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ANÁLISIS FUNCIONAL EN ESTRUCTURAS ASIMÉTRICAS

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Resumen

Esta tesis, titulada “Análisis funcional en estructuras asimétricas”, consiste en el estudio de distintos tipos de estructuras de naturaleza asimétrica, ya sea en relación a una estructura métrica, algebráica, diferencial, o a una combinación de ellas, teniendo las asimetrías métricas un rol protagónico. Algunos ejemplos de estas estructuras incluyen a los *espacios normados asimétricos*, en los cuales un espacio vectorial real E es dotado de una función p que satisface todas salvo una de las condiciones necesarias para ser una norma, lo cual permite que los valores de $p(v)$ y $p(-v)$ puedan diferir para algunos puntos $v \in E$. Esta noción puede generalizarse aún más relajando la estructura algebráica del espacio vectorial, reemplazando el grupo aditivo por un monoide (el cual puede no poseer inversos aditivos), y restringiendo el producto por escalar a los escalares no-negativos. Esto da origen a la noción de *cono normado*. Otro ejemplo interesante son las variedades de Finlser, las cuales tienen la misma estructura diferencial de una variedad suave de dimensión finita, pero cada uno de los espacios tangentes está dotado de una norma asimétrica, en contraste de las normas generadas por productos internos usados en las variedades Riemannianas. Estos ejemplos pueden ser estudiados en el contexto de los *espacios cuasi-métricos*, los cuales son una generalización de los espacios métricos en la que la función distancia cumple todas las condiciones necesarias para ser una métrica a excepción de la simetría, en el sentido de que la distancia entre dos puntos a y b puede no coincidir con la distancia entre b y a. Quitar esta hipótesis permite una mayor flexibilidad en las situaciones que pueden ser modeladas con este concepto, pero tiene la desventaja de que pueden perderse herramientas y resultados conocidos en espacios métricos.

Nuestro estudio de estas estructuras asimétricas se realiza usando herramientas y nociones inspiradas por el análisis funcional clásico. En especial, la idea de utilizar alguna estructura en un espacio de funciones a valores reales $\mathcal{F}(X)$ sobre un conjunto X para determinar alguna propiedad sobre el conjunto X está presente en la mayoría de los resultados de este trabajo.

El Capítulo 2 contiene todas las definiciones y nociones preliminares necesarias. El Capítulo 3 presenta una caracterización de casi isometrías entre variedades de Finsler, la cual fue publicada en [17]. El Capítulo 4 trata sobre una generalización del concepto de espacio Lipschitz-libre al contexto de los espacios cuasi-métricos, la cual se publicó en [18]. El Capítulo 5 se divide en dos secciones: la Sección 5.1 da una generalización del clásico teorema de Myers-Nakai al caso asimétrico de las variedades de Finsler. Este resultado requirió definir nuevas estructuras asimétricas (llamadas *semianillo-cónico* y *álgebra normada asimétrica extendida*), las cuales son luego utilizadas en la Sección 5.2 para demostrar un teorema tipo Banach-Stone abstracto para espacios métricos.

Palabras clave: Espacios cuasi-métricos, espacios normados asimétricos, cono normado, variedad de Finsler, isometrías, casi isometrías, funciones Lipschitz, funciones semi-Lipschitz, espacio libre asimétrico, teoremas tipo Banach-Stone.

Abstract

This thesis, entitled "Functional analysis in asymmetric structures", consists of the study of different types of structures of asymmetric nature, related to metric, algebraic and differential structures, as well as combinations of them, with metric asymmetries playing a central role. Some examples of these structures include *asymmetric normed spaces*, where a real linear space E is endowed with a function p satisfying all but one of the conditions required to be a norm, which allows for the values of $p(v)$ and $p(-v)$ to differ for some points $v \in E$. This notion can be generalized further by relaxing the algebraic structure of the linear space, by replacing the additive group with a monoid (which may lack additive inverses), and restricting scalar multiplication to non negative scalars. This gives rise to the notion of *normed cones*. Another interesting example are Finsler manifolds, which share the differential structure of finite-dimensional smooth manifolds, but each tangent space in the tangent bundle of the manifold is endowed with an asymmetric norm, as opposed to the inner product norms used in Riemannian manifolds. These examples, as well as most of the structures studied in this work, can be viewed in the framework of *quasi-metric spaces*, which are a generalization of metric spaces in which the distance function satisfies all the conditions required to be a metric except one: the distance function does not need to be symmetric, in the sense that the distance between two points a and b may differ from the distance between b and a. Dropping the assumption of symmetry from the definition of a metric offers a greater degree of flexibility regarding situations that can be modeled using this concept, at the expense of losing some of the tools and results known for metric spaces.

Our study of these asymmetric structures is carried out using tools and notions inspired by classical functional analysis. In particular, the idea of employing some structure on a space of real-valued functions $\mathcal{F}(X)$ over a set X to determine some property on the set X itself is present in most of the results of this work.

Chapter 2 contains the necessary preliminary definitions and notions. Chapter 3 contains a characterization of almost isometries between Finsler manifolds, which was published in [17]. Chapter 4 deals with the construction of a generalization of Lipschitz-free spaces in the framework of quasi-metric spaces, which was published in [18]. Chapter 5 is divided in two sections: Section 5.1 gives a generalization of the classical Myers-Nakai theorem, which characterizes isometries between Riemannian manifolds, to the asymmetric case of Finsler manifolds. This result required the definition of new asymmetric structures (namely, *conic-semirings* and *extended asymmetric normed algebras*), which are then used in Section 5.2 to prove an abstract Banach-Stone type theorem for metric spaces.

Keywords: Quasi-metric spaces, asymmetric normed spaces, normed cones, Finsler man-

ifolds, isometries, almost isometries, Lipschitz functions, semi-Lipschitz functions, asymmetric free space, Banach-Stone type theorems.

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Chapter 1

Introduction

The study of metric spaces is a well-explored field in mathematics, paramount for the training of mathematicians with interest in analysis. The definition of metrics encompasses three axioms, namely, separation of points, symmetry, and the famous triangle inequality. These three axioms are strong enough to produce a robust theory, and at the same time easy to verify in order to give metric spaces plethora of applications. Still, in some cases (both theoretical and applied) the axioms of metrics are too restrictive, which motivates considering relaxations.

Relaxations can be made in several directions, by addressing different aspects of the definition of metrics, and in particular, one of the three aforementioned axioms. One of the earliest examples of such a relaxation came under the name of *pseudometric* spaces [32], where the distance function is no longer required to separate points (i.e., two different points can be at distance zero). In the context of linear spaces, this leads to the notion of *seminorms.* Another known example, under the name of *p-metric* spaces or *quasimetric* spaces, relaxes the triangular inequality instead [1]. This notion has been studied in the context of Banach spaces under the name of quasi-Banach spaces and p-Banach spaces [2].

Unfortunately, notations and terminology of generalized metric spaces are not yet standard. What some authors call a pseudometric, some others call a semimetric (in an effort to use a similar name to the one already widely established on normed spaces). The situation is even more complicated with terms such as *quasi-metric*, which for most authors refers to a distance function that does not need to be symmetric, while other authors (e.g. [1]) use the term quasimetric to refer to a relaxation of the triangle inequality. On top of that, linear versions of these notions do not often use the same terminology, as most of them were developed earlier along with the theory of Banach spaces. Such is the case of pseudometric spaces and seminormed spaces, and quasi-metric spaces (in the asymmetric sense) and its linear version, called asymmetric normed spaces, since the term quasi-normed space and quasi-Banach space were already established, and mean something completely different.

Therefore, readers should be constantly alerted when dealing with these types of notions, and authors should be as clear as possible with all definitions. Fortunately, a mayor part of this document involves only one of the above mentioned relaxations of metric spaces, so possible ambiguities are kept to a minimal. Any such troubles will be properly addressed as needed, mainly on Chapter 2.

This work deals with different types of structures of asymmetric nature, related to metric, algebraic and differential structures, as well as combinations of them. Most of these notions can be studied within the framework of *quasi-metric spaces*, that is, sets endowed with distance functions satisfying the axioms of a metric, except symmetry. This difference might seem small at first but, as we shall see on Chapter 2, it has deep consequences even in the most fundamental aspects of metric spaces, such as their topology. Even though quasi-metric spaces have appeared in the literature as early as the 1930s (see [44]), their structure is not completely understood yet.

Dropping the assumption of symmetry from the definition of a metric can be seen as a natural relaxation of the core concept of metric spaces. Indeed, by looking at some of the real world scenarios (be it in physics, resource allocation, measuring travel distances and times, etc) that inspired the definition and study of metrics, one may argue that the symmetry requirement is somewhat "unnatural". This was noted by M. Gromov in his celebrated book *"Metric Structures for Riemannian and Non-Riemannian Spaces"*, where he stated the following (in regards to the symmetry requirement in the definition of metrics): *"...This unpleasantly limits many applications: the effort of climbing up to the top of a mountain, in real life as well as in mathematics, is not at all the same as descending back to the starting point."*.

The definition of quasi-metric spaces is instinctively associated with its natural class of morphisms, known as *semi-Lipschitz functions*. The study of this class of functions quickly becomes quite interesting: as we shall see in Chapter 2, the set of real-valued semi-Lipschitz functions on a quasi-metric space (X, d), denoted by $\mathrm{SLIP}(X)$, is often not a linear space, but only a convex cone. We highlight the importance of this fact, as it reveals how our starting "metric asymmetry" is then reflected as an "algebraic asymmetry", since the cone $\mathrm{SLIP}(X)$ lacks the "additive inverse" property of a linear space. This sort of "propagation" of asymmetries into natural function spaces makes the endeavor of developing tools of functional analysis for these spaces more challenging.

This document is organized as follows: Chapter 2 contains all the preliminary definitions and notions necessary for a reader, from whom we will not assume acquaintance with quasi-metric spaces and related topics. This chapter also contains the basic definitions of Finsler manifolds and Lipschitz-free spaces, which will be necessary for the remaining chapters.

Chapters 3, 4 and 5 present different, self-contained results, which are not required to be read in order.

Chapter 2

Preliminaries

As already mentioned in Chapter 1, dropping the assumption of symmetry from the definition of a metric offers a major degree of flexibility regarding situations that can be modeled using this concept. On the other hand, the advantages of working with symmetric distances cannot be denied. As we shall see in this Chapter, symmetry plays a very important role in many notions of crucial importance to the theory of metric spaces, such as completeness and Lipschitzianity. Nevertheless, some other notions remain "well-behaved-enough" under the loss of symmetry, resulting in an interesting and new framework, which retains enough key similarities with the classic symmetric theory as to not be completely detached from it, while exhibiting profound and remarkable differences.

This section contains basics of the theory of quasi-metric spaces, as well as more advanced results in the literature that will be required for Chapters 3 to 5.

2.1 Quasi-metric spaces

Before giving the abstract definition of quasi-metric spaces, some concrete examples are in order. Suppose one wants to walk up a relatively steep hill. The Euclidean distance between our starting point A at the bottom, and a point B at the top of the hill will be the same no matter if we are going up or down, which is of course factually correct, but it does not reflect other variables of the problem that might be of interest, such as the effort involved (going from A to B should be harder than the other way around), the time required to make the trip, and so on. If one wants to include such information in the definition of distance, we must look further than the classical notion of metrics. The same problem of traveling through uneven terrain can be approached from a geometrical standpoint, as at each point p of the terrain, the effort required to move would depend not only in the point p, but in the direction v of movement. This can be modeled using *Finsler manifolds*, which we will visit on Section 2.4.

More applied examples justifying the need for asymmetric distances can be found in transportation problems using roads and streets of a city, which are often not bidirectional, and even when they are, the most optimal route from point A to point B need not coincide with the best route from B to A. This implies that quantities such as traveled distance, travel times, fuel consumption, etc, may not be symmetric with respect to the starting and ending points. More generally, any problem that can be modeled using directed graphs admits a potential asymmetry, as the *shortest path distance* commonly used on directed graphs is not symmetric.

2.1.1 Basic definitions

Throughout this work we will denote by $\mathbb{R}_+$ the set of non-negative real numbers. The maximum and minimum between two real numbers a and b are denoted by $a \vee b$ and $a \wedge b$, respectively. We will also use the convention $\inf \emptyset = +\infty$.

Definition 2.1 (Quasi-metric space) *A* quasi-metric space *is a pair* (X, d)*, where* $X \neq \emptyset$ *and*

$$d : X \times X \to [0, \infty)$$

is a function, called quasi-metric *(or* quasi-distance*), satisfying:*

(i) $\forall x, y, z \in X$*:* $d(x, y) \leq d(x, z) + d(z, y)$ *(*triangular inequality*);*

(ii) $\forall x, y \in X$*:* $x = y \iff d(x, y) = 0$.

Note that a quasi-metric does not possess the symmetric property $d(x, y) = d(y, x)$ of a distance.
If we replace the last condition by

$$\text{(ii)}' \qquad x = y \quad \iff \quad \begin{cases} d(x,y) = 0 \\ d(y,x) = 0 \end{cases}$$

then we say that d is a *quasi-hemi-metric*, in which case the distance between two different points is allowed to be 0, as long as the distance in the "opposite orientation" is strictly positive. Throughout this paper, we shall treat both variants of quasi-metric spaces. The terminology of quasi-metric space will thus refer to a pair (X, d) where d is either a quasi-distance or a quasi-hemi-distance.

In this work we shall also consider *extended quasi-metrics* $\tilde{d} : X \times X \to [0, \infty]$, that is, quasi-metrics that satisfy the same two conditions above, but are also allowed to take the value $+\infty$.

As it is the case in the classical theory of metric spaces, the special case when the distance function is associated with a norm deserves to be addressed on its own right.

Definition 2.2 *Given a real vector space* E*, we denote by* $\| \cdot \| : E \to \mathbb{R}_+$ *a norm on* E *and by* $\| \cdot | : E \to \mathbb{R}_+$ *an* asymmetric norm *on* E*, that is, a function satisfying:*

(i) $\forall x, y \in E$: $\|x+y| \leq \|x| + \|y|$;

(ii) $\forall x \in E$: $x = 0 \iff \|x| = 0$;

(iii) $\forall x \in E$, $\forall r > 0$: $\|r\,x| = r\,\|x|$.

If we replace the second condition by

$$\text{(ii)}' \quad x = 0 \quad \iff \quad \begin{cases} \|x| = 0 \\ \|-x| = 0 \end{cases}$$

then we say that $\|\cdot| : E \to \mathbb{R}_+$ *is an* asymmetric hemi-norm *on* E*.*

The terminology of *asymmetric normed space* refers to pairs $(E, \|\cdot|)$ having either asymmetric norms or asymmetric-hemi norms on E. The symbol $\|\cdot|$, using two vertical bars on the left and only one bar on the right side, serves as a reminder of the asymmetric nature of these type of functionals, in the sense that the values $\|x|$ and $\|-x|$ may not coincide.

We may also consider, keeping the same notation, *extended asymmetric hemi-norms*, allowing $\|\cdot|$ to take the value $+\infty$. Finally, we denote by u the asymmetric hemi-norm on $\mathbb{R}$ defined by

$$u(x) = \max\{x, 0\}, \quad \text{for every } x \in \mathbb{R}. \tag{2.1}$$

If X is a vector space equipped with an (extended) asymmetric (hemi-)norm $\|\cdot|$, then the function

$$d(x, y) := \|y - x|, \quad \text{for all } x, y \in X \tag{2.2}$$

is an (extended) quasi-(hemi-)metric on X that satisfies:

$$d(x+z, y+z) = d(x, y) \quad \text{and} \quad d(rx, ry) = rd(x, y), \tag{2.3}$$

for all $x, y, z \in X$ and $r \in \mathbb{R}_+$.

For a general quasi-metric space (X, d), and $x, y \in X$ the *reverse quasi-metric* $\bar{d}$ is defined by

$$\bar{d}(x, y) = d(y, x).$$

Remark 2.3 (Terminology alert I) *The reader should be alerted that terminology may slightly vary according to the authors. Some authors allow the quasi-hemi-metric and the asymmetric hemi-norm to also take negative values. They also use the terms hemi-metric and hemi-norm to refer to what we call quasi-hemi-metric and asymmetric hemi-norm, respectively (see, for instance, [26]). In this work, the adjective quasi refers to the asymmetry of the metric, and the adjective hemi to the fact that distinct elements* x, y *in* X *may have a quasi-distance* $d(x, y)$ *equal to* 0*.*

Two quasi-metric spaces can be completely identified via isometries, as follows.

Definition 2.4 (Isometry) *A bijective mapping* Φ *between extended quasi-metric spaces* (X, d) *and* (Y, ρ) *is called an* isometry *if for every* $x_1, x_2 \in X$*, it holds*

$$\rho(\Phi(x_1), \Phi(x_2)) = d(x_1, x_2).$$

This notion of isometry is the most straight-forward generalization of the notion of isometries between metric spaces, but it is not the only one. The following notion, relevant to some applications, presents itself as a weaker generalization of isometries in the classical sense. First, we need to define the *triangular function* associated with a quasi-metric.

Definition 2.5 (Triangular function) *Let* (X, d) *be a quasi-metric space. The* triangular function $\mathrm{Tr}_X : X \times X \times X \to [0, +\infty)$ *(associated to the quasi-metric space* X*) is defined by*

$$\mathrm{Tr}_X(x_1, x_2, x_3) = d(x_1, x_2) + d(x_2, x_3) - d(x_1, x_3).$$

Definition 2.6 (Almost isometries) *A bijection* $\tau : X \to Y$ *between the quasi-metric spaces* (X, d_X) *and* (Y, d_Y) *is called:*

(i) an almost isometry*, if it preserves the respective triangular functions, that is*

$$\mathrm{Tr}_Y(\tau(x_1), \tau(x_2), \tau(x_3)) = \mathrm{Tr}_X(x_1, x_2, x_3), \quad \text{for all } x_1, x_2, x_3 \in X \tag{2.4}$$

(ii) a strict almost isometry*, if it satisfies* (2.4) *and there exists a constant* $c \geq 1$ *such that*

$$\frac{1}{c}\, d_X(x_1, x_2) \leq d_Y(\tau(x_1), \tau(x_2)) \leq c\, d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X.$$

Clearly, every isometry is a (strict) almost isometry, and in metric spaces every almost isometry is in fact an isometry. The following proposition gives a useful characterization of almost isometries.

Proposition 2.7 (Characterization of almost isometries) *Given quasi-metric spaces* (X, d_X) *and* (Y, d_Y)*, a bijection* $\tau : X \to Y$ *is an almost isometry if and only if there exists a function* $\phi : X \to \mathbb{R}$ *such that for any* $x_1, x_2 \in X$

$$d_Y(\tau(x_1), \tau(x_2)) = d_X(x_1, x_2) + \phi(x_1) - \phi(x_2).$$

Moreover, the function ϕ *can be determined up to an additive constant by*

$$\phi(x) = d_Y(\tau(x), \tau(x_0)) - d_X(x, x_0), \quad \text{for any fixed } x_0 \in X.$$

In a Chapter 3, we will show that strict almost isometries can be identified in terms of the function ϕ used in Proposition 2.7

2.1.2 Topologies of a quasi-metric space

A particularly useful tool when dealing with quasi-metric spaces is the fact that every quasi-metric has associated several (symmetric) metrics. Every quasi-metric can be *symmetrized* in the sense of the following definition.

Definition 2.8 (Symmetrized distance) *Let (X,d) be a quasi-metric space. Then*

$$d^s(x,y) = \max\{d(x,y), d(y,x)\} \qquad \text{and} \qquad d^{s_1}(x,y) = d(x,y) + d(y,x) \tag{2.5}$$

are two natural symmetrizations of the quasi-distance d, which are equivalent to each other:

$$d^s(x,y) \le d^{s_1}(x,y) \le 2\, d^s(x,y), \quad \text{for all } x,y \in X.$$

If d is an extended quasi-metric, then so is its reverse $\bar{d}$ and consequently the symmetrizations d^s and d^{s_1} give rise to extended metrics. In case that X is a vector space and d satisfies (2.3), the above symmetrizations preserve the invariance by translations and homothety.

Definition 2.9 *Every (possibly extended) quasi-metric space (X,d) can be endowed with three "natural" topologies:*

(i) The forward topology $\mathcal{T}(d)$*, generated by the family of open* forward-*balls*

$$\{B_d(x,r)\colon x \in X, r > 0\},$$

where $B_d(x,r) := \{y \in X\colon d(x,y) < r\}$, for all $x \in X$ and $r > 0$.

(ii) The backward topology $\mathcal{T}(\bar{d})$*, generated by the family of* backward-*balls*

$$\{B_{\bar{d}}(x,r)\colon x \in X, r > 0\},$$

where $B_{\bar{d}}(x,r) := \{y \in X\colon d(y,x) < r\}$, for all $x \in X$ and $r > 0$.

(iii) The symmetric topology $\mathcal{T}(d^s)$*, generated by the family of sets*

$$\{B_d(x,r) \cap B_{\bar{d}}(x,r)\colon x \in X, r > 0\}.$$

The symmetric topology being generated by the symmetrized distance d^s or d^{s_1} defined in (2.5) is obviously a metric topology. On the other hand, $\mathcal{T}(d)$ and $\mathcal{T}(\bar{d})$ are not in principle metric topologies. Nevertheless, they are both first countable topologies, since they have local bases consisting of balls of rational radii. Notice also that $\mathcal{T}(d^s)$ is finer that both $\mathcal{T}(d)$ and $\mathcal{T}(\bar{d})$, and therefore, any d-open (respectively $\bar{d}$-open) set is also open in the symmetrized topology.

In what follows, unless stated otherwise, for any the quasi-metric space (X,d), the **default topology will be the forward topology**, which is either a T_1-topology (when d is a quasi-metric) or a T_0-topology (when d is a quasi-hemi-metric).

Definition 2.10 *Let us consider $\mathbb{R}$ with the asymmetric distance d_u defined by*

$$d_u(x,y) = u(y-x) = \max\{y-x, 0\}.$$

It is easy to check that $\mathcal{T}(d_u)$ has a local basis of the form $\{(-\infty, x_0+\varepsilon) : \ \varepsilon > 0\}$ for each $x_0 \in \mathbb{R}$, while $\mathcal{T}(\bar{d_u})$ has a local basis consisting of sets of the form $(x_0 - \varepsilon, +\infty)$, and $\mathcal{T}(d_u^s)$ is the usual topology of $\mathbb{R}$. Notice that d_u is issued from the asymmetric hemi-norm $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$, see (2.1) and (2.2). The forward topology of this space behaves in a remarkably different way than the usual (metric) topology of $\mathbb{R}$. For example, the unit ball $\overline{B}(0,1) = \{y \in \mathbb{R} : d_u(0,y) \leq 1\} = (-\infty, 1]$ is not $\mathcal{T}(d_u)$-closed because $(1,\infty)$ is not $\mathcal{T}(d_u)$-open.

Remark 2.11 *The topology $\mathcal{T}(d_u)$ is a particular case of the left order topology for totally ordered sets (see [40] for more details on left and right order topologies).*

Another interesting fact about the space $(\mathbb{R}, d_u)$, is that it characterizes semi-continuity.

Proposition 2.12 *Let (X,τ) be a topological space, and let $f : X \to \mathbb{R}$ be a function. Then, f is continuous for d_u (that is, continuous for the forward topology $\mathcal{T}(d_u)$) if and only if f is upper semicontinuous for the usual topology of $\mathbb{R}$. The same assertion holds for the backward topology and lower semicontinuity.*

Proof. Let us note that every non trivial open set in $(\mathbb{R}, d_u)$ is of the form $(-\infty, \lambda)$, for $\lambda \in \mathbb{R}$. Then, both notions of continuity are fulfilled when the super-level sets $S(\lambda) = f^{-1}([\lambda, +\infty))$ are closed in (X,τ) for all $\lambda \in \mathbb{R}$.

□

The following example reveals that the topology of a quasi-metric space, which is T_1, may not be T_2.

Example 2.13 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of distinct elements and consider the space*

$$X = \{x_n \ : n \in \mathbb{N}\} \cup \{\bar{x}, \bar{y}\},$$

where $\bar{x}$ and $\bar{y}$ are different from each other and from any element of the sequence. Then the function d defined on $X \times X$ by $d(\bar{x}, x_n) = d(\bar{y}, x_n) = 1/n$, for every $n \in \mathbb{N}$, and $d(x,y) = 1$ for all other cases where $x \neq y$, is a quasi-metric on X. In this case, the forward topology $\mathcal{T}(d)$ cannot be T_2, since $\{x_n\}_n$ converges to both $\bar{x}$ and $\bar{y}$. Notice that the symmetrized distance d^s is discrete, with $d^s(x,y) = 1$, whenever $x \neq y$.

2.1.3 Cones and conic norms

As stated earlier, the particular case where a quasi-metric space is endowed with linear structure is of paramount importance. *Asymmetric normed spaces* have been the focus of the theory of asymmetric functional analysis so far, and we refer the reader to the book [15] for an in-depth and mostly self-contained review on the subject. However, as important as asymmetric normed spaces are, we shall approach them from a more general point of view, as we firmly believe there is a more general and fundamental object behind asymmetric normed spaces. This object, known in the literature by the names of *normed cone*, *semi-linear space*, *quasi-normed cone*, among others, has been known for a while (see for instance, [9] and [31]), but it has not received the same attention and development as asymmetric normed spaces. We shall see that this type of spaces appear naturally when studying quasi-metric spaces when we delve into the asymmetric equivalent of Lipschitz functions in Section 2.2. For the time being, let us start the basic definitions and notions needed to understand normed cones as a generalization of normed linear spaces.

First, we shall recall from [41] the notion of an abstract cone. To this end, let us first recall that a *monoid* is a semigroup $(X, +)$ with neutral element 0.

Definition 2.14 (Abstract cone) *A* cone *on $\mathbb{R}_+$ is a triple $(C, +, \cdot)$ such that $(C, +)$ is an abelian monoid (with neutral element 0), and $\cdot$ is a mapping from $\mathbb{R}_+ \times C$ to C such that for all $x, y \in C$ and $r, s \in \mathbb{R}_+$:*

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;

(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$ *and* $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;

(iii) $1 \cdot x = x$ *and* $0 \cdot x = 0$.

Note that this definition does not include the existence of additive inverses. However, when such an inverse exists for some $x \in C$, it is unique, and we denote it by $-x$. This purely algebraic definition can be seen as a "natural" generalization of real linear spaces, where the Abelian group is replaced by a monoid, and the field of scalars is reduced to $\mathbb{R}_+$. Removing the possibility of negative scalars is necessary, as the notion of abstract cone does not ensure the existence of additive inverses. However, when the monoid C is actually a commutative group, the scalar product can be extended to $\mathbb{R}$ in the natural way, and we end up with a linear space in the usual sense.

Definition 2.15 (Subcone) *A* subcone *of a cone $(C, +, \cdot)$ is a cone $(S, +|_S, \cdot|_S)$ such that S is a subset of C and $+|_S$ and $\cdot|_S$ are, respectively, the restriction of $+$ to $S \times S$ and of $\cdot$ to $\mathbb{R}_+ \times S$.*

Definition 2.16 (Cancellative cone) *A cone $(C, +, \cdot)$ is called* cancellative *if for any $x, y, z \in C$,*

$$x + z = y + z \implies x = y.$$

It follows readily that every cone that embeds into a linear space is cancellative. Before we proceed, let us give two examples of abstract cones which are not cancellative.

Example 2.17 (Non-cancellative cone) (i) *Consider a cone C and let $\mathcal{S}(C)$ be the set of subcones of C, under the usual operations of subset addition and scalar product. Then $\mathcal{S}(C)$ may not be cancellative.*
Indeed, for $C = \mathbb{R}^2$, let us consider the following elements of $S(C)$:

$$X = \{(x, 0) \,:\, x \in \mathbb{R}\}, \quad Y = \{(0, x) \,:\, x \in \mathbb{R}\} \quad \text{and} \quad Z = \{(x, x) \,:\, x \in \mathbb{R}\}.$$

It follows that $X + Z = Y + Z$ but $X \neq Y$.

(ii) For a nonempty set X, consider the set of non-negative functions $\mathbb{R}_+^X$, with the operations $\lambda \odot f = f^\lambda$ (product with external scalar) and $f \oplus g = f \cdot g$ (addition). Then $\mathbb{R}_+^X$ is not cancellative.

Definition 2.18 (Cone morphisms) *A* linear mapping *from a cone $(C_1, +, \cdot)$ to a cone $(C_2, +, \cdot)$ is a mapping $f \colon C_1 \to C_2$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)$ for any $x, y \in C_1$ and any $\alpha, \beta \in \mathbb{R}_+$.*

Remark 2.19 (Compatibility of cone morphisms) *Let f be a linear mapping between two cones C_1 and C_2. Then if $H_i := \{x \in C_i : -x \in C_i\}$ denotes the linear part of the cone C_i, for $i \in \{1, 2\}$, then it is straightforward to see that for every $x \in H_1$, $f(-x) = -f(x)$. In particular, the restriction of f onto H_1 yields a linear mapping between the linear spaces H_1 and H_2.*

We shall now introduce the notion of a *conic-norm*, which will be relevant for our developments.

Definition 2.20 (Conic norm) *A* conic-norm *on a cone $(C, +, \cdot)$ is a function $\| \cdot | \colon C \to \mathbb{R}_+$ such that for all $x, y \in C$ and $r > 0$:*

(i) $\|x + y| \leq \|x| + \|y|$;

(ii) $\|x| = 0 \iff x = 0$;

(iii) $\|r \cdot x| = r\|x|$.

The pair $(C, \| \cdot |)$ is called normed cone. *If we replace condition (ii) by*

$$\text{(ii)}' \qquad x = 0 \iff \forall z \in C,\ [x + z = 0 \implies \|x| = \|z| = 0]\,,$$

then we say that $\|\cdot| \colon C \to \mathbb{R}_+$ is a conic hemi-norm. *A cone equipped with either a conic-norm or a conic hemi-norm will be called* normed cone. *This is in accordance with the terminology* asymmetric normed space, *which refers to a vector space equipped with either an asymmetric norm or an asymmetric hemi-norm. (The asymmetry is now stemming from the use of a cone, rather than a vector space. Notice however that C is not necessarily a cancellative cone.)*

Example 2.21 *Consider the pair $(\mathbb{R}^2, \| \cdot |)$, with*

$$\|(x_1, x_2)| := u(x_1) + u(x_2),$$

where u is the canonical asymmetric hemi-norm of $\mathbb{R}$ given by $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$. By restricting $\| \cdot |$ to any cone $C \subseteq \mathbb{R}^2$, we obtain a conic-hemi-norm. The case of the third quadrant $C = \mathbb{R}_-^2$ corresponds to an example of normed cone with the trivial conic hemi-norm equal to 0 everywhere.

Remark 2.22 (Terminology alert II) *The reader should again be alerted that some authors ([41],* e.g.*) employ the term of* quasi-norm *to refer to what we call "conic hemi-norm". We*

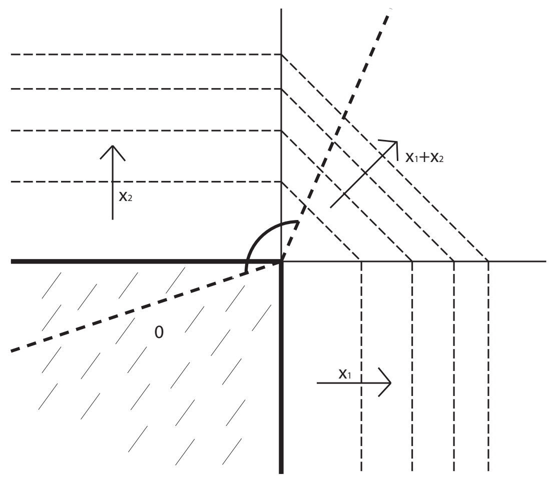


Figure 2.1: Illustration of Example 2.21

decided to opt for the term "conic hemi-norm" because it is more suggestive. At the same time, the term "quasi-norm" might have a different meaning in the theory of asymmetric Banach spaces ([2], e.g.*). The asymmetric aspect of the conic-norm is inherent to the definition of a cone, and therefore does not require the prefix "quasi".*

Remark 2.23 (Conic-norm vs asymmetric norm) *If the cone happens to be a linear space X, then the conic-norm corresponds to an asymmetric norm on X, and instead of the term "normed cone" we use the term* asymmetric normed space*, as in [15]. The same applies to the case of conic hemi-norms and asymmetric hemi-norms. Given an asymmetric normed space $(X, \|\cdot|)$, one can define the* reverse *norm of an element $x \in X$ as $\|-x|$, and the (symmetric) norms (symmetrizations of $\|\cdot|$)*

$$\|x\|_s := \max\{\|x|, \|-x|\} \qquad \text{and} \qquad \|x\|_{s_1} := \|x| + \|-x|.$$

It is clear that the above norms are equivalent.

An extended quasi-metric d on a cone $(C, +, \cdot)$ is called *invariant* if it satisfies

$$d(x+z, y+z) = d(x,y) \quad \text{and} \quad d(rx, ry) = rd(x,y), \tag{2.6}$$

which is the case whenever the extended quasi-metric d is induced by a conic-norm which is the restriction of an asymmetric norm of a vector space that contains C. An extended quasi-metric d on a cone $(C, +, \cdot)$ is called *subinvariant* if $d(x+z, y+z) \leq d(x,y)$ instead of the first part of (2.6). More generally, the following result, established in [23, Proposition 1], states that given a normed cone $(C, \|\cdot|)$, there is a natural way to generate an extended quasi-metric d_e.

Proposition 2.24 (Extended quasi-metrics generated by conic-norms) *Let $\|\cdot|$ be a conic-(hemi-)norm on a cone $(C, +, \cdot)$. Then the function d_e defined on $C \times C$ by*

$$d_e(x,y) = \inf_{\substack{z \in C \\ y = x+z}} \|z|,$$

is a subinvariant extended quasi(-hemi)-metric on C. If the cone $(C, +, \cdot)$ is cancellative, then d_e is invariant.

Moreover, for $x \in C$, $r \in \mathbb{R}_+ \backslash \{0\}$ and $\varepsilon > 0$, we have

$$rB_{d_e}(x, \varepsilon) = rx + \{y \in C \colon \|y| < r\varepsilon\},$$

and the translations are $\mathcal{T}(d_e)$-open.

Remark 2.25 (i) *The quasi-metric d_e might take infinite values as long as C is not a linear space (the infimum may be taken over the empty set).*

(ii) *If C is a cancellative cone, then the infimum in the above definition becomes superfluous, and if C is a linear space, the definition of d_e coincides with the definition of the quasi-metric given in* (2.2).

(iii) *The quasi-metric induced by the reverse norm coincides with the one obtained by the reverse quasi-metric. The same is true for the symmetrized metric which coincides with the metric obtained by the symmetrization of the asymmetric norm.*

Using the extended quasi-metric of Definition 2.24, we define an equivalence between normed cones.

Definition 2.26 (Isomorphisms between normed cones) *A bijective mapping $\Phi : X \to Y$ between two normed cones is called an* isometric isomorphism *if it is linear (*c.f. *Definition 2.18) and an isometry between the corresponding extended quasi-metrics, that is,*

$$d_e(\Phi x_1, \Phi x_2) = d_e(x_1, x_2), \quad \textit{for all } x_1, x_2 \in X.$$

Note that this is equivalent to the relation $\|\Phi x| = \|x|$, for all $x \in X$.

2.1.4 Completeness in quasi-metric spaces

It is a well known fact that completeness is a crucial notion in the study of metric spaces, and it is a necessary hypothesis for many of the most important results on said theory. Sadly, completeness is one of the concepts that does not have one straight-forward generalization to quasi-metric spaces. In fact, there are several quasi-metric generalizations of completeness, each one preserving a relevant aspect of metric completeness, and all being non equivalent to each other. Proposition 2.27 illustrates that. The following definitions are not meant to be compared by the reader, as only one of them will be relevant for this work. For an in-depth discussion on the subject, we refer the reader to [15].

Let (X, d) be a quasi-metric space. We say that a sequence (x_n) in (X, d) is

- left d-Cauchy if for every $\varepsilon > 0$ there exists $x \in X$ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \ d(x, x_n) < \varepsilon;$$

- d^s-Cauchy if it is a Cauchy sequence in the metric space (X, d^s), that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k \geq n_0,\ d(x_n, x_k) < \varepsilon;$$

- left K-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k,\ n_0 \leq k \leq n \implies d(x_k, x_n) < \varepsilon;$$

- weak left K-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ d(x_{n_0}, x_n) < \varepsilon.$$

This notions, along with their backward variants, can be used to define notions of completeness, by asking for the chosen class of Cauchy sequences to be convergent. Moreover, by choosing the topology under which the convergence in considered, even more notions are. As for the relation between some of this notions, the following proposition sheds some light into the situation.

Proposition 2.27 *[15, Proposition 1.2.1] Let (X, d) be a quasi-metric space. The aforementioned notions of Cauchy sequence are related as follows:*

$$\begin{aligned} d^s\text{-Cauchy} &\Rightarrow \quad \text{left } K\text{-Cauchy} \\ &\Rightarrow \quad \text{weakly left } K\text{-Cauchy} \Rightarrow \text{left } d\text{-Cauchy}. \end{aligned}$$

No one of the above implications is reversible.

As a consequence, the notions of quasi-metric completeness that can be defined using the aforementioned notions of Cauchy sequences will also differ. From all the possible choices, two will be of use for our purposes.

Definition 2.28 (Forward completeness) *A sequence (x_n) in a (possibly extended) quasi-metric space (X, d) is said to be* forward-Cauchy *if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n_0 \leq n \leq m$,*

$$d(x_n, x_m) < \varepsilon.$$

A space where every forward-Cauchy sequence is forward-convergent (i.e, there exists $x_0 \in X$ such that $d(x_0, x_n) \to 0$) is called forward complete.

Definition 2.29 (Bicompleteness) *A (possibly extended) quasi-metric space (X, d) is called* bicomplete *if the (extended) metric space (X, d^s) is complete, meaning that any d^s-Cauchy sequence in X is d^s-converging in X. If X is a linear space and d is the quasi-metric induced by an asymmetric norm $\| \cdot |$, we say $(X, \| \cdot |)$ is a bi-Banach space, whenever X is complete under the symmetrized metric d^s.*

Definition 2.30 (Bicompletion of a quasi-metric space) *Let (X,d) be an (extended) quasi-metric space. A* bicompletion *of (X,d) is an (extended) quasi-metric space $(\tilde{X},\tilde{d})$, along with a mapping*

$$\iota : (X,d) \to (\tilde{X},\tilde{d})$$

such that:

(i) ι is an isometric embedding;

(ii) $\iota(X)$ is dense in $\tilde{X}$ for the symmetrized topology;

(iii) $\left(\tilde{X},\tilde{d}\right)$ is bicomplete.

An important result regarding bicompleteness of normed cones (and therefore of asymmetric normed spaces) is the existence and uniqueness of the bicompletion, see [37, Theorem 3.13]. This result, once again, generalizes the usual completion of normed linear spaces.

Proposition 2.31 (Uniqueness of bicompletion for cancellative normed cones) *Let $(C,\|\cdot|)$ be a cancellative normed cone. Then there exists a unique (up to an isometric isomorphism) bicompletion of $(C,\|\cdot|)$, which is also a normed cone, and the embedding into the bicompletion is linear. If C is a linear space, then its bicompletion is an asymmetric normed space.*

The reason for choosing bicompleteness among all other possible notions of quasi-metric completeness is its compatibility with linear and conic structures, as shown by the proposition above. We shall also see on Section 2.2 that this compatibility extends to the notion of *dual cones*, which play a crucial role in our results.

2.1.5 Index of symmetry

Not every quasi-metric space exhibits the same behavior with respect to its reverse quasi-metric. As a trivial example, we have metric spaces, where d and $\bar{d}$ are the same. The opposite occurs on Definition 2.10, where d_u and $\bar{d_u}$ are never comparable, as whenever $d_u(x,y) > 0$, $\bar{d_u}(x,y) = 0$. Intermediate examples can be constructed by taking $\alpha \in (0,\infty)$, and defining following the quasi-metric on $\mathbb{R}$

$$\rho_\alpha(x,y) = \begin{cases} \alpha(y-x) & \text{if } y \geq x \\ x-y & \text{if } x \geq y \end{cases},$$

so that ρ_α and $\bar{\rho_\alpha}$ are "equivalent".

The following notion to quantify how asymmetric a quasi-metric space is was introduced independently by Shen and Zhao in [39] and by Bachir and Flores in [7].

Definition 2.32 (Index of symmetry) *Let (X,d) be a quasi-metric space. The index of symmetry of (X,d) is defined by*

$$c(X,d) = \inf_{d(x,y)>0} \frac{d(y,x)}{d(x,y)} \in [0,1].$$

When there is no risk of confusion, we simply write $c(X)$. If (X,d) is an extended quasi-metric space, with at least two points at infinite distance, the index of symmetry will be considered 0*.*

Clearly, the class of quasi-metric spaces with index of symmetry 1 is exactly the class of metric spaces. It is also easy to check that $c(\mathbb{R}, d_u) = 0$ and that $c(\mathbb{R}, \rho_\alpha) = \min\{\alpha, \alpha^{-1}\}$.

The index of symmetry can be used to classify asymmetric normed spaces in terms their duals. A particularly useful result is presented in Subsection 2.3. For an in depth study on the classification of asymmetric normed spaces using their index of symmetry, we refer the reader to [7].

2.2 Semi-Lipschitz functions

This Section deals with the most fundamental concept and tool of this work. Semi-Lipschitz functions are the natural generalization of Lipschitz functions to quasi-metric spaces, albeit not without it subtleties. Before we give the definition of semi-Lipschitz functions, we will first take a closer look at the quasi-metric space of Definition 2.10. This space, which we would continue to denote $(\mathbb{R}, d_u)$, will serve as our "model" quasi-metric space. For convenience of the reader, we recall the definitions of the function u and d_u.

$$u(x) = \max\{x, 0\}, \quad \text{for every } x \in \mathbb{R}. \tag{2.7}$$

$$d_u(x,y) = u(y-x) = \max\{y-x, 0\}, \quad \text{for every } x, y \in \mathbb{R}. \tag{2.8}$$

Mind the order of the variables x and y on the leftmost and rightmost sides of eq. (2.8). The fact that u is an asymmetric hemi-norm on $\mathbb{R}$ implies that order in which we write the difference between y and x (as opposed to $x - y$) does matter. We chose this order because it aligns with the notion that, in a (symmetric) normed linear space, the norm of a vector v represents the length of an "arrow" going from the origin to the position of v. Since we usually write the norm of v as $\|v\|$ instead of $\|-v\|$, we argue that "correct" orientation is the one that results in $d(0, v) = \|v - 0\|$.

Another important aspect of the space $(\mathbb{R}, d_u)$ is that it "splits" the usual metric d of $\mathbb{R}$ in the sense that the symmetrization of d_u coincides with d, and moreover, only one of the terms of the symmetrization is non zero, that is,

$$d(x,y) = \begin{cases} d_u(x,y) & \text{if } y \geq x \\ d_u(y,x) & \text{if } y \leq x \end{cases} \tag{2.9}$$

Moreover, as shown in Proposition 2.12, the forward topology of this quasi-metric space characterizes the usual notion of upper semicontinuity for real-valued functions, in the sense that, for every topological space X, a function $f : X \to \mathbb{R}$ is upper semicontinuous if and only if $f : X \to (\mathbb{R}, d_u)$ is continuous. It can easily be proved that the backward topology $\mathcal{T}(\overline{d_u})$ holds the same property for lower semicontinuous functions.

The quasi-metric space $(\mathbb{R}, d_u)$ will show its relevance once we proceed to define real-valued semi-Lipschitz functions.

As for the general definition of semi-Lipschitz function, it comes as a straightforward generalization of the usual definition of Lipschitz functions between metric spaces.

Definition 2.33 *Let (X, d) and (Z, ρ) be quasi-metric spaces. A function $f : X \to Z$ is said to be semi-Lipschitz if there exists a constant $L \geq 0$ such that for every $x, y \in X$ we have*

$$\rho(f(x), f(y)) \leq Ld(x, y). \tag{2.10}$$

The infimum of all constants satisfying eq. (2.10) is called the *semi-Lipschitz constant* of f, and is denoted by $\mathrm{SLIP}_{d,\rho}(f)$, or simply $\mathrm{SLIP}(f)$ when there is no risk of confusion regarding the quasi-metrics used to compute the constant.

$$\mathrm{SLIP}(f) := \inf \big\{ L > 0 : \ eq.\ (2.10) \text{ holds} \big\}.$$

In addition, we say that a function g is λ-semi-Lipschitz whenever $\mathrm{SLIP}(g) \leq \lambda$.

Proposition 2.34 *Let (X, d) and (Z, ρ) be quasi-metric spaces and let $f : X \to Z$ be a semi-Lipschitz function. Then,*

$$\mathrm{SLIP}(f) = \sup_{d(x,y)>0} \frac{\rho(f(x), f(y))}{d(x, y)}.$$

Remark 2.35 *Notice that, unlike in metric spaces, the conditions "$d(x, y) > 0$" and "$x \neq y$" are not equivalent. However, this two conditions coincide whenever d is a quasi-metric (as opposed to a quasi-hemi-metric).*

Given two quasi-metric spaces (X, d) and (Z, ρ), the set of semi-Lipschitz functions from X to Z will be denoted by $\mathrm{SLIP}\left((X, d), (Z, \rho)\right)$, or simply $\mathrm{SLIP}(X, Z)$ when there is no risk of confusion regarding the respective quasi-hemi-metrics. If X and Z are *pointed* quasi-metric spaces with base points x_0 and z_0, we will consider the subset

$$\mathrm{SLIP}_0(X, Z) = \{f \in \mathrm{SLIP}(X, Z) \ : \ f(x_0) = z_0\}.$$

Normed cones will always be assumed to have their respective origin as base point. The special case when $Z = \mathbb{R}$ shall be addressed later.

Proposition 2.36 *Let (X, ρ) be a quasi-metric space with index of symmetry $c(X) = \alpha > 0$. Then, for every quasi-metric space (Z, d), every semi-Lipschitz function $f : X \to Z$ is $\alpha^{-1}\mathrm{SLIP}_{\bar{\rho},d}(f)$-semi-Lipschitz. In other words,*

$$\mathrm{SLIP}_{\bar\rho,d}(f) \le \alpha^{-1}\mathrm{SLIP}_{\rho,d}(f).$$

Moreover, we have

$$\alpha\mathrm{SLIP}_{\bar\rho,d}(f) \le \mathrm{SLIP}_{\rho,d}(f) \le \alpha^{-1}\mathrm{SLIP}_{\bar\rho,d}(f).$$

Proof. The first inequality follows directly by taking $f \in \mathrm{SLIP}_{\rho,d}(X,Z)$, writing the semi-Lipschitz inequality and using the index of symmetry of X to bound ρ with $\bar\rho$. The same argument yields the remaining inequality. □

Remark 2.37 *This proposition holds regardless of the symmetry index of the target space* (Z,d).

The next step is to define real-valued semi-Lipschitz functions in a useful manner. A first approach could be to endow $\mathbb{R}$ with its usual metric d and see what happens when we apply it to Definition 2.33. Let (X,ρ) be a quasi-metric space, and let $f:(X,\rho)\to(\mathbb{R},d)$ be a semi-Lipschitz function, which means that for every x,y in X, we have

$$d(f(x),f(y)) \le L\rho(x,y),$$

for some constant $L\ge 0$. Since $d(\alpha,\beta)=|\beta-\alpha|$, we can replace f by $-f$ in the inequality above, obtaining

$$d(-f(x),-f(y)) \le L\rho(y,x),$$

which implies the function $-f$ is also semi-Lipschitz, with the same constant as f. This choice of quasi-metric for $\mathbb{R}$ would lead us to a set of semi-Lipschitz functions $\mathrm{SLIP}(X,\mathbb{R})$ which does not reflect any of the possible asymmetries present in the space (X,ρ), which is not very promising for our purposes, as we seek to utilize this class of functionals to study the quasi-metric structure of (X,ρ). Nevertheless, we shall keep this notion under a different name.

Definition 2.38 *We say a function $f:(X,\rho)\to(M,d)$ from a quasi-metric space (X,ρ) into a metric space (M,d) is* ***Lipschitz*** *if it is semi-Lipschitz in the sense of Definition 2.33. In this case, both semi-Lipschitz constants $\mathrm{SLIP}_{\rho,d}(f)$ and $\mathrm{SLIP}_{\bar\rho,d}(f)$ coincide, and we denote it instead by $\mathrm{LIP}(f)$, which will be referred as the Lipschitz constant of f. When $(M,d)=(\mathbb{R},|\cdot|)$, we adopt the notation $\|f\|_{\mathrm{LIP}}=\mathrm{LIP}(f)$.*

A more interesting approach is to endow $\mathbb{R}$ with a quasi-metric, and by the arguments presented at the beginning of this Section, the natural candidates are d_u and its reverse, $\overline{d_u}$. The question then becomes, which one to choose? We are not aware of any discussion regarding this choice in the literature, besides the one presented by the author and his collaborators in [17] and [18]. To the best of our knowledge, all other sources implicitly use the quasi-metric $\overline{d_u}$ on $\mathbb{R}$, resulting in inequalities usually written as follows, for a function $f:(X,\rho)\to\mathbb{R}$ from a quasi-metric space (X,ρ),

$$f(x)-f(y) \le L\rho(x,y). \tag{2.11}$$

Perhaps this inequality was first introduced by simply removing the absolute value from the usual inequality for real-valued Lipschitz functions, but if we rewrite it in the language of quasi-metric spaces using the conic-hemi-norm u, we obtain

$$\begin{aligned} f(x) - f(y) \leq L\rho(x,y) \quad &\iff \quad u(f(x) - f(y)) \leq L\rho(x,y) \\ &\iff \quad d_u(f(y), f(x)) \leq L\rho(x,y) \\ &\iff \quad \overline{d_u}(f(x), f(y)) \leq L\rho(x,y). \end{aligned}$$

In other words, $f : (X, \rho) \to (\mathbb{R}, \overline{d_u})$ is semi-Lipschitz according to Definition 2.33.

The choice between using d_u and $\overline{d_u}$ may seem arbitrary, due to the fact that a function $f : (X, \rho) \to (\mathbb{R}, d_u)$ is semi-Lipschitz according to Definition 2.33 if and only if it is semi-Lipschitz on $(X, \bar{\rho})$ according to (2.11). This is also equivalent to $-f$ being semi-Lipschitz on (X, ρ) according to (2.11). Therefore, the difference between using d_u or $\bar{d_u}$is equivalent to either a change of orientation of the quasi-metric (replace ρ by $\bar{\rho}$) or of the sign of the values of f (replace f by $-f$). In view of this, it could seem that the reasonable thing to do would be to adopt the same convention that has been used for decades in the literature. Nevertheless, this poses practical disadvantages that justify our choice of orientation for Definition 2.33:

(i) If $(X, \| \cdot |)$ is a normed cone, the norm $\| \cdot |$ may not be semi-Lipschitz according to (2.11), while $-\| \cdot |$ is always semi-Lipschitz according to (2.11).

(ii) In general, if (X, ρ) is a quasi-metric space, the functions of the form $\rho(x_0, \cdot)$ that characterize forward convergence (in the sense that $\{x_n\}_n \longrightarrow x_0$ in the forward topology if and only if $\rho(x_0, x_n) \longrightarrow 0$) may not be semi-Lipschitz according to (2.11), while $-\rho(x_0, \cdot)$ and $\rho(\cdot, x_0)$ will be so.

(iii) The notion of dual space of a normed cone (or asymmetric normed space) currently used in the literature is based on the conic-hemi-norm u, in such a way that the real-valued linear functions belonging to said dual spaces (which will be defined in Section 2.3) are always semi-Lipschitz when endowing $\mathbb{R}$ with the quasi-hemi-metric d_u (with respect to Definition 2.33).

Therefore, if we wish for the theory of quasi-metric spaces to be fully compatible with the theory of normed cones and asymmetric normed spaces, the definition of real-valued semi-Lipschitz functions must follow the orientation given in Definition 2.33. From now on, **whenever we write a semi-Lipschitz inequality for a real-valued function $f : (X, d) \to \mathbb{R}$, we will simply write**

$$f(y) - f(x) \leq Ld(x, y), \tag{2.12}$$

where the conic hemi-norm $u(\cdot)$ is omitted on the left side of (2.12), as $t = u(t)$ for any $t \in \mathbb{R}_+$ and the inequality is trivially satisfied whenever the left side is negative.

A criterion to determine whether a function is semi-Lipschitz can be established.

Proposition 2.39 *Let (X, d) be a quasi-metric space and $f : X \to \mathbb{R}$.*

(i) If d is a quasi-metric, then f is semi-Lipschitz if and only if

$$\mathrm{SLIP}(f) = \sup_{x \neq y} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} = \sup_{x \neq y} \frac{f(x) - f(y)}{d(y,x)} < \infty.$$

(ii) If d is a quasi-hemi-metric, then f is semi-Lipschitz if and only if $\mathrm{SLIP}(f) < \infty$*. In this case,*

$$\mathrm{SLIP}(f) = \sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} = \sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y,x)}.$$

Remark 2.40 *Let* (X,d) *be a quasi-metric space and* $f : X \to \mathbb{R}$*. If for all* $x, y \in X$ *we have* $f(x) \leq f(y)$ *whenever* $d(y,x) = 0$ *(usually called d-monotonicity), then the following equality holds:*

$$\sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} = \sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y,x)}. \tag{2.13}$$

It follows readily from Definition 2.33 that every semi-Lipschitz function is d-monotonic, and therefore it satisfies (2.13)*.*

Example 2.41 *(i) If* $f : X \to \mathbb{R}$ *is not semi-Lipschitz or d-monotonic, then the equality* (2.13) *is not necessarily true. For example, let* $X = \{a,b\}$ *with* $a, b \in \mathbb{R}$*, consider* $d : X \times X \to [0,\infty)$ *the quasi-hemi-metric given by* $d(a,b) = 1$ *and* $d(b,a) = 0$*, and let* $f : X \to \mathbb{R}$ *defined as* $f(a) = 1$ *and* $f(b) = 0$*. Then f is not semi-Lipschitz,*

$$\sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y,x)} = -1 \quad \text{and} \quad \sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} = 0.$$

(ii) The equality (2.13) *could be true without f being semi-Lipschitz. For instance, let* $X = \{a,b,c\}$ *with* $a, b, c \in \mathbb{R}$*, consider* $d : X \times X \to [0,\infty)$ *the quasi-hemi-metric given by*

$$d(x,y) = \begin{cases} 1, & \text{if } x = a, y = b \\ 1, & \text{if } x = b, y = c \\ 2, & \text{if } x = a, y = c \\ 0, & \text{otherwise} \end{cases},$$

and let $f : X \to \mathbb{R}$ *defined as* $f(a) = 2$*,* $f(b) = 1$ *and* $f(c) = 1$*. Then f is not semi-Lipschitz, since* $f(a) - f(b) = 1$ *and* $d(b,a) = 0$*. However,*

$$\sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y,x)} = 0 \quad \text{and} \quad \sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} = 0.$$

Now that we have settled the definition of semi-Lipschitz functionals, we can fix the notation for the space of real-valued semi-Lipschitz functions. For any quasi-metric space (X,d), we write $\mathrm{SLIP}(X)$ (respectively $\mathrm{SLIP}_0(X)$) to denote the space $\mathrm{SLIP}(X,\mathbb{R})$ (respectively $\mathrm{SLIP}_0(X,\mathbb{R})$), where $\mathbb{R}$ is endowed with its usual quasi-metric d_u. From this point forward, we will use the symbol $\| \cdot |_S$ to denote the semi-Lipschitz constant of a real-valued function. The reason, as the notation suggests, is the following.

Proposition 2.42 *Let (X, d) be a pointed quasi-metric space. Then, $(\mathrm{SLIP}_0(X), \| \cdot |_S)$ is a cancellative normed cone.*

Proposition 2.43 *Let (X, d) be a quasi-metric space. Then, a function $f : X \to \mathbb{R}$ is Lipschitz (in the sense of Definition 2.38) if and only if both f and $-f$ are semi-Lipschitz.*

Remark 2.44 *In general, $\mathrm{SLIP}(X)$ and $\mathrm{SLIP}_0(X))$ need not be linear spaces. This can be seen by taking $(X, d) = (\mathbb{R}, d_u)$, and considering the function $f(x) = d_u(0, x)$, which belongs to $\mathrm{SLIP}_0(X)$, while $-f$ does not. A more nuanced example will be given in Chapter 3 (see forthcoming Example 3.8). Notice that we can always define the (possibly trivial) subcone of Lipschitz functions, which is always a linear space.*

Next, we state some notable properties of semi-Lipschitz functionals that will be useful later. Some proofs are omitted, as they follow directly from the definitions involved.

Proposition 2.45 *Let (X, d) ,(Y, q) and (Z, ρ) be quasi-metric spaces. The following properties hold.*

- *Every semi-Lipschitz function is continuous for the respective forward topologies.*
- *If $f \in \mathrm{SLIP}(X, Y)$ and $g \in \mathrm{SLIP}(Y, Z)$, then $g \circ f$ belongs to $\mathrm{SLIP}(X, Z)$.*
- $\mathrm{SLIP}_{d,q}(X, Y) \subseteq \mathrm{SLIP}_{d^s,q}(X, Y) = \mathrm{LIP}_{d^s,q^s}(X, Y)$.

Remark 2.46 *The last property of Proposition 2.45 can be interpreted as "semi-Lipschitz functions are always Lipschitz in the symmetrized spaces".*

Proposition 2.47 *Let (X, ρ) be a quasi-metric space, and let $f : (X, \rho) \to (\mathbb{R}, d_u)$ be a semi-Lipschitz function. Then, f is upper-semicontinuous with respect to the forward topology of (X, ρ).*

Proof. It follows directly from Proposition 2.45 and the fact that the forward topology of $(\mathbb{R}, d_u)$ characterizes upper semi-continuity (see Proposition 2.12). □

Proposition 2.48 *Let (X, ρ) be a quasi-metric space. Then, for every $x \in X$, the function $\rho(x, \cdot) : (X, \rho) \to (\mathbb{R}, d_u)$ is semi-Lipschitz and has semi-Lipschitz constant equal to 1.*

Proof. Let $y, z \in X$. By the triangular inequality, $\rho(x, y) \le \rho(x, z) + \rho(z, y)$, which can be rewritten as

$$\rho(x, y) - \rho(x, z) \le \rho(z, y),$$

which implies $\rho(x, \cdot)$ is semi-Lipschitz of constant less or equal to 1 for any $x \in X$. Moreover, the constant is achieved when $z = x$. □

Proposition 2.49 *Real-valued semi-Lipschitz functions are stable with respect to the max/min operations. Moreover, if* $f, g \in \mathrm{SLIP}(X)$ *for a quasi-metric space* X*, then*

$$\max\{\mathrm{SLIP}(f \wedge g), \mathrm{SLIP}(f \vee g)\} \leq \max\{\mathrm{SLIP}(f), \mathrm{SLIP}(g)\}.$$

Proof. Let (X, d) be a quasi-metric space, and let $f, g \in \mathrm{SLIP}(X)$. Let us consider the function $h = f \vee g$ and two points $x, y \in X$. Without loss of generality, we may assume $f(x) \geq g(x)$. Then,

$$\begin{aligned} h(x) - h(y) &= f(x) \vee g(x) - [f(y) \vee g(y)] \\ &= f(x) - [f(y) \vee g(y)] \\ &\leq f(x) - f(y) \\ &\leq \mathrm{SLIP}(f) d(y, x) \\ &\leq \max\{\mathrm{SLIP}(f), \mathrm{SLIP}(g)\} d(y, x). \end{aligned}$$

Notice that the last inequality holds regardless of our assumption on the values of $f(x)$ and $g(x)$. It follows that $h = f \vee g$ is semi-Lipschitz of constant at most $\max\{\mathrm{SLIP}(f), \mathrm{SLIP}(g)\}$. A similar argument can be made for the function $f \wedge g$. □

Using this property, we have in particular the following definition.

Definition 2.50 *Let* (X, d) *be a quasi-metric space. The space of real-valued semi-Lipschitz functions with semi-Lipschitz constant less or equal to 1 is denoted by*

$$\mathrm{SLIP}_1(X) = \{f : X \to \mathbb{R} \, : \, \mathrm{SLIP}(f) \leq 1\}.$$

Thanks to Proposition 2.49, we know $\mathrm{SLIP}_1(X)$ has a natural lattice structure. Furthermore, it is also closed under convex combinations. Thus, following [14], we say that $\mathrm{SLIP}_1(X)$ has a *convex lattice* structure. If (Y, ρ) is another quasi-metric space, we say that a bijection $T : \mathrm{SLIP}_1(Y) \to \mathrm{SLIP}_1(X)$ is a *convex lattice isomorphism* if T preserves both order and convex combinations, that is,

- $Tf \geq Tg$ if and only if $f \geq g$ for all $f, g \in \mathrm{SLIP}_1(Y)$, and
- $T(\lambda f + (1 - \lambda)g) = \lambda Tf + (1 - \lambda)Tg$ for all $f, g \in \mathrm{SLIP}_1(Y)$ and $\lambda \in [0, 1]$.

Remark 2.51 *Note that any order-preserving bijection between lattices is automatically a lattice isomorphism, so any convex lattice isomorphism satisfies* $T(f \wedge g) = Tf \wedge Tg$ *and* $T(f \vee g) = Tf \vee Tg$ *for all* $f, g \in \mathrm{SLIP}_1(Y)$.

The following result, taken from [14, Theorem 3.1] (and modified to fit our orientation for semi-Lipschitz functionals) reveals the importance of the convex lattice structure $\mathrm{SLIP}_1(X)$ for the study of the quasi-metric structure of a bicomplete quasi-metric space.

Theorem 2.52 (Representation of almost isometries between quasi-metric spaces) *Let* (X, d) *and* (Y, ρ) *be bicomplete quasi-metric spaces, and let* $T : \mathrm{SLIP}_1(Y) \to \mathrm{SLIP}_1(X)$ *be a convex lattice isomorphism. Then there exist* $\alpha > 0$*, a homeomorphism* $\tau : (X, d) \to (Y, \rho)$ *and a quasi-metric* d' *on* X*, such that*

- (X, d) *and* (X, d') *are almost-isometric, and* $d'(x, x') = d(x, x') + T0(x') - T0(x)$*, where* $T0$ *is the image via* T *of the constant function of value* 0*.*
- $\tau : (X, \alpha \cdot d') \to (Y, \rho)$ *is an isometry.*
- *For every* $f \in \mathrm{SLIP}_1(Y)$ *we have that* $Tf = c \cdot (f \circ \tau) + \phi$*, where* $c = \alpha^{-1}$ *and* $\phi = T0$*.*

Therefore, two bicomplete quasi-metric spaces are almost isometric up to a multiplicative constant whenever the respective spaces of real-valued 1-semi-Lipschitz functions are *isomorphic as convex lattices*, and that the isomorphism is a composition operator associated with the almost isomerty.

In Chapter 3, we will show a result of similar flavor to the above, for the case that the quasi-metric spaces are *Finsler manifolds* (see forthcoming Definition 2.66).

2.3 Duality on normed cones and asymmetric normed spaces

If one wants to define duality between normed cones or asymmetric normed spaces, the natural notion would be to consider a space of real-valued functions which preserve the structure of the normed cone, which encompasses algebraic and quasi-metric structures, as well as the relation between both. Therefore, we need functionals that are both conic and quasi-metric morphisms. The following proposition helps us characterize said functionals.

Proposition 2.53 (Linear functionals over a normed cone) *Let* $(C, \| \cdot |)$ *be a normed cone and* $\varphi : C \to \mathbb{R}$ *a linear functional. Then the following are equivalent:*

(i) φ *is upper semicontinuous (in short, usc);*

(ii) φ *belongs to* $\mathrm{SLIP}_0(C, d_e)$*, where* d_e *is the (extended) quasi-metric induced by the conic-norm* $\| \cdot |$ *(c.f. Proposition 2.24);*

(iii) *there exists* $M \geq 0$ *such that* $\varphi(x) \leq M \|x|$*, for all* $x \in C$*.*

Proof. Let us show that (i) implies (iii). Assume that the linear functional φ is usc. Then there exists $\alpha > 0$ such that $\varphi\left(B(0, \alpha)\right) \subseteq (-\infty, 1)$. Set $M = 2/\alpha$. Then for every $x \in C$ with $\|x| \neq 0$, we have $\tilde{x} = \frac{\alpha x}{2\|x|} \in B(0, \alpha)$, hence $\varphi\left(\tilde{x}\right) < 1$ and $\varphi(x) < M\|x|$. If $x \in C$ with $\|x| = 0$, then for every $r > 0$ we have $\|rx| = 0$ and $\varphi(rx) < 1$, which implies $\varphi(x) < \frac{1}{r}$ and necessarily $\varphi(x) \leq 0$.

Let us now show that (iii) implies (ii). We need to establish the inequality $\varphi(x) - \varphi(y) \leq L\, d_e(y, x),\ \forall x, y \in C$, for some $L \geq 0$. If $d_e(y, x) = \infty$, the inequality becomes trivial. If not, then $x \in y + C$, so we can write $x = y + z$, and then $\varphi(x) - \varphi(y) = \varphi(z) \leq M\|z|$. By taking infimum of all z such that $x = y + z$, we get that $\varphi(x) - \varphi(y) \leq M d_e(y, x)$, that is, φ is semi-Lipschitz.

Let us finally assume (ii) and recall that the forward topology on $(C, \|\cdot|)$ is first countable. Then take $\{x_n\}_n \subseteq C$ such that $d_e(x, x_n) \to 0$. Since φ is semi-Lipschitz, we have $\varphi(x_n) - \varphi(x) \leq L d_e(x, x_n)$ for some $L \geq 0$, which yields that $\varphi(x) \geq \limsup \varphi(x_n)$. □

Remark 2.54 *Each one of the above statements is also equivalent to φ being lower semi-continuous (in short, lsc) for the reverse extended quasi-metric $\bar{d}_e$:*

Indeed, assume there exists $M \geq 0$ such that $\varphi(x) \leq M\|x|$ for all $x \in C$, and consider a sequence $\{z_n\}_n$ and z in C such that $\bar{d}_e(z, z_n) \to 0$. Then $d_e(z_n, z) \to 0$, which yields the existence of a sequence $\{y_n\}_n \subset C$ such that $y_n + z_n = z$ and $\|y_n| \to 0$. Since φ is linear, $\varphi(z) = \varphi(z_n) + \varphi(y_n) \leq \varphi(z_n) + M\|y_n|$, which yields that φ is lsc for $\bar{d}_e$.

On the other hand, if φ is lsc for $\bar{d}_e$, an analogous argument to Proposition 2.53 ((i) $\Longrightarrow$ (iii)) leads to the same conclusion, that is, the existence of $M \geq 0$ such that $\varphi(x) \leq M\|x|$ for all $x \in C$.

Definition 2.55 (Dual normed cone) *Let $(C, \|\cdot|)$ be a normed cone. We define the* dual cone *of C as*

$$C^* := \{\varphi : C \to \mathbb{R} : \varphi \text{ usc, linear}\} = \{\varphi \in \mathrm{SLIP}_0(C) : \varphi \text{ linear}\}.$$

For any $\varphi \in C^*$, the dual conic-norm *is defined by*

$$\|\varphi|^* := \sup_{\|x|\leq 1} \max\{\varphi(x), 0\} = \sup_{\|x|\leq 1} \varphi(x).$$

It is easy to check that $\|\cdot|^*$ is a conic-norm on C^* (obviously $\|\varphi|^* \geq 0$, since $\varphi(0) = 0$). Moreover, if $(C, \|\cdot|)$ is a normed cone with conic-hemi-norm, then $\|\cdot|^*$ is a conic-hemi-norm on C^*.

The next result follows directly from the proof of Proposition 2.53.

Proposition 2.56 *Let $(C, \|\cdot|)$ be a normed cone, and $\varphi \in C^*$. Then*

$$\|\varphi|^* = \inf\{M > 0 : \varphi(x) \leq M\|x|, \text{ for all } x \in C\}.$$

As in the case of normed spaces, there is a direct relation between the semi-Lipschitz constant and the dual norm of a linear functional:

Corollary 2.57 (Dual conic-norm and semi-Lipschitz constant) *Let $(C, \|\cdot|)$ be a normed cone, and $\varphi \in C^*$. Then $\|\varphi|^* = \|\varphi|_S$ and the subcone of linear functionals of* $\mathrm{SLIP}_0(C)$ *(linear semi-Lipschitz functions) is isometrically isomorphic to $(C^*, \|\cdot|^*)$ (linear usc functions).*

Proof. The inequality $\|\varphi|_S \leq \|\varphi|^*$ follows from Proposition 2.53 (see (ii)⇒(iii)). For the opposite inequality, since φ is semi-Lipschitz and $\varphi(0) = 0$ we get:

$$\varphi(x) = \varphi(x) - \varphi(0) \leq \|\varphi|_S \, d_e(0, x) = \|\varphi|_S \, \|x|,$$

yielding by Proposition 2.56 that $\|\varphi|^* \leq \|\varphi|_S$. The proof is complete. □

A very curious fact about asymmetric normed spaces is that, in general, their duals are not linear spaces. In fact, it was proved in [7] (see Corollary 2) that having a linear dual cone is exclusive to asymmetric normed spaces with positive index of symmetry.

Proposition 2.58 *Let $(E, \|\cdot|)$ be an asymmetric normed space. Then, the dual cone E^* is a linear space if and only if $c(E) > 0$.*

A non linear version of this result will be presented in Chapter 4 (see forthcoming Proposition 4.19.

Remarkably, this property does not hold for normed cones, in the sense that a normed cone $(C, \|\cdot|)$ can have index of symmetry $c(C) = 0$ and still have its dual C^* be a linear space.

Example 2.59 *Consider the asymmetric normed space $(\mathbb{R}, u)$. It is easy to check that the dual cone of $(\mathbb{R}, u)$ is isometrically isomorphic to the cone $(\mathbb{R}_+, u)$, which is not a linear space. If we compute the dual cone of $(\mathbb{R}_+, u)$, we find out that it is isometrically isomorphic to the original asymmetric normed space $(\mathbb{R}, u)$. Therefore, we have found a normed cone $(\mathbb{R}_+, u)$ of symmetry index 0, whose dual cone is a linear space (of index 0 as well).*

It is worth noting that the index of symmetry is not very well suited to study normed cones, as every normed cone which is not a linear space has symmetry index 0.

Proposition 2.60 *Let $(C, \|\cdot|)$ be a normed cone with non-zero symmetry index. Then C is a linear space.*

Proof. Choose $v \in C$ such that it does not have an additive inverse in C. If we try to compute $d_e(v, 0)$, we see that the infimum used in the definition of the extended quasi-metric is taken over an empty set, and therefore, $d_e(v, 0) = +\infty$. □

For T_1 and finite dimensional asymmetric normed spaces, the situation is much simpler.

Proposition 2.61 (Dual of a finite-dimensional linear space) *Let $(E, \|\cdot|)$ be a T_1 asymmetric normed space of finite dimension. Then there exists $M > 0$ such that*

$$\|{-x}| \leq M\|x|, \text{ for all } x \in E. \tag{2.14}$$

Furthermore, $(E, \|\cdot|)^*$ is also an asymmetric normed space satisfying that for every $\varphi \in (E, \|\cdot|)^*$, $-\varphi \in (E, \|\cdot|)^*$ and $\| - \varphi|^* \leq M\|\varphi|^*$. In particular, $(E^*, \|\cdot|)$ is a linear space (not only a normed cone).

Proof. Let $B = \{x \in E : \|x| \leq 1\}$ be the unit ball of E. Since in finite dimensions all asymmetric norms inducing a T_1-topology are equivalent (see [22, Corollary 11] or [7, Theorem 3] for example), it follows that B is closed convex and $0 \in \operatorname{int} B$. Thus we can assure the existence of $M > 0$ such that $\left\| \frac{-x}{\|x|} \right| \leq M$, for all $x \in E$ with $\|x| \neq 0$, which yields $\|-x| \leq M\|x|$, for all $x \in E$. Now, if $\varphi \in (E, \| \cdot |)^*$ then

$$-\varphi(x) = \varphi(-x) \leq \|\varphi|^* \|-x| \leq M \|\varphi|^* \|x|, \text{ for all } x \in E$$

and

$$\|-\varphi|^* \left(= \sup_{\|x| \leq 1} -\varphi(x) \right) \leq M \|\varphi|^*, \text{ for all } \varphi \in (E, \| \cdot |)^*.$$

The proof is complete. □

Remark 2.62 *If E is infinite-dimensional, then* (2.14) *may not be fulfilled. For example, let*

$$E = \{f \in \mathcal{C}([0,1]) : \int_0^1 f(t)\, dt = 0\}$$

and $\|f| := \max_{t \in [0,1]} \max\{f(t), 0\}$. *Consider the sequence of functions $\{f_n\}_n \subset E$ defined as*

$$f_n(x) = \begin{cases} \frac{1}{n}\,, & \text{if } \ 0 \leq x < \frac{1}{n^2} \\ \frac{n}{2-n^2}x + \frac{1-n^2}{2n-n^3}\,, & \text{if } \ \frac{1}{n^2} \leq x < 1 - \frac{1}{n^2} \\ -n^3 x - n(1-n^2)\,, & \text{if } \ 1 - \frac{1}{n^2} \leq x \leq 1 \end{cases} \quad (n \in \mathbb{N}).$$

Then $\|f_n| = 1/n$ for each $n \geq 2$ and $\|-f_n| = n \longrightarrow \infty$, which contradicts (2.14).

In addition, E^* is a normed cone (and not a vector space). To see this, let $\delta_1 : \mathcal{C}([0,1]) \to \mathbb{R}$ be defined as $\delta_1(f) = f(1)$. Then $\{f_n\}_n \longrightarrow 0$, $\delta_1(f_n) = -n \longrightarrow -\infty$ and $\delta_1(0) = 0$, which shows that the linear functional δ_1 is not lower semicontinuous in $(E, \| \cdot |)$.

Next, we present some results regarding weak topologies in asymmetric normed spaces.

Remark 2.63 (Continuity of evaluation functionals) *Let $(E, \| \cdot |)$ be an asymmetric normed space with dual E^*. For every $x \in E$, the evaluation functional $\widehat{x} : E^* \to \mathbb{R}$ defined as $\widehat{x}(\varphi) = \varphi(x)$ is linear and $\| \cdot |^*$-continuous. Indeed, we have*

$$\widehat{x}(\varphi) = \varphi(x) \leq \|\varphi|^* \|x| \quad \textit{and} \quad -\widehat{x}(\varphi) = -\varphi(x) = \varphi(-x) \leq \|\varphi|^* \|-x|,$$

which yields $|\widehat{x}(\varphi)| \leq \max\{\|x|, \|-x|\} \|\varphi|^*$, which implies $\widehat{x}$ is continuous.

Definition 2.64 (Asymmetric weak topologies) *Let E be an asymmetric normed space with dual E^*.*

(i) The weak *topology w on E is defined as the coarsest topology for which every $\phi \in E^*$ remains upper semicontinuous.*

(ii) The weak-star *topology w^* on E^* is defined as the coarsest topology that makes every evaluation functional $\{\widehat{x} : E^* \to (\mathbb{R}, | \cdot |),\ x \in E\}$ continuous (notice by Remark 2.63 that $\widehat{x}$ is always $\| \cdot |^*$-continuous, where $\| \cdot |^*$ is the conic norm of E^*).*

Therefore the weak-star topology w^* on E^* is weaker than the forward $\|\cdot|^*$-topology.

Some results regarding duality of asymmetric normed spaces that were not found in the literature, and were developed as part of this Thesis, will be presented in Subsection 4.1 of Chapter 4.

2.4 Finsler Manifolds

One of the most studied examples of quasi-metric spaces are *Finsler Manifolds*, due to the variety of physical phenomena that can be modeled using them. For more details on this, we refer the reader to the first chapter of the book [8].

Definition 2.65 (Minkowski norm) *Let E be a finite-dimensional real vector space. A functional $F : E \to [0, +\infty)$ is called a* Minkowski norm *on E if the following conditions are satisfied:*

(i) Positive homogeneity: $F(\lambda v) = \lambda F(v)$ for every $v \in E$ and $\lambda \geq 0$.

(ii) Regularity: F is continuous on E and C^∞-smooth on $E \setminus \{0\}$.

(iii) Strong convexity: for every $v \in E \setminus \{0\}$, the quadratic form associated to the second derivative of the function F^2 at v, that is,

$$g_v = \frac{1}{2} d^2[F^2](v),$$

is positive definite on E.

Every Minkowski norm satisfies in addition the following conditions (see [8, Theorem 1.2.2] e.g.):

(iv) Positivity: $F(v) = 0$ if and only if $v = 0$.

(v) Triangle inequality: $F(u + v) \leq F(u) + F(v)$, for every $u, v \in E$.

Thus, a Minkowski norm is an asymmetric norm in the sense of Definition 2.2. We say F is *symmetric* (or *absolutely homogeneous*) if

$$F(\lambda v) = |\lambda| F(v) \text{ for any } \lambda \in \mathbb{R} \text{ and } v \in E,$$

in which case F is a norm in the usual sense.

It is clear that every norm associated to an inner product is a Minkowski norm. In general, a Minkowski norm does not need to be symmetric, and there are indeed very interesting examples of asymmetric Minkowski norms, such as, for example, Randers spaces ([8]) or more generally Finsler manifolds.

Definition 2.66 (Finsler manifold) *A Finsler manifold is a pair* $(\mathcal{X}, F)$ *such that* $\mathcal{X}$ *is a finite-dimensional* C^∞*-smooth manifold and* $F : T\mathcal{X} \to [0, \infty)$ *is a continuous function defined on the tangent bundle* $T\mathcal{X}$*, satisfying*

(i) F *is a* C^∞*-smooth on* $T\mathcal{X} \setminus \{0\}$.

(ii) For every $x \in \mathcal{X}$, $F(x, \cdot) : T_x\mathcal{X} \to [0, \infty)$ *is a Minkowski norm on the tangent space* $T_x\mathcal{X}$.

The Finsler structure F is said to be *reversible* if, for every $x \in \mathcal{X}$, $F(x, \cdot)$ is symmetric. Clearly, any Riemannian manifold is a reversible Finsler manifold, where the symmetric Minkowski norm on each tangent space is given by an inner product.

Definition 2.67 (Finsler distance d_F) *Let* $(\mathcal{X}, F)$ *be a connected Finsler manifold. The* Finsler distance d_F *on* $\mathcal{X}$ *is defined by*

$$d_F(x, y) = \inf\{\ell_F(\sigma) : \sigma \text{ is a piecewise } C^1 \text{ path from } x \text{ to } y\},$$

where the Finsler length of a piecewise C^1 *path* $\sigma : [a, b] \to \mathcal{X}$ *is defined as:*

$$\ell_F(\sigma) = \int_a^b F(\sigma(t), \dot{\sigma}(t))dt,$$

where $\dot{\sigma}$ *is the derivative of* σ*. The Finsler distance* d_F *is a* T_1*-quasi-metric on* $\mathcal{X}$ *for any connected Finsler manifold* $(\mathcal{X}, F)$ *(see e.g. [8, Section 6.2]).*

Remark 2.68 (Topology of a Finsler manifold) *Even if the forward and backward distances of a connected Finsler manifold* $\mathcal{X}$ *differ, they do induce the same topology on* $\mathcal{X}$*, which coincides with the manifold topology (see [8, Chapter 6.2]). Therefore, for Finsler manifolds, the three topologies of Definition 2.9 are the same.*

Definition 2.69 (Finsler isometry) *A mapping* $\tau : (\mathcal{X}, F) \to (\mathcal{Y}, G)$ *between Finsler manifolds is said to be a* Finsler isometry *if it is a diffeomorphism which preserves the Finsler structure, that is, for every* $x \in \mathcal{X}$ *and every* $v \in T_x\mathcal{X}$*:*

$$F(x, v) = G(\tau(x), d\tau(x)(v)).$$

A classical result due to Myers and Steenrod [35] asserts that a mapping between Riemannian manifolds is a Riemannian isometry if and only if it is a metric isometry for the corresponding Riemannian distances. This was extended by Deng and Hou in [20] to the context of Finsler manifolds:

Theorem 2.70 (Characterization of isometries for Finsler manifolds) *Let* $(\mathcal{X}, F)$ *and* $(\mathcal{Y}, G)$ *be connected Finsler manifolds. Then* $\tau : (\mathcal{X}, F) \to (\mathcal{Y}, G)$ *is a Finsler isometry if and only if it is bijective and an isometry for the corresponding Finsler distances.*

A weaker result, established in [30] (see Lemma 3.1 and Proposition 3.2 therein), holds for almost isometries. Given a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ and a Finsler structure F on $\mathcal{X}$, we denote by $\tau_*(F)$ the Finsler structure on $\mathcal{Y}$ obtained as the push-forward of F by τ, that is, for every $y \in \mathcal{Y}$ and every $w \in T_y\mathcal{Y}$:

$$\tau_*(F)(y, w) = F(\tau^{-1}(y), d\tau^{-1}(y)(w)).$$

Proposition 2.71 (Characterization of almost isometries for Finsler manifolds) *Let $(\mathcal{X}, F)$ and $(\mathcal{Y}, G)$ be connected Finsler manifolds, and let $\tau : \mathcal{X} \to \mathcal{Y}$ be an almost isometry induced by a function $\phi : \mathcal{X} \to \mathbb{R}$ (in the sense of Proposition 2.7). Then τ and ϕ are smooth, and $G = \tau_*(F) - d(\phi \circ \tau^{-1})$. Conversely, if $G = \tau_*(F) - d(\phi \circ \tau^{-1})$, then τ is an almost isometry.*

In what follows, for simplicity, the term Finsler manifold will also refer to the pair $(\mathcal{X}, d_{\mathcal{X}})$, where $(\mathcal{X}, F)$ is a Finsler Manifold and $d_{\mathcal{X}}$ is the Finsler distance induced by F.

A remarkable property of Finsler manifolds is that Lipschitz functions can be approximated using functions of class C^1. The following result [25, Theorem 8] ensures this property.

Theorem 2.72 (Smooth approximation of Lipschitz functions in Finsler manifolds) *Let $(\mathcal{X}, F)$ be a connected, second countable Finsler manifold, $f : \mathcal{X} \to \mathbb{R}$ a Lipschitz function (in the sense of Definition 2.38), $\varepsilon : \mathcal{X} \to (0, +\infty)$ a continuous function and $r > 0$. Then, there exists a C^1-smooth Lipschitz function $g : \mathcal{X} \to \mathbb{R}$ such that:*

(i) $|g(x) - f(x)| \leq \varepsilon(x)$ *for all* $x \in \mathcal{X}$ *;*

(ii) $\|g\|_{\text{LIP}} \leq \|f\|_{\text{LIP}} + r$.

2.5 Lipschitz-free spaces

Lipschitz-free Banach spaces, also known by the name of Arens-Eells spaces [6], have been an active field of research in the areas of functional analysis and non linear geometry of Banach spaces. The surge in interest on this object was initiated with the seminal paper of Godefroy and Kalton [29], and has continued to this day. In this brief section, we give the basic definitions and most fundamental properties of Lipschitz-free spaces. For an in depth review on the subject, we refer the reader to the book [42].

Let (M, d) be a metric space with a distinguished point $x_0 \in M$. We will refer to M as a pointed metric space. For each pointed metric space, we can define its associated space of real-valued Lipschitz functions.

$$\text{LIP}_0(M) = \{f : X \to \mathbb{R} \ : \ f \text{ is Lipschitz and } f(x_0) = 0\}.$$

It is a well known fact that $\text{LIP}_0(M)$ is a Banach space when endowed with the norm $\|f\|_{\text{LIP}}$.

For each $x \in M$, the evaluation functional $\delta_x : \mathrm{LIP}_0(M) \to \mathbb{R}$ belongs to the dual space $\mathrm{LIP}_0(M)^*$. Moreover, the subset of evaluation mappings $\delta(M) \subseteq \mathrm{LIP}_0(M)^*$ is isometric to (M, d).

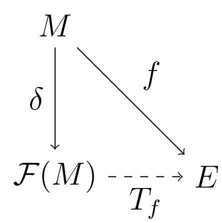
Definition 2.73 *The Lipschitz-free space over a pointed metric space (M, d), denoted by $\mathcal{F}(M)$, is defined as the closed linear span of $\delta(M)$. That is,*

$$\mathcal{F}(M) = \overline{\mathrm{span}}\left(\{\delta_m \,:\, m \in M\}\right) \subseteq \mathrm{LIP}_0(M)^*.$$

Proposition 2.74 *The Lipschitz-free space $\mathcal{F}(M)$ is an isometric predual to* $\mathrm{LIP}_0(M)$*, whose weak-star topology coincides with the topology of point wise convergence on bounded sets.*

One of the main features of the Lipschitz-free spaces is the following "universal property".

Proposition 2.75 *Let (M, D) be a pointed metric space, E a normed space and $f : M \to E$ a Lipschitz functional such that $f(x_0) = 0$. There exists a unique linear and continuous extension $T_f : \mathcal{F}(M) \to E$, such that $\|T_f\| = \|f\|_{\mathrm{LIP}}$ and $T_f \circ \delta = f$. In other words, the extension T_f has the same norm as f, and the following diagram commutes.*



The following definitions, which will be useful for Chapters 4 and 5, are taken from [42].

Definition 2.76 *Let (M, d) be a metric space. We say a function $f : M \to \mathbb{R}$ is locally flat if for every $p \in M$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $a, b \in B_d(p, \delta)$ implies $|f(b) - f(a)| \leq \varepsilon d(a, b)$.*

Definition 2.77 *We say a function $f : M \to \mathbb{R}$ is flat at infinity if for every $\varepsilon > 0$, there exists a compact set $K \subset M$ such that*

$$a, b \notin K \implies |f(b) - f(a)| \leq \varepsilon d(a, b).$$

Definition 2.78 *Let (M, d) be a pointed and boundedly compact metric space. The set of little Lipschitz functions* $\mathrm{lip}(M)$ *consists of all Lipschitz functions which are locally flat and flat at infinity. We denote by* $\mathrm{lip}_0(M)$ *the subset of little Lipschitz functions which vanish at the base point.*

Proposition 2.79 *[42, Proposition 4.7] Let (M, d) be a compact metric space. Then,* $\mathrm{lip}(M)$ *is a lattice and an algebra, and it is also closed under bounded inversions. In particular, for every $f, g \in$* $\mathrm{lip}(M)$*,*

(i) $f \vee g$, $f \wedge g$ and fg belong to $\mathrm{lip}(M)$*,*

(*ii*) *if* f *is bounded away from* 0*, then* $\frac{1}{f}$ *belongs to* $\mathrm{lip}(M)$*.*

We conclude this subsection with a remarkable biduality result for the space $\mathrm{lip}_0(M)$.

Definition 2.80 *Let* (M, d) *be a pointed metric space. We say a linear subspace* E *of* $\mathrm{LIP}_0(M)$ *separates points uniformly if there exists a constant* $C \in (0, 1]$ *such that for every* $p, q \in M$ *there exists a function* $f \in E$ *such that* $\|f\|_{\mathrm{LIP}} leq 1$ *and* $|f(p) - f(q)| = Cd(p, q)$*. The greatest constant* C *satisfying this condition is called the separation factor of* E*.*

Theorem 2.81 *[42, Theorem 4.35] Let* (M, d) *be a pointed complete metric space such that* $\mathrm{lip}_0(M)$ *separates points uniformly. Then,* $\mathrm{lip}_0(M)^*$ *is linearly homeomorphic to* $\mathcal{F}(M)$ *and* $\mathrm{lip}_0(M)^{**}$ *is linearly homeomorphic to* $\mathrm{LIP}_0(M)$*. If* M *is boundedly compact, or the separation factor is* 1*, then both isomorphisms are isometries.*

A recent result gives a geometrical characterization of the uniform separation property. A metric space is called *purely* 1*-unrectifiable* if it contains no bi-Lipschitz images of compact, positive measure subsets of $\mathbb{R}$.

Theorem 2.82 *[3, Theorem A] Let* (M, d) *be a compact metric space. Then,* $\mathrm{lip}_0(M)$ *separates points uniformly if and only if* (M, d) *is purely* 1*-unrectifiable.*

Chapter 3

Almost isometries between Finsler manifolds

This chapter corresponds to the article [17], where a new result on almost isometries between Finsler manifolds was established, using a space of smooth real-valued functions. The chapter is organized as follows: first, some new results regarding smooth semi-Lipschitz functions and almost isometries of Finsler manifolds are presented in Section 3.1. Section 3.2 contains the proof of the main result of this chapter (Theorem 3.14), which is divided in 3 stages in Subsections 3.2.1, 3.2.2 and 3.2.3. Characterization of isometries and almost isometries following Theorem 3.14 are presented in Subsection 3.2.4.

3.1 Smooth semi-Lipschitz functions and almost isometries

We begin this chapter by drawing inspiration from Theorem 2.52, which in broad terms, states that almost isometries between bicomplete quasi-metric spaces X and Y can always be described using the convex lattices $\mathrm{SLIP}_1(X)$ and $\mathrm{SLIP}_1(Y)$. The following proposition shows that semi-Lipschitz functional with constant strictly less than 1 play a special role in terms of the almost isometries they define.

Proposition 3.1 (Characterization of strict almost isometries) *Let $\tau : X \to Y$ be an almost isometry between the quasi-metric spaces (X, d_X) and (Y, d_Y). Let $\phi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be the functions associated to τ and respectively, to τ^{-1} in the sense of Proposition 2.7, that is,*

$$d_Y(\tau(x_1), \tau(x_2)) = d_X(x_1, x_2) + \phi(x_1) - \phi(x_2),$$
$$d_X(\tau^{-1}(y_1), \tau^{-1}(y_2)) = d_Y(y_1, y_2) + \psi(y_1) - \psi(y_2).$$

Then τ is a strict almost isometry if, and only if, $\|\phi|_S < 1$ and $\|\psi|_S < 1$.

Proof. Suppose first that $\tau : X \to Y$ is a strict almost isometry, and consider $c > 1$ such that

$$c^{-1}\, d_X(x,x') \le d_Y(\tau(x),\tau(x')) \le c\, d_X(x,x') \quad \text{for all } x,x' \in X.$$

Since $\phi(x') - \phi(x) = d_X(x,x') - d_Y(\tau(x),\tau(x'))$, whenever $d_X(x,x') > 0$ we have that

$$\frac{\phi(x') - \phi(x)}{d_X(x,x')} = 1 - \frac{d_Y(\tau(x),\tau(x'))}{d_X(x,x')} \le 1 - c^{-1}.$$

Thus $\|\phi|_S \le 1 - c^{-1} < 1$. By considering τ^{-1}, we also obtain that $\|\psi|_S \le 1 - c^{-1} < 1$.

Conversely, let $0 < \alpha < 1$ such that $\|\phi|_S \le \alpha$ and $\|\varphi|_S \le \alpha$. Then for $d_X(x,x') > 0$ we have that

$$\frac{d_Y(\tau(x),\tau(x'))}{d_X(x,x')} = 1 - \frac{\phi(x') - \phi(x)}{d_X(x,x')} \ge 1 - \alpha = \frac{1}{c},$$

where $c = (1-\alpha)^{-1}$. The other inequality follows in the same way. □

As stated on Proposition 2.71, almost isometries between connected Finsler manifolds are always, and the functional ϕ inducing the almost isometry (in the sense of Proposition 2.7) is smooth as well. Given that the objective of this chapter is to study strict almost isometries between Finsler manifolds, Propositions 2.7 and 3.1 point towards the following class of real-valued functions.

Definition 3.2 *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a connected Finsler manifold. The space of C^1-smooth (forward) semi-Lipschitz functions with semi-Lipschitz constant* strictly *less than* 1 *will be denoted by*

$$SC^1_{1^-}(\mathcal{X}) := \{f \in C^1(X,\mathbb{R}) \; : \; \|f|_S < 1\}.$$

When the Finsler manifold $(\mathcal{X}, d_{\mathcal{X}})$ is reversible, we write $C^1_{1^-}(\mathcal{X})$ instead of $SC^1_{1^-}(\mathcal{X})$.

The set $SC^1_{1^-}(\mathcal{X})$ (respectively, the set $C^1_{1^-}(\mathcal{X})$ in the reversible case) is convex and partially ordered, but in contrast to $\mathrm{SLIP}_1(\mathcal{X})$, it is not a lattice, since differentiability is often lost when taking suprema and infima. Therefore, for the study of Finsler manifolds, we shall consider the structure $SC^1_{1^-}(\mathcal{X})$ as a *convex partially ordered* set. We shall now define the notion of isomorphism for the aforementioned structures.

Definition 3.3 (Isomorphism between convex partially ordered sets) *Given connected Finsler manifolds $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$, we say that a bijection*

$$T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X})$$

is an isomorphism of convex partially ordered sets *if*

(i) $Tf \ge Tg$ if and only if $f \ge g$ for all $f,g \in SC^1_{1^-}(\mathcal{Y})$, and

(ii) $T(\lambda f + (1-\lambda)g) = \lambda Tf + (1-\lambda)Tg$ for all $f,g \in SC^1_{1^-}(\mathcal{Y})$ and $\lambda \in [0,1]$.

We shall now define the norm and the asymmetric norm of the derivative $df(x)$ of a smooth function $f \in C^1(\mathcal{X})$, at a point x of a Finsler manifold $\mathcal{X}$.

Definition 3.4 (Norm and asymmetric norm of the derivative $df(x)$) *Let $(\mathcal{X}, F)$ be a connected Finsler manifold and $f : \mathcal{X} \to \mathbb{R}$ a C^1-smooth function. The norm of the derivative of f at the point $x \in \mathcal{X}$ is defined by:*

$$\|df(x)\|_F = \sup\{|df(x)(v)| \;:\; v \in T_x\mathcal{X},\ F(x,v) = 1\}.$$

In the same way, the asymmetric norm of $df(x)$ is defined by:

$$\|df(x)|_F = \sup\{df(x)(v) \;:\; v \in T_x\mathcal{X},\ F(x,v) = 1\}.$$

It is clear that, in the case of a reversible Finsler manifold, the norm and the asymmetric norm of $df(x)$ coincide. In general, we have that $\|df(x)|_F \leq \|df(x)\|_F$.

It is proved in [25, Theorem 5] that, for a C^1-smooth function f defined on a connected Finsler manifold, the Lipschitz constant of f coincides with the supremum of the norm of its derivative. In fact, the same proof of [25, Theorem 5] gives also the corresponding one-sided result:

Proposition 3.5 ($\|f|_S = \|df|_{S,\infty}$) *Let $(\mathcal{X}, F)$ be a connected Finsler manifold and $f : \mathcal{X} \to \mathbb{R}$ a C^1-smooth function. Then*

$$\|f\|_{\mathrm{LIP}} = \|df\|_\infty := \sup\{\|df(x)\|_F \;:\; x \in \mathcal{X}\} \in [0, \infty],$$

where

$$\|f\|_{\mathrm{LIP}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_F(x,y)} \quad \text{is the Lipschitz constant of } f.$$

Similarly,

$$\|f|_S = \|df|_{S,\infty} := \sup\{\|df(x)|_F \;:\; x \in \mathcal{X}\} \in [0, \infty],$$

where $\|f|_S$ is the semi-Lipschitz constant of f with respect to the Finsler quasi-metric $d_\mathcal{X}$.

As a direct consequence we obtain the following alternative description of $SC^1_{1^-}(\mathcal{X})$:

Corollary 3.6 (The convex partially ordered set $SC^1_{1^-}(\mathcal{X})$) *Let $(\mathcal{X}, d_\mathcal{X})$ be a connected Finsler manifold. Then*

$$SC^1_{1^-}(\mathcal{X}) = \{f \in C^1(X, \mathbb{R}) \;:\; \|f|_S < 1\} = \{f \in C^1(X, \mathbb{R}) \;:\; \|df|_{S,\infty} < 1\}.$$

Using the above result, we can easily see that, in the case of compact manifolds, every almost isometry is strict.

Proposition 3.7 (Almost isometries for compact Finsler manifolds) *Let $(\mathcal{X}, F)$ and $(\mathcal{Y}, G)$ be connected and compact Finsler manifolds. Then every almost isometry $\tau : \mathcal{X} \to \mathcal{Y}$ is strict.*

Proof. Consider the function $\phi : \mathcal{X} \to \mathbb{R}$ associated to τ in the sense of Proposition 2.7. By Proposition 2.71 we have that $G = \tau_*(F) - d(\phi \circ \tau^{-1})$. Then for every $x \in \mathcal{X}$ and every $v \in T_x\mathcal{X}$:

$$G(\tau(x), d\tau(x)(v)) = F(x, v) - d\phi(x)(v).$$

As a consequence, if $F(x, v) = 1$, since we have that $d\tau(x)(v) \neq 0$, and then $G(\tau(x), d\tau(x)(v)) > 0$, it follows that $d\phi(x)(v) < 1$.

For every $x \in \mathcal{X}$, the indicatrix $S_x := \{v \in T_x\mathcal{X} \ : \ F(x, v) = 1\}$ is compact. Therefore, for each fixed $x_0 \in \mathcal{X}$ we can choose a compact neighborhood W^{x_0} such that the portion of the indicatrix bundle over W^{x_0} is a compact set. That is, the set

$$B_{x_0} = \{(x, v) \in T\mathcal{X} \ : \ x \in W^{x_0};\ v \in T_x\mathcal{X},\ F(x, v) = 1\}$$

is compact, and furthermore $d\phi(x)(v) < 1$ for every $(x, v) \in B_{x_0}$. Then $\|d\phi(x)|_S < 1$ for every $x \in W^{x_0}$. Now, from the compactness of $\mathcal{X}$ we obtain that $\|d\phi|_{S,\infty} < 1$. Then by Corollary 3.6 we have that $\|\phi|_S < 1$. Finally, considering τ^{-1} and using Proposition 3.1 we obtain the result. □

We next give a simple example of non-strict almost isometry:

Example 3.8 (Nonstrict almost isometry) *Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$. We consider on $\mathcal{X}$ the usual Finsler structure $F_{\mathcal{X}}(x, v) = |v|$ and we define on $\mathcal{Y}$ the Finsler structure $F_{\mathcal{Y}}(x, v) = |v| - d\phi(x)(v)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is given by*

$$\phi(x) := \int_0^x \frac{t^2}{1 + t^2}\, dt.$$

Note that $(\mathcal{Y}, F_{\mathcal{Y}})$ is a Randers space, since $|\phi'(x)| < 1$ for every $x \in \mathbb{R}$. It is easy to see that the associated Finsler distances are $d_{\mathcal{X}}(x, x') = |x - x'|$ and $d_{\mathcal{Y}}(x, x') = |x - x'| + \phi(x) - \phi(x')$. In this way we obtain that the identity map $\tau : \mathcal{X} \to \mathcal{Y}$ given by $\tau(x) = x$ is an almost isometry from $(\mathcal{X}, d_{\mathcal{X}})$ to $(\mathcal{Y}, d_{\mathcal{Y}})$. Nevertheless in this case we have that $\|\phi|_S = 1$. Therefore by Proposition 3.1 the almost isometry τ is not strict.

The Finsler manifold $(\mathcal{Y}, d_{\mathcal{Y}})$ also provides a concrete example of a semi-Lipschitz function f on a T_1-quasi-metric space such tat $-f$ is not semi-Lipschitz (the example mentioned after Remark 2.44 makes use of the fact that d_u does not separate points, while in this case, $d_{\mathcal{Y}}(x, y) > 0$ whenever $x \neq y$). Let us consider the function $f(x) = -x$. By Proposition 3.5, we can use the asymmetric norm of the derivative of f to compute the corresponding semi-Lipschitz constant. Let $t \in \mathcal{Y}$. Then,

$$\|df(t)|_F = \sup\{v \cdot df(t) \ : \ F_{\mathcal{Y}}(t, v) = 1\} = \sup\{-v \ : \ |v| - v d\phi(t) = 1\} \leq 1,$$

and therefore, $\|f|_S \leq 1$. On the other hand, for $-f$, we have

$$\|d(-f)(t)|_F = \sup\{\ v \ : \ |v| - v d\phi(t) = 1\},$$

which grows to $+\infty$ when $t \to +\infty$. It follows that $\| - f|_S$ can not be finite.

The following proposition shows that the elements of $SC^1_{1^-}(\mathcal{X})$ can be used to describe open sets of $\mathcal{X}$. The proof is omitted, as it follows from standard smooth manifold arguments.

Proposition 3.9 *Let $(\mathcal{X}, F)$ be a Finsler manifold and $\mathcal{U}$ an open subset of $\mathcal{X}$. Then, there exists a smooth function $f : \mathcal{X} \to [0, \infty)$ such that*

$$\mathcal{U} = \{x \in \mathcal{X} : \ f(x) > 0\}.$$

Moreover, f can be chosen so that $\|df\|_\infty < 1$, and therefore $f \in SC^1_{1^-}(\mathcal{X})$.

Let us now recall the approximation result from [25, Theorem 8] already mentioned as Theorem 2.72 in Chapter 2. An adaptation of this result (stated below as Corollary 3.11) will be one of the key elements of the main result of this Chapter.

Theorem 3.10 (Smooth approximation of Lipschitz functions in Finsler manifolds) *Let $(\mathcal{X}, F)$ be a connected, second countable Finsler manifold, $f : \mathcal{X} \to \mathbb{R}$ a Lipschitz function, $\varepsilon : \mathcal{X} \to (0, +\infty)$ a continuous function and $r > 0$. Then, there exists a C^1-smooth Lipschitz function $g : \mathcal{X} \to \mathbb{R}$ such that:*

(i) $|g(x) - f(x)| \leq \varepsilon(x)$ *for all* $x \in \mathcal{X}$ *;*

(ii) $\|g\|_{\mathrm{LIP}} \leq \|f\|_{\mathrm{LIP}} + r$.

By replacing the Lipschitz functions by semi-Lipschitz functions in Proposition 6, Lemma 7 and Theorem 8 of [25], we obtain the following corollary:

Corollary 3.11 (Smooth approximation of semi-Lipschitz functions in Finsler manifolds) *Let $(\mathcal{X}, F)$ be a connected, second countable Finsler manifold, $f : \mathcal{X} \to \mathbb{R}$ a semi-Lipschitz function, $\varepsilon : \mathcal{X} \to (0, +\infty)$ a continuous function and $r > 0$. Then, there exists a C^1-smooth semi-Lipschitz function $g : \mathcal{X} \to \mathbb{R}$ that approximates f in the following sense:*

(i) $|g(x) - f(x)| \leq \varepsilon(x)$ *for all* $x \in \mathcal{X}$ *;*

(ii) $\|g|_S \leq \|f|_S + r$.

The proof of Corollary 3.11 (which is based to results analogous to Proposition 6 and Lemma 7 of [25]) is omitted, since all arguments are straightforward adaptations of the aforementioned ones, by replacing Lipschitz bounds with semi-Lipschitz ones. Nevertheless, we state the asymmetric version of Proposition 6 of [25], as it will be of use later.

Proposition 3.12 *Let $\mathcal{X}$, $\mathcal{Y}$ be connected Finsler manifolds. A function $f : \mathcal{X} \to \mathcal{Y}$ is C-semi-Lipschitz if and only if it is **locally** C-semi-Lipschitz, that is, for every $x \in \mathcal{X}$ there exists a neighborhood U of x such that $f|_U$ is C-semi-Lipschitz.*

The following proposition shows that given two connected Finsler manifolds $\mathcal{X}$ and $\mathcal{Y}$, each strict almost isometry between $\mathcal{X}$ and $\mathcal{Y}$ (with respect to their Finsler distances) induces an isomorphism of convex partially ordered sets between $SC^1_{1^-}(\mathcal{Y})$ and $SC^1_{1^-}(\mathcal{X})$.

Proposition 3.13 *Let $\mathcal{X}$, $\mathcal{Y}$ be connected Finsler manifolds and $\tau : \mathcal{X} \to \mathcal{Y}$ a strict almost isometry with respect to their Finsler distances induced by a function $\phi : X \to \mathbb{R}$ (in the sense of Proposition 2.7). Then the mapping*

$$\begin{cases} T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X}) \\ Tf = f \circ \tau + \phi \end{cases}$$

is an isomorphism of convex partially ordered sets.

Proof. Consider the mapping $Tf = f \circ \tau + \phi$. Note that the convexity and order-preserving properties of T are immediate, so we only need to check that T is a well-defined bijection. To this end, note first that if $\|f|_S \leq 1$, then $\|Tf|_S \leq 1$, since:

$$Tf(x') - Tf(x) = f(\tau(x')) - f(\tau(x)) + \phi(x') - \phi(x) \leq d_{\mathcal{Y}}(\tau(x), \tau(x')) + \phi(x') - \phi(x) = d_{\mathcal{X}}(x, x').$$

We shall now prove that if $f \in SC^1_{1^-}(\mathcal{Y})$ then $\|Tf|_S < 1$. Note that $T0 = \phi$ and from Proposition 3.1 we have that $\|\phi|_S < 1$. Choose $\lambda \in (0,1)$ such that $\|\lambda^{-1} f|_S < 1$. Then

$$\begin{aligned} \|Tf|_S &= \left\|T\left((\lambda\lambda^{-1})f + (1-\lambda)0\right)\right|_S = \left\|\lambda T\left(\lambda^{-1} f\right) + (1-\lambda)\, T0\right|_S \\ &\leq \lambda \left\|T\left(\lambda^{-1} f\right)\right|_S + (1-\lambda)\, \|T0|_S \leq \lambda + (1-\lambda)\|T0|_S < 1. \end{aligned} \tag{3.1}$$

This shows that $T\left(SC^1_{1^-}(\mathcal{Y})\right) \subset SC^1_{1^-}(\mathcal{X})$ and T is well-defined. An analogous argument holds for the inverse mapping $T^{-1} g = g \circ \tau^{-1} - \phi \circ \tau^{-1}$, so we conclude that T is a bijection. □

3.2 Main result

The main result of this Section is the converse of Proposition 3.13 which eventually provides a functional characterization of strict almost isometries between connected, second countable and bicomplete Finsler manifolds, which becomes a characterization of all almost isometries in the compact setting (see forthcoming Corollaries 3.35–3.36).

Theorem 3.14 (Main result) *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ connected, second countable Finsler manifolds which are bicomplete (with their respective Finsler distances). Assume there exists an isomorphism of convex partially ordered sets $T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X})$. Then, there exist $\alpha > 0$, a quasi-metric d'_X on X and a bijection $\tau : \mathcal{X} \to \mathcal{Y}$ such that:*

(i) *$(\mathcal{X}, d_{\mathcal{X}})$ is almost isometric to $(\mathcal{X}, d'_{\mathcal{X}})$.*

(ii) *$(\mathcal{X}, \alpha \cdot d'_{\mathcal{X}})$ is isometric to $(\mathcal{Y}, d_{\mathcal{Y}})$ via τ.*

(iii) *$\mathcal{X}$ is diffeomorphic to $\mathcal{Y}$ via τ.*

(iv) *$\forall f \in SC^1_{1^-}(\mathcal{Y})$, $Tf = c \cdot (f \circ \tau) + \phi$, with $c = \alpha^{-1}$ and $\phi = T0$.*
(In particular, ϕ is smooth and $\|\phi|_S < 1$).

The proof of the above theorem will be given in Subsection 3.2.3. Before, we shall need to establish several intermediate results. The proof is divided in three steps, presented in Subsections 3.2.1, 3.2.2 and 3.2.3.

In what follows, we assume that:

$(\mathcal{H}_1)$ $(\mathcal{X}, d_\mathcal{X})$, $(\mathcal{Y}, d_\mathcal{Y})$ are connected, second countable and bicomplete Finsler manifolds;

$(\mathcal{H}_2)$ T is an isomorphism of the convex partially ordered sets $SC^1_{1^-}(\mathcal{Y})$ and $SC^1_{1^-}(\mathcal{X})$.

3.2.1 Order and topology

The following definition introduces some useful notation and describes a certain type of open subsets of the Finsler manifolds that are naturally associated with the class of smooth semi-Lipschitz functions.

Definition 3.15 (Open sets related to the order structure) *Let $h \in SC^1_{1^-}(\mathcal{Y})$. We define*

$$\begin{aligned} SC^1_{1^-}(\mathcal{Y})_h =& \{f \in SC^1_{1^-}(\mathcal{Y}) :\ f \geq h\}, \\ SC^1_{1^-}(\mathcal{X})_{Th} =& \{g \in SC^1_{1^-}(\mathcal{X}) :\ g \geq Th\} = T(SC^1_{1^-}(\mathcal{Y})_h). \end{aligned}$$

Furthermore, for any $f \in SC^1_{1^-}(\mathcal{Y})_h$, we denote:

$$\begin{aligned} \mathrm{supp}_h(f) =& \overline{\{y \in \mathcal{Y} :\ f(y) > h(y)\}} \quad \textit{and} \quad \mathcal{V}_h^f = \mathrm{int}\,(\mathrm{supp}_h(f)), \\ \mathrm{supp}_{Th}(Tf) =& \overline{\{x \in \mathcal{X} :\ Tf(x) > Th(x)\}} \quad \textit{and} \quad \mathcal{U}_{Th}^{Tf} = \mathrm{int}\,(\mathrm{supp}_{Th}(Tf)), \end{aligned}$$

where closure and interior are taken in the symmetric topologies of $(\mathcal{Y}, d_\mathcal{Y})$ and $(\mathcal{X}, d_\mathcal{X})$.

Before we proceed, let us introduce the notion of *bump* function on a Finsler manifold $\mathcal{X}$.

Definition 3.16 ((Smooth semi-Lipschitz) bump functions) *Let $\mathcal{X}$ be Finsler manifold. A nonnegative smooth semi-Lipschitz function $b : \mathcal{X} \to \mathbb{R}_+$ is called a bump function on $\mathcal{X}$ centered at a point $x_0 \in \mathcal{X}$, provided $b(x_0) > 0$ and $\mathrm{supp}(b) \subset B_\mathcal{X}(x_0, r)$ for some $r > 0$.*

It is well-known that for every $x_0 \in \mathcal{X}$ and $r > 0$ there exists a bump function $b \in SC^1_{1^-}(\mathcal{X})_0$ with $\mathrm{supp}(b) \subset B_\mathcal{X}(x_0, r)$ and $b(x_0) > 0$.

We are now ready to describe a basis for the topologies in $\mathcal{Y}$ and $\mathcal{X}$ respectively, which will play an important role in the sequel.

Proposition 3.17 (Topology basis for $\mathcal{X}$ and $\mathcal{Y}$) *Let $\mathcal{X}, \mathcal{Y}$ be two Finsler manifolds and let us fix a function $h \in SC^1_{1^-}(\mathcal{Y})$. Then the families*

$$\mathcal{B}_h(\mathcal{Y}) = \{\mathcal{V}_h^f :\ f \in SC^1_{1^-}(\mathcal{Y})_h\} \quad \textit{and} \quad \mathcal{B}_h(\mathcal{X}) = \{\mathcal{U}_{Th}^{Tf} :\ f \in SC^1_{1^-}(\mathcal{Y})_h\}$$

are basis for the topologies of $(\mathcal{Y}, d_\mathcal{Y})$ and $(\mathcal{X}, d_\mathcal{X})$ respectively.

PROOF. Given $y_0 \in \mathcal{Y}$ and a ball $B_{\mathcal{Y}} := B_{\mathcal{Y}}(y_0, r)$ for the distance $d^s_{\mathcal{Y}}$ centered at y_0 and of radius $r > 0$, we take a bump function $b \in SC^1_{1^-}(\mathcal{Y})_0$ such that $\mathrm{supp}(b) \subset B_{\mathcal{Y}}$, $b(y_0) > 0$ and $\|b|_S + \|h|_S < 1$. Defining $f = h + b$, we get that $y_0 \in \mathcal{V}^f_h \subset B_{\mathcal{Y}}$.

Given $x_0 \in \mathcal{X}$ and a ball $B_{\mathcal{X}}$ for $d^s_{\mathcal{X}}$ containing x_0, take $b \in SC^1_{1^-}(\mathcal{X})_0$ such that $\mathrm{supp}(b) \subset B_{\mathcal{X}}$, $b(x_0) > 0$ and $\|b|_S + \|Th|_S < 1$. Since $Th + b \geq Th$ and T is an isomorphism of convex partially ordered sets, there exists $f \in SC^1_{1^-}(\mathcal{Y})_h$ such that $Tf = Th + b$. Therefore, $x_0 \in \mathcal{U}^{Tf}_{Th} \subset B_{\mathcal{X}}$. □

The following proposition is straightforward. It asserts the existence of a natural bijection between the basis $\mathcal{B}_h(\mathcal{Y})$ and $\mathcal{B}_h(\mathcal{X})$:

Proposition 3.18 *Let T be as in $(\mathcal{H}_2)$. Then for every $h \in SC^1_{1^-}(\mathcal{Y})$, the mapping $\mathcal{I}_h : \mathcal{B}_h(\mathcal{Y}) \to \mathcal{B}_h(\mathcal{X})$ given by $T(\mathcal{V}^f_h) = \mathcal{U}^{Tf}_{Th}$ is a bijection.*

Remark 3.19 *The aforementioned basis seems to depend on the choice of the function h. Nonetheless, we shall show in forthcoming Proposition 3.25 and respectively, Corollary 3.28, that the basis $\mathcal{B}_h(\mathcal{X})$, $\mathcal{B}_h(\mathcal{Y})$ and, respectively, the bijection $\mathcal{I}_h$ do not depend on the choice of h.*

Next, we show that for each $h \in SC^1_{1^-}(\mathcal{Y})$, the bijection $\mathcal{I}_h$ preserves the order structure of $(\mathcal{B}_h(\mathcal{Y}), \subset)$ and $(\mathcal{B}_h(\mathcal{X}), \subset)$. To this end, following [11] we introduce the following notation:

1. $f \sqcap_h g = \{u \in SC^1_{1^-}(\mathcal{Y})_h : u \leq f,\ u \leq g\}$.
2. $f \sqsubset_h g$ if for any $u \in SC^1_{1^-}(\mathcal{Y})_h,\ u \sqcap_h g = \{h\} \implies u \sqcap_h f = \{h\}$.
3. $Tf \sqcap_{Th} Tg = \{v \in SC^1_{1^-}(\mathcal{X})_{Th} : v \leq Tf,\ v \leq Tg\}$.
4. $Tf \sqsubset_{Th} Tg$ if for any $v \in SC^1_{1^-}(\mathcal{X})_{Th},\ v \sqcap_{Th} Tg = \{Th\} \implies v \sqcap_{Th} Tf = \{Th\}$.

The following proposition gives more insight to the above notation. The proof follows the ideas of [11].

Proposition 3.20 *Let $h \in SC^1_{1^-}(\mathcal{Y})$ and $f, g \in SC^1_{1^-}(\mathcal{Y})_h$. Then*

(i) $f \sqcap_h g = \{h\} \iff \mathcal{V}^f_h \cap \mathcal{V}^g_h = \emptyset$.

(ii) $f \sqsubset_h g \iff \mathcal{V}^f_h \subset \mathcal{V}^g_h \iff \mathrm{supp}_h(f) \subset \mathrm{supp}_h(g)$.

(iii) $Tf \sqcap_{Th} Tg = \{Th\} \iff \mathcal{U}^{Tf}_{Th} \cap \mathcal{U}^{Tg}_{Th} = \emptyset$.

(iv) $Tf \sqsubset_{Th} Tg \iff \mathcal{U}^{Tf}_{Th} \subset \mathcal{U}^{Tg}_{Th} \iff \mathrm{supp}_{Th}(Tf) \subset \mathrm{supp}_{Th}(Tg)$.

PROOF. (i) If $\mathcal{V}_h^f \cap \mathcal{V}_h^g = \emptyset$ and $u \in f \sqcap_h g$, then $u(y) \leq f(y) \wedge g(y)$ for all $y \in (\mathcal{V}_h^f)^c \cup (\mathcal{V}_h^g)^c = \mathcal{Y}$, so $u = h$.
Conversely, suppose that $\mathcal{V}_h^f \cap \mathcal{V}_h^g \neq \emptyset$ and let $y \in \mathcal{V}_h^f \cap \mathcal{V}_h^g$. Since $y \in \mathrm{supp}_h(f)$, there exists a sequence $\{y_n\} \subset \{z : f(z) > h(z)\}$, with $y_n \to y$. Then, there exists $N \in \mathbb{N}$ such that $y_N \in \mathcal{V}_h^f \cap \mathcal{V}_h^g$ and $f(y_N) > h(y_N)$. Since $y_N \in \mathrm{supp}_h(g)$, there exists a sequence $(y^j) \subset \{z : g(z) > h(z)\}$ such that $y^j \to y_N$, and there is $J \in \mathbb{N}$ such that $y^J \in (f-h)^{-1}(0,\infty) \cap \mathcal{V}_h^f \cap \mathcal{V}_h^g$, so $f(y^J) \wedge g(y^J) > h(y^J)$. Taking a suitable bump function $b \in SC^1_{1^-}(\mathcal{Y})_0$ with positive value at y^J, we get that $h \lneq b + h \in f \sqcap_h g$, and therefore $f \sqcap_h g \neq \{h\}$.

(ii) Let $f, g \in SC^1_{1^-}(\mathcal{Y})_h$. Then we have

$$
\begin{aligned}
f \sqsubset_h g &\iff \left[\forall u \in SC^1_{1^-}(\mathcal{Y})_h,\ u \sqcap_h g = \{h\} \implies u \sqcap_h f = \{h\}\right], \\
&\iff \left[\forall u \in SC^1_{1^-}(\mathcal{Y})_h,\ \mathcal{V}_h^u \cap \mathcal{V}_h^g = \emptyset \implies \mathcal{V}_h^u \cap \mathcal{V}_h^f = \emptyset\right], \\
&\iff \left[\forall u \in SC^1_{1^-}(\mathcal{Y})_h,\ \mathcal{V}_h^g \subset (\mathcal{V}_h^u)^c \implies \mathcal{V}_h^f \subset (\mathcal{V}_h^u)^c\right].
\end{aligned}
\tag{3.2}
$$

Clearly, $\mathcal{V}_h^f \subset \mathcal{V}_h^g$ implies $f \sqsubset_h g$. On the other hand, since $\mathcal{B}_h(\mathcal{Y})$ is a basis, we can express $\overline{\mathcal{V}_h^g}$ as an intersection of sets of the form $(\mathcal{V}_h^u)^c$. Let $\mathcal{H} \subset SC^1_{1^-}(\mathcal{Y})_h$ such that $\overline{\mathcal{V}_h^g} = \bigcap_{u \in \mathcal{H}} (\mathcal{V}_h^u)^c$.
Then, using (3.2) we deduce:

$$
f \sqsubset_h g \implies \forall u \in \mathcal{H},\ \mathcal{V}_h^g \subset (\mathcal{V}_h^u)^c \implies \mathcal{V}_h^f \subset (\mathcal{V}_h^u)^c,
$$

so that

$$
\mathcal{V}_h^f \subset \overline{\mathcal{V}_h^g} \implies \overline{\mathcal{V}_h^f} \subset \overline{\mathcal{V}_h^g} \implies \mathcal{V}_h^f \subset \mathcal{V}_h^g,
$$

since the elements of $\mathcal{B}_h(\mathcal{Y})$ are regular open sets (that is, they coincide with the interior of their closure).

The proofs of (iii) and (iv) are analogous to the proofs of (i) and (ii) respectively, and will be omitted. □

We have shown that inclusions between members of $\mathcal{B}_h(\mathcal{Y})$ (and $\mathcal{B}_h(\mathcal{X})$) can be described using the relation $\sqsubset_h$ on $SC^1_{1^-}(\mathcal{Y})_h$ (respectively $\sqsubset_{Th}$ on $SC^1_{1^-}(\mathcal{X})_h$), which depends only on the convex and order structure of $SC^1_{1^-}(\mathcal{Y})$ (respectively $SC^1_{1^-}(\mathcal{X})$), so we can use the isomorphism T to relate inclusions between sets of each basis.

Proposition 3.21 *Let $h \in SC^1_{1^-}(\mathcal{Y})$ and $f, g \in SC^1_{1^-}(\mathcal{Y})_h$. Then*

$$
f \sqsubset_h g \iff Tf \sqsubset_{Th} Tg.
$$

Therefore,

$$
\mathcal{V}_h^f \subset \mathcal{V}_h^g \iff \mathcal{U}_{Th}^{Tf} \subset \mathcal{U}_{Th}^{Tg}.
$$

PROOF. Let $f, g \in SC^1_{1^-}(\mathcal{Y})_h$. Using the properties of T, we obtain:

$$
\begin{aligned}
f \sqsubset_h g &\iff \left[\forall u \in SC^1_{1^-}(\mathcal{Y})_h,\ u \sqcap_h g = \{h\} \implies u \sqcap_h f = \{h\}\right],\\
&\iff \left[\begin{array}{l} \forall u \in SC^1_{1^-}(\mathcal{Y})_h,\ \{v \in SC^1_{1^-}(\mathcal{Y})_h :\ v \leq u \wedge g\} = \{h\} \\ \implies \{v \in SC^1_{1^-}(\mathcal{Y})_h :\ v \leq u \wedge f\} = \{h\} \end{array}\right],\\
&\iff \left[\begin{array}{l} \forall Tu \in SC^1_{1^-}(\mathcal{X})_{Th},\ \{Tv \in SC^1_{1^-}(\mathcal{X})_{Th} :\ Tv \leq Tu \wedge Tg\} = \{Th\} \\ \implies \{Tv \in SC^1_{1^-}(\mathcal{X})_{Th} :\ Tv \leq Tu \wedge Tf\} = \{Th\} \end{array}\right],\\
&\iff Tf \sqsubset_{Th} Tg.
\end{aligned}
$$

The second part of the statement follows directly using Proposition 3.20. □

Corollary 3.22 *For any $h \in SC^1_{1^-}(\mathcal{Y})$, the mapping $\mathcal{I}_h$ from Proposition 3.18 is an order-preserving bijection, that is, for any $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{B}_h(\mathcal{Y})$*

$$\mathcal{V}_1 \subset \mathcal{V}_2 \iff \mathcal{I}_h(\mathcal{V}_1) \subset \mathcal{I}_h(\mathcal{V}_2).$$

Next, we show that local inequalities between elements of $SC^1_{1^-}(\mathcal{Y})$ can be characterized via its convex-order structure. In what follows, we fix $h \in SC^1_{1^-}(\mathcal{Y})$ and for any $g \in SC^1_{1^-}(\mathcal{Y})_h$ and $\lambda \in [0,1]$, we set:

$$g_\lambda := \lambda g + (1-\lambda)h.$$

Proposition 3.23 (Characterization of dominance on $\mathcal{V}^f_h$) *Let $h \in SC^1_{1^-}(\mathcal{Y})$ and $\varphi, \psi, f \in SC^1_{1^-}(\mathcal{Y})_h$. Then we have:*

$$\varphi \geq \psi \text{ on } \mathcal{V}^f_h \iff \forall \lambda \in [0,1],\ \forall u \sqsubset_h f,\ \psi_\lambda \sqcap_h u \subset \varphi_\lambda \sqcap_h u.$$

PROOF. The "only if" implication is straightforward. For the "if" implication, suppose there is $y_0 \in \mathcal{V}^f_h$ such that $\psi(y_0) > \varphi(y_0)$. Then there is a symmetric ball B containing y_0 such that $\psi > \varphi$ on B. Furthermore, we can take $u \in SC^1_{1^-}(\mathcal{Y})_h$ defined by $u = h + b$, with $b : \mathcal{Y} \to [0, \varepsilon]$ a C^1 Lipschitz bump function supported on B such that $b(y_0) = \varepsilon$, for some $\varepsilon > 0$. With this, $u(y_0) := \alpha > h(y_0)$ and $y_0 \in \overline{\mathcal{V}^u_h} \subset B \subset \mathcal{V}^f_h$. Without lost of generality, $\psi(y_0) > \alpha$. Let $\lambda \in [0,1]$ such that $\psi_\lambda(y_0) > \alpha > \varphi_\lambda(y_0)$, and let $\eta : Y \to [0,1]$ be a C^1 Lipschitz bump function such that $\eta|_{\overline{\{\psi_\lambda < u\}}} = 0$, $\eta(y_0) = 1$, and define $v = \eta b + h$. Note that v is semi-Lipschitz, since for $y, y' \in \mathcal{Y}$ we have:

$$
\begin{aligned}
v(y') - v(y) &= \eta(y')b(y') + h(y') - \eta(y)b(y) - h(y)\\
&= \eta(y')b(y') - \eta(y')b(y) + \eta(y')b(y) - \eta(y)b(y) + h(y') - h(y)\\
&\leq \|\eta\|_\infty(b(y') - b(y)) + \|b\|_\infty(\eta(y') - \eta(y)) + \|h|_S\, d_{\mathcal{Y}}(y, y')\\
&\leq (\|b\|_{\mathrm{LIP}} + \varepsilon\|\eta\|_{\mathrm{LIP}} + \|h|_S)\, d_{\mathcal{Y}}(y, y').
\end{aligned}
$$

Choose $t \in (0,1]$ such that $v_t \in SC^1_{1^-}(\mathcal{Y})$. Since for $g \in SC^1_{1^-}(\mathcal{Y})$ and $\lambda \in [0,1]$,

$$(g_\lambda)_t = t g_\lambda + (1-t)h = \lambda t g + (1 - \lambda t)h = g_{\lambda t},$$

we get that $v_t \in SC^1_{1^-}(\mathcal{Y})_h$, $v_t \sqsubset_h f$ (since $\mathrm{supp}_h(v_t) \subset \mathrm{supp}(b) \subset B \subset \mathrm{supp}_h(f)$), $v_t(y) \leq \psi_{\lambda t}(y)$ for all $y \in \mathcal{Y}$, and finally $v_t(y_0) = u_t(y_0) > \varphi_{\lambda t}(y_0)$. Therefore, $v_t \in (\psi_{\lambda t} \sqcap_h v_t) \setminus (\varphi_{\lambda t} \sqcap_h v_t)$, a contradiction. □

We now state the following useful lemma.

Lemma 3.24 (Transfer principle) *Let $h \in SC^1_{1^-}(\mathcal{Y})$ and $\varphi, \psi, f \in SC^1_{1^-}(\mathcal{Y})_h$. Then:*

$$\varphi \geq \psi \text{ on } \mathcal{V}_h^f \iff T\varphi \geq T\psi \text{ on } \mathcal{U}_{Th}^{Tf}.$$

Proof. It follows from Proposition 3.23, since the right side of the equivalence depends only on the convex and order structure of $SC^1_{1^-}(\mathcal{Y})$, which is preserved by T, so for any $u, v, f, g \in SC^1_{1^-}(\mathcal{Y})_h$ we have:

$$u \sqsubset_h f \iff Tu \sqsubset_h Tf \quad \text{and} \quad v \in f \sqcap_h g \iff Tv \in Tf \sqcap_h Tg.$$

Therefore $(T\varphi)_\lambda = T(\varphi_\lambda)$, for any $\varphi \in SC^1_{1^-}(\mathcal{Y})_h$ and $\lambda \in [0,1]$. □

Next, we show that the basis $\mathcal{B}_h(\mathcal{Y})$, $\mathcal{B}_h(\mathcal{X})$ and the bijection $\mathcal{I}_h$ are independent of h.

Proposition 3.25 (Independence of the topological basis from h) *Let $h \in SC^1_{1^-}(\mathcal{Y})$. Then*
(i). $\mathcal{B}_h(\mathcal{Y}) = \mathcal{B}_0(\mathcal{Y}) := \mathcal{B}(\mathcal{Y})$
(ii). $\mathcal{B}_h(\mathcal{X}) = \mathcal{B}_0(\mathcal{X}) := \mathcal{B}(\mathcal{X})$.

Proof. (i). Let $\mathcal{V}_0^f \in \mathcal{B}(\mathcal{Y})$. Since $\|h|_S < 1$, there is $\lambda \in (0,1]$ such that $\lambda f + h \in SC^1_{1^-}(\mathcal{Y})$, so $\{f > 0\} = \{\lambda f + h > h\}$, and therefore $\mathcal{V}_0^f = \mathcal{V}_h^{\lambda f + h} \in \mathcal{B}_h(\mathcal{Y})$. Conversely, let $\mathcal{V}_h^g \in \mathcal{B}_h(\mathcal{Y})$. Since the set $\{g > h\} := \{y \in \mathcal{Y} : g(y) > h(y)\}$ is open, by Proposition 3.9 there exists $f \in SC^1_{1^-}(\mathcal{Y})_0$ such that $\{f > 0\} = \{g > h\}$, so $\mathcal{V}_h^g = \mathcal{V}_0^f \in \mathcal{B}(\mathcal{Y})$.

(ii). Let $\mathcal{U}_{T0}^{Tf} \in \mathcal{B}(\mathcal{X})$, and $g \in SC^1_{1^-}(\mathcal{X})_0$ such that $\{Tf > T0\} = \{g > 0\}$. Take $\lambda \in (0,1]$ such that $\lambda g + Th \in SC^1_{1^-}(\mathcal{X})$, and since $\lambda g + Th \geq Th$, there exists $\tilde{f} \in SC^1_{1^-}(\mathcal{Y})_h$ such that $T\tilde{f} = \lambda g + Th$. Therefore, $\{Tf > T0\} = \{T\tilde{f} > Th\}$ and $\mathcal{U}_{T0}^{Tf} = \mathcal{U}_{Th}^{T\tilde{f}} \in \mathcal{B}_h(\mathcal{X})$. Conversely, let $\mathcal{U}_{Th}^{Tf} \in \mathcal{B}_h(\mathcal{X})$ and $g \in SC^1_{1^-}(\mathcal{X})_0$ such that $\{Tf > Th\} = \{g > 0\}$. Taking $\lambda \in (0,1]$ such that $\lambda g + T0 \in SC^1_{1^-}(\mathcal{X})$, we get that $\mathcal{U}_{Th}^{Tf} = \mathcal{U}_{T0}^{\lambda g + T0} \in \mathcal{B}(\mathcal{X})$. □

The following result completes the transfer principle of Lemma 3.24:

Proposition 3.26 *Let $h \in SC^1_{1^-}(\mathcal{Y})$ and $U \in \mathcal{B}(\mathcal{X})$. Then $\mathcal{V} = \mathcal{I}_h^{-1}(\mathcal{U})$ is the only element in $\mathcal{B}(\mathcal{Y})$ such that for any $\varphi, \psi \in SC^1_{1^-}(\mathcal{Y})_h$*

$$\varphi \geq \psi \text{ on } \mathcal{V} \iff T\varphi \geq T\psi \text{ on } \mathcal{U}. \tag{3.3}$$

Proof. Lemma 3.24 ensures that $\mathcal{V}$ satisfies (3.3). Let $\tilde{\mathcal{V}} \neq \mathcal{V}$ in $\mathcal{B}(\mathcal{Y})$ satisfying the same property. Without loss of generality, $\mathcal{V} \setminus \tilde{\mathcal{V}} \neq \emptyset$. Since both sets are regular open sets, there exists $y \in \mathcal{V} \setminus \tilde{\mathcal{V}}$ and $\varepsilon > 0$ such that the symmetric ball $B(y, \varepsilon) := B$ is contained in $\mathcal{V} \setminus \tilde{\mathcal{V}}$. Given $y_0 \in B$, we can take $\varphi, \psi \in SC^1_{1^-}(\mathcal{Y})_h$ such that $\mathrm{supp}_h(\varphi) \cup \mathrm{supp}_h(\psi) \subset B$, $\varphi(y_0) < \psi(y_0)$. Therefore $\varphi \not\geq \psi$ on $B \subset \mathcal{V}$, but $\varphi \geq \psi$ on $\tilde{\mathcal{V}}$, which contradicts (3.3). □

Using the above, we show the independence of the bijection (*c.f.* Proposition 3.18) from h for a particular case. (The general case will be given in Corollary 3.28.)

Proposition 3.27 *Let $h_1, h_2 \in SC^1_{1^-}(\mathcal{Y})$, such that $h_1 \le h_2$. Then $\mathcal{I}_{h_1} = \mathcal{I}_{h_2}$.*

Proof. Let $\mathcal{U} \in \mathcal{B}(\mathcal{X})$, and let $\varphi, \psi \in SC^1_{1^-}(\mathcal{Y})_{h_2}$ such that $\varphi \ge \psi$ on $\mathcal{V}_2 := \mathcal{I}^{-1}_{h_2}(\mathcal{U})$. By Lemma 3.24, $T\varphi \ge T\psi$ on $\mathcal{U}$, and since $h_1 \le h_2$, $\varphi, \psi \in SC^1_{1^-}(\mathcal{Y})_{h_1}$, and by Lemma 3.24 $\varphi \ge \psi$ on $\mathcal{V}_1 := \mathcal{I}^{-1}_{h_1}(\mathcal{U})$. Hence, by Proposition 3.26, $\mathcal{V}_1 = \mathcal{V}_2$, and therefore, $\mathcal{I}_{h_1} = \mathcal{I}_{h_2}$. □

Corollary 3.28 (Independence of the bijection from h) *Let $h \in SC^1_{1^-}(\mathcal{Y})$. Then*

$$\mathcal{I}_h = \mathcal{I}_0 := \mathcal{I}.$$

Proof. Consider $h \vee 0 \in \mathrm{SLIP}_1(\mathcal{Y})$ and note that $\|h \vee 0|_S \le \|h|_S < 1$. Take $\eta > 0$ such that $\|h|_S + \eta < 1$ and $g : \mathcal{Y} \to \mathbb{R}$ a semi-Lipschitz C^1-smooth approximation given by Corollary 3.11, using $\varepsilon = \frac{\eta}{2}$ and $r = \eta$. Replacing g by $g + \varepsilon$ we get an approximation from above of $h \vee 0$, that is:

$$g \ge h \vee 0, \quad \|g|_S \le \|h \vee 0|_S + \eta < 1 \quad \text{and} \quad g(y) - (h \vee 0)(y) \le \eta, \ \forall y \in \mathcal{Y}.$$

It follows that $g \in SC^1_{1^-}(\mathcal{Y})$, $g \ge h$ and $g \ge 0$. By Proposition 3.27, $\mathcal{I}_h = \mathcal{I}_g = \mathcal{I}_0$. □

Thanks to this result, we can simply work with the basis $\mathcal{B}(\mathcal{Y})$ and $\mathcal{B}(\mathcal{X})$ (without fixing a function h) and with the bijection $\mathcal{I} : \mathcal{B}(\mathcal{Y}) \to \mathcal{B}(\mathcal{X})$. The following lemma, established in [12, Lemma 6] and [13, Lemma 2] is paramount for our considerations.

Lemma 3.29 (Key Lemma) *Let (X, d_X) and (Y, d_Y) be complete metric spaces, and let $B(X)$ and $B(Y)$ be bases for their topologies. If $\mathcal{I} : B(Y) \to B(X)$ is an inclusion-preserving bijection, then there exist dense subsets $X' \subset X$, $Y' \subset Y$ and a homeomorphism $\tau : X' \to Y'$ such that for every $x \in X'$ and $\mathcal{V} \in B(Y)$ it holds:*

$$\tau(x) \in \mathcal{V} \iff x \in \mathcal{I}(\mathcal{V}).$$

Since we deal with Finsler manifolds $\mathcal{X}$ and $\mathcal{Y}$ which are bicomplete, we can apply Lemma 3.29 to the underlying complete metric spaces $(\mathcal{X}, d^s_{\mathcal{X}})$ and $(\mathcal{Y}, d^s_{\mathcal{Y}})$ to obtain:

Corollary 3.30 (Homeomorphism of dense subsets) *Let $\mathcal{X}$, $\mathcal{Y}$ bicomplete Finsler manifolds and T as in $(\mathcal{H}_2)$. Then there exist dense subsets for the symmetrized topologies $\mathcal{X}' \subset \mathcal{X}$, $\mathcal{Y}' \subset \mathcal{Y}$ and an homeomorphism $\tau : \mathcal{X}' \to \mathcal{Y}'$ such that for any $x \in \mathcal{X}'$ and $\mathcal{V} \in \mathcal{B}(\mathcal{Y})$,*

$$\tau(x) \in \mathcal{V} \iff x \in \mathcal{I}(\mathcal{V}). \tag{3.4}$$

3.2.2 Pointwise behaviour of the isomorphism

The following result will allow us to deduce information about the pointwise behavior of the isomorphism T.

Corollary 3.31 *Let $f, g \in SC^1_{1^-}(\mathcal{Y})$, $\mathcal{X}' \subset \mathcal{X}$ the dense subset of Corollary 3.30 and $x_0 \in \mathcal{X}'$. Then,*

$$f(\tau(x_0)) = g(\tau(x_0)) \iff Tf(x_0) = Tg(x_0),$$

where $\tau : \mathcal{X}' \to \mathcal{Y}'$ is the homeomorphism of Corollary 3.30.

Proof. We need to ensure that we can apply Lemma 3.24. To this end, take $\varepsilon > 0$ such that $\|f|_S \vee \|g|_S + \varepsilon < 1$, and let h be a C^1-smooth semi-Lipschitz approximation of $f \wedge g$ such that $h \leq f \wedge g$ and $\|h|_S \leq \|f|_S \vee \|g|_S + \varepsilon < 1$. Set $y_0 = \tau(x_0)$. It suffices to prove that $Tf(x_0) > Tg(x_0)$ implies $f(y_0) > g(y_0)$. Let us assume, towards a contradiction, that $Tf(x_0) > Tg(x_0)$ and $f(y_0) \leq g(y_0)$. Since $Tf > Tg$ holds true in a neighborhood of x_0, it follows from Lemma 3.24 that $f \geq g$ is also satisfied in a neighborhood of y_0. Therefore, $f(y_0) = g(y_0)$ and y_0 is a local minimum of the function $f - g$. In particular, we deduce that $df(y_0) = dg(y_0)$.

Let now $\varphi \in SC^1_{1^-}(\mathcal{Y})_h$ such that $\varphi(y_0) = f(y_0)$ and $d\varphi(y_0) \neq df(y_0)$. Choose $m \in \mathbb{N}$ such that $f \geq g$ is satisfied in the symmetric ball $B(y_0, \frac{1}{m})$. Then, for every $n \geq m$ we can choose $y_n \in B(y_0, \frac{1}{n}) \cap \mathcal{Y}'$ and $r_n > 0$ such that $\varphi > f$ on $B(y_n, r_n)$. By Lemma 3.24, we deduce that $T\varphi \geq Tf$ on $\mathcal{I}(B(y_n, r_n))$. Denoting $x_n = \tau^{-1}(y_n)$, and applying Corollary 3.30, we obtain sequences (y_n) and (x_n) converging to y_0 and x_0 respectively, and such that $T\varphi(x_n) \geq Tf(x_n)$ for all $n \geq m$, which yields $T\varphi(x_0) \geq Tf(x_0)$. The same argument can now be repeated (choosing $y_j \in B(y_0, \frac{1}{j}) \cap \mathcal{Y}'$ such that $\varphi < g$ in a neighborhood of y_j) to obtain $T\varphi(x_0) \leq Tg(x_0)$, which gives a contradiction.

The other implication follows by the same argument. □

We shall now show that the convexity property of the isomorphism T determines how it acts on the constant functions.

Proposition 3.32 (Action of T on the constant functions) *Let $g \in SC^1_{1^-}(\mathcal{Y})$. Then $Tg - T0$ is constant if and only if g is constant. Moreover, there exists $\alpha > 0$ such that*

$$T\lambda = T0 + \alpha^{-1}\lambda, \quad \forall \lambda \in \mathbb{R}.$$

Proof. Let $\lambda \in \mathbb{R}$ and $g^\lambda \in SC^1_{1^-}(\mathcal{Y})$ such that $Tg^\lambda = T0 + \lambda$.

Let us first assume that $\lambda \geq 1$. Then by convexity property of the isomorphism T we deduce:

$$T\left(\lambda^{-1}g^\lambda\right) = T\left(\lambda^{-1}g^\lambda + \lambda^{-1}(\lambda-1)0\right) = \lambda^{-1}Tg^\lambda + \lambda^{-1}(\lambda-1)T0 = \lambda^{-1}T0 + 1 + \lambda^{-1}(\lambda-1)T0 = T0 + 1.$$

It follows that $Tg^1 := T0 + 1 = T(\lambda^{-1}g^\lambda)$, therefore, since T is bijective, $\lambda g^1 = g^\lambda$ for all $\lambda \geq 1$, so $\|g^1|_S \leq \lambda^{-1}$ for all $\lambda \geq 1$. This latter yields that the function g^1 is constant, that is, there exists $\alpha \in \mathbb{R}$ such that $g^1 = \alpha$, whence $g^\lambda = \alpha\lambda$ for all $\lambda \geq 1$. Since $Tg^1 = T0 + 1 > T0$, it follows that $\alpha > 0$.

Let us now consider the case $\lambda \in [0, 1)$. Then $T\left(\lambda g^1\right) = \lambda Tg^1 + (1-\lambda)T0 = \lambda + T0 = T\left(g^\lambda\right)$, therefore, $\lambda g^1 = \lambda\alpha = g^\lambda$ for all $\lambda \in [0, 1)$. It follows that $g^\lambda = \lambda\alpha$ for any $\lambda \geq 0$. In particular, $T\left(g^{\alpha^{-1}\lambda}\right) = T0 + \alpha^{-1}\lambda = T\lambda$ for any $\lambda \geq 0$.

Finally, using again convexity of T we get:

$$T0 = T\left(\frac{1}{2}\lambda + \frac{1}{2}(-\lambda)\right) = \frac{1}{2}T\lambda + \frac{1}{2}T(-\lambda) = \frac{1}{2}\left(T0 + \alpha^{-1}\lambda\right) + \frac{1}{2}T(-\lambda),$$

which yields $T(-\lambda) = T0 - \alpha^{-1}\lambda$, for every $\lambda \geq 0$. □

Combining Proposition 3.32 and Corollary 3.31, we obtain

Corollary 3.33 *Let $f \in SC^1_{1^-}(\mathcal{Y})$, $\mathcal{X}' \subset \mathcal{X}$ the dense subset from Corollary 3.30 and $x_0 \in \mathcal{X}'$. Denoting by $c = \alpha^{-1} = T1 - T0$ and $\phi = T0$, we have that*

$$Tf(x_0) = c \cdot f(\tau(x_0)) + \phi(x_0).$$

Proof. Applying Corollary 3.31 to f and the constant function of value $f(\tau(x_0))$, we get

$$Tf(x_0) = Tg(x_0) = T0(x_0) + \alpha^{-1}f(\tau(x_0)) = cf(\tau(x_0)) + \phi(x_0).$$

□

3.2.3 Proof of the main result

Recalling the notation of the statement of Theorem 3.14 we set $c := \alpha^{-1} = T1 - T0$ and $\phi = T0$. Since $\|\phi|_S < 1$, in particular $\phi(x_1) - \phi(x_2) < d_{\mathcal{X}}(x_2, x_1)$ for all $x_1, x_2 \in \mathcal{X}$ such that $x_1 \neq x_2$. It is easy to check that we can use ϕ to define a quasi-metric on $\mathcal{X}$ as in Proposition 2.7, obtaining that $d'_{\mathcal{X}}(x_1, x_2) = d_{\mathcal{X}}(x_1, x_2) + \phi(x_1) - \phi(x_2)$ is a quasi-metric on $\mathcal{X}$ such that $(\mathcal{X}, d_{\mathcal{X}})$ is almost isometric to $(\mathcal{X}, d'_{\mathcal{X}})$. In order to modify the isomorphism T, we define the following mappings:

- $R : SC^1_{1^-}(\mathcal{X}, d_{\mathcal{X}}) \to SC^1_{1^-}(\mathcal{X}, d'_{\mathcal{X}})$ by $R(g) = g - \phi$;
- $S : SC^1_{1^-}(\mathcal{X}, d'_{\mathcal{X}}) \to SC^1_{1^-}(\mathcal{X}, \alpha d'_{\mathcal{X}})$ by $S(h) = \alpha h$; and
- $\hat{T} : SC^1_{1^-}(\mathcal{Y}, d_{\mathcal{Y}}) \to SC^1_{1^-}(\mathcal{X}, \alpha d'_{\mathcal{X}})$ by $\hat{T}(f) = S \circ R \circ T(f)$.

Thanks to Proposition 3.13 the mapping R is well-defined: indeed, the same arguments used in Proposition 3.13 are valid for the quasi-metric $d_{\mathcal{X}'}$ (which comes from a Finsler structure, thanks to Proposition 2.71). We shall prove that both $\hat{T}$ and $\hat{T}^{-1}$ act as composition operators whenever their images are evaluated on the dense sets $\mathcal{X}'$ and $\mathcal{Y}'$ of Corollary 3.30 respectively. Indeed, let $f \in SC^1_{1^-}(\mathcal{Y})$ and $x_0 \in \mathcal{X}'$. Applying Corollary 3.33, we have:

$$\hat{T}f(x_0) = S \circ R \circ T(f)(x_0) = \alpha\left(\alpha^{-1} f(\tau(x_0)) + \phi(x_0) - \phi(x_0)\right) = f(\tau(x_0)).$$

On the other hand, for $g \in SC^1_{1^-}(\mathcal{X}, \alpha d'_{\mathcal{X}})$ and $y_0 \in \mathcal{Y}'$, we have

$$\hat{T}^{-1}g(y_0) = T^{-1} \circ R^{-1} \circ S^{-1}(g)(y_0) = T^{-1}\left(\alpha^{-1}g + \phi\right)(y_0).$$

Since $\alpha^{-1}g + \phi \in SC^1_{1^-}(\mathcal{X}, d_{\mathcal{X}})$, there exists $f \in SC^1_{1^-}(\mathcal{Y}, d_{\mathcal{Y}})$ such that $Tf = \alpha^{-1}g + \phi$. Then, denoting $x_0 = \tau^{-1}(y_0)$ we obtain that $\alpha^{-1}g(x_0) + \phi(x_0) = Tf(x_0) = \alpha^{-1}f(y_0) + \phi(x_0)$, whence $f(y_0) = g(x_0)$. Finally

$$\hat{T}^{-1}g(y_0) = T^{-1}(Tf)(y_0) = g(x_0) = g(\tau^{-1}(y_0)).$$

Let us now prove that $\tau : (\mathcal{X}', \alpha d'_{\mathcal{X}}) \to (\mathcal{Y}', d_{\mathcal{Y}})$ is an isometry. To this end, let $x_1, x_2 \in \mathcal{X}'$, $y_1 = \tau(x_1)$ and $y_2 = \tau(x_2)$. Take $\lambda \in (0,1)$ and $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$, and consider the function $f_\lambda(\cdot) = \lambda d_{\mathcal{Y}}(y_1, \cdot)$. Note that $\|f_\lambda|_S = \lambda < 1$, so we can apply Corollary 3.11 (smooth approximation of semi-Lipschitz functions), obtaining $g \in C^1(\mathcal{Y})$ such that $|g(y) - f_\lambda(y)| < \varepsilon$ for all $y \in \mathcal{Y}$ and $\|g|_S \leq \lambda + \varepsilon < 1$. The second condition guarantees that $g \in SC^1_{1^-}(\mathcal{Y}, d_{\mathcal{Y}})$. From the first condition it follows that $|g(y_1)| < \varepsilon$ and $g(y_2) > \lambda d_{\mathcal{Y}}(y_1, y_2) - \varepsilon$. We deduce:

$$\alpha d'_{\mathcal{X}}(x_1, x_2) \geq \hat{T}g(x_2) - \hat{T}g(x_1) = g(y_2) - g(y_1) \geq \lambda d_{\mathcal{Y}}(y_1, y_2) - 2\varepsilon$$

for any $\varepsilon > 0$ such that $\varepsilon + \lambda < 1$. Consequently, $\alpha d'_{\mathcal{X}}(x_1, x_2) \geq \lambda d_{\mathcal{Y}}(y_1, y_2)$, for any $\lambda \in (0,1)$. Therefore

$$\alpha d'_{\mathcal{X}}(x_1, x_2) \geq d_{\mathcal{Y}}(y_1, y_2).$$

A similar argument holds for the reverse inequality. Take $\lambda \in (0,1)$, $\varepsilon > 0$ such that $\lambda + \varepsilon < 1$, and consider $f_\lambda(\cdot) = \lambda d_{\mathcal{X}}(x_1, \cdot)$. Applying again Corollary 3.11 we get $g \in C^1(\mathcal{X})$ such that $|g(x) - f_\lambda(x)| < \varepsilon$ for all $x \in \mathcal{X}$ and $\|g|_S \leq \lambda + \varepsilon < 1$. Consider $\tilde{g} = \alpha(g - \phi) + \alpha\lambda\phi(x_1) \in C^1(\mathcal{X})$. Moreover, $\tilde{g} \in SC^1_{1^-}(\mathcal{X}, \alpha d'_{\mathcal{X}})$, since $\tilde{g} = S \circ R(g) + \alpha\lambda\phi(x_1)$. Let us now note that

$$|\tilde{g}(x_2) - \lambda\alpha d'_{\mathcal{X}}(x_1, x_2)| = |\alpha(g(x_2) - \lambda d_{\mathcal{X}}(x_1, x_2)) - \alpha\phi(x_2) - \lambda\alpha\phi(x_2)| \leq \alpha\varepsilon + \alpha(1-\lambda)|\phi(x_2)|,$$

which together with $|\tilde{g}(x_1)| = |\alpha g(x_1) - \alpha\phi(x_1) + \alpha\lambda\phi(x_1)| \leq \alpha\varepsilon + \alpha(1-\lambda)|\phi(x_1)|$, yields

$$\begin{aligned} d_{\mathcal{Y}}(y_1, y_2) &\geq \hat{T}^{-1}\tilde{g}(y_2) - \hat{T}^{-1}\tilde{g}(y_1) = \tilde{g}(x_2) - \tilde{g}(x_1) \\ &\geq \lambda\alpha d'_{\mathcal{X}}(x_1, x_2) - 2\alpha\varepsilon - \alpha(1-\lambda)(|\phi(x_1)| + |\phi(x_2)|), \end{aligned}$$

for any $\varepsilon > 0$ such that $\varepsilon + \lambda < 1$. Hence,

$$d_{\mathcal{Y}}(y_1, y_2) \geq \alpha d'_{\mathcal{X}}(x_1, x_2) - \alpha(1-\lambda)(|\phi(x_1)| + |\phi(x_2)|),$$

for any $\lambda \in (0,1)$, and therefore $d_{\mathcal{Y}}(y_1, y_2) \geq \alpha d'_{\mathcal{X}}(x_1, x_2)$.

We conclude that $\tau : (\mathcal{X}', \alpha d'_{\mathcal{X}}) \to (\mathcal{Y}', d_{\mathcal{Y}})$ is an isometry. It is easy to check that $(\mathcal{X}, \alpha d'_{\mathcal{X}})$ is also bicomplete, as $(d'_{\mathcal{X}})^s \leq 2d^s_{\mathcal{X}}$. Then, the isometry τ between the symmetrizations of $(\mathcal{X}', \alpha d'_{\mathcal{X}})$ and $(\mathcal{Y}', d_{\mathcal{Y}})$ extends to an isometry between $(\mathcal{X}, \alpha d'_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$. By continuity, we obtain that for any $f \in SC^1_{1^-}(\mathcal{Y})$ and $x \in \mathcal{X}$,

$$Tf(x) = c \cdot f(\tau(x)) + \phi(x).$$

Moreover, since τ is an almost isometry between the Finsler manifolds $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, \alpha^{-1} d_{\mathcal{Y}})$, both τ and ϕ are smooth, thanks to Proposition 2.71. □

3.2.4 Characterizations of isometries and strict almost isometries

Let us recall from [14] the following definition:

Definition 3.34 (almost unital isomorphism) *An isomorphism of convex partially ordered sets*

$$T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X})$$

is called almost unital if $T1 - T0 = 1$.

Applying the results of the previous section we obtain:

Corollary 3.35 (Characterization of strict Finsler almost isometries) *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be connected, second countable Finsler manifolds, which are bicomplete (with their respective Finsler distances). Then, there is a strict almost isometry between $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ if and only if there exists an almost unital isomorphism*

$$T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X}).$$

In particular, for any such isomorphism, there exist a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ and a smooth function $\phi \in SC^1_{1^-}(\mathcal{X})$ such that $Tf = f \circ \tau + \phi$ for all $f \in SC^1_{1^-}(\mathcal{Y})$.

Proof. The "if" implication follows directly from Theorem 3.14 and Definition 3.34, since by Proposition 3.32 we deduce that $c := \alpha^{-1} = T1 - T0 = 1$. The "only if" part follows from Proposition 3.13, using again the fact that Proposition 3.32 holds with $\alpha = 1$ and consequently the isomorphism T is almost unital. □

Using Proposition 3.7, we obtain the following characterization of almost isometries between compact Finsler manifolds:

Corollary 3.36 (Characterization of almost isometries between compact Finsler manifolds) *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be compact, connected, second countable Finsler manifolds, which are bicomplete (with their respective Finsler distances). Then $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are almost isometric if and only if there exists an almost unital isomorphism*

$$T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X}).$$

In particular, for any such isomorphism, there exist a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ and a smooth function $\phi \in SC^1_{1^-}(\mathcal{X})$ such that $Tf = f \circ \tau + \phi$ for all $f \in SC^1_{1^-}(\mathcal{Y})$.

If we focus on isometries, we obtain:

Corollary 3.37 (Characterization of Finsler isometries) *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be connected, second countable Finsler manifolds which are bicomplete (with their respective Finsler distances). Then, $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are* isometric *if and only if there exists an isomorphism $T : SC^1_{1^-}(\mathcal{Y}) \to SC^1_{1^-}(\mathcal{X})$ such that*

$$\|Tf|_S = \|f|_S\,, \quad \textit{for all} \;\; f \in SC^1_{1^-}(\mathcal{Y}).$$

Moreover, for any such isomorphism, there exist a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ and $\beta \in \mathbb{R}$ such that $Tf = f \circ \tau + \beta$ for all $f \in SC^1_{1^-}(\mathcal{Y})$.

Proof. If τ is an isometry between $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$, then $f \mapsto T(f) := f \circ \tau$ is an isomorphism between convex, partially ordered structures that satisfies $\|Tf|_S = \|f|_S$ for all $f \in SC^1_{1^-}(\mathcal{Y})$.

Conversely, we can apply Theorem 3.14 to the isomorphism T, and since $\|T0|_S = 0$, the function $T0$ is a constant, so the quasi-metric $d'_{\mathcal{X}}$ induced by $T0$ is the same as $d_{\mathcal{X}}$. In addition, α must be 1 for T to preserve semi-Lipschitz constants, and therefore $\tau : (X, d_{\mathcal{X}}) \to (Y, d_{\mathcal{Y}})$ is an isometry. □

In the particular case of reversible Finsler manifolds, Theorem 3.14 can be restated as follows.

Corollary 3.38 *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be connected, second countable, reversible complete Finsler manifolds and $T : C^1_{1^-}(\mathcal{Y}) \to C^1_{1^-}(\mathcal{X})$ be an isomorphism of convex partially ordered sets. Then, there exist $\alpha > 0$, $\beta \in \mathbb{R}$ and a bijection $\tau : \mathcal{X} \to \mathcal{Y}$ such that:*

(i) $(\mathcal{Y}, d_{\mathcal{Y}})$ and $(\mathcal{X}, \alpha d_{\mathcal{X}})$ are isometrically diffeomorphic via τ.

(ii) For every $f \in C^1_{1^-}(Y)$ we have $Tf = c \cdot (f \circ \tau) + \beta$, where $c = \alpha^{-1}$ and $\beta = T0$.

Proof. It follows from Theorem 3.14. (Since all involved distances are symmetric, ϕ must be constant.) □

Therefore we obtain the following characterization of isometries for reversible Finsler Manifolds.

Corollary 3.39 (Characterization of isometries for reversible Finsler manifolds) *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ connected, second countable, reversible complete Finsler manifolds. Then the manifolds $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}})$ are isometric if and only if there exists an almost unital isomorphism $T : C^1_{1^-}(\mathcal{Y}) \to C^1_{1^-}(\mathcal{X})$. Moreover, for any such isomorphism there exists a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ and $\beta \in \mathbb{R}$ such that*

$$Tf = f \circ \tau + \beta \quad \textit{for all} \;\; f \in C^1_{1^-}(\mathcal{Y}).$$

Note that the isomorphism of partially ordered sets in the above Corollary preserves Lipschitz constants, and can be replaced by $\tilde{T} = T - \beta$ in order to extend linearly to the spaces of C^1-smooth Lipschitz functions, denoted by $C^1_{\mathrm{LIP}}(\mathcal{Y})$ and $C^1_{\mathrm{LIP}}(\mathcal{X})$ respectively. Therefore, for the particular case of reversible Finsler manifolds we can reformulate Corollary 3.39 as follows:

Corollary 3.40 *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ connected, second countable, reversible complete Finsler manifolds. Then $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are isometric if and only if there exists a linear, order and semi-norm preserving bijection $T : (C^1_{\mathrm{LIP}}(\mathcal{Y}), \|\cdot\|_{\mathrm{LIP}}) \to (C^1_{\mathrm{LIP}}(\mathcal{X}), \|\cdot\|_{\mathrm{LIP}})$. Moreover, for any such bijection there exists a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ such that $Tf = f \circ \tau$ for all $f \in C^1_{\mathrm{LIP}}(\mathcal{Y})$.*

The same idea can be applied to non reversible manifolds, using the normed cones of C^1-smooth semi-Lipschitz functions $C^1_{\mathrm{SLIP}}(\mathcal{Y})$ and $C^1_{\mathrm{SLIP}}(\mathcal{X})$:

Corollary 3.41 *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ connected, second countable and bicomplete Finsler manifolds. Then $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are isometric if and only if there exists a linear, order and asymmetric-norm preserving bijection $T : (C^1_{\mathrm{SLIP}}(\mathcal{Y}), \|\cdot|_S) \to (C^1_{\mathrm{SLIP}}(\mathcal{X}), \|\cdot|_S)$. Moreover, for any such bijection there exists a diffeomorphism $\tau : \mathcal{X} \to \mathcal{Y}$ such that $Tf = f \circ \tau$ for all $f \in C^1_{\mathrm{SLIP}}(\mathcal{Y})$.*

Chapter 4

The semi-Lipschitz free space

The objective of this chapter is to present the construction of a generalization of the Lipschitz free space in the framework of quasi-metric spaces, published in [18], as well as related results related to the subject. Section 4.2 is devoted to the construction of the semi-Lipschitz free space, and its main properties are showcased in Section 4.3. In Section 4.4 we introduce the notion of asymmetrizations of a metric spaces, and we explore its relation with the semi-Lipschitz free space. Section 4.5 contains examples of semi-Lipschitz free spaces for concrete quasi-metric spaces. At the end of the Chapter, we introduce the notion of locally flat semi-Lipschitz functions, and present an illustrative examples where the cone of locally flat semi-Lipschitz functions forms a predual of the semi-Lipschitz free space, mirroring the result known for Lipschitz free spaces.

A first immediate challenge in endeavoring to construct an asymmetric version of Lipschitz free spaces is the lack of linear structure of the cone $\mathrm{SLIP}_0(X)$. In order to address this difficulty, a considerable amount of bibliographical research was required, in order to find the right tools and framework needed for this idea to work. The bulk of those efforts are condensed in Chapter 2, but some "gaps" had to be filled. We start this chapter with some propositions and lemmas that were not found in the literature, and had to be developed in order to construct the semi-Lipschitz free space.

4.1 Additional results regarding duality in asymmetric normed spaces

In what follows, we shall make use of the notation $\langle y^*, y\rangle = y^*(y)$.

Lemma 4.1 *Let E be an asymmetric normed space with dual E^*, and $\varphi : E^* \to \mathbb{R}$ a linear w^*-continuous functional. Then there exists $x_\varphi \in E$ such that $\varphi(x^*) = x^*(x_\varphi)$ for all $x^* \in E^*$.*

Proof. Since φ is w^*-continuous, the set $\varphi^{-1}(-1, 1)$ is a w^*-neighborhood of 0, so there exist

$x_1, ..., x_n \in E$ such that

$$\{x_i^* \in E^* : \max_{i=1,...,n} |\langle x^*, x_i \rangle| < 1\} \subseteq \varphi^{-1}(-1, 1),$$

which yields

$$\bigcap_{i=1}^{n} \mathrm{Ker}(\widehat{x}_i) \subseteq \mathrm{Ker}(\varphi). \tag{4.1}$$

The above kernels are contained in the cone E^*. We can linearly extend φ and the evaluation functionals $\widehat{x}_1, ..., \widehat{x}_n$ from the normed cone E^* to the linear space $\mathrm{span}(E^*) \subseteq \mathbb{R}^E$. This operation preserves the inclusion (4.1) on the linear space $\mathrm{span}(E^*)$. It follows that the extension $\widehat{x}_\varphi$ of φ is a linear combination of the extensions of $\widehat{x}_1, ..., \widehat{x}_n$. □

The following result is analogous for the classical one in the operator theory (see [38, Theorem 4.10]).

Lemma 4.2 *Let $(E, \|\cdot|_E)$, $(F, \|\cdot|_F)$ be asymmetric normed spaces, E^* and F^* their respective dual cones and $T : F^* \to E^*$ a linear bounded operator (meaning that there exists $K \geq 0$ such that $\|Ty^*|_F \leq K\|y^*|_E$ for all $x \in E$). If T is (w^*-w^*)-continuous, then there exists a linear bounded operator $S : E \to F$ such that $T = S^*$, in the sense that*

$$\langle y^*, Sx \rangle = \langle Ty^*, x \rangle, \quad \textit{for all } x \in E \textit{ and } y^* \in F^*.$$

Furthermore, if T is a bijective isometry, so is S.

Proof. Let $x \in E$, and define $f : F^* \to \mathbb{R}$ as $f(y^*) = \widehat{x}(Ty^*) = y^*(y_x) = \hat{y}_x(y^*)$, which is w^*-continuous, and therefore by Lemma 4.1 there exists y_x such that $\widehat{x}(T) = \widehat{y_x}$ and $y^*(y_x) = \hat{x}Ty^*$, and define $Sx = y_x$, which is linear and bounded, since

$$\|Sx|_F = \|y_x|_F = \|\widehat{y_x}| = \|\widehat{x} \circ T| = \sup_{\|y^*| \leq 1} (\widehat{x} \circ T)(y^*) \leq \|x|_E \|T|.$$

And $S^* = T$, as

$$\langle S^* y^*, x \rangle = \langle y^*, Sx \rangle = \langle \widehat{x} \circ T, y^* \rangle = \langle Ty^*, x \rangle$$

for all $x \in E$ and $y^* \in F^*$, so $S^* = T$. Finally, if T is an isometry then

$$\|Sx|_F = \sup_{\|y^*| \leq 1} \langle y^*, Sx \rangle = \sup_{\|y^*| \leq 1} \langle Ty^*, x \rangle = \sup_{\|y^*| \leq 1} \langle x^*, x \rangle = \sup_{\|x^*| \leq 1} \langle Ty^*, x \rangle,$$

where the first equality follows as a corollary of the Hahn-Banach theorem for asymmetric normed spaces ([15, Corollary 2.2.4]). □

The following proposition shows that an asymmetric normed space and its bicompletion have the same dual. This fact will be relevant for our main result.

Proposition 4.3 (Unique extension of a linear usc functional) *Let $(E, \|\cdot|)$ be an asymmetric normed space, $D \subseteq E$ a subspace that is dense in the symmetrization of the induced quasi-metric, and $\varphi : D \to \mathbb{R}$ a linear usc functional. Then φ has a unique linear usc extension to E.*

Proof. Thanks to the Hahn-Banach theorem [15, Theorem 2.2.1], φ has at least one linear usc extension to E. Let us assume, towards a contradiction, that φ has two different extensions ϕ_1 and ϕ_2, with $\phi_1(x) < \phi_2(x)$ for some $x \in E$. Since D is dense for the symmetrized extended quasi-metric (*c.f.* Definition 2.8), there is a sequence $\{x_n\}_n \subseteq D$ such that $x_n \to x$ in both d_e and $\bar{d}_e$. Since ϕ_1 and ϕ_2 are usc for d_e, we deduce that they are also lsc for $\bar{d}_e$ (see Remark 2.54). Moreover, both functionals coincide on the sequence $\{x_n\}_n$. We deduce:

$$\limsup_n \phi_2(x_n) \leq \phi_1(x) < \phi_2(x) \leq \liminf_n \phi_2(x_n),$$

which is a contradiction. Therefore $\phi_1 = \phi_2$. □

Proposition 4.4 (Dual of an asymmetric normed space) *Let $(E, \|\cdot|)$ be an asymmetric normed space and $(\tilde{E}, \|\cdot|_\sim)$ its bicompletion. Then, the respective dual cones are isometrically isomorphic.*

Proof. We already know that the extension mapping from E^* to $\tilde{E}^*$ is a bijection, in virtue of Proposition 4.3. To check that it is an isometry, we only need to check that $\|\phi_{|E}|^* \geq \|\phi|^*$ for any $\phi \in \tilde{E}^*$, as the reverse inequality is obvious. Let $B_{\tilde{E}}$ be the unit ball of $\tilde{E}$ for the forward distance, and consider $\phi \in \tilde{E}^*$ and a sequence $\{z_n\}_n$ on $B_{\tilde{E}}$ such that $\phi(z_n) \to \|\phi|^* := \sup_{z \in B_{\tilde{E}}} \phi(z)$. Since E is dense for the symmetrized topology in $\tilde{E}$ (by definition), for each $n \in \mathbb{N}$ there exists a sequence $\{x_n^j\}_j \subseteq B_E$ such that $\{x_n^j\}$ converges to z_n in the symmetrized distance of $\tilde{E}$. In particular, $\{x_n^j\}_j$ converges for both quasi-metrics d_e and $\bar{d}_e$. Since ϕ is lsc for $\bar{d}_e$, we have that $\phi(z_n) \leq \liminf_j \phi(x_n^j)$, for every $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|\phi|^* < \varepsilon + \phi(z_{n_0})$, and consequently

$$\|\phi|^* < \varepsilon + \liminf_j \phi\left(x_{n_0}^j\right) \leq \varepsilon + \|\phi_{|E}|^*.$$

This completes the proof. □

Finally, the following lemma will be of use for Subsection 4.5.

Lemma 4.5 ($(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})^* = (\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$) *Let $\mathcal{L}^1(\mathbb{R})$ be endowed with the asymmetric norm*

$$\|f|_{1,+} := \int_{\mathbb{R}} f^+ d\lambda,$$

where $f^+(x) = \max\{f(x), 0\}$ and λ denotes the Lebesgue measure. Then, the dual of $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ is isometrically isomorphic to $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$, where $\mathcal{L}^\infty_+(\mathbb{R})$ denotes the cone of non negative functions in $\mathcal{L}^\infty(\mathbb{R})$.

Proof. The facts that $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ is an asymmetric normed space and $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$ is a normed cone are straightforward. Take $\varphi \in (\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})^*$. Then $\varphi : \mathcal{L}^1(\mathbb{R}) \to \mathbb{R}$ is linear and ($\|\cdot|_{1,+}$-u)-continuous (see Example 4.25). Then, by Proposition 2.45, φ is continuous for the symmetrized norms in both spaces, therefore

$$|\varphi(f)| \leq \|\varphi\|^* \max\left\{\|f^+|_{1,+},\ \|{-f^+}|_{1,+}\right\} \leq \|\varphi\|^* \|f\|_1,$$

where $\|\cdot\|^*$ denotes the dual norm of the normed space $(\mathcal{L}^1(\mathbb{R}), (\|\cdot|_{1,+})^s)$ and $\|\cdot\|_1$ is the usual norm on $\mathcal{L}^1(\mathbb{R})$. It follows that φ is $(\|\cdot\|_1$-$|\cdot|)$-continuous, and therefore there exists $g \in \mathcal{L}^\infty(\mathbb{R})$ such that $\varphi(f) = \int gf d\lambda$ for all $f \in \mathcal{L}^1(\mathbb{R})$.

We claim that $g \geq 0$ almost everywhere:
Indeed, suppose, towards a contradiction, that there exists a set E of measure $0 < \lambda(E) < \infty$ such that $g < 0$ on E. Consider the sequence $f_n = -n\mathbf{1}_E$ (where $\mathbf{1}_E$ is the characteristic function of E), which clearly belongs to $\mathcal{L}^1(\mathbb{R})$. On the other hand, since $\|f_n|_{1,+} = 0$ for all $n \in \mathbb{N}$, the function f_n belongs to the unit ball of the asymmetric norm $\|\cdot|_{1,+}$. Then, as $n \longrightarrow +\infty$, we deduce

$$\varphi(f_n) = \int gf_n d\lambda = \int_{E^c} gf_n d\lambda + \int_E gf_n d\lambda = n\int_E (-g) d\lambda \longrightarrow +\infty.$$

Therefore, φ can not be $(\|\cdot|_{1,+}$-$u)$-continuous, a contradiction.

Notice now that any $g \in \mathcal{L}^\infty_+(\mathbb{R})$ defines a linear $(\|\cdot|_{1,+}$-$u)$-continuous functional φ in the same manner:

$$\varphi(f) = \int_{\mathbb{R}} g\, f\, d\lambda \leq \int_{\mathbb{R}} gf^+ d\lambda \leq \|g\|_\infty \int_{\mathbb{R}} f^+ = \|g\|_\infty \|f|_{1,+},$$

which yields that $\|\varphi|^* \leq \|g\|_\infty$. On the other hand, take $\varepsilon > 0$ and a set E of finite measure such that $g(x) \geq \|g\|_\infty - \varepsilon$ on E. Then consider the function $f = \dfrac{\mathrm{sgn}(g)}{\lambda(E)}\mathbf{1}_E$, where $\mathrm{sgn}(g)$ denotes the sign of g, and note that $\|f|_{1,+} \leq 1$. Then

$$\varphi(f) = \frac{1}{\lambda(E)}\int_E g d\lambda \geq \frac{1}{\lambda(E)}\int_E [\|g\|_\infty - \varepsilon] d\lambda = \|g\|_\infty - \varepsilon.$$

It follows that $\|\varphi|^* = \|g\|_\infty$, and therefore, we can identify the dual of $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ to $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$ by an isometric isomorphism. □

4.2 Construction of the semi-Lipschitz free space

Throughout this Chapter, (X, d) will denote a quasi-metric space, with d being possibly a quasi-hemi-metric, and with base point $x_0 \in X$. We are ready to proceed to the construction of the (asymmetric) semi-Lipschitz free space. For every $x \in X$ we consider the corresponding *evaluation* mapping

$$\delta_x : \mathrm{SLIP}_0(X) \to \mathbb{R} \text{ defined by } \delta_x(f) = f(x), \ \forall f \in SLip_0(X).$$

Notice that δ_x is a linear mapping over the cone $SLip_0(X)$ (*c.f.* Definition 2.18). We can also define the linear mapping $-\delta_x$ by $-\delta_x(f) := -f(x)$, for all $f \in SLip_0(X)$.

Proposition 4.6 (δ_x belongs to the linearity part of $(SLip_0(X))^*$) *For each $x \in X$, both the evaluation functional $\delta_x : \mathrm{SLIP}_0(X) \to \mathbb{R}$ and its opposite $-\delta_x$ belong to the dual cone* $(\mathrm{SLip}_0(X), \|\cdot|_S)^*$.

Proof. Let $x \in X$. Since δ_x is linear, we only need to check that it is bounded from above on the unit ball of $SLip_0(X)$. Indeed, for any $f \in \mathrm{SLIP}_0(X)$, we have $f(x) = f(x) - f(x_0) \leq d(x_0, x) \, \|f|_S$, therefore $\delta_x \in \mathrm{SLip}_0(X)^*$. Using the same argument, we get that $-f(x) \leq d(x, x_0) \, \|f|_S$. $\square$

Remark 4.7 *The fact that both δ_x and $-\delta_x$ are semi-Lipschitz yields that δ_x is actually a Lipschitz function on $(SLip_0(X), \| \cdot |)$ of constant $\|\delta_x\|_{\mathrm{LIP}} = \max\{d(x, x_0), d(x_0, x)\}$.*

Proposition 4.8 (Isometric injection of X into $SLip_0(X)^*$) *The mapping*

$$\delta : (X, d) \to (\mathrm{SLIP}_0(X)^*, \| \cdot |^*)\,,$$

defined by $\delta(x) = \delta_x$ is (injective and) an isometry *onto its image. Therefore, for any $x, y \in X$, we have:*

$$d(x, y) = \|\delta_y - \delta_x|^*.$$

Proof. Let us take $x, y \in X$. First of all, it is worth noting that the quasi-metric generated by the conic-norm is extended (Proposition 2.24) and that $\|\delta_y - \delta_x|^*$ is well defined (by Proposition 4.6). Note also that any dual cone is cancellative, since it is contained in a linear space of real-valued functions. To prove injectivity of δ, consider $x, y \in X$ such that $\delta_x = \delta_y$. Then we take the functions $f(\cdot) = d(x, \cdot) - d(x, x_0)$ and $g(\cdot) = d(y, \cdot) - d(y, x_0)$. Since $\delta_x(f) = \delta_y(f)$, and $\delta_x(g) = \delta_y(g)$, we conclude that both $d(x, y)$ and $d(y, x)$ must be zero, therefore $x = y$ (Definition 2.1(iii)).

By Remark 2.25(ii), for any $x, y \in X$ we have that $d_e(\delta_x, \delta_y) = \|\delta_y - \delta_x|^*$. Then, for any $x, y \in X$,

$$\begin{aligned} d_e(\delta_x, \delta_y) = \sup_{\|f|_S \leq 1} (\delta_y - \delta_x)(f) &= \sup_{\|f|_S \leq 1} \big\{f(y) - f(x)\big\} \\ &\leq \sup_{\|f|_S \leq 1} \|f|_S \, d(x, y) = d(x, y). \end{aligned}$$

Conversely, by taking $f(\cdot) = d(x, \cdot) - d(x, x_0)$ it follows (see Proposition 2.48) that

$$f(y) - f(x) = d(x, y) \text{ and } f(y) - f(x) = (\delta_y - \delta_x)(f) \leq \|\delta_y - \delta_x|_* = d_e(\delta_x, \delta_y).$$

Then the result holds. $\square$

We now take the asymmetric normed space $(\mathrm{span}\,(\delta(X))\,, \| \cdot |^*)$ (which is contained in the normed cone $(\mathrm{SLIP}_0(X), \| \cdot |^*)$), and we define the (asymmetric) *semi-Lipschitz free space* to be the bicompletion of $(\mathrm{span}\,(\delta(X))\,, \| \cdot |^*)$.

Definition 4.9 (The semi-Lipschitz free space) *Let (X, d) be a quasi-metric space with base point x_0. The* semi-Lipschitz free space over (X, d), *denoted by $\mathcal{F}_a(X)$, is the (unique) bicompletion of the asymmetric normed space* $(\mathrm{span}\,(\delta(X))\,, \| \cdot |^*)$, *where $\|\cdot|^*$ is the restriction of the norm of* $\mathrm{SLIP}_0(X)^*$.

4.3 Main properties

We are now ready to establish our main result which is analogous of the fundamental property of the Lipschitz-free space of a metric space: being a predual of the space of Lipschitz functions vanishing at the base point.

Theorem 4.10 ($\mathcal{F}_a(X)^* = SLip_0(X)$) *Let (X, d) be a quasi-metric space with base point x_0. Then the dual cone of $\mathcal{F}_a(X)$ is isometrically isomorphic to* $\mathrm{SLIP}_0(X)$.

Proof. Thanks to Proposition 4.4, we only need to check that the dual cone of $(\mathrm{span}\,(\delta(X))\,, \|\cdot^*)$ is isometrically isomorphic to $\mathrm{SLIP}_0(X)$. To this end, we define the mapping

$$\Phi : \mathrm{SLIP}_0(X) \to (\mathrm{span}\,(\delta(X))\,, \|\cdot|^*)^*\,,$$

with

$$\Phi(f)\left(\sum_i \lambda_i\,\delta_{x_i}\right) = \sum_i \lambda_i f\,(x_i)$$

for any linear combination of evaluation functionals. First, we check that Φ is well defined: Φ is obviously linear and we next demonstrate the condition (ii) of Proposition 2.53. For any $f \in \mathrm{SLIP}_0(X)$ and any $\sum_i \lambda_i\,\delta_{x_i} \in \mathrm{span}\,(\delta(X))$, we have

$$\Phi(f)\left(\sum_i \lambda_i \delta_{x_i}\right) = \sum_i \lambda_i f(x_i) = \left(\sum_i \lambda_i \delta_{x_i}\right)(f) \;\le\; \left\|\sum_i \lambda_i \delta_{x_i}\right|^* \|f|_S.$$

Therefore $\|f|_S \ge \|\Phi(f)|^{**}$, where $\|\cdot|^{**}$ is the norm on $(\mathrm{span}\,(\delta(X))\,, \|\cdot|^*)^*$. Conversely, consider $f \in \mathrm{SLIP}_0(X)$. Then, by Proposition 2.39, we have

$$\begin{aligned}\|f|_S &= \sup_{d(y,x)>0} \frac{\max\{f(x)-f(y),0\}}{d(y,x)} \\ &= \sup_{d(y,x)>0} \frac{\max\{\Phi(f)\,(\delta_x-\delta_y)\,,0\}}{\|\delta_x-\delta_y|^*} \le \|\Phi(f)|^{**},\end{aligned}$$

from which we deduce that Φ is an isometry. Since Φ is obviously linear and injective, it remains only to establish surjectivity. This follows from the fact that any $\varphi \in (\mathrm{span}\,(\delta(X))\,, \|\cdot|^*)^*$ can be seen as $\Phi\,(\varphi\circ\delta)$, with $\varphi\circ\delta$ being semi-Lipschitz on X: indeed, for every $x, y \in X$ we have:

$$\varphi\,(\delta(x)) - \varphi\,(\delta(y)) = \varphi\,(\delta_x - \delta_y) \le \|\varphi|^{**}\|\delta_x-\delta_y|^* = \|\varphi|^{**}\,d(y,x).$$

This shows that $\varphi\circ\delta$ belongs to $SLip_0(X)$ and Φ is surjective. □

Remark 4.11 (Compatibility with the classical theory of metric free spaces) *If (X, d) is a metric space, then* $\mathrm{SLIP}_0(X) = \mathrm{LIP}_0(X)$. *Moreover, every linear usc functional on a normed space is continuous; thus, the dual cone of a normed linear space is the same as the usual dual. We deduce that $\mathcal{F}_a(X) = \mathcal{F}(X)$.*

Remark 4.12 *For a quasi-metric space* (X,d), *it is easy to check that the space of semi-Lipschitz functions for the reverse quasi-metric* $\mathrm{SLIP}_0(X,\bar{d})$ *is exactly* $-\mathrm{SLIP}_0(X,d)$, *and that* $\|f|_S = \|{-f}|_{\bar{S}}$ *for any* $f \in \mathrm{SLIP}_0(X,d)$, *where* $\|{-f}|_{\bar{S}}$ *denotes the semi-Lipschitz constant of* $-f$ *on* $(X,\bar{d})$. *Using this isometry, we can identify the dual cones of* $\mathrm{SLIP}_0(X,\bar{d})$ *by the isometry* Ψ *defined by* $\Psi(\mu)(f) = \mu(-f)$ *for all* $f \in \mathrm{SLIP}_0(X,d)$, *and therefore we obtain that* $\mathcal{F}_a(X,d) = \Psi(\mathcal{F}_a(X,\bar{d}))$ *and that* $\|\Psi(\mu)|^*_{\bar{d}} = \|{-\mu}|^*$, *where* $\|\cdot|^*_{\bar{d}}$ *is the norm of* $\mathcal{F}_a(X,\bar{d})$.

4.3.1 Linearization of semi-Lipschitz functions: a universal property

As normed cones can be endowed with extended quasi-metrics (see Proposition 2.24), we can apply Definition 2.33 to the case of semi-Lipschitz functions with values in a normed cone. Let $(C,\|\cdot|)$ be a normed cone, and denote as $d_e^c(u,v)$ its corresponding extended quasi-metric (as per Proposition 2.24). For the sake of convenience, we rewrite some of our previous definitions for functions with values in C.

As in Proposition 2.39, a function $f : X \to C$ is semi-Lipschitz if and only if $\|f|_S < \infty$. Moreover, if d is a quasi-metric and $f : X \to C$ is semi-Lipschitz, then

$$\|f|_S = \sup_{x\neq y} \frac{\max\{d_e^c(f(y),f(x)),0\}}{d(y,x)} = \sup_{x\neq y} \frac{d_e^c(f(y),f(x))}{d(y,x)} < \infty.$$

The same as Proposition 2.39 applies to the case that d is a quasi-hemi-metric.

Given a quasi-metric space (X,d) with base point x_0, for the following result consider the isometric injection $\delta : (X,d) \to (\mathrm{SLIP}_0(X)^*, \|\cdot|^*)$ of Proposition 4.8. We next show that the semi-Lipschitz free space over a quasi-metric space (X,d) with base point x_0 is characterized by the following universal property, which is an analog of the Lipschitz case (see [29, Lemma 2.2]).

Theorem 4.13 (Linearization of semi-Lipschitz functions) *Let* (X,d) *be a quasi-metric space with base point* x_0. *Suppose that* $(C,\|\cdot|)$ *is a normed cone and* $f \in \mathrm{SLIP}_0(X,C)$. *Then there exists a unique linear map* $T_f : \mathcal{F}_a(X) \to C$ *extending* f, *i.e.* $T_f \circ \delta = f$ *and* $\|T_f| = \|f|_S$.

Proof. If $f \in \mathrm{SLIP}_0(X,C)$, then $T_f \colon \mathcal{F}_a(X) \to C^{**}$ defined by

$$T_f(\gamma)(\phi) = \gamma(\phi \circ f) \qquad (\gamma \in \mathcal{F}_a(X),\ \phi \in C^*)$$

belongs to the set of bounded linear mappings from $\mathcal{F}_a(X)$ into C^{**}, and

$$\begin{aligned}\|T_f| = \sup_{\|\gamma|^*\leq 1} \|T_f(\gamma)|^{**} &= \sup_{\|\gamma|^*\leq 1}\sup_{\|\phi|^*\leq 1} T_f(\gamma)(\phi)\\ &= \sup_{\|\phi|^*\leq 1}\sup_{\|\gamma|^*\leq 1} \gamma(\phi\circ f) = \sup_{\|\phi|^*\leq 1} \|\phi\circ f|_S \leq \|f|_S.\end{aligned}$$

Observe that the last inequality is accomplished by taking into account that ϕ is linear and

$$\sup_{\|\phi|^*\leq 1} \|\phi \circ f|_S = \sup_{\|\phi|^*\leq 1} \sup_{d(y,x)>0} \left\{ \frac{(\phi \circ f)(x) - (\phi \circ f)(y)}{d(y,x)} \right\}$$
$$= \sup_{\|\phi|^*\leq 1} \sup_{d(y,x)>0} \left\{ \frac{\phi(f(x) - f(y))}{d(y,x)} \right\} \leq \sup_{\|\phi|^*\leq 1} \|\phi|^* \, \|f|_S = \|f|_S.$$

(By abuse of notation, we still denote by $\|T_f| = \sup_{\|\gamma|^*\leq 1} \|T_f(\gamma)|^*$ the conic-norm of the linear function $T_f \colon \mathcal{F}_a(X) \to C^{**}$). Furthermore, if $i_C \colon C \to C^{**}$ is the canonical injection, we have

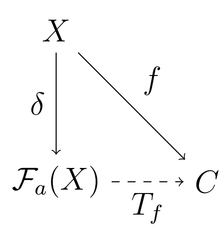
$$\langle T_f(\delta(x)), \phi \rangle = T_f(\delta(x))(\phi) = \delta(x)(\phi \circ f)$$
$$= \phi(f(x)) = i_C(f(x))(\phi) = \langle i_C(f(x)), \phi \rangle$$

for every $x \in X$ and $\phi \in C^*$, and hence $T_f(\delta(x)) = i_C(f(x)) \in i_C(C)$ for every $x \in X$. This yields that $T_f(\gamma) \in i_C(C)$ for every $\gamma \in \mathcal{F}_a(X)$. Identifying $i_C(f(x)) \in i_C(C)$ with $f(x) \in C$, we have $T_f \in \mathcal{L}(\mathcal{F}_a(X), C)$ and $T_f \circ \delta = f$. So, since $T_f \circ \delta = f$ and δ is an isometry (Proposition 4.8), we deduce that

$$\|f|_S = \sup_{d(y,x)>0} \left\{ \frac{d_e^c(f(y), f(x))}{d(y,x)} \right\}$$
$$= \sup_{d(y,x)>0} \left\{ \frac{\|T_f(\delta(x)) - T_f(\delta(y))|}{d(y,x)} \right\} = \sup_{d(y,x)>0} \left\{ \frac{\|T_f(\delta(x) - \delta(y))|}{d(y,x)} \right\}$$
$$\leq \sup_{d(y,x)>0} \left\{ \frac{\|T_f|\|\delta(x) - \delta(y)|^*}{d(y,x)} \right\} = \|T_f| \sup_{d(y,x)>0} \left\{ \frac{\|\delta(x) - \delta(y)|^*}{\|\delta(x) - \delta(y)|^*} \right\} = \|T_f|.$$

Thus $\|T_f| = \|f|_S$. Assume now that there exists a linear bounded mapping $S_f : \mathcal{F}_a(X) \to C$ such that $S_f \circ \delta = f$. Then it is clear that $S_f(\delta(x)) = T_f(\delta(x))$ for all $x \in X$ and, by the definition of $\mathcal{F}_a(X)$, it follows that $S_f = T_f$. □

Remark 4.14 (Universal property) *Equivalently, the condition $T_f \circ \delta = f$ means that the following diagram commutes*



Furthermore, as a consequence of the universal property that we have just proved, it is not difficult to establish that the mapping $f \mapsto T_f$ is an isometric isomorphism of $\mathrm{SLIP}_0(X, C)$ *into the cone of bounded linear mappings $L(\mathcal{F}_a(X), C)$, which constitutes another proof of Theorem 4.10 for the particular case $C = \mathbb{R}$. Indeed, we already know that the mapping $f \mapsto T_f$ is an isometry of* $\mathrm{SLIP}_0(X, \mathbb{R})$ *onto $\mathcal{F}_a(X)^*$. Now, given $T \in L(\mathcal{F}_a(X), C)$, we can define a mapping $f \colon X \to C$ by $f(x) = T(\delta(x))$ for all $x \in X$. Since*

$$d_e^c(f(y), f(x)) = d_e^c(T(\delta(y)), T(\delta(x))) \leq \|T| \, \|\delta(x) - \delta(y)|^* = \|T| \, d(y,x)$$

for all $x, y \in X$, the function f is in $\mathrm{SLIP}_0(X, C)$. *By the universal property of $\mathcal{F}_a(X)$, there is a unique operator $T_f \in L(\mathcal{F}_a(X), C)$ such that $T_f \circ \delta = f$. Hence $T = T_f$ and thus the mapping $f \mapsto T_f$ is a surjective isometry.*

The proof of the following result is immediate from Theorem 4.13.

Corollary 4.15 (Linearization of quasi-metric morphisms) *Let (X_1, d_1) and (X_2, d_2) be two pointed quasi-metric spaces, and $f \in \mathrm{SLIP}_0(X_1, X_2)$. Then there is a unique linear map $\hat{T}_f : \mathcal{F}_a(X_1) \to \mathcal{F}_a(X_2)$ such that $\hat{T}_f \circ \delta_{X_1} = \delta_{X_2} \circ f$, i.e. the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad f \quad} & X_2 \\ \delta_{X_1} \Big\downarrow & & \delta_{X_2} \Big\downarrow \\ \mathcal{F}_a(X_1) & \underset{\hat{T}_f}{\dashrightarrow} & \mathcal{F}_a(X_2) \end{array}$$

commutes, and $\|\hat{T}_f| = \|f|_S$, where δ_{X_1} and δ_{X_2} are the isometric injections of the quasi-metric spaces (X_1, d_1) and (X_2, d_2) to their free spaces (c.f. *Proposition 4.8).*

For the following proposition, we refer to the reader to [33] for a survey on the extensions of semi-Lipschitz functions on quasi-metric spaces.

Proposition 4.16 (The free space of a quasi-metric subspace) *Let (X, d) be a quasi-metric space with base point x_0, and consider (M, d) a subspace of (X, d) such that $x_0 \in M$. Then $\mathcal{F}_a(M)$ is isometrically isomorphic to a subspace of $\mathcal{F}_a(X)$.*

Proof. Let $\hat{T}_i : \mathcal{F}_a(M) \to \mathcal{F}_a(X)$ be the linearization given by Corollary 4.15 of the identity mapping $i : M \to X$. Since $\|\hat{T}_i| = \|i|_S = 1$, we know that $\|\hat{T}_i(Q)|^*_{\mathcal{F}_a(X)} \leq \|Q|^*_{\mathcal{F}_a(M)}$. For the opposite inequality, consider $Q \in \mathrm{span}\,(\delta(M))$. Clearly, $\hat{T}_i(Q) = Q \in \mathrm{span}\,(\delta(X))$. Then, for any $f \in \mathrm{SLIP}(M)$, the expression $\tilde{f}(x) = \inf_{m \in M}\{f(m) + \|f|_S d(m, x)\}$, $x \in X$ (which is an adaptation of the McShane extension of Lipschitz maps), provides a semi-Lipschitz extension with the same associated conic-norm $\|f|_S$. It follows that

$$\|Q|^*_{\mathcal{F}_a(M)} = \sup_{\substack{\|f|_S \leq 1 \\ f \in \mathrm{SLIP}(M)}} \langle Q, f \rangle \leq \sup_{\substack{\|f|_S \leq 1 \\ f \in \mathrm{SLIP}(X)}} \langle Q, f \rangle = \|Q|^*_{\mathcal{F}_a(X)} = \|\hat{T}_i(Q)|^*_{\mathcal{F}_a(X)}.$$

By continuity of $\hat{T}_i$ (and density of $\mathrm{span}\,(\delta(M))$ in $\mathcal{F}_a(M)$), we can extend the previous inequality to any $Q \in \mathcal{F}_a(M)$, which concludes the proof. □

4.3.2 Preservation of index of symmetry

Proposition 4.17 *Let (X, d) be a quasi-metric space with index of symmetry $c(X) > 0$. Then,* $\mathrm{SLIP}_0(X)$ *is an asymmetric normed space, with the same index of symmetry as (X, d).*

Proof. The fact that $\mathrm{SLIP}_0(X)$ is a linear space follows from Proposition 2.36, noting that for real-valued functions, $\mathrm{SLIP}_{d,d_u}(-f) = \mathrm{SLIP}_{\bar{d},d_u}(f)$. The fact that the index of symmetry of $\mathrm{SLIP}_0(X)$ is no greater than $c(X)$ also follows from Proposition 2.36. To see that the index of symmetry of $\mathrm{SLIP}_0(X)$ can not be lower than $c(X)$, fix $y \in X$ and consider the function $f_y(x) = d(y,x)$, which is semi-Lipschitz of constant 1 (see Proposition 2.48). If we assume $c(\mathrm{SLIP}_0(X)) = \alpha < c(X)$, then f_y will be an α-semi-Lipschitz function on $(X, \bar{d})$, and therefore

$$f_y(x) - f_y(y) = d(y,x) \leq \alpha \bar{d}(y,x) = \alpha d(x,y),$$

for any $x, y \in X$, which contradicts the definition of $c(X)$. □

Proposition 4.18 (Preservation of index of symmetry) *Let (X,d) be a quasi-metric space. Then, the semi-Lipschitz free space $\mathcal{F}_a(X)$ has the same index of symmetry as (X,d).*

Proof. It is clear that, since $\mathcal{F}_a(X)$ contains an isometric copy of (X,d), its index of symmetry is no greater than $c(X)$. This implies that if $c(X) = 0$, then so is $c\left(\mathcal{F}_a(X)\right)$. On the other hand, if $c(X) > 0$, Proposition 4.17 implies that $c(X)\|f|_S \leq \|-f|_S \leq c(X)^{-1}\|f|_S$ for every $f \in \mathrm{SLIP}_0(X)$. This inequality, together with the definition of the norm in $\mathcal{F}_a(X)$ yield that $c(X)\|\mu|^* \leq \|-\mu|^* \leq c(X)^{-1}\|\mu|^*$ for all $\mu \in \mathcal{F}_a(X)$. □

This result, albeit simple, can be extremely useful, as it allows us to apply results concerning the index of symmetry obtained in the context of asymmetric normed spaces (such as the ones showed in [7]) to any quasi-metric space. The following proposition follows directly from one such result (see Proposition 2.58) and Proposition 4.18.

Proposition 4.19 *Let (X,d) be a quasi-metric space. Then, the following are equivalent:*

- *(i)* $c(X) > 0$
- *(ii)* $\mathrm{SLIP}_0(X)$ *is a linear space.*

4.3.3 Relation with molecules and the Kantorovich-Rubinstein norm

Given (X,d) a quasi-metric space (always with a base point $x_0 \in X$), we next give a description of the closed unit ball of $\mathcal{F}_a(X)$ by means of the semi-Lipschitz evaluation functionals (often called molecules)

$$M_{(x,y)} = \frac{\delta(x) - \delta(y)}{d(y,x)},$$

where $x, y \in X$ such that $d(y,x) > 0$. Let $\widehat{\mathcal{M}}_X$ denote the set $\{M_{(x,y)} : x, y \in X \text{ with } d(y,x) > 0\}$.

Before going to this, if (X,d) is an asymmetric locally convex space, it is worth noting that the *asymmetric polar* of a subset $Y \subset X$ in the case of the asymmetric dual X^* can be defined as [15, p. 161]

$$Y^\alpha = \{\varphi \in X^* : \varphi(y) \leq 1, \text{ for all } y \in Y\}.$$

Analogously, we can define the *asymmetric polar* of a subset W of the dual X^* by [15, p. 165]

$$W_\alpha = \{x \in X : \varphi(x) \le 1, \text{ for all } \varphi \in W\}.$$

Proposition 4.20 *Let (X, d) be a quasi-metric space with base point x_0. The closed unit ball of $\mathcal{F}_a(X)$ coincides with $\left(\{M_{(x,y)} : x, y \in X : d(y,x) > 0\}^\alpha\right)_\alpha$.*

Proof. Let $B_{\mathrm{SLIP}_0(X)}$, $B_{\mathcal{F}_a(X)}$ and $B_{\mathcal{F}_a(X)^*}$ denote respectively the closed unit balls of $\mathrm{SLIP}_0(X)$, $\mathcal{F}_a(X)$ and $\mathcal{F}_a(X)^*$, and consider the isometry $\Phi : \mathrm{SLIP}_0(X) \to (\mathrm{span}\,(\delta(X)), \|\cdot|^*)^*$ defined in the proof of Theorem 4.10 as

$$\Phi(f)\left(\sum_i \lambda_i \delta_{x_i}\right) = \sum_i \lambda_i f(x_i)$$

for any linear combination of evaluation functionals. If $f \in \mathrm{SLIP}_0(X)$, the condition $\|f|_S \le 1$ is equivalent to $\frac{f(x)-f(y)}{d(y,x)} \le 1$, for all $x, y \in X$ with $d(y,x) > 0$ (by Proposition 2.39). Since Φ is an isometry, $\|f|_S \le 1$ also yields $\Phi(f)(M_{(x,y)}) \le 1$, for all $M_{(x,y)} \in \widehat{\mathcal{M}}_X$. Hence

$$\begin{aligned} B_{\mathcal{F}_a(X)^*} &= \{\Phi(f) : f \in \mathrm{SLIP}_0(X), \Phi(f)(M_{(x,y)}) \le 1, \ \forall M_{(x,y)} \in \widehat{\mathcal{M}}_X\} \\ &= \{F \in \mathcal{F}_a(X)^* : F(M_{(x,y)}) \le 1, \ \forall M_{(x,y)} \in \widehat{\mathcal{M}}_X\} = (\widehat{\mathcal{M}}_X)^\alpha \end{aligned}$$

and thus

$$\Phi(B_{\mathrm{SLIP}_0(X)})_\alpha = \left((\widehat{\mathcal{M}}_X)^\alpha\right)_\alpha.$$

Moreover,

$$\begin{aligned} \left((\widehat{\mathcal{M}}_X)^\alpha\right)_\alpha &= \Phi(B_{\mathrm{SLIP}_0(X)})_\alpha \\ &= \{\gamma \in \mathcal{F}_a(X) : \Phi(f)(\gamma) \le 1, \ \forall f \in B_{\mathrm{SLIP}_0(X)}\} \\ &= \{\gamma \in \mathcal{F}_a(X) : \gamma(f) \le 1, \ \forall f \in B_{\mathrm{SLIP}_0(X)}\} \\ &= \{\gamma \in \mathcal{F}_a(X) : \|\gamma|^*(= \sup_{\|f|_S \le 1} \gamma(f) \ ) \le 1\} = B_{\mathcal{F}_a(X)}. \end{aligned}$$

The proof is complete. □

Remark 4.21 *Let (X, d) be a quasi-metric space and $\overline{x} \notin X$. Then setting $\tilde{X} = X \cup \{\overline{x}\}$ and extending d from $X \times X$ to $\tilde{X} \times \tilde{X}$ by $\tilde{d}(x, \overline{x}) = \tilde{d}(\overline{x}, x) = 1$ and $\tilde{d}(\overline{x}, \overline{x}) = 0$, we obtain a new quasi-metric space $(\tilde{X}, \tilde{d})$ with base point $x_0 \equiv \overline{x}$. Then the above construction will correspond to an asymmetric version of the Arens-Eells approach (*c.f. *[6]).*

Let us consider another conic-norm on $\mathrm{span}\,(\delta(X))$ (and on $\mathcal{F}_a(X)$) which is based on a variant of the so-called Kantorovich-Rubinstein norm (see [16, Section 8.4.5]).

Example 4.22 (Kantorovich-Rubinstein conic-norm) *Let X be a vector space equipped with a quasi-metric d and a base point x_0. For $\gamma, \overline{\gamma} \in \mathrm{span}\,(\delta(X))$ take the representation $\gamma - \overline{\gamma} = \sum_{i=1}^n \lambda_i(\hat{y}_i - \hat{z}_i)$, where possibly some $\hat{y}_i$ or $\hat{z}_i$ are equal to $\widehat{x}_0 = 0$, and set*

$$d_{KR}(\gamma, \overline{\gamma}) := \inf\{d(\lambda_1 z_1, \lambda_1 y_1) + \ldots + d(\lambda_n z_n, \lambda_n y_n)\}.$$

Then $\|\gamma|_{KR} := d_{KR}(\widehat{x}_0, \gamma)$ *is an asymmetric norm on* $\operatorname{span}(\delta(X))$ *and*

$$d_{KR}(\widehat{x}, \hat{y}) = d(y, x), \text{ for all } x, y \in X.$$

Moreover, $\|\gamma|_{KR}$ *coincides with the restriction of the conic norm* $\|\cdot|^*$ *of* $\mathrm{SLIP}_0(X)^*$ *to* $\operatorname{span}(\delta(X))$ *and thus extends to* $\mathcal{F}_a(X)$*. Indeed, if* $\|\cdot|'$ *is a conic-norm on* $\operatorname{span}(\delta(X))$ *satisfying* $\|\delta(x)-\delta(y)|' \leq d(y,x)$*, for all* $x, y \in X$*, then every* $\gamma = \lambda_1(\hat{y_1}-\hat{z_1})+\ldots+\lambda_n(\hat{y_n}-\hat{z_n})$ *accomplishes*

$$\begin{aligned}\|\gamma|' = \|\lambda_1(\hat{y_1}-\hat{z_1})+\ldots+\lambda_n(\hat{y_n}-\hat{z_n})|' &\leq \|\lambda_1(\hat{y_1}-\hat{z_1})|'+\ldots+\|\lambda_n(\hat{y_n}-\hat{z_n})|' \\ &\leq d(\lambda_1 z_1, \lambda_1 y_1)+\ldots+d(\lambda_n z_n, \lambda_n y_n),\end{aligned}$$

which shows that $\|\gamma|' \leq \|\gamma|_{KR}$*. Particularly, we deduce from this that* $\|\gamma|^* \leq \|\gamma|_{KR}$ *(since the conic-norm* $\|\cdot|^*$ *on* $\mathcal{F}_a(X)$ *satisfies* $\|\delta(x)-\delta(y)|^* = d(y,x)$*, for all* $x, y \in X$*). Hence* $d(y,x) = \|\delta(x)-\delta(y)|^* \leq \|\delta(x)-\delta(y)|_{KR} \leq d(y,x)$*, for all* $x, y \in X$*, which implies that*

$$\|\delta(x)-\delta(y)|_{KR} = d(y,x), \quad \text{for all } x, y \in X.$$

Consider now the mapping $L : X \to (\operatorname{span}(\delta(X)), \|\cdot|_{KR})$ *sending* x *to* $\delta(x)$*, which is clearly an isometric embedding. By the universality property of* $\mathcal{F}_a(X)$ *(see Theorem 4.13), we know that* L *extends to* $\tilde{L} : \mathcal{F}_a(X) \to (\operatorname{span}(\delta(X)), \|\cdot|_{KR})$ *and* $\|\cdot|_{KR} \leq \|\cdot|^*$*, so the conic-norms* $\|\cdot|_{KR}$ *and* $\|\cdot|^*$ *are the same.*

4.4 Canonical asymmetrizations and free spaces

We shall start this section by noting how, given a metric space (X, D), certain subcones of $\mathrm{LIP}_0(X)$ can be used to endow the Lipschitz-free space $\mathcal{F}(X)$ of asymmetric hemi-norms. Following Definition 2.14, all cones considered in this Section will be assumed to be convex.

Remark 4.23 (Asymmetrizations in $\mathcal{F}(X)$) *There is a natural way to asymmetrize the norm* $\|\cdot\|_{\mathcal{F}}$ *of the free space* $\mathcal{F}(X)$ *of a given metric space* (X, D)*, based on the dual space* $L := \mathrm{LIP}_0(X)$*. Let us denote by* $\langle\cdot,\cdot\rangle$ *the duality map of the duality pair* $(L, \mathcal{F}(X))$*. Then the norm* $||\cdot||_{\mathcal{F}}$ *of* $\mathcal{F}(X)$ *can be represented as follows:*

$$\|Q\|_{\mathcal{F}} := \sup_{\substack{\phi \in L \\ \|\phi\|_L \leq 1}} \langle \phi, Q\rangle, \quad \text{for every } Q \in \mathcal{F}(X). \tag{4.2}$$

Consider any generating closed cone P *of* L *(i.e.,* $L = \operatorname{span}(P) = P - P$*) that satisfies:*

$$\forall \phi \in L,\ \exists \phi_1, \phi_2 \in P : \begin{cases} \phi = \phi_1 - \phi_2 \\ \max\{||\phi_1||_L, ||\phi_2||_L\} \leq ||\phi||_L \leq ||\phi_1||_L + ||\phi_2||_L \end{cases}. \tag{4.3}$$

We set:

$$\|Q|_{\mathcal{F}_P} := \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} \langle \phi, Q\rangle, \quad \text{for every } Q \in \mathcal{F}(X). \tag{4.4}$$

Notice that for any $Q \in \mathcal{F}(X)$ *we have* $\max\{\|Q|_{\mathcal{F}_P}, \|{-Q}|_{\mathcal{F}_P}\} \leq \|Q\|_{\mathcal{F}}$. *Since the supremum in* (4.2) *is attained at some* $\phi \in L$ *with* $||\phi||_L = 1$ *(by Hahn-Banach theorem), using the decomposition (4.3) we deduce:*

$$\|Q\|_{\mathcal{F}} = \langle \phi, Q \rangle = \langle \phi_1, Q \rangle + \langle \phi_2, -Q \rangle \leq \|Q|_{\mathcal{F}_P} + \|{-Q}|_{\mathcal{F}_P}. \tag{4.5}$$

This shows that condition $(ii)'$ *of Definition 2.2 holds and* (4.4) *defines an asymmetric hemi-norm* $\| \cdot |_{\mathcal{F}_P}$ *on the vector space* $\mathcal{F}(X)$.

We shall refer to the asymmetric norm $\| \cdot |_{\mathcal{F}_P}$ *defined in* (4.4) *as the* P-asymmetrization *of the free space* $\mathcal{F}(X)$, *for which we implicitly assume that* (4.3) *holds. We shall mainly deal with the case where* P *is the cone of positive Lipschitz functions, that is:*

$$P = L_+ := \{\phi \in L : \phi \geq 0\}.$$

In this case, we denote the arising asymmetric norm by $\| \cdot |_{\mathcal{F}_+}$. *Notice that if* $\phi (= \phi^+ - \phi^-) \in L$ *then both its positive part* ϕ^+ *and its negative part* ϕ^- *are also in* L *and they satisfy* $|\phi^+(x) - \phi^+(y)| \leq |\phi(x) - \phi(y)|$ *and* $|\phi^-(x) - \phi^-(y)| \leq |\phi(x) - \phi(y)|$, *for all* $x, y \in X$, *which leads to* (4.3).

More generally, a P*-asymmetrization of* $\mathcal{F}(X)$ *is called canonical, if* P *is of the form*

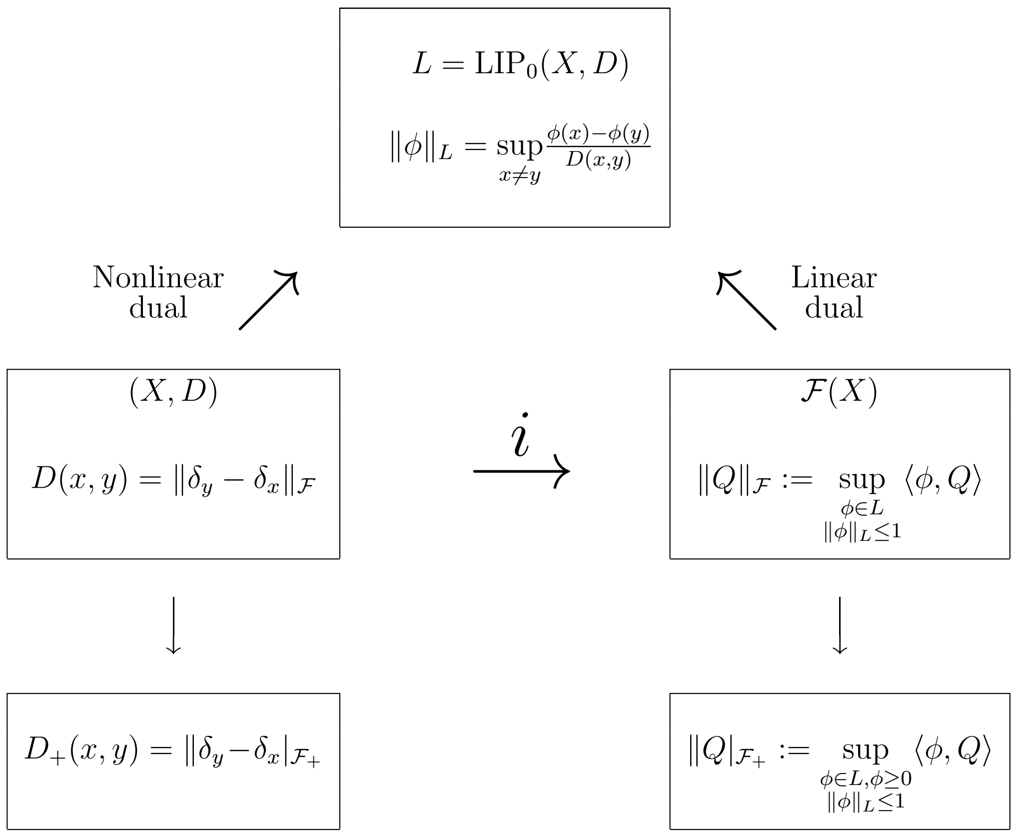
$$P := \{\phi \in L : \, T\phi \geq 0\},$$

where T *is a* linear isometry *that identifies* L *with some Banach lattice in a canonical way.*

Definition 4.24 (Canonical asymmetrization of a metric space) *Let* (X, D) *be a metric space with a base point* $x_0 \in X$. *Every* P*-asymmetrization of the free space* $\mathcal{F}(X)$ *(c.f. Remark 4.23) induces, via the isometric injection of* X *into* $\mathcal{F}(X)$, *an asymmetrization of the distance* D, *given by:*

$$D_P(x, y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} (\phi(y) - \phi(x)), \qquad \text{for all } x, y \in X.$$

The quasi-(hemi-)distance D_P *is called the* P*-asymmetrization of* (X, D). *If* $\| \cdot |_{\mathcal{F}_P}$ *is a canonical asymmetrization of* $\mathcal{F}(X)$, *then* D_P *will be called a* canonical asymmetrization *of* D. *In case* $P = L_+$, *the canonical asymmetrization will be denoted by* D_+. *The diagram below illustrates the situation.*



Let us illustrate the above notion of canonical asymmetrization by means of the following simple example.

Example 4.25 (Canonical asymmetrizations of $\mathbb{R}$) *Let us consider $\mathbb{R}$ as a metric space, with its usual distance $D(x, y) = |y - x|$, for all $x, y, \in \mathbb{R}$ and $x_0 = 0$ as a distinguished point. It is well-known ([27, 42]) that the free space $\mathcal{F}(\mathbb{R})$ can be identified with the space of Lebesgue integrable functions $\mathcal{L}^1(\mathbb{R})$, provided we identify the space $L = (\mathrm{LIP}_0(X, D), \|\cdot\|_L)$ of real-valued Lipschitz functions vanishing at 0 with the Banach space $\mathcal{L}^\infty(\mathbb{R})$ (essentially bounded Lebesgue measurable functions) via the canonical linear isometry $T\phi = \phi'$ (a.e.), for all $\phi \in L$ (*c.f. *Rademacher theorem). Then taken either*

$$P = L_+ = \{\phi \in L : \phi \geq 0\} \quad \text{or, respectively,} \quad P = \{\phi \in L : \phi' \geq 0\},$$

leads to two different canonical asymmetrizations of $\mathbb{R}$ (via the asymmetrizations $\|\cdot|_{\mathcal{F}_+}$ and respectively $\|\cdot|_{\mathcal{F}_P}$ of its free space). The first asymmetrization is given by the formula

$$D_+(x, y) = \|\delta(y) - \delta(x)|_{\mathcal{F}_+} = \sup_{\substack{\phi \in L_+ \\ \|\phi\|_L \leq 1}} (\phi(y) - \phi(x)) \,.$$

Notice that $D_+(x, y) \leq \max\{|y - x|, |y|\}$. It can be easily seen that if $y > x > 0$ or $y < x < 0$, then $D_+(x, y) = |y - x|$ (take $\phi_*(t) = |t|$ in L_+ with $||\phi_*||_L = 1$). However, $D_+(1, n) = n - 1$, while $D_+(n, 1) = 1$ for every $n \geq 2$.

The second asymmetrization, thanks to the monotonicity of every ϕ in P, yields that for all $x, y \in X$

$$D_P(x,y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} = \sup_{\substack{\phi\in L,\, \phi'\geq 0\\ \|\phi\|_L\leq 1}} (\phi(y)-\phi(x))$$
$$= \max\{y-x,\, 0\} = u(y-x) = d_u(x,y),$$

where $u(\cdot)$ is the asymmetric hemi-norm given by $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$ and d_u the corresponding quasi-hemi-distance.

Proposition 4.26 (Asymmetrization vs symmetrization) *Assume that (X, D_P) is a P-asymmetrization of a metric space (X, D) (c.f. Definition 4.24). Then the symmetrizations D_P^s and $D_P^{s_0}$ are bi-Lipschitz equivalent to the initial distance D, and consequently, the Banach spaces $\mathrm{LIP}_0(X, D)$, $\mathrm{LIP}_0(X, D_P^s)$ and $\mathrm{LIP}_0(X, D_P^{s_0})$ are isomorphic.*

Proof. It suffices to prove the result for D_P^s. Take $x, y \in X$. Let $\hat{\phi}$ be a function in $L = \mathrm{LIP}_0(X, D)$ with $\|\hat{\phi}\|_L \leq 1$ such that

$$D(x,y) = \sup_{\substack{\phi\in L\\ \|\phi\|_L\leq 1}} (\phi(y)-\phi(x)) = \hat{\phi}(y) - \hat{\phi}(x).$$

Let $\hat{\phi}_1$ and $\hat{\phi}_2$ be functions in P such that $\hat{\phi} = \hat{\phi}_1 - \hat{\phi}_2$, with the inequality $\max\{\|\hat{\phi}_1\|_L, \|\hat{\phi}_2\|_L\} \leq \|\hat{\phi}\|_L = 1$. Then

$$D(x,y) = (\hat{\phi}_1(y) - \hat{\phi}_1(x)) + (\hat{\phi}_2(x) - \hat{\phi}_2(y))$$
$$\leq \sup_{\substack{\psi\in P\\ \|\psi\|_L\leq 1}} (\psi(y)-\psi(x)) + \sup_{\substack{\psi\in P\\ \|\psi\|_L\leq 1}} (\psi(x)-\psi(y)),$$

which coincides with $D_P(x,y) + D_P(y,x) = D_P^s(x,y)$. Furthermore, it is clear that

$$D_P^s(x,y) = D_P(x,y) + D_P(y,x) \leq 2D(x,y).$$

Thus, the distances D_P^s and D are equivalent, and $\mathrm{LIP}_0(X, D)$ is linear isomorphic to $\mathrm{LIP}_0(X, D_P^s)$. □

4.4.1 Relation with the semi-Lipschitz free space

Let $X = (X, D)$ be a metric space with a base point $x_0 \in X$ and denote by

$$L = (\mathrm{LIP}_0(X, D), \|\cdot\|_L)$$

its nonlinear dual. Let $P \subseteq L$ be a cone satisfying (4.3), that is, for every $\phi \in L$ there exists $\phi_1, \phi_2 \in P$ such that $\phi = \phi_1 - \phi_2$ and $\max\{||\phi_1||_L, ||\phi_2||_L\} \leq ||\phi||_L \leq ||\phi_1||_L + ||\phi_2||_L$. Let us denote by D_P the P-asymmetrization of X (*c.f.* Definition 4.24). We also denote by

$$SL = (\mathrm{SLIP}_0(X, D_P), \|\cdot|_S)$$

the nonlinear asymmetric dual of (X, D_P), that is, the normed cone of semi-Lipschitz functions on (X, D_P).

Lemma 4.27 (Isometric injection of P into SL) *For every metric space (X, D) and every P-asymmetrization (X, D_P):*

(i) *there exists an isometric injection of P into SL;*

(ii) *there is a non-expansive injection of SL into L.*

Proof. Let $\phi \in SL$ and $x, y \in X$. Then

$$\phi(y) - \phi(x) \leq \|\phi|_S D_P(x,y) = \|\phi|_S \|\delta_y - \delta_x|_{\mathcal{F}_P} \leq \|\phi|_S \|\delta_y - \delta_x\|_{\mathcal{F}} = \|\phi|_S D(x,y),$$

which yields that $\phi \in \mathrm{LIP}_0(X, D)$ and $\|\phi\|_L \leq \|\phi|_S$. This proves (ii).

Let now $\phi : (X, D) \to \mathbb{R}$ be in P with $\|\phi\|_L \neq 0$. Then $\phi_1 = \frac{\phi}{\|\phi\|_L}$ is also in P and $\|\phi_1\|_L = 1$. Given $x, y \in X$, we deduce:

$$\begin{aligned} D_P(x,y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} &= \sup_{\substack{\psi \in P \\ \|\psi\|_L \leq 1}} (\psi(y) - \psi(x)) \\ &\geq \phi_1(y) - \phi_1(x) = \frac{1}{\|\phi\|_L}(\phi(y) - \phi(x)), \end{aligned}$$

which yields $\phi(y) - \phi(x) \leq \|\phi\|_L D_P(x,y)$. Therefore, $\phi \in SL$ and $\|\phi|_S \leq \|\phi\|_L$. Combining with (ii) we deduce $\|\phi\|_L = \|\phi|_S$ and (i) follows. □

Let us set

$$F = \mathrm{span}\{\delta(x) : x \in X\} \subset SL^* \quad \text{and} \quad \widehat{F} = \mathrm{span}\{\widehat{\delta}(x) : x \in X\} \subset L^* \tag{4.6}$$

where δ (respectively, $\widehat{\delta}$) is the canonical injection of (X, D_P) into SL^* (respectively, of (X, D) into L^*). There is a canonical bijection between F and $\widehat{F}$, under which a general element $Q = \sum_{i=1}^n \lambda_i \delta(x_i)$ of F is identified with the element $\widehat{Q} = \sum_{i=1}^n \lambda_i \widehat{\delta}(x_i)$ of $\widehat{F}$. Using this bijection, we have the following result.

Proposition 4.28 ($||\cdot||_{\mathcal{F}}$ is equivalent to the symmetrization of $\|\cdot|_{\mathcal{F}_a}$) *For any $Q \in F$ it holds:*

$$\max\{\|Q|_{\mathcal{F}_a}, \|{-Q}|_{\mathcal{F}_a}\} \leq \|\widehat{Q}\|_{\mathcal{F}} \leq \|\widehat{Q}|_{\mathcal{F}_P} + \| - \widehat{Q}|_{\mathcal{F}_P} \leq 2\max\{\|Q|_{\mathcal{F}_a}, \|{-Q}|_{\mathcal{F}_a}\}.$$

Proof. Let $F = \mathrm{span}\,(\delta(X))$ and $Q \in F$. Then Q is of the form $Q = \sum_{i=1}^k \lambda_i \delta(x_i)$ for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ and $x_i \in X$, $i = 1, \ldots, n$, and

$$\|\widehat{Q}|_{\mathcal{F}_P} = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} \langle \phi, \widehat{Q} \rangle = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} \sum_{k=1}^n \lambda_i \phi(x_i) \leq \sup_{\substack{\varphi \in SL \\ \|\phi|_S \leq 1}} \sum_{k=1}^n \lambda_i \varphi(x_i) := \|Q|_{\mathcal{F}_a}.$$

We also obtain $\| - \widehat{Q}|_{\mathcal{F}_P} \leq \|{-Q}|_{\mathcal{F}_a}$. Now, if $\varphi \in SL$ satisfies $\|\phi|_S \leq 1$, then by Lemma 4.27(ii) we deduce that $\varphi \in L$ and $\|\varphi\|_L \leq \|\varphi|_S \leq 1$. Hence

$$\|Q|_{\mathcal{F}_a} \leq \sup_{\substack{\phi \in L \\ \|\phi\|_L \leq 1}} \sum_{k=1}^n \lambda_i \phi(x_i) = \|\widehat{Q}\|_{\mathcal{F}}$$

and $\|{-Q}|_{\mathcal{F}_a} \le \|\widehat{Q}\|_{\mathcal{F}}$, which yields

$$\max\{\|Q|_{\mathcal{F}_a}, \|{-Q}|_{\mathcal{F}_a}\} \le \|\widehat{Q}\|_{\mathcal{F}} \le \|\widehat{Q}|_{\mathcal{F}_P} + \| - \widehat{Q}|_{\mathcal{F}_P},$$

where the last inequality follows from (4.5). The result follows. $\square$

In the sequel, we shall identify F with $\widehat{F}$, defined in (4.6). Under this identification, the norm $\|\cdot\|_{\mathcal{F}}$ can be considered to be also defined on F. Under this convention, the statement of Proposition 4.28 reads as follows: the norm $||\cdot||_{\mathcal{F}}$ is equivalent to the symmetrization of $\|\cdot|_{\mathcal{F}_a}$ and consequently

$$\mathcal{F}_a(X, D_P) = \overline{F}^{\|\cdot\|_{\mathcal{F}_a^s}} = \overline{F}^{\|\cdot\|_{\mathcal{F}}} = \mathcal{F}(X, D), \tag{4.7}$$

which yields that $\mathcal{F}_a(X, D_P)$ and $\mathcal{F}(X, D)$ can be identified as sets. Moreover

$$D_P(x, y) = \|\delta(y) - \delta(x)|_{\mathcal{F}_P} = \|\delta(y) - \delta(x)|_{\mathcal{F}_a}.$$

Hence the following result holds.

Theorem 4.29 (Compatibility I) *Let (X, D) be a metric space with a P-asymmetrization. Then, the symmetrizations of $(\mathcal{F}(X, D), \|\cdot|_{\mathcal{F}_P})$ and $(\mathcal{F}_a(X), \|\cdot|_{\mathcal{F}_a})$ are both isomorphic to the Lipschitz-free space $(\mathcal{F}(X, D), \|\cdot\|_{\mathcal{F}})$.*

The following diagram illustrates the situation described by Theorem 4.29.

$$\begin{array}{ccccc}
F = \operatorname{span}(\delta(X)) & \sqsubseteq & \mathcal{F}_a(X) & \sqsubseteq & (SL)^* \\
\updownarrow & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}_a^s}\text{-dense} & & \\
\widehat{F} = \operatorname{span}\left(\widehat{\delta}(X)\right) & \sqsubseteq & \mathcal{F}(X) & \sqsubseteq & L^* \\
 & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}}\text{-dense} & &
\end{array}$$

Let us now study the inverse procedure: we start with a quasi-metric space (X, d) and consider a symmetrization D of its distance (where D is either d^s or d^{s_0}, see Remark 2.8). It is easily seen that every $\phi \in \mathrm{SLIP}_0(X, d)$ satisfies $\phi \in \mathrm{LIP}_0(X, D)$ and $\|\phi\|_L \le \|\phi|_S$. Therefore, $P := \mathrm{SLIP}_0(X, d)$ can be viewed as a cone in $\mathrm{LIP}_0(X, D)$ and be used to define an asymmetric norm $\|\cdot|_{\mathcal{F}_P}$ on $\mathcal{F}(X, D)$ and consequently a quasi-metric D_P on X. In this setting, forthcoming Proposition 4.30 establishes a compatibility result under the following assumption:

There exists $\alpha \ge 1$ such that for every $\phi \in \mathrm{SLIP}_0(X, d)$

$$\Big(\|\phi\|_L \le \Big) \ \|\phi|_S \le \alpha \|\phi\|_L \, . \tag{4.8}$$

Proposition 4.30 (Compatibility II) *Let (X,d) be a quasi-metric space with symmetrized distance D and assume (4.8) holds. We set $P := \mathrm{SLIP}_0(X,d)$ and define for every $Q \in \mathcal{F}(X,D)$*

$$\|Q|_{\mathcal{F}_P} := \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} \langle Q,\phi\rangle\,.$$

Then for all $Q \in \mathrm{span}(\delta(X))$

$$\|Q|_{\mathcal{F}_a} \le \|Q|_{\mathcal{F}_P} \le \alpha\|Q|_{\mathcal{F}_a}\,. \tag{4.9}$$

In particular, setting for $x,y\in X$

$$D_P(x,y) := \|\delta_y - \delta_x|_{\mathcal{F}_P}$$

we obtain that for all $x,y\in X$ it holds:

$$d(x,y) \le D_P(x,y) \le \alpha d(x,y)\,. \tag{4.10}$$

Terminology (*equivalence of asymmetric norms/quasi-metrics*). We interpret relation (4.9) as an equivalence relation for the asymmetric norms $\|Q|_{\mathcal{F}_a}$ and $\|Q|_{\mathcal{F}_P}$. Similarly, relation (4.10) means that the quasi-distances d and D_P are equivalent.

Proof. The equivalence between the asymmetric norms $\|\cdot|_{\mathcal{F}_a}$ and $\|\cdot|_{\mathcal{F}_P}$ on the vector space $\mathrm{span}(\delta(X))$ follows directly from their definitions and the inequalities $\|\phi\|_L \le \|\phi|_S \le \alpha\|\phi\|_L$. □

Remark 4.31 *The equivalence between the quasi-metric d and the canonical asymmetrization D_P of the symmetrized distance D yields an equivalence between D and the symmetrization $(D_P)^s$ of D_P.*

If in addition to (4.8), we assume that $P = \mathrm{SLIP}_0(X,d)$ induces an asymmetrization on the free space $\mathcal{F}(X,D)$, that is, for every $\phi \in \mathrm{LIP}_0(X,D)$ there exist $\phi_1,\phi_2\in P$ such that

$$\phi = \phi_1 - \phi_2 \qquad \text{and} \qquad \max\{||\phi_1||_L, ||\phi_2||_L\} \le ||\phi||_L \le ||\phi_1||_L + ||\phi_2||_L$$

then the equivalence between D and $(D_P)^s$ extends to the corresponding free spaces (see Remark 4.23). In particular, the following result holds.

Proposition 4.32 (Compatibility III) *Let (X,d) be a quasi-metric space and $D = d^s$ or $D = d^{s_0}$. Assume that the cone $P = \mathrm{SLIP}_0(X,d)$ of $\mathrm{LIP}_0(X,D)$ induces an asymmetrization in $\mathcal{F}(X,D)$ and (4.8) holds. Then the asymmetric free spaces $\mathcal{F}_a(X,d)$ and $\mathcal{F}_a(X,D_P)$ coincide (as sets) with the free space $\mathcal{F}(X,D)$, that is:*

$$\mathcal{F}_a(X,d) = \mathcal{F}(X,D) = \mathcal{F}_a(X,D_P).$$

Moreover:

(i). *The quasi-metrics d and D_P are equivalent and the same applies to the (symmetric) metrics D and $(D_P)^s$, $(D_P)^{s_0}$ (symmetrizations of D_P).*

(ii). *The asymmetric norms $\|\cdot|_{\mathcal{F}_a(X,d)}$, $\|\cdot|_{\mathcal{F}_P}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent.*

(iii). *The symmetrizations of $\|\cdot|_{\mathcal{F}_a(X,d)}$, $\|\cdot|_{\mathcal{F}_P}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent to $\|\cdot\|_{\mathcal{F}}$.*

Proof. We have already seen that $\mathcal{F}(X,D) = \mathcal{F}_a(X,D_P)$ (as sets), see (4.7). By Proposition 4.30, the asymmetric norms $\|\cdot|_{\mathcal{F}_a(X,d)}$ and $\|\cdot|_{\mathcal{F}_P}$ are equivalent on $F = \mathrm{span}\{\delta(x) : x \in X\}$, therefore

$$\mathcal{F}_a(X,d) = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_a(X,d)}} = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_P}} = \mathcal{F}(X,D).$$

Assertions (i) follow directly from Proposition 4.30. For (ii) it remains to prove that $\|\cdot|_{\mathcal{F}_a(X,d)}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent. We established in (4.10) that the quasi-distances d and D_P are equivalent. This yields that the normed cones $\mathrm{SLIP}_0(X,d)$ and $\mathrm{SLIP}_0(X,D_P)$ are isomorphic, which leads to an isomorphism of the corresponding semi-Lipschitz free spaces. The equivalence between the symmetrizations of the asymmetric norms asserted in (iii) now follows from (ii). Thanks to Theorem 4.29 they are also equivalent to $\|\cdot\|_{\mathcal{F}}$. □

Remark 4.33 *If the value of α associated to the assumption* (4.8) *is equal to* 1*, all of the aforementioned equivalences of Proposition 4.32 become equalities.*

4.4.2 Properties (S) and (S*)

We have shown that the P-asymmetrization of a metric space (X,D) gives rise to a quasi-metric space, for which the symmetrization of its asymmetric free space is isomorphic to the free space $(\mathcal{F}(X), \|\cdot\|_{\mathcal{F}})$. In this subsection we shall be interested in cases in which the aforementioned isomorphism is in fact an isometry.

Definition 4.34 *Let (X,D) be a metric space, $L = \mathrm{LIP}_0(X,D)$ and $P \subset L$ be a cone. We say that the metric space (X,D) satisfies:*

(i) property (**S**) *with respect to P, if P induces a nontrivial asymmetrization D_P on X and*

$$SL = \mathrm{SLIP}_0(X,D_P) = P.$$

(ii) property ($\mathbf{S}^*$) *(respectively,* ($\mathbf{S}^*_0$)*) with respect to P if, in addition to (i), it holds:*

$$\|Q\|_{\mathcal{F}} = \|Q|_{\mathcal{F}_P} + \|{-Q}|_{\mathcal{F}_P} \ \ (\textit{respectively, } \ \|Q\|_{\mathcal{F}} = \max\{\|Q|_{\mathcal{F}_P}, \|{-Q}|_{\mathcal{F}_P}\})$$

for every $Q \in \mathcal{F}(X,D)$.

The following proposition is straightforward.

Proposition 4.35 *Let (X,D) be a metric space.*

(i). *If (X,D) satisfies* (**S**) *with respect to P, then $(\mathcal{F}(X,D), \|\cdot|_{\mathcal{F}_P})$ and $(\mathcal{F}_a(X,D_P), \|\cdot|_{\mathcal{F}_a})$ are identical.*

(ii). *If (X,D) satisfies* ($\mathbf{S}^*$) *(resp.* ($\mathbf{S}^*_0$)*) with respect to P, then the d^s-symmetrization (resp. d^{s_0}-symmetrization) of $(\mathcal{F}_a(X,D_P), \|\cdot|_{\mathcal{F}_a})$ given in* (2.5) *is isometrically isomorphic to $(\mathcal{F}(X,D), \|\cdot\|_{\mathcal{F}})$.*

Before we proceed, let us give examples of metric spaces for which the above properties fail.

Example 4.36 **(i)** *(Property* (**S**) *fails) Let $X = \mathbb{R}$ with the usual distance $D(t,s) = |s-t|$, for $t, s \in \mathbb{R}$. Let L be the space of Lipschitz functions on $\mathbb{R}$ vanishing at 0 and set*

$$P := \{\phi \in L : \int_{\mathbb{R}} \phi \in [0, +\infty]\}.$$

Then P contains the cone L_+ of non-negative Lipschitz functions vanishing at 0, and consequently $L = P - P$ and (4.3) holds. It is easy to see that

$$D_P(t,s) = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} (\phi(s) - \phi(t)) = |s-t| = D(t,s).$$

Therefore, $SL = L \neq P$ and (**S**) *fails.*

(ii) *(Property* (**S**) *holds but properties* ($\mathbf{S}^*$) *and* ($\mathbf{S}_\mathbf{0}^*$) *fail) We consider again $X = \mathbb{R}$ equipped with its usual distance D and L be the space of Lipschitz functions on $\mathbb{R}$ vanishing at 0. We now set*

$$P = L_+ := \{\phi \in L : \ \phi \geq 0\}.$$

It follows easily that

$$\begin{aligned} D_+(s,t) &= \sup_{\substack{\phi \in L_+ \\ \|\phi\|_L \leq 1}} (\phi(t) - \phi(s)) \\ &= \begin{cases} |t-s|\,, & \text{if } \ 0 \leq s \leq t \ \text{ or } \ s \leq t \leq 0 \\ \min\{\, t, s-t\,\}\,, & \text{if } \ 0 \leq t \leq s \\ \min\{|s|, s-t\}\,, & \text{if } \ t \leq s \leq 0 \\ |t|\,, & \text{if } \ t \leq 0 \leq s \ \text{ or } \ s \leq 0 \leq t \end{cases}. \end{aligned}$$

Let us show that property (**S**) *holds. Indeed, for $s \neq 0$ we have $D_+(0,s) = s$ and $D_+(s,0) = 0$. By Lemma 4.27(i), $P \subset SL \subset L$. Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be any function vanishing at 0 and assume that for some $s \neq 0$ we have $\varphi(s) < 0$. Then $\varphi(0) - \varphi(s) > 0$ and $D_+(s,0) = 0$ reveals that φ cannot be in SL, showing that* (**S**) *holds.*

Taking now any two integers $n, k \geq 2$ we have $D_+(n,-k) = k$, $D_+(-k,n) = n$ and $D(n,-k) = n+k$, which shows that ($\mathbf{S}_\mathbf{0}^*$) *fails. On the other hand, $D_+(1,n) = n-1 = D(1,n)$ and $D_+(n,1) = 1$ which shows that* ($\mathbf{S}^*$) *fails.*

A typical example of a metric space for which (S^*) holds is the set of real numbers $\mathbb{R}$ viewed as a pointed metric space, for the cone $P = \{\phi \in L : \phi' \geq 0\}$, see forthcoming Lemma 4.46. To obtain additional examples of metric spaces satisfying ($\mathbf{S}^*$), let us first recall definitions and results due to Godard [27], regarding $\mathbb{R}$-trees.

Definition 4.37 ($\mathbb{R}$-tree) *An $\mathbb{R}$-tree is a metric space T satisfying the following two conditions:*

(i) For any points $x, y \in T$, there exists a unique isometry $\phi := \phi_{xy}$ of the closed interval $[\,0\,, d(x,y)\,]$ into T such that $\phi(0) = x$ and $\phi(d(x,y)) = y$.
(The range of this isometry is called segment and is denoted by $[x,y]$.)

(ii) *Any one-to-one continuous mapping* $\varphi : [0,1] \to T$ *has same range as the isometry* $\phi_{a,b}$ *associated to the points* $a = \varphi(0)$ *and* $b = \varphi(1)$.

Our aim is to prove that subsets of (pointed) $\mathbb{R}$-trees satisfy property ($\mathbf{S}^*$). The base point of an $\mathbb{R}$-tree is denoted by 0. Then one defines a partial order $\preccurlyeq$ on T, by setting $x \preccurlyeq y$ if $x \in [0, y]$.
A subset A of T is said to be *measurable* whenever $\phi_{xy}^{-1}(A)$ is Lebesgue-measurable for any x and y in T. If A is measurable and S is the segment $[x, y]$, we write $\lambda_S(A)$ for $\lambda\left(\phi_{xy}^{-1}(A)\right)$, where λ is the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{R}$ be the family of all subsets of T which can be written as a finite union of disjoints segments, and for $R = \bigcup_{k=1}^{n} S_k \in \mathcal{R}$, we set

$$\lambda_R(A) = \sum_{k=1}^{n} \lambda_{S_k}(A).$$

Then,

$$\lambda_T(A) = \sup_{R \in \mathcal{R}} \lambda_R(A),$$

defines a measure (called the length measure) on the σ-algebra of T-measurable sets such that

$$\int_{[x,y]} f(u) d\lambda_T(u) = \int_0^{d(x,y)} f(\phi_{xy}(t)) dt$$

for any $f \in L_1(T)$ and $x, y \in T$.

Definition 4.38 (measure on an $\mathbb{R}$-tree) *Let* T *be a pointed* $\mathbb{R}$*-tree, and let* A *be a closed subset of* T. *We denote by* μ_A *the measure defined by*

$$\mu_A = \lambda_A + \sum_{a \in A} L(a) \delta_a,$$

where λ_A *is the restriction of the length measure* λ_T *to* A, δ_a *is the Dirac measure on* a *and* $L(a) = \inf_{x \in A \cap [0,a)} d(a, x)$.

Proposition 4.39 *[27, Proposition 2.3] Let* T *be a pointed* $\mathbb{R}$*-tree, and let* A *be a closed subset of* T *containing* 0. *Then,* $\mathcal{L}^1(\mu_A)^*$ *is isometrically isomorphic to* $\mathcal{L}^\infty(\mu_A)$.

Definition 4.40 (Differentiation on an $\mathbb{R}$-tree) *Let* T *be a pointed* $\mathbb{R}$*-tree,* A *a closed subset of* T *containing* 0 *and* $f : A \to \mathbb{R}$. *For* $a \in A$, *let* $\tilde{a}$ *be the unique point in* $[0, a]$ *such that* $d(a, \tilde{a}) = L(a)$. *If* $L(a) > 0$, *we say that* f *is differentiable at* a *with derivative*

$$f'(a) = \frac{f(a) - f(\tilde{a})}{L(a)}.$$

If $L(a) = 0$, *we say that* f *is differentiable at* a, *whenever the limit*

$$\lim_{\substack{x \to \tilde{a} \\ x \in [0,a) \cap A}} \frac{f(a) - f(x)}{d(x, a)}$$

exists, and we denote by $f'(a)$ *the value of this limit.*

Proposition 4.41 *[27, pp. 4313-4314] Let f be a real-valued Lipschitz function defined on an $\mathbb{R}$-tree T. Then, f is differentiable almost everywhere on T and*

$$f(x) - f(0) = \int_{[0,x]} f' d\lambda_T,$$

for all $x \in T$.

The following theorem characterizes subsets of $\mathbb{R}$-trees in terms of their Lipschitz-free spaces.

Theorem 4.42 *[27, Theorem 4.2] Let (X, D) be a complete pointed metric space. Then the following assertions are equivalent:*

(i). $\mathcal{F}(X)$ is isometrically isomorphic to a subspace of an $\mathcal{L}^1$-space;

(ii). (X, D) isometrically embeds into an $\mathbb{R}$-tree.

We are now ready to prove our result on $\mathbb{R}$-trees.

Proposition 4.43 *Let (X, D) be a subset of an $\mathbb{R}$-tree T. Then, (X, D) satisfies property (S^*) with respect to the cone*

$$P = \{\phi \in \mathrm{LIP}_0(X, D) \, : \, \phi' \geq 0\},$$

Proof. Thanks to Theorem 4.42, we may use Godard's embedding, denoted by Φ_*, to isometrically identify $\mathcal{F}(X, D)$ with a subspace Y of $\mathcal{L}^1(T)$, by sending $\delta_x \in \mathcal{F}(X, D)$ to $\Phi_*(\delta_x) = \mathbb{1}_{[0,x]} \in \mathcal{L}^1(T)$. This embedding is the restriction to $\mathcal{F}(X, D)$ of the pre-adjoint of the (weak-star to weak-star continuous) isometry $\Phi : \mathcal{L}^\infty(T) \to \mathrm{LIP}_0(T)$ defined by $\Phi(g)(x) = \int_{[0,x]} g d\mu_X$ for any $x \in T$.

Let $\iota : (X, D) \to (Y, \| \cdot \|_1)$ be the isometric injection induced by Godard's embedding Φ. We keep the same notation $\| \cdot |_{\mathcal{F}_P}$ for the asymmetric hemi-norm induced in Y by this embedding. The P-asymmetrization of the norm of Y is given by

$$\|f|_{\mathcal{F}_P} = \sup_{\substack{\phi \geq 0 \\ \|\phi\|_\infty \leq 1}} \langle \phi, f \rangle = \sup_{\substack{\phi \geq 0 \\ \|\phi\|_\infty \leq 1}} \int_X f \phi d\mu_X = \int_X f^+ := \|f|_{1,+},$$

for all $f \in Y$, where $f^+(t) = \max\{f(t), 0\}$ for any $t \in T$. Therefore, $D_P(y, x) = \|\iota(x) - \iota(y)|_{\mathcal{F}_P} = 0$ whenever $\iota(x) \leq \iota(y)$ almost everywhere, which is equivalent to $x \preccurlyeq y$ in the order of T. Then, for $\varphi \in SL = \mathrm{SLIP}_0(X, D_P)$ and $x, y \in X$ such that $x \preccurlyeq y$, we have $\varphi(x) - \varphi(y) \leq \|\varphi|_S D_P(y, x) = 0$, and therefore $x \preccurlyeq y$ yields $\varphi(x) \leq \varphi(y)$.
It is easy to check that $\Phi^{-1}(\varphi) = \varphi' \in \mathcal{L}_\infty(T)$ for all $\varphi \in L$. The monotonicity property of semi-Lipschitz functions proved above yields that $\varphi' \geq 0$, so φ belongs to the cone P. Therefore, $SL \subset P$ and in view of Lemma 4.27(i) we deduce that $SL = P$, hence (X, D) satisfies property (**S**).

Let $g \in \mathcal{F}(X, D)$, and $f = \Phi_*(g)$. Then

$$\begin{aligned} \|g\|_{\mathcal{F}} = \|f\|_1 &= \sup_{\|\phi\|_\infty \leq 1} \langle \phi, f \rangle = \langle f, \text{sgn}(f) \rangle \\ &= \langle f^+, \text{sgn}(f) \rangle - \langle f^-, \text{sgn}(f) \rangle = \|f|_{1,+} + \| - f|_{1,+} \\ &= \|g|_{\mathcal{F}_P} + \| - g|_{\mathcal{F}_P}, \end{aligned}$$

where sgn(f) denotes the sign of f. We conclude that (X, D) satisfies property $(\mathbf{S}^*)$. □

Combining Propositions 4.35 and 4.43, we obtain

Proposition 4.44 *Let (X, D) be a subset of an $\mathbb{R}$-tree. Then, there exists a canonical asymmetrization D_P of D such that the symmetrization of the semi-Lipschitz free space $\mathcal{F}_a(X, D_P)$ is isometrically isomorphic to $\mathcal{F}(X, D)$.*

4.5 Examples of semi-Lipschitz free spaces

Let us now illustrate the semi-Lipschitz free space for three concrete examples of quasi-metric spaces: a finite quasi-metric space consisting of three points, the set of natural numbers $\mathbb{N}$ with a discrete quasi-metric and the set of real numbers $\mathbb{R}$ under the quasi-hemi-metric defined by the canonical conic hemi-norm u. We also include an example-scheme stemming from canonical asymmetrizations of subsets of $\mathbb{R}$-trees.

4.5.1 A 3-point quasi-metric space

Let $X = \{x_0, x_1, x_2\}$ be a set of three points, endowed with the following quasi-metric (in a general form):

$$\begin{array}{lll} \rho(x_0, x_1) = a_{01} & \rho(x_1, x_0) = a_{10} & \rho(x_0, x_2) = a_{02} \\ \rho(x_2, x_0) = a_{20} & \rho(x_1, x_2) = a_{12} & \rho(x_2, x_1) = a_{21} \end{array}$$

Taking x_0 as base point, it is clear that the set of semi-Lipschitz functions vanishing at x_0 can be algebraically identified with $\mathbb{R}^2$, i.e. any function $g : X \to \mathbb{R}$ with $g(x_0) = 0$ is in $\text{SLIP}_0(X)$, with associated semi-Lipschitz norm equal to

$$\|g|_S = \max\left\{ \frac{g_1 - g_2}{a_{21}}, \frac{g_2 - g_1}{a_{12}}, \frac{g_1}{a_{01}}, \frac{g_2}{a_{02}}, \frac{-g_1}{a_{10}}, \frac{-g_2}{a_{20}} \right\},$$

where $g_1 = g(x_1)$ and $g_2 = g(x_2)$. Therefore, the unit ball B of $\text{SLIP}_0(X, \rho) \simeq \mathbb{R}^2$ is in the polygon generated by the linear inequalities defined in terms of the asymmetric norm. The dual cone of $(\text{SLIP}_0(X), \| \cdot |_S)$ is the vector space $\mathbb{R}^2$ endowed with the asymmetric norm determined by the Minkowski gauge of the asymmetric polar B^o of the unit ball B of $\text{SLIP}_0(X, \rho)$, that is

$$B^o = \{X \in \mathbb{R}^2 : \langle g, X \rangle < 1, \ \forall g \in B\}.$$

Since the evaluation functionals $\delta(x_1), \delta(x_2)$ generate the vector space $\mathbb{R}^2$, it follows that $\mathcal{F}_a(X,\rho)$ is isomorphic to $\mathbb{R}^2$, with the asymmetric norm determined by the aforementioned Minkowski gauge. Furthermore, for any $g \in \mathrm{SLIP}_0(X)$, its linearization $T_g\colon \mathcal{F}_a(X) \to \mathbb{R}$ is given by

$$T_g\left(\lambda_1 \hat{x_1} + \lambda_2 \hat{x_2}\right) = \lambda_1 g(x_1) + \lambda_2 g(x_2),$$

with $\lambda_i \in \mathbb{R}$, $i = 1, 2$. Notice that the unit balls of $\mathrm{SLIP}_0(X,\rho)$ and its dual cone have at most 6 extreme points (see Figure 4.1).

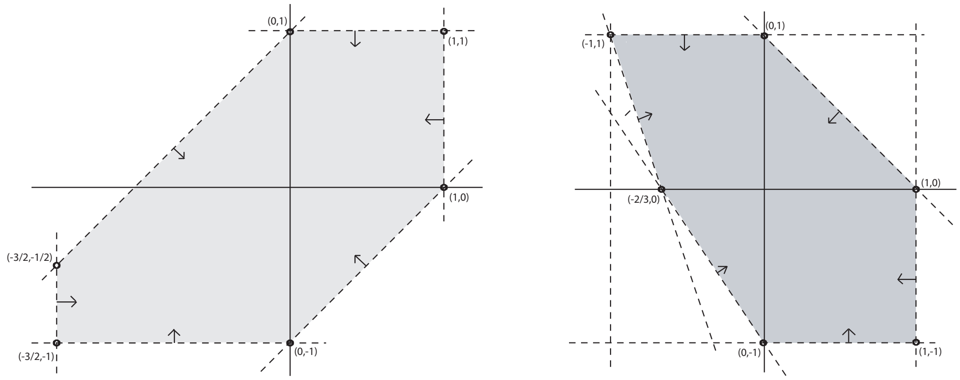


Figure 4.1: Representation of the unit ball of $\mathrm{SLIP}_0(X,\rho)$ and its asymmetric polar, respectively, with $X = \{x_0, x_1, x_2\}$, $\rho(x_1, x_0) = \frac{3}{2}$ and $\rho(x_i, x_j) = 1$ for $i \neq j$ with $(i,j) \neq (1,0)$

4.5.2 $\mathbb{N}$ as a quasi-metric space

We now consider the set of natural numbers $\mathbb{N}$ (including 0) endowed with the quasi-metric defined by

$$d(n,m) = \begin{cases} 1, & \text{if } m \notin \{0,n\} \\ 0, & \text{if } m \in \{0,n\} \end{cases}.$$

We fix as a base point $x_0 = 0$. Let $y = \left(y(n)\right)_n \in \mathrm{SLIP}_0(\mathbb{N}, d)$. Then $y(0) = 0$ and the semi-Lipschitz condition implies that the sequence $\left(y(n)\right)_n$ is non-negative: indeed

$$y(0) - y(n) = -y(n) \leq \|y|_S \, d(n,0) = 0$$

and

$$y(n) - y(0) = y(n) \leq \|y|_S \, d(0,n) = \|y|_S.$$

Therefore we have $\left(y(n)\right)_n \in \ell^\infty(\mathbb{N})$ and $\|y|_S \geq \|y\|_\infty$. Moreover,

$$\|y|_S = \sup_{d(n,m)>0} \frac{y(m) - y(n)}{1} \leq \sup_{d(n,m)>0} y(m) = \|y\|_\infty,$$

since $y(n) \geq 0$ for all $n \in \mathbb{N}$. It is easy to check than any bounded non-negative sequence satisfies the semi-Lipschitz condition, so it follows that $\mathrm{SLIP}_0(\mathbb{N}, d)$ is $(\ell^\infty_+(\mathbb{N}), \|\cdot\|_\infty)$, the positive cone of $\ell^\infty(\mathbb{N})$. The dual norm on $\ell^\infty_+(\mathbb{N})^*$ is given by

$$\|\varphi|^* = \sup_{\substack{(y_n) \in \ell^\infty_+(\mathbb{N}) \\ \|(y_n)\|_\infty \leq 1}} \varphi((y_n)).$$

The set of evaluation functionals $\{\delta(n) \,:\, n \in \mathbb{N}\} \subset \ell^\infty_+(\mathbb{N})^*$ can be identified with the canonical basis of $\ell^1(\mathbb{N})$, so the linear span of $\delta(\mathbb{N})$ is the set of finitely supported sequences $c_{00}(\mathbb{N})$. On this set, the dual norm of $\mathrm{SLIP}_0(\mathbb{N},d)^*$ becomes

$$\|(x_n)|^* = \sum_{n\in\mathbb{N}} \max\{x_n, 0\} = \sum_{n\in\mathbb{N}} x_n^+ := \|(x_n)|_{1,+},$$

since the supremum on the dual norm is taken over the positive cone of $\ell^\infty(\mathbb{N})$ (and it is attained at the sequence $(\mathrm{sgn}(x_n) \vee 0)$). It is easy to check that the symmetrization of the asymmetric norm $\|\cdot|_{1,+}$ is equivalent to the usual norm of $\ell^1(\mathbb{N})$, and therefore the asymmetric normed space $(\ell^1(\mathbb{N}), \|\cdot|_{1,+})$ satisfies the conditions to be the bicompletion of $(c_{00}(\mathbb{N}), \|\cdot|_{1,+})$. Therefore, the semi-Lipschitz free space $\mathcal{F}_a(\mathbb{N},d)$ is isometrically isomorphic to $(\ell^1(\mathbb{N}), \|\cdot|_{1,+})$ and the linearization T_y of a function $y = (y(n))_n \in \mathrm{SLIP}_0(\mathbb{N},d)$ can be obtained from

$$T_y(e_n) = y(n), \; n = 1, 2, \ldots$$

where e_n is the n-th element of the canonical basis of $\ell^1(\mathbb{N})$.

It is well known that the free space $\mathcal{F}(\mathbb{N}, D)$ of $\mathbb{N}$ equipped with the distance

$$D(m,n) = \begin{cases} 2, & \text{if } n \notin \{0, m\} \\ 1, & \text{if } n = 0 \text{ or } m = 0 \\ 0, & \text{if } n = m \end{cases}$$

is isometric to $\ell^1(\mathbb{N})$ (see, for instance, [27, 28, 42]), and

$$L = \mathrm{LIP}_0(\mathbb{N}, D) = \{y = y_n \in \mathbb{R}^{\mathbb{N}} : \|y\|_L := \frac{y(n) - y(m)}{D(m,n)} < \infty\}$$

is isometric to $\ell^\infty(\mathbb{N})$. Given $m, n \in \mathbb{N}$, then the canonical asymmetrization of D (Definition 4.24) is

$$D_+(m,n) = \|\widehat{\delta}(n) - \widehat{\delta}(m)|_{\mathcal{F}_+} = \sup_{\substack{y \in l^\infty(\mathbb{N})_+ \\ \|y\|_\infty \le 1}} \langle y, e_n - e_m\rangle = \sup_{0 \le y_n \le 1} \sum_{k \ge 0} y_k x_k,$$

where $x_n = 1$, $x_m = -1$, and $x_k = 0$, for $k \notin \{n, m\}$. According to Theorem 4.29, notice that $\mathcal{F}(\mathbb{N}, D) = \mathcal{F}(\mathbb{N}, d)$ (as a set), with $d = D_+$, $\mathrm{SLIP}_0(\mathbb{N}, d) = \ell^\infty_+(\mathbb{N}) = \mathrm{SLIP}_0(\mathbb{N}, D_+)$ and $\|x|_{\mathcal{F}_a} = \|x|_{\mathcal{F}_+} = \sum_{n\ge 0} x_n^+$.

4.5.3 The quasi-metric space $(\mathbb{R}, d_u)$

Note that since the symmetrized distance d_u^s is equal to the usual metric of $\mathbb{R}$ (which can be seen as a pointed $\mathbb{R}$-tree), $\mathcal{F}(\mathbb{R}, u)$ can be obtained from Proposition 4.35. We hereby include a direct self-contained proof, which does not rely on Godard's work on $\mathbb{R}$-trees. Let us start with some preliminary results.

Lemma 4.45 (Semi-Lipschitz functions in $(\mathbb{R}, u)$) *Let $f \in \mathrm{SLIP}_0(\mathbb{R}, u)$. Then f is a non-decreasing function in $\mathrm{LIP}_0(\mathbb{R})$.*

Proof. By Proposition 2.45, f is Lipschitz on $(\mathbb{R}, u^s) = (\mathbb{R}, |\cdot|)$, and therefore is differentiable almost everywhere. Note that if $x \le y$, then $d_u(y,x) = 0$, so $f(x) \le f(y)$. As f is non-decreasing, $f' \ge 0$. □

We are now ready to establish the following result.

Lemma 4.46 $(\mathrm{SLIP}_0(\mathbb{R},u), \|\cdot|_S)$ *and* $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$ *are isometrically isomorphic as normed cones.*

Proof. Consider the mapping $T : (\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty) \to (\mathrm{SLIP}_0(\mathbb{R},u), \|\cdot|_S)$ defined by

$$Tg(x) = \int_0^x g d\lambda = \int \mathbf{1}_{[0,x]} g,$$

which is surjective by the previous analysis. This mapping is well defined since for $x \ge y$ we have $Tg(x) - Tg(y) = \int_y^x g d\lambda \le \|g\|_\infty (x-y) = \|g\|_\infty d_u(y,x)$. If $x < y$ then

$$Tg(x) - Tg(y) = -\int_x^y g d\lambda \le 0 = d_u(y,x).$$

This also proves that $\|Tg|_S \le \|g\|_\infty$. On the other hand, consider $x \in \mathbb{R}$ a point of differentiability of Tg. Then

$$Tg'(x) = \lim_{y \searrow x} \frac{Tg(y) - Tg(x)}{y - x} \le \sup_{x<y} \frac{Tg(y) - Tg(x)}{y - x} = \|Tg|_S,$$

and since clearly $(Tg)' = g$, we conclude that $\|g\|_\infty \le \|Tg|_S$ and that T is an isometric isomorphism. □

For the following result, if $f \in \mathcal{L}^1(\mathbb{R})$ recall the notation $\|f|_{1,+} = \int_{\mathbb{R}} f^+ d\lambda$, where $f^+(x) = \max\{f(x), 0\}$ and λ denotes the Lebesgue measure, which was used in Lemma 4.5.

Theorem 4.47 *The semi-Lipschitz free space* $\mathcal{F}_a((\mathbb{R},u))$ *of the asymmetric hemi-normed space* $(\mathbb{R},u)$ *is isometrically isomorphic to* $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$.

Proof. By Lemma 4.5, we know that $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ is the asymmetric predual of $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|)$. Therefore we only need to check that the isometry $T : (\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty) \to (\mathrm{SLIP}_0(\mathbb{R},u), \|\cdot|_S)$ defined in the previous proof is $(w^*$-$w^*)$-continuous, in which case Lemma 4.2 will give us an isometry between the preduals $\mathcal{F}_a(\mathbb{R},u)$ and $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$. So, let (g_α) be a net on $\mathcal{L}^\infty_+(\mathbb{R})$ converging to g in the w^* topology induced by the predual $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$, and take $x \in \mathbb{R}$ and the corresponding $\widehat{x} \in \mathcal{F}_a(\mathbb{R},u)$. Then

$$\langle Tg_\alpha, \hat{x}\rangle = \int_0^x g_\alpha = \langle g_\alpha, \mathbf{1}_{[0,x]}\rangle \longrightarrow \langle g, \mathbf{1}_{[0,x]}\rangle, \tag{4.11}$$

by the w^* convergence of (g_α). Now, for an arbitrary $\mu \in \mathcal{F}_a(X)$ we can take a sequence $(\mu_n) \subset \mathrm{span}(\delta(\mathbb{R}))$ such that $\mu_n \to \mu$ in the symmetrized topology of $\mathrm{SLIP}_0(\mathbb{R},u)^*$, and therefore

$$\langle Tg_\alpha, \mu\rangle = \lim_n \langle Tg_\alpha, \mu_n\rangle, \tag{4.12}$$

where the last convergence is with respect to the usual norm on $\mathbb{R}$, thanks to the symmetrized-$|\cdot|$ continuity of semi-Lipschitz functions. Equations (4.11) and (4.12) yield that $\langle Tg_\alpha, \mu\rangle \to \langle Tg, \mu\rangle$ for the norm topology in $\mathbb{R}$, so T is (w^*-w^*)-continuous, and by Lemma 4.2 there exists an isometric isomorphism between $(\mathcal{F}_a(\mathbb{R}, u), \|\cdot|^*)$ and $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$. □

As we show in Example 4.25, $d_u(x, y) = u(y - x)$ is a canonical asymmetrization of $D(x, y) = |y - x|$ for the cone $P = \{\phi \in L : \phi' \geq 0\}$. Notice that the canonical asymmetrization D_+, based on the cone $P = L_+$ gives a different asymmetrization.

4.5.4 Canonic asymmetrization of subsets of $\mathbb{R}$-trees.

Propositions 4.35 and 4.43 provide a variety of examples of quasi-metric spaces (X, d) whose corresponding semi-Lipschitz free spaces are isometrically isomorphic to subspaces of $(\mathcal{L}^1(T), \|\cdot|_{1,+})$, where T is an $\mathbb{R}$-tree containing the symmetrized space (X, d^s). We can obtain more specific examples by applying the following recent result from [4, Theorem 1.1], which gives a characterization of all complete metric spaces whose Lipschitz-free space is isometric to a subspace of $\ell^1(\Gamma)$ for some set Γ.

Theorem 4.48 *Let (X, D) be a complete pointed metric space. Then the following are equivalent:*

- (i) *$\mathcal{F}(X)$ is isometrically isomorphic to a subspace of $\ell^1(\Gamma)$ for some set Γ;*
- (ii) *(X, D) is a subset of an $\mathbb{R}$-tree such that $\lambda(X) = 0$ and $\lambda(\overline{\mathrm{Br}(X)}) = 0$, where λ is the length measure and $\mathrm{Br}(X)$ is the set of branching points of X.*

Since every metric space as above satisfies property $(\mathbf{S}^*)$ (*c.f.* Proposition 4.43), we deduce that the corresponding semi-Lipschitz free space are isometrically isomorphic to $(\ell^1(\Gamma), \|\cdot|_{1,+})$ for some set Γ.

A careful reader might have observed that in all examples presented in this section the semi-Lipschitz free space of the given quasi-metric space can be easily obtained from the Lipschitz free space of its symmetrization. We shall now show that this is always the case, provided assumption $(\mathcal{H})$ below holds. (This is the case in all of the aforementioned examples).

Using the same notation as in the second part of Subsection 4.4.1, let (X, d) be a quasi-metric space and (X, D) its symmetrization (D is either d^s or d^{s_0}). Then $P := \mathrm{SLIP}_0(X, d)$ is a cone in $\mathrm{LIP}_0(X, D)$ and $\|\phi\|_L \leq \|\phi|_S$ for all $\phi \in P$. Let us assume:

$(\mathcal{H})$ For every $\phi \in \mathrm{LIP}_0(X, D)$ there exist $\phi_1, \phi_2 \in \mathrm{SLIP}_0(X, d)$ such that

$$\phi = \phi_1 - \phi_2 \qquad \text{and} \qquad \max\{\|\phi_1|_S, \|\phi_2|_S\} \leq \|\phi\|_L.$$

Notice that since $\|\phi_i\|_L \leq \|\phi_i\|_S$, for $i \in \{1, 2\}$ and in view of the triangular inequality

$$\|\phi\|_L = \|\phi_1 - \phi_2\| \leq ||\phi_1||_L + ||\phi_2||_L$$

we observe that $(\mathcal{H})$ yields in particular that P induces a canonical asymmetrization in $\mathcal{F}(X, D)$ (in the sense of Remark 4.23).

Proposition 4.49 *Let (X, d) be a quasi-metric space and assume $(\mathcal{H})$ holds. Then the semi-Lipschitz free space $\mathcal{F}_a(X, d)$ coincides (as a set) with the free space $\mathcal{F}(X, D)$ of the symmetrized space (X, D) and is endowed with the asymmetric norm*

$$\|Q| = \sup_{\substack{\|\phi|_S \le 1 \\ \phi \in \mathrm{SLIP}_0(X,d)}} \langle Q, \phi \rangle\,, \quad \textit{for all } Q \in \mathcal{F}_a(X, d).$$

Proof. Following the method used in Subsection 4.4.1, we start by identifying the sets

$$F = \mathrm{span}\{\delta(x) : x \in X\} \subset \mathrm{SLIP}_0(X, d)^*$$

and

$$\widehat{F} = \mathrm{span}\{\widehat{\delta}(x) : x \in X\} \subset \mathrm{LIP}_0(X, D)^*$$

where δ and $\widehat{\delta}$ are the canonical injections of (X, d) into $\mathrm{SLIP}_0(X, d)^*$ and of (X, D) into $\mathrm{LIP}_0(X, D)^*$, respectively. To conclude, it suffices to prove that the d^s-symmetrization $\|\cdot\|^s$ of the asymmetric norm $\|\cdot|$ is equivalent to $\|\cdot\|_{\mathcal{F}}$. Consider $Q \in F$. Since $\|\phi|_S \ge \|\phi\|_L$ for any $\phi \in \mathrm{SLIP}_0(X, d)$, it follows (by the definition of each norm) that $\|Q| \le \|Q\|_{\mathcal{F}}$, so $\|Q\|^s \le 2\|Q\|_{\mathcal{F}}$. Conversely, take ϕ in the unit ball of $\mathrm{LIP}_0(X, D)$ such that $\|Q\|_{\mathcal{F}} = \langle Q, \phi\rangle$, and consider $\phi_1, \phi_2 \in \mathrm{SLIP}_0(X, d)$ such that $\phi = \phi_1 - \phi_2$, with $\max\{\|\phi_1|_S, \|\phi_2|_S\} \le \|\phi\|_L \le 1$. Then

$$\|Q\|_{\mathcal{F}} = \langle Q, \phi\rangle = \langle Q, \phi_1\rangle + \langle -Q, \phi_2\rangle \le \|Q| + \|-Q| := \|Q\|^s.$$

The result follows from the fact that $\mathcal{F}_a(X, d) = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_a}} = \overline{F}^{\|\cdot\|_{\mathcal{F}}} = \mathcal{F}(X, D)$. $\square$

4.6 Locally flat semi-Lipschitz functions

In this brief section we explore the idea of generalizing the notion of locally flat Lipschitz functions to quasi-metric spaces. The following definitions follows the ideas found in Chapter 4 of [42].

Definition 4.50 *Let (X, d) be a quasi-metric space. We say a function $f : X \to (\mathbb{R}, d_u)$ is locally flat if for every $p \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $a, b \in B_d(p, \delta)$ implies $f(b) - f(a) \le \varepsilon d(a, b)$.*

Definition 4.51 *We say a function $f : X \to (\mathbb{R}, d_u)$ on a quasi-metric space (X, d) is flat at infinity if for every $\varepsilon > 0$, there exists a forward-compact set $K \subset X$ such that*

$$a, b \notin K \implies f(b) - f(a) \le \varepsilon d(a, b).$$

Definition 4.52 *Let (X, d) be a pointed and boundedly compact (for the forward topology) quasi-metric space. The set of little semi-Lipschitz functions* $\mathrm{slip}(X)$ *consists of all semi-Lipschitz functions which are locally flat and flat at infinity. We denote by* $\mathrm{slip}(X)_0$ *the subset of little semi-Lipschitz functions which vanish at the base point.*

It is clear that for any boundedly compact pointed quasi-metric space, $(\mathrm{slip}_0(X), \| \cdot |_S)$ is a subcone of $\mathrm{SLIP}_0(X)$. It is natural then to ask whether, under the right assumptions, $\mathrm{slip}_0(X)$ is a predual to the semi-Lipschitz free space $\mathcal{F}_a(X)$. We currently do not know whether such a duality result (mirroring the symmetric case, see Theorem 2.81) can be asserted. Nevertheless, we present an illustrative example where the desired duality relation holds.

Consider the set of natural numbers $\mathbb{N}$ (including 0) endowed with the quasi-hemi-metric defined by

$$d(n,m) = \begin{cases} 1, & \text{if } m \notin \{0,n\} \\ 0, & \text{if } m \in \{0,n\} \end{cases}. \tag{4.13}$$

and take $x_0 = 0$ as base point. As seen in Subsection 4.5.2, the space of semi-Lipschitz functions over this quasi-metric space can be identified with the positive cone of $\ell^\infty(\mathbb{N})$, meaning that $\mathrm{SLIP}_0(\mathbb{N}, \| \cdot \|_S)$ is isometrically isomorphic (in the sense of normed cones) to $(\ell^\infty(\mathbb{N})_+, \| \cdot \|_\infty)$. We have also shown in Subsection 4.5.2 that $\mathcal{F}_a(\mathbb{N}, d)$ is isometrically isomorphic (in the sense of asymmetric normed spaces) to $\ell^1(\mathbb{N})$, endowed with the following asymmetric norm:

$$\|(x_n)|^* = \sum_{n \in \mathbb{N}} \max\{x_n, 0\} = \sum_{n\in\mathbb{N}} x_n^+ := \|(x_n)|_{1,+}.$$

We proceed to show that $\mathrm{slip}_0(\mathbb{N}, d)$ is isometrically isomorphic (in the sense of normed cones) to the positive cone of $c_0(\mathbb{N})$, endowed with the supremum norm.

Proposition 4.53 $\mathrm{slip}_0(\mathbb{N}, d)$ *is isometrically isomorphic to* $(c_0(\mathbb{N})_+, \| \cdot \|_\infty)$.

Proof. First of all, since the space $(\mathbb{N}, d)$ is discrete, the local flatness condition is superfluous. Secondly, the fact that any $f \in \mathrm{slip}_0(\mathbb{N}, d)$ must be non negative follows from the semi-Lipschitz inequality (using 0 as one of the points). Finally, it is clear that $(\mathbb{N}, d)$ is boundedly compact, and that the condition of flatness at infinity is equivalent to belonging to $c_0(\mathbb{N})$. Indeed, assuming flatness at infinity, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_n - f_m \leq \varepsilon d(m,n)$ for all $m, n \geq N$. From this inequality we can deduce $|f_n - f_m| \leq \varepsilon \max\{d(m,n), d(n,m)\} = \varepsilon$, and we are done. The remaining implication is trivial, which finishes the proof. □

Proposition 4.54 *The dual cone of* $(c_0(\mathbb{N})_+, \| \cdot \|_\infty)$ *is isometrically isomorphic to the asymmetric normed space* $(\ell^1(\mathbb{N}), \| \cdot |_{1,+})$.

Proof. Let $\varphi \in (c_0(\mathbb{N})_+, \| \cdot \|_\infty)^*$. By linearity, it is clear that φ must be of the form $\varphi((f_n)) = \langle \varphi_n, f_n \rangle$ for a sequence $(\varphi_n)_{n\in\mathbb{N}}$. Fix $\varphi \in (c_0(\mathbb{N})_+, \| \cdot \|_\infty)^*$, and for any $k \in \mathbb{N}$, let a^k denote the truncation to the k-th coordinate of the sequence $a_n = \mathrm{sgn}((\varphi_n)^+)$. Then, by continuity of φ, we have that

$$\varphi(a^k) = \sum_{i=1}^{k} (\varphi_i)^+ \leq \|\varphi|^* \|a^k\|_\infty = \|\varphi|^*$$

It follows that (φ_n) belongs to $\ell^1(\mathbb{N})$ and that $\|\varphi|^* \geq \|(\varphi_n)|_{1,+}$. The opposite inequality for the norms follows immediately using the sequence $(a^k)_{k\in\mathbb{N}}$ in the definition of dual norm of $(c_0(\mathbb{N})_+, \|\cdot\|_\infty)^*$, and so does the fact that every element in $\ell^1(\mathbb{N})$ defines a linear functional in $(c_0(\mathbb{N})_+, \|\cdot\|_\infty)^*$. □

Proposition 4.55 *Consider the natural numbers* $\mathbb{N}$ *(including 0), endowed with the quasi-hemi-metric* d *given in* (4.13)*. Then, the space of little semi-Lipschitz functions* $\mathrm{slip}_0(\mathbb{N})$ *is a predual for the semi-Lipschitz free space* $\mathcal{F}_a(\mathbb{N})$*, which is in turn a predual for* $\mathrm{SLIP}_0(\mathbb{N})$*. In particular,*

$$\mathrm{slip}_0(\mathbb{N})^{**} = \mathrm{SLIP}_0(\mathbb{N}).$$

Remark 4.56 *This biduality relation is particularly interesting, as the primal* $(c_0(\mathbb{N})_+, \|\cdot\|_\infty)$ *is strictly a cone, the first dual* $(\ell^1(\mathbb{N}), \|\cdot|_{1,+})$ *is a linear space, and the bidual* $(\ell^\infty(\mathbb{N})_+, \|\cdot\|_\infty)$ *is once again strictly a cone.*

Chapter 5

Asymmetric structures and Banach-Stone type results

This chapter deals with the idea of employing asymmetric structures in order to determine quasi-metric and metric structures. This follows the general idea behind the classical Banach-Stone Theorem, which states that compact and Hausdorff topological spaces are completely determined by the Banach space structure of the corresponding space of continuous functions. The term *Banach-Stone type theorem* is commonly used to describe results that assert that a certain structure on a set X is determined by the structure of a suitable space of real-valued functions over X. For more examples on Banach-Stone type theorems, we refer the reader to the survey [24]. In Section 5.1, we present one such result, which generalizes the classical Myers-Nakai Theorem to the case of non reversible Finsler manifolds. In order to achieve this, new structures on subsets of smooth functions have to be defined. In Section 5.2, we make use of the newly defined structures to obtain an abstract Banach-Stone type theorem for metric spaces, which encompasses several known results, as well as some new ones.

5.1 A Myers-Nakai theorem for non reversible Finsler manifolds

The classical Myers-Nakai theorem asserts that the Riemannian structure of a Riemannian Manifold M is determined by the normed algebra structure of the associated space of real-valued, bounded functions of class C^1 with bounded derivatives, denoted by $C_b^1(M)$, when $C_b^1(M)$ is endowed with the natural norm $\|f\| = \|f\|_\infty \vee \|df\|_\infty$. More precisely, the theorem can be stated as follows.

Theorem 5.1 (Myers-Nakai theorem for Riemannian manifolds) *Let M and N be two connected Riemannian manifolds. Then M and N are isometric as Riemannian manifolds if and only if the spaces $C_b^1(M)$ and $C_b^1(N)$ are isometrically isomorphic as normed algebras. Moreover, every isometric isomorphism of normed algebras $T : C_b^1(N) \to C_b^1(M)$ must be of the form $Tf = f \circ \tau$, for a Riemannian isometry $\tau : M \to N$.*

This result was proved by Myers in [34] for compact manifolds, and later by Nakai in [36] for the general case. This theorem is closely related to a previous result of Myers and Steenrod (see [35]), where it was shown that, between connected Riemannian manifolds, Riemannian isometries coincide with metric isometries, when viewing each manifold as a metric space. As mentioned in Chapter 2, this result was generalized to Finsler manifolds by Deng and Hou (see Theorem 2.70).

Another aspect to keep in mind is that, for a Riemannian manifold, a smooth function has bounded derivative if and only if it is Lipschitz, so the algebra $C_b^1(M)$ can be seen as a subspace of $\mathrm{LIP}(M)$, with norm given by $\|f\| = \|f\|_\infty \vee \|f\|_{\mathrm{LIP}}$. The same assertion holds true for **reversible** Finsler manifolds, and this was used by Garrido et al. to generalize the Myers-Nakai theorem to the context of reversible Finsler manifolds.

The aim of this section is to provide a further generalization to the case of general (asymmetric) Finsler manifolds. Proposition 3.5 gives a hint of to this possibility, as it allows us to see the space of C^1-smooth functions with bounded derivative (with respect to Definition 3.4) as a subset of $\mathrm{SLIP}(\mathcal{X})$, for any connected Finsler manifold $\mathcal{X}$. This observation provides a good starting point to study the possibility of extending the Myers-Nakai theorem to Finsler manifolds, but at the same time, it points to an imminent obstacle.

5.1.1 Algebraic challenges

As discussed in Chapter 2, the set $\mathrm{SLIP}(\mathcal{X})$ needs not be a linear space (see Remark 2.44). This is the case even for quasi-metric spaces as "well behaved" as Finsler manifolds, as shown by Example 3.8. Therefore, our first step will be to define a suitable asymmetric analogue for the space $C_b^1(M)$ used in Theorem 5.1.

Definition 5.2 *Let $(\mathcal{X}, F)$ be a second countable and connected Finsler manifold. Consider the following sets of C^1-smooth, bounded, semi-Lipschitz and real-valued functions on $\mathcal{X}$:*

$$\begin{aligned} SC_b^1(\mathcal{X}) &:= \{f \in C^1(\mathcal{X}) \,:\, \|f\|_\infty < \infty,\, \|f|_S < \infty\} = C^1(\mathcal{X}) \cap L^\infty(\mathcal{X}) \cap \mathrm{SLIP}(\mathcal{X}), \\ SC_b^1(\mathcal{X})_+ &:= \{f \in SC_b^1(\mathcal{X}) \,:\, f \geq 0\}. \end{aligned}$$

Clearly, both sets are subcones of $\mathrm{SLIP}(\mathcal{X})$, and can be endowed with the natural conic-norm $\|f| = \|f\|_\infty \vee \|f|_S$. Note that, thanks to Proposition 3.5, an equivalent definition for both spaces and their conic-norm can be given by asking for the differential of each function to be bounded (in the sense of Definition 3.4).

Given that, in general, the cone $SC_b^1(\mathcal{X})$ fails to be a linear space, we cannot expect $SC_b^1(\mathcal{X})$ to be an algebra in the usual sense. Moreover, the lack of additive inverses makes it impossible for $SC_b^1(\mathcal{X})$ to be closed under multiplication.

Remark 5.3 *In general, $SC_b^1(\mathcal{X})$ is not closed under pointwise multiplication.*

Proof. Consider a function $f \in SC_b^1(\mathcal{X})$ such that $-f$ does not belong to $SC_b^1(\mathcal{X})$ (as in Example 3.8). Then, the product of the constant function of value (-1) and f does not belong to $SC_b^1(\mathcal{X})$. □

Despite this negative result, we can avoid this problem by restricting ourselves to functions with non negative values.

Proposition 5.4 *Let $(\mathcal{X}, F)$ be a Finsler manifold. The set $SC_b^1(\mathcal{X})_+$ is closed under pointwise multiplication. Moreover, for any $f, g \in SC_b^1(\mathcal{X})_+$, we have*

$$\|fg| \leq 2\|f|\|g|. \tag{5.1}$$

Proof. It is clear that inequality (5.1) implies that the function fg belongs to $SC_b^1(\mathcal{X})_+$. Moreover, it is also clear that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty \leq \|f|\|g|$, so we only need to prove that $\|fg|_S \leq 2\|f|\|g|$. To this end, let $x, y \in \mathcal{X}$, and recall the notation $u(t) = \max\{t, 0\}$ for $t \in \mathbb{R}$.

$$\begin{aligned}
f(x)g(x) - f(y)g(y) &= f(x)\left(g(x) - g(y)\right) + g(y)\left(f(x) - f(y)\right) \\
&\leq f(x)u\left(g(x) - g(y)\right) + g(y)u\left(f(x) - f(y)\right) \\
&\leq \|f\|_\infty u\left(g(x) - g(y)\right) + \|g\|_\infty u\left(f(x) - f(y)\right) \\
&\leq \|f\|_\infty \|g|_S d_\mathcal{X}(y, x) + \|g\|_\infty \|f|_S d_\mathcal{X}(y, x) \\
&= (\|f\|_\infty \|g|_S + \|g\|_\infty \|f|_S) d_\mathcal{X}(y, x) \\
&\leq 2(\|f\|_\infty \|g|_S \vee \|g\|_\infty \|f|_S) d_\mathcal{X}(y, x) \\
&\leq 2 d_\mathcal{X}(y, x) \left[(\|f\|_\infty \vee \|f|_S)\|g|_S\right] \vee \left[(\|g\|_\infty \vee \|g|_S)\|f|_S\right] \\
&= 2\left(\|f|\|g|_S\right) \vee \left(\|g|\|f|_S\right) d_\mathcal{X}(y, x) \\
&\leq 2\|f|\|g| d_\mathcal{X}(y, x).
\end{aligned}$$

Note that the first inequality above may not hold if either f or g were allowed to take negative values. □

Therefore, the set $SC_b^1(\mathcal{X})_+$ is:

(*i*) A cone for the operations of addition and scalar multiplication (using non negative scalars).

(*ii*) Closed under the operation of pointwise multiplication, which distributes over addition and scalar multiplication.

(*iii*) A normed cone when endowed with $\|\cdot|$, with the property that $\|fg| \leq 2\|f|\|g|$ for all $f, g \in SC_b^1(\mathcal{X})_+$.

Inspired by this properties, we introduce a new definition.

5.1.2 New asymmetric structures

First, we recall the algebraic definition of a semiring: a semiring is a commutative monoid endowed with a compatible multiplication operation that distributes over the addition of the monoid.

Definition 5.5 (Conic-semiring) *A conic-semiring is an abstract cone (as per Definition 2.14) endowed with a multiplication that makes it a semiring. If the cone is endowed with a conic for which there exists a constant $K \geq 0$ such that $\|fg| \leq K\|f|\|g|$ for all f, g in the cone, we will call it a normed conic-semiring. A normed conic-semiring will be called unital if it has a multiplicative unit.*

Using this definition, we have that $SC_b^1(\mathcal{X})_+$ is a normed conic-semiring when endowed with its natural operations and conic-norm.

Just like the notion of abstract cone is the asymmetric version of real linear spaces (by using monoids instead of groups and $\mathbb{R}_+$ instead of $\mathbb{R}$ for scalars), the notion of semiring can be seen as the asymmetric version of rings (replacing the additive group with a monoid). By combining these two ideas, we can view conic-semirings as an asymmetric version of algebras.

Let us recall that our objective here was to find a suitable asymmetric version of normed algebras, which is a notion that involves both algebraic and metric structures. With this in mind, we can see Definition 5.5 as a generalization which forgoes the former (linear and ring structure) in order to preserve the latter (having a well defined norm). We will also explore the opposite idea, that is, sacrificing properties of the norm in order to preserve the algebraic structure. Recall that an **extended** asymmetric norm has the same properties as an asymmetric hemi-norm (see Definition 2.2), but is allowed to take the value $+\infty$. We emphasize that extended (symmetric) normed spaces have been studied in the literature (for example, in [10]) but, to the best of our knowledge, it has not yet been combined with asymmetric structures. This relaxation of asymmetric normed spaces will allow us to study linear spaces where the asymmetric norm is not always well defined (in the sense of taking finite values).

Definition 5.6 (Finite subcone) *Let $(E, \|\cdot|)$ be an extended asymmetric normed space. The subset $F = \{x \in E \ : \ \|x| < +\infty\}$ (which is always a cone) is called the finite subcone of E.*

Definition 5.7 (Extended asymmetric normed algebra) *An algebra $\mathcal{A}$ endowed with an extended asymmetric norm $\|\cdot|$ will be called an extended asymmetric normed algebra if the finite subcone satisfies a submultiplicative condition for the norm, i.e., there exists $K \geq 0$ such that $\|fg| \leq K\|f|\|g|$ for all $f, g \in \mathcal{A}$ such that $\|f|, \|g| < \infty$.*

This new definition is clearly connected with conic-semirings:

Proposition 5.8 *Let $(\mathcal{A}, \|\cdot|)$ be an extended asymmetric normed algebra. Then, the finite subcone of $\mathcal{A}$ is a normed conic-semiring when endowed with the norm of $\mathcal{A}$.*

One can also define an extended asymmetric normed algebra from a given normed conic-semiring, but an additional assumption is needed, as not every abstract cone can be linearly embedded into a linear space (see Example 2.17). A sufficient condition to ensure that an abstract cone can be embedded into a (real) linear space is that the cone is cancellative (see Definition 2.16). When working with a cancellative cone C that linearly embeds into $\mathbb{R}^X$ for some set X, $\mathrm{span}(C)$ will denote the linear span of the image of C in $\mathbb{R}^X$.

Proposition 5.9 *Let $(C, \| \cdot |)$ be a cancellative normed conic-semiring that linearly embeds into $\mathbb{R}^X$ for a set X. Set $\mathcal{A} = \mathrm{span}(C)$, and for any $a \in \mathcal{A}$, define*

$$\|a|_{\mathcal{A}} = \begin{cases} \|a| & \text{if } a \in C \\ +\infty & \text{if } a \notin C \end{cases}.$$

Then, $(\mathcal{A}, \| \cdot |_{\mathcal{A}})$ is an extended asymmetric normed algebra, and the finite subcone of $\mathcal{A}$ coincides with C. In this case, we will say that the extended asymmetric normed algebra $\mathcal{A}$ is generated by the normed conic-semiring C.

Proof. Let us verify that $\mathcal{A}$ is an algebra. Since it is by definition a linear space, we only need to check that it is closed under multiplication. Let $x, y \in \mathcal{A}$. Since C is a cone, $\mathrm{span}(C)$ can be written as $C - C = \{c_1 - c_2 \ : \ c_1, c_2 \in C\}$, so we can write

$$xy = (x_1 - x_2)(y_1 - y_2) = x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 = (x_1y_1 + x_2y_2) - (x_1y_2 + x_2y_1),$$

with x_i and y_i in C, for $i = \{1, 2\}$. It follows that $xy \in C - C = \mathcal{A}$. The remaining of the properties hold by definition. □

Following Proposition 5.9, we will work with the following space.

Definition 5.10 *Let $(\mathcal{X}, F)$ be a connected Finsler manifold. We define $\mathcal{A}(\mathcal{X})$ to be the extended asymmetric normed algebra generated by the conic-semiring $SC_b^1(\mathcal{X})_+$,*

$$\mathcal{A}(\mathcal{X}) = \mathrm{span}\left(SC_b^1(\mathcal{X})_+\right).$$

The extended asymmetric norm of an element f of this algebra is given by $\|f| = \|f\|_\infty \vee \|f|_S$ if $f \in SC_b^1(\mathcal{X})_+$, and $+\infty$ otherwise.

Remark 5.11 *Notice that, since every function in $SC_b^1(\mathcal{X})$ is bounded, we can always write $f = (f + \|f\|_\infty) - \|f\|_\infty$, which belongs to $\mathcal{A}(\mathcal{X})$. Therefore, $SC_b^1(\mathcal{X})$ and $SC_b^1(\mathcal{X})_+$ have the same linear span (although they do not induce the same norm on $\mathcal{A}(\mathcal{X})$). It follows that if $\mathcal{X}$ is a reversible Finsler manifold (in particular, if $\mathcal{X}$ is Riemannian), then $\mathcal{A}(\mathcal{X}) = C_b^1(\mathcal{X})$.*

Definition 5.12 *A linear function T between two extended asymmetric normed algebras $(\mathcal{A}_2, \| \cdot |_2)$ and $(\mathcal{A}_1, \| \cdot |_1)$ is said to be forward continuous (or simply continuous) if there exists a constant $K \geq 0$ such that*

$$\|Tf|_1 \leq K\|f|_2,$$

for all $f \in \mathcal{A}_2$. The least constant K satisfying the inequality above is called the norm of T, denoted by $\|T|$.

Note that a continuous linear function necessarily sends the finite subcone of its domain into the finite subcone of its range.

Definition 5.13 *Given two extended asymmetric normed algebras* $(\mathcal{A}_1, \|\cdot|_1)$ *and* $(\mathcal{A}_2, \|\cdot|_2)$, *a mapping* $T : \mathcal{A}_2 \to \mathcal{A}_1$ *is called an extended asymmetric normed algebra isomorphism provided:*

(i) T *is linear and bijective,*

(ii) T *is bicontinuous, i.e.,* T *and* T^{-1} *are continuous in the sense of Definition 5.12.*

(iii) $T(fg) = Tf \cdot Tg$ *for all* $f, g \in \mathcal{A}_2$.

The isomorphism T *is called an extended asymmetric normed algebra isometry if*

$$\|Tf|_1 = \|f|_2$$

for all $f \in \mathcal{A}_2$, *or equivalently, if* $\|T| = \|T^{-1}| = 1$.

In what follows, we will work with extended asymmetric normed algebras associated with Finsler manifolds. Whenever we mention the dual $\mathcal{A}^*$ of an extended asymmetric normed algebra $\mathcal{A}$, it will be its dual cone when viewing $\mathcal{A}$ as an extended asymmetric normed space. Given a functional $\varphi \in \mathcal{A}^*$, we say φ is *multiplicative* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. If the algebra $\mathcal{A}$ is contained in $\mathbb{R}^X$ for some set X (i.e., $\mathcal{A}$ is an algebra of functions over X), we will say a functional $\varphi \in \mathcal{A}^*$ is *positive* if $\varphi(f) \geq 0$ whenever $f \geq 0$ as a function.

5.1.3 Main result

Our aim is to prove that for every pair of connected, second countable and forward complete Finsler manifolds $\mathcal{X}$ and $\mathcal{Y}$, every extended asymmetric normed algebra isometry between $\mathcal{A}(\mathcal{Y})$ and $\mathcal{A}(\mathcal{X})$ induces an isometry between $\mathcal{X}$ and $\mathcal{Y}$.

Definition 5.14 *Let* $(\mathcal{X}, F)$ *be a Finsler manifold. We define the* ***structure space*** *of* $\mathcal{X}$ *as*

$$\mathcal{S}(\mathcal{X}) := \{\varphi : \mathcal{A}(\mathcal{X}) \to \mathbb{R} \ : \ \varphi \text{ is linear, multiplicative and forward continuous}\} \subset \mathcal{A}(\mathcal{X})^*.$$

Remark 5.15 *Every* $\varphi \in \mathcal{S}(\mathcal{X})$ *is actually continuous. Indeed, to see that* $-\varphi$ *is upper semicontinuous, we need to given a bound for* $-\varphi(f)$ *for any* $f \in \mathcal{A}(\mathcal{X})$. *By denoting the constant function of value* 1 *as* $\mathbb{1}$, *we have*

$$-\varphi(f) = \varphi(-\mathbb{1})\varphi(f) \leq \varphi(-\mathbb{1})\|f|\|\varphi|^* \leq K\|f|.$$

As a consequence, an equivalent definition of the structure space $\mathcal{S}(\mathcal{X})$ *could be given by requiring each functional to be continuous instead of usc.*

Proposition 5.16 *Every* $\varphi \in \mathcal{S}(\mathcal{X})$ *is positive.*

Proof. We will use the fact that $\mathcal{A}(\mathcal{X})$ is closed under bounded inversions, that is, whenever $f \in \mathcal{A}(\mathcal{X})$ satisfies $f \geq 1$, then $f^{-1} \in \mathcal{A}(\mathcal{X})$. Indeed, if we take $f \in \mathcal{A}(\mathcal{X})$ such that $f \geq 1$, then it is clear that f^{-1} is of class C^1. Moreover, we have that the derivative of $-f^{-1}$ at a point x equals $f(x)^{-2}df(x)$, so by Proposition 3.5, we have that the function $-f^{-1}$ is semi-Lipschitz. It follows that $f^{-1} \in -SC^1_b(\mathcal{X}) \subseteq \mathcal{A}(\mathcal{X})$.

Using this, and the fact that $\varphi(\mathbb{1}) = 1$, we can show that $\alpha = \varphi(f)$ belongs to $\overline{f(\mathcal{X})}^{|\cdot|}$ for every $f \in \mathcal{A}(\mathcal{X})$. Otherwise, there would exist $\varepsilon > 0$ such that no sequence in $f(\mathcal{X})$ accumulates in $B(\alpha, \varepsilon)$, which implies $(f-\alpha) \geq \varepsilon$. Noting that both $(f-\alpha)$ and $(f-\alpha)^2$ belong to $\mathcal{A}(\mathcal{X})$, we arrive at a contradiction:

$$1 = \varphi((f-\alpha)^2) \cdot \varphi((f-\alpha)^{-2}) = 0.$$

□

Proposition 5.17 *The set of evaluation mappings $\delta(\mathcal{X}) = \{\delta_x : x \in \mathcal{X}\}$ is dense in $\mathcal{S}(\mathcal{X})$ for the w^* topology of $\mathcal{A}(\mathcal{X})^*$.*

Proof. Let $\varphi \in \mathcal{S}(\mathcal{X})$, and consider a basic w^*-neighborhood of φ:

$$W = \{\psi \in \mathcal{S}(\mathcal{X}) : |\varphi(f_i) - \psi(f_i)| < \varepsilon \ \forall i \in \{1, ..., n\}\},$$

with $\varepsilon > 0$, $n \in \mathbb{N}$ and $f_1, ..., f_n \in \mathcal{A}(\mathcal{X})$. Suppose that $\delta_x \notin W$ for every $x \in \mathcal{X}$. Then, the function $g = \sum_{i=1}^n (f_i - \varphi(f_i))^2$ satisfies $g \geq \varepsilon^2$, but $\varphi(g) = 0$, which contradicts the positivity of φ, as $\varphi(g - \varepsilon^2) = -\varepsilon^2 < 0$, since $\varphi(\mathbb{1}) = 1$. □

From this point forward, we will assume $(\mathcal{X}, F)$ to be a connected, forward complete and second countable Finsler manifold with Finsler distance $d_{\mathcal{X}}$.

Proposition 5.18 *The mapping $\delta^{-1} : \delta(\mathcal{X}) \to \mathcal{X}$ that maps each δ_x to its corresponding $x \in \mathcal{X}$ is w^* continuous.*

Proof. Consider a net $(\delta_{x_\lambda}) \subset \delta(\mathcal{X})$ converging to δ_{x_0} in the weak star topology of $\mathcal{A}(\mathcal{X})^*$, and consider the function $f(x) = \min\{d(x_0, x), 1\}$, which is semi-Lipschitz and bounded, but it fails to be of class C^1. To remedy this, for $\varepsilon > 0$, consider a smooth approximation g of f obtained using Corollary 3.11, such that $|\min\{d(x_0, x), 1\} - g(x)| \leq \varepsilon$. Then, if we evaluate the net (δ_{x_λ}) on the function $g \in \mathcal{A}(\mathcal{X})$, we obtain that $\delta_{x_\lambda}(g) = g(x_\lambda)$ converges (in absolute value) to $\delta_{x_0}(g) = g(x_0) \leq \varepsilon$, and since $|\min\{d_{\mathcal{X}}(x_0, x), 1\} - g(x)| \leq \varepsilon$, we conclude that $d_{\mathcal{X}}(x_0, x_\lambda)$ converges to 0. □

Proposition 5.19 *Consider the structure space $\varphi \in S(\mathcal{X})$. Then, the following are equivalent:*

1. *φ has a countable neighborhood basis in $S(\mathcal{X})$ for the w^* topology of $\mathcal{A}(\mathcal{X})$.*
2. *There exists $x \in \mathcal{X}$ such that $\varphi = \delta_x$.*

Proof. (1) $\implies$ (2) Since $(\mathcal{X}, F)$ is forward complete, the Hopf-Rinow theorem (see Theorem 6.6.1 of [8]) asserts that forward bounded and closed subsets of $\mathcal{X}$ are compact, which implies that the function $f(x) = d_{\mathcal{X}}(p, x)$ is proper for any $p \in \mathcal{X}$. By taking a C^1-smooth semi-Lipschitz approximation by above of f (using Corollary 3.11), we obtain a proper function $g \in SC_b^1(\mathcal{X})_+$. Suppose now that $\varphi \in S(\mathcal{X}) \setminus \delta(\mathcal{X})$ has a countable neighborhood basis in $S(\mathcal{X})$ for the w^* topology. Then by Proposition 5.17, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$, with no convergent subsequences, such that $\delta_{x_n} \to \varphi$ in the w^* topology. Since g is proper, $\lim g(x_n) = +\infty$, so there exists a subsequence x_{n_k} such that $g(x_{n_k}) + 1 < g(x_{n_{k+1}})$. Now we can choose a C^1-smooth function $\theta : \mathbb{R} \to [0, 1]$, with bounded derivative (so that $\theta \circ g \in SC_b^1(\mathcal{X})_+$), such that $\theta(g(x_{n_{2k+1}})) = 1$ and $\theta(g(x_{n_{2k}})) = 0$ for every $k \in \mathbb{N}$, which is a contradiction, as the sequence $\delta_{x_{n_k}}(g)$ is not convergent.
(2) $\implies$ (1) If $\varphi = \delta_x$ for some $x \in \mathcal{X}$, then consider any countable neighborhood basis (V_n) of x in $\mathcal{X}$. Then the family of closures $\{\mathrm{cl}_{S(\mathcal{X})}(V_n) : n \in \mathbb{N}\}$ is a countable neighborhood basis of δ_x in $S(\mathcal{X})$. □

Lemma 5.20 *For each $x, y \in \mathcal{X}$,*

$$\min\{d_{\mathcal{X}}(x, y), 1\} \leq \|\delta_y - \delta_x|^* \leq d_{\mathcal{X}}(x, y)$$

where $\|\cdot|^*$ is the norm on $\mathcal{A}(\mathcal{X})^*$.

Proof. First, let us note that $\delta_y - \delta_x$ belongs to the dual cone $\mathcal{A}(\mathcal{X})^*$, as every evaluation functional is Lipschitz. Indeed, $|\delta_x(f)| = |f(x)| \leq \|f|_\infty \leq \|f|$ for all $f \in \mathcal{A}(\mathcal{X})$, and therefore both δ_x and $-\delta_x$ belong to $\mathcal{A}(\mathcal{X})^*$. As any function $f \in \mathcal{A}(\mathcal{X})$ with finite norm is semi-Lipschitz, we immediately have that $\|\delta_y - \delta_x|^* \leq d_{\mathcal{X}}(x, y)$ (by definition of dual norm). On the other hand, fix $x, y \in \mathcal{X}$ and for $\varepsilon > 0$, consider the function $f(u) = \min\{d_{\mathcal{X}}(x, u), 1\}$ and a C^1-smooth and semi-Lipschitz approximation by above g of f, such that $0 \leq g(u) - f(u) \leq \varepsilon$ and $\|g|_S \leq \|f|_S + \varepsilon = 1 + \varepsilon$. Replacing g with $\tilde{g} = (1/(1 + \varepsilon))g$, we have that $\|\tilde{g}| \leq 1$, so

$$\begin{aligned}
\|\delta_y - \delta_x|^* \geq (\delta_y - \delta_x)(\tilde{g}) &= \frac{1}{1+\varepsilon}(g(y) - g(x)) \\
&\geq \frac{1}{1+\varepsilon}(f(y) - g(x)) \\
&\geq \frac{1}{1+\varepsilon}(f(y) - f(x) - \varepsilon) \\
&= \frac{1}{1+\varepsilon}(\min\{d_{\mathcal{X}}(x, y), 1\} - \varepsilon),
\end{aligned}$$

for every $\varepsilon > 0$, and therefore $\|\delta_y - \delta_x|^* \geq \min\{d_{\mathcal{X}}(x, y), 1\}$. □

Theorem 5.21 *Let $(\mathcal{X}, F_{\mathcal{X}})$ and $(\mathcal{Y}, F_{\mathcal{Y}})$ be connected, second countable and forward complete Finsler manifolds, and $T : \mathcal{A}(\mathcal{Y}) \to \mathcal{A}(\mathcal{X})$. Then, the following are equivalent:*

1. *T is an extended asymmetric normed algebra isomorphism.*

2. *There exists a bi-semi-Lipschitz diffeomorphism $h : (\mathcal{X}, d_{\mathcal{X}}) \to (\mathcal{Y}, d_{\mathcal{Y}})$ such that $Tf = f \circ h$ for all $f \in \mathcal{A}(\mathcal{Y})$, and the semi-Lipschitz constants of h and h^{-1} are bounded by $\|T^*|$ and $\|(T^*)^{-1}|$, respectively.*

Proof. (1) $\implies$ (2) Suppose $T : \mathcal{A}(\mathcal{Y}) \to \mathcal{A}(\mathcal{X})$ is an extended asymmetric normed algebra isomorphism, and consider the dual map $T^* : \mathcal{A}(\mathcal{X})^* \to \mathcal{A}(\mathcal{Y})^*$, defined in the usual way, so that $\langle T^*\varphi, f\rangle = \langle \varphi, Tf\rangle$ for all $f \in \mathcal{A}(\mathcal{Y})$ and $\varphi \in \mathcal{A}(\mathcal{X})^*$. Clearly, T^* maps continuous multiplicative functionals into continuous multiplicative functionals, and since T^* is by definition w^* to w^* continuous, it follows that T^* restricts to an homeomorphism between $\mathcal{S}(\mathcal{X})$ and $S(\mathcal{Y})$. Consider now the natural embeddings $\delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{S}(\mathcal{X})$ and $\delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{S}(\mathcal{Y})$. By Lemma 3.4, $T^*(\delta_{\mathcal{X}}(\mathcal{X})) = \delta_{\mathcal{Y}}(\mathcal{Y})$. Now, we can define $h(x) = (\delta_{\mathcal{Y}})^{-1} \circ T^* \circ \delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{Y}$, which is an homeomorphism from $\mathcal{X}$ onto $\mathcal{Y}$. Moreover, for any $f \in \mathcal{A}(\mathcal{Y})$, and any $x \in \mathcal{X}$

$$\begin{aligned} Tf(x) &= \langle \delta_x, Tf\rangle = \langle T^*\delta_x, f\rangle \\ &= \langle \delta_{h(x)}, f\rangle = f(h(x)) = (f \circ h)(x). \end{aligned}$$

Note that since $f \circ h$ is C^1-smooth for any non negative and compactly supported $f \in C^1(\mathcal{Y})$, we can deduce that h is a diffeomorphism, and the same can be said about h^{-1}.
To see that h is bi-semi-Lipschitz, it suffices to prove that it is locally bi-semi-Lipschitz (see Proposition 3.12). To this end, fix $p \in \mathcal{X}$. Consider the open neighborhood of p $U_p = B^s(p, 1) \cap h^{-1}(B^s(h(p), 1))$, where $B^s(q, r)$ denotes a symmetrized ball of center q and radius r. Then, for any $x_1, x_2 \in U_p$, we have $d_{\mathcal{X}}(x_1, x_2) < 1$ and $d_{\mathcal{Y}}(h(x_1), h(x_2)) < 1$. Then, using Lemma 5.20 we have that

$$d_{\mathcal{Y}}(h(x_1), h(x_2)) \leq \|\delta_{h(x_2)} - \delta_{h(x_1)}|^* = \|T^*\delta_{x_2} - T^*\delta_{x_1}|^* \leq \|T^*|\|\delta_{x_2} - \delta_{x_1}|^* \leq \|T^*|d_{\mathcal{X}}(x_1, x_2).$$

The same argument can be used for h^{-1}, obtaining that h is bi-semi-Lipschitz. The remaining implication is direct. □

Theorem 5.22 (Non reversible Myers-Nakai Theorem) *Let $(\mathcal{X}, F_{\mathcal{X}})$ and $(\mathcal{Y}, F_{\mathcal{Y}})$ be connected, second countable and forward complete Finsler manifolds. Then, $(\mathcal{X}, F_{\mathcal{X}})$ and $(\mathcal{Y}, F_{\mathcal{Y}})$ are Finsler isometric if and only if there exist an extended asymmetric normed algebra isometry $T : \mathcal{A}(Y) \to \mathcal{A}(\mathcal{X})$. Moreover, every extended asymmetric normed algebra isometry between $\mathcal{A}(Y)$ and $\mathcal{A}(\mathcal{X})$ is of the form $Tf = f \circ h$, where $h : \mathcal{X} \to Y$ is a Finsler isometry.*

5.2 An abstract Banach-Stone type theorem

Many of the ideas used in Section 5.1 do not seem to rely on specific properties of Finsler manifolds and semi-Lipschitz functions. In fact, many of the key notions used to prove Theorem 5.22, such as, the definition of the structure space, the weak star topology of the dual of an asymmetric normed space and the embedding $\delta : X \to \mathcal{S}(X)$, make sense for a general extended asymmetric normed algebra $\mathcal{A}$ of functions over X. This Section presents a modified, more general version of Theorem 5.22, which can be applied to many spaces of continuous functions over metric spaces, provided some hypothesis are met.

We will start this section by introducing two known classes of functions with Lipschitz-like behavior, pointwise Lipschitz functions and functions with bounded metric slope. The latter turns out to be an asymmetric object, and can be studied using the tools developed in Subsection 5.1.2. After that, we prove the more general version of the main result of this Section, which has the weaker conclusion, as expected. After proving this version in Subsection 5.2.2, we deduce a version with a stronger conclusion, which requires additional requirements on the functions spaces to be used. Finally, we present an intermediate version, which can be applied to the classes of functions introduced in Subsection 5.2.1.

5.2.1 Point-wise Lipschitz functions and metric slopes

Let (X, d) be a metric space. For a given continuous function $f : X \to \mathbb{R}$, the *pointwise Lipschitz constant* at a non-isolated point $x_0 \in X$ is

$$\mathrm{Lip} f(x_0) := \lim_{r\to 0} \sup_{0<s<r} \sup_{d(x,x_0)<s} \frac{|f(x_0) - f(x)|}{s} = \limsup_{\substack{x\to x_0\\ x\neq x_0}} \frac{|f(x_0) - f(x)|}{d(x, x_0)},$$

and the *asymptotic Lipschitz constant* at a point $x_0 \in X$ is

$$\mathrm{Lip}_a f(x_0) := \limsup_{y,z\to x_0} \frac{|f(z) - f(y)|}{d(z, y)}.$$

If f is a *differentiable* function, then $\mathrm{Lip} f(x_0) = \|\nabla f(x_0)\|$. However, the pointwise Lipschitz constant is always non-negative, so it is not so useful if we want to determine descent or ascent directions (and detect minima or maxima). To this end, an asymmetric object is needed.

Definition 5.23 (Metric slope) *For a function $f : X \to \mathbb{R}$, the metric slope at a point $x_0 \in X$ is*

$$|\partial f|^-(x_0) := \limsup_{\substack{x\to x_0\\ x\neq x_0}} \frac{\max\{f(x_0) - f(x), 0\}}{d(x_0, x)} = \begin{cases} 0 & \text{if } x_0 \text{ is a local minimizer of } f, \\ \limsup_{\substack{x\to x_0\\ x\neq x_0}} \frac{f(x_0)-f(x)}{d(x_0,x)} & \text{otherwise.} \end{cases}$$

Notice that, if x_0 is a local minimizer of f, then $|\partial f|^-(x_0) = 0$. However, $|\partial f|^-(x_0) = 0$ at a point $x_0 = 0$ does not necessarily imply that x_0 is a minimum of f (for example, $f(x) = -x^2$ on $\mathbb{R}$).

An *ascendent* metric slope, denoted by $|\partial f|^{+}(x_0)$, can be defined in a similar manner, replacing $f(x_0) - f(x)$ by $f(x) - f(x_0)$ in the numerator, and

$$|\partial f|^{+}(x_0) = |\partial(-f)|^{-}(x_0).$$

The metric slope, also called *local slope*, *descendent slope* or *calmness rate*, was introduced in [19] (see also [5]) in connection with steepest descent evolutionary problems. The notation should not be confused with the relaxed slope $|\partial^{-} f|$ defined in [5, Section 2.3].

The following Figure shows how the notion of metric slope can differ from the pointwise Lipschitz constant.

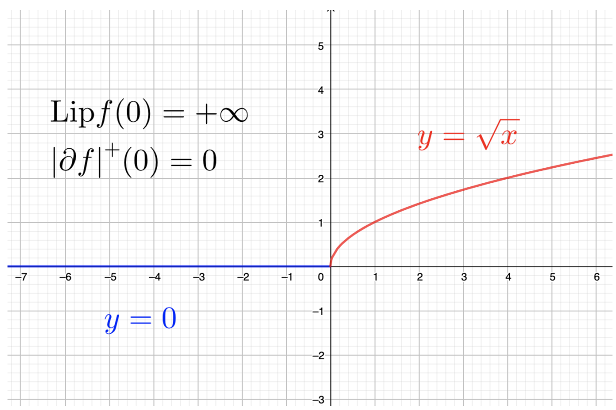


Figure 5.1: Lipschitz function for which $0 = |\partial f|^{+}(0) < \mathrm{Lip} f(0) = +\infty$.

If $f : [a, b] \to \mathbb{R}$ is differentiable at $x_0 \in [a, b]$, then $|f'(x_0)| = |\partial f|^{-}(x_0) = |\partial f|^{+}(x_0)$. We shall make use of the following proposition, which proof follows directly from the definitions involved.

Proposition 5.24 *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ a function. Then,*

$$\mathrm{Lip} f(x_0) = \max\{|\partial f|^{-}(x_0), |\partial f|^{+}(x_0)\}. \tag{5.2}$$

For a Lipschitz function $f : X \to \mathbb{R}$ defined on a metric space, with Lipschitz constant $\mathrm{Lip}(f)$, we have

$$|\partial f|^{-}(x) \leq \max\left\{|\partial f|^{+}(x), |\partial f|^{-}(x)\right\} = \mathrm{Lip} f(x) \leq \mathrm{Lip}_a f(x) \leq \mathrm{LIP}(f), \tag{5.3}$$

where all inequalities might be strict. See Figure 5.2 for $|\partial f|^{-}(x) \neq \mathrm{Lip} f(x)$ and [21, Example 2.7] for $\mathrm{Lip} f(x) \neq \mathrm{Lip}_a f(x)$.

These notions can be easily brought to the general context of quasi-metric spaces. The objects defined above will appear as particular cases of this general setting.

Recall that a sequence (x_n) on a quasi-metric space (X, d) converges to x in the *forward topology* if and only if $d(x, x_n)$ converges to 0 in $(\mathbb{R}, |\cdot|)$.

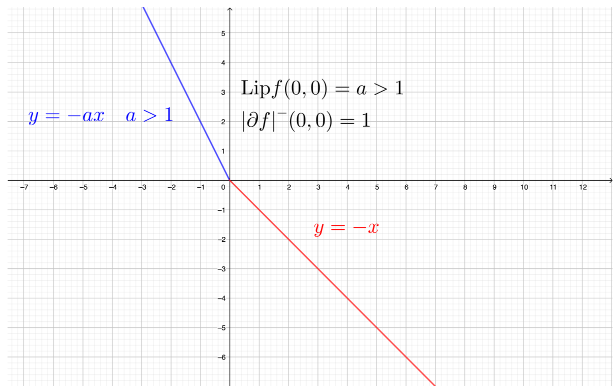


Figure 5.2: Lipschitz function for which $0 < |\partial f|^-(0) = 1 < \mathrm{Lip}f(0) = a = |\partial f|^+(0)$.

Definition 5.25 (Pointwise Semi-Lipschitz functions) *Let (X,d) and (Y,ρ) be quasi-metric spaces and $f : X \to Y$ be a function. f is said to be pointwise semi-Lipschitz at a non-isolated point $x_0 \in X$ if there exists $\alpha \geq 0$ and $\delta > 0$ such that*

$$d(x_0, x) < \delta \implies \rho(f(x_0), f(x)) \leq \alpha d(x_0, x). \tag{5.4}$$

The least constant satisfying inequality (5.4) is called the pointwise semi-Lipschitz constant of f at x_0, and will be denoted by $\mathrm{SLip}f(x_0)$. If x_0 is an isolated point, we define $\mathrm{SLip}f(x_0) = 0$. A function $f : X \to Y$ is called pointwise semi-Lipschitz if $\mathrm{SLip}f(x) < +\infty$ for every $x \in X$.

Remark 5.26 *It is easy to check that this definition encompasses both, the definitions of the pointwise Lipschitz constant and the metric slope in the following way:*

- *If (X,d) is a metric space and $(Y,\rho) = (\mathbb{R}, |\cdot|)$, then*

  $$\mathrm{SLip}f(x) = \mathrm{Lip}f(x)$$

  for any $x \in X$.

- *If (X,d) is a metric space and $(Y,\rho) = (\mathbb{R}, d_u)$, then*

  $$\mathrm{SLip}f(x) = |\partial f|^+(x) = |\partial(-f)|^-(x)$$

  for any $x \in X$. Then, for $f : X \to \mathbb{R}$, the formula (5.2) can be rewritten as

  $$\mathrm{Lip}f(x) = \max\{\mathrm{SLip}f(x), \mathrm{SLip}(-f)(x)\}.$$

In what follows, if we consider real-valued functions $f : X \to \mathbb{R}$, when computing $\mathrm{SLip}f$ we will assume that $\mathbb{R}$ is endowed with the quasi-metric d_u.

Remark 5.27 *Any locally flat semi-Lipschitz function f (see Definition 4.50) has* $\mathrm{SLip} f = 0$ *at all points.*

Definition 5.28 *We say a bijection $\tau : X \to Y$ is a pointwise semi-Lipschitz homeomorphism if both τ and τ^{-1} are pointwise semi-Lipschitz. An analogous definition is used for pointwise Lipschitz homeomorphisms.*

Remark 5.29 *For a function $f : X \to \mathbb{R}$ on a quasi-metric space X, a straightforward computation shows that f is upper semicontinuous at any point $x \in X$ where* $\mathrm{SLip} f(x) < \infty$*. This is also true for a function τ between two quasi-metric spaces (X, d) and (Y, ρ), in which case τ is forward-forward continuous at every point where* $\mathrm{SLip}\ \tau(x) < +\infty$*.*

The fact that pointwise semi-Lipschitz functions do not need to be continuous shows us that this notion does not coincide with the one of pointwise Lipschitz functions.

Example 5.30 Let us consider the function $f : (\mathbb{R}, |\cdot|) \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{5.5}$$

In this case, $\mathrm{Lip} f(0) = +\infty$, while $\mathrm{SLip} f(x) = 0$ at every $x \in \mathbb{R}$. In other words, pointwise semi-Lipschitz functions need not be pointwise Lipschitz.

Remark 5.31 *Example 5.30 shows that the notions of pointwise semi-Lipschitz functions and pointwise Lipschitz differ even when the domain is a metric space, as opposed of what happens with (globally) semi-Lipschitz functions, which are always Lipschitz when the domain is a metric space.*

Remark 5.32 For a function $f : (X, d_X) \to (Y, d_Y)$ between quasi-metric spaces, the following inequalities hold.

- $\mathrm{SLip}_{d_X^s, d_Y} f(x) \leq \mathrm{SLip}_{d_X, d_Y} f(x) \leq \mathrm{SLip}_{d_X, d_Y^s} f(x)$, where SLip_{d_1, d_2} is computed using the quasi-metric d_1 on X and the quasi-metric d_2 on Y.
- $\mathrm{SLip}_{d_1, d_2} f(x) \leq \mathrm{SLIP}_{d_1, d_2} f$, for $d_1 = \{d_X, d_X^s\}$ and $d_2 = \{d_Y, d_Y^s\}$.

We recall the definition of the space $D(X)$ of pointwise Lipschitz functions, which was studied in [21]. For metric spaces (X, d_X) and (Y, d_Y), consider

$$D(X) = \{f : X \longrightarrow \mathbb{R} : \sup_{x \in X} \mathrm{Lip} f(x) < +\infty\},$$
$$D(X, Y) = \{f : X \longrightarrow Y : \sup_{x \in X} \mathrm{Lip} f(x) < +\infty\}.$$

For the asymmetric cases, we can define the following functional spaces.

Definition 5.33 *Let (X, d_X) and (Y, d_Y) be quasi-metric spaces. Consider*

$$D_{\mathrm{SL}}(X,Y) = \{f : X \longrightarrow Y : \sup_{x\in X} \mathrm{SLip} f(x) < +\infty\}.$$

*If (X,d) is a **metric** space and $Y = \mathbb{R}$ we denote*

- ◇ $D^\infty(X) = \{f \in D(X) : \sup_{x\in X} |f(x)| < +\infty\}$.
- ◇ $D_{\mathrm{SL}}(X) = \{f : X \longrightarrow \mathbb{R} : \sup_{x\in X} \mathrm{SLip} f(x) = \sup_{x\in X} |\partial f|^+(x) < +\infty\}$.
- ◇ $D^\infty_{\mathrm{SL}}(X) = \{f \in D_{\mathrm{SL}}(X) : \sup_{x\in X} |f(x)| < +\infty\}$.

We remark the convention that, when computing $\mathrm{Lip} f(x)$*, $\mathbb{R}$ is assumed to carry the usual metric, and when computing* $\mathrm{SLip} f(x)$*, $\mathbb{R}$ is endowed with the quasi-metric d_u.*

With these definitions, the conclusion of Example 5.30 can be restated as *"$D_{\mathrm{SL}}(X)$ is not a vector space, even when X is a metric space"*. However, the function used in Example 5.30 is not continuous. It is natural to ask whether such an example of a function in $D_{\mathrm{SL}}(X) \setminus D(X)$ must be discontinuous. In what follows we provide an example of a metric space X and a continuous function $f : X \to \mathbb{R}$ such that $f \in D_{\mathrm{SL}}(X) \setminus D(X)$.

Example 5.34 *Consider on the interval $X = [0,1]$ the snowflake distance $d(x,y) = |x-y|^{1/2}$. Select a point $a \in X$, and choose a sequence of different points (a_n) in X converging to a, and a sequence of small enough radii (r_n), such that $0 < r_n < \frac{1}{n} d(a_n, a)$, the open balls $B_d(a_n, r_n)$ are pairwise disjoint, and each $x \neq a$ has a neighborhood V^x which meets only a finite number of balls $B_d(a_n, r_n)$.*

Note that any Lipschitz function $f : ([0,1], |\cdot|) \to \mathbb{R}$ satisfies that $\mathrm{Lip}_d f(x) = 0$ for every $x \in X$. Indeed, if

$$|f(x) - f(y)| \leq K|x-y|$$

Then

$$\limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)} \leq \limsup_{y\to x} \frac{K|x-y|}{|x-y|^{1/2}} = 0.$$

Then, if for each n we fix some $k_n > 0$, we can select a continuous function $f_n : X \to \mathbb{R}$, which is Lipschitz for the euclidean distance in X, such that $0 \leq f_n \leq k_n d(a_n, a)$ on X, $f_n(a_n) = k_n d(a_n, a)$, and the support of f_n is contained into $B_d(a_n, r_n)$. Now consider

$$f = \sum_{n=1}^{\infty} f_n.$$

The function f is well-defined and continuous on $X \setminus \{a\}$, since the sum is locally finite. If we assume in addition that

$$\lim_{n\to\infty} = k_n d(a_n, a) = 0$$

we obtain that f is continuous on X. Furthermore, we have that $\mathrm{Lip} f(x) = 0$ *for each $x \neq a$. Thus $|\partial f|^+(x) = 0$ for $x \neq a$. Since a is a local (and global) minimum of f, we see that also $|\partial f|^+(a) = 0$.*

On the other hand,

$$\mathrm{Lip} f(a) = \limsup_{n\to\infty} \frac{|f(a) - f(a_n)|}{d(a_n, a)} = \sup_n k_n.$$

If we choose a constant sequence $k_n = k$ we obtain an example where

$$\sup_{x\in X} \mathrm{SLip} f(x) = 0 < k = \sup_{x\in X} \mathrm{Lip} f(x) < +\infty.$$

On the other hand, if we choose a sequence (k_n) tending to infinity, for example $k_n = \frac{1}{\sqrt{d(a_n,a)}}$, we obtain an example where $\mathrm{SLip} f(x) = 0$ *and* $\mathrm{Lip} f(x) < +\infty$ *for every $x \in X$, but*

$$\sup_{x\in X} \mathrm{Lip} f(x) = +\infty.$$

Before continuing, we recall the Banach-Stone type theorem obtained in [21, Theorem 4.6]. We need a definition first.

Definition 5.35 *A metric space (X, d) is said to be locally radially quasi-convex if for every $x \in X$, there exists a neighborhood U_x of x and a constant $K_x > 0$ such that for each $y \in U_x$ there exists a rectifiable curve γ in U_x connecting x and y and such that $\ell(\gamma) \leq K_x d(x, y)$.*

Theorem 5.36 *Let (X, d_X) and (Y, d_Y) be complete locally radially quasi-convex metric spaces. The following are equivalent.*

- *(X, d_X) and (Y, d_Y) are pointwise Lipschitz homeomorphic.*
- *$D^\infty(X)$ and $D^\infty(X)$ are isomorphic as algebras.*

If one wanted a result in the same line, but using $D^\infty_{\mathrm{SL}}(X)$ instead, major adjustments would have to be made, as $D^\infty_{\mathrm{SL}}(X)$ is not a linear space. But, as we saw in Theorem 5.22, this obstacle can be overcome. The following subsections deal with the issue of formulating such a result in an abstract way, making use of the definitions and tools learned from Section 5.1.

5.2.2 Topological version

Definition 5.37 *Let $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ be extended asymmetric normed algebras of real-valued functions over X and Y, respectively. An isomorphism $T : \mathcal{A}(Y) \to \mathcal{A}(X)$ is called positive if $Tf \geq 0$ whenever $f \geq 0$.*

We shall start by stating the main result.

Theorem 5.38 *Consider the real line $\mathbb{R}$ endowed with its usual metric. Let (X, d_X) and (Y, d_Y) be complete* ***metric spaces****, let $\mathcal{F}(X, Y)$ be a subset of $C(X, Y)$, and let $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ be subcones of $C(X, \mathbb{R})$ and $C(Y, \mathbb{R})$, respectively, such that:*

(i) *The subcones* $\mathcal{G}_+^\infty(X) := \mathcal{G}(X) \cap L^\infty(X) \cap [0,\infty)^X$ *and* $\mathcal{G}_+^\infty(Y) := \mathcal{G}(Y) \cap L^\infty(Y) \cap [0,\infty)^Y$ *are endowed with conic norms, which are finer than* $\|\cdot\|_\infty$*, and which make* $\mathcal{G}_+^\infty(X)$ *and* $\mathcal{G}_+^\infty(Y)$ *into* ***unital*** *normed conic-semirings under the usual addition and multiplication of real-valued functions.*

(ii) $h \in \mathcal{F}(X,Y)$ *if and only if* $f \circ h \in \mathcal{G}_+^\infty(X)$ *for all* $f \in \mathcal{G}_+^\infty(Y)$*.*

(iii) $\mathcal{G}_+^\infty(X)$ *(respectively* $\mathcal{G}_+^\infty(Y)$*) is uniformly separating for* (X, d_X) *(respectively* (Y, d_Y)*), in the sense that, for every pair of subsets* A *and* B *of* X*, with* $d_X(A,B) > 0$*, there exists some* $f \in \mathcal{G}_+^\infty(X)$ *such that* $\overline{f(A)}^{|\cdot|} \cap \overline{f(B)}^{|\cdot|} = \emptyset$*.*

Denote $\mathcal{A}(X) = \operatorname{span}(\mathcal{G}_+^\infty(X))$*, endowed with the extended asymmetric norm induced by* $\mathcal{G}_+^\infty(X)$*, and its natural algebra structure, and let* $T : \mathcal{A}(Y) \to \mathcal{A}(X)$ *be a positive isomorphism of extended asymmetric normed algebras. Then, there exists* $\tau \in \mathcal{F}(X,Y)$*, with* $\tau^{-1} \in \mathcal{F}(Y,X)$*, and such that*

$$Tf = f \circ \tau$$

for all $f \in \mathcal{A}(Y)$*.*

Remark 5.39 *In this result, the set* $\mathcal{F}(X,Y)$ *has the role of "space of morphisms" between* X *and* Y*. The functional spaces* $\mathcal{G}(X)$ *and* $\mathcal{G}(Y)$ *need not be of the "same nature" as* $\mathcal{F}(X,Y)$*, and this will often be the case in our examples (see forthcoming Corollary 5.46).*

Let us begin with the proof of Theorem 5.38. Thanks to Proposition 5.9, we know that $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are extended asymmetric normed algebras. Therefore, we can readily define the structure space.

$$\mathcal{S}(X) := \{\varphi : \mathcal{A}(X) \to \mathbb{R} \ : \ \varphi \text{ is linear, multiplicative, continuous and } \textbf{positive}\} \subset \mathcal{A}(X)^*.$$

Notice that, unlike in Section 5.1, we include positivity in the definition of the structure space.

Proposition 5.40 *The set of evaluation functionals* $\delta(X) = \{\delta_x : \mathcal{A}(X) \to \mathbb{R} \ : x \in X\}$ *is contained in* $\mathcal{S}(X)$*.*

Proof. It is clear that every δ_x is linear and positive on $\mathcal{A}(X)$. Upper semi continuity is deduced from the fact that the extended asymmetric norm on $\mathcal{A}(X)$ is finer than the supremum norm, and continuity follows from the same argument used in Remark 5.15. $\square$

Proposition 5.41 (X, d_X) *is homeomorphic to* $(\delta(X), w^*)$*.*

Proof. We start by noting that this proposition can be reduced to a well known result in general topology, as the weak-star topology coincides with the product topology of $\mathbb{R}^X$ (see for instance Theorem 8.16 of [43]). Nevertheless, we include the proof for the sake of completeness. We start by proving that $\delta : X \to \delta(X)$ is open. Let $U \subseteq X$ be an open set, and fix a point $\delta_x \in \delta(U)$. Since x does not belong to the closed set U^c, and we can use hypothesis *(iii)* of Theorem 5.38 to obtain a function $f \in \mathcal{G}_+^\infty(X)$ that separates $\{x\}$ from U^c, that is, $f(x) \notin \overline{f(U^c)}$, which implies the existence of $\varepsilon > 0$ such that $B(f(x), \varepsilon) \cap f(U^c) = \emptyset$.

Consider now the weak-star neighborhood of 0 defined by the separating function f and the radius ε of the ball, that is, $W = \{\delta_z \in \mathcal{S}(X) : |\delta_z(f)| < \varepsilon\}$. Then, the set $\delta_x + W$ is a weak-star neighborhood of δ_x contained in $\delta(U)$. On the other hand, continuity of the mapping δ follows directly from the fact that the functions in $\mathcal{G}(X)$ (and therefore $\mathcal{A}(X)$) are continuous. □

Remark 5.42 *It is worth noting that an asymmetric normed space E is not in general a topological vector space, as multiplication by scalars may fail to be continuous at points of the form $(0, x) \in \mathbb{R} \times E$. Fortunately, addition remains continuous, and therefore, translations of open sets are still open.*

Proposition 5.43 *$\delta(X)$ is weak-star dense in $\mathcal{S}(X)$.*

Proof. Consider a basic weak-star neighborhood of a function $\varphi \in \mathcal{S}(X)$:

$$W = \{\psi \in \mathcal{S}(X) : |\psi(f_i) - \varphi(f_i)| < \varepsilon, \ i = 1, ..., n\}.$$

Assume that $W \cap \delta(X) = \emptyset$, and consider the function $g = \sum_{i=1}^{n} (f_i - \varphi(f_i))^2 \in \mathcal{A}(X)$. Then, $g(x) \geq n\varepsilon^2 > 0$ and $\varphi(g) = 0$, which contradicts the positivity of φ. Therefore, $W \cap \delta(X) \neq \emptyset$ and $\delta(X)$ is weak star dense in $\mathcal{S}(X)$. □

Lemma 5.44 *The following are equivalent:*

- *(i) $\varphi \in \mathcal{S}(X)$ has a countable neighborhood basis.*
- *(ii) There exists $x \in X$ such that $\varphi = \delta_x$*

Proof. Assume (ii), and consider a countable neighborhood basis (V_n) for $x \in X$ such that $\varphi = \delta_X$. Proposition 5.41 implies that $(\overline{\delta(V_n)}^{w^*})$ is a weak-star neighborhood basis for φ. Conversely, assume $\varphi \in \mathcal{S}(X) \setminus \delta(X)$ has such a neighborhood basis. By Proposition 5.43, there exist a sequence (x_n) in X such that δ_{x_n} converges to φ in the weak-star topology of $\mathcal{S}(X)$. This implies, by completeness of (X, d), that (x_n) has no Cauchy sub-sequence, otherwise such a sub-sequence would be convergent to $x \in X$, which would contradict the fact that $\varphi \notin \delta(X)$, as the weak-star topology of the dual cone of an extended asymmetric normed space is T_1. Therefore, there exists $\varepsilon > 0$ and a sub-sequence (x_{n_k}) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ whenever $k \neq j$. Define $A = \{x_{n_k} : k \text{ is odd}\}$ and $B = \{x_{n_k} : k \text{ is even}\}$. Since $\mathcal{G}_+^{\infty}(X)$ is uniformly separating, we can find $f \in \mathcal{G}_+^{\infty}(X)$ such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$, but, since (x_n) converges to φ in the weak star topology, we have that $f(x_n)$ converges (in the $|\cdot|$-topology of $\mathbb{R}$) to $\varphi(f)$, which then must belong to $\in \overline{f(A)} \cap \overline{f(B)}$. □

We can now proceed to the proof of Theorem 5.38.

Proof. Let $T : \mathcal{A}(Y) \to \mathcal{A}(X)$ be a positive isomorphism of extended asymmetric normed algebras, and consider the dual mapping $T^* : \mathcal{A}(X)^* \to \mathcal{A}(Y)^*$ defined by the formula

$$\langle T^*\varphi, f\rangle = \langle \varphi, Tf\rangle \text{ for all } f \in \mathcal{A}(Y).$$

This definition automatically yields that T^* is well defined, linear, bijective and w^*-w^* continuous. The algebraic properties of T, along with the assumed positivity, guarantee that T^* sends positive and multiplicative functionals to positive and multiplicative functionals. Moreover, Lemma 5.44 ensures that T^* sends $\delta(X)$ into $\delta(Y)$. Then, we can define $\tau : X \to Y$ as $\tau(x) = \delta_Y^{-1} T^*(\delta_X(x))$, where δ_X and δ_Y are the corresponding embeddings of X and Y into $\mathcal{A}(X)$ and $\mathcal{A}(Y)$. Injectivity of τ follows from the fact that T is surjective and that $\mathcal{G}_+^\infty(X)$ is separating for d_X, and surjectivity follows directly from the properties of T^*. Finally, hypothesis (ii) guarantees that $\tau \in \mathcal{F}(X, Y)$. The same argument for the isomorphism T^{-1} yields that $\tau^{-1} \in \mathcal{F}(Y, X)$. The formula $Tf = f \circ \tau$ follows from the definition of τ. □

Before we can apply this result to the spaces $D^\infty(X)$ and $D_{\mathrm{SL}}^\infty(X) \cap C(X)$, we need to ensure that hypothesis (ii) of Theorem 5.38 is satisfied. Let us denote by $D_+^\infty(X)$ and $D_{\mathrm{SL}+}^\infty(X)$ the respective cones of non-negative functions.

Lemma 5.45 *Let (X, d_X) and (Y, d_Y) be metric spaces and $h : X \to Y$. The following are equivalent:*

(1) *$h : X \to Y$ is pointwise Lipschitz.*

(2) *For every $f \in D_+^\infty(Y)$, $f \circ h \in D_+^\infty(X)$.*

(3) *For every $f \in D_{\mathrm{SL}+}^\infty(Y)$, $f \circ h \in D_{\mathrm{SL}+}^\infty(X)$.*

Proof. We start with (1) implies (3). Fix $f \in D_{\mathrm{SL}+}^\infty(Y)$, $x_0 \in X$ and let $y_0 = h(x_0)$. Since $f \in D_{\mathrm{SL}}(Y)$, there exists $\alpha > 0$ and $\delta > 0$ such that, whenever $d_Y(y_0, y) < \delta$, we have

$$d_u(f(y_0), f(y)) \leq \alpha d_Y(y_0, y).$$

As $h \in D(X, Y)$, there exists $\beta > 0$ and $\delta' > 0$ such that, whenever $d_X(x_0, x) < \delta'$, we have

$$d_Y(h(y_0), h(y)) \leq \beta d_X(x_0, x) \leq \beta \delta'.$$

Now, we take δ'' such that $0 < \delta'' < \min\{\delta', \frac{\delta}{\beta}\}$. It follows that if $d_X(x_0, x) < \delta''$, then

$$d_Y(h(x_0), h(x)) \leq \beta d_X(x_0, x) < \beta \delta'' \leq \delta,$$

which implies that

$$d_u(f(h(x_0)), f(h(x))) \leq \alpha \beta d_X(x_0, x),$$

so $f \circ h \in D_{\mathrm{SL}}(X)$. Since $f \circ h$ is clearly bounded and non-negative, we conclude $f \circ h \in D_{\mathrm{SL}+}^\infty(X)$. The same argument can be used to prove that (1) implies (2).
To see that (3) implies (1), for every $q \in Y$ consider the function $f_q(y) = \min\{d_Y(q, y), 1\}$, which is Lipschitz of constant 1, which implies that $\mathrm{SLip} f_q(y) \leq 1$ for all $y \in Y$ (see inequality (5.3)). In particular, for every $x_0 \in X$, we can take $y_0 = h(x_0)$, and have $\mathrm{SLip} f_{y_0}(h(x_0)) \leq 1$, which implies $f_{y_0} \circ h$ belongs to $D_{\mathrm{SL}+}^\infty(X)$ and therefore,

$$\mathrm{SLip}(f_{y_0} \circ h)(x_0) = \limsup_{x \to x_0} \frac{d_u(f_{y_0} \circ h(x_0), f_{y_0} \circ h(x))}{d_X(x_0, x)} < +\infty.$$

On the other hand, when x is close enough to x_0:

$$d_u(f_{y_0} \circ h(x_0), f_{y_0} \circ h(x)) = d_u(0, d_Y(h(x_0), h(x))) = d_Y(h(x_0), h(x)).$$

It follows that

$$\limsup_{x \to x_0} \frac{d_Y(h(x_0), h(x))}{d_X(x_0, x)} = \mathrm{SLip}(f_{y_0} \circ h)(x_0) < +\infty.$$

□

Corollary 5.46 *Theorem 5.38 can be applied to the following classes of spaces of real-valued functions $\mathcal{G}^\infty(X)$.*

(*a*) $C_b(X)$ *of bounded, continuous functions on a completely metrizable topological space X, endowed with the supremum norm. In this case, τ will be an homemorphism.*

(*b*) $D^\infty(X)$, *of bounded pointwise Lipschitz functions with bounded pointwise Lipschitz constant on a complete metric space X, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|\mathrm{Lip}(f)\|_\infty\}$. In this case, τ will be a pointwise Lipschitz homeomorphism.*

(*c*) $D^\infty_{\mathrm{SL}}(X) \cap C(X)$, *of bounded* ***continuous*** *functions with bounded metric slope on a complete metric space X, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|\mathrm{SLip}(f)\|_\infty\}$. In this case, τ will be a pointwise Lipschitz homeomorphism.*

(*d*) $C^1_b(\mathcal{X})$, *of bounded functions with bounded derivative on a connected, reversible and complete Finsler manifold $\mathcal{X}$, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|df\|_\infty\}$. In this case, τ will be a bi-Lipschitz diffeomorphism.*

(*e*) $SC^1_b(\mathcal{X})$, *of bounded semi-Lipschitz functions of class C^1 on a connected and bicomplete Finsler manifold, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|df|_\infty\}$. In this case, τ will be a* ***semi****-Lipschitz diffeomorphism. (To apply Theorem 5.38, the metric $d_{\mathcal{X}}$ has to be the symmetrization of the Finsler quasi-metric. We remark that the separation property holds due to every semi-Lipschitz function on a Finsler manifold being continuous.)*

(*f*) $\mathrm{LIP}^\infty(X)\}$, *of bounded Lipschitz functions on a complete metric space, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \mathrm{LIP}(f)$. In this case, τ will be a Lipschitz homeomorphism.*

(*g*) $\mathrm{lip}(X)$ *of little Lipschitz functions on a compact and purely 1-unrectifiable metric space X (see Theorem 2.82), endowed with the norm $\|f\| = \max\{\|f\|_\infty, \mathrm{LIP}(f)\}$. In this case, τ will be a Lipschitz homeomorphism.*

5.2.3 Lipschitz version

Several of the examples of Corollary 5.46 could be improved upon, for example, adding some form of quantitative control over the homeomorphism τ. This approach makes sense when τ is, for instance, a Lipschitz homeomorphism, but not when τ is only a topological homeomorphism. In order to refine these results, we need to add stronger hypothesis that will yield stronger conclusions, at the expense of reducing the scope of the result.

Theorem 5.47 *Let (X, d_X) and (Y, d_Y) be complete* ***metric spaces****, let $\mathcal{F}(X,Y)$ be a subset of $C(X,Y)$, and let $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ be subcones of $\mathrm{LIP}(X)$ and $\mathrm{LIP}(Y)$, respectively, such that:*

- (i) *For $Z \in \{X, Y\}$, the subcone $\mathcal{G}_+^\infty(Z) := \mathcal{G}(Z) \cap L^\infty(Z) \cap [0,\infty)^Z$ is endowed with a conic norm $\|\cdot|_Z$ which satisfies $\|\cdot|_Z \geq \max\{\mathrm{LIP}(\cdot), \|\cdot\|_\infty\}$, and which makes $\mathcal{G}_+^\infty(Z)$ into a unital normed conic-semiring under the usual addition and multiplication of real-valued functions.*
- (ii) *$h \in \mathcal{F}(X,Y)$ if and only if $f \circ h \in \mathcal{G}_+^\infty(X)$ for all $f \in \mathcal{G}_+^\infty(Y)$.*
- (iii) *$\mathcal{G}_+^\infty(X)$ (respectively $\mathcal{G}_+^\infty(Y)$) is uniformly separating for (X, d_X) (respectively (Y, d_Y)), in the sense that, for every pair of subsets A and B of X, with $d_X(A,B) > 0$, there exists some $f \in \mathcal{G}_+^\infty(X)$ such that $\overline{f(A)}^{|\cdot|} \cap \overline{f(B)}^{|\cdot|} = \emptyset$.*
- (iv) *There exists a constant $C \geq 1$ such that for every pair of points $w, z \in Y$, there exists a function $f \in \mathcal{G}_+^\infty(Y)$ with $\|f| \leq C$ such that $f(z) - f(w) = d_Y(w,z)$.*

Denote $\mathcal{A}(X) = \mathrm{span}(\mathcal{G}_+^\infty(X))$, endowed with the extended asymmetric norm associated with $\mathcal{G}_+^\infty(X)$, and its natural algebra structure, and let $T : \mathcal{A}(Y) \to \mathcal{A}(X)$ be a positive isomorphism of extended asymmetric normed algebras. Then, there exists $\tau \in \mathcal{F}(X,Y)$, with $\tau^{-1} \in \mathcal{F}(Y,X)$, and such that

$$Tf = f \circ \tau$$

for all $f \in \mathcal{G}^\infty(Y)$, and such that $\mathrm{LIP}(\tau) \leq \mathbf{C}\|\mathbf{T}|$.

Proof. Clearly, all hypothesis for Theorem 5.38 are met. It only remains to prove the bound on the Lipschitz constant of τ. Take two points $a, b \in X$, and let us estimate $d_Y(\tau(a), \tau(b))$. Condition (iv) allows us to take $f \in \mathcal{G}_+^\infty(Y)$ with $\mathrm{LIP}(f) \leq C$ such that $f(\tau(a)) - f(\tau(b)) = d_Y(\tau(a), \tau(b))$. The composition formula yields that

$$d_Y(\tau(a), \tau(b)) = Tf(a) - Tf(b).$$

Since the function Tf is Lipschitz and $\mathrm{LIP}(Tf) \leq \|Tf|$ (by hypothesis (i)), we have that

$$d_Y(\tau(a), \tau(b)) \leq \|Tf| d_X(a,b) \leq \|f| \|T| d_X(a,b) \leq C\|T| d_X(a,b),$$

which implies τ is $C\|T|$-Lipschitz. □

Remark 5.48 *In what follows, the least constant C satisfying condition (iv) of Theorem 5.47 will be called the* ***separation constant*** *of the family $\mathcal{G}_+^\infty(Y)$.*

Corollary 5.49 *For the following classes of spaces of real-valued functions $\mathcal{G}^\infty(X)$, we can obtain a quantitative bound on the homeomorphism $\tau : X \to Y$ obtained in terms of the separation constant C and the norm of the isomorphism T. Let us denote $K = C \max\{\|T|, \|T^{-1}|\}$.*

(a) $C_b^1(\mathcal{X})$, *of bounded functions with bounded derivative on a connected, complete and reversible Finsler manifold* $\mathcal{X}$, *endowed with the norm* $\|f\| = \max\{\|f\|_\infty, \|df\|_\infty\}$, *obtaining a Lipschitz diffeomorphism* τ *satisfying*

$$\max\{\|d\tau\|_\infty, \|d\tau^{-1}\|_\infty\} \leq K.$$

(b) $\mathrm{LIP}^\infty(X)$, *of bounded Lipschitz functions on a complete metric space, endowed with the norm* $\|f\| = \max\{\|f\|_\infty, \mathrm{LIP}(f)\}$, *obtaining a Lipschitz homeomorphism* τ *satisfying*

$$\max\{\mathrm{LIP}(\tau), \mathrm{LIP}(\tau^{-1})\} \leq K.$$

(c) $\mathrm{lip}(X)$ *of little Lipschitz functions on a compact and purely* 1*-unrectifiable metric space* X, *endowed with the norm* $\|f\| = \max\{\|f\|_\infty, \mathrm{LIP}(f)\}$, *obtaining a Lipschitz homeomorphism* τ *satisfying*

$$\max\{\mathrm{LIP}(\tau), \mathrm{LIP}(\tau^{-1})\} \leq K.$$

In all the examples above the separation constant is $C = 1$. In the Lipschitz case, this can be proved using distance functions. For the case of Riemannian and Finsler manifolds, this can be achieved by using smooth approximations. For locally flat Lipschitz functions, it can be deduced from the fact that, for boundedly compact metric spaces, the separation factor (in the sense of Definition 2.80) is always 1 (see [42, Corollary 4.40]).

It follows that in all three cases of Corollary 5.49, we have $K = 1$ whenever the isomorphism T is in fact an isometry, which implies τ is also an isometry.

Remark 5.50 *In all cases mentioned in Corollary 5.49, the hypothesis of positivity of* T *is unnecessary. Indeed, all algebras mentioned above are known to be closed under bounded inversions (see Proposition 2.79 for the case of little Lipchitz functions). This can be used to prove that every* φ *in the structure space* $\mathcal{S}(X)$ *is positive (using the same argument as in Proposition 5.16), which guarantees that the dual operator* T^* *sends* $\mathcal{S}(X)$ *into* $\mathcal{S}(Y)$*, thus eliminating the need for positivity of* T.

5.2.4 Pointwise Lipschitz version

Our last result deals with pointwise Lipschitz functions and functions with bounded metric slopes. In order to obtain a bound on the pointwise Lipschitz constant of the desired homeomorphism between metric spaces (X, d_X) and (Y, d_Y), we will need an additional hypothesis.

Definition 5.51 *A metric space* (X, d) *is called* ***uniformly locally radially quasi-convex*** *if there exists a constant* $K > 0$ *such that for every* $x_0 \in X$ *there exists a neighborhood* U_{x_0} *of* x_0 *such that for every* $y \in U_{x_0}$ *there exists a rectifiable curve* γ *in* U_x *connecting* x *and* y *and such that* $\ell(\gamma) \leq K d(x, y)$.

Notice that, unlike in Definition 5.35, Definition 5.51 requires for the constant K to be uniform over X.

Finally, we will use the following lemma, which is an adaptation of the one presented in [21, Lemma 2.3] for functions with bounded pointwise Lipschitz constant. The statement and proof of the lemma have been modified in order to work with continuous functions with bounded pointwise semi-Lipschitz constant.

Lemma 5.52 *Let (X,d) be a metric space and let $f \in D_{\mathrm{SL}}(X) \cap C(X)$. Let $x, y \in X$ and suppose there exists a rectifiable curve $\gamma : [a,b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then,*

$$f(y) - f(x) \leq \|\mathrm{SLip}(f)\|_\infty \ell(\gamma),$$

where $\ell(\gamma)$ denotes the length of the curve γ.

Proof. Let $K = \|\mathrm{SLip}(f)\|_\infty < +\infty$. For $\varepsilon > 0$, let us denote $K' = K + \varepsilon$. Since $\mathrm{SLip}\, f(\gamma(a)) < K'$, there exists $\delta > 0$ such that, whenever $d(\gamma(a), x) < \delta$, we have that

$$f(\gamma(a)) - f(x) \leq K' d(\gamma(a), x).$$

By continuity of γ, there exists $t^* \in (a,b]$ such that $d(\gamma(a), \gamma(t^*)) < \delta$, and therefore

$$f(\gamma(a)) - f(\gamma(t^*)) \leq K' d(\gamma(a), \gamma(t^*)) \leq K' \ell(\gamma|_{[a,t^*]}).$$

Let us consider the set

$$A = \{t \in (a,b] \ : \ f(\gamma(a)) - f(\gamma(t)) \leq K' \ell(\gamma|_{[a,t]})\},$$

which is clearly non empty (as $t^* \in A$) and bounded by above, so we can consider $s = \sup(A)$. Let us check that s belongs to A. By definition of s, there exists a sequence $(t_n) \subset A$ such that $(t_n) \to s$ and $f(\gamma(a)) - f(\gamma(t_n)) \leq K' \ell(\gamma|_{[a,t_n]})$. By continuity of f, we conclude that $f(\gamma(a)) - f(\gamma(s)) \leq K' \ell(\gamma|_{[a,s]})$. Next, we shall prove that $s = b$. If this were not the case, we would have $a < s < b$, and since $\mathrm{SLip}\, f(\gamma(s)) < K'$, we can take $t^* \in (s,b]$ satisfying

$$f(\gamma(s)) - f(\gamma(t^*)) \leq K' \ell(\gamma|_{[s,t^*]}).$$

Then,

$$\begin{aligned} f(\gamma(a)) - f(\gamma(t^*)) &= f(\gamma(a)) - f(\gamma(s)) + f(\gamma(s)) - f(\gamma(t^*)) \\ &\leq K' \ell(\gamma|_{[a,s]}) + K' \ell(\gamma|_{[s,t^*]}) \\ &= K' \ell(\gamma|_{[a,t^*]}), \end{aligned}$$

which implies $t^* \in A$, contradicting the fact that $s = \sup(A)$. Having proved that $s = b$, the fact that $s \in A$ yields the desired result. □

Theorem 5.53 . *Let (X, d_X) and (Y, d_Y) be complete **uniformly locally radially quasi-convex** metric spaces, and let $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ be subcones of $C(X) \cap D_{\mathrm{SL}}(X)$ and $C(Y) \cap D_{\mathrm{SL}}(Y)$, respectively, such that:*

(i) For $Z \in \{X, Y\}$, the subcone $\mathcal{G}_+^\infty(Z) := \mathcal{G}(Z) \cap L^\infty(Z) \cap [0,\infty)^Z$ is endowed with a conic norm $\|\cdot|_Z$ which satisfies $\|\cdot|_Z \geq \max\{\|\mathrm{SLip}(\cdot)\|_\infty, \|\cdot\|_\infty\}$, and which make $\mathcal{G}_+^\infty(Z)$ into a unital normed conic-semiring under the usual addition and multiplication of real-valued functions.

(ii) $h : X \to Y$ *is pointwise Lipschitz if and only if* $f \circ h \in \mathcal{G}_+^\infty(X)$ *for all* $f \in \mathcal{G}_+^\infty(Y)$.

(iii) $\mathcal{G}_+^\infty(X)$ *(respectively* $\mathcal{G}_+^\infty(Y)$*)) is uniformly separating for* (X, d_X) *(respectively* (Y, d_Y)*), in the sense that, for every pair of subsets* A *and* B *of* X*, with* $d_X(A, B) > 0$*, there exists some* $f \in \mathcal{G}_+^\infty(X)$ *such that* $\overline{f(A)}^{|\cdot|} \cap \overline{f(B)}^{|\cdot|} = \emptyset$.

(iv) There exists a constant $C \geq 1$ *such that for every pair of points* $w, z \in Y$*, there exists a function* $f \in \mathcal{G}_+^\infty(Y)$ *with* $\|f| \leq C$ *such that* $f(z) - f(w) = d_Y(w, z)$.

Denote $\mathcal{A}(X) = \mathrm{span}(\mathcal{G}_+^\infty(X))$*, endowed with the extended asymmetric norm associated with* $\mathcal{G}_+^\infty(X)$*, and its natural algebra structure, and let* $T : \mathcal{A}(Y) \to \mathcal{A}(X)$ *be a positive isomorphism of extended asymmetric normed algebras. Then, there exists a pointwise Lipschitz homeomorphism* $\tau : X \to Y$ *such that*

$$Tf = f \circ \tau$$

for all $f \in \mathcal{G}^\infty(Y)$*, and such that*

$$\|\mathrm{Lip}(\tau)\|_\infty \leq K_X C \|T|,$$

where $K_X > 0$ *is the constant associated with he uniform local radial quasi-convexity of* X.

Proof. Clearly, all hypothesis for Theorem 5.38 are met. It only remains to prove the bound on the pointwise Lipschitz constant of τ. Fix a non isolated point $x_0 \in X$, and let U_{x_0} and K be the neighborhood and constant given by the uniform local radial quasi-convexity of X. Then, for any point $x \in U_{x_0}$, let $\gamma_x : [a, b] \to U_{x_0}$ be a rectifiable curve such that $\gamma_x(a) = x_0$ and $\gamma_x(b) = x$. Using hypothesis *(iv)* of Theorem 5.53, take $f \in \mathcal{G}_+^\infty(Y)$ with $\|f| \leq C$ such that $f(\tau(x)) - f(\tau(x_0)) = d_Y(\tau(x_0), \tau(x))$. Using the composition formula, we get

$$d_Y(\tau(x_0), \tau(x)) = Tf(x_0) - Tf(x)$$

Next, we apply Lemma 5.52 to the function Tf and the curve γ_x connecting x_0 and x, obtaining that

$$d_Y(\tau(x_0), \tau(x)) \leq \|\mathrm{SLip}(Tf)\|_\infty \ell(\gamma_x).$$

Since $\|\mathrm{SLip}(Tf)\|_\infty \leq \|Tf| \leq \|T|\|f| \leq C\|T|$ and $\ell(\gamma_x) \leq K d_X(x_0, x)$, we conclude that

$$d_Y(\tau(x_0), \tau(x)) \leq KC\|T| d_X(x_0, x) \text{ for any } x \in U_{x_0},$$

which implies $\mathrm{Lip}(\tau)(x_0) \leq KC\|T|$.

□

Remark 5.54 *This result is valid for functions with values in* $\mathbb{R}$ *with either the usual metric* d *or the quasi-metric* d_u*, in which cases* $\mathrm{SLip}(\cdot) = \mathrm{Lip}(\cdot)$ *and* $\mathrm{SLip}(\cdot) = |\partial(\cdot)|^+$*, respectively.*

Corollary 5.55 *Theorem 5.53 can be applied to the following spaces, provided* X *is a complete and uniformly locally radially quasi-convex:*

(a) $\mathcal{G}(X) = D(X)$ *of functions with bounded pointwise Lipschitz constant.*

(b) $\mathcal{G}(X) = D_{\mathrm{SL}}(X) \cap C(X)$ *of continuous functions with bounded metric slope.*

In both cases, the homeomorphism τ is pointwise Lipschitz, with

$$\|\mathrm{Lip}(\tau)\|_\infty \leq K_X C \|T|,$$

where $K_X > 0$ is the constant associated with he uniform local radial quasi-convexity of X.

Remark 5.56 *In all cases mentioned in Corollary 5.55, the hypothesis of positivity of T is unnecessary. It was shown in [21] that the algebra $D^\infty(X)$ is closed under bounded inversions, which is also known for $\mathrm{LIP}^\infty(X)$. Following the arguments of Proposition 5.16, it can be proven that every φ in the structure space $\mathcal{S}(X)$ is positive which guarantees that the dual operator T^* maps $\mathcal{S}(X)$ into $\mathcal{S}(Y)$, thus eliminating the need for positivity of T. The same argument works for $D_{\mathrm{SL}}(X)$.*

Corollary 5.57 *If the separation constant C in hypothesis (iv) of Theorem 5.47 is 1 and T is an isometric isomorphism of extended asymmetric normed algebras, then τ is an isometry. If the separation constant C in hypothesis (iv) of Theorem 5.53 is 1, as well as the constants K_X and K_Y associated with the uniform local radial quasi-convexity of X and Y, respectively, and T is an isometric isomorphism of extended asymmetric normed algebras, then τ is a pointwise isometry.*

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