r colours are needed in a vertex cover formed by monochromatic paths.

The $r = 3$ case of the conjecture of Erdős, Gyárfás, and Pyber [11] was solved asymptotically by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [20] in 2011. Unfortunately, the previously mentioned conjecture was proven false by Pokrovskiy [30] in 2014 for all $r \geq 3$. He presented a counterexample in which r vertex disjoint monochromatic cycles can cover, at most, all vertices except one. Pokrovskiy [30] also conjectured that for each r there is a constant c_r, such that in every r-edge-coloured complete graph K_n, there are r vertex-disjoint monochromatic cycles covering $n - c_r$ vertices in K_n. For $r = 3$ Pokrovskiy [30] proved that $c_3 \leq 43000$ and Letzter [28], independently, proved that $c_3 \leq 60$. Pokrovskiy [31] conjectured $c_3 = 1$.

The best known general upper bound for the minimum number of monochromatic cycles required to partition any r-edge-coloured complete graph is $100r\log(r)$, established by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [19] in 2006. Therefore, the gap between the upper and lower bound for the minimum number of cycles needed to partition the vertex set of an r-edge-coloured complete graph remains a factor of $\log r$.

There exist results for other host structures such as hypergraphs [16, 17, 35] and infinite graphs [10, 17, 33, 38]. Also, there are results using other subgraphs to cover such as graphs of bounded degree [14, 36] and connected components [11, 12, 24].

In 1997, Gyárfás, Jagota, and Schelp [18] took another direction and proved the following: Assuming $n \geq 5$ and that G is a graph obtained from K_n by deleting at most $m = \lfloor n/2 \rfloor$ edges, then for every 2-edge-colouring of G, $V(G)$ can be partitioned into a red and a blue path. This was one of the first approaches of monochromatic covering with a non-complete host graph.

There exist more specific results of monochromatic cycle (and path) partitioning for few colours on non-complete host graphs. For two colours, in 2015, Schaudt and Stein [37] proved that any 2-edge-coloured complete k-partite graph G on n vertices, with $k \geq 3$ such that the largest partition class of G contains at most $n/2$ vertices, can be covered with two vertex-disjoint monochromatic paths of distinct colours. Moreover, the same authors proved in [37] that, under the same conditions, if the graph G is large enough, then it can be covered with 14 vertex-disjoint monochromatic cycles. For three colours, Lang, Schaudt, and Stein [27] proved in 2017 that every 3-edge-coloured complete bipartite graph $K_{n,n}$ contains 5 vertex-disjoint monochromatic cycles such that they cover all but $o(n)$ vertices. The same authors in [27] proved that there exists $n_0 \in \mathbb{N}$ such that for every complete bipartite graph $K_{n,n}$ with $n \geq n_0$ there exists a partition of $V(K_{n,n})$ into 18 monochromatic cycles.

Monochromatic cycle partitioning for r-edge-coloured complete bipartite graphs has also been studied to a large extent. In 1989, Gyárfás [15] proved that for any r-edge-coloured balanced complete bipartite graph $K_{n,n}$, the minimum number of monochromatic paths needed to cover its vertex set is bounded by a function of r. Moreover, in 1997, Haxell [23] proved a stronger result. For any r-edge-coloured $K_{n,n}$ the number of monochromatic cycles needed to partition its vertex set is upper bounded by a function of r. Furthermore, for large r the needed number of cycles to partition the vertex set of any r-edge-coloured $K_{n,n}$ can be upper bounded by $c(r\log(r))^2$. Peng, Rödl, and Ruciński [29] showed a better upper bound, lowering the number of needed cycles to $O(r^2\log r)$. This result of Peng, Rödl, and Ruciński was improved in 2018 by Stein and Lang [27] who proved that $4r^2$ monochromatic cycles suffice to partition the vertex set of a large enough bipartite graph.

Other parameters have been studied to bound the number of monochromatic cycles needed to partition the vertex set of a graph. In 1963, Pósa [32] proved that the vertex set of every graph G can be partitioned into at most $\alpha(G)$ cycles where $\alpha(G)$ denotes the independence number of G. In 2010, Sárközy [34] showed that the vertex set of any r-edge-coloured graph G can be partitioned into at most $25(\alpha r)^2\log(\alpha r)$ monochromatic cycles and conjectured this number can be lowered to $\alpha(G)r$. The counterexample provided by Pokrovskiy mentioned above disproves this conjecture. Nevertheless, the $r=2$ case of this conjecture is true in an asymptotic sense, as Balogh, Barát, Gerbner, Gyárfás, and Sárközy [5] showed in 2014.

Minimum degree is another parameter that has been studied to bound the cycle partition number. In 1952, Dirac [9] proved a classic result. For every graph G with $n\geq 3$ vertices, if $\delta(G)\geq n/2$, then there exists a Hamiltonian cycle, which is a cycle that contains every vertex in $V(G)$. This can be considered a monochromatic cycle partition in a 1-edge-coloured graph.

Following with the study of minimum degree conditions, in 2014, Balogh, Barát, Gerbner, Gyárfás, and Sárközy [5] conjectured the following: For any 2-edge-colouring of the edges of any n-vertex graph G of minimum degree $3n/4$, there are two distinctly coloured monochromatic cycles which together partition the vertices of G. If this result is true, it would be tight. In support of their conjecture, they proved an approximate version in which G has minimum degree $3n/4+o(n)$ and the cycles are allowed to miss $o(n)$ vertices. In 2017, DeBiasio and Nelsen [8] showed that under this (stronger) degree condition a complete partition is possible. Finally, in 2019, Letzter [28] resolved the full conjecture for all sufficiently large n.

After these advances, Pokrovskiy [31] conjectured that analogous results are true for graphs of lower minimum degree. In particular, he conjectured that for a 2-edge-coloured graph G with $\delta(G)\geq 2n/3$ a partition into 3 monochromatic cycles is possible. Similarly, he also conjectured that for a 2-edge-coloured graph G with $\delta(G)\geq n/2$ a partition into 4 monochromatic cycles is possible. In 2022, Allen, Böttcher, Lang, Skokan, and Stein [2] proved the first of these conjectures approximately (using $\delta(G)\geq(2/3+\varepsilon)n$).

In 2021, Korándi, Lang, Letzter, and Pokrovskiy [26] proved the following theorem.

Theorem 1.1 (Theorem 1.2 in [26]) *For $r\geq 2$, let n be sufficiently large. Then any r-edge-coloured graph G on n vertices with $\delta(G)\geq n/2+1200r\log(n)$ admits a partition into 10^7r^2*

monochromatic cycles.

They also provide a construction which shows that the number of cycles of Theorem 1.1 is essentially best possible. Notice that this minimum degree condition allows us to find a Hamiltonian cycle in G, so for $r = 1$, we can find a monochromatic cycle partition of one cycle.

The main contribution made by Korándi, Lang, Letzter and Pokrovskiy was providing a minimum degree threshold such that, for graphs that satisfy such threshold, there exists a monochromatic cycle partition of size $O(r^2)$. Such threshold cannot be reduced too much since there are graphs with slightly less degree than $n/2 + 1200r\log(n)$ that cannot be partitioned into $O(r^2)$ monochromatic cycles. This last fact is supported by the next proposition from [26].

Proposition 1.2 (Proposition 1.1 in [26]) *There exists n_0 such that there exists a 2-edge-coloured graph G on $n \geq n_0$ vertices with $\delta(G) \geq n/2 + \log(n)/(16\log(\log(n)))$ whose vertices cannot be partitioned into fewer than $\log(n)/(32\log(\log(n)))$ monochromatic cycles.*

The proof of Proposition 1.2 presented in [26] is done through the construction of a graph that cannot be covered by less than $\log(n)/(32\log(\log(n)))$. Unfortunately, Proposition 1.2 cannot be used to bound the minimum degree needed in bipartite graphs since the construction involved is not bipartite.

There exists an interesting result on minimum degree conditions for cycle partitioning in (uncoloured) balanced bipartite graphs. Define an n-ladder to be the balanced bipartite graph L_n with vertex sets $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ such that $(ai, bj) \in E(L_n)$ if and only if $|i - j| \leq 1$. So L_n consists of two vertex disjoint n-paths $a_1b_2a_3b_4\ldots$ and $b_1a_2b_3a_4\ldots$ together with *rungs* formed by the matching $a_1b_1, \ldots, a_nb_n$. In 2002, Czygrinow and Kierstead [7] proved that for sufficiently large n, every balanced bipartite graph $G = (U, V)$ with $|U| = |V| = n$ and $\delta(G) \geq n/2 + 1$ contains a spanning ladder. This implies the existence of a Hamiltonian cycle, a cycle partition of size 1.

Following the work carried out in [26], the main contribution of this thesis is that we establish minimum degree conditions to bound the smallest number of monochromatic cycles needed to partition the vertex set of an r-edge-coloured balanced bipartite graph given that the vertex set of the graph is large enough. This is summarized in the next theorem.

Theorem 1.3 *For each $r \geq 2$ and each $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ any r-edge-coloured balanced bipartite graph G on n vertices with $\delta(G) \geq (1/4 + \eta)n$ admits a partition into 10^7r^2 monochromatic cycles.*

The minimum degree bound of Theorem 1.3 is not far from the very best. The following example shows that, given a graph G such that its minimum degree is slightly less than $n/4$ it is not possible to partition the vertex set $V(G)$ into $f(r)$ monochromatic cycles where f is any increasing function.

We construct a balanced bipartite graph G on n vertices with parts A and B as follows. Consider two copies of $K_{(1/4-\eta)n,(1/4+\eta)n}$, $G_1 = \{C_1, D_1\}$ and $G_2 = \{C_2, D_2\}$ such that $|C_1| = |C_2| = (1/4-\eta)n$ and $|D_1| = |D_2| = (1/4+\eta)n$. Then, we define $A := C_1 \cup D_2$, $B := D_1 \cup C_2$

and $E(G) = E(G_1) \cup E(G_2)$. It is clear that G is a balanced bipartite graph and $\delta(G) = (1/4 - \eta)n$. We colour $E(G)$ with only one colour (say, red) as in Figure 1.1.

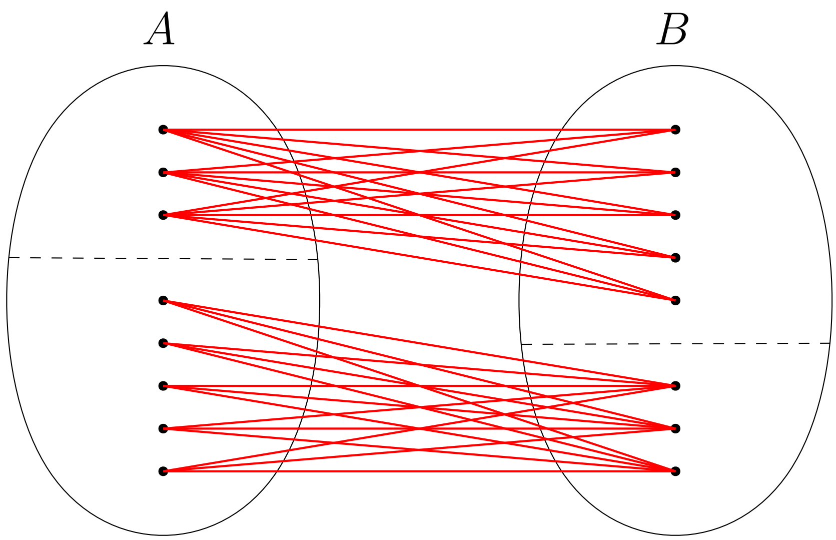

Figure 1.1: Diagram of graph with $\delta(G) = (1/4 - \eta)n$ which cannot be covered with less than $O(n)$ monochromatic cycles.

The dashed lines represent the separation between the copies of $K_{(1/4-\eta)n,(1/4+\eta)n}$. We know that cycles will cover the same number of vertices in A and B so for each of the copies of $K_{(1/4-\eta)n,(1/4+\eta)n}$ there will be $2\eta n$ uncovered vertices. We have to add them as isolated vertices to the cycle partition, resulting in at least $O(n)$ monochromatic cycles.

The following corollary presents an extension to Theorem 1.3 in which we can partition the vertex set of slightly unbalanced r-edge-coloured bipartite graphs into $O(r^2)$ monochromatic cycles.

Corollary 1.4 *For each $r \geq 2$ and each $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ any r-edge-coloured bipartite graph $G = \{A, B\}$ on n vertices with $\delta(G) \geq (1/4 + \eta)n$ and* $\max\{|A|, |B|\} - \min\{|A|, |B|\} \in O(r^2)$ *admits a partition in* $10^7 r^2 + \max\{|A|, |B|\} - \min\{|A|, |B|\}$ *monochromatic cycles.*

We defer the proof of Corollary 1.4 to Annex.

1.2. Overview of the proof

Following very closely the proof of Theorem 1.1 by Korándi, Lang, Letzter, and Pokrovskiy in [26], we start with a graph G to which we apply a modified version of Szemerédi's regularity lemma [25] to obtain a regular partition $\{V_0, V_1, \ldots, V_m\}$ of the vertices of G, and define the corresponding reduced graph $\mathcal{G}$. Then, we choose $O(r^2)$ monochromatic components such that its union $\mathcal{H}$ contains a perfect matching $\mathcal{M}$. The graph $\mathcal{H}$ will contain a perfect matching even after removing some vertices. This property will be introduced as being "birobustly matchable".

Using a method proposed by Łuczak [39] we turn the matching $\mathcal{M}$ into $O(r^2)$ disjoint monochromatic cycles $\mathcal{C}^{\mathcal{H}}$ covering almost all of G. The uncovered vertices will be added to $\mathcal{C}^{\mathcal{H}}$ using the blow-up lemma, finding monochromatic spanning paths in the regular pairs associated to the matching $\mathcal{M}$. To do this we have to prepare the graph. First, we cover the vertices that do not have regular behaviour with $O(r^2)$ cycles, $\mathcal{C}^b$. Then we want to extend the cycles of $\mathcal{C}^{\mathcal{H}}$ to the remaining vertices. The remaining vertices in the clusters might be unbalanced after removing the vertices that are already covered. Thus, we extend $\mathcal{C}^{\mathcal{H}}$ at the right location by finding space to allocate the extension of the cycles. This is possible because of the *birobustly matchable* property mentioned earlier.

We end the proof by applying the blow-up lemma to add the remaining vertices to $\mathcal{C}^{\mathcal{H}}$. We end up with a monochromatic cycle partition $\mathcal{C}^{\mathcal{H}} \cup \mathcal{C}^b$ that contains $O(r^2)$ cycles. Finally, we prove a lemma that joins the two concepts, birobustly matchable graphs and balanced bipartite graphs with large minimum degree, and allows us to conclude the result.

Chapter 2

Notation and previous results

In this chapter, we establish the foundation for our investigation by introducing the essential notation, definitions, and previous results that will serve as the framework for our subsequent analysis.

2.1. Initial definitions

A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V. We call the elements in V *vertices* of G and the elements in E *edges* of the graph G.

Let $G = (V, E)$ be a (non-empty) graph. Two vertices x, y of G are *adjacent*, or *neighbours*, if $\{x, y\}$ is an edge of G. The set of neighbours (or *neighbourhood*) of a vertex v in G is denoted by $N_G(v)$. The *degree* of a vertex v in the graph G is defined as $\deg_G(v) := |N_G(v)|$.

The *neighbourhood of a set* S in the graph G is defined as the set $N_G(S) := \bigcup_{v \in S} N_G(v)$. The *degree of a vertex* v *to a set* W in a graph G is defined as $\deg_G(v, W) := |N_G(v) \cap W|$

In the last definitions, we drop the index G, leaving just $N(v)$, $\deg(v)$, $N(S)$ and $\deg(v, W)$ if it is clear the underlying graph we are studying.

2.2. Matching theorem

Matchings are a keystone in the construction of large monochromatic cycles in this work. Thus, we present a well-known result that we use to construct matchings in graphs.

Theorem 2.1 (Hall [22]) *Let H be a bipartite graph with bipartition $\{X, Y\}$. Then there exists a matching that covers X if and only if for each subset S of X it holds that $|S| \leq |N(S)|$.*

2.3. Regularity

A crucial tool in extremal graph theory, also used in this thesis, is regularity. Regularity (and more specifically, Szemerédi's regularity lemma) is a cornerstone of extremal graph theory, allowing any graph to be approximated by random graphs. This "random" behaviour is really powerful in the sense that reasonable properties with the random framework help prove the result in a more general one. Here we introduce the necessary concepts to understand and work with regularity.

Given a graph G and vertex sets $V, W \subseteq V(G)$ such that $V \cap W = \emptyset$ we define the density of the pair (V, W) by $d(V, W) = \frac{|E(G)|}{|V||W|}$. The pair (V, W) is called ε-regular, if every pair of subsets $X \subseteq V$ and $Y \subseteq W$ with $|X| \geq \varepsilon |V|$ and $|Y| \geq \varepsilon |W|$ satisfy that $|d(V, W) - d(X, Y)| \leq \varepsilon$.

We say that a vertex $v \in V$ has *typical degree* in (V, W), if $\deg(v, W) \geq (d(V, W) - \varepsilon)|W|$. Notice that all but at most $\varepsilon|V|$ vertices in V have typical degree in (V, W).

Now we present a different version of Szemerédi's regularity lemma. This lemma is a cornerstone result in extremal combinatorics and is widely used to have a rough idea of how a large graph should look like.

Lemma 2.2 (Szemerédi's Regularity Lemma, degree form with r colors and a prepartition [25]) *For every $\varepsilon > 0$ and integers r, ℓ, there is an $M = M(\varepsilon, r, \ell)$ such that the following holds. Let G be a graph on $n \geq 1/\varepsilon$ vertices whose edges are coloured with r colours, let $\{W_1, \dots, W_{\ell'}\}$ be an equipartition of $V(G)$ for some $1 \leq \ell' \leq \ell$, and let $d > 0$. Then there is a partition $\{V_0, \dots, V_m\}$ of $V(G)$ and a subgraph G' of G with vertex set $V(G) \setminus V_0$ such that the following conditions hold.*

a) $1/\varepsilon \leq m \leq M$,

b) $|V_0| \leq \varepsilon n$ and $|V_1| = \ldots = |V_m| \leq \varepsilon n$,

c) for every $i \in [m]$, there is $j \in [\ell']$ with $V_i \subseteq W_j$,

d) for every $j \in [\ell']$, there are equally many $i \in [m]$ with $V_i \subseteq W_j$,

e) $deg_{G'}(v) \geq deg_G(v) - (rd + \varepsilon)n$ for each $v \in V(G) \setminus V_0$,

f) $G'[V_i]$ contains no edges for $i \in [m]$, and

g) all pairs (V_i, V_j) are ε-regular in G' for $i \neq j \in [m]$ and have in each colour either density 0 or density at least d.

Let G be an r-edge-coloured graph with a partition $\{V_0, \dots, V_m\}$ obtained from Lemma 2.2 with parameters ε and d. We define the (ε, d)-reduced graph $\mathcal{G}$ with respect to the partition $\{V_0, \dots, V_m\}$ to be a graph with vertex set $V(\mathcal{G}) = \{x_1, \dots, x_m\}$ where two vertices x_i and x_j are connected by an edge of colour c, if (V_i, V_j) is an ε-regular pair of density at least d in colour c (if this holds for multiple colours, we choose one of them arbitrarily). Note that if G is balanced ℓ-partite with partition $\{W_1, \dots, W_\ell\}$, then $\mathcal{G}$ is a balanced ℓ-partite graph as well. It is often convenient to refer to a *cluster* V_i via its corresponding vertex in the reduced graph, i.e. $V_i = V(x_i)$.

Proposition 2.3 (Degree and edges variation in reduced graphs. Proposition 3.3 in [26]) *Let G be an r-edge-coloured graph and $\mathcal{G}$ be an (ε, d)-reduced graph obtained from Lemma 2.2. Then the following properties hold:*

a) If $deg_G(v) \geq cn$ for some $v \in V_i, i \in [m]$, then $deg_{\mathcal{G}}(x_i) \geq (c - rd - \varepsilon)m$.

b) If $deg_G(v) \geq cn$ for all but ηn vertices $v \in V(G)$, then $deg_{\mathcal{G}}(x) \geq (c - rd - \varepsilon)m$ for all but $(\eta + \varepsilon)m$ vertices $x \in V(\mathcal{G})$.

c) If $\bigcup_{x_i \in X} V_i$ induces at least cn^2 edges in G for some $X \subseteq V(\mathcal{G})$, then X induces at least $(c - rd - \varepsilon)m^2$ edges.

2.4. Robustly matchable graphs and matching lemmas

As we mentioned, Korándi, Lang, Letzter, and Pokrovskiy worked on a more general framework than the one presented in this thesis. Here we mention previous definitions and results from [26] to reference them when needed.

Definition 2.4 (Perfect b-matching. Definition 3.5 in [26]) *Let $b : V(G) \to \mathbb{N}$ be a function. A perfect b-matching of G is a non-negative function $\omega : E(G) \to \mathbb{N}$, such that $\sum_{u \in N(v)} \omega(uv) = b(v)$ for every vertex $v \in V(G)$. When b is a constant function equal to τ, we call ω a perfect τ-matching.*

Lemma 2.5 (Lemma 3.8 in [26]) *Every (μ, ν)-robustly 2-matchable graph H with $\mu \leq \nu < 1/1000$ contains a perfect 2-matching.*

Lemma 2.6 (Lemma 3.9 in [26]) *Suppose H is a (μ, ν)-robustly 2-matchable graph on n vertices and let $\varepsilon > 0$. Suppose H' is a spanning subgraph of H such that $\deg_{H'}(v) \geq \deg_H(v) - \varepsilon n$ for every vertex v. Then H' is a $(\mu + \varepsilon, \nu - \varepsilon)$-robustly 2-matchable graph whose type coincides with that of H.*

Lemma 2.7 (Lemma 3.10 in [26]) *Suppose H is an r-edge-coloured (μ, ν)-robustly 2-matchable graph on n vertices. Let $\mathcal{H}$ be the (ε, d)-reduced graph of H obtained from Lemma 2.2 with some parameters $\varepsilon, d > 0$ and $\ell = 2$ (and the corresponding bipartition if H is a robustly 2-matchable of type 2 graph). Then $\mathcal{H}$ is $(\mu + rd + 2\varepsilon, \nu - rd - 2\varepsilon)$-robustly 2-matchable. Moreover, the type of $\mathcal{H}$ coincides with the type of H.*

Lemma 2.8 (Lemma 3.11 in [26]) *Let t, γ be constants, and let H be a (μ, ν)-robustly 2-matchable graph on m vertices such that $m/t \leq \gamma \leq \mu \leq \nu/4 < 1/4000$. Then H has a perfect b-matching for every function $b : V(H) \to \mathbb{N}$ that satisfies*

a) $(1 - \gamma)t \leq b(x) \leq t$ for every $x \in V(H)$,

b) $\sum_{x \in V(\Psi)} b(x)$ is even for every component Ψ of H, and

c) if H is of type 2 with bipartition $\{X, Y\}$ then $\sum_{x \in X} b(x) = \sum_{y \in Y} b(y)$.

2.5. Auxiliary lemmas for paths and cycles

Finding long paths or paths that behave in a particular way can be very difficult. Here we present some results that will be helpful when we need to obtain some specific kind of paths, cycles or vertex sets.

Lemma 2.9 (Long paths in regular pairs. Lemma 3.1 in [26]) *For every $d \in (0,1)$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following statement holds. Suppose that (V_1, V_2) is an ε-regular pair of density $d = d(V_1, V_2)$ with $|V_1| = |V_2| = n$ in a graph G. For $i \in \{1,2\}$, let $v_i \in V_i$ and let $U_i \subseteq V_i$ be a set of size at least $n/6$ which contains at least $2\varepsilon n$ neighbours of v_{3-i}.*
Then for every $2 \leq k \leq (1-24\varepsilon) \cdot min\{|U_1|, |U_2|\}$, there is a v_1-v_2-path of order $2k$ in $G[U_1 \cup \{v_1\}, U_2 \cup \{v_2\}]$.

If, aditionally, $\delta(G[U_1, U_2]) \geq 5\varepsilon n$, then $G[U_1 \cup \{v_1\}, U_2 \cup \{v_2\}]$ contains a v_1-v_2-path of order $2k$ for every k such that $2 \leq k \leq \min\{|U_1 \cup \{v_1\}|, |U_2 \cup \{v_2\}|\}$.

Lemma 2.10 (Set-avoiding paths. Lemma 3.4 in [26]) *For every $d \in (0,1)$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following statement holds. Let $G = (V, E)$ be an r-edge-coloured graph on n vertices with a partition $\{V_0, \ldots, V_m\}$ and an (ε, d)-reduced graph $\mathcal{G}$ obtained from Lemma 2.2. Suppose that $W \subseteq V$ is a vertex set such that $|W \cap V_i| \leq (d/2) \cdot |V_i|$ for every $i \in [m]$. Let $x_i x_j$, $x_{i'} x_{j'} \in E(\mathcal{G})$ be two edges in a component of colour c.*

Then for any two vertices $v \in V_i$ and $w \in V_{j'}$ of typical degree in colour c in (V_i, V_j) and $(V_{i'}, V_{j'})$, G contains a c-coloured v-w-path P of order at most $2m$ that avoids all vertices of W.

Lemma 2.11 (Erdős, Gyárfás & Pyber [11]) *Let H be an r-coloured bipartite graph with bipartition $\{A, B\}$. Suppose that $|A| \geq 100^3 r^3 |B|$ and that every vertex in B has at least $|A|/100$ neighbours in A. Then there is a set of at most $100r^2$ monochromatic pairwise vertex-disjoint proper cycles and edges that together cover all vertices of B.*

2.6. Graphs and probabilities

The probabilistic method allows us to consider the existence of certain kind of graphs (or subgraphs) with specific conditions without explicitly finding them (for further references, see [3]). Here we work with a result that will be recurrent in the next chapter to guarantee the existence of a set of vertices with good properties.

Proposition 2.12 (Proposition 3.14 in [26]) *Let G be a graph on n vertices with an (ε, d)-regular partition $\{V_0, \ldots, V_m\}$ as provided by Lemma 2.2. Also, let p be a positive parameter, and let $B \subseteq V = V(G)$ be a vertex set satisfying $V_0 \subseteq B$ and $|B \cap V_i| \leq 10p|V_i|$ for every $i \in [m]$. If $m \log(n)/\sqrt{n} < p < 1/100$ and $\varepsilon < 1/10$, then there is a set $A \subseteq V \setminus B$ with the following properties.*

a) $|A| \geq (p/2)n$,

b) $|A \cap V_i| \leq 2p|V_i|$ *for every* $i \in [m]$,

c) $\deg(v, A \cap V_i) \geq (p/2)\deg(v, V_i)$ *for every* $v \in V$ *and* $i \in [m]$ *with* $\deg(v, V_i) > 30p|V_i|$,

d) $\deg(v, A) \geq |A|/100$ *for every vertex* $v \in V$ *with* $\deg(v, V \setminus B) > n/40$.

2.7. Main covering theorems

Now, we mention the main results obtained in [26] to highlight them and also to compare them in the next section with the results presented in this thesis.

Theorem 2.13 (Theorem 4.1 in [26]) *For every $r \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ and $\mu > 0$ such that for every $n \geq n_0$ the following statement holds. Let G be an r-edge-coloured graph on n vertices with minimum degree $\delta(G) \geq n/2 + 1200r \log n$. Then the vertices of G can be partitioned into at most $400r + 2$ monochromatic cycles and a $(\mu, 20\mu)$-robustly 2-matchable graph H on at least $n/2$ vertices.*

Theorem 2.14 (Theorem 4.2 in [26]) *For every $r \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ and $\mu > 0, \nu > 0$ such that, for every $n \geq n_0$, every r-edge-coloured (μ, ν)-robustly 2-matchable graph on n vertices can be partitioned into $(1/\mu + 200)r^2$ cycles.*

2.8. Similarities and differences

We now present the comparison of the work presented in this thesis with their analogous results in [26].

We start by changing the initial definition of what is the main tool for partitioning a graph into monochromatic cycles. In the case of [26], this is the definition of *robustly 2-matchable* graph.

Definition 2.15 ((μ, ν)-robustly 2-matchable graphs. Definition 3.7 in [26]) *A graph H on n vertices is (μ, ν)-robustly 2-matchable if any of the following two conditions holds.*

1. *$\delta(H) \geq (1/2 - \mu)n$ and every set of $(1/2 - \nu)n$ vertices spans at least νn^2 edges.*
2. *H is a balanced bipartite graph with parts A, B (of size $n/2$) such that*
   - *$\delta(H) \geq (1/32 - \mu)n$, and*
   - *all but at most $(1/64 + \mu)n$ vertices in H have degree at least $(1/3 - \mu)n$.*

If condition 1 is satisfied, then the graph is said to be (μ, ν)-robustly 2-matchable of type 1. Likewise, if condition 2 is satisfied, the graph is said to be (μ, ν)-robustly 2-matchable of type 2.

We replace this definition with our own definition of what is a (μ, ν)-*birobustly matchable* graph. This is presented as Definition 3.1.

We adapt the conditions of a (μ, ν)-robustly 2-matchable graph of type 1 to conditions that adequately apply to a balanced bipartite graph. In particular, this means setting the minimum degree of the graph to almost half of the degree that a complete balanced bipartite graph would have, just as in Definition 2.15, $\delta(G) \geq (1/2 - \mu)n$, in the (μ, ν)-birobustly matchable graph definition, we set $\delta(G) \geq (1/4 - \mu/2)n$.

We also adapt that "every set of $(1/2-\nu)n$ vertices spans at least νn^2 edges" from Definition 2.15 to make sense in the bipartite graph. This is done by adding the requirement to (μ, ν)-birobustly matchable graphs that any set of vertices that is large enough (we keep the $(1/2-\nu)n$ threshold) and has $\sqrt{\nu}n$ vertices in each part spans at least νn^2 edges. We will refer to this condition as the *edge spanning condition* throughout this thesis.

We mostly compare the proofs of the results of this thesis with their analogues from [26] on (μ, ν)-robustly 2-matchable graphs of type 1 due to the fact that we adapted that type of graphs to define (μ, ν)-birobustly matchable graphs.

Following, there is the presentation of Lemma 3.2, an analogue to Lemma 2.5 for (μ, ν)-birobustly matchable graphs. The objective of these two Lemmas is, under the assumption that $\mu \leq \nu < 1/1000$, to find a perfect 2-matching in every (μ, ν)-robustly 2-matchable graph, and a perfect matching in every (μ, ν)-birobustly matchable graph for Lemma 2.5 and Lemma 3.2, respectively.

The proof of Lemma 2.5 presented in [26] applies Theorem 2.1 to any (μ, ν)-robustly 2-matchable graph H of type 1 (with $\mu \leq \nu < 1/1000$) remembering that, by Definition 2.5, for any independent set S, $|S| \leq (1/2-\nu)n \leq (1/2-\mu)n \leq \delta(G) \leq |N(S)|$. In our case Lemma 3.2 is proved by contradiction. Given a (μ, ν)-birobustly matchable graph $H = \{X, Y\}$, we assume that there exists a set $S \subseteq X$ which prevents the use of Theorem 2.1. Then, since H has large minimum degree, after some computation, we conclude that the existence of S contradicts the edge spanning condition for H to be a (μ, ν)-birobustly matchable graph.

Lemma 3.3 is the analogue of Lemma 2.7 in [26]. As before, we consider a (μ, ν)-birobustly matchable graph H instead of a (μ, ν)-robustly 2-matchable graph. We show that if $\mathcal{H} = \{\mathcal{X}, \mathcal{Y}\}$ is the (ε, d)-reduced graph of an r-edge-coloured (μ, ν)-birobustly matchable graph H, with $|\mathcal{H}| = m$, then $\mathcal{H}$ is a $(\mu+2rd+2\varepsilon, \nu-rd-\varepsilon)$-birobustly matchable graph. We follow the same strategy as in [26], using Proposition 2.3, but adapting some calculations. By Proposition 2.3a) $\delta(\mathcal{H}) \geq (1/4-\mu)m$ implies that $\delta(\mathcal{H}) \geq (1/4-\mu/2-rd-\varepsilon)m$ and by Proposition 2.3c), for every set $Z \subseteq V(\mathcal{H})$ such that $|Z| \geq (1/2-\nu)m$, $|Z \cap \mathcal{X}|, |Z \cap \mathcal{Y}| \geq \sqrt{\nu}m$ spans at least $(\nu-rd-\varepsilon)m^2$ edges. In Lemma 2.7 the minimum degree of $\mathcal{H}$ is $(1/2-\mu-rd-\varepsilon)m$, and the condition of set Z is only that $|Z| \geq (1/2-\nu)m$ to span $(\nu-rd-\varepsilon)m^2$ edges.

For Lemma 3.4, our version of Lemma 2.8, the main objective is to prove, under some assumptions, the existence of a perfect b-matching. In Lemma 2.8, the perfect b-matching is found in a (μ, ν)-robustly 2-matchable graph, whereas in Lemma 3.4 we find a perfect b-matching in a (μ, ν)-birobustly matchable graph.

We start in the same way as in the proof of Lemma 2.8 but we elaborate more in the details of the proof. Given a (μ, ν)-birobustly matchable graph H, the second condition of Lemma 3.4 indicates that $\sum_{x \in V(\Psi)} b(x)$ is even for every component Ψ of the graph H. Therefore, we can associate each vertex with odd $b(x)$ with another vertex with odd $b(x)$. For each pair of these vertices, we choose one path whose endpoints are the two associated vertices. The family of these paths is defined as $\mathcal{P}$. Then, we define $\omega_0(e)$ as the number of paths in $\mathcal{P}$ that contain e, and $b_0(x)$ as $\sum_{y \in N(x)} \omega_0(xy)$. Then, $b_0(x)$ is odd if and only if $b(x)$ is odd. Defining $b_1(x) = b(x) - b_0(x)$ we get that $b_1(x)$ is always even.

Using the third condition of Lemma 3.4, we obtain that $\sum_{x \in X} b_1(x) = \sum_{y \in Y} b_1(y)$. In our proof, we transform our graph H into H', where each vertex $v \in H$ is replaced by a cluster $W(x)$ of $b_1(v)$ vertices (instead of $b_1(v)/2$, like it is done in [26]), and for each edge $uv \in E(H)$, the graph $H'[W(u), W(v)]$ is complete. The fact that $\sum_{x \in X} b_1(x) = \sum_{y \in Y} b_1(y)$ shows that H' is a balanced bipartite graph. We obtain that $\delta(H') \geq (1/4 - \gamma/2 - \mu/2)|H'|$ instead of the degree in the proof of Lemma 2.8, where $\delta(H') \geq (1/4 - \gamma - \mu)|H'|$. The remaining step is done using a different strategy. We count the vertices in each cluster to contradict the existence of a set that blocks the use of Lemma 2.1. In [26], instead of counting the vertices of each cluster, the size of independent sets is bounded, so Lemma 2.1 can be applied. In any case, this allows us to find a perfect matching in H' which implies the existence of a perfect b-matching in H.

We define the conditions (3.7) in a very similar way as in [26], but we choose $d = \mu/r$ so that we can set the value and determine the constants involving d.

The next result in this thesis is Theorem 3.5. This is the analogue to Theorem 2.14 in [26]. We start with the exact same opening but using a graph G that is birobustly matchable instead of robustly 2-matchable. We apply Lemma 3.3 to obtain the corresponding (ε, d)-reduced graph $\mathcal{G}$ and we define $\mathcal{H}$ as the subgraph of $\mathcal{G}$ that consists of all the edges contained in monochromatic components of order at least $m\mu/r$. Then we present the proof of Claim 3.6. This claim has an analogue role to Claim 2.6 has in this proof. We present the proof of this Claim within the proof of Theorem 3.5 because it is easier to work with the graph being already defined by (3.7) than with a graph that satisfies fewer hypotheses like it is done in [26].

The proof of Claim 3.6 consists in verifying the two conditions of Definition 3.1 for $\mathcal{H}$ to be a $(6\mu, \nu - 5\mu)$-birobustly matchable graph. Here, $\mathcal{H}$ is a graph such that $|\mathcal{H}| = m$, and $\deg_{\mathcal{H}}(v) \geq deg_{\mathcal{G}}(v) - \mu m$, where $\mathcal{G}$ is a $(4\mu, \nu - 3\mu)$ birobustly matchable graph. This Claim has the same role as Lemma 2.6. The minimum degree condition is verified by a simple calculation, just as in the proof of Lemma 2.6 in [26]. The proof of the edge spanning condition is more technical than its counterpart. In [26], the result follows from noticing that if each vertex loses at most εn neighbours, for n vertices, the amount of lost edges is at most εn^2, concluding their result. In our case, we separate two cases. Because of how we defined the edge spanning condition in Definition 3.1 we have to verify that, for any set $Z \subseteq \mathcal{H}$ such that $|Z| \geq (1/2 - (\nu - 5\mu))m$ and $|Z \cap \mathcal{X}|, |Z \cap \mathcal{Y}| \geq (\sqrt{\nu - 5\mu})m$, $|E[Z]| \geq (\nu - 5\mu)m^2$. We study the cases when relying on the good properties (3.8) and (3.9) that $\mathcal{H}$ has because of how it is defined and the relation between the parameters by (3.7).

We follow the same structure, find a perfect matching $\mathcal{M}$ (instead of a 2-perfect matching) contained in $\mathcal{H}$, and then we prove Claim 3.7. This proof is almost identical to the proof of Claim 4.3 in [26] (using the numeration of [26]). We The only difference is that we can bound $|B|$ with $2\varepsilon n$ instead of $3\varepsilon n$, but that makes no difference in this proof. Claim 3.7 proves the existence of a monochromatic cycle cover for *bad* vertices, these are vertices which do not have typical degree in each regular pair that corresponds to an edge of $\mathcal{M}$. Using the fact that bad vertices are few per cluster ($|B \cap V_i| \leq \varepsilon |V_i|$) we obtain that the number of bad vertices is at most $2\varepsilon n$. Then, using Proposition 2.12 and Lemma 2.11 we can conclude the result.

Next is Claim 3.8, which is essentially a rearrangement of the proof of Claim 4.4 in [26] (using the numeration of [26]). There is no notable difference other than the presentation of the proof. We perform an algorithm over the monochromatic components of the graph $\mathcal{H}$ which is performed on every monochromatic component ϕ of $\mathcal{H}$. For each edge $x_i x_j$ of ϕ, choose a "representative" edge $uv \in E(G)$ such that u and v are typical vertices in the regular pair $(V(x_i), V(x_j))$, and the colour of uv is the same as the colour of $x_i x_j$. Then, use Lemma 2.10 to extend this edge to a long monochromatic path. This result gives us an structure of vertex-disjoint monochromatic cycles that cover most of the good vertices of the graph G.

Claim 3.9 is an original addition to this thesis. We present this proof as a tool to simplify the proof of the following result, Claim 3.10. We rely on the large minimum degree and edge spanning condition of $\mathcal{H}$, which is a $(6\mu, \nu - 5\mu)$-birobustly matchable graph, to verify that $\mathcal{H}$ cannot have 3 components. Finally, we verify that $\mathcal{H}$ cannot have two components, because there would be edges between the two of them, due to the edge spanning condition of $\mathcal{H}$, which concludes the proof.

The proof of Claim 3.10 starts exactly as its counterpart, Claim 4.5 in [26] (using the numeration of [26]), but take a closer look at the details of the proof. We verify the three conditions to use Lemma 3.4 with our function b. The proof is similar to the one presented in [26] with the difference of using Claim 3.9 for the third condition. The other two conditions are verified in more detail, but not in a different way from [26]. That is, bounding the function b using the different hypotheses of Claim 3.10 and doing some calculations.

Next, we define ω, which will help us to choose a length for the paths will find in G between regular pairs in $\mathcal{H}$. The main difference is that, as our matching is a perfect matching and not a perfect 2-matching, $\deg_{\mathcal{M}}(x_i) = 1$ for every $x_i \in \mathcal{H}$. So, we redefine ω to be consistent with this fact.

Finally, having proved all the claims up to Claim 3.11, we can apply the results directly. There is no difference with the proof of the analogue result, Claim 4.6 in [26] (using the numeration from [26]).

Section 3.3 is an analogue of Section 5 in [26]. In Section 3.3, we compare Theorem 3.12 with Theorem 2.13. The proof of Theorem 2.13 starts assuming that G, the graph we want to partition, is not a $(\mu, 20\mu)$-robustly 2-matchable graph of type 1. Due to the minimum degree of G, there exists a set of size $(1/2 - 20\mu)n$ that spans fewer than $20\mu n^2$ edges. From there, G can be partitioned into two sets that can be turned into a $(\mu, 20\mu)$-robustly 2-matchable graph of type 2 by removing few cycles, which are constructed inductively. Our Theorem is simpler (in a technical sense) because it only involves some computations to verify that a balanced bipartite graph of sufficient minimum degree is a birobustly matchable graph. We do not need to remove cycles beforehand, or proving that there exists a specific partition of our graph with certain properties, as is done in the proof of Theorem 2.13 in [26].

Finally, in the last section of Chapter 3, we combine Theorem 3.12 and Theorem 3.5 to provide the details and conclude the proof of Theorem 1.3. This is not done in [26].

Chapter 3

Proof of the principal theorem

In this chapter we prove Theorem 1.3. For this purpose, the first section is devoted to the introduction of a family of graphs called *birobustly matchable graphs* and to the study some useful properties about those graphs. In the second section we prove that we can find a monochromatic cycle partition on these graphs under certain conditions. After that, in the third section we show that every balanced bipartite graph with enough minimum degree is a birobustly matchable graph. Finally, we join the main results obtained through the chapter and we prove Theorem 1.3.

3.1. Birobustly matchable graphs

In order to prove Theorem 1.3 we will define what is a *birobustly matchable graph*. Afterwards, we prove some useful lemmas which will lead us to the proof of the principal result.

Definition 3.1 *We call a balanced bipartite graph H with bipartition $\{X, Y\}$ on n vertices a (μ, ν)-birobustly matchable graph if the next two conditions hold.*

1. $\delta(H) \geq (1/4 - \mu/2)n$, *and*
2. *every set $Z \subseteq X \cup Y$ of at least $(1/2 - \nu)n$ vertices with $|Z \cap X| \geq \sqrt{\nu}n$ and $|Z \cap Y| \geq \sqrt{\nu}n$ spans at least νn^2 edges.*

An example of a (μ, ν)-birobustly matchable graph can be constructed as follows. Consider $\mu \leq \nu$, a balanced bipartite graph G with bipartition $\{X, Y\}$ and a set $Z \subseteq V(G)$ such that $|Z \cap X| = (1/4 - \nu)n$ and $|Z \cap Y| = n/4$. Now, add to $E(G)$ every edge from $Z \cap X$ to $Y \setminus (Z \cap Y)$, every edge from $Z \cap Y$ to $X \setminus (Z \cap X)$ and every edge from $Y \setminus (Z \cap Y)$ to $X \setminus (Z \cap X)$. Finally, add any νn^2 edges between $Z \cap X$ and $Z \cap Y$.
Now, the graph G satisfies condition 1 of Definition 3.1 by construction.

- $d(v) = n/2$ for every $v \in (X \setminus (Z \cap X)) \cup (Y \setminus (Z \cap Y))$,
- $d(v) = n/4 + \nu n$ for every $v \in Z \cap Y$, and
- $d(v) = n/4$ for every $v \in Z \cap X$.

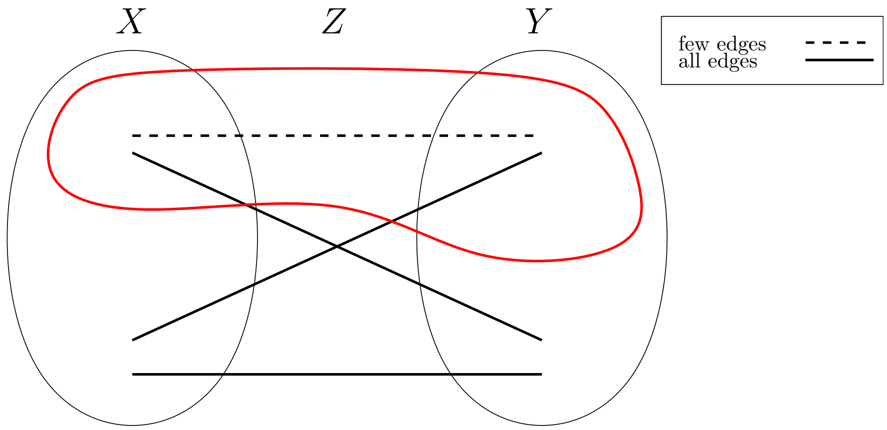

Figure 3.1: Diagram of the example of a birobustly matchable graph.

Condition 2 of Definition 3.1 is also satisfied. Note that, by construction, Z spans νn^2 edges. Adding vertices only adds edges, so that is not a problem. Changing the set Z would exchange its vertices with the ones in $(X \setminus (Z \cap X)) \cup (Y \setminus (Z \cap Y))$ which are connected to every vertex in their counterpart. This implies that the number of edges only increases when compared to $E[Z]$. In any case, any set of at least $(1/2 - \nu)n$ vertices span at least νn^2 edges.

The next lemma is our modified version of Lemma 2.5. Some obvious differences are that now we are verifying the existence of a perfect matching (instead of a perfect 2-matching) on a birobustly matchable graph (instead of a robustly 2-matchable graph). More importantly, the main difference is how we verify conditions in order to use Theorem 2.1. We take advantage of the large degree of the birobustly matchable graphs as well as the edge spanning condition.

Lemma 3.2 *Every (μ, ν)-birobustly matchable graph H with $\mu \leq \nu < 1/1000$ contains a perfect matching.*

Proof. Let $H = \{X, Y\}$ satisfy the hypotheses of the lemma. By Theorem 2.1 it suffices to prove that for all $S \subseteq X$ we have that $|S| \leq |N(S)|$. Let us assume this assumption is false, i.e., there exists $S \subseteq X$ such that $|S| > |N(S)|$.
Note that

$$|S| > |N(S)| = |Y| - |Y \setminus N(S)| = n/2 - |Y \setminus N(S)|$$

Since $|S| \leq |X| = n/2$, it follows that $|Y \setminus N(S)| > 0$.

Note that no vertex $v \in Y \setminus N(S)$ has any neighbour in S. Also as H is (μ, ν)-birobustly matchable for every $v \in H$, $\deg(v) \geq (1/4 - \mu/2)n$. So, $(1/4 + \mu/2)n \geq |S| > |N(S)|$, and therefore,

$$|Y \setminus N(S)| > n/2 - (1/4 + \mu/2)n = (1/4 - \mu/2)n \geq \sqrt{\nu} n. \tag{3.1}$$

Thus, since H is (μ, ν)-birobustly matchable, there are no edges between S and $Y \setminus N(S)$ and

$$|S| > |N(S)| \geq \delta(H) \geq (1/4 - \mu/2)n > \sqrt{\nu} n \tag{3.2}$$

it follows that $|S \cup (Y \setminus N(S))| < (1/2 - \nu)n$, but this is a contradiction with (3.1) and (3.2). $\square$

The next lemma is our version of Lemma 2.7. Aside from proving the lemma for birobustly matchable graphs instead of proving it for robustly 2-matchable graphs this lemma is proved following the same structure as Lemma 2.7.

Lemma 3.3 *Suppose H is an r-edge-coloured (μ, ν)-birobustly matchable graph on n vertices. Let $\mathcal{H}$ be the (ε, d)-reduced graph of H obtained from Lemma 2.2 with parameters ε and $d > 0$. Then $\mathcal{H}$ is $(\mu + 2rd + 2\varepsilon, \nu - rd - \varepsilon)$-birobustly matchable.*

Proof. Let us assume that H is a (μ, ν)-birobustly matchable graph with bipartition $\{X, Y\}$. Suppose that $\mathcal{H}$ has m vertices and a bipartition $\{\mathcal{X}, \mathcal{Y}\}$. Proposition 2.3*a*) guarantees that $\delta(\mathcal{H}) \geq (1/4 - \mu/2 - rd - \varepsilon)m$, and Proposition 2.3*c*) implies that every set $Z \subseteq \mathcal{H}$ such that $|Z| \geq (1/2 - \nu)m$, $|Z \cap \mathcal{X}| \geq \sqrt{\nu}m$ and $|Z \cap \mathcal{Y}| \geq \sqrt{\nu}m$ induces at least $(\nu - rd - \varepsilon)m^2$ edges. Therefore $\mathcal{H}$ is $(\mu + 2rd + 2\varepsilon, \nu - rd - \varepsilon)$-birobustly matchable.

□

Lemma 3.4 is our version of Lemma 2.8. Aside from changing the studied graph to a (μ, ν)-birobuslty matchable graph we change some details of the proof.

- Here we transform each vertex $x \in X$ into a set $W(x)$ of size $b_1(x)$ (instead of $b_1(x)/2$).
- We use the structure of the reduced graph in a different way to achieve the condition needed to use Theorem 2.1.
- We add the third condition here to balance out the function b between both sides.

The main objective of the lemma remains the same. Find a perfect b-matching under certain conditions.

Lemma 3.4 *Let t, γ be constants, and let H be a (μ, ν)-birobustly matchable graph on m vertices with bipartition $\{X, Y\}$ such that $m/t \leq \gamma \leq \mu \leq \nu/4 < 1/4000$. Then H has a perfect b-matching for every function $b : V(H) \to \mathbb{N}$ that satisfies*

a) $(1 - \gamma)t \leq b(x) \leq t$ *for every* $x \in V(H)$,

b) $\sum_{x \in V(\Psi)} b(x)$ *is even for every component* Ψ *of* H, *and*

c) $\sum_{x \in X} b(x) = \sum_{y \in Y} b(y)$.

Proof. As, by *b*), $\sum_{x \in V(\Psi)} b(x)$ is even for every component Ψ, we can associate each vertex with odd $b(x)$ with another vertex which also has odd $b(x)$ in the same component. Consider a family $\mathcal{P}$ that contains one path in H between each such pair of vertices.
Note that

$$|\mathcal{P}| \leq m. \tag{3.3}$$

Let $\omega_0 : E(H) \to \mathbb{N}$ be the function for which $\omega_0(e)$ is the number of paths in $\mathcal{P}$ containing e. Then, $b_0(x) = \sum_{y \in N(x)} \omega_0(xy)$ is odd if and only if $b(x)$ is odd. Let us elaborate this a bit further.

We have two options for any path $P \in \mathcal{P}$ such that $x \in P$. Either it ends at x or it does not. If it ends at x, then P adds 1 to $b_0(x)$ and if it does not end at x then P adds 2 to $b_0(x)$. Therefore, $b_0(x)$ is odd if and only if there exists a path $P \in \mathcal{P}$ such that x is an endpoint of P. Finally, by our choice of $\mathcal{P}$, this occurs if and only if $b(x)$ is odd.

So $b_1(x) = b(x) - b_0(x)$ is even for every $x \in V(H)$. Thus, using *a*) and the fact that $m \leq \gamma t$ by assumption, it follows that for every vertex x,

$$(1-2\gamma)t \leq (1-\gamma)t - m \leq b_1(x) \leq t, \tag{3.4}$$

where the second inequality comes from the fact that $b_0(x) \leq m$ as $b_0(x)$ counts the number of paths between vertices with odd b that pass through x. This number is at most m because of (3.3).

Also note that

$$\sum_{x\in X} b_1(x) = \sum_{x\in X} b(x) - \sum_{e\in E(H)} \omega_0(e) = \sum_{y\in Y} b(y) - \sum_{e\in E(H)} \omega_0(e) = \sum_{y\in Y} b_1(y),$$

where the second equality is obtained using c) and the first equality arises from

$$\sum_{x\in X} b_1(x) = \sum_{x\in X} b(x) - \sum_{x\in X}\sum_{y\in N(x)} \omega_0(xy) = \sum_{x\in X} b(x) - \sum_{e\in E(H)} \omega_0(e).$$

Let H' denote the graph obtained from H by replacing each vertex $x \in V(H)$ by a set $W(x)$ of size $b_1(x)$ and replacing each edge $xy \in E(H)$ by a complete bipartite graph with bipartition $\{W(x), W(y)\}$. We claim that

$$H' \text{ has a perfect matching } \omega'. \tag{3.5}$$

Note that assuming (3.5), we can obtain a b-matching in H. Indeed, let $\omega_1 : E(H) \to \mathbb{N}$ be a function such that $\omega_1(xy) := \sum_{x'\in W(x), y'\in W(y)} \omega'(x'y')$. Then ω_1 is a perfect b_1-matching in H, and defining $\omega(xy) := \omega_0(xy) + \omega_1(xy)$ we get a perfect b-matching in H, which finishes the proof.

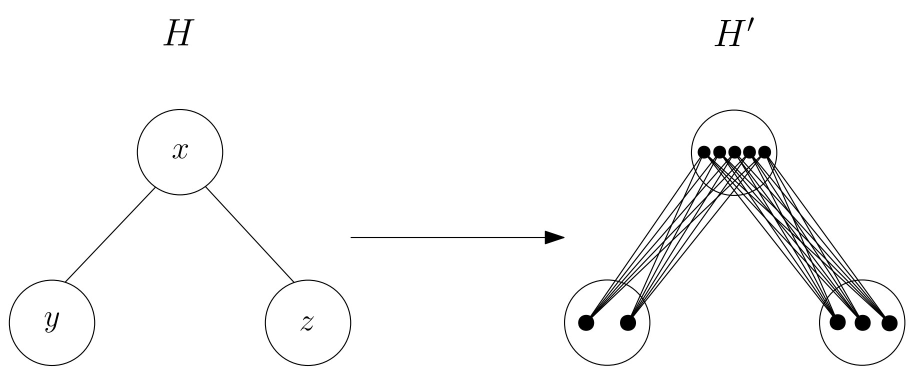

Figure 3.2: Example of H' with $V(H) = \{x, y, z\}$, $E(H) = \{(x, y), (x, z)\}$ and $b_1(x) = 5$, $b_1(y) = 2$, $b_1(z) = 3$

It remains to prove (3.5). For this let us consider that $|H'| = n$ where $n := \sum_{x\in V(H)} b_1(x)$, and by a),

$$(1-2\gamma)tm \leq n \leq tm. \tag{3.6}$$

We will show that H' has a perfect matching ω'. Since H is bipartite with bipartition $\{X, Y\}$, H' is also bipartite with bipartition $\{X', Y'\}$ such that $X' := \{v \in W(x) : x \in X\}$ and $Y' := \{v \in W(y) : y \in Y\}$. Note that $|X'| = |Y'| = n/2$ since $\sum_{x\in X} b_1(x) = \sum_{y\in Y} b_1(y)$.

As H is a (μ, ν)-birobustly matchable graph of type 1, $\delta(H) \geq (1/4 - \mu/2)m$, and by (3.6) we readily get

$$\begin{aligned}\delta(H') &\geq \delta(H) \cdot \min_{x \in V(H)} b_1(x) \\ &\geq (1/4 - \mu/2)m \cdot (1-2\gamma)t \\ &\geq (1/4 - \gamma/2 - \mu/2 + \gamma\mu)n \\ &\geq (1/4 - \gamma/2 - \mu/2)n.\end{aligned}$$

Let us consider a set $S' \subseteq X'$. We are using Theorem 2.1 to cover X' by vertex-disjoint edges in H'.
If $|S'| > |N(S')|$, we have

$$\begin{aligned}|S'| + |Y' \setminus N(S')| &= |S'| + |Y'| - |N(S')| \\ &= |Y'| + (|S'| - |N(S')|) \\ &> |Y'| = n/2.\end{aligned}$$

Note that as $|S' \cup (Y' \setminus N(S'))| > n/2$, $(S' \cup (Y' \setminus N(S'))) \cap X' \neq \emptyset$ and $(S' \cup (Y' \setminus N(S'))) \cap Y' \neq \emptyset$. Also note that, using that $b_1(x) \leq t$, $\forall x \in V(H)$, these vertices come from a set $Z \subseteq V(H)$ that has more than $\dfrac{n/2}{t}$ vertices (in particular, $|Z \cap X| \geq \sqrt{\nu}n$ and $|Z \cap Y| \geq \sqrt{\nu}n$ by an argument similar to the one presented in Lemma 3.2). Moreover, this quantity is bounded

$$\begin{aligned}n/(2t) &\geq (1-2\gamma)m/2 \\ &= (1/2 - \gamma)m \\ &\geq (1/2 - \nu)m.\end{aligned}$$

The first inequality arises as we stated earlier that $n \geq (1-2\gamma)mt$ and the last one derives from using the fact that $\gamma \leq \nu/4$. But H is a (μ, ν)-birobustly matchable graph. Therefore, any set of at least $(1/2 - \nu)m$ vertices spans at least νm^2 edges.
Thus, there exist edges in $H[Z]$ and that implies the existence of edges between S' and $Y' \setminus N(S')$, which is a contradiction. Proving that $|S'| \leq |N(S')|$ for any independent set $S' \subseteq X'$.
Then, using Theorem 2.1, we conclude that there exists a matching ω' which covers X'. As $|X'| = |Y'| = n/2$, ω' is a perfect matching of H'. By what we mentioned earlier, this concludes the proof.

□

3.2. Monochromatic cycle partition of birobustly matchable graphs

Since we are interested in finding monochromatic cycle partitions, this section is dedicated to showing that we are able to find such partitions in birobustly matchable graphs.

We define the conditions that our parameters must fulfil in order to satisfy the hypotheses of the following demonstrations. This will allow us to focus on the demonstration itself, rather than on verifying that the hypotheses are satisfied during each proof.

$$\left.\begin{aligned}
&\nu < \frac{1}{1000},\\
&\mu < \min\left\{\frac{1}{700000}, \frac{\nu}{20}\right\},\\
&d = \frac{\mu}{r},\\
&\varepsilon < \min\left\{\frac{1}{10^{13}r^6}, \frac{\mu^4}{20^4}, \frac{d^2}{4000}, \varepsilon_{2.9}(d), \varepsilon_{2.10}(d)\right\},\\
&n > \max\{\tfrac{4}{\varepsilon}(M_{2.2}(\varepsilon, r, 2))^4, n_{2.9}(\varepsilon), n_{2.10}(\varepsilon)\},
\end{aligned}\right\} \tag{3.7}$$

where $\varepsilon_{2.9}$ and $n_{2.9}$ are the constants ε and n_0 obtained by using Lemma 2.9 with $d = \mu/r \in (0,1)$, $\varepsilon_{2.10}$ and $n_{2.10}$ are the constants ε and n_0 obtained by using Lemma 2.10 with $d = \mu/r \in (0,1)$, and $M_{2.2}$ is the constant M obtained by using Lemma 2.2 with ε defined as above, r as the number of colours, and $\ell = 2$ (because our graph is bipartite).

The proof of the following theorem is very close to the one presented in [26] for Theorem 2.14. Nevertheless, many of the technicalities, bounds and arguments have been subtly modified to fit the framework established in this thesis. For instance, in Claim 3.6 we prove that the graph $\mathcal{H}$ is $(6\mu, \nu - 5\mu)$-birobustly matchable instead of $(4\mu, \nu - 3\mu)$-robustly 2-matchable. In particular, this means that we have to verify that new conditions are met, which are not studied in [26]. In Claim 3.7 we bound the number of bad vertices by $2\varepsilon n$ instead of $3\varepsilon n$, even though this does not change the final bound of the Claim. We organize the proof of Claim 3.8 in way that seems more natural which is present the steps of an algorithm, then verify that the steps can be performed, and finally we conclude that the algorithm outputs the object we are looking for. We also introduce Claim 3.9 as a tool to prove Claim 3.10.

Theorem 3.5 *For every $0 < \nu < 1/1000$ and $r \in \mathbb{N}$ there exist $\mu > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, every r-edge-coloured (μ, ν)-birobustly matchable graph on n vertices can be partitioned into $(1/\mu + 100)r^2$ monochromatic cycles.*

Proof. Let G be an r-edge-coloured (μ, ν)-birobustly matchable graph with bipartition $\{X, Y\}$. We apply Lemma 2.2 with ε, r and $\ell = 2$ to obtain a partition $V_0, V_1, V_2, \ldots, V_m$ of $V(G)$ as detailed in Lemma 2.2. Let $\mathcal{G}$ be the corresponding (ε, d)-reduced graph. As G is balanced bipartite, $\mathcal{G}$ is also balanced bipartite, and we denote its bipartition by $\{\mathcal{X}, \mathcal{Y}\}$. Note that $\mathcal{G}$ has $m \leq M_{2.2}(\varepsilon, r, 2)$ vertices.

By Lemma 3.3, and as $\varepsilon \leq \mu/2$ and $d \leq \mu/r$, by (3.7), we know that $\mathcal{G}$ is $(4\mu, \nu - 3\mu)$-birobustly matchable. Let $\mathcal{H}$ denote the subgraph of $\mathcal{G}$ that consists of all edges contained in monochromatic components of order at least $m\mu/r$. Then

$$\mathcal{H} \text{ is the union of at most } r^2/\mu \text{ monochromatic components} \tag{3.8}$$

and

$$\deg_{\mathcal{H}}(x) \geq \deg_{\mathcal{G}}(x) - \mu m \text{ for every vertex } x \text{ in } \mathcal{G}. \tag{3.9}$$

Claim 3.6 *$\mathcal{H}$ is $(6\mu, \nu - 5\mu)$-birobustly matchable.*

As $\mathcal{G}$ is a $(4\mu, \nu - 3\mu)$-birobustly matchable graph, then

$$\delta(\mathcal{H}) \geq \delta(\mathcal{G}) - \mu m \geq (1/4 - (4\mu + 2\mu)/2)m = (1/4 - 6\mu/2)m, \tag{3.10}$$

as needed.
Consider any set $Z \subseteq V(\mathcal{H})$ such that

$$|Z \cap \mathcal{Y}| \geq |Z \cap \mathcal{X}| \geq (\sqrt{\nu - 5\mu})m \text{ and } |Z| = (1/2 - (\nu - 5\mu))m. \tag{3.11}$$

(The case where $|Z \cap \mathcal{X}| \geq |Z \cap \mathcal{Y}|$ is analogous.)

We separate two cases.
First, if $|Z \cap \mathcal{X}| \geq (\sqrt{\nu - 3\mu})m$, we can use the fact that $\mathcal{G}$ is a $(4\mu, \nu - 3\mu)$-birobustly matchable graph so $\mathcal{H}[Z]$ spans at least $(\nu - 3\mu)m^2$ edges in $\mathcal{G}$ by Definition 3.1. But,

$$|E(\mathcal{H})| = \sum_{v \in \mathcal{H}} \deg_{\mathcal{H}}(v)/2 \geq \sum_{v \in \mathcal{G}} (\deg_{\mathcal{G}}(v) - \mu m)/2 = |E(\mathcal{G})| - \mu m^2/2$$

where the first inequality holds because $\mathcal{H}$ is spanning. So, $\mathcal{H}[Z]$ spans at least $(\nu - 5\mu)m^2$ edges.
The other case is when $(\sqrt{\nu - 5\mu})m \leq |Z \cap \mathcal{X}| < (\sqrt{\nu - 3\mu})m$. In this case, each vertex $v \in Z \cap \mathcal{X}$ is connected (in $\mathcal{H}$) to at least $\deg_{\mathcal{H}}(v) - |\mathcal{Y} \setminus Z|$ vertices of $Z \cap \mathcal{Y}$.

Aditionally, we can bound $|\mathcal{Y} \setminus Z|$ with the information we have. First, note that

$$((1/2 - (\nu - 5\mu)) - \sqrt{\nu - 3\mu})m \leq |Z \cap \mathcal{Y}| \leq ((1/2 - (\nu - 5\mu)) - \sqrt{\nu - 5\mu})m \tag{3.12}$$

because $(1/2 - (\nu - 5\mu))m = |Z| = |Z \cap \mathcal{Y}| + |Z \cap \mathcal{X}|$. Therefore,

$$((\nu - 5\mu) + \sqrt{\nu - 5\mu})m \leq |\mathcal{Y} \setminus Z| \leq ((\nu - 5\mu) + \sqrt{\nu - 3\mu})m. \tag{3.13}$$

Bounding the number of edges we obtain

$$\begin{aligned}
|E(Z \cap \mathcal{X}, Z \cap \mathcal{Y})| &\geq (\delta(\mathcal{H}) - |\mathcal{Y} \setminus Z|) \cdot |Z \cap \mathcal{X}| \\
&\geq ((1/4 - 6\mu/2) - (\nu - 5\mu) - \sqrt{\nu - 3\mu})m \cdot (\sqrt{\nu - 5\mu})m \\
&\geq (1/4 + 2\mu - \nu - \sqrt{\nu}) \cdot (\sqrt{\nu - 5\mu})m^2 \\
&\geq (1/4 - 2\sqrt{\nu}) \cdot (\sqrt{\nu - 5\mu})m^2 \\
&\geq (\nu - 5\mu)m^2
\end{aligned}$$

where the second inequality follows from (3.10) and (3.13) and the fourth inequality holds because $\nu < \sqrt{\nu}$ and the last one follows from (3.7). As Z is an arbitrary set we conclude that $\mathcal{H}$ is a $(6\mu, \nu - 5\mu)$-birobustly matchable graph. With this the proof of Claim 3.6 is finished.

As $20\mu \leq \nu \leq 1/1000$, by (3.7), Lemma 3.2 implies that $\mathcal{H}$ contains a perfect matching denoted by $\mathcal{M}$.

Let us call a vertex $v \in V_i$ ($i \in [m]$) *good* if v has typical degree in the regular pair (V_i, V_j) that corresponds to the respective edge of $\mathcal{M}$. In other words, if $x_i x_j \in \mathcal{M}$ is of colour c then v is *good* if $\deg_c(v, V_j) \geq (d - \varepsilon)|V_j|$. We call all other vertices of G *bad*.

Claim 3.7 *There is a collection $\mathcal{C}^b$ of at most $100r^2$ vertex-disjoint monochromatic proper cycles and edges in G covering all bad vertices such that*

$$|V_i \cap V(\mathcal{C}^b)| \leq 5\sqrt{\varepsilon}|V_i| \qquad \text{for every } i \in [m]. \tag{3.14}$$

Now, we prove Claim 3.7. Let B be the set of bad vertices (note that $V_0 \subseteq B$). By the definition of typical degree, and because $\mathcal{M}$ is a perfect matching, we know that $|B \cap V_i| \leq \varepsilon|V_i|$ for every $i \in [m]$. In particular,

$$|B| \leq \varepsilon|V_i| \cdot m + |V_0| \leq \varepsilon|V_i| \cdot m + \varepsilon n \leq 2\varepsilon n. \tag{3.15}$$

This together with (3.7) means that we can apply Proposition 2.12 with $p = 2\sqrt{\varepsilon}$ to obtain a set A of size $|A| \geq \sqrt{\varepsilon}n \geq 2 \cdot 100^3 r^3 \varepsilon n \geq 100^3 r^3 |B|$ such that $|A \cap V_i| \leq 4\sqrt{\varepsilon}|V_i|$ for every $i \in [m]$, where the second inequality is a consequence of (3.7), and each vertex $v \in G$ with $\deg_G(v, V \setminus B) > n/40$ has at least $|A|/100$ neighbours in A. As $\delta(G) \geq (1/4 - \mu/2)n$ and $|B| \leq 2\varepsilon n$, this actually holds for every vertex of G, and in particular for every vertex in B. But then Lemma 2.11 provides a set $\mathcal{C}^b$ of at most $100r^2$ disjoint monochromatic proper cycles and edges covering B. Note that the vertices of $\mathcal{C}^b$ are contained in $A \cup B$, meaning that

$$|V_i \cap V(\mathcal{C}^b)| \leq |V_i \cap A| + |V_i \cap B| \leq 4\sqrt{\varepsilon}|V_i| + \varepsilon|V_i| \leq 5\sqrt{\varepsilon}|V_i| \tag{3.16}$$

so (3.14) clearly holds. This proves Claim 3.7.

Claim 3.8 *There is a collection $\mathcal{C}^{\mathcal{H}}$ of at most r^2/μ vertex-disjoint monochromatic proper cycles and edges in G, all disjoint from $\mathcal{C}^b$, such that*

1. *for every edge $e = x_i x_j$ of $\mathcal{H}$, there is an edge $u_e v_e$ of colour $c(e)$ in $\mathcal{C}^{\mathcal{H}}$ between vertices $u_e \in V_i$ and $v_e \in V_j$ that have typical degree in the regular pair (V_i, V_j), and*
2. *$|V_i \cap V(\mathcal{C}^{\mathcal{H}})| \leq \varepsilon|V_i|$ for every $i \in [m]$.*

To prove Claim 3.8 let ϕ be a monochromatic component of $\mathcal{H}$ of colour c and let $e_1, e_2, \ldots, e_s \in E(\mathcal{H})$ be its edges. We will apply an algorithm over ϕ which consists of the following two steps:

1. For $i = 1, \ldots, s$ let $e_i = y_i z_i$, and pick $u_i \in V(y_i)$ and $v_i \in V(z_i)$ that are not yet used, but have typical degree in the regular pair $(V(y_i), V(z_i))$, and $u_i v_i$ is a c-coloured edge in G.
2. For $i = 1, \ldots, s$ use Lemma 2.10 to find a c-coloured $v_i - u_{i+1}$ path P_i in G of order at most $2m$ that avoids all previously used vertices (except v_i and u_{i+1}), where $u_{s+1} = u_1$.

Let us verify that the algorithm can perform its steps.

For step 1, note that as $(V(y_i), V(z_i))$ is a regular pair, each one of the sets have at least $(1 - \varepsilon)|V(y_i)|$ typical vertices, of which at most $\varepsilon|V(y_i)|$ have been used in former steps and at most $5\sqrt{\varepsilon}|V_i|$ are in $\mathcal{C}^b$ by (3.14). Then there is an edge between unused typical vertices in colour c because $\varepsilon < 1/100$, by (3.7), and $(V(y_i), V(z_i))$ is ε-regular.

For step 2, we apply Lemma 2.10 with the set W consisting of the vertices of $\mathcal{C}^b$ in $V(y_i) \cup V(z_i)$, as well as all previously used vertices except v_i and u_{i+1}. This is possible because, using that at most $\varepsilon|V(y_i)|$ vertices of $V(y_i)$ are used by the algorithm and (3.14),

$$|W \cap V_i| \leq |W| < 12\sqrt{\varepsilon}|V_i| \leq (d/2)|V_i| \text{ for every } i \in [m]. \tag{3.17}$$

The last inequality comes from (3.7).

As the algorithm works, we define $C_\phi = u_1v_1P_1u_2v_2P_2 \ldots u_sv_sP_su_1$. By construction, every edge u_iv_i satisfies condition 1 for e_i. This implies that the condition 1 is satisfied in ϕ with C_ϕ. Repeating this for every monochromatic component $\phi \in \mathcal{H}$ gives us at most $r^2\mu$ disjoint monochromatic cycles satisfying condition 1 for every edge of $E(\mathcal{H})$. Now as the edges and paths produced by these steps use at most $|E(\mathcal{H})| \cdot 2m \leq m^3 \leq \varepsilon|V_i|$ vertices in G, condition 2 is also satisfied. Thus, ending the proof of the Claim.

Note that $\mathcal{C}^b$ and $\mathcal{C}^{\mathcal{H}}$ together contain at most $(1/\mu + 100)r^2$ cycles.

In order to verify the necessary conditions to use Lemma 3.4 we prove the following claim. This is a small difference from the proof of Theorem 2.14. We first verify that there cannot be more than 2 components by taking advantage of the large degree of $\mathcal{H}$. Finally, we verify that there is only one component by using the edge spanning condition of birobustly matchable graphs.

Claim 3.9 *$\mathcal{H}$ (uncoloured) is a connected graph.*

To prove Claim 3.9 notice that as $\mathcal{H}$ is a $(6\mu, \nu - 5\mu)$ the next two conditions hold.

1. $\delta(\mathcal{H}) \geq (1/4 - 3\mu)m$, and
2. every set $Z \subseteq \mathcal{X} \cup \mathcal{Y}$ of at least $(1/2 - (\nu - 5\mu))m$ vertices with $|Z \cap \mathcal{X}| \geq \sqrt{(\nu - 5\mu)}m$ and $|Z \cap \mathcal{Y}| \geq \sqrt{(\nu - 5\mu)}m$ spans at least $(\nu - 5\mu)m^2$ edges.

First, let us verify that there can be at most two components. Let $a \in \mathcal{X}$ be a vertex. If $\mathcal{H}$ is disconnected and has more than two connected components, then there are vertices $b, c \in \mathcal{X}$ such that there is no path in $\mathcal{H}$ connecting any two of a, b and c. If that is the case, then

$$N_{\mathcal{H}}(a) \cap N_{\mathcal{H}}(b) \cap N_{\mathcal{H}}(c) = \emptyset. \tag{3.18}$$

Because of 1, that would mean that

$$|\mathcal{Y}| \geq |N_{\mathcal{H}}(a) \cup N_{\mathcal{H}}(b) \cup N_{\mathcal{H}}(c)| = |N_{\mathcal{H}}(a)| + |N_{\mathcal{H}}(b)| + |N_{\mathcal{H}}(c)| \geq (3/4 - 9\mu)m. \tag{3.19}$$

As $9\mu < m/4$, because of (3.7), this is a contradiction because $|\mathcal{Y}| = m/2$.

Now, let us verify that $\mathcal{H}$ cannot have 2 components. Assume the opposite. That is, exists $\mathcal{A}, \mathcal{B} \subseteq V(\mathcal{H})$ such that $\mathcal{A}$ and $\mathcal{B}$ are components, partition $V(\mathcal{H})$ and $E[\mathcal{A}, \mathcal{B}] = \emptyset$. Note that it must hold either $|\mathcal{A} \cap \mathcal{X}| \geq |\mathcal{A} \cap \mathcal{Y}|$ or $|\mathcal{A} \cap \mathcal{X}| \leq |\mathcal{A} \cap \mathcal{Y}|$. Let us assume the first one holds (the procedure is analogous in the other case).
If $|\mathcal{A} \cap \mathcal{X}| \geq |\mathcal{A} \cap \mathcal{Y}|$ then $|\mathcal{B} \cap \mathcal{X}| \leq |\mathcal{B} \cap \mathcal{Y}|$, because $|\mathcal{X}| = |\mathcal{Y}| = m/2$.
Notice that $|\mathcal{A} \cap \mathcal{X}| + |\mathcal{B} \cap \mathcal{Y}| > m/2 > (1/2 - (\nu - 5\mu))m$ which, by condition 2, implies that there are edges between the two components, which is a contradiction. Thus, $\mathcal{H}$ is a

connected graph, proving Claim 3.9.

Defining $\mathcal{C}^0 := \mathcal{C}^b \cup \mathcal{C}^{\mathcal{H}}$ we know that

$$|V_i \cap V(\mathcal{C}^0)| \leq |V_i \cap V(\mathcal{C}^b \cup \mathcal{C}^{\mathcal{H}})| \leq (5\sqrt{\varepsilon} + \varepsilon)|V_i| \leq 6\sqrt{\varepsilon}|V_i| \tag{3.20}$$

for every $i \in [m]$. The rest of the proof will extend the cycles in $\mathcal{C}^{\mathcal{H}}$ so that they cover all the remaining vertices with $\mathcal{C}^{\mathcal{H}}$ as the base structure of our cycle cover. That means we will use Lemma 2.9 to replace each edge $u_e v_e$ (corresponding to some $e = x_i x_j$ in $\mathcal{H}$) with a $u_e - v_e$ path P_e in (V_i, V_j). In order to do this, let us define ℓ such that

$$(1 - \varepsilon^{1/4})|V_i| \leq \ell \leq (1 - \varepsilon^{1/4})|V_i| + 1 \text{ and } \ell \text{ is an integer.} \tag{3.21}$$

This will be the length of our new $u_e - v_e$ paths. We intend to cover at least ℓ vertices in each cluster by the paths corresponding to the edges of the perfect matching $\mathcal{M}$. This leaves $b(x_i) = |V_i \setminus V(\mathcal{C}^0)| - \ell$ vertices in V_i. Note that

$$0 \leq \varepsilon^{1/4}|V_i| - |V_i \cap V(\mathcal{C}^0)| \leq b(x_i) \leq \varepsilon^{1/4}|V_i| \tag{3.22}$$

vertices in each V_i. The last inequality comes from (3.20) and (3.7).

Claim 3.10 *$\mathcal{H}$ contains a perfect b-matching $\omega_0 : E(\mathcal{H}) \to \mathbb{N}$*

We now prove Claim 3.10. Since, by (3.20) and (3.7), $|V_i \cap V(\mathcal{C}^0)| \leq 6\sqrt{\varepsilon}|V_i| \leq \mu\varepsilon^{1/4}|V_i|$, using (3.22) we have

$$(1 - \mu)\varepsilon^{1/4}|V_i| \leq \varepsilon^{1/4}|V_i| - |V_i \cap \mathcal{C}^0| \leq b(x_i) \leq \varepsilon^{1/4}|V_i| \tag{3.23}$$

Now, we use Lemma 3.4 setting

- $\gamma = \mu$, and
- $t = \varepsilon^{1/4}|V(x_i)|$ which is constant as a function of x_i.

In particular, we want to verify if

$$m/(\varepsilon^{1/4}|V(x_i)|) \leq \mu \leq 6\mu \leq (\nu - 5\mu)/4 < 1/4000. \tag{3.24}$$

The inequalities above hold because of (3.7).

We need to verify that b satisfies the conditions needed for Lemma 3.4.

1. By (3.23), $(1 - \gamma)t \leq b(x) \leq t$ for every $x \in V(H)$.
2. We can see the following

$$\begin{aligned} \sum_{x \in \mathcal{X}} b(x) &= \sum_{x \in \mathcal{X}} |V(x) \setminus V(\mathcal{C}^0)| - \ell \\ &= \sum_{y \in \mathcal{Y}} |V(y) \setminus V(\mathcal{C}^0)| - \ell \\ &= \sum_{y \in \mathcal{Y}} b(y). \end{aligned}$$

The second equality comes from the fact that $\mathcal{H}$ is a balanced bipartite graph.

3. As $\mathcal{H}$ is a connected balanced bipartite graph, it is clear that $\sum_{x\in V(\Psi)} b(x)$ is even for every component Ψ of H, as it is $\sum_{x\in V(\Psi)} b(x) = \sum_{x\in\mathcal{X}} b(x) + \sum_{y\in\mathcal{Y}} b(y) = 2\cdot\sum_{x\in\mathcal{X}} b(x)$.

Thus, $\mathcal{H}$ contains a perfect b-matching. This finishes the proof of Claim 3.10.

Let ω_0 be the perfect b-matching guaranteed by Claim 3.10. Define $\omega : E(\mathcal{H}) \to \mathbb{N}$ as

$$\omega(x_ix_j) = \begin{cases} \omega_0(x_ix_j) & \text{for } x_ix_j \notin \mathcal{M} \\ \omega_0(x_ix_j) + \ell & \text{for } x_ix_j \in \mathcal{M} \end{cases}$$

Note that for every edge $x_ix_j \in E(\mathcal{H})$, $\omega(x_ix_j)$ is integral because ω_0 is integral since it is the value assigned by the perfect b-matching and ℓ is integral because of how we defined it. Then for every vertex $x_i \in \mathcal{H}$, we have

$$\sum_{x_j\in N_{\mathcal{H}}(x_i)} \omega(x_ix_j) = |V_i \setminus V(\mathcal{C}^0)| \qquad \text{and} \qquad \sum_{x_j\in N_{\mathcal{H}\setminus\mathcal{M}}(x_i)} \omega(x_ix_j) \le b(x_i) \le \varepsilon^{1/4}|V_i|. \quad (3.25)$$

The proof of the next claim is exactly as its analogue from the proof of Theorem 2.14. As we have proved every necessary claim and lemma to reach this point, this last step can be applied exactly as in [26].

Claim 3.11 (Claim 4.6 of [26]) *For every edge $e = x_ix_j$ in $E(\mathcal{H})$, there is a $u_e - v_e$ path P_e of colour $c(e)$ in $G[V_i, V_j]$ that contains exactly $\omega(e)+1$ vertices in each of V_i and V_j. Moreover, these paths can be chosen so that they are internally vertex-disjoint from each other and from $\mathcal{C}^0$.*

In order to prove Claim 3.11 let us first apply Proposition 2.12 with $p = 2\sqrt{\varepsilon}$ and $B = V(\mathcal{C}^0)$ to get a set $A = A^1$ with the properties given in the statement of the proposition and then apply it again with the same p and $B = V(\mathcal{C}^0) \cup A^1$ to get another such set A^2. This is possible because $V_0 \subseteq V(\mathcal{C}^0) \subseteq V(\mathcal{C}^0) \cup A^1$ holds, and we also have $|V(\mathcal{C}^0) \cap V_i| \le 6\sqrt{\varepsilon}|V_i|$. Thus

$$|A^1 \cap V_i| \le 4\sqrt{\varepsilon}|V_i| \text{ and } |A^2 \cap V_i| \le 4\sqrt{\varepsilon}|V_i| \text{ for every } i \in [m] \quad (3.26)$$

as a consequence of Proposition 2.12. Let $A_i^b = A^b \cap V_i$ for every $i \in [m]$ and $b \in [2]$. Then

1. $|A_i^b| \le 4\sqrt{\varepsilon}|V_i|$ for every $i \in [m]$ and $b \in [2]$, and

2. for every edge x_ix_j in $\mathcal{H}$ of colour c and every vertex $v \in V_j$ with typical degree in the regular pair (V_i, V_j), we have $\deg_c(v, A_i^b) \ge 6\varepsilon|V_i|$ for $b \in [2]$.

Let us elaborate on 2. Note that every such vertex v of typical degree satisfies $\deg_c(v, V_i) \ge (d-\varepsilon)|V_i| > 60\sqrt{\varepsilon}|V_i|$ (using $d > 61\sqrt{\varepsilon}$, from (3.7)) so by Proposition 2.12

$$\deg_c(v, A_i^b) \ge \sqrt{\varepsilon}(d-\varepsilon)|V_i| > 60\varepsilon|V_i|. \quad (3.27)$$

This last inequality holds because of (3.7).

Now, consider $\mathcal{C}^{\mathcal{H}}$ as mentioned in Claim 3.8. Let $e_1, \dots, e_s$ be the edges of $\mathcal{H} \setminus \mathcal{M}$. We will find the $u_k - v_k$ paths P_k (where u_kv_k is the edge in $\mathcal{C}^{\mathcal{H}}$ corresponding to e_k) one by one so that

$$\text{for every } k, \text{ the vertex set of } \mathcal{P}_k := \bigcup_{j=1}^{k-1} P_j \text{ is disjoint from each } A_i^2, \text{ and} \tag{3.28}$$

$$\text{intersects each } A_i^1 \text{ in at most } k-1 \text{ vertices.} \tag{3.29}$$

We are going to use an inductive argument. Suppose we have already found $P_1, \ldots, P_{k-1}$. Let us also assume that $u_k \in V_i$ and $v_k \in V_j$ (so $e_k = x_i x_j$), and let c be the colour of e_k.

If $\omega(e_k) = 0$ we can take $P_k = u_k v_k$.

If $\omega(e_k) = 1$, then notice that $\deg_c(u_k, A_j^1 \setminus V(P_k)) \geq 4\varepsilon|V_j| - k \geq \varepsilon|V_j|$ (using $\varepsilon|V_i| > \varepsilon n/(2m) > m^2$, from (3.7)) and similarly, $\deg_c(v_k, A_i^1 \setminus V(\mathcal{P}_k)) \geq \varepsilon|V_i|$. Hence, as $A_j^1 \setminus V(P_k) \subseteq V_j$, $A_i^1 \setminus V(\mathcal{P}_k) \subseteq V_i$ and (V_i, V_j) is an ε-regular pair with density at least d in the colour c, by Lemma 2.2 we can find adjacent vertices $u \in A_i^1 \setminus V(\mathcal{P}_k)$ and $v \in A_j^1 \setminus V(\mathcal{P}_k)$ such that $\mathcal{P}_k = u_k v u v_k$ is a c-coloured path, as needed.

To verify the remaining cases, suppose $\omega(e_k) > 1$. Let $W = V(\mathcal{C}^0) \cup A^1 \cup A^2 \cup V(\mathcal{P}_k)$ be the set of "forbidden" vertices. We will need a pair of neighbours $u \in A_i^1 \setminus V(\mathcal{P}_k)$ and $v \in A_j^1 \setminus V(\mathcal{P}_k)$ of v_k and u_k respectively, but now we are going to apply Lemma 2.9 to connect them with a $u - v$ path of the right length that avoids W.

We have seen above that $\deg_c(u_k, A_j^1 \setminus V(\mathcal{P}_k)) \geq \varepsilon|V_j|$. Also,

$$|V_i \setminus W| \geq |V_i| - |V_i \cap V(\mathcal{C}^0)| - |A_i^1| - |A_i^2| - b(x_i) \geq (1 - 14\sqrt{\varepsilon} - \varepsilon^{1/4})|V_i| \geq |V_i|/2$$

by (3.7). Following the same argument shown in the previous case, by Lemma 2.2, there is a neighbour $v \in A_j^1 \setminus V(\mathcal{P}_k)$ of u_k such that $\deg_c(v, V_i \setminus W) \geq (d - \varepsilon)|V_i|/2 \geq \varepsilon|V_i|$. Where the last inequality follows from (3.7). Similarly, there is a neighbour $u \in A_i^1 \setminus V(\mathcal{P}_k)$ of v_k such that $\deg_c(u, V_j \setminus W) \geq \varepsilon|V_i|$. As $\omega(e_k) \leq \varepsilon^{1/4}|V_i| \leq (1 - \sqrt{\varepsilon})|V_i \setminus W|$, we can apply Lemma 2.9 (with $U_1 = V_i \setminus W$ and $U_2 = V_j \setminus W$) to find a c-coloured $v - u$ path P' of order $2\omega(e_k)$ that is internally vertex-disjoint from W. Therefore, $P_k = u_k v P' u v_k$ is a path satisfying our requirements.

Finally, let $e_{s+1}, \ldots, e_{s+t}$ be the edges of $\mathcal{M}$. Note that each vertex of $\mathcal{H}$ is incident to exactly one of these edges. Using the same notation as before, we will find the final $u_k - v_k$ paths P_k.

Fix k and let $U_i = V_i \setminus (V(\mathcal{C}^0) \cup V(\mathcal{P}_k))$. Using $|A_i^2| \leq \ell/2$ and the assumption that, by (3.7), ε is small enough, it is easy to verify from the definitions that we have $|U_i| \geq \omega(e_k) \geq \ell/2 \geq |V_i|/3$. We can also state that $|U_j| \geq \omega(e_k) \geq |V_j|/3$ for U_j defining it analogously to U_i.

Now we are going to find a $u_k - v_k$ path P_k of order $2(\omega(e_k)+1)$. As $\min\{|U_i \cup \{u_k\}|, |U_j \cup \{v_k\}|\} \geq \omega(e_k) + 1$, we just need to verify that $\delta(G[U_i, U_j]) \geq 5\varepsilon|V_i|$ to use Lemma 2.9.
As e_k is the last edge at x_i, $A_i^2 \subseteq U_i$. Then, by condition 2, we obtain $\deg_c(v, U_i) \geq 6\varepsilon|V_i| - k \geq 5\varepsilon|V_i|$ for every $v \in U_j$, and similarly, $\deg_c(u, U_j) \geq 5\varepsilon|V_j|$ for every $u \in U_i$, as needed. This concludes the proof of Claim 3.11.

Now replace every edge $u_e v_e$ with the path obtained in Claim 3.11 in the appropriate cycle of $\mathcal{C}^{\mathcal{H}}$. This gives us $(1/\mu + 100)r^2$ monochromatic cycles that cover all vertices in V_0, and $|V_i|$ vertices in each V_i. This last fact is given by how ω is defined. Therefore, we find a monochromatic cycle partition of G, as needed. □

3.3. Turning balanced bipartite graphs into birobustly matchable graphs

As the family of birobustly matchable graphs does not obviously seem to be big, we connect the family of birobustly matchable graphs with the family of balanced bipartite graphs through the next theorem.

Even though Theorem 3.12 is much simpler, it has an equivalent role in the proof of Theorem 1.3 as Theorem 2.13 has in the proof of Theorem 1.1 but without having to consider any additional cycles. The simplicity of Theorem 3.12 lies in the fact that we verify the two conditions of Definition 3.1 by relying on the large minimum degree from the hypothesis, rather than having to construct the robustly 2-matchable graph through a more technical prodecure.

Theorem 3.12 *For any $\mu > 0$, every balanced bipartite graph G with minimum degree $\delta(G) \geq n/4 + 11\nu n/2$ is a (μ, ν)-birobustly matchable graph.*

Proof. Let G be a balanced bipartite graph such that $\delta(G) \geq n/4 + 11\nu n/2$.
As the minimum degree condition is satisfied for G to be a (μ, ν)-birobustly matchable graph (regardless of the chosen μ), we are going to focus on the edge spanning condition.
Let us consider a set W and a parameter ω such that

$$|W| = (1/2 - \nu)n \text{ and } |W \cap B| \geq |W \cap A| = \omega \geq \sqrt{\nu} n. \tag{3.30}$$

This also means that

$$|W \cap B| = (1/2 - \nu)n - \omega. \tag{3.31}$$

As $|W \cap B| \geq |W \cap A|$ it holds that

$$\sqrt{\nu} n \leq \omega \leq (1/4 - \nu/2)n. \tag{3.32}$$

Now, note that for every vertex $v \in A \cap W$

$$|N_{B \cap W}(v)| \geq \delta(G) - |B \setminus W| \geq (1/4 + 11\nu/2)n - (\nu n + \omega) = (1/4 + 9\nu/2)n - \omega \tag{3.33}$$

where the second inequality comes from (3.31). Now, we can bound the number of edges induced by W.
Let $v \in W \cap A$, then

$$\begin{aligned} |E[W]| &\geq |W \cap A| \cdot |N_{B \cap W}(v)| \\ &\geq \omega \cdot ((1/4 + 9\nu/2)n - \omega). \end{aligned}$$

We define $f(\omega)$ as the quadratic equation of ω on the previous line. By (3.32) the minimum value of f is given by

$$\min(f) = \min\{f(\sqrt{\nu}n), f((1/4 - \nu/2)n)\}.$$

We study these two cases separately.

1. $\min(f) = f(\sqrt{\nu}n)$.

   It suffices to prove that

   $$(\sqrt{\nu}n) \cdot ((1/4 + 9\nu/2)n - \sqrt{\nu}n) \geq \nu n^2$$

   which is equivalent to prove that

   $$(1/4 + 9\nu/2) \geq 2\sqrt{\nu}$$

   and this holds by (3.7).

2. $\min(f) = f((1/4 - \nu/2)n)$.

   As before, it suffices to prove that

   $$((1/4 - \nu/2)n) \cdot ((1/4 + 9\nu/2)n - (1/4 - \nu/2)n) \geq \nu n^2$$

   but this is equivalent to

   $$5(1/4 - \nu/2) \geq 1$$

   which, as before, holds by (3.7).

These cases prove that $|E[W]| \geq \nu n^2$. Therefore, G is (μ, ν)-birobustly matchable. □

3.4. Proof of Theorem 1.3

Finally, having all the necessary tools to complete the proof of Theorem 1.3 we present it.

Proof. Let $r \geq 2$ and $\eta > 0$. Let G be an r-edge-coloured balanced bipartite graph on n vertices such that $\delta(G) \geq (1/4 + \eta)n$. By Theorem 3.5, choosing $\nu = \min\{1/1001, 2\eta/11\}$ we know that there exist $\mu \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, every r-edge-coloured (μ, ν)-birobustly matchable graph on n vertices can be partitioned into $(1/\mu+100)r^2$ monochromatic cycles. Nevertheless, by Theorem 3.12 we know that for any $\mu > 0$, G is a (μ, ν)-birobustly matchable graph. Finally, choosing n_0 provided by Theorem 3.5 we know that G admits a partition into $10^7 r^2$ monochromatic cycles, concluding the proof. □

Bibliography

[1] Peter Allen. Covering two-edge-coloured complete graphs with two disjoint monochromatic cycles. *Combinatorics, Probability and Computing*, 17(4):471–486, 2008. doi: 10.1017/S0963548308009164.

[2] Peter Allen, Julia Böttcher, Richard Lang, Jozef Skokan, and Maya Stein. Partitioning a 2-edge-coloured graph of minimum degree $2n/3 + o(n)$ into three monochromatic cycles. 2022. doi: 10.48550/arXiv.2204.00496.

[3] Noga Alon and Joel H. Spencer. *The Probabilistic Method.* Wiley, New York, second edition, 2004.

[4] Jacqueline Ayel. *Sur l'existence de deux cycles supplémentaires unicolores, disjoints et de Couleurs différentes dans un graphe Complet Bicolore.* PhD thesis, Université scientifique et médicale de Grenoblé, 1979.

[5] József Balogh, János Barát, Dániel Gerbner, András Gyárfás, and Gábor N. Sárközy. Partitioning 2-edge-colored graphs by monochromatic paths and cycles. *Combinatorica*, 34(5):507–526, 2014. doi: 10.1007/s00493-014-2935-4.

[6] Stéphane Bessy and Stéphan Thomassé. Partitioning a graph into a cycle and an anticycle, a proof of Lehel's conjecture. *Journal of Combinatorial Theory, Series B*, 100(2): 176–180, 2010. ISSN 0095-8956. doi: https://doi.org/10.1016/j.jctb.2009.07.001.

[7] A. Czygrinow and H.A. Kierstead. 2-factors in dense bipartite graphs. *Discrete Mathematics*, 257(2-3):357–369, November 2002. doi: 10.1016/s0012-365x(02)00435-1.

[8] Louis DeBiasio and Luke L. Nelsen. Monochromatic cycle partitions of graphs with large minimum degree. *Journal of Combinatorial Theory, Series B*, 122:634–667, 2017. doi: 10.1016/j.jctb.2016.08.006.

[9] G. A. Dirac. Some theorems on abstract graphs. *Proceedings of the London Mathematical Society*, s3-2(1):69–81, 1952. doi: 10.1112/plms/s3-2.1.69.

[10] Márton Elekes, Dániel T. Soukup, Lajos Soukup, and Zoltán Szentmiklóssy. Decompositions of edge-colored infinite complete graphs into monochromatic paths. *Discrete Mathematics*, 340(8):2053–2069, 2017. doi: https://doi.org/10.1016/j.disc.2016.09.028.

[11] P. Erdős, A. Gyárfás, and L. Pyber. Vertex coverings by monochromatic cycles and trees. *Journal of Combinatorial Theory, Series B*, 51(1):90–95, 1991. doi: 10.1016/0095-8956(91)90007-7.

[12] Shinya Fujita, Michitaka Furuya, András Gyárfás, and Ágnes Tóth. Partition of graphs and hypergraphs into monochromatic connected parts. *The Electronic Journal of Combinatorics*, 19(3), 2012. doi: 10.37236/2121.

[13] László Gerencsér and András Gyárfás. On Ramsey-type problems. *Ann. Univ. Eötvös,*

Sect.Math., 10:167–170, 1967.

[14] Andrey Grinshpun and Gábor N. Sárközy. Monochromatic bounded degree subgraph partitions. *Discrete Mathematics*, 339(1):46–53, 2016. doi: 10.1016/j.disc.2015.07.005.

[15] András Gyárfás. Covering complete graphs by monochromatic paths. *In: Irregularities of Partitions, Springer Verlag. Algorithms and Combinatorics*, 8:89–91, 1989.

[16] András Gyárfás and Gábor Sárközy. Monochromatic loose-cycle partitions in hypergraphs. *The Electronic Journal of Combinatorics*, 21(2), 2014. doi: 10.37236/4062.

[17] András Gyárfás and Gábor N. Sárközy. Monochromatic path and cycle partitions in hypergraphs. *The Electronic Journal of Combinatorics*, 20(1), 2013. doi: 10.37236/2631.

[18] András Gyárfás, A. Jagota, and RH Schelp. Monochromatic path covers in nearly complete graphs. *The Journal of Combinatorial Mathematics and Combinatorial Computing*, 25:129–144, 1997.

[19] András Gyárfás, Miklós Ruszinkó, Gábor N. Sárközy, and Endre Szemerédi. An improved bound for the monochromatic cycle partition number. *Journal of Combinatorial Theory, Series B*, 96(6):855–873, 2006. doi: 10.1016/j.jctb.2006.02.007.

[20] András Gyárfás, Miklós Ruszinkó, Gábor N. Sárközy, and Endre Szemerédi. Partitioning 3-colored complete graphs into three monochromatic cycles. *The Electronic Journal of Combinatorics*, 18(1), 2011. doi: 10.37236/540.

[21] A. Gyárfás. Vertex coverings by monochromatic paths and cycles. *Journal of Graph Theory*, 7(1):131–135, 1983. doi: 10.1002/jgt.3190070116.

[22] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, s1-10(1):26–30, 1935. doi: 10.1112/jlms/s1-10.37.26.

[23] P.E. Haxell. Partitioning complete bipartite graphs by monochromatic cycles. *Journal of Combinatorial Theory, Series B*, 69(2):210–218, 1997. doi: 10.1006/jctb.1997.1737.

[24] P.E. Haxell and Y. Kohayakawa. Partitioning by monochromatic trees. *Journal of Combinatorial Theory, Series B*, 68(2):218–222, 1996. doi: 10.1006/jctb.1996.0065.

[25] János Komlós, Ali Shokoufandeh, Miklós Simonovits, and Endre Szemerédi. The regularity lemma and its applications in graph theory. *Theoretical Aspects of Computer Science*, page 84–112, 2002. doi: 10.1007/3-540-45878-6_3.

[26] Dániel Korándi, Richard Lang, Shoham Letzter, and Alexey Pokrovskiy. Minimum degree conditions for monochromatic cycle partitioning. *Journal of Combinatorial Theory, Series B*, 146:96–123, 2021. doi: 10.1016/j.jctb.2020.07.005.

[27] Richard Lang and Maya Stein. Local colourings and monochromatic partitions in complete bipartite graphs. *European Journal of Combinatorics*, 60:42–54, 2017. doi: 10.1016/j.ejc.2016.09.003.

[28] Shoham Letzter. Monochromatic cycle partitions of 2-coloured graphs with minimum degree $3n/4$. *The Electronic Journal of Combinatorics*, 26(1), 2019. doi: 10.37236/7239.

[29] Yuejian Peng, Vojtech Rödl, and Andrzej Ruciński. Holes in graphs. *The Electronic Journal of Combinatorics*, 9(1), 2001. doi: 10.37236/1618.

[30] Alexey Pokrovskiy. Partitioning edge-coloured complete graphs into monochromatic cycles and paths. *Journal of Combinatorial Theory, Series B*, 106:70–97, 2014. doi:

10.1016/j.jctb.2014.01.003.

[31] Alexey Pokrovskiy. Partitioning a graph into a cycle and a sparse graph. *Discrete Mathematics*, 346(1):113161, 2023. doi: 10.1016/j.disc.2022.113161.

[32] Lajos Pósa. On the circuits of finite graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 8:355–361, 1963.

[33] Richard Rado. Monochromatic paths in graphs. In *Advances in Graph Theory*, pages 191–194. Elsevier, 1978. doi: 10.1016/s0167-5060(08)70507-7.

[34] Gábor N. Sárközy. Monochromatic cycle partitions of edge-colored graphs. *Journal of Graph Theory*, 66(1):57–64, 2010. doi: 10.1002/jgt.20492.

[35] Gábor N. Sárközy. Improved monochromatic loose cycle partitions in hypergraphs. *Discrete Mathematics*, 334:52–62, 2014. doi: 10.1016/j.disc.2014.06.025.

[36] Gábor N. Sárközy, Stanley M. Selkow, and Fei Song. An improved bound for vertex partitions by connected monochromatic k-regular graphs. *Journal of Graph Theory*, 73 (2):127–145, 2012. doi: 10.1002/jgt.21661.

[37] Oliver Schaudt and Maya Stein. Partitioning two-coloured complete multipartite graphs into monochromatic paths and cycles. *Electronic Notes in Discrete Mathematics*, 50: 313–318, 2015. doi: 10.1016/j.endm.2015.07.052.

[38] Dániel T. Soukup. *Colouring problems of Erdős and Rado on infinite graphs*. PhD thesis, University of Toronto, 2015.

[39] Tomasz Łuczak. $\mathrm{R}(\mathrm{C}_n, \mathrm{C}_n, \mathrm{C}_n) \leq (4 + o(1))n$. *Journal of Combinatorial Theory, Series B*, 75(2):174–187, 1999. doi: 10.1006/jctb.1998.1874.

[40] Tomasz Łuczak, Vojtěch Rödl, and Endre Szemerédi. Partitioning two-coloured complete graphs into two monochromatic cycles. *Combinatorics, Probability and Computing*, 1998. doi: 10.1017/S0963548398003599.

Annex

Proof of Corollary 1.4

Here we present the proof of Corollary 1.4.

Proof. Here, $n_{1.3}(\eta, r)$ will be the number n_0 provided by Theorem 1.3 such that for every $n \geq n_0$, an r-edge-coloured balanced bipartite graph on n vertices with $\delta(G) \geq (1/4 + \eta)n$ contains a monochromatic cycle partition of size $10^7 r^2$.

Let $G = \{A, B\}$ be an r-edge-coloured bipartite graph such that $\delta(G) \geq (1/4 + \eta)n$ with $\eta > 0$. Let us assume $|A| = n/2 + \omega$ and $|B| = n/2 - \omega$ such that $\omega \in O(r^2)$. Since G is bipartite, any cycle that is not an isolated vertex will cover the same number of vertices in A and B. Now, choose any set $S \subseteq A$ such that $|S| = 2\omega$. Choose any $\eta' > 0$ such that $\eta > \eta'$. Note that we can define $n_0 = \max\{n_{1.3}(\eta', r) + 2\omega, 2\omega/(\eta - \eta')\}$. This is possible because, by our hypotheses, $\omega \in O(r^2)$. This implies that for every $n \geq n_0$, $(1/4+\eta)n - 2\omega \geq (1/4+\eta')n$. Then, $\delta(G[A \setminus S, B]) \geq (1/4 + \eta')n$ and $G[A \setminus S, B]$ is a balanced bipartite graph. Thus, as $|G[A \setminus S, B]| \geq n_0(\eta', r)$, by Theorem 1.3, $V(G[A \setminus S, B])$ contains a monochromatic cycle partition of size $10^7 r^2$. We denote such partition as $\mathcal{C}$.

Finally, we define $\mathcal{C}_2 = \mathcal{C} \cup S$ which, clearly, is a monochromatic cycle partition for $V(G)$. Since $|S| = 2\omega$, $\mathcal{C}_2$ has size $10^7 r^2 + \max\{|A|, |B|\} - \min\{|A|, |B|\}$, concluding the proof. $\square$