UNIVERSIDAD DE CHILE
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CONSTRUCTION OF SOLUTIONS TO LIOUVILLE TYPE EQUATIONS ON THE FLAT TORUS

**TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA
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SANTIAGO DE CHILE
MAYO DE 2011

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Resumen

En esta tesis doctoral se construyen soluciones para ecuaciones diferenciales parciales elípticas con no-linealidades exponenciales en el toro plano. La motivación proviene de ecuaciones de tipo Liouville en el estudio de la teoría de vórtices de Chern-Simons periódicos.

En el primer capítulo mostramos el problema de vórtices de Chern-Simons periódicos, mencionando algunos resultados conocidos y deducimos su relación con la ecuación de campo medio (mean field equation). Mencionamos los resultados obtenidos para esta ecuación. Para una ecuación de tipo Liouville con una fuente singular se menciona el resultado conseguido.

El segundo capítulo recopila algunos elementos que serán usados en los capítulos posteriores. Estos son nociones de valores críticos, la función de Green para el laplaciano en el toro y la ecuación de Liouville.

En el tercer capítulo construimos soluciones para la ecuación de campo medio. A través de una reducción de Lyapunov-Schmidt aseguramos la existencia de una familia de soluciones que se concentran en puntos distintos del dominio, los cuales son caracterizados por un funcional en dimensión finita. En particular, recuperamos un resultado de Chen y Lin. Además, deducimos el mismo resultado bajo una condición de punto crítico más débil.

En el cuarto capítulo realizamos una construcción análoga para una ecuación de tipo Liouville con una fuente singular. Bajo la condición que el peso de la fuente sea suficientemente grande aseguramos la existencia de una familia de soluciones que se concentran en un número de puntos del dominio, menor estricto que el peso de la fuente más uno. Estos puntos resultan ser distintos entre sí y distintos del punto donde está ubicada la fuente.

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Dedicado a mis padres Carmen y Enrique

Outline

This dissertation is organized as follows

Chapter 1: In first chapter, we present the self-dual Chern-Simons-Higgs vortex equation on a flat two-torus. We mention some known results. Also, we show its relation with the mean field equation in the existence of non-topological type solutions. Motivated by this fact, we mention our results for mean field equation and compare with some previous results. A result for a Liouville type equation on the flat two-torus with a singular source is present at the end of this chapter. For a more complete description, we refer the reader to corresponding chapters.

Chapter 2: This chapter is concerning to some topics, which will be useful in the sequel chapters. We present the notions of critical value, the Green's function and the Liouville equation.

Chapter 3: Here, we study the mean field equation on a flat two-torus with periodic boundary conditions. By a "Lyapunov-Schmidt" reduction we have re-obtained the existence of blowing up solutions due to C.-C. Chen and C.-S. Lin. Moreover, under weaker non-degeneracy conditions used by Chen and Lin, we are able to assure the existence of blowing up solutions. The blow up points are characterized as critical points, satisfying some stability condition, of a finite-dimensional functional. The results of this chapter were obtained in collaboration with Dr. Pierpaolo Esposito at University of Rome III, in Rome, and Dr. Manuel del Pino at the University of Chile, in Santiago, and are in progress.

Chapter 4: This chapter deals with an analogous construction for a Lioville type equation with singular source. The assumption that the weight of the source is sufficiently large, allows us the chance to conclude the existence of blowing up solutions with exactly m points of the domain, different one from each other and from the source. The m should be less than the weight plus one. These results, which are the most relevant part of this dissertation, were obtained in collaboration with Dr. Manuel del Pino at the University of Chile, in Santiago, and are contained in the research paper *Singular Limits for Liouville-type equations on the flat torus*, submitted for publication in Calculus of Variation and Partial Differential Equations.

Contents

Chapter 1

Introduction

In recent years the Chern-Simons vortex theory has been extensively studied for its possible application to the physics of high critical temperature superconductivity (see Dunne [34] and references therein). In the study of this theory, some problems can be proposed in terms of elliptic partial differential equations with exponential nonlinearity. Sometimes called Liouville type equation after [52]. Particularly, the self-dual Chern-Simons-Higgs vortex equation on a flat 2-torus Ω can be written as follows

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2}\, e^{u}(1-e^{u}) - 4\pi \sum_{j=1}^{\ell} n_j \delta_{p_j}, & \text{in } \Omega, \\ \qquad u \quad \text{doubly periodic on} & \partial\Omega, \end{cases} \tag{1.0.1}$$

where $\varepsilon > 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(\beta/\alpha) > 0$,

$$\Omega = \{z = s\alpha + t\beta \in \mathbb{C} \mid 0 < s, t < 1\},$$

δ_p denote a Dirac mass in p, $p_j \in \Omega$, $n_j \in \mathbb{N}$, $j = 1, \ldots, \ell$ and $p_j \neq p_k$ if $j \neq k$. This problem was proposed in [45, 46] in an attempt to explain superconductivity of type 2. Here, $2\varepsilon > 0$ is the Chern-Simons parameter and the points p_j, $j = 1, \ldots, \ell$ are called *vortices*.

Observe that taking $u = u_0 + v$, (1.0.1) is equivalent to

$$\begin{cases} -\Delta v = \frac{1}{\varepsilon^2}\, e^{u_0+v}(1-e^{u_0+v}) - \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ \qquad u \quad \text{doubly periodic on} & \partial\Omega, \end{cases} \tag{1.0.2}$$

where u_0 is the unique function satisfying

$$\begin{cases} -\Delta u_0 = \frac{4\pi N}{|\Omega|} - 4\pi \sum_{j=1}^{\ell} n_j \delta_{p_j}, & \text{in } \Omega, \\ \quad u_0 \quad \text{doubly periodic on} & \partial\Omega, \\ \quad \int_\Omega u_0 = 0 & \end{cases}$$

$\sum_{j=1}^{\ell} n_j = N$ and $|\Omega|$ is the Lebesgue measure of Ω. Note that if v is a solution of (1.0.2), then, by integration over Ω, we obtain

$$\int_\Omega e^{u_0+v}(1-e^{u_0+v}) = 4\pi N \varepsilon^2. \tag{1.0.3}$$

Also, we have

$$\int_\Omega \left(e^{u_0+v} - \frac{1}{2}\right)^2 = \frac{|\Omega|}{4} - 4\pi N \varepsilon^2.$$

Thus, a necessary condition for (1.0.2) to admit a solution is that $|\Omega| > 16\pi N\varepsilon^2$. Concerning the asymptotic behavior of the solutions of (1.0.2) (for $\varepsilon > 0$ small), we see that by the condition (1.0.3), we are lead to expect two classes of solutions. Namely, those solutions v_ε satisfying:

$$e^{u_0+v_\varepsilon} \to 1 \quad \text{a.e. in } \Omega, \text{ as } \varepsilon \to 0 \tag{1.0.4}$$

which, we are called of *topological-type*; and those satisfying:

$$e^{u_0+v_\varepsilon} \to 0 \quad \text{a.e. in } \Omega, \text{ as } \varepsilon \to 0 \tag{1.0.5}$$

called of *non-topological-type.*

Existence results have been shown in [6, 65] for an arbitrary number of prescribed vortices. It is well-known [6] that there exists a constant $\varepsilon_c > 0$ satisfying $|\Omega| \geq 16\pi N\varepsilon_c^2$, such that if $\varepsilon > \varepsilon_c$ then (1.0.2) has no solution, while if $0 < \varepsilon \leq \varepsilon_c$, there are at least two solutions of (1.0.2). One of which is the maximal solution, see [6], and the other one can be obtained through the min-max variational method, see [65]. In fact, (1.0.2) admits a variational structure, in the sense that weak solutions for (1.0.2) are the critical points of the following energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2}\int_\Omega \left(e^{u_0+u} - 1\right)^2 + \frac{4\pi N}{|\Omega|}\int_\Omega u, \qquad u \in \mathcal{H}(\Omega), \tag{1.0.6}$$

where

$$\mathcal{H}(\Omega) = \{u \in H^1_{\text{loc}}(\mathbb{R}^2) \mid u \text{ is doubly periodic with periodic cell domain } \Omega\}.$$

Thus, the maximal solution is a local minimum for J_ε in $\mathcal{H}(\Omega)$. Furthermore, as $\varepsilon \to 0$, the maximal solution tends to 0 uniformly in any compact subset of $\Omega \setminus \{p_1, \ldots, p_\ell\}$. Hence, the maximal solution is of topological-type. But the second solution has a different asymptotic behavior. For $N \geq 3$, it is proved in [22] that as $\varepsilon \to 0$, the mountain pass solution blows up at a point $q \neq p_j$ for any $j = 1, \ldots, \ell$. For $N = 1$, Tarantello showed in [65] that the mountain pass solution does not blow up, while in the case $N = 2$, whether the mountain pass solution blows up or not depends on whether a minimization problem has no minimizer. Indeed, define

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log\left(\int_\Omega e^{u_0+u}\right), \tag{1.0.7}$$

and

$$E = \left\{u \in \mathcal{H}(\Omega) \,:\, \int_\Omega u = 0\right\}.$$

The existence of bubbling solution for

$$\begin{cases} -\Delta v = \frac{1}{\varepsilon^2}\, e^{u_0+v}(1 - e^{u_0+v}) - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ \qquad v \quad \text{doubly periodic on} & \partial\Omega, \end{cases} \tag{1.0.8}$$

namely, equation (1.0.2) with $N = 2$, is related to the following minimization problem

$$\inf\{I(u) \,:\, u \in E\}. \tag{1.0.9}$$

Nolasco and Tarantello [60] proved the following result:

Theorem 1.0.1. *There exists an* $\varepsilon_0 > 0$ *(*$32\pi\varepsilon_0^2 < |\Omega|$*) such that for every* $0 < \varepsilon < \varepsilon_0$ *problem* (1.0.8) *admits a solution*

$$v_\varepsilon = w_\varepsilon + c_\varepsilon, \qquad \text{with} \quad \int_\Omega w_\varepsilon = 0$$

for some constant c_ε *satisfying* $c_\varepsilon \to -\infty$ *as* $\varepsilon \to 0$*. And up to subsequence, one of the following holds*

(a) if (1.0.9) *is achieved, then* $w_\varepsilon \to w$ *in* $C^q(\Omega)$ *for any* $q \geq 0$ *as* $\varepsilon \to 0$*, and* w *is a minimizer;*

(b) if (1.0.9) *is not achieved, then there exists a* $p_0 \in \Omega$*, satisfying* $u_0(p_0) = \max_\Omega u_0$ *and*

$$\frac{e^{u_0+w_\varepsilon}}{\int_\Omega e^{u_0+w_\varepsilon}} \rightharpoonup \delta_{p_0},$$

in sense of measure as $\varepsilon \to 0$*.*

Also, there are several results concerning to problem (1.0.2) in [22, 24, 55, 56]. In [9, 10, 44, 61, 62], many results are shown in the existence of planar Chern-Simons vortices which is the equation (1.0.1) in the whole plane with an appropriate decay behavior at infinity instead of doubly periodic conditions.

The mean field equation related to (1.0.9) is

$$-\Delta u = \lambda \left(\frac{e^{u_0+v}}{\int_\Omega e^{u_0+v}} - \frac{1}{|\Omega|} \right), \qquad \text{in } \Omega. \tag{1.0.10}$$

Thus, if (1.0.9) is achieved, then w_ε converges to a solution w of (1.0.10) with $\lambda = 8\pi$. In general, namely for any N, equation (1.0.2) is related with the mean field equation as we will show next. If v is a solution of (1.0.2) then writing $v = w + c$, where

$$\int_\Omega w = 0 \qquad \text{and} \qquad c = \frac{1}{|\Omega|}\int_\Omega v,$$

we get the following identity

$$e^{2c}\int_\Omega e^{2(u_0+w)} - e^c \int_\Omega e^{u_0+w} + 4\pi N\varepsilon^2 = 0.$$

Hence, necessarily,

$$\left(\int_\Omega e^{u_0+w}\right)^2 - 16\pi N\varepsilon^2 \int_\Omega e^{2(u_0+w)} \geq 0 \tag{1.0.11}$$

and

$$e^c = \frac{\int_\Omega e^{u_0+w} \pm \sqrt{\left(\int_\Omega e^{u_0+w}\right)^2 - 16\pi N\varepsilon^2 \int_\Omega e^{2(u_0+w)}}}{2\int_\Omega e^{2(u_0+w)}}. \tag{1.0.12}$$

The two possible choice of "plus" or minus sign in (1.0.12) is another indication for multiple existence for (1.0.2). In [65], the topological-type solutions of (1.0.2) are characterized and satisfy (1.0.12) with the "plus" sing. On the other hand, in order to find non-topological-type solutions of (1.0.2), we should impose that (1.0.12) holds with the "minus" sign. Thus, denote $c_-(w)$ the choice of (1.0.12) with the minus sign and observe that

$$e^{c_-(w)} = \frac{8\pi N\varepsilon^2}{\int_\Omega e^{u_0+w}\left(1 + \sqrt{1 - 16\pi N\varepsilon^2 \frac{\int_\Omega e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right)}.$$

Hence, it holds that for any constant $\alpha \in \mathbb{R}$

$$e^{c_-(w+\alpha)} = e^{-\alpha} e^{c_-(w)}.$$

Since $\Delta v = \Delta w$ and replacing $e^{c_-(w)}$ in the equation (1.0.2), we will find non-topological type solutions of (1.0.2) if we are able to solve the following problem

$$\begin{cases} -\Delta w = \frac{1}{\varepsilon^2} e^{u_0+w+c_-(w)}(1 - e^{u_0+w+c_-(w)}) - \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ \quad w \quad \text{doubly periodic on} \quad \partial\Omega, \\ \int_\Omega w = 0. \end{cases}$$

Note that we have

$$\frac{1}{\varepsilon^2} e^{u_0+w+c_-(w)}(1 - e^{u_0+w+c_-(w)}) = \frac{8\pi N\, e^{u_0+w}}{\int_\Omega e^{u_0+w} \left(1 + \sqrt{1 - 16\pi N \varepsilon^2 \frac{\int_\Omega k^2 e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right)} \times \left(1 - \frac{8\pi N\, \varepsilon^2\, e^{u_0+w}}{\int_\Omega e^{u_0+w} \left(1 + \sqrt{1 - 16\pi N \varepsilon^2 \frac{\int_\Omega k^2 e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right)}\right).$$

Whence, as $\varepsilon \to 0$, one naturally ends up with the mean field equation

$$\begin{cases} -\Delta w = \frac{4\pi N\, e^{u_0+w}}{\int_\Omega e^{u_0+w}} - \frac{4\pi N}{|\Omega|}, & \text{in } \Omega, \\ \quad w \quad \text{doubly periodic on} \quad \partial\Omega, \\ \int_\Omega w = 0. \end{cases}$$

Namely, equation (1.0.10) with $\lambda = 4\pi N$. Let us observe that we could consider that e^{u_0} as a function k, with $k \geq 0$.

Motivated by the existence of non-topological solutions to problem (1.0.2), we have studied two related elliptic partial differential equations with exponential nonlinearity. First, we have addressed existence issues for mean field equations on a flat two-torus, and we believe that this approach give us a way to carry out the existence of non-topological solutions of (1.0.2). On the other hand, due to the presence of exponential nonlinearity and singular sources, we have studied a Liouville type equation on the torus involving a singular source.

Second chapter is devoted to introduce some elements which will be useful in the sequel chapters. These are the notions of critical value, which allow us to assure the existence of solutions to considered problems. They apply to a functional which involve the Green's function which is also considered in this chapter. Due to the presence of exponential non-linearities, we will review the Liouville equation, showing the main tools in the construction of approximations of the solutions.

Third chapter is concerning to the problem

$$-\Delta u = \lambda \left(\frac{k\, e^u}{\int_\Omega k\, e^u} - \frac{1}{|\Omega|} \right), \tag{1.0.13}$$

in a flat two-torus with periodic boundary conditions, where $\lambda > 0$, k is a C^3 non-negative, not identically zero doubly periodic function and $|\Omega|$ is the measure of Ω. By a "Lyapunov-Schmidt" reduction we have re-obtained the existence of blowing up solutions due to C.-C. Chen and C.-S. Lin [18]. Moreover, under weaker non-degeneracy conditions of [18], we are able to assure the existence of blowing up solutions. The blow up points are characterized as critical points, satisfying some stability condition, of a finite-dimensional functional φ_m, involving the function k and the Green's function of $-\Delta$ with respect to doubly periodic conditions on $\partial\Omega$. In fact, taking $\xi = (\xi_1, \dots, \xi_m)$ we have that

$$\varphi_m(\xi) = -2\sum_{j=1}^{m} \log k(\xi_j) - \sum_{l\neq j} G(\xi_l, \xi_j).$$

and $G = G(x, y)$ satisfy

$$\begin{cases} -\Delta_x G(\cdot, y) = 8\pi\delta_y - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ \quad G(\cdot, y) \quad \text{is doubly periodic on } \partial\Omega, \\ \int_\Omega G(x, y)\, dx = 0. \end{cases}$$

where δ_p denote a Dirac mass in $p \in \Omega$. Define the function H by

$$G(x, y) = -4\log|x - y| + H(x, y).$$

Let us observe that $H(x, x)$ is constant for all $x \in \Omega$, when Ω is a flat two-torus. An admissibility condition in terms of k, the Green's function G and its regular part H should be satisfied in an appropriate region containing the critical points. That is,

$$V(\xi) = 4\pi\sum_{j=1}^{m} \Delta\rho_j(\xi_j) \neq 0, \tag{1.0.14}$$

for all $\xi = (\xi_1, \dots, \xi_m) \in \mathcal{D}$, where $\mathcal{D} \subset \Omega^m$ contains the critical points and the stability condition take place, and where

$$\rho_j(x) := k(x)\exp\left(H(x, \xi_j) + \sum_{l\neq j} G(x, \xi_l)\right). \tag{1.0.15}$$

Our approach allows us to know when either $\lambda \to 8\pi m^+$ or $\lambda \to 8\pi m^-$. Indeed, it should be satisfied $\operatorname{sgn}(\lambda - 8\pi m) = \operatorname{sgn} V(\xi)$ for all $\xi \in \mathcal{D}$. *Stable critical points* and *non-trivial critical values* of φ_m give us the stability conditions on critical points enough to conclude the results. The second one allows us to considered the case $k \equiv 1$. The solutions are constructed using a family of solutions of the Liouville equation in $\mathbb{R}^2$, suitable scaled, translated and projected in order to have the boundary conditions. Solutions are found as a small perturbation of these initial approximation. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration points and the high of the bubbles. Similarly to [27], we identify an extra element of the approximate kernel, which introduces another parameter to be adjusted in the problem, related to the high of the bubbles. An important element in the reduction procedure is the invertibility of the linearized operator in suitable L^∞-weighted spaces. We remark that in case of the choice $k = e^{u_0}$, the admissibility condition is not satisfy when $N = 2m$. An higher order expansion is then needed in the study of existence of solutions to problem (1.0.10), which blow up at points outside of the set $\{p_1, \dots, p_\ell\}$. We conjecture that a similar procedure could be used to address the existence of non-topological solutions to problem (1.0.2).

The fourth chapter is devoted to the Liouville equation on the torus with a singular source, that is

$$\begin{cases} -\Delta u = \varepsilon^2 e^u - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 e^u + \frac{4\pi N}{|\Omega|} - 4\pi N \delta_p, & \text{in } \Omega, \\ u \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega u = 0, \end{cases} \tag{1.0.16}$$

where $p \in \Omega$ and $N > 0$. We stress that in some sense, problem (1.0.16) is similar to (1.0.2), due to the presence of exponential nonlinearity and the singular source. The assumption $m < N+1$, $m \in \mathbb{N}$ on the weight of the source, allows us the chance to conclude the existence of blowing up solutions with exactly m points of Ω, different one from each other and from p. Observe that (1.0.16) is equivalent to

$$\begin{cases} -\Delta u = \varepsilon^2 k(x) e^u - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(z) e^{u(z)}\, dz, & \text{in } \Omega, \\ u \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega u = 0 \end{cases} \tag{1.0.17}$$

where $k = \exp(-\frac{N}{2}G(\cdot,p))$, so that k is positive everywhere except at $x = p$ and $k(x) \sim |x-p|^{2N}$, as $x \to p$. By a "Lyapunov-Schmidt" reduction we have found conditions under which there exists a family of solutions to (1.0.17), $\{u_\varepsilon\}_\varepsilon$ such that

$$\varepsilon^2 k e^{u_\varepsilon} \rightharpoonup 8\pi \sum_{i=1}^{m} \delta_{q_i} \qquad \text{as } \varepsilon \to 0 \text{ in measure sense.}$$

These conditions are satisfied for the problem (1.0.16) whenever $1 \le m < N+1$, yielding thus the result. In particular, if $k \in C^2(\bar\Omega)$ and $\inf_\Omega k > 0$ then such a family of solutions does exist for any $m \ge 1$. Note that $\inf_\Omega \exp(-\frac{N}{2}G(\cdot,p)) = 0$. Similarly, as above in the problem (1.0.13), the location of points q_i, $i = 1,\dots,m$ is characterized as a critical point of a functional φ_m. The notion of non-trivial critical value gives us the chance to get the existence of blowing up solutions of problem (1.0.16), where the concentration points are different from p. This fact is analogous to the corresponding version of the Liouville equation on bounded domains with Dirichlet boundary conditions shown in [31]. The solutions are constructed using a family of solutions of the Liouville equation in $\mathbb{R}^2$, suitable scaled and projected to make it up to a good order for the boundary conditions. Solutions are found as a small additive perturbation of these initial approximation. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration point. An important element in the reduction procedure is the bounded invertibility of the linearized operator in suitable L^∞-weighted spaces. We stress that here we only need to adjust the location of blow up points.

Chapter 2

Preliminaries

In this chapter, we give some definitions and show some topics which we shall use in the following chapters. For instance, we present the notions of critical value, the Green's function and the Liouville equation.

2.1 Critical values

In this section we will see two different notions of critical values. These are *stable critical value* and *non-trivial critical value.*

Definition 2.1.1. Let S, ∂Q and Q be compact subsets of a domain D. We will say that S *links Q via ∂Q by homotopy in D* if $\partial Q \subset Q$, $S \cap \partial Q = \emptyset$ and $\gamma(1, Q) \cap S \neq \emptyset$ for any $\gamma \in \Gamma$, where

$$\Gamma := \{\gamma \in C([0,1] \times Q, D) \mid \gamma(0, \cdot) = \mathrm{Id}_Q,\ \gamma(t, \cdot) = \mathrm{Id}_{\partial Q} \forall t \in [0,1]\}.$$

Now, let us recall the following notion of stability of critical values introduced in [43] and used also in [37]. Let $F : D \to \mathbb{R}$ be a C^1-function. We say that:

Definition 2.1.2. c is a *stable critical value* of F in D, if there exist compact subsets S, ∂Q and Q of D such that S links Q via ∂Q by homotopy in D,

$$\max_{\partial Q} F < \min_{S} F$$

and the set $\{x \in D \mid c - \varepsilon \leq F(x) \leq c + \varepsilon\}$ is complete for some $\varepsilon > 0$, where

$$c := \inf_{\gamma \in \Gamma} \max_{x \in Q} F(\gamma(1, x)).$$

An important consequence is that if $\mathcal{C}$ is a stable critical value of F then any C^1 small perturbation of F has a critical value.

On the other hand, we also consider the role of *non-trivial critical values* of a functional φ_m, in existence of blowing-up solutions of considered problems in this thesis. Let Ω^m denote the cross product of m copies of Ω and let $\tilde{\Omega} \subset \bar{\Omega}$ set we always assume non-empty. Let $\varphi : \mathcal{D} \to \mathbb{R}$ be a C^1-function.

Definition 2.1.3. Let $\mathcal{D}$ be an open set in Ω^m compactly contained in $\tilde{\Omega}^m$ with smooth boundary. We will say that φ_m *links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B and B_0* if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that

$$\sup_{y \in B_0} \varphi_m(y) < \mathcal{C} \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} \varphi_m(\Phi(y)), \tag{2.1.1}$$

where $\Psi(1,\cdot) = \Phi$, $\Psi \in \Gamma$

$$\Gamma = \{\Psi \in C([0,1] \times B, \mathcal{D}) \mid \Psi(0,\cdot) = \mathrm{Id}_B,\ \Psi(t,\cdot)|_{B_0} = \mathrm{Id}_{B_0} \text{ for all } t \in [0,1]\}$$

and for all $y \in \partial\mathcal{D}$ such that $\varphi_m(y) = \mathcal{C}$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla\varphi_m(y) \cdot \tau_y \neq 0\,. \tag{2.1.2}$$

Furthermore, we call $\mathcal{C}$ a *non-trivial critical level* of φ_m in $\mathcal{D}$.

Note that under these conditions a critical point $\bar{y} \in \mathcal{D}$ of φ_m with $\varphi_m(\bar{y}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_m shows. Condition (2.1.1) is a general way of describing a change of topology in the level sets $\{\varphi_m \leq c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{C}$, while (2.1.2) prevents intersection of the level set $\mathcal{C}$ with the boundary. It is easy to check that the above conditions hold if

$$\inf_{x\in\mathcal{D}} \varphi_m(x) < \inf_{x\in\partial\mathcal{D}} \varphi_m(x), \quad \text{or} \quad \sup_{x\in\mathcal{D}} \varphi_m(x) > \sup_{x\in\partial\mathcal{D}} \varphi_m(x)\,,$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_m. The level $\mathcal{C}$ may be taken in these cases respectively as that of the minimum and the maximum of φ_m in $\mathcal{D}$. These hold also if φ_m is C^1-close to a function with a non-degenerate critical point in $\mathcal{D}$.

This local notion of nontrivial critical value in (2.1.1)-(2.1.2) was introduced in [28] in the analysis of concentration phenomena in nonlinear Schrödinger equations. And it was also used in [31, 38].

2.2 Green's function on the Torus

Given $z \in \mathbb{C}$ it is possible define a function, say G^*, which allows us to show an explicit formula for the Green's function. This function is the well-known Nerón's function in the theory of elliptic curves [49]. We denote $e(z) = e^{2\pi i z}$. See [21] for a proof.

Lemma 2.2.1. *Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(\beta/\alpha) > 0$. The function G^* defined by*

$$\begin{aligned} G^*(z) := \mathrm{Im}&\left(\frac{|z|^2 - \bar{\alpha}z^2/\alpha}{2(\alpha\bar{\beta} - \bar{\alpha}\beta)} - \frac{z}{2\alpha} + \frac{\beta}{12\alpha}\right) \\ &- \frac{1}{2\pi}\log\left|\left(1 - e\left(\frac{z}{\alpha}\right)\right) \times \prod_{n=1}^{\infty}\left(1 - e\left(\frac{n\beta + z}{\alpha}\right)\right)\left(1 - e\left(\frac{n\beta - z}{\alpha}\right)\right)\right|, \end{aligned} \tag{2.2.1}$$

is both α and β periodic, namely, $G^*(z) = G^*(z+\alpha) = G^*(z+\beta)$ for all $z \in \mathbb{R}^2$ and satisfies

$$-\Delta G^* = \sum_{z\in\alpha\mathbb{Z}+\beta\mathbb{Z}} \delta_z - \frac{1}{|\Omega|}, \quad \text{in } \mathbb{R}^2, \quad \int_\Omega G^*(x)\,dx = 0,$$

where δ_p is a Dirac mass at $p \in \Omega|\Omega| = \mathrm{Im}(\bar{\alpha}\beta)$ is the area of the open cell

$$\Omega = \{z = s\alpha + t\beta \in \mathbb{C} \mid 0 < s, t < 1\}.$$

Note that $G^*(z) = G^*(-z)$ for all $z \in \mathbb{C} \setminus (\alpha\mathbb{Z} + \beta\mathbb{Z})$. Also, we can express this function in the following form

$$G^*(z) = -\frac{1}{2\pi}\log|z| + \frac{|z|^2}{4|\Omega|} + H^*(z),$$

where H^* is an harmonic function in $(\mathbb{C} \setminus (\alpha\mathbb{Z} + \beta\mathbb{Z})) \cup \{0\}$. Now, we observe that the Green's function satisfy $G(x, y) = 8\pi G^*(x-y)$, for $x, y \in \Omega$. In fact, we can consider $G(\cdot, y) : \bar{\Omega}\setminus\{y\} \to \mathbb{R}$ with $y \in \Omega$. Furthermore, $G(x, y) = G(y, x)$ for all $x, y \in \bar{\Omega}$, $x \neq y$ and we can express G in the following form

$$G(x, y) = -4\log|x - y| + H(x, y), \tag{2.2.2}$$

where $H(x, y) = \frac{2\pi}{|\Omega|}|x - y|^2 + 8\pi H^*(x - y)$. Denote Γ the function given by $\Gamma(z) = -4\log|z|$. Note that H satisfies

$$\begin{cases} -\Delta_x H(\cdot, \xi) = \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ \Gamma(\cdot - \xi) + H(\cdot, \xi) \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega [\Gamma(\cdot - \xi) + H(\cdot, \xi)] = 0. \end{cases} \tag{2.2.3}$$

and

$$H(x, x) \equiv -4\log\frac{2\pi}{|\alpha|} + 8\pi\,\mathrm{Im}\,\frac{\beta}{12\alpha} - 4\log\left|\prod_{n=1}^{\infty}\left(1 - e\left(\frac{n\beta}{\alpha}\right)\right)^2\right|,$$

that is to say, $H(x, x) = 8\pi H^*(0)$ for all $x \in \Omega$. Due to definition of H^*, the function H has a singularity in (x, y) if $x - y \in (\alpha\mathbb{Z} + \beta\mathbb{Z}) \setminus \{0\}$.

Remark 2.2.1. An important fact is that for any $\varphi \in C^2(\bar{\Omega})$ we have the following integral representation formula

$$\varphi(x) = \frac{1}{|\Omega|}\int_\Omega \varphi - \frac{1}{8\pi}\int_\Omega G(x, y)\,\Delta\varphi(y)\,dy + \frac{1}{8\pi}\int_{\partial\Omega}\left[G(x, y)\frac{\partial\varphi(y)}{\partial\nu} - \varphi(y)\frac{\partial G(x, y)}{\partial\nu}\right]d\sigma(y) \tag{2.2.4}$$

for all $x \in \Omega$.

Let us introduce the projection operator P into the doubly periodic functions with zero average: let Pu be the unique solution of

$$\begin{cases} \Delta Pu = \Delta u - \frac{1}{|\Omega|}\int_\Omega \Delta u, & \text{in } \Omega, \\ \quad Pu \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega Pu = 0. \end{cases} \tag{2.2.5}$$

Using the Green's function, we know that Pu has the following integral representation

$$Pu(x) = -\frac{1}{8\pi}\int_\Omega G(x, y)\,\Delta u(y)\,dy. \tag{2.2.6}$$

This operator will be used in order to satisfy the boundary conditions in the construction of an ansatz for solutions.

2.3 Liouville equation

Our main goal is to study elliptic equations involving exponential nonlinearities. In order to analyze such elliptic problems, we review the Liouville equation which provides a "basic cell" to construct approximations of a solution in both considered problems.

To this purpose, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by means of transformation $(x, y) \in \mathbb{R}^2 \mapsto z = x + iy \in \mathbb{C}$. Hence, for any holomorphic function $f = f(z)$ in $\mathbb{C}$ there holds

$$\Delta \log(1 + |f(z)|^2) = 4\frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$

Thus, if f is univalent in $\mathbb{C}$, then

$$u(z) = \log \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2} \tag{2.3.1}$$

satisfies the equation

$$-\Delta u = e^u, \qquad \text{in } \mathbb{R}^2. \tag{2.3.2}$$

In [52], it was shown that the expression (2.3.1) gives all solutions for (2.3.2). We shall restrict our attention to the solutions of the Liouville equation with the finite energy condition $e^u \in L^1(\mathbb{R}^2)$, namely, the problem

$$\begin{cases} -\Delta u = e^u, & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty \end{cases}$$

which, by the Liouville formula, are given by the family of functions

$$U_{\delta,\xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2}, \tag{2.3.3}$$

where $\delta > 0$ and $\xi \in \mathbb{R}^2$. See [16, 25]. Besides, it follows that

$$\int_{\mathbb{R}^2} e^{U_{\delta,\xi}} = 8\pi, \qquad U_{\delta,\xi}(x) \to -\infty \quad \text{as} \quad |x| \to +\infty.$$

and

$$e^{U_{\delta,\xi}} \rightharpoonup 8\pi\delta_\xi \qquad \text{as } \delta \to 0, \qquad \text{in measure sense.}$$

Also, note that given a small number $r > 0$,

$$\sup_{B(\xi,r)} U_{\delta,\xi} \to +\infty, \qquad \text{as} \quad \delta \to 0.$$

Due to these all properties, we shall use functions $U_{\delta,\xi}$ for the construction of an approximate solution of the problems.

Chapter 3

Mean Field Equation on the Torus

3.1 Introduction

In this chapter we consider the problem

$$-\Delta u = \lambda \left(\frac{k\, e^u}{\int_\Omega k\, e^u} - \frac{1}{|\Omega|} \right), \tag{3.1.1}$$

in a flat two-torus $\Omega = \{z = s\alpha + t\beta \in \mathbb{R}^2 \mid 0 < s, t < 1\}$, with periodic boundary conditions on $\partial\Omega$, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(\beta/\alpha) > 0$, $\lambda > 0$, k is a C^3 non-negative, not identically zero doubly periodic function and $|\Omega|$ is the measure of Ω. This equation and its variants arises in many different disciplines in mathematics. In the study of existence of metrics conformal to the standard ones on $\Omega = \mathbb{S}^2$ having a prescribed Gaussian curvature k, equation (3.1.1) appears with $\lambda = 8\pi$. This is the Nirenberg problem. For a compact Riemann surface is called the Kazdan-Warner problem. There are several results related to these problems, some of them are due to Kazdan and Warner [47], Chang and Yang [12] and Chang, Gursky, and Yang [14]. For bounded domains of $\mathbb{R}^2$, a version of (3.1.1) arises in statistical mechanics and it is referred as a "mean field equation". These results are due to Caglioti, Lions, Marchioro, and Pulvirenti [7, 8] and Kiessling [15, 48]. In our particular case, when Ω is a flat two-torus, equation (3.1.1) is related to double vortex condensates in the relativistic Chern-Simons-Higgs model, as shown by Nolasco and Tarantello [60]. For the mathematical theory of the relativistic Chern-Simons- Higgs model, see [6, 9, 10, 22, 23, 24, 44, 45, 46, 55, 56, 60, 61, 62, 63, 65, 68, 69].

Observe that (3.1.1) admits a variational structure, in the sense that weak solutions for (3.1.1) are the critical points of the following energy functional

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \log \left(\int_\Omega k(x) e^u \right), \qquad u \in H^1(\Omega). \tag{3.1.2}$$

For $\lambda < 8\pi$, J_λ is bounded from below and the infimum of J_λ can be achieved by the well-known inequality due to Moser and Trudinger. For $\lambda \geq 8\pi$ the existence problem of (3.1.1) is much harder. By variational methods, Struwe and Tarantello [63] were able to obtain nontrivial solutions of (3.1.1) for $8\pi < \lambda < 4\pi^2$ when $k \equiv 1$ and Ω is the flat torus with fundamental domain $[0, 1] \times [0, 1]$. Also, Ding, Jost, Li, and Wang [33] proved the existence of solutions to (3.1.1) for $8\pi < \lambda < 16\pi$ when Ω is a compact Riemann surface with genus $g \geq 1$. Lin [53] proved, for the case $\Omega = \mathbb{S}^2$ and $8\pi < \lambda < 16\pi$, nonvanishing of the Leray-Schauder degree to equation (3.1.1), and consequently, the existence of solutions follows for the case of genus 0.

In general case, the existence of solutions of (3.1.1) for this equation in a Riemann surface has been addressed by C.C. Chen and C. S. Lin in [17, 18]. They completed the program initiated by Li [50], who proposed the problem of studying the existence of solutions of (3.1.1) by the Leray-Schauder degree. Since the equation (3.1.1) is invariant under adding a constant we look for a solutions in the subspace

$$E = \left\{ u \in H^1(\Omega) : \int_\Omega u = 0 \right\}.$$

By the results of Brezis and Merle [5] and Li and Shafrir [51], it follows that for any integer $m \geq 0$ and for any compact set $I \subset (8m\pi, 8(m+1)\pi)$, solutions of (3.1.1) belonging to E are uniformly bounded for any positive C^1 function k and $\lambda \in I$. Thus, the Leray-Schauder degree $d(\lambda)$ of (3.1.1) can be defined in the space of functions with vanishing mean value for $\lambda \neq 8\pi m$. Furthermore, $d(\lambda)$ is independent of both the function k and the parameter λ whenever $\lambda \in (8m\pi, 8(m+1)\pi)$, and it is known that $d(\lambda) = 1$ for $\lambda \in (0, 8\pi)$. Set

$$d_m^+ = \lim_{\lambda \to 8\pi m^+} d(\lambda) \qquad \text{and} \qquad d_m^- = \lim_{\lambda \to 8\pi m^-} d(\lambda).$$

An important fact is that the gap of $d_m^+ - d_m^-$ is due to the occurrence of blow-up solutions when $\lambda \to 8\pi m$, that is, there is a sequence of solutions u_n of (3.1.1) and $u_n \in E$ with $\lambda = \lambda_n$ such that $\max_\Omega u_n \to +\infty$ and $\lambda_n \to 8\pi m$. By a result of Li [50], u_n blows-up at exactly m points $\{p_1, \dots, p_m\}$, namely, there is a small $r > 0$ such that $\sup_{B(p_j,r)} u_n \to +\infty$. These points are called either *blowing-up points* or concentration points. The location of the concentration points are characterized as a critical point of a functional defined explicitly in terms of k and the Green's function $G = G(x, y)$ of $-\Delta$ on Ω, i.e., given $y \in \Omega$

$$\begin{cases} -\Delta_x G(\cdot, y) = 8\pi\delta_y - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ \int_\Omega G(x, y)\, dx = 0, \end{cases}$$

where δ_p denote a Dirac mass in $p \in \Omega$. Let us denote the regular part of the Green's function $\tilde{H}$ by

$$G(x, y) = -4 \log \operatorname{dist}(x, y) + \tilde{H}(x, y).$$

Hence, the concentration points $\xi = (\xi_1, \dots, \xi_m)$ of a multiple blowing-up solutions are a critical point of

$$\varphi_m(\xi) = -\sum_{j=1}^m [2 \log k(\xi_j) + \tilde{H}(\xi_j, \xi_j)] - \sum_{l \neq j} G(\xi_l, \xi_j). \tag{3.1.3}$$

Since Chen and Lin were interested in the computation of the Leray-Schauder topological degree, they constructed all possible solutions with exactly m blow-up points and compute their Morse index. It turns out that the gap $d_m^+ - d_m^-$ is equal to the sum of the Morse indices of all possible blow-up solutions of (3.1.1) when $\lambda \to 8\pi m$ from the above. In the construction of blowing-up solutions they obtained the following result. See [17, 18].

Theorem 3.1.1. *Let k be a C^3 positive function on Ω, where Ω is a compact Riemann surface with $|\Omega| = 1$. Assume that*

1. *the function φ_m is a Morse function on $\Omega^m \setminus \mathcal{E}_m$ with N critical points, where*
$$\mathcal{E}_m = \{(x_1, \dots, x_m) \in \Omega^m \mid x_i = x_j \text{ for some } i \neq j\};$$
2. *the quantity*
$$\mathcal{L}(\xi) = \sum_{j=1}^m [\Delta(\log k)(\xi_j) + 8\pi m - 2K(\xi_j)]\, k(\xi_j) e^{\tilde{H}(\xi_j, \xi_j) + \sum_{l=1, l \neq j}^m G(\xi_j, \xi_l)}, \quad \xi = (\xi_1, \dots, \xi_m)$$

does not vanish for any critical point of φ_m, where K is the Gaussian curvature of Ω.

Then there exists a family of solutions which blows-up at m points.

We shall restrict our attention to recover the existence result of a family of solutions which blows-up at exactly m points when Ω is a flat two-torus, under weaker assumptions. In particular, we will show some conditions under which there exists a family of solutions $\{u_\lambda\}_\lambda$ which blows-up at exactly m different points $q_1, \dots, q_m \in \tilde{\Omega}$, in the following sense, as $\lambda \to 8\pi m$

$$\frac{\lambda k\, e^{u_\lambda}}{\int_\Omega k\, e^{u_\lambda}} \rightharpoonup 8\pi \sum_{i=1}^{m} \delta_{q_i}, \qquad \text{in sense of measures in } \bar{\Omega}. \tag{3.1.4}$$

Here, $\tilde{\Omega} = \{q \in \Omega : k(q) > 0\}$ set we always assume non-empty. Precisely, we are interested in lift the non-degeneracy assumption on critical points. Let us observe that $K \equiv 0$ when Ω is a torus.

Let us mention that under the assumption of nondegenerate critical points of the analogue φ_m, Baraket and Pacard [1] prove the existence of blowing up solutions of (3.1.1) in a bounded domain of $\mathbb{R}^2$ with $k \equiv 1$. Also, in bounded domains of $\mathbb{R}^2$ existence results were shown in [31, 37] under an assumption of topologically nontrivial critical point.

In our approach, namely, when Ω is a flat two-torus we take

$$G(x, y) = 4 \log \frac{1}{|x - y|} + H(x, y)$$

as we have studied in chapter 2 section 2.2. Let us observe that $H(x, x)$ is constant for all $x \in \Omega$, when Ω is a flat two-torus, and $H(x, x) = \tilde{H}(x, x)$. Thus, we consider

$$\varphi_m(\xi) = -2 \sum_{j=1}^{m} \log k(\xi_j) - \sum_{l \neq j} G(\xi_l, \xi_j).$$

An observation we make is that in any compact subset of $\tilde{\Omega}^m$, we may define, without ambiguity,

$$\varphi_m(x_1, \dots, x_m) = -\infty \quad \text{if } \operatorname{dist}(x_i - x_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) = 0 \text{ for some } i \neq j.$$

Furthermore, $H^1(\Omega) = \mathcal{H}(\Omega)$ with $\mathcal{H}(\Omega)$ defined in the introduction. Denote for $\xi = (\xi_1, \dots, \xi_m) \in \tilde{\Omega} \setminus \mathcal{E}_m$

$$V(\xi) = 4\pi \sum_{j=1}^{m} \Delta \rho_j(\xi_j), \tag{3.1.5}$$

where

$$\rho_j(x) := k(x) \exp\left(H(x, \xi_j) + \sum_{l \neq j} G(x, \xi_l) \right) \tag{3.1.6}$$

and now

$$\mathcal{E}_m = \{(x_1, \dots, x_m) \in \bar{\Omega}^m \mid \operatorname{dist}(x_i - x_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) = 0 \text{ for some } i \neq j\}.$$

We shall use notions of critical value introduced in chapter 1.

Theorem 3.1.2. *Let $\mathcal{C}$ be a stable critical value of φ_m in a domain $\mathcal{D}$ compactly contained in $\tilde{\Omega}^m \setminus \mathcal{E}_m$. If $V(\xi) \neq 0$, for all $\xi = (\xi_1, \dots, \xi_m) \in \bar{\mathcal{D}}$. Then, there exists a family of solutions u_λ to* (3.1.1) *and m different points $q_i \in \Omega$, $i = 1, \dots, m$ satisfying* (3.1.4). *Furthermore, $\varphi_m(q) = \mathcal{C}$ and $\nabla \varphi_m(q) = 0$.*

We remark that it holds

1. if $V(\xi) > 0$ then the blowing-up solutions exist for $\lambda > 8\pi m$, and
2. if $V(\xi) < 0$ then the blowing-up solutions exist for $\lambda < 8\pi m$.

Here, we are considering a weaker assumption at critical points of φ_m.

We will consider a different kind of critical value, which also lifts the nondegeneracy assumptions of [18] on critical points of φ_m. Thus, we could consider the case $k \equiv 1$. More precisely, we consider the role of *non-trivial critical values* of φ_m, in existence of blowing-up solutions of (3.1.1). In the next result we assume $k \geq 0$, $k \not\equiv 0$, k is doubly periodic on $\partial\Omega$ and $k \in C(\bar{\Omega}) \cap C^3(\tilde{\Omega})$.

Theorem 3.1.3. *Let $m \geq 1$ and assume that there is a domain $\mathcal{D}$ compactly contained in $\tilde{\Omega}^m \backslash \mathcal{E}_m$, where φ_m has a non-trivial level $\mathcal{C}$. If $V(\xi) \neq 0$ for all $\xi \in \bar{\mathcal{D}}$, then there exists a solution u_λ to* (3.1.1) *and m different points $\xi_j \in \tilde{\Omega}$, $i = 1, \ldots, m$ satisfying* (3.1.4)*. Furthermore, $\varphi_m(\xi) = \mathcal{C}$ and $\nabla\varphi_m(\xi) = 0$.*

Let us mention that the problem (3.1.1) and related ones with *singular data* have attracted great attention. Several results for have been addressed in [2, 4, 66], concerning to the profile of blowing up solutions and quantization of blow-up levels. Chen and Lin have begun in [19], the study of mean field equation with singular data from the point of view of the topological Leray-Schauder degree and estimates of blowing up solutions. On the other hand, existence results have been achieved in works [31, 32, 35, 36]. In this situation, our approach does not apply directly, since $V(\xi) = 0$ for all possible points ξ. An higher order expansion is then needed in the study of existence of solutions to (3.1.1) with singular sources.

Let us consider the particular case when $k \equiv 1$ in Ω, namely,

$$\begin{cases} -\Delta u = \lambda\left(\dfrac{e^u}{\int_\Omega e^u} - \dfrac{1}{|\Omega|}\right), & \text{in } \Omega, \\ \quad u \quad \text{is doubly periodic on } \partial\Omega & \\ \int_\Omega u = 0. & \end{cases} \tag{3.1.7}$$

We get the following result.

Theorem 3.1.4. *Given any $m \geq 1$ there exists a family of solution to* (3.1.7) *and m different points such that u_λ concentrates at those points as* (3.1.4)*, as $\lambda \to 8\pi m^+$.*

The solutions are constructed using a family of solutions of the Liouville equation in $\mathbb{R}^2$, suitable scaled, translated and projected in order to have the boundary conditions. Usually, in other related problems of asymptotic analysis, solutions are found as a small additive perturbation of the initial approximation. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the parameters involve in the approximation, typically the location of concentration point. In our case, we also have to consider the high of the bubbles. Similarly to [27], we identify an extra element of the approximate kernel of the linearized operator, which introduces another parameter to be adjusted in the problem, related to all high of the bubbles. An important element in the reduction procedure is the invertibility of the linearized operator in suitable L^∞-weighted spaces. However, in our problem, this is not enough. Indeed, in order to perform a precise expansion of the reduced functional in C^1 sense, we need to improve the main term in the ansatz, adding one term in the expansion of the solution (see section 3.4). This fact is basically due to presence of the extra parameter to be adjusted and the estimate (3.3.18).

3.2 Approximation of the solution

In this section we will provide an approximation for the solution of problem (3.1.1) on the torus. We will use the Green's function as shown in chapter 2.

Consider as "basic cells" the function $U_{\delta,\xi}$ given by (2.3.3), with $\xi \in \Omega$. We would like to consider $U_{\delta,\xi}$ as the approximation of a solution around ξ. In order to satisfy the boundary conditions, we take $PU_{\delta,\xi}$, where P is the projection operator introduced in (2.2.5). First, let us find out the behavior of $PU_{\delta,\xi}$ away from ξ and around ξ. We obtain the following characterization.

Lemma 3.2.1. *Given $\xi \in \Omega$, the function $PU_{\delta,\xi}$, where $U_{\delta,\xi}$ is given by* (2.3.3)*, satisfies*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \log(8\delta^2) + H(x,\xi) + \alpha_{\delta,\xi} + O(\delta^2) \tag{3.2.1}$$

uniformly in C^2-sense on compact subsets of Ω, where

$$\alpha_{\delta,\xi} = \frac{1}{|\Omega|}\int_\Omega \log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\,dy.$$

In particular,

$$PU_{\delta,\xi}(x) = G(x,\xi) + \alpha_{\delta,\xi} + O(\delta^2), \tag{3.2.2}$$

where the term $O(\cdot)$ is uniform in C^2-sense on compact subsets of $\Omega \setminus \{\xi\}$ and C^1-sense on compact subsets of $\bar{\Omega} \setminus \{\xi\}$.

Proof: First, observe that

$$U_{\delta,\xi}(x) = \log(8\delta^2) + 4\log\frac{1}{|x-\xi|} + O(\delta^2) \tag{3.2.3}$$

uniformly in C^2-sense for x on compact subsets of $\bar{\Omega} \setminus \{\xi\}$. Let us take

$$\varphi(x) = U_{\delta,\xi}(x) - \log(8\delta^2) + H(x,\xi) + \alpha_{\delta,\xi}.$$

Then, by (2.2.3)

$$\int_\Omega \varphi = \int_\Omega \left[\log\frac{1}{(\delta^2+|x-\xi|^2)^2} + H(x,\xi)\right] dx + \int_\Omega \log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\,dy = 0$$

since $\int_\Omega G(\cdot,\xi) = 0$. Now, $\Delta\varphi = \Delta U_{\delta,\xi} + \frac{8\pi}{|\Omega|}$. Hence, we get

$$\int_\Omega G(x,y)\Delta\varphi(y)\,dy = \int_\Omega G(x,y)\Delta U_{\delta,\xi}(y)\,dy.$$

Thus, by the integral representation formula (2.2.4) we deduce that

$$\varphi(x) = PU_{\delta,\xi}(x) + \frac{1}{8\pi}\int_{\partial\Omega}\left[G(x,y)\frac{\partial\varphi(y)}{\partial\nu} - \varphi(y)\frac{\partial G(x,y)}{\partial\nu}\right] d\sigma(y)$$

for all $x \in \Omega$. On the other hand, we have

$$\int_{\partial\Omega} G(x,y)\frac{\partial\varphi(y)}{\partial\nu}\,d\sigma(y) = -\int_{\partial\Omega} G(x,y)\frac{\partial}{\partial\nu}\left[\log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\right] d\sigma(y),$$

since $G(x,\cdot)$ and $G(\cdot,\xi)$ are doubly periodic functions on $\partial\Omega$ and

$$\int_{\partial\Omega} G(x,y)\frac{\partial G}{\partial \nu}(y,\xi)\,d\sigma(y)=0.$$

Similarly, we get

$$\int_{\partial\Omega}\varphi(y)\frac{\partial G(x,y)}{\partial\nu}\,d\sigma(y)=-\int_{\partial\Omega}\log\left[\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\right]\frac{\partial G(x,y)}{\partial\nu}\,d\sigma(y).$$

Therefore,

$$\begin{aligned}PU_{\delta,\xi}(x)=U_{\delta,\xi}(x)-\log(8\delta^2)+H(x,\xi)+\frac{1}{|\Omega|}\int_\Omega\log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\,dy\\+\frac{1}{8\pi}\int_{\partial\Omega}\left[G(x,y)\frac{\partial}{\partial\nu}\left\{\log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\right\}\right.\\\left.-\log\left\{\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\right\}\frac{\partial G(x,y)}{\partial\nu}\right]\,d\sigma(y).\end{aligned}$$

Note that

$$\log\frac{(\delta^2+|x-\xi|^2)^2}{|x-\xi|^4}=\frac{2\delta^2}{|x-\xi|^2}=O(\delta^2)$$

uniformly for x over compact subsets of $\bar\Omega\setminus\{\xi\}$ and

$$\Delta(PU_{\delta,\xi}-\varphi)=\frac{1}{|\Omega|}\int_\Omega e^{U_{\delta,\xi}}-\frac{8\pi}{|\Omega|}=O(\delta^2).$$

Thus, we conclude (3.2.1) in C^2-sense on compact subset of Ω. From (3.2.1), (3.2.3) and

$$\Delta(PU_{\delta,\xi}-G(\cdot,\xi)-\alpha_{\delta,\xi})=-e^{U_{\delta,\xi}}+\frac{1}{|\Omega|}\int_\Omega e^{U_{\delta,\xi}}-\frac{8\pi}{|\Omega|}=O(\delta^2)\tag{3.2.4}$$

uniformly on compact subset of $\Omega\setminus\{\xi\}$, we get (3.2.2) in C^2- sense uniformly on compact subset of $\Omega\setminus\{\xi\}$. Finally, if we consider a point $x\in\partial\Omega$ then we can extend the function $PU_{\delta,\xi}$ to $B(x,r)$ periodically, for $r>0$ small and thus (3.2.4) is satisfied in weak sense. By regularity theory, we conclude that (3.2.2) is satisfied in C^1-sense on compact subset of $\bar\Omega\setminus\{\xi\}$. This complete the proof. □

Observe that

$$\alpha_{\delta,\xi}=-\frac{4\pi}{|\Omega|}\delta^2\log\delta+O(\delta^2).$$

In fact, we decompose

$$\begin{aligned}\int_\Omega\log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\,dy&=\int_{B(\xi,r)}\log\frac{(\delta^2+|y-\xi|^2)^2}{|y-\xi|^4}\,dy+O(\delta^2)\\&=-4\pi\delta^2\log\delta+O(\delta^2).\end{aligned}$$

Given m a positive integer let us consider $\xi_j\in\Omega$, $j=1,\dots,m$ distinct points with $k(\xi_j)>0$ and $\delta_j>0$, $j=1,\dots,m$. In order to have a good approximation we will assume that

$$\delta_j^2=\delta^2\rho_j(\xi_j),\quad\forall\, j=1,\dots,m,\tag{3.2.5}$$

and

$$\exists\, C > 1 \,:\, |\lambda - 8\pi m| \le C\delta^2 |\log \delta|. \tag{3.2.6}$$

where $\delta > 0$ and ρ_j is given by (3.1.6). Denote $U_j := U_{\delta_j,\xi_j}$ and $W_j = PU_j$, $j = 1, \dots, m$, where $U_{\delta,\xi}$ are given by (2.3.3) and P is the projection operator defined by (2.2.5). Thus, our first approximation is

$$W(x) = W_1(x) + \cdots + W_m(x), \qquad x \in \Omega. \tag{3.2.7}$$

We look for a solution u of (3.1.1) of the form $u = W + \phi$. Now, in terms of ϕ, the problem (3.1.1) becomes $\phi \in E$ such that

$$L(\phi) = -[R + N(\phi)], \qquad \text{in } \Omega, \tag{3.2.8}$$

where

$$L(\phi) = \Delta\phi + \lambda \frac{ke^W}{\int_\Omega ke^W}\left(\phi - \frac{\int_\Omega ke^W \phi}{\int_\Omega ke^W}\right) \tag{3.2.9}$$

$$R = \Delta W + \lambda \left(\frac{ke^W}{\int_\Omega ke^W} - \frac{1}{|\Omega|}\right), \tag{3.2.10}$$

and

$$N(\phi) = \lambda \left(\frac{ke^{W+\phi}}{\int_\Omega ke^{W+\phi}} - \frac{ke^W\phi}{\int_\Omega ke^W} + \frac{ke^W \int_\Omega ke^W \phi}{\left(\int_\Omega ke^W\right)^2} - \frac{ke^W}{\int_\Omega ke^W}\right). \tag{3.2.11}$$

Let us observe that

$$\int_\Omega R = \int_\Omega L(\phi) = \int_\Omega N(\phi) = 0.$$

Furthermore, in order to get the invertibility of the linear operator L in suitable function spaces, let us consider the weighted norm

$$\|h\|_* = \sup_{x\in\Omega} \left[\sum_{j=1}^m \frac{\delta_j^\sigma}{(\delta_j^2 + |x - \xi_j|^2)^{1+\sigma/2}}\right]^{-1} |h(x)|, \tag{3.2.12}$$

for any $h \in L^\infty(\Omega)$ and where $0 < \sigma < 1$ is a small fixed constant. Let us see how well W solves the above problem in $\|\cdot\|_*$.

Lemma 3.2.2. *Assume* (3.2.5) *and* (3.2.6). *There exists a constant $C > 0$ independent of δ such that*

$$\|R\|_* \le C\delta \tag{3.2.13}$$

and also we have that

$$\|R - R_0\|_* \le C\delta^{2-\sigma}|\log\delta|, \tag{3.2.14}$$

where

$$R_0(x) = \sum_{j=1}^m e^{U_j(x)} \frac{\nabla\rho_j(\xi_j)}{\rho_j(\xi_j)} \cdot (x - \xi_j). \tag{3.2.15}$$

Proof: First, from Lemma 3.2.1 we note that for any $j \in \{1, \dots, m\}$

$$W_j(x) = U_j(x) - \log(8\delta_j^2) + H(x, \xi_j) + O(\delta^2|\log\delta|)$$

uniformly for x on compact subsets of Ω and

$$W_j(x) = G(x, \xi_j) + O(\delta^2|\log\delta|)$$

uniformly for x on compact subsets of $\bar\Omega \setminus \{\xi_j\}$. Hence, for $r>0$ small fixed we have that

$$\begin{aligned}\int_\Omega ke^W &= \sum_{j=1}^m \int_{B(\xi_j,r)} ke^{\sum_{l=1}^m W_l} + O(1)\\ &= \sum_{j=1}^m \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} ke^{U_j+H(\cdot,\xi_j)+\sum_{l\neq j} G(\cdot,\xi_l)+O(\delta^2|\log\delta|)} + O(1)\\ &= \sum_{j=1}^m \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} e^{U_j}\rho_j(1+O(\delta^2|\log\delta|)) + O(1)\\ &= \sum_{j=1}^m \frac{1}{\delta_j^2}[\pi\rho_j(\xi_j)+O(\delta^2|\log\delta|)] + O(1)\\ &= \frac{\pi m}{\delta^2} + O(|\log\delta|),\end{aligned}$$

since

$$\begin{aligned}\int_{B(\xi_j,r)} \frac{\delta_j^2\rho_j(x)}{(\delta_j^2+|x-\xi_j|^2)^2}\,dx &= \int_{B(0,\frac{r}{\delta_j})} \frac{1}{(1+|y|^2)^2}\,\rho_j(\xi_j+\delta_j y)\,dy\\ &= \int_{B(0,\frac{r}{\delta_j})} \frac{1}{(1+|y|^2)^2}\left[\rho_j(\xi_j)+\nabla\rho_j(\xi_j)\cdot\delta_j y + O(\delta_j^2|y|^2)\right] dy\\ &= \pi\rho_j(\xi_j) + O(\delta^2|\log\delta|)\end{aligned}$$

Thus, we get that

$$\begin{aligned}R &= \sum_{j=1}^m \left[\Delta U_j - \frac{1}{|\Omega|}\int_\Omega \Delta U_j\right] + \lambda\frac{ke^W}{\int_\Omega ke^W} - \frac{\lambda}{|\Omega|}\\ &= -\sum_{j=1}^m e^{U_j} + \lambda\frac{k\exp\left(\sum_{j=1}^m W_j\right)}{\pi m\delta^{-2}+O(|\log\delta|)} + \frac{1}{|\Omega|}\sum_{j=1}^m\int_\Omega e^{U_j} - \frac{\lambda}{|\Omega|}.\end{aligned}$$

Let us observe that if $|x-\xi_j|>r$ then $e^{U_j(x)} = O(\delta^2)$ and

$$\int_\Omega e^{U_j} = 8\pi + O(\delta^2).$$

Hence, if $|x-\xi_j|>r$ for all $j=1,\dots,m$ then by Lemma 3.2.1 and (3.2.5)-(3.2.6) we get that $W(x)=O(1)$ and

$$R(x) = O(\delta^2) + \frac{8\pi m-\lambda}{|\Omega|} = O(\delta^2|\log\delta|).$$

Now, if $|x-\xi_j|<r$ for some $j\in\{1,\dots,m\}$ then by Lemma 3.2.1 and (3.2.5)-(3.2.6)

$$\begin{aligned}R(x) &= -e^{U_j(x)} + \lambda e^{U_j(x)}\,\frac{k(x)e^{-\log(8\delta_j^2)+H(x,\xi_j)+\sum_{l\neq j}G(x,\xi_l)+O(\delta^2|\log\delta|)}}{\pi m\delta^{-2}+O(|\log\delta|)} + \frac{8\pi m-\lambda}{|\Omega|} + O(\delta^2)\\ &= e^{U_j(x)}\left[-1+\frac{\lambda\rho_j(x)[1+O(\delta^2|\log\delta|)]}{8\pi m\rho_j(\xi_j)+O(\delta^2|\log\delta|)}\right] + O(\delta^2|\log\delta|)\\ &= e^{U_j(x)}\left[\frac{\rho_j(x)-\rho_j(\xi_j)}{\rho_j(\xi_j)} + \frac{(\lambda-8\pi m)\rho_j(x)}{8\pi m\rho_j(\xi_j)} + O(\delta^2|\log\delta|)\right] + O(\delta^2|\log\delta|)\\ &= e^{U_j(x)}\left[\frac{\nabla\rho_j(\xi_j)}{\rho_j(\xi_j)}\cdot(x-\xi_j) + O(|x-\xi_j|^2) + O(\delta^2|\log\delta|)\right] + O(\delta^2|\log\delta|).\end{aligned}$$

Finally, from the definition of $\|\cdot\|_*$ we conclude (3.2.13) and (3.2.14). $\square$

Let us stress that by doubly periodic conditions on $\partial\Omega$ all points ξ_j are interior and thus, without loss of generality we shall always assume $\xi_j \in \Omega$. Furthermore, a posteriori we shall give an explicit relation between λ and δ, in order to find a solution to (3.2.8) (see proof of Theorem 3.1.2).

3.3 The linear operator

In this section, we will prove the invertibility of the linear operator L, by using the L^∞-norm introduce in (3.2.12), under suitable orthogonal conditions.

Let us consider the following linear operator in $\mathbb{R}^2$

$$L_0(\phi) = \Delta\phi + \frac{8}{(1+|y|^2)^2}\phi.$$

It is well-known that the bounded solutions of $L_0(\phi) = 0$ in $\mathbb{R}^2$ are precisely linear combinations of

$$Y_i(y) = \frac{4y_i}{1+|y|^2}, \qquad i = 1, 2, \qquad \text{and} \qquad Y_0(y) = 2\,\frac{1-|y|^2}{1+|y|^2}.$$

See [1] for a proof. On the other hand, let us observe that formally the operator L, scaled and centered at ξ_j/δ_j by setting $y = (x-\xi_j)/\delta_j$, approaches

$$\tilde{L}_0(\phi) = \Delta\phi + \frac{8}{(1+|y|^2)^2}\left(\phi - \frac{1}{\pi m}\int_{\mathbb{R}^2} \frac{\phi(z)}{(1+|z|^2)^2}\,dz\right).$$

It turns out that the bounded solutions of $\tilde{L}_0(\phi) = 0$ in $\mathbb{R}^2$ are linear combinations of Y_j, $j = 0, 1, 2$ and the constant functions. This exhibits a difference in comparison with some results related to linearized operators in Liouville type equations with Dirichlet or Neumann boundary conditions in a domain, [31, 37, 38, 40], where the approximate kernel is span by the translations Y_j, $j = 1, 2$ and dilations Y_0, and the invertibility is obtained avoiding the dilations Y_0. Here, we have the constants functions in the approximate kernel and concerning to the invertibility of the operator L it is not possible to avoid the dilations.

Let us introduce the functions

$$Z_{ij}(x) = Y_i\left(\frac{x-\xi_j}{\delta_j}\right), \qquad i = 0, 1, 2,\ j = 1, \dots, m$$

for $x \in \Omega$. Consider the linear problem of finding a function $\phi \in E \cap W^{2,2}(\Omega)$ and scalars c_{ij}, $i = 1, 2$, $j = 1, \dots, m$ and c_0 such that

$$\begin{cases} L(\phi) \;=\; h + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\Delta P Z_{ij} + c_0 \Delta P Z, & \text{in } \Omega, \\ \int_\Omega \Delta P Z_{ij}\phi = 0, \quad \text{for all } i = 1, 2,\ j = 1, \dots, m, & \\ \int_\Omega \phi \Delta P Z = 0 & \end{cases} \tag{3.3.1}$$

where $h \in C^{0,\alpha}(\Omega)$, $\int_\Omega h = 0$, $\|h\|_* < +\infty$, $Z = \sum_{l=1}^m Z_{0l}$ and PZ_{ij}, $i = 1, 2$, $j = 1, \dots, m$, $PZ = \sum_{l=1}^m PZ_{0l}$ are the projection of Z_{ij} and Z respectively, namely,

$$\begin{cases} \Delta P Z_{ij} = \Delta Z_{ij} - \frac{1}{|\Omega|}\int_\Omega \Delta Z_{ij}, & \text{in } \Omega, \\ P Z_{ij} \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega P Z_{ij} = 0. & \end{cases} \tag{3.3.2}$$

Let us stress that the orthogonality conditions in the above problem are taken with respect to the elements of the approximate kernel due to translations and an extra element which involves dilations. Similar situation appears also in [27].

First, we will prove an a priori estimate for the problem (3.3.1) with $c_{ij} = 0$ for all $i = 1, 2$, $j = 1, \dots, m$ and $c_0 = 0$. Specifically, we consider the problem

$$\begin{cases} L(\phi) \,=\, h, & \text{in } \Omega, \\ \int_\Omega \Delta P Z_{ij} \phi = 0, & \text{for all } i = 1, 2,\ j = 1, \dots, m, \\ \int_\Omega \phi \Delta P Z = 0 \end{cases} \tag{3.3.3}$$

Proposition 3.3.1. *Let $d > 0$ be fixed. There exist positive numbers δ_0 and C, such that for any points $\xi_j \in \Omega$, $j = 1, \dots, m$, which satisfy*

$$dist(\xi_l - \xi_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) \geq d \qquad for \quad l \neq j\ , \tag{3.3.4}$$

$\delta_j > 0$ satisfying (3.2.5) and (3.2.6), and any solution ϕ to problem (3.3.3), one has

$$\|\phi\|_\infty \leq C \left(\log \frac{1}{\delta} \right) \|h\|_* \qquad for\ all\ \delta < \delta_0. \tag{3.3.5}$$

Proof: The proof of estimate (3.3.5) consists of some steps. Let us assume the opposite, namely, the existence of sequences $\delta^n \to 0$, points $\xi_j^n \in \Omega$, $\delta_j^n = \delta^n \rho_j(\xi_j^n)$, functions h_n with $|\log \delta^n|\, \|h_n\|_* = o(1)$ as $n \to +\infty$, $\phi_n \in E$ with $\|\phi_n\|_\infty = 1$ and

$$\begin{cases} L(\phi_n) \,=\, h_n, & \text{in } \Omega, \\ \int_\Omega \phi_n \Delta P Z_{ij} = 0, & \text{for all } i = 1, 2,\ j = 1, \dots, m, \\ \int_\Omega \phi_n \Delta P Z = 0. \end{cases}$$

Without loss of generality, we assume that $\xi_j^n \to \xi_j^*$ as $n \to +\infty$ and $\xi_j^* \in \Omega$ for all $j = 1, \dots, m$, by the doubly periodic boundary conditions. Also, observe that there is a constant $C_0 > 0$ such that $C_0^{-1} \leq \rho_j(\xi_j^n) \leq C_0$, by using (3.3.4). Let us denote

$$\psi_n := \phi_n - \frac{\int_\Omega k e^W \phi_n}{\int_\Omega k e^W} \qquad \text{and} \qquad K = \lambda \frac{k e^W}{\int_\Omega k e^W}.$$

Then ψ_n satisfies

$$\begin{cases} \Delta \psi_n + K \psi_n \,=\, h_n, & \text{in } \Omega, \\ \int_\Omega \psi_n \Delta P Z_{ij} = 0, & \text{for all } i = 1, 2,\ j = 1, \dots, m, \\ \int_\Omega \psi_n \Delta P Z = 0. \end{cases}$$

Claim 3.3.1. *There is a constant $\sigma_0 > 0$ such that $\|\psi_n\|_\infty > \sigma_0$ for all $n \geq 1$ up to subsequences. Furthermore, ψ_n converges to a constant $\tilde{c}$ as $n \to +\infty$ in $C^{2,\alpha}$ sense over compact subsets of $\Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$.*

Proof: Assume that $\|\psi_n\|_\infty \to 0$ as $n \to +\infty$. Hence, we have that

$$\left\| \phi_n - \frac{\int_\Omega k e^W \phi_n}{\int_\Omega k e^W} \right\|_\infty \to 0 \qquad \text{as} \quad n \to +\infty.$$

Since

$$\left|\frac{\int_\Omega ke^W\phi_n}{\int_\Omega ke^W}\right| \le \|\phi_n\|_\infty \le 1,$$

we conclude that ϕ_n converges uniformly to a constant in Ω. But $\int_\Omega \phi_n = 0$ pass to the limit and we get a contradiction, since $\|\phi_n\|_\infty = 1$ and $\|\phi_n\|_\infty \to 0$ as $n \to +\infty$. On the other hand, given $r > 0$, we observe that

$$\Delta\psi_n = O([\delta^n]^2) + o(1), \qquad \text{uniformly for } x \in \Omega \setminus \cup_{j=1}^m B(\xi_j^n, r)$$

since if $|x - \xi_j^n| > r$ for all $j = 1, \dots, m$ then we have that

$$K(x) = \frac{\lambda k(x)\exp(\sum_{j=1}^m G(x,\xi_l) + O([\delta^n]^2|\log\delta^n|))}{\pi m[\delta^n]^{-2} + O(|\log\delta^n|)} = O([\delta^n]^2)$$

and

$$|h_n(x)| \le \sum_{j=1}^m \frac{[\delta_j^n]^\sigma}{([\delta_j^n]^2 + |x-\xi_j^n|^2)^{1+\sigma/2}}\|h_n\|_* \le C[\delta^n]^\sigma\|h_n\|_*.$$

Therefore, passing to a subsequence $\psi_n \to \psi$ as $n \to +\infty$ in $C^{2,\alpha}$ sense over compact subsets of $\Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$. Since $\|\psi_n\|_\infty \le 2\|\phi_n\|_\infty \le 2$, it follows that ψ is bounded and can be extended continuously to Ω and satisfies

$$\begin{cases} \Delta\psi = 0, & \text{in } \Omega,\\ \psi & \text{is doubly periodic on on } \partial\Omega. \end{cases}$$

Therefore, $\psi \equiv \tilde c$ in Ω. It follows that

$$\tilde c = -\lim_{n\to+\infty}\frac{\int_\Omega ke^W\phi_n}{\int_\Omega ke^W}, \qquad \text{since} \qquad \frac{1}{|\Omega|}\int_\Omega \psi_n = -\frac{\int_\Omega ke^W\phi_n}{\int_\Omega ke^W}.$$

□

Now, consider the function $\Psi_{n,j}(y) = \psi_n(\xi_j^n + \delta_j^n y)$. Then, $\Psi_{n,j}$ satisfies

$$\Delta\Psi_{n,j} + \tilde K_{n,j}(y)\Psi_{n,j} = \tilde h_{n,j}(y) \qquad \text{in } \Omega_{n,j} \equiv (\delta_j^n)^{-1}(\Omega - \xi_j^n),$$

where $\tilde K_{n,j}(y) := (\delta_j^n)^2 K(\xi_j^n + \delta_j^n y)$ and $\tilde h_{n,j}(y) = (\delta_j^n)^2 h_n(\xi_j^n + \delta_j^n y)$.

Claim 3.3.2. *There holds that $\Psi_{n,j}$ converges uniformly over compact subsets of $\mathbb{R}^2$ to $a_{0j}Y_0$, as $n \to +\infty$, for some constant $a_{0j} \in \mathbb{R}$, $j = 1, \dots, m$. Furthermore,*

$$\sum_{j=1}^m a_{0j} = 0. \tag{3.3.6}$$

Proof: Let observe that uniformly for y over compact subsets of $\mathbb{R}^2$ we have that as $n \to +\infty$

$$\tilde K_{n,j}(y) = \frac{8}{(1+|y|^2)^2}(1+o(1)) \quad \text{and} \quad |\tilde h_{n,j}(y)| \le C\|h_n\|_*.$$

Hence, we get that elliptic estimates imply that as $n \to +\infty$, $\Psi_{n,j}$ converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded solution Ψ_j of

$$\Delta\Psi_j + \frac{8}{(1+|y|^2)^2}\Psi_j = 0.$$

Recall that $\lambda_n \to 8\pi m$ as $n \to +\infty$, by using (3.2.6). We know that for some constants $a_{ij} \in \mathbb{R}$, $i = 0, 1, 2$ it holds $\Psi_j(y) = \sum_{i=0}^{2} a_{ij} Y_i(y), \quad y \in \mathbb{R}^2$. On the other hand, we have that for all $i = 1, 2$, $j = 1, \ldots, m$

$$0 = \int_\Omega \psi_n \Delta P Z_{ij} = \int_\Omega \psi_n \Delta Z_{ij} - \frac{1}{|\Omega|} \int_\Omega \psi_n \int_\Omega \Delta Z_{ij}.$$

Then, we estimate

$$\begin{aligned}\int_\Omega \psi_n \Delta Z_{ij} &= -\int_\Omega \psi_n(x) \frac{32(\delta_j^n)^3 (x - \xi_j^n)_i}{((\delta_j^n)^2 + |x - \xi_j^n|^2)^3}\, dx \\ &= -32 \int_{B(0, \frac{r}{\delta_j^n})} \Psi_{n,j}(y)\, \frac{y_i}{(1 + |y|^2)^3}\, dy + O([\delta^n]^3)\end{aligned}$$

and

$$\int_\Omega \Delta Z_{ij} = -32 \int_{B(0, \frac{r}{\delta_j^n})} \frac{y_i}{(1 + |y|^2)^3}\, dy + O([\delta^n]^3) = o(1)$$

as $n \to +\infty$. Therefore, by dominated convergence we get that

$$\int_{\mathbb{R}^2} \Psi_j(y)\, \frac{y_i}{(1 + |y|^2)^3}\, dy = 0, \qquad i = 1, 2$$

and we conclude that $a_{1j} = a_{2j} = 0$ for all $j = 1, \ldots, m$. Thus, $\Psi_{n,j}$ converges uniformly over compact subset of $\mathbb{R}^2$ to Ψ_j as $n \to +\infty$ for all $j = 1, \ldots, m$, and as claimed

$$\Psi_j(y) = a_{0j} Y_0(y) = 2a_{0j}\, \frac{1 - |y|^2}{1 + |y|^2}, \qquad y \in \mathbb{R}^2.$$

Let us observe that

$$0 = \int_\Omega \psi_n \Delta P Z = \sum_{l=1}^{m} \int_\Omega \psi_n \Delta P Z_{0l} = \sum_{l=1}^{m} \left[\int_\Omega \psi_n \Delta Z_{0l} - \frac{1}{|\Omega|} \int_\Omega \psi_n \int_\Omega \Delta Z_{0l} \right].$$

Hence, we have that

$$\begin{aligned}\int_\Omega \psi_n \Delta Z_{0j} &= -\int_\Omega \psi_n(x) \frac{16(\delta_j^n)^2 [(\delta_j^n)^2 - |x - \xi_j^n|^2]}{((\delta_j^n)^2 + |x - \xi_j^n|^2)^3}\, dx \\ &= -16 \int_{B(0, \frac{r}{\delta_j^n})} \Psi_{n,j}(y)\, \frac{1 - |y|^2}{(1 + |y|^2)^3}\, dy + O([\delta^n]^2)\end{aligned}$$

and

$$\int_\Omega \Delta Z_{0j} = -16 \int_{B(0, \frac{r}{\delta_j^n})} \frac{1 - |y|^2}{(1 + |y|^2)^3}\, dy + O([\delta^n]^2) = O([\delta^n]^2)$$

as $n \to +\infty$. Therefore, by dominated convergence we get that

$$\lim_{n \to +\infty} \int_\Omega \psi_n \Delta P Z_{0j} = -16 \int_{\mathbb{R}^2} \Psi_j(y)\, \frac{1 - |y|^2}{(1 + |y|^2)^3}\, dy,$$

and we conclude that

$$0 = \sum_{l=1}^{m} \int_{\mathbb{R}^2} \Psi_l(y)\, \frac{1 - |y|^2}{(1 + |y|^2)^3}\, dy = \sum_{l=1}^{m} 2a_{0l} \int_{\mathbb{R}^2} \frac{(1 - |y|^2)^2}{(1 + |y|^2)^4}\, dy.$$

Thus, the claim follows since $\int_{\mathbb{R}^2} \frac{(1 - |y|^2)^2}{(1 + |y|^2)^4}\, dy > 0$. □

On the other hand, from the equation of ψ_n and by (2.2.2) we have the following integral representation

$$\psi_n(x) = \frac{1}{|\Omega|}\int_\Omega \psi_n + \frac{1}{8\pi}\int_\Omega \left[4\log\frac{1}{|x-y|} + H(x,y)\right][K(y)\psi_n(y) - h_n(y)]\,dy. \tag{3.3.7}$$

Claim 3.3.3. *There holds $\tilde c = 0$ and hence, for any $r > 0$ small ($r < d$)*

$$\|\psi_n\|_{L^\infty(\Omega\setminus\cup_{j=1}^m B(\xi_j^n, r))} \to 0 \qquad \text{as } n \to +\infty.$$

Proof: Let us estimate the right hand side of (3.3.7). First, we estimate the integrals involving h_n. Observe that for $R > 0$ fixed we have that

$$\int_{B(0,R\delta_j^n)} |\log|x||\,dx = \frac{\pi R^2(\delta_j^n)^2}{2} - \pi R^2(\delta_j^n)^2 \log R\delta_j^n.$$

Hence, we get that for $x \in \Omega$

$$\begin{aligned}\left|\int_{B(x,R\delta_j^n)} \log\frac{1}{|x-z|} h_n(z)\,dz\right| &\le \|h_n\|_* \int_{B(x,R\delta_j^n)} \left|\log\frac{1}{|x-z|}\right| \sum_{l=1}^m \frac{(\delta_j^n)^\sigma}{((\delta_j^n)^2 + |z-\xi_l^n|^2)^{1+\sigma/2}}\,dz \\ &\le \frac{C}{(\delta^n)^2}\|h_n\|_* \int_{B(0,R\delta_j^n)} |\log|z||\,dz \\ &\le C|\log\delta^n|\,\|h_n\|_*.\end{aligned}$$

Now, if $|x-y| > R\delta_j^n$ then for some constant $C > 0$ we have that $C^{-1}\delta^n < |y-x| < C/\delta^n$ and $|\log|x-y|| \le C|\log\delta^n|$. Thus, we get that

$$\begin{aligned}\left|\int_{\Omega\setminus B(x,R\delta_j^n)} \log\frac{1}{|x-y|} h_n(y)\,dy\right| &\le C\left(\log\frac{1}{\delta^n}\right)\int_{\Omega\setminus B(x,R\delta_j^n)} |h_n(y)|\,dy \\ &\le C|\log\delta^n|\,\|h_n\|_*.\end{aligned}$$

Therefore, we conclude that

$$\left|\int_\Omega \log\frac{1}{|x-y|} h_n(y)\,dy\right| \le C\left(\log\frac{1}{\delta^n}\right)\|h_n\|_*$$

uniformly for $x \in \Omega$. Analogously,

$$\left|\int_\Omega H(x,y)h_n(y)\,dy\right| \le C\left(\log\frac{1}{\delta^n}\right)\|h_n\|_*$$

uniformly for $x \in \Omega$. Now, if $|y-\xi_l^n| > r$ for all $l = 1,\dots,m$ and $|x-\xi_j^n| < R\delta_j^n$ for some $j \in \{1,\dots,m\}$ then $K(y) = O([\delta^n]^2)$ and $|x-y| > R\delta_j^n$. Hence, we get that $|\log|x-y|| < C\log\frac{1}{\delta^n}$ and

$$\int_\Omega \log\frac{1}{|x-y|}K(y)\psi_n(y)\,dy = \sum_{l=1}^m \int_{B(\xi_l^m,r)} \log\frac{1}{|x-y|}K(y)\psi_n(y)\,dy + O\left([\delta^n]^2|\log\delta^n|\right).$$

Now, for any $l \in \{1,\dots,m\}$ we have that

$$\int_{B(\xi_l^n,r)} \log\frac{1}{|x-y|}K(y)\psi_n(y)\,dy = \int_{B(0,\frac{r}{\delta_l^n})} \log\frac{1}{|x-(\xi_l^n+\delta_l^n z)|}\tilde K_l(z)\Psi_{n,l}(z)\,dz.$$

Recall that

$$|\tilde{K}_l(z)\Psi_{n,l}(z)| \le \frac{C}{(1+|z|^2)^2}, \qquad \text{for all } z \in B(0, \frac{r}{\delta_l^n}),$$

since $\|\Psi_{n,l}\|_{L^\infty(\Omega_{n,j})} = 1$ and for $|z| < R$

$$\tilde{K}_l(z) = [\delta^n]^2 e^{U_j(\xi_j^n + \delta_j^n z)} \frac{\lambda_n \rho_j(\xi_j^n + \delta_j^n z)}{8\pi m \rho_j(\xi_j)} (1 + O([\delta^n]^2 |\log \delta^n|)) = \frac{8}{(1+|z|^2)^2}(1+o(1)).$$

Also, we know that for any l, $\Psi_{n,l}$ converges uniformly to $a_{0l}Y_0$ on compact subsets of $\mathbb{R}^2$. So, taking $x = \xi_j^n$ and if $l \neq j$ then by dominated convergence we get that

$$\lim_{n\to+\infty} \int_{B(\xi_l^n, r)} \log \frac{1}{|\xi_j^n - y|} K(y)\psi_n(y)\, dy = \log \frac{1}{|\xi_j^* - \xi_l^*|} \int_{\mathbb{R}^2} \frac{8}{(1+|z|^2)^2}\, a_{0l} Y_0(z)\, dz = 0,$$

since $\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3}\, dy = 0$. For $l = j$, we again take $x = \xi_j^n$ and we obtain

$$\begin{aligned} \int_{B(\xi_j^n, r)} \log \frac{1}{|\xi_j^n - y|} K(y)\psi_n(y)\, dy &= \int_{B(0, \frac{r}{\delta_j^n})} \log \frac{1}{|\delta_j^n z|} \tilde{K}_j(z)\Psi_{n,j}(z)\, dz \\ &= \int_{B(0, \frac{r}{\delta_j^n})} \log \frac{1}{|z|} \tilde{K}_j(z)\Psi_{n,j}(z)\, dz \\ &\quad - \log \delta_j^n \int_{B(0, \frac{r}{\delta_j^n})} \tilde{K}_j(z)\Psi_{n,j}(z)\, dz \end{aligned}$$

Similarly, we have that

$$\int_\Omega H(x,y)K(y)\psi_n(y)\, dy = \sum_{l=1}^m \int_{B(\xi_l^n, r)} H(x,y)K(y)\psi_n(y)\, dy + O\left([\delta^n]^2\right)$$

and

$$\int_{B(\xi_l^n, r)} H(x,y)K(y)\psi_n(y)\, dy = \int_{B(0, \frac{r}{\delta_l^n})} H(x, \xi_l^n + \delta_l^n z)\tilde{K}_l(z)\Psi_{n,l}(z)\, dz.$$

Hence, we get that for any $l \in \{1, \ldots, m\}$

$$\lim_{n\to+\infty} \int_{B(\xi_l^n, r)} H(\xi_j^n, y)K(y)\psi_n(y)\, dy = H(\xi_j^*, \xi_l^*) \int_{\mathbb{R}^2} \frac{8}{(1+|z|^2)^2}\, a_{0l} Y_0(z)\, dz = 0.$$

Observe that by (3.3.4) we have $G(\xi_j^*, \xi_l^*) = -4 \log |\xi_j^* - \xi_l^*| + H(\xi_j^*, \xi_l^*) \in \mathbb{R}$. Therefore, by the integral representation of ψ_n we have obtained that

$$\psi_n(\xi_j^n) = \frac{1}{|\Omega|} \int_\Omega \psi_n + \frac{1}{2\pi} \int_{B(0, \frac{r}{\delta_j^n})} \log \frac{1}{|z|} \tilde{K}_j(z)\Psi_{n,j}\, dz - \log \delta_j^n \int_{B(\xi_j^n, r)} K(y)\psi_n(y)\, dy + o(1), \quad (3.3.8)$$

as $n \to +\infty$. On the other hand, let us note that integrating the equation of ψ_n, we get that $\int_\Omega K\psi_n = 0$, since $\int_\Omega h_n = 0$, and hence,

$$\sum_{l=1}^m \int_{B(\xi_l^m, r)} K(y)\psi_n(y)\, dy = -\int_{\Omega \setminus \cup_{l=1}^m B(\xi_l^n, r)} K\psi_n = O([\delta^n]^2).$$

Also, by dominated convergence we know that

$$\lim_{n\to+\infty}\int_{B(0,\frac{r}{\delta_j^n})}\log\frac{1}{|z|}\tilde{K}_j(z)\Psi_{n,j}(z)\,dz=\int_{\mathbb{R}^2}\log\frac{1}{|z|}\,\frac{8}{(1+|z|^2)^2}\,a_{0l}Y_0(z)\,dz.$$

Taking the sum of equations (3.3.8), since $\Psi_{n,j}(0)=\psi_n(\xi_j^n)$ and letting $n\to+\infty$, we find that, we get that

$$\sum_{j=1}^m\Psi_j(0)=m\tilde{c}+\frac{1}{2\pi}\sum_{j=1}^m\int_{\mathbb{R}^2}\log\frac{1}{|z|}\frac{8}{(1+|z|^2)^2}\Psi_j(z)\,dz.$$

Recall that $\Psi_j(y)=\tilde{a}_{0j}\frac{1-|y|^2}{1+|y|^2}$ with $\tilde{a}_{0j}=2a_{0j}$. Hence, it follows that $\tilde{c}=0$, since $\sum_{j=1}^m\tilde{a}_{0j}=0$. Therefore, the conclusion follows. □

Claim 3.3.4. *There holds $a_{0j}=0$ for all $j=1,\dots,m$. And by claim 3.3.2 it follows that*

$$\|\Psi_{n,j}\|_{L^\infty(B(0,R))}=\|\psi_n\|_{L^\infty(B(\xi_j^n,R\delta_j^n))}\to 0,\qquad as\quad n\to+\infty.$$

Proof: To this aim let us construct a suitable test function and from the assumption on h_n, $|\log\delta^n|\,\|h_n\|_*=o(1)$ we get the additional orthogonality relation

$$\int_{\mathbb{R}^2}\frac{8(1-|z|^2)}{(1+|z|^2)^3}\,\Psi_j(z)\,dz=0,\tag{3.3.9}$$

which implies $a_{0j}=0$ as claimed. We will use an idea developed first in [37] and then exploited in [38, 39, 40]. We will omit the subscript n in ξ_j^n and δ_j^n. Define the functions $\eta_{n,j}$ and $\tilde{\eta}_{n,j}$ for $x\in\Omega$ given by

$$\eta_{n,j}(x)=\frac{4}{3}\log(\delta_j^2+|x-\xi_j|^2)\frac{\delta_j^2-|x-\xi_j|^2}{\delta_j^2+|x-\xi_j|^2}+\frac{8}{3}\frac{\delta_j^2}{\delta_j^2+|x-\xi_j|^2}$$

and

$$\tilde{\eta}_{n,j}(x)=-\frac{2\delta_j^2}{\delta_j^2+|x-\xi_j|^2}.$$

Let us note that $\eta_{n,j}$ and $\tilde{\eta}_{n,j}$ satisfy

$$\Delta\eta_{n,j}+\frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\eta_{n,j}=2\frac{8\delta_j^2(\delta_j^2-|x-\xi_j|^2)}{(\delta_j^2+|x-\xi_j|^2)^3}$$

and

$$\Delta\tilde{\eta}_{n,j}+\frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\tilde{\eta}_{n,j}=-\frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}.$$

Consider the test function $P\tilde{Z}_{n,j}$, where $\tilde{Z}_{n,j}=\eta_{n,j}+\frac{2}{3}H(\xi_j,\xi_j)\tilde{\eta}_{n,j}$. From the representation formula (2.2.4) we get that

$$P\tilde{Z}_{n,j}-\tilde{Z}_{n,j}-\frac{2}{3}H(\cdot,\xi_j)=O(\delta^2|\log\delta|),\tag{3.3.10}$$

in C^2-sense over compact subset of Ω. Recall that ψ_n satisfies

$$\Delta\psi_n+K\psi_n=h,\qquad \text{in}\quad\Omega.$$

Multiply this equation by $P\tilde{Z}_{n,j}$ and integrate on Ω, then we get that

$$\begin{aligned}
\int_\Omega hP\tilde{Z}_{n,j} &= \int_\Omega \psi_n \left(\Delta P\tilde{Z}_{n,j} + KP\tilde{Z}_{n,j}\right) \\
&= \int_\Omega \psi_n \Delta\tilde{Z}_{n,j} - \frac{1}{|\Omega|}\int_\Omega \psi_n \int_\Omega \Delta\tilde{Z}_{n,j} + \int_\Omega K\psi_n P\tilde{Z}_{n,j} \\
&= \int_\Omega \psi_n \left(2\frac{8\delta_j^2(\delta_j^2 - |x-\xi_j|^2)}{(\delta_j^2 + |x-\xi_j|^2)^3} - \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left[\tilde{Z}_{n,j} + \frac{2}{3}H(\xi_j,\xi_j)\right]\right) \\
&\quad - \frac{1}{|\Omega|}\int_\Omega \psi_n \int_\Omega \Delta\tilde{Z}_{n,j} + \int_\Omega K\psi_n P\tilde{Z}_{n,j} \\
&= 2\int_\Omega \psi_n \frac{8\delta_j^2(\delta_j^2 - |x-\xi_j|^2)}{(\delta_j^2 + |x-\xi_j|^2)^3} + \int_\Omega \psi_n \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left(P\tilde{Z}_{n,j} - \tilde{Z}_{n,j} - \frac{2}{3}H(\xi_j,\xi_j)\right) \\
&\quad - \frac{1}{|\Omega|}\int_\Omega \psi_n \int_\Omega \Delta\tilde{Z}_{n,j} + \int_\Omega \left(K - \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\right)\psi_n P\tilde{Z}_{n,j}
\end{aligned}$$

From (3.3.10) we get that $P\tilde{Z}_{n,j} = \tilde{Z}_{n,j} + O(1) = O(|\log\delta_n|)$, then

$$\int_\Omega hP\tilde{Z}_{n,j} = O(|\log\delta_n|\,\|h_n\|_*) = o(1)$$

as $n \to +\infty$. From the definition of $\tilde{Z}_{n,j}$, we have that

$$\begin{aligned}
\int_\Omega \Delta\tilde{Z}_{n,j} &= \int_\Omega \left(e^{U_j}Z_{0j} - e^{U_j}\eta_{n,j} + \frac{2}{3}H(\xi_j,\xi_j)\left[-e^{U_j} - e^{U_j}\tilde{\eta}_{n,j}\right]\right) \\
&= -\int_\Omega e^{U_j}\frac{4}{3}\log(\delta_j^2 + |x-\xi_j|^2)\frac{\delta_j^2 - |x-\xi_j|^2}{\delta_j^2 + |x-\xi_j|^2}\,dx + O(1) \\
&= -\frac{64}{3}\log\delta_j \int_{B(0,\frac{r}{\delta_j})} \frac{1-|y|^2}{(1+|y|^2)^3}\,dy + \frac{32}{3}\int_{B(0,\frac{r}{\delta_j})} \frac{1-|y|^2}{(1+|y|^2)^3}\log(1+|y|^2)\,dy + O(1) \\
&= O(1)
\end{aligned}$$

as $n \to +\infty$. On the other hand, we have that $\Psi_{n,j}(z) = \psi_n(\xi_j^n + \delta_j^n z)$ then as $n \to +\infty$

$$\begin{aligned}
\int_\Omega \psi_n(x)\frac{8\delta_j^2(\delta_j^2 - |x-\xi_j|^2)}{(\delta_j^2+|x-\xi_j|^2)^3}\,dx &= \int_{B(\xi_j,r)} \psi_n(x)\frac{8\delta_j^2(\delta_j^2 - |x-\xi_j|^2)}{(\delta_j^2+|x-\xi_j|^2)^3}\,dx + O(\delta_n^2) \\
&= \int_{B\left(0,\frac{r}{\delta_n}\right)} \Psi_{n,j}(z)\frac{8(1-|z|^2)}{(1+|z|^2)^3}\,dx + O(\delta_n^2)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \psi_n \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left(P\tilde{Z}_{n,j} - \tilde{Z}_{n,j} - \frac{2}{3}H(\xi_j,\xi_j)\right) \\
&\qquad = \int_\Omega \psi_n \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left(P\tilde{Z}_{n,j}(x) - \tilde{Z}_{n,j}(x) - \frac{2}{3}H(x,\xi_j)\right)dx \\
&\qquad\quad + \frac{2}{3}\int_\Omega \psi_n \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left(H(x,\xi_j) - H(\xi_j,\xi_j)\right)dx \\
&\qquad = \frac{2}{3}\int_{B(\xi_j,r)} \psi_n \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left(H(x,\xi_j) - H(\xi_j,\xi_j)\right)dx \\
&\qquad\quad + O(\delta_n^2|\log\delta_n|) \\
&\qquad = O(\delta_n)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \left(K - \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\right)\psi_n P\tilde{Z}_{n,j}\\
&= \int_{B(\xi_j,r)} \left(K - \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\right)\psi_n P\tilde{Z}_{n,j} + O(\delta_n^2|\log\delta_n|) + \sum_{l=1,l\neq j}^m \int_{B(\xi_l,r)} K\psi_n P\tilde{Z}_{n,j}\\
&= \int_{B(\xi_j,r)} e^{U_j}[O(|x-\xi_j|)+O(\delta^2|\log\delta|)]\psi_n P\tilde{Z}_{n,j} + O(\delta_n^2|\log\delta_n|)\\
&\quad + \sum_{l=1,l\neq j}^m \int_{B(0,\frac{r}{\delta_n})} \tilde{K}_l\Psi_{n,l}(z)P\tilde{Z}_{n,j}(\xi_l+\delta_l z)\,dz\\
&= O\left(\delta_n|\log\delta_n|\int_{B(0,\frac{r}{\delta_n})}\frac{1}{(1+|z|)^3}\,dz\right) + O(\delta_n^2|\log\delta_n|) + o(1)\\
&= o(1),
\end{aligned}$$

since if $l \neq j$ then we find that

$$\begin{aligned}
P\tilde{Z}_{n,j}(\xi_l+\delta_l z) &= \tilde{Z}_{n,j}(\xi_l+\delta_l z) + \frac{2}{3}H(\xi_l+\delta_l z,\xi_j) + O(\delta^2|\log\delta|)\\
&= \frac{2}{3}G(\xi_l+\delta_l z,\xi_j) + O(\delta^2|\log\delta|)\\
&= \frac{2}{3}G(\xi_l,\xi_j) + O(\delta)
\end{aligned}$$

for all $|z| < \frac{r}{\delta_l}$ and

$$\int_{B(0,\frac{r}{\delta_l})} \tilde{K}_l\Psi_{n,l}(z)P\tilde{Z}_{n,j}(\xi_l+\delta_l z)\,dz = o(1)$$

thanks to dominated convergence. Therefore, we conclude (3.3.9) and hence, $a_{0j} = 0$ for all $j = 1,\dots,m$. Also, it follows that $\Psi_{n,j}$ converges to zero uniformly over compact subset of $\mathbb{R}^2$ for all $j = 1,\dots,m$, namely, $\|\psi_n\|_{L^\infty(B(\xi_j^n,R\delta_j^n))} = \|\Psi_{n,j}\|_{L^\infty(B(0,R))} \to 0$, as $n \to +\infty$. □

Let us denote $\tilde{L}(\phi) = \Delta\phi + K\phi$.

Claim 3.3.5. *The operator $\tilde{L}$ satisfies the maximum principle in $\Omega\setminus\cup_{j=1}^m B(\xi_j^n, R\delta_j^n))$ for R large enough.*

Proof: First, we have that

$$K(x) = \frac{\lambda k e^W}{\int_\Omega k e^W} = \sum_{j=1}^m e^{U_j}\left[1 + O(|x-\xi_j|) + O(\delta_j^2|\log\delta_j|)\right].$$

Hence, we get that there is a constant D_0 such that

$$K(x) \le D_0 \sum_{j=1}^m \frac{\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2} \qquad \text{for all } x\in\Omega.$$

Now, consider the function

$$\tilde{Z}(x) = -\sum_{j=1}^m Y_0\left(\frac{a(x-\xi_j)}{\delta_j}\right) = 2\sum_{j=1}^m \frac{a^2|x-\xi_j|^2-\delta_j^2}{\delta_j^2+a^2|x-\xi_j|^2},$$

with $0 < a < \frac{1}{\sqrt{2D_0}}$. Then,

$$-\Delta \tilde{Z}(x) = 2\sum_{j=1}^{m} \frac{8\delta_j^2 a^2 (a^2|x-\xi_j|^2 - \delta_j^2)}{(\delta_j^2 + a^2|x-\xi_j|^2)^3},$$

and for $|x - \xi_j| > R\delta_j$ we have that

$$\begin{aligned}-\Delta \tilde{Z}(x) &\geq 16\sum_{j=1}^{m} \frac{\delta_j^2 a^2}{(\delta_j^2 + a^2|x-\xi_j|^2)^2} \frac{a^2R^2 - 1}{a^2R^2 + 1} \\ &\geq 4\sum_{j=1}^{m} \frac{a^2\delta_j^2 R^4}{(1 + a^2R^2)^2} \frac{1}{|x-\xi_j|^4} \geq \frac{1}{a^2}\sum_{j=1}^{m} \frac{\delta_j^2}{|x-\xi_j|^4},\end{aligned}$$

where $\sqrt{\frac{5}{3}}a < R$, so $a^2R^2 > \frac{5}{3} > 1$ and $\frac{a^2R^2-1}{a^2R^2+1} > \frac{1}{4}$.

On the other hand, $\tilde{Z}(x) \leq 2$ and

$$K(x)\tilde{Z}(x) \leq 2D_0 \sum_{j=1}^{m} \frac{\delta_j^2}{(\delta_j^2 + |x-\xi_j|^2)^2} \leq 2D_0 \sum_{j=1}^{m} \frac{\delta_j^2}{|x-\xi_j|^4} < 0.$$

By the choice of a we have that

$$\tilde{L}(\tilde{Z}) = \Delta\tilde{Z} + K\tilde{Z} \leq \left(-\frac{1}{a^2} + 2D_0\right) \sum_{j=1}^{m} \frac{\delta_j^2}{|x-\xi_j|^4} < 0$$

and

$$\tilde{Z}(x) \geq 2\sum_{j=1}^{m} \frac{a^2R^2\delta_j^2 - \delta_j^2}{\delta_j^2 + a^2R^2\delta_j^2} \geq 2m\frac{a^2R^2 - 1}{1 + a^2R^2} > \frac{m}{2} > 0$$

for $|x - \xi_j| > R\delta_j$. Therefore, $\tilde{L}(\tilde{Z}) < 0$ and $\tilde{Z} > 0$ in $\Omega \setminus \cup_{j=1}^m B(\xi_j, R\delta_j)$ and we conclude that $\tilde{L}$ satisfies the maximum principle, namely, if $\tilde{L}(\phi) \leq 0$ in $\Omega \setminus \cup_{j=1}^m B(\xi_j, R\delta_j)$ and $\phi \geq 0$ on $\partial(\Omega \setminus \cup_{j=1}^m B(\xi_j, R\delta_j))$ then $\phi \geq 0$ in $\Omega \setminus \cup_{j=1}^m B(\xi_j, R\delta_j)$. Note that we have that the maximum principle also in the region $\cup_{j=1}^m [B(\xi_j, r/2) \setminus B(\xi_j, R\delta_j)]$, with $r < d$. □

Claim 3.3.6. *There exists a constant $C > 0$ such that*

$$\|\psi\|_{L^\infty(\cup_{j=1}^m [B(\xi_j, r/2)\setminus B(\xi_j, R\delta_j)])} \leq C[\|\psi\|_i + \|h\|_*],$$

where

$$\|\psi\|_i = \|\psi\|_{L^\infty(\cup_{j=1}^m [\partial B(\xi_j, R\delta_j) \cup \partial B(\xi_j, r/2)])}.$$

Proof: First, let us consider the functions η_j given by

$$\begin{aligned}-\Delta\eta_j &= \tfrac{2\delta_j^\sigma}{|x-\xi_j|^{2+\sigma}}, \qquad R\delta_j < |x-\xi_j| < r, \\ \eta_j(x) &= 0, \quad \text{for} \quad |x-\xi_j| = R\delta_j,\ |x-\xi_j| = r.\end{aligned}$$

A direct computations shows that

$$\eta_j(s) = -\frac{2\delta_j^\sigma}{\sigma^2 s^\sigma} + a_j \log s + b_j, \quad s = |x - \xi_j|$$

with

$$a_j = \frac{1}{\log \frac{R\delta_j}{r}} \frac{2\delta_j^\sigma}{\sigma^2}\left(\frac{1}{R^\sigma \delta_j^\sigma} - \frac{1}{r^\sigma}\right) \quad \text{and} \quad b_j = \frac{2\delta_j^\sigma}{\sigma^2 r^\sigma} + \frac{\log r}{\log \frac{R\delta_j}{r}} \frac{2\delta_j^\sigma}{\sigma^2}\left(\frac{1}{R^\sigma \delta_j^\sigma} - \frac{1}{r^\sigma}\right).$$

Observe that

$$0 \le \eta_j(s) \le -\frac{2\delta_j^\sigma}{\sigma^2 r^\sigma} + a_j \log R\delta_j + b_j = a_j \log \frac{R\delta_j}{r} \le \frac{2}{\sigma^2 R^\sigma}.$$

Now, consider the function

$$\tilde{\phi} = 2\|\psi\|_i \tilde{Z} + \|h\|_* \sum_{j=1}^m \eta_j.$$

Hence, we get that

$$\begin{aligned}
\tilde{L}(\tilde{\phi}) &\le \|h\|_* \sum_{j=1}^m \tilde{L}(\eta_j) \le \|h\|_* \sum_{j=1}^m \left[-\frac{2\delta_j^\sigma}{|x-\xi_j|^{2+\sigma}} + \frac{2D_0}{\sigma^2 R^\sigma} \sum_{l=1}^m \frac{\delta_l^2}{(\delta_l^2 + |x-\xi_l|^2)^2}\right] \\
&= \|h\|_* \sum_{j=1}^m \left[-\frac{2\delta_j^\sigma}{|x-\xi_j|^{2+\sigma}} + \frac{2D_0 m}{\sigma^2 R^\sigma} \frac{\delta_j^2}{(\delta_j^2 + |x-\xi_j|^2)^2}\right] \\
&\le -\|h\|_* \sum_{j=1}^m \frac{\delta_j^\sigma}{(\delta_j^2 + |x-\xi_j|^2)^2}
\end{aligned}$$

for R large enough ($2D_0 m \le \sigma^2 R^\sigma$). Also, we have that $2\tilde{Z} \ge m \ge 1$ and $\tilde{\phi}(x) \ge |\psi(x)|$ for all $x \in \cup_{j=1}^m [\partial B(\xi_j, R\delta_j) \cup \partial B(\xi_j, r/2)]$. By the maximum principle we conclude that $|\psi(x)| \le \tilde{\phi}(x)$ for all $x \in \cup_{j=1}^m [B(\xi_j, R\delta_j) \cup B(\xi_j, r/2)]$.. Therefore, the claim follows. □

Recall that by claim 3.3.4, $\|\psi_n\|_{L^\infty(B(\xi_j^n, R\delta_j^n))} = o(1)$ as $n \to +\infty$ for all $j = 1, \ldots, m$ and $\|\psi_n\|_{L^\infty(\Omega \setminus \cup_{j=1}^m B(\xi_j^n, r/2))} = o(1)$ as $n \to +\infty$. Hence, we conclude that $\|\psi_n\|_\infty = o(1)$ as $n \to +\infty$ which is a contradiction since by claim 3.3.1 $\|\psi_n\|_\infty > \sigma_0$. This completes the proof. □

Our main result for the problem (3.3.1) states its invertibility in the following way.

Proposition 3.3.2. *Let $d > 0$ be fixed. There exist positive numbers δ_0 and C, such that for any points $\xi_j \in \Omega$, $j = 1, \ldots, m$ satisfying* (3.3.4) *and $\delta_j > 0$ satisfying* (3.2.5) *and* (3.2.6)*, there is a unique solution $\phi \in E \cap W^{2,2}(\Omega)$ to problem* (3.3.1) *for all $\delta < \delta_0$. Moreover,*

$$\|\phi\|_\infty \le C\left(\log \frac{1}{\delta}\right)\|h\|_*, \quad |c_{ij}| \le C\|h\|_*, \quad i = 1, 2, j = 1, \ldots, m, \quad \text{and} \quad |c_0| \le C\|h\|_*. \tag{3.3.11}$$

Proof: First, observe that $\Delta Z_{ij} = -e^{U_j} Z_{ij}$ for all $i = 0, 1, 2$, $j = 1, \ldots, m$

$$\int_\Omega \Delta Z_{0j} = O(\delta_j^2) \quad \text{and} \quad \int_\Omega \Delta Z_{ij} = O(\delta_j^3), \quad i = 1, 2.$$

Since $\Delta P Z_{ij} = \Delta Z_{ij} - \frac{1}{|\Omega|}\int_\Omega \Delta Z_{ij}$, it follows that $\|\Delta P Z_{ij}\|_* \le C$ for all $i = 0, 1, 2$, $j = 1, \ldots, m$. Thus, by Proposition 3.3.1, any solution to (3.3.1) satisfies

$$\|\phi\|_\infty \le C\left(\log \frac{1}{\delta}\right)\left[\|h\|_* + \sum_{i=1}^2 \sum_{j=1}^m |c_{ij}| + |c_0|\right].$$

Let us estimates the values of constants $|c_{ij}|$. We test equation (3.3.1) against PZ_{ij}, $i = 1, 2$. Hence,

$$\langle L(\phi), PZ_{ij}\rangle = \langle h, PZ_{ij}\rangle + \sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\langle \Delta PZ_{kl}, PZ_{ij}\rangle + c_0\langle \Delta PZ, PZ_{ij}\rangle,$$

where $\langle f, g\rangle = \int_\Omega fg$. Note that $\langle L(\phi), PZ_{ij}\rangle = \langle \phi, L(PZ_{ij})\rangle$. Furthermore, we have that

$$\langle \Delta PZ, PZ_{ij}\rangle = \sum_{l=1}^{m}\langle \Delta PZ_{0l}, PZ_{ij}\rangle.$$

Hence, we get that for $i = 1, 2$

$$\langle \phi, L(PZ_{ij})\rangle = \langle h, PZ_{ij}\rangle + \sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\langle \Delta PZ_{kl}, PZ_{ij}\rangle + c_0\sum_{l=0}^{m}\langle \Delta PZ_{0l}, PZ_{ij}\rangle.$$

Given $i, k = 0, 1, 2$, $j, l = 1, \ldots, m$ let us estimate $\langle \Delta PZ_{kl}, PZ_{ij}\rangle$. Observe that

$$PZ_{ij} = Z_{ij} + O(\delta_j), \quad i = 1, 2, j = 1, \ldots, m \tag{3.3.12}$$

and

$$PZ_{0j} = Z_{0j} + 2 + O(\delta_j^2|\log \delta_j|), \quad j = 1, \ldots, m, \tag{3.3.13}$$

uniformly on compact subsets of Ω, where

$$Z_{ij}(x) = \frac{4\delta_j(x-\xi_j)_i}{\delta_j^2 + |x-\xi_j|^2} \quad \text{and} \quad Z_{0j}(x) = -2 + \frac{4\delta_j^2}{\delta_j^2 + |x-\xi_j|^2}.$$

Hence, we have that

$$\langle \Delta PZ_{kl}, PZ_{ij}\rangle = \int_\Omega \Delta Z_{kl} PZ_{ij}.$$

For $i = 1, 2$ we get that

$$\langle \Delta PZ_{kl}, PZ_{ij}\rangle = -\int_\Omega e^{U_l} Z_{kl}(Z_{ij} + O(\delta_j))$$

so, if $l \neq j$ then (for $r < d/2$)

$$\int_\Omega e^{U_l} Z_{kl} Z_{ij} = \int_{B(\xi_l, r)} e^{U_l} Z_{kl} Z_{ij} + O(\delta_l^2) = O(\delta_j) + O(\delta_l^2)$$

since $e^{U_l} = O(\delta_l^2)$ in $\Omega \setminus B(\xi_l, r)$ and $Z_{ij} = O(\delta_j)$ in $B(\xi_l, r)$. Now, if $l = j$ then

$$\int_\Omega e^{U_j} Z_{kj} Z_{ij} = \int_{B(\xi_j, r)} e^{U_j} Z_{kj} Z_{ij} + O(\delta_j^2) \quad \text{and if } k \neq i \text{ then} \quad \int_{B(\xi_j, r)} e^{U_j} Z_{kj} Z_{ij} = 0.$$

If $l = j$ and $k = i$, we then get that

$$\langle \Delta PZ_{ij}, PZ_{ij}\rangle = -\int_\Omega e^{U_j} Z_{ij}^2 + O(\delta_j) = -\int_{B(\xi_j, r)} e^{U_j} Z_{ij}^2 + O(\delta_j^2) + O(\delta_j)$$

and

$$\int_{B(\xi_j, r)} e^{U_j} Z_{ij}^2 = \int_{B(0, \frac{r}{\delta_j})} \frac{8}{(1+|y|^2)^2} Y_i^2(y)\, dy = 128 \int_{B(0, \frac{r}{\delta_j})} \frac{y_i^2}{(1+|y|^2)^4}\, dy$$
$$= \frac{32}{3}\pi + O(\delta_j^4).$$

Now, similarly as above, for $i=0$ we get that if $j\neq l$ then

$$\langle \Delta PZ_{kl}, PZ_{0j}\rangle = O(\delta^2|\log\delta|), \quad k=0,1,2, j=1,\dots,m.$$

If $j=l$ and $k=1,2$ then

$$\langle \Delta PZ_{kl}, PZ_{0j}\rangle = O(\delta^2|\log\delta|).$$

And if $j=l$, $k=0$ then

$$\int_{B(\xi_l,r)} e^{U_l} Z_{0l}(Z_{0l}+2) = \int_{B(0,\frac{r}{\delta_l})} \frac{8}{(1+|y|^2)^2} Y_0(y)[Y_0(y)+2] = \frac{32\pi}{3} + O(\delta^4)$$

Thus, we conclude that for all $i,k=0,1,2$, $j,l=1,\dots,m$,

$$\langle \Delta PZ_{kl}, PZ_{ij}\rangle = \begin{cases} -\frac{32\pi}{3} + O(\delta), & \text{if } j=l,\ i=k \\ O(\delta), & \text{otherwise.}\end{cases}$$

Hence, we get that for $i=1,2$

$$\begin{aligned}|c_{ij}|\,|\langle \Delta PZ_{ij}, PZ_{ij}\rangle| \le{}& C\|\phi\|_\infty \, \|L(PZ_{ij})\|_* + C\|h\|_*\, \|PZ_{ij}\|_\infty + \sum_{kl\neq ij} |c_{kl}|\,|\langle \Delta PZ_{kl}, PZ_{ij}\rangle| \\ &+ |c_0| \sum_{l=0}^{m} |\langle \Delta PZ_{0l}, PZ_{ij}\rangle| \\ \le{}& C\left[\|\phi\|_\infty \, \|L(PZ_{ij})\|_* + \|h\|_* + \delta \sum_{kl\neq ij} |c_{kl}| + \delta|c_0|\right] \\ \le{}& C\Bigg[\left(\log\frac{1}{\delta}\right)\left(\|h\|_* + \sum_{k=1}^{2}\sum_{l=1}^{m}|c_{kl}| + |c_0|\right)\|L(PZ_{ij})\|_* + \|h\|_* \\ &+ \delta\left(\sum_{k=1}^{2}\sum_{l=1}^{m}|c_{kl}| + |c_0|\right)\Bigg].\end{aligned}$$

Let us estimate $\|L(PZ_{ij})\|_*$, for $i=1,2$. We know that

$$L(PZ_{ij}) = \Delta Z_{ij} - \frac{1}{|\Omega|}\int_\Omega \Delta Z_{ij} + \frac{\lambda k e^W}{\int_\Omega k e^W}\left(PZ_{ij} - \frac{\int_\Omega k e^W PZ_{ij}}{\int_\Omega k e^W}\right)$$

and

$$\int_\Omega k e^W PZ_{ij} = \int_\Omega k e^W [Z_{ij} + O(\delta)] = \int_{B(\xi_j,r)} k e^W Z_{ij} + O\left(\frac{1}{\delta}\right),$$

since $\int_{\Omega\setminus B(\xi_j,r)} k e^W = O(\delta^{-2})$ and $Z_{ij} = O(\delta)$ in $\Omega\setminus B(\xi_j,r)$. Now, we have that

$$\int_{B(\xi_j,r)} k e^W Z_{ij} = \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} e^{U_j}\rho_j(1+O(\delta^2|\log\delta|))\, Z_{ij}$$

and

$$\begin{aligned}\int_{B(\xi_j,r)} e^{U_j}\rho_j Z_{ij} &= \int_{B(\xi_j,r)} \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\,\frac{4\delta_j(x-\xi_j)_i}{\delta_j^2+|x-\xi_j|^2}\rho_j(x)\,dx \\ &= \int_{B(0,\frac{r}{\delta_j})} \frac{32 y_i}{(1+|y|^2)^3}[\rho_j(\xi_j) + O(\delta_j|y|)]\,dy \\ &= O(\delta).\end{aligned}$$

Hence, we conclude that

$$\int_\Omega ke^W PZ_{ij} = O\Big(\frac{1}{\delta}\Big) \qquad \text{and} \qquad \frac{\int_\Omega ke^W PZ_{ij}}{\int_\Omega ke^W} = O(\delta).$$

Since $\int_\Omega \Delta Z_{ij} = O(\delta^3)$, we get that

$$L(PZ_{ij}) = -e^{U_j} Z_{ij} + \frac{\lambda k e^W}{\int_\Omega ke^W}\big(Z_{ij} + O(\delta)\big) + O(\delta^3).$$

Recall that $Z_{ij} = O(\delta)$ in $\Omega \setminus B(\xi_j, r)$, then $L(PZ_{ij}) = O(\delta^3)$ in $\Omega \setminus \cup_{l=1}^m B(\xi_l, r)$.

On the other hand, recall that for $x \in B(\xi_l, r)$ we have that

$$\frac{\lambda k e^W}{\int_\Omega ke^W} = e^{U_l(x)}\left[1 + \frac{\rho_l(x) - \rho_l(\xi_l)}{\rho_l(\xi_l)} + O(\delta^2|\log\delta|)\right].$$

Then, for $l \neq j$ we obtain that

$$L(PZ_{ij}) = e^{U_l} O(\delta) + O(\delta^3)$$

in $B(\xi_l, r)$, and for $x \in B(\xi_j, r)$ we find that

$$\begin{aligned} L(PZ_{ij})(x) &= e^{U_j}\left[-Z_{ij} + \big(1 + O(|x-\xi_j|) + O(\delta^2|\log\delta|)\big)\big(Z_{ij} + O(\delta)\big)\right] + O(\delta^3)\\ &= e^{U_j}\left[Z_{ij}\big\{O(|x-\xi_j|) + O(\delta^2|\log\delta|)\big\} + O(\delta)\big)\right] + O(\delta^3)\\ &= e^{U_j}\left[O(|x-\xi_j|) + O(\delta)\right] + O(\delta^3). \end{aligned}$$

Thus, from the definition of *-norm we conclude that $\|L(PZ_{ij})\|_* \le C\delta$ for $i = 1, 2$ $j = 1, \dots, m$.

Now, since $|\langle \Delta PZ_{ij}, Z_{ij}\rangle| \ge 8\pi$, it follows that

$$\begin{aligned} |c_{ij}| &\le C\left[\delta|\log\delta|\left(\|h\|_* + \sum_{k=1}^2\sum_{l=1}^m |c_{kl}| + |c_0|\right) + \|h\|_* + \delta\left(\sum_{k=1}^2\sum_{l=1}^m |c_{kl}| + |c_0|\right)\right]\\ &\le C\left[\delta|\log\delta|\left(\sum_{k=1}^2\sum_{l=1}^m |c_{kl}| + |c_0|\right) + \|h\|_*\right]. \end{aligned} \tag{3.3.14}$$

Let us estimate $|c_0|$. We test equation (3.3.1) against PZ and similarly as above, we get that

$$\langle \phi, L(PZ)\rangle = \langle h, PZ\rangle + \sum_{k=1}^2\sum_{l=1}^m\sum_{j=1}^m c_{kl}\langle \Delta PZ_{kl}, PZ_{0j}\rangle + c_0\sum_{j=1}^m\sum_{l=1}^m \langle \Delta PZ_{0j}, PZ_{0l}\rangle$$

and

$$\begin{aligned} |c_0|\left|\sum_{j=1}^m\sum_{l=1}^m \langle \Delta PZ_{0j}, PZ_{0l}\rangle\right| &\le C\|\phi\|_\infty\,\|L(PZ)\|_* + C\|h\|_*\,\|PZ\|_\infty\\ &\quad + \sum_{k=1}^2\sum_{j=1}^m\sum_{l=1}^m |c_{kl}|\,|\langle \Delta PZ_{kl}, PZ_{0j}\rangle|\\ &\le C\Bigg[\left(\log\frac{1}{\delta}\right)\left(\|h\|_* + \sum_{k=1}^2\sum_{l=1}^m |c_{kl}| + |c_0|\right)\|L(PZ)\|_* + \|h\|_*\\ &\quad + \delta\sum_{k=1}^2\sum_{l=1}^m |c_{kl}|\Bigg]. \end{aligned}$$

Let us estimate $\|L(PZ)\|_*$. By using (3.3.13), we have that

$$\int_\Omega ke^W PZ_{0j} = \int_\Omega ke^W \left[Z_{0j} + 2 + O(\delta^2|\log\delta|)\right] = \int_{B(\xi_j,r)} ke^W(Z_{0j}+2) + O(|\log\delta|),$$

since $\int_{\Omega\setminus B(\xi_j,r)} ke^W = O(\delta^{-2})$ and $Z_{0j}+2 = O(\delta_j^2)$ in $\Omega\setminus B(\xi_j,r)$. So, we estimate

$$\int_{B(\xi_j,r)} ke^W(Z_{0j}+2) = \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} e^{U_j}\rho_j(1+O(\delta^2|\log\delta|))(Z_{0j}+2)$$

and

$$\begin{aligned}\int_{B(\xi_j,r)} e^{U_j}\rho_j(Z_{0j}+2) &= \int_{B(0,\frac{r}{\delta_j})} \frac{32}{(1+|y|^2)^3}\rho_j(\xi_j+\delta_j y)\,dy\\ &= \int_{B(0,\frac{r}{\delta_j})} \frac{32}{(1+|y|^2)^3}\left[\rho_j(\xi_j) + \nabla\rho_j(\xi_j)\cdot\delta_j y + O(\delta_j^2|y|^2)\right]dy\\ &= 16\pi\rho_j(\xi_j) + O(\delta_j^2).\end{aligned}$$

Also, we have that

$$\int_\Omega ke^W PZ_{0j} = \frac{1}{8\delta_j^2}[16\pi\rho_j(\xi_j) + O(\delta_j^2)] = \frac{2\pi}{\delta^2} + O(|\log\delta|).$$

Then, it follows that

$$\frac{\int_\Omega ke^W PZ_{0j}}{\int_\Omega ke^W} = \frac{\frac{2\pi}{\delta^2} + O(|\log\delta|)}{\frac{\pi m}{\delta^2} + O(|\log\delta|)} = \frac{2}{m} + O(\delta^2|\log\delta|).$$

Since, $\int_\Omega \Delta Z_{0j} = O(\delta_j^2)$ we get that

$$\begin{aligned}L(PZ) &= \sum_{j=1}^m\left(\Delta Z_{0j} - \frac{1}{|\Omega|}\int_\Omega \Delta Z_{0j}\right) + \frac{\lambda ke^W}{\int_\Omega ke^W}\left(\sum_{j=1}^m PZ_{0j} - \sum_{j=1}^m \frac{\int_\Omega ke^W PZ_{0j}}{\int_\Omega ke^W}\right)\\ &= -\sum_{j=1}^m e^{U_j}Z_{0j} + \frac{\lambda ke^W}{\int_\Omega ke^W}\left(\sum_{j=1}^m (Z_{0j}+2) - 2 + O(\delta^2|\log\delta|)\right) + O(\delta^2).\end{aligned}$$

Now, we know that $Z_{0j}+2 = O(\delta_j^2)$ in $\Omega\setminus B(\xi_j,r)$. Then, $L(PZ) = O(\delta^2)$ in $\Omega\setminus\cup_{j=1}^m B(\xi_j,r)$. If $x\in B(\xi_j,r)$ for some $j\in\{1,\dots,m\}$ then

$$\begin{aligned}L(PZ) &= e^{U_j}\left[-Z_{0j} + \left(1 + O(|x-\xi_j|) + O(\delta^2|\log\delta|)\right)\left(Z_{0j} + O(\delta^2|\log\delta|)\right)\right] + O(\delta^2)\\ &= e^{U_j}\left[Z_{0j}\left\{O(|x-\xi_j|) + O(\delta^2|\log\delta|)\right\} + O(\delta^2|\log\delta|)\right] + O(\delta^2).\end{aligned}$$

From the definition of $\|\cdot\|_*$, we conclude that $\|L(PZ)\|_* \le C\delta$.

Now, since $\langle \Delta PZ_{0j}, PZ_{0j}\rangle = -\frac{32\pi}{3} + O(\delta)$ and $\langle \Delta PZ_{0l}, PZ_{0j}\rangle = O(\delta)$ for all $j\neq l$, we get that $|\langle \Delta PZ, PZ\rangle| \ge 8\pi m$ for all $\delta>0$ small enough and it follows that

$$|c_0| \le C\left[\delta\log\frac{1}{\delta}\left(\sum_{k=1}^2\sum_{l=1}^m |c_{kl}| + |c_0|\right) + \|h\|_*\right]. \tag{3.3.15}$$

Combining (3.3.14) and (3.3.15) we obtain that $|c_{ij}| \le C\|h\|_*$ for all $i = 1, 2,\ j = 1, \dots, m$ and $|c_0| \le C\|h\|_*$. It follows that $\|\phi\|_\infty \le C(\log \frac{1}{\delta})\|h\|_*$ and the priori estimate has been thus proven. It only remains to prove the solvability assertion. To this purpose we consider the space

$$H = \left\{ \phi \in E \,:\, \int_\Omega \Delta P Z_{ij}\, \phi = 0,\ i = 1, 2, j = 1, \dots, m,\ \int_\Omega \Delta P Z\, \phi = 0 \right\},$$

endowed with the usual inner product $[\phi, \psi] = \int_\Omega \nabla\phi\nabla\psi$. Problem (3.3.1) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi, \psi] = \int_\Omega \left[\frac{\lambda k e^W}{\int_\Omega k e^W} \left(\phi - \frac{\int_\Omega k e^W \phi}{\int_\Omega k e^W} \right) - h \right] \psi, \qquad \text{for all } \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = \mathcal{K}(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where $\mathcal{K}$ is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = \mathcal{K}(\phi)$ has only the zero solution in H. This last equation is equivalent to (3.3.1) with $h \equiv 0$. Thus, existence of a unique solution follows from the a priori estimate (3.3.11). This completes the proof.

□

Remark 3.3.1. Given $f \in L^2_\#(\Omega) := \{u \in L^2(\Omega) \,:\, \int_\Omega u = 0\}$ denote $u = \Delta^{-1} f$ such that $u \in E$ and $\Delta u = f$ in Ω. Then, $\Delta^{-1} : L^2_\#(\Omega) \to E \hookrightarrow L^2_\#(\Omega)$ is compact and we have that

$$\mathcal{K}(\phi) = \Delta^{-1} \left[\frac{\lambda k e^W}{\int_\Omega k e^W} \left(\phi - \frac{\int_\Omega k e^W \phi}{\int_\Omega k e^W} \right) \right], \qquad \tilde{h} = \Delta^{-1}(-h).$$

Thus, $\mathcal{K} : L^2_\#(\Omega) \to L^2_\#(\Omega)$ is compact.

The result of latter proposition implies that the unique solution $\phi = T(h)$ of (3.3.1) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < +\infty$, into L^∞ (with $\int_\Omega h = \int_\Omega \phi = 0$).

It is possible to show that T is differentiable with respect to either $\beta = \xi_{kl}$, $k = 1, 2$, $l = 1, \dots, m$ or $\beta = \delta$. From equation (3.3.1), we formally get that $X = \partial_\beta \phi$ should satisfy

$$L(X) = \tilde{h}(\phi) + \sum_{i,j} d_{ij} \Delta P Z_{ij} + d_0 \Delta P Z,$$

where

$$\begin{aligned} \tilde{h}(\phi) = &- \partial_\beta \left(\frac{\lambda k e^W}{\int_\Omega k e^W} \right) \phi + \partial_\beta \left[\frac{\lambda k e^W}{\left(\int_\Omega k e^W\right)^2} \right] \int_\Omega k e^W \phi + \frac{\lambda k e^W}{\left(\int_\Omega k e^W\right)^2} \int_\Omega k e^W \partial_\beta W \phi \\ &+ \sum_{i,j} c_{ij} \partial_\beta (\Delta P Z_{ij}) + c_0 \partial_\beta (\Delta P Z), \end{aligned}$$

and $d_{ij} = \partial_\beta c_{ij}$, $d_0 = \partial_\beta c_0$. The orthogonality conditions now become

$$\int_\Omega X \Delta P Z_{ij} = - \int_\Omega \phi\, \partial_\beta (\Delta P Z_{ij}) \qquad \text{and} \qquad \int_\Omega X \Delta P Z = - \int_\Omega \phi\, \partial_\beta (\Delta P Z).$$

We will recast X. We consider the function

$$Y = X + \sum_{i=1}^{2}\sum_{j=1}^{m} b_{ij} PZ_{ij} + b_0 PZ,$$

where the coefficients b_{ij}, $i = 1, 2$ $j = 1, \dots, m$ and b_0 are chosen in order to satisfy the orthogonality conditions

$$\int_\Omega Y\,\Delta PZ_{pq} = 0, \qquad p = 1, 2,\ q = 1, \dots, m \qquad \text{and} \qquad \int_\Omega Y\,\Delta PZ = 0.$$

Note that

$$\int_\Omega X = \int_\Omega Y = 0.$$

Let us observe that b_{ij}, $i = 1, 2$ $j = 1, \dots, m$ and b_0 are well defined, since they satisfy an almost diagonal system. Also, we get that

$$|b_{ij}| \le \frac{C}{\delta}\left(\log\frac{1}{\delta}\right)\|h\|_* \qquad \text{and} \qquad |b_0| \le \frac{C}{\delta}\left(\log\frac{1}{\delta}\right)\|h\|_*.$$

Indeed, consider the vectors $\mathbf{v} = (b_{11}, \dots, b_{1m}, b_{21}, \dots, b_{2m}, b_0)$,

$$\mathbf{w} = \Big(\int_\Omega \phi\,\partial_\beta(\Delta PZ_{11}), \dots, \int_\Omega \phi\,\partial_\beta(\Delta PZ_{1m}), \int_\Omega \phi\,\partial_\beta(\Delta PZ_{21}), \dots,$$
$$\int_\Omega \phi\,\partial_\beta(\Delta PZ_{2m}), \int_\Omega \phi\,\partial_\beta(\Delta PZ)\Big)$$

and the matrix $A = (a_{ij})_{1\le i,j\le m}$ given by

$$a_{ij} = \begin{cases} \int_\Omega \Delta PZ_{1i}\,PZ_{1j}, & 1 \le i, j \le m \\ \int_\Omega \Delta PZ_{1i}\,PZ_{2j}, & 1 \le i \le m,\ m+1 \le j \le 2m \\ \int_\Omega \Delta PZ_{2i}\,PZ_{1j}, & m+1 \le i \le 2m,\ 1 \le j \le m \\ \int_\Omega \Delta PZ_{2i}\,PZ_{2j}, & m+1 \le i, j \le 2m \\ \int_\Omega \Delta PZ\,PZ_{1j}, & i = 2m+1,\ 1 \le j \le 2m \\ \int_\Omega \Delta PZ\,PZ_{2j}, & i = 2m+1,\ m+1 \le j \le 2m \\ \int_\Omega \Delta PZ_{1i}\,PZ, & j = 2m+1,\ 1 \le i \le m \\ \int_\Omega \Delta PZ_{2i}\,PZ, & j = 2m+1,\ m+1 \le i \le 2m \\ \int_\Omega \Delta PZ\,PZ, & i = 2m+1, j = 2m+1 \end{cases}.$$

We know that $a_{ij} = O(\delta)$ if $i \ne j$ and

$$a_{ii} = \begin{cases} -\frac{32\pi}{3} + O(\delta), & i = 1, \dots, 2m \\ -\frac{32\pi m}{3} + O(\delta), & i = 2m+1 \end{cases}.$$

Thus, we have that $A\mathbf{v} = \mathbf{w}$ has a unique solution and we get that A is an almost diagonal matrix, so, it is invertible (for δ small enough) and we can deduce that

$$\|\mathbf{v}\|_{\mathbb{R}^{2m+1}} \le C\|\mathbf{w}\|_{\mathbb{R}^{2m+1}} \le C\max\left\{\left|\int_\Omega \phi\,\partial_\beta(\Delta PZ_{ij})\right| \,:\, i = 0, 1, 2,\ j = 1, \dots, m\right\}$$

since $PZ = \sum_{j=1}^m PZ_{0j}$. Also, we obtain

$$\left|\int_\Omega \phi\, \partial_\beta(\Delta PZ_{ij})\right| \le C\|\phi\|_\infty \,\|\partial_\beta(\Delta PZ_{ij})\|_*$$

and

$$\partial_\beta(\Delta PZ_{ij}) = -e^{U_j}[\partial_\beta U_j Z_{ij} + \partial_\beta Z_{ij}] + \frac{1}{|\Omega|}\int_\Omega e^{U_j}[\partial_\beta U_j Z_{ij} + \partial_\beta Z_{ij}].$$

For $\beta = \xi_{kl}$ we get that $\partial_{\xi_{kl}}(\rho_j(\xi_j)) = \rho_j(\xi_j)\partial_{2k}G(\xi_j,\xi_l)$ if $j \ne l$ and $\partial_{\xi_{kl}}(\rho_l(\xi_l)) = \partial_k\rho_l(\xi_l)$, since $\partial_{2k}H(\xi_l,\xi_l) = 0$. Hence, we deduce that

$$\partial_{\xi_{kl}}U_l(x) = \frac{4(x-\xi_l)_k}{\delta_l^2 + |x-\xi_l|^2} + \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{|x-\xi_l|^2-\delta_l^2}{\delta_l^2+|x-\xi_l|^2},$$

if $j \ne l$

$$\partial_{\xi_{kl}}U_j(x) = \partial_{2k}G(\xi_j,\xi_l)\frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2},$$

$$\partial_{\xi_{kl}}Z_{0l}(x) = \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{4\delta_l^2|x-\xi_l|^2}{(\delta_l^2+|x-\xi_l|^2)^2} + \frac{8\delta_l^2(x-\xi_l)_k}{(\delta_l^2+|x-\xi_l|^2)^2},$$

if $j \ne l$

$$\partial_{\xi_{kl}}Z_{0j}(x) = \partial_{2k}G(\xi_j,\xi_l)\frac{4\delta_j^2|x-\xi_j|^2}{(\delta_j^2+|x-\xi_j|^2)^2},$$

and for $i = 1, 2$

$$\partial_{\xi_{kl}}Z_{il}(x) = \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{2\delta_l(x-\xi_l)_i}{\delta_l^2+|x-\xi_l|^2}\frac{|x-\xi_l|^2-\delta_l^2}{\delta_l^2+|x-\xi_l|^2} - \delta_{ik}\frac{4\delta_l}{\delta_l^2+|x-\xi_l|^2} + \frac{8\delta_l(x-\xi_l)_i(x-\xi_l)_k}{(\delta_l^2+|x-\xi_l|^2)^2},$$

if $j \ne l$

$$\partial_{\xi_{kl}}Z_{ij}(x) = \partial_{2k}G(\xi_j,\xi_l)\frac{2\delta_j(x-\xi_j)_i}{\delta_j^2+|x-\xi_j|^2}\frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2},$$

since (3.2.5). Note that $\|Z_{ij}\|_\infty \le C$ for all $i = 0,1,2$ $j = 1,\dots,m$, $\|\partial_{\xi_{kl}}U_l\|_\infty \le \frac{C}{\delta}$, $\|\partial_{\xi_{kl}}U_j\|_\infty \le C$ for all $j \ne l$, $\|\partial_{\xi_{kl}}Z_{il}\|_\infty \le \frac{C}{\delta}$ for all $i = 0,1,2$ and $\|\partial_{\xi_{kl}}Z_{ij}\|_\infty \le C$, for all $i = 0,1,2$, $j \ne l$. Then, for all $j \ne l$

$$\|e^{U_j}\partial_{\xi_{kl}}U_jZ_{ij}\|_\infty \le \|e^{U_j}\|_*\|\partial_{\xi_{kl}}U_jZ_{ij}\|_\infty \le C$$

and

$$\|e^{U_j}\partial_{\xi_{kl}}Z_{ij}\|_\infty \le \|e^{U_j}\|_*\|\partial_{\xi_{kl}}Z_{ij}\|_\infty \le C.$$

Now, if $j = l$ then we get

$$\|e^{U_l}\partial_{\xi_{kl}}U_lZ_{il}\|_\infty \le \|e^{U_l}\|_*\|\partial_{\xi_{kl}}U_lZ_{il}\|_\infty \le \frac{C}{\delta}$$

and

$$\|e^{U_l}\partial_{\xi_{kl}}Z_{il}\|_\infty \le \|e^{U_l}\|_*\|\partial_{\xi_{kl}}Z_{il}\|_\infty \le \frac{C}{\delta}$$

for all $i = 0,1,2$. Let us estimates the integrals, if $i \ne k$ then

$$\int_\Omega e^{U_l}\partial_{\xi_{kl}}U_l\, Z_{il} = \int_{\Omega\setminus B(\xi_l,r)} e^{U_l}\partial_{\xi_{kl}}U_l\, Z_{il} = O(\delta^3)$$

and

$$\int_\Omega e^{U_l} \partial_{\xi_{kl}} Z_{il} = \int_{\Omega \setminus B(\xi_l, r)} e^{U_l} \partial_{\xi_{kl}} Z_{il} = O(\delta^3),$$

for $i = 0$ we have that

$$\int_\Omega e^{U_l} \partial_{\xi_{kl}} U_l \, Z_{0l} = \int_{B(\xi_l, r)} e^{U_l} \partial_{\xi_{kl}} U_l \, Z_{il} + O(\delta^2) = O(1)$$

and

$$\int_\Omega e^{U_l} \partial_{\xi_{kl}} Z_{0l} = \int_{B(\xi_l, r)} e^{U_l} \partial_{\xi_{kl}} Z_{il} + O(\delta^4) = O(1).$$

Now, if $i = k$ then

$$\int_\Omega e^{U_l} \partial_{\xi_{kl}} U_l \, Z_{il} = \int_{B(\xi_l, r)} e^{U_l} \frac{16\delta_l (x - \xi_l)_k^2}{(\delta_l^2 + |x - \xi_l|^2)^2} \, dx + O(\delta^3)$$

and

$$\int_\Omega e^{U_l} \partial_{\xi_{kl}} Z_{il} = \int_{B(\xi_l, r)} e^{U_l} \left[-\frac{4\delta_l}{\delta_l^2 + |x - \xi_l|^2} + \frac{8\delta_l (x - \xi_l)_k^2}{(\delta_l^2 + |x - \xi_l|^2)^2} \right] dx + O(\delta^3).$$

Hence, we get that

$$\begin{aligned} \int_\Omega e^{U_l} \left[\partial_{\xi_{kl}} U_l \, Z_{il} + \partial_{\xi_{kl}} Z_{il} \right] &= \int_{B(\xi_l, r)} e^{U_l} \left[-\frac{4\delta_l}{\delta_l^2 + |x - \xi_l|^2} + \frac{24\delta_l (x - \xi_l)_k^2}{(\delta_l^2 + |x - \xi_l|^2)^2} \right] dx + O(\delta^3) \\ &= O(\delta^3). \end{aligned}$$

Therefore, definition of *-norm we get that

$$\|\partial_{\xi_{kl}} (\Delta P Z_{ij})\|_* \le \frac{C}{\delta}, \quad \text{for all } i = 0, 1, 2, \; k = 1, 2, \; j, l = 1, \ldots, m.$$

Now, for $\beta = \delta$ we get that

$$\partial_\delta U_j(x) = \frac{2}{\delta} \frac{|x - \xi_l|^2 - \delta_l^2}{\delta_l^2 + |x - \xi_l|^2}, \qquad \partial_\delta Z_{0j}(x) = \frac{1}{\delta} \frac{8\delta_j^2 |x - \xi_j|^2}{\delta_j^2 + |x - \xi_j|^2},$$

and for $i = 1, 2$

$$\partial_\delta Z_{ij}(x) = \frac{1}{\delta} \frac{4\delta_j (x - \xi_j)_i}{\delta_j^2 + |x - \xi_j|^2} \frac{|x - \xi_j|^2 - \delta_j^2}{\delta_j^2 + |x - \xi_j|^2}.$$

Hence, similarly as above, we get that $\|\partial_\delta(\Delta P Z_{ij})\|_* \le \frac{C}{\delta}$ for all $i = 0, 1, 2, \; j = 1, \ldots, m$. Thus, we conclude that

$$|b_{ij}| \le \frac{C}{\delta} \|\phi\|_\infty \le \frac{C}{\delta} \left(\log \frac{1}{\delta} \right) \|h\|_* \qquad \text{for all } i = 0, 1, 2, \; j = 1, \ldots, m$$

and

$$|b_0| \le \frac{C}{\delta} \|\phi\|_\infty \le \frac{C}{\delta} \left(\log \frac{1}{\delta} \right) \|h\|_*.$$

Hence, the function X above can be uniquely expressed as

$$X = T(f) - \sum_{i=1}^{2} \sum_{j=1}^{m} b_{ij} P Z_{ij} - b_0 P Z,$$

namely, $Y = T(f)$, where

$$f = \tilde{h}(\phi) + \sum_{i=1}^{2}\sum_{j=1}^{m} b_{ij} L(PZ_{ij}) + b_0 L(PZ).$$

This computations is not just formal. Arguing directly by definition it shows that indeed $\partial_\beta \phi = X$ for either $\beta = \xi_{kl}$ or $\beta = \delta$. Moreover, we find that

$$\|f\|_* \le \|\tilde{h}(\phi)\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m} |b_{ij}|\, \|L(PZ_{ij})\|_* + |b_0|\, \|L(PZ)\|_*.$$

From the definition of $\tilde{h}(\phi)$ we have that

$$\begin{aligned}\|\tilde{h}(\phi)\|_* \le \|\partial_\beta K\|_*\, \|\phi\|_\infty + \left\| \partial_\beta \left(\frac{K}{\int_\Omega ke^W} \right) \right\|_* \|\phi\|_\infty \int_\Omega ke^W + \|K\|_* \|\partial_\beta W\|_\infty \|\phi\|_\infty \\ + \sum_{i,j} |c_{ij}|\, \|\partial_\beta(\Delta PZ_{ij})\|_* + |c_0|\, \|\partial_\beta(\Delta PZ)\|_*.\end{aligned}$$

In order to have good estimates, we need to know the derivatives $\partial_\beta W$ whether either $\beta = \xi_{kl}$ or $\beta = \delta$. Using the integral representation (2.2.4), it is possible to show that

$$\begin{aligned}\partial_{\xi_{kl}} W(x) = \frac{4(x-\xi_l)_k}{\delta_l^2 + |x-\xi_l|^2} - \frac{\partial_k \rho_l(\xi_l)}{\rho_l(\xi_l)} \frac{2\delta_l^2}{\delta_l^2 + |x-\xi_l|^2} + \partial_{2k} H(x,\xi_l) \\ - \sum_{j=1, j\ne l}^{m} \partial_{2k} G(\xi_j, \xi_l) \frac{2\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + O(\delta^2 |\log\delta|)\end{aligned} \tag{3.3.16}$$

and

$$\partial_\delta W(x) = -\frac{1}{\delta} \sum_{j=1}^{m} \frac{4\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + O(\delta|\log\delta|). \tag{3.3.17}$$

Hence, it readily follows that $\|\partial_\beta W\|_\infty \le \frac{C}{\delta}$ for either $\beta = \xi_{kl}$ or $\beta = \delta$. Now, we find that

$$\partial_\beta K = K\left(\partial_\beta W - \frac{\int_\Omega ke^W \partial_\beta W}{\int_\Omega ke^W} \right) \quad \text{and} \quad \|\partial_\beta K\|_* \le 2\|K\|_* \|\partial_\beta W\|_\infty \le \frac{C}{\delta},$$

since $\|K\|_* \le C$. Also, we have that

$$\left\| \partial_\beta \left[\frac{K}{\int_\Omega ke^W} \right] \right\|_* \le 3 \frac{\|K\|_*}{\int_\Omega ke^W} \|\partial_\beta W\|_\infty \le \frac{C}{\delta} \frac{1}{\int_\Omega ke^W}.$$

From Proposition 3.3.2 and the previous estimates for $\|\partial_\beta(\Delta PZ_{ij})\|_*$ and $\|\partial_\beta(\Delta PZ)\|_*$ we get that

$$\|\tilde{h}(\phi)\|_* \le \frac{C}{\delta}\|\phi\|_\infty + \frac{C}{\delta}\|h\|_* \le \frac{C}{\delta}\left(\log\frac{1}{\delta}\right)\|h\|_*.$$

Recall that $\|L(PZ_{ij})\|_* \le C\delta$ for $i = 1, 2$, $j = 1, \ldots, m$ and $\|L(PZ)\|_* \le C\delta$. Hence,

$$\begin{aligned}\|X\|_\infty &\le \|T(f)\|_\infty + \sum_{i=1}^{2}\sum_{j=1}^{m} |b_{ij}|\, \|PZ_{ij}\|_\infty + |b_0|\, \|PZ\|_\infty \\ &\le C\left(\log\frac{1}{\delta}\right)\|f\|_* + \frac{C}{\delta}\|\phi\|_\infty.\end{aligned}$$

Therefore, we conclude that for either $\beta = \xi_{kl}$ or $\beta = \delta$ with $k = 1, 2$, $l = 1, \dots, m$

$$\|\partial_\beta T(h)\|_\infty \le \frac{C}{\delta}\left(\log\frac{1}{\delta}\right)^2 \|h\|_*. \tag{3.3.18}$$

From previous estimates and arguments we deduce that $\partial_\delta T$ is differentiable with respect to δ. We formally differentiate the equation

$$\partial_\delta T(h) = T(f) - \sum_{i=1}^{2}\sum_{j=1}^{m} b_{ij} PZ_{ij} - b_0 PZ$$

with respect to δ. Similarly as above, we estimate every term and finally, we obtain that

$$\|\partial_{\delta\delta} T(h)\|_\infty \le \frac{C}{\delta^2}\left(\log\frac{1}{\delta}\right)^3 \|h\|_*. \tag{3.3.19}$$

3.4 The nonlinear problem

In what follows we will solve a nonlinear problem. Recall that our goal is to solve (3.2.8). Instead of solve directly the problem (3.2.8) we shall solve an intermediate problem. First, we construct a function ϕ_0 which will be the main order in the remainder term, namely, we look for a solution $u = W + \phi$ and we expand $\phi = \phi_0 + \phi_1$.

Note that as $\delta \to 0$ for $|y| < R'$, $R' > 0$ some large constant, we have that

$$\delta_j R(\xi_j + \delta_j y) = \delta_j R_0(\xi_j + \delta_j y) + o(1) = \frac{8}{(1+|y|^2)^2}\frac{\nabla\rho_j(\xi_j)}{\rho_j(\xi_j)}\cdot y + o(1).$$

In fact, $\xi_j + \delta_j y \in B(\xi_j, r)$ for some $r > 0$ fixed and

$$\delta_j R(\xi_j + \delta_j y) = \delta_j e^{U_j(\xi_j+\delta_j y)}\left[\frac{\nabla\rho_j(\xi_j+\delta_j y)}{\rho_j(\xi_j)}\cdot \delta_j y + O(\delta_j^2|y|^2) + O(\delta^2|\log\delta|)\right] + \delta_j O(\delta^2|\log\delta|).$$

Hence, roughly speaking ϕ_0 should satisfy $\delta_j\phi_0(\xi_j + \delta_j y) \sim \Psi(y)$ around ξ_j, where Ψ is a solution to

$$\Delta\Psi + \frac{8}{(1+|y|^2)^2}\left(\Psi - \frac{1}{\pi m}\int_{\mathbb{R}^2}\frac{8}{(1+|z|^2)^2}\Psi(z)\,dz\right) = -\frac{8}{(1+|y|^2)^2}\frac{\nabla\rho_j(\xi_j)}{\rho_j(\xi_j)}\cdot y,$$

up to orthogonal conditions, in $\mathbb{R}^2$.

Consider the function

$$\Psi_i(y) = \frac{2y_i}{1+|y|^2}\log(1+|y|^2), \qquad i = 1, 2.$$

This function satisfies in $\mathbb{R}^2$

$$\Delta\Psi_i + \frac{8}{(1+|y|^2)^2}\Psi_i = -\frac{8y_i}{(1+|y|^2)^2} + \frac{24y_i}{(1+|y|^2)^3}$$

and

$$\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2}\Psi_i(y)\,dy = 0.$$

Note that Ψ_i is bounded in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2}\left[-\frac{8y_i}{(1+|y|^2)^2} + \frac{24y_i}{(1+|y|^2)^3}\right]Y_i(y)\,dy = 0.$$

Let us define for $x \in \Omega$, $i = 1, 2$, $j = 1, \ldots, m$ the function

$$\psi_{ij}(x) = \delta_j \Psi_i\left(\frac{x - \xi_j}{\delta_j}\right) = \frac{2\delta_j^2 (x - \xi_j)_i}{\delta_j^2 + |x - \xi_j|^2}\left[-2\log\delta_j + \log(\delta_j^2 + |x - \xi_j|^2)\right]$$

which satisfies

$$\Delta\psi_{ij} + e^{U_j}\psi_{ij} = -e^{U_j}(x - \xi_j)_i + \frac{3}{4}\,\delta_j e^{U_j} Z_{ij}.$$

Thus, we define the function

$$\phi_0(x) = \sum_{i=1}^{2}\sum_{j=1}^{m} \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\, P\psi_{ij}(x), \qquad x \in \Omega, \tag{3.4.1}$$

where $P\psi_{ij}$ is the projection of ψ_{ij} into the doubly periodic functions with zero average. Observe that

$$L(\phi_0) = \sum_{i=1}^{2}\sum_{j=1}^{m} \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\, L(P\psi_{ij})$$

and

$$\begin{aligned} L(P\psi_{ij}) &= \Delta\psi_{ij} - \frac{1}{|\Omega|}\int_\Omega \Delta\psi_{ij} + \frac{\lambda k e^W}{\int_\Omega k e^W}\left(P\psi_{ij} - \frac{\int_\Omega k e^W P\psi_{ij}}{\int_\Omega k e^W}\right) \\ &= -e^{U_j}\left[(x - \xi_j)_i - \frac{3}{4}\delta_j Z_{ij} + \psi_{ij}\right] + \frac{1}{|\Omega|}\int_\Omega e^{U_j}\left[(\cdot - \xi_j)_i - \frac{3}{4}\delta_j Z_{ij} + \psi_{ij}\right] \\ &\quad + \frac{\lambda k e^W}{\int_\Omega k e^W}\left(P\psi_{ij} - \frac{\int_\Omega k e^W P\psi_{ij}}{\int_\Omega k e^W}\right). \end{aligned}$$

Hence, we get that

$$L(\phi_0) = -\tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m}\left(-\frac{3}{4}\delta_j\right)\frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\Delta P Z_{ij}, \tag{3.4.2}$$

where

$$\tilde{R}_0 = R_0 - \frac{1}{|\Omega|}\int_\Omega R_0 + \sum_{i=1}^{2}\sum_{j=1}^{m}\frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\left[e^{U_j}\psi_{ij} - \frac{1}{|\Omega|}\int_\Omega e^{U_j}\psi_{ij}\right] - \frac{\lambda k e^W}{\int_\Omega k e^W}\left(\phi_0 - \frac{\int_\Omega k e^W \phi_0}{\int_\Omega k e^W}\right)$$

A function u of the form $u = W + \phi_0 + \phi_1$ satisfy (3.1.1) if and only if $\phi_1 \in E$ and

$$L(\phi_0 + \phi_1) = -[R + N(\phi_0 + \phi_1)]$$

or equivalently $\phi_1 \in E$ and

$$L(\phi_1) = -[R + L(\phi_0) + N(\phi_0 + \phi_1)],$$

where L, R and N are given by (3.2.9), (3.2.10) and (3.2.11) respectively. Also, we have that

$$N(\phi_0 + \phi_1) = N(\phi_0) + \Lambda(\phi_1) + N_0(\phi_1),$$

where

$$\Lambda(\phi_1) = \lambda\left(\frac{k e^{W+\phi_0}\phi_1}{\int_\Omega k e^{W+\phi_0}} - \frac{k e^W \phi_1}{\int_\Omega k e^W} - \frac{k e^{W+\phi_0}\int_\Omega k e^{W+\phi_0}\phi_1}{\left(\int_\Omega k e^{W+\phi_0}\right)^2} + \frac{k e^W \int_\Omega k e^W \phi_1}{\left(\int_\Omega k e^W\right)^2}\right), \tag{3.4.3}$$

$$N_0(\phi_1) = \lambda \left(\frac{ke^{W+\phi_0+\phi_1}}{\int_\Omega ke^{W+\phi_0+\phi_1}} - \frac{ke^{W+\phi_0}\phi_1}{\int_\Omega ke^{W+\phi_0}} + \frac{ke^{W+\phi_0}\int_\Omega ke^{W+\phi_0}\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} - \frac{ke^{W+\phi_0}}{\int_\Omega ke^{W+\phi_0}} \right). \tag{3.4.4}$$

Observe that

$$\int_\Omega \tilde{R}_0 = \int_\Omega \Lambda(\phi_1) = \int_\Omega N_0(\phi_1) = 0.$$

We consider the following auxiliary non linear problem

$$\begin{cases} L(\phi_1) = -[R_1 + \Lambda(\phi_1) + N_0(\phi_1)] + \sum_{i=1}^2 \sum_{j=1}^m c_{ij}^{(1)} \Delta PZ_{ij} + c_0^{(1)} \Delta PZ, & \text{in } \Omega, \\ \int_\Omega \Delta PZ_{ij}\phi_1 = 0, \quad \text{for all } i = 1,2, j = 1,\dots,m, \quad \int_\Omega \Delta PZ\phi_1 = 0, \end{cases} \tag{3.4.5}$$

where

$$R_1 = R - \tilde{R}_0 + N(\phi_0), \tag{3.4.6}$$

for some $\phi \in E \cap W^{2,2}(\Omega)$ and $c_{ij}^{(1)}, c_0^{(1)} \in \mathbb{R}$, $i = 1,2$, $j = 1,\dots,m$.

Lemma 3.4.1. *Let $m > 0$, $d > 0$. Then there exist $\delta_0 > 0$, $C > 0$ such that for $0 < \delta < \delta_0$ and for any points $\xi_1,\dots,\xi_m \in \Omega$, satisfying* (3.3.4) *and $\delta_j > 0$ satisfying* (3.2.5) *and* (3.2.6)*, problem* (3.4.5) *admits a unique solution ϕ_1, $c_{ij}^{(1)}$, $i = 0,1,2$, $j = 1,\dots,m$ and $c_0^{(1)}$ such that*

$$\|\phi_1\|_\infty \le C\delta^{2-\sigma}|\log\delta|^2. \tag{3.4.7}$$

Furthermore, the function $(\delta,\xi) \mapsto \phi(\delta,\xi) \in E \cap L^\infty(\Omega)$ is C^1 in $\xi = (\xi_1,\dots,\xi_m)$ and C^2 in δ. Moreover, we have the following estimates

$$\|\partial_\beta \phi_1\|_\infty \le C\delta^{1-\sigma}|\log\delta|^3, \qquad \text{for either } \beta = \xi_{kl} \text{ or } \beta = \delta \tag{3.4.8}$$

and

$$\|\partial_{\delta\delta}\phi_1\|_\infty \le C\delta^{-\sigma}|\log\delta|^4. \tag{3.4.9}$$

Proof: In terms of the operator T defined in Proposition 3.3.2, problem (3.4.5) becomes

$$\phi_1 = T(-[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]) := \mathcal{A}(\phi_1). \tag{3.4.10}$$

For a given number $\nu > 0$, let us consider

$$\mathcal{F}_\nu = \{\phi \in C(\bar{\Omega}) \,:\, \|\phi\|_\infty \le \nu\delta^{2-\sigma}|\log\delta|^2\}.$$

From Proposition 3.3.2, we get that for any $\phi \in \mathcal{F}_\nu$,

$$\begin{aligned} \|\mathcal{A}(\phi)\|_\infty &\le C|\log\delta| \|R_1 + \Lambda(\phi_1) + N_0(\phi_1)\|_* \\ &\le C|\log\delta| \left[\|R_1\|_* + \|\Lambda(\phi_1)\|_* + \|N_0(\phi_1)\|_*\right]. \end{aligned}$$

Let us estimate $\|R_1\|_*$. We have that

$$\begin{aligned} \|R_1\|_* &\le \|R - \tilde{R}_0\|_* + \|N(\phi_0)\|_* \\ &\le \|R - R_0\|_* + \|R_0 - \tilde{R}_0\|_* + \|N(\phi_0)\|_*. \end{aligned}$$

We know that, from Lemma 3.2.2, $\|R - R_0\|_* \le C\delta^{2-\sigma}|\log\delta|$. Also, we have that

$$R_0 - \tilde{R}_0 = \frac{1}{|\Omega|}\int_\Omega R_0 - \sum_{i=1}^2 \sum_{j=1}^m \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} \left[e^{U_j}\psi_{ij} + \frac{1}{|\Omega|}\int_\Omega e^{U_j}\psi_{ij} \right] + \frac{\lambda ke^W}{\int_\Omega ke^W} \left(\phi_0 - \frac{\int_\Omega ke^W \phi_0}{\int_\Omega ke^W} \right).$$

Observe that from the integral representation (2.2.4) we get that

$$P\psi_{ij} = \psi_{ij} + O(\delta^2|\log\delta|) \tag{3.4.11}$$

uniformly o compact subsets of Ω. Let us estimate the integral

$$\int_\Omega ke^W\phi_0 = \sum_{i=1}^{2}\sum_{j=1}^{m}\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\int_\Omega ke^W P\psi_{ij}.$$

Hence, we get that

$$\begin{aligned}\int_\Omega ke^W P\psi_{ij} &= \int_\Omega ke^W(\psi_{ij} + O(\delta^2|\log\delta|))\\ &= \int_{B(\xi_j,r)} ke^W\psi_{ij} + O(|\log\delta|),\end{aligned}$$

since $\int_\Omega ke^W = O(\delta^{-2})$ and $\psi_{ij} = O(\delta^2|\log\delta|)$ in $\bar\Omega\setminus B(\xi_j,r)$. Hence, we get that

$$\int_{B(\xi_j,r)} ke^W\psi_{ij} = \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} e^{U_j}\rho_j\psi_{ij}[1+O(\delta^2|\log\delta|)]$$

and

$$\begin{aligned}\int_{B(\xi_j,r)} e^{U_j}\rho_j\psi_{ij} &= \int_{B(0,\frac{r}{\delta_j})}\frac{8}{(1+|y|^2)^2}\rho_j(\xi_j+\delta_j y)\delta_j\Psi_{ij}(y)\,dy\\ &= \delta_j\int_{B(0,\frac{r}{\delta_j})}\frac{8}{(1+|y|^2)^2}\frac{2y_i}{1+|y|^2}\log(1+|y|^2)[\rho_j(\xi_j)+O(\delta_j|y|)]\,dy\\ &= O(\delta_j^2).\end{aligned}$$

Thus, we obtain that

$$\int_\Omega ke^W P\psi_{ij} = O(|\log\delta|) \qquad \text{for all } i=1,2,\ j=1,\dots,m$$

and

$$P\psi_{ij} - \frac{\int_\Omega ke^W P\psi_{ij}}{\int_\Omega ke^W} = \psi_{ij} + O(\delta^2|\log\delta|),$$

since $\int_\Omega ke^W = O(\delta^{-2})$. Now, note that

$$\int_\Omega R_0 = O(\delta^2) \qquad \text{and} \qquad \int_\Omega e^{U_j}\psi_{ij} = O(\delta^4|\log\delta|).$$

If $|x-\xi_j| > r$ for all $j=1,\dots,m$ then $(R_0-\tilde R_0)(x) = O(\delta^2)$. Now, if $|x-\xi_j| < r$ for some

$j \in \{1, \ldots, m\}$ then

$$
\begin{aligned}
(R_0 - \tilde{R}_0)(x) = O(\delta^2) &- \sum_{i=1}^{2}\sum_{l=1}^{m} \frac{\partial_i \rho_l(\xi_l)}{\rho_l(\xi_l)} \left[e^{U_l}\psi_{il} + O(\delta^4|\log\delta|)\right] \\
&+ \frac{\lambda k e^W}{\int_\Omega k e^W}\left(\sum_{i=1}^{2}\sum_{l=1}^{m} \frac{\partial_i \rho_l(\xi_l)}{\rho_l(\xi_l)}\psi_{il} + O(\delta^2|\log\delta|)\right) \\
&= e^{U_j}\left[1 + O(|x-\xi_j|) + O(\delta^2|\log\delta|)\right]\left[\sum_{i=1}^{2} \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\psi_{ij} + O(\delta^2|\log\delta|)\right] \\
&- \sum_{i=1}^{2} \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} e^{U_j}\psi_{ij} + O(\delta^2) \\
&= \sum_{i=1}^{2} \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} e^{U_j}\left[\psi_{ij}\left\{O(|x-\xi_j|) + O(\delta^2|\log\delta|)\right\} + O(\delta^2|\log\delta|)\right] + O(\delta^2).
\end{aligned}
$$

Thus, we conclude that $\|R_0 - \tilde{R}_0\|_* \le C\delta^{2-\sigma}$. On the other hand, by definition of N we estimate

$$
\|N(\phi_0)\|_* \le C\|K\|_*\|\phi_0\|_\infty \le C\|\phi_0\|_\infty^2.
$$

Hence, there is a constant C independent of ν such that

$$
\|N(\phi_0)\|_* \le C\delta^2.
$$

Therefore, we conclude that $\|R_1\|_* \le C\delta^{2-\sigma}|\log\delta|$.

Now, we estimate the linear term and we obtain that

$$
\|\Lambda(\phi)\|_* \le C\|\phi_0\|_*\|\phi\|_* \le C\nu\delta^{3-\sigma}|\log\delta|^2.
$$

Furthermore, we get that

$$
\|N_0(\phi)\|_* \le C\left\|\frac{\lambda k e^{W+\phi_0}}{\int_\Omega k e^{W+\phi_0}}\right\|_* \|\phi\|_\infty^2 \le C\|\phi\|_\infty^2 \le C\nu^2\delta^{4-2\sigma}|\log\delta|^4.
$$

Hence, we get that for any $\phi \in \mathcal{F}_\nu$

$$
\begin{aligned}
\|\mathcal{A}(\phi)\|_\infty &\le C|\log\delta|\left[\delta^{2-\sigma}|\log\delta| + \|\phi_0\|_\infty\|\phi\|_\infty + \|\phi\|^2\right] \\
&\le C|\log\delta|\left[\delta^{2-\sigma}|\log\delta| + \nu\delta^{3-\sigma}|\log\delta|^2 + \nu^2\delta^{4-2\sigma}|\log\delta|^4\right] \\
&\le C\delta^{2-\sigma}|\log\delta|^2\left[1 + \nu\delta|\log\delta| + \nu^2\delta^{2-\sigma}|\log\delta|^3\right].
\end{aligned}
$$

Given $\phi_1, \phi_2 \in \mathcal{F}_\nu$, we have that

$$
\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) = T\left(-[\Lambda(\phi_1) - \Lambda(\phi_2) + N_0(\phi_1) - N_0(\phi_2)]\right)
$$

and

$$
\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_\infty \le C|\log\delta|[\|\Lambda(\phi_1) - \Lambda(\phi_2)\|_* + \|N_0(\phi_1) - N_0(\phi_2)\|_*].
$$

We know that $\Lambda(\phi_1) - \Lambda(\phi_2) = \Lambda(\phi_1 - \phi_2)$, so

$$
\|\Lambda(\phi_1) - \Lambda(\phi_2)\|_* \le C\|\phi_0\|_\infty\|\phi_1 - \phi_2\|_\infty \le C\delta\|\phi_1 - \phi_2\|_\infty.
$$

From the definition of N_0, it follows that

$$\|N_0(\phi_1) - N_0(\phi_2)\|_* \leq C(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty$$

for some constant C independent of ν. Therefore, we conclude that

$$\begin{aligned}\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_\infty &\leq C|\log\delta| \left[\delta\|\phi_1 - \phi_2\|_\infty + \nu\delta^{2-\sigma}|\log\delta|^2\|\phi_1 - \phi_2\|_\infty\right] \\ &\leq C\delta|\log\delta| \left[1 + \nu\delta^{1-\sigma}|\log\delta|\right]\|\phi_1 - \phi_2\|_\infty.\end{aligned}$$

It follows that for all δ sufficiently small $\mathcal{A}$ is a contraction mapping of $\mathcal{F}_\nu$ (for ν large enough), therefore a unique fixed point ϕ_1 of $\mathcal{A}$ exists in $\mathcal{F}_\nu$.

Let us now discuss the differentiability of ϕ_1 depending on (δ, ξ), i.e., $(\delta, \xi) \mapsto \phi_1(\delta, \xi) \in C(\bar{\Omega})$. Since R_1 depends continuously in (δ, ξ), using fixed point characterization (3.4.10) we deduce that the mapping $(\delta, \xi) \mapsto \phi_1$ is also continuous. Then, formally for $\beta = \xi_{kl}$ or $\beta = \delta$, we get that

$$\partial_\beta\phi_1 = \partial_\beta T(-[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]) + T(-\partial_\beta[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]).$$

From the definition of R_1, we have that

$$\partial_\beta R_1 = \partial_\beta R - \partial_\beta \tilde{R}_0 + \partial_\beta[N(\phi_0)], \qquad \text{for } \beta = \xi_{kl} \text{ or } \beta = \delta,$$

and

$$\begin{aligned}\partial_\beta[N(\phi)] = N(\phi)\partial_\beta W &+ \lambda\left(\frac{ke^{W+\phi}}{\int_\Omega ke^{W+\phi}} - \frac{ke^W}{\int_\Omega ke^W}\right)\partial_\beta\phi - \lambda\left(\frac{ke^{W+\phi}\int_\Omega ke^{W+\phi}\partial_\beta W}{\left(\int_\Omega ke^{W+\phi}\right)^2}\right. \\ &- \frac{ke^W\int_\Omega ke^W\partial_\beta W}{\left(\int_\Omega ke^W\right)^2} - \frac{ke^W\phi\int_\Omega ke^W\partial_\beta W}{\left(\int_\Omega ke^W\right)^2} - \frac{ke^W\int_\Omega ke^W\partial_\beta W\phi}{\left(\int_\Omega ke^W\right)^2} \\ &\left.+2\frac{ke^W\left(\int_\Omega ke^W\partial_\beta W\right)\left(\int_\Omega ke^W\phi\right)}{\left(\int_\Omega ke^W\right)^3}\right) - \lambda\left(\frac{ke^{W+\phi}\int_\Omega ke^{W+\phi}\partial_\beta\phi}{\left(\int_\Omega ke^{W+\phi}\right)^2} - \frac{ke^W\int_\Omega ke^W\partial_\beta\phi}{\left(\int_\Omega ke^W\right)^2}\right).\end{aligned}$$

So, we conclude that

$$\|\partial_\beta[N(\phi_0)]\|_* \leq C\left[\|\partial_\beta W\|_\infty\|\phi_0\|_\infty^2 + \|\phi_0\|_\infty\|\partial_\beta\phi_0\|_\infty\right].$$

Similarly, we get that

$$\|\partial_\beta[N_0(\phi_1)]\|_* \leq C\left[\|\partial_\beta W + \partial_\beta\phi_0\|_\infty\|\phi_1\|_\infty^2 + \|\phi_1\|_\infty\|\partial_\beta\phi_1\|_\infty^2\right]. \tag{3.4.12}$$

On the other hand, we have that

$$\begin{aligned}\partial_\beta[\Lambda(\phi_1)] = \Lambda(\phi_1)\partial_\beta W + \lambda\Bigg(&\frac{ke^{W+\phi_0}\partial_\beta\phi_0\phi_1}{\int_\Omega ke^{W+\phi_0}} - \frac{ke^{W+\phi_0}\phi_1\int_\Omega ke^{W+\phi_0}\partial_\beta\phi_0}{\int_\Omega ke^{W+\phi_0}}\\ &- \frac{ke^{W+\phi_0}\partial_\beta\phi_0\int_\Omega ke^{W+\phi_0}\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} - \frac{ke^{W+\phi_0}\int_\Omega ke^{W+\phi_0}\partial_\beta\phi_0\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2}\\ &+2\frac{ke^{W+\phi_0}\left(\int_\Omega ke^{W+\phi_0}\phi_1\right)\left(\int_\Omega ke^{W+\phi_0}\partial_\beta\phi_0\right)}{\left(\int_\Omega ke^{W+\phi_0}\right)^3}\Bigg) + \lambda\left(\frac{ke^{W+\phi_0}}{\int_\Omega ke^{W+\phi_0}} - \frac{ke^W}{\int_\Omega ke^W}\right)\partial_\beta\phi_1\\ &- \lambda\left(\frac{ke^{W+\phi_0}\int_\Omega ke^{W+\phi_0}\partial_\beta W}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} - \frac{ke^W\int_\Omega ke^W\partial_\beta W}{\left(\int_\Omega ke^W\right)^2}\right)\phi_1\\ &- \lambda\left(\frac{ke^{W+\phi_0}\int_\Omega ke^{W+\phi_0}\partial_\beta W\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} - \frac{ke^W\int_\Omega ke^W\partial_\beta W\phi_1}{\left(\int_\Omega ke^W\right)^2}\right)\\ &- \lambda\left(\frac{ke^{W+\phi_0}\int_\Omega ke^{W+\phi_0}\partial_\beta\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} - \frac{ke^W\int_\Omega ke^W\partial_\beta\phi_1}{\left(\int_\Omega ke^W\right)^2}\right)\\ &+ 2\lambda\left(\frac{ke^{W+\phi_0}\left(\int_\Omega ke^{W+\phi_0}\phi_1\right)\left(\int_\Omega ke^{W+\phi_0}\partial_\beta W\right)}{\left(\int_\Omega ke^{W+\phi_0}\right)^3} - \frac{ke^W\left(\int_\Omega ke^W\phi_1\right)\left(\int_\Omega ke^W\partial_\beta W\right)}{\left(\int_\Omega ke^W\right)^3}\right)\end{aligned}$$

and we conclude that

$$\|\partial_\beta(\Lambda(\phi_1))\|_* \le C\left[\|\partial_\beta W\|_\infty\|\phi_0\|_\infty\|\phi_1\|_\infty + \|\partial_\beta\phi_0\|_\infty\|\phi_1\|_\infty + \|\phi_0\|_\infty\|\partial_\beta\phi_1\|_\infty\right]. \qquad (3.4.13)$$

Let us estimate $\|\partial_\beta R - \partial_\beta \tilde{R}_0\|_*$ and $\|\partial_\beta\phi_0\|_\infty$ for $\beta = \xi_{kl}$ and $\beta = \delta$. First, note that

$$\partial_\beta R - \partial_\beta\tilde{R}_0 = \partial_\beta R - \partial_\beta R_0 + \partial_\beta R_0 - \partial_\beta\tilde{R}_0.$$

Hence, for $\beta = \xi_{kl}$ we get that

$$\partial_{\xi_{kl}}R = \Delta\partial_{\xi_{kl}}W + \frac{\lambda ke^W}{\int_\Omega ke^W}\left(\partial_{\xi_{kl}}W - \frac{\int_\Omega ke^W\partial_{\xi_{kl}}W}{\int_\Omega ke^W}\right).$$

Using the expansion (3.3.16), we find that

$$\int_\Omega ke^W\partial_{\xi_{kl}}W = O\left(\frac{|\log\delta|}{\delta}\right).$$

On the other hand, we know that

$$\Delta\partial_{\xi_{kl}}W = \sum_{j=1}^m\left[-e^{U_j}\partial_{\xi_{kl}}U_j + \frac{1}{|\Omega|}\int_\Omega e^{U_j}\partial_{\xi_{kl}}U_j\right].$$

It follows that

$$\int_\Omega e^{U_j}\partial_{\xi_{kl}}U_j = O(\delta^2).$$

On the other hand, if $j \ne l$ then

$$\partial_{\xi_{kl}}\left(\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\right) = \partial_{2k}(\partial_{1i}G(\xi_j,\xi_l)) \qquad \text{and} \qquad \partial_{\xi_{kl}}\left(\frac{\partial_i\rho_l(\xi_l)}{\rho_l(\xi_l)}\right) = \frac{\partial_{ik}\rho_l(\xi_l)}{\rho_l(\xi_l)} - \frac{\partial_i\rho_l(\xi_l)\partial_k\rho_l(\xi_l)}{[\rho_l(\xi_l)]^2}.$$

Hence, we have that

$$\partial_{\xi_{kl}} R_0(x) = e^{U_l}\Bigg[\frac{\nabla(\partial_k\rho_l)(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l) + \frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l)\left(\frac{4(x-\xi_l)_k}{\delta_l^2+|x-\xi_l|^2}\right.$$
$$\left. - \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{2\delta_l^2}{\delta_l^2+|x-\xi_l|^2}\right) - \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\Bigg] + \sum_{j\neq l} e^{U_j}\Bigg[\nabla_x(\partial_{2k}G)(\xi_j,\xi_l)\cdot(x-\xi_j)$$
$$+ \partial_{2k}G(\xi_j,\xi_l)\frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_j)\frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2}\Bigg].$$

Now, we estimate in the following form, if $|x-\xi_j|>r$ for all $j=1,\dots,m$ then uniformly

$$\partial_{\xi_{kl}} R = O(\delta^2) \qquad \text{and} \qquad \partial_{\xi_{kl}} R_0 = O(\delta^2).$$

If $|x-\xi_j|<r$ for some $j\in\{1,\dots,m\}$ then we have two cases. Suppose that $j=l$, then

$$\partial_{\xi_{kl}} R(x) = -\,e^{U_l}\left[\frac{4(x-\xi_l)_k}{\delta_l^2+|x-\xi_l|^2} + \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{|x-\xi_l|^2-\delta_l^2}{\delta_l^2+|x-\xi_l|^2}\right] + O(\delta^2)$$
$$+ e^{U_l}\left[1+\frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l) + O(|x-\xi_l|^2) + O(\delta|\log\delta|)\right]$$
$$\times\left[\frac{4(x-\xi_l)_k}{\delta_l^2+|x-\xi_l|^2} - \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{2\delta_l^2}{\delta_l^2+|x-\xi_l|^2} + \partial_{2k}H(x,\xi_l) + O(\delta|\log\delta|)\right]$$

and

$$\partial_{\xi_{kl}} R_0(x) = e^{U_l}\Bigg[-\frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)} + \frac{\nabla(\partial_k\rho_l)(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l) + \frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l)\left(\frac{4(x-\xi_l)_k}{\delta_l^2+|x-\xi_l|^2}\right.$$
$$\left. - \frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{2\delta_l^2}{\delta_l^2+|x-\xi_l|^2}\right)\Bigg] + O(\delta^2).$$

Hence, we get that

$$(\partial_{\xi_{kl}} R - \partial_{\xi_{kl}} R_0)(x) = e^{U_l}\Bigg[\partial_{2k}H(x,\xi_l) + \partial_{2k}H(x,\xi_l)\frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l)$$
$$- \frac{\nabla(\partial_k\rho_l)(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_l) + O(|x-\xi_l|) + O(\delta|\log\delta|)\Bigg] + O(\delta^2).$$

Now, if $j\neq l$ then we have that

$$\partial_{\xi_{kl}} R(x) = -\,e^{U_j}\partial_{2k}G(\xi_j,\xi_l)\frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2} + e^{U_j}\Bigg[1+\frac{\nabla\rho_j(\xi_j)}{\rho_j(\xi_j)}\cdot(x-\xi_j) + O(|x-\xi_j|^2)$$
$$+ O(\delta|\log\delta|)\Bigg]\left[\partial_{2k}G(x,\xi_l) - \partial_{2k}G(\xi_j,\xi_l)\frac{2\delta_j^2}{\delta_j^2+|x-\xi_j|^2} + O(\delta|\log\delta|)\right] + O(\delta^2)$$

and

$$\partial_{\xi_{kl}} R_0(x) = e^{U_j}\left[\nabla_x(\partial_{2k}G)(\xi_j,\xi_l)\cdot(x-\xi_l) + \partial_{2k}G(\xi_j,\xi_l)\frac{\nabla\rho_l(\xi_l)}{\rho_l(\xi_l)}\cdot(x-\xi_j)\frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2}\right]$$
$$+ O(\delta^2).$$

Thus, we get that for $|x-\xi_j|<r$, $j\neq l$

$$\partial_{\xi_{kl}}R(x)-\partial_{\xi_{kl}}R_0(x)=e^{U_j}\left[O(|x-\xi_j|^2)+O(\delta|\log\delta|)\right]+O(\delta^2).$$

Therefore, from the definition of *-norm we conclude that

$$\|\partial_{\xi_{kl}}R-\partial_{\xi_{kl}}R_0\|_*\le C\delta|\log\delta|$$

Now, let us estimate $\|\partial_{\xi_{kl}}R_0-\partial_{\xi_{kl}}\tilde R_0\|_*$. From the definition we know that

$$\begin{aligned}\partial_{\xi_{kl}}R_0-\partial_{\xi_{kl}}\tilde R_0=&\frac{1}{|\Omega|}\int_\Omega\partial_{\xi_{kl}}R_0+\sum_{i=1}^2\sum_{j=1}^m\partial_{\xi_{kl}}\left(\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\right)\left[-e^{U_j}\psi_{ij}+\frac{1}{|\Omega|}\int_\Omega e^{U_j}\psi_{ij}\right]\\&+\sum_{i=1}^2\sum_{j=1}^m\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\partial_{\xi_{kl}}\left[-e^{U_j}\psi_{ij}+\frac{1}{|\Omega|}\int_\Omega e^{U_j}\psi_{ij}\right]\\&+\partial_{\xi_{kl}}\left(\frac{\lambda ke^W}{\int_\Omega ke^W}\right)\left(\phi_0-\frac{\int_\Omega ke^W\phi_0}{\int_\Omega ke^W}\right)+\frac{\lambda ke^W}{\int_\Omega ke^W}\partial_{\xi_{kl}}\left(\phi_0-\frac{\int_\Omega ke^W\phi_0}{\int_\Omega ke^W}\right).\end{aligned}$$

Furthermore, we have that

$$\partial_{\xi_{kl}}\phi_0=\sum_{i=1}^2\sum_{j=1}^m\left[\partial_{\xi_{kl}}\left(\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\right)P\psi_{ij}+\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\partial_{\xi_{kl}}\left(P\psi_{ij}\right)\right].$$

By the integral representation formula (2.2.4) we get that

$$\partial_{\xi_{kl}}P\psi_{ij}=\partial_{\xi_{kl}}\psi_{ij}+O(\delta^2|\log\delta|)\tag{3.4.14}$$

uniformly on compact subsets of Ω for all $i=1,2$, $j=1,\dots,m$. So,

$$\begin{aligned}\partial_{\xi_{kl}}\psi_{il}(x)=&\left[\frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\frac{2\delta_l^2(x-\xi_l)_i|x-\xi_l|^2}{(\delta_l^2+|x-\xi_l|^2)^2}+\frac{4\delta_l^2(x-\xi_l)_i(x-\xi_l)_k}{(\delta_l^2+|x-\xi_l|^2)^2}\right]\left[\log\left(\frac{\delta_l^2+|x-\xi_l|^2}{\delta_l^2}\right)-1\right]\\&-\delta_{ik}\frac{2\delta_l^2}{\delta_l^2+|x-\xi_l|^2}\log\left(\frac{\delta_l^2+|x-\xi_l|^2}{\delta_l^2}\right),\end{aligned}$$

where δ_{ik} is the Kronecker's delta, and for $j\neq l$

$$\partial_{\xi_{kl}}\psi_{ij}(x)=\partial_{2k}G(\xi_j,\xi_l)\frac{2\delta_j^2(x-\xi_j)_i|x-\xi_j|^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left[\log\left(\frac{\delta_j^2+|x-\xi_j|^2}{\delta_j^2}\right)-1\right].$$

Hence and similarly as above we find that from the definition of *-norm

$$\|\partial_{\xi_{kl}}R_0-\partial_{\xi_{kl}}\tilde R_0\|_*\le C\delta|\log\delta|.$$

Now, it follows that

$$\|\partial_{\xi_{kl}}R-\partial_{\xi_{kl}}\tilde R_0\|_*\le\|\partial_{\xi_{kl}}R-\partial_{\xi_{kl}}R_0\|_*+\|\partial_{\xi_{kl}}R_0-\partial_{\xi_{kl}}\tilde R_0\|_*\le C\delta|\log\delta|.$$

On the other hand, we have that

$$\|\partial_{\xi_{kl}}\phi_0\|_\infty\le C\sum_{i=1}^2\sum_{j=1}^m\left[\|P\psi_{ij}\|_\infty+\|\partial_{\xi_{kl}}P\psi_{ij}\|_\infty\right]\le C'.$$

From the definition of R_1, we have that

$$\begin{aligned}\|\partial_{\xi_{kl}} R_1\|_* &\leq \|\partial_{\xi_{kl}} R - \partial_{\xi_{kl}} \tilde{R}_0\|_* + \|\partial_{\xi_{kl}} [N(\phi_0)]\|_* \\ &\leq C\delta|\log\delta| + C\left[\|\partial_{\xi_{kl}} W\|_\infty \|\phi_0\|_\infty^2 + \|\phi_0\|_\infty \|\partial_{\xi_{kl}} \phi_0\|_\infty\right] \\ &\leq C\delta|\log\delta|.\end{aligned}$$

Thus, using estimates (3.3.18), (3.4.12) and (3.4.13) for $\beta = \xi_{kl}$ we get that

$$\begin{aligned}\|\partial_{\xi_{kl}} \phi_1\|_* &\leq \|\partial_{\xi_{kl}} T(-[R_1 + \Lambda(\phi_1) + N_0(\phi_1)])\|_\infty + \|T(-\partial_{\xi_{kl}}[R_1 + \Lambda(\phi_1) + N(\phi_0)]\|_\infty \\ &\leq C\Bigg[\frac{|\log\delta|^2}{\delta}\big(\|R_1\|_* + \|\Lambda(\phi_1)\|_* + \|N_0(\phi_1)\|_*\big) + |\log\delta|\big(\|\partial_{\xi_{kl}} R_1\|_* \\ &\qquad + \|\partial_{\xi_{kl}}[\Lambda(\phi_1)]\|_* + \|\partial_{\xi_{kl}}[N_0(\phi_1)]\|_*\big)\Bigg] \\ &\leq C\Bigg[\frac{|\log\delta|^2}{\delta}\big(\delta^{2-\sigma}|\log\delta| + \|\phi_0\|_\infty\|\phi_1\|_\infty + \|\phi_1\|_\infty^2\big) + |\log\delta|\big(\delta|\log\delta| \\ &\qquad + \|\partial_{\xi_{kl}} W\|_\infty \|\phi_0\|_\infty \|\phi_1\|_\infty + \|\partial_{\xi_{kl}}\phi_0\|_\infty\|\phi_1\|_\infty + \|\phi_0\|_\infty \|\partial_{\xi_{kl}}\phi_1\|_\infty \\ &\qquad + \|\partial_{\xi_{kl}}(W + \phi_0)\|_\infty \|\phi_1\|_\infty^2 + \|\phi_1\|_\infty \|\partial_{\xi_{kl}}\phi_1\|_\infty\big)\Bigg] \\ &\leq C\left[\delta^{1-\sigma}|\log\delta|^3 + \delta|\log\delta|\,\|\partial_{\xi_{kl}}\phi_1\|_\infty\right]\end{aligned}$$

and we conclude (3.4.8) for $\beta = \xi_{kl}$.

On the other hand, using similar arguments as above, there holds that

$$\|\partial_\delta R - \partial_\delta \tilde{R}_0\|_* \leq \|\partial_\delta R - \partial_\delta R_0\|_* + \|\partial_\delta R_0 - \partial_\delta \tilde{R}_0\|_* \leq C\delta^{1-\sigma}$$

and $\|\partial_\delta \phi_0\|_\infty \leq C$. Hence, we get that

$$\|\partial_\delta R_1\|_* \leq C\delta^{1-\sigma}.$$

Thus, using estimates (3.3.18), (3.4.12) and (3.4.13) for $\beta = \delta$ we conclude (3.4.8) for $\beta = \delta$.

We proceed in the same way for the second derivatives with respect to δ. So, we have that

$$\begin{aligned}\partial_{\delta\delta}\phi_1 = \partial_{\delta\delta} T(-[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]) + 2\partial_\delta T(-\partial_\delta[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]) \\ + T(-\partial_{\delta\delta}[R_1 + \Lambda(\phi_1) + N_0(\phi_1)]).\end{aligned}$$

Using previous estimates, similar arguments and (3.3.19) we obtain that

$$\|\partial_{\delta\delta}\phi_1\|_\infty \leq C\delta^{-\sigma}|\log\delta|^4.$$

The above computations can be made rigorous by using the implicit function theorem and the fixed point representation (3.4.10) which guarantees C^2 regularity in δ and C^1 regularity in ξ. □

3.5 The finite dimensional variational reduction

In view of Lemma 4.4.1, given any points $\xi_j \in \Omega$ satisfying (3.3.4) and any $\delta_j > 0$, $j = 1, \ldots, m$ satisfying (3.2.5)-(3.2.6), we consider $\phi_1(\delta, \xi)$, $c_{ij}^{(1)}(\delta, \xi)$, $i = 1, 2$, $j = 1, \ldots, m$, and $c_0^{(1)} = (\delta, \xi)$ where $\xi = (\xi_1, \ldots, \xi_m)$, to be the unique solution to (3.4.5) satisfying (3.4.7), (3.4.8) and (3.4.9).

After problem (3.4.5) has been solved, we observe that from the choice of R_1, $\Lambda(\phi_1)$ and $N_0(\phi_1)$ we get that

$$L(\phi_0+\phi_1) = -[R+N(\phi_0+\phi_1)] + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\Delta PZ_{ij} + c_0\Delta PZ, \qquad \text{in } \Omega, \tag{3.5.1}$$

where

$$c_{ij} = -\frac{3}{4}\,\delta_j\frac{\partial_i\rho_j(\xi_j)}{\rho(\xi_j)} + c_{ij}^{(1)} \qquad \text{for all} \quad i=1,2, j=1,\dots,m$$

and $c_0 = c_0^{(1)}$, and ϕ_0 is given by (3.4.1). Hence, we find a solution to (3.2.8) and then to the original problem if δ and ξ are such that

$$\begin{aligned} c_{ij}(\delta,\xi) &= 0, \qquad \text{for all } i=1,2, j=1,\dots,m \\ c_0(\delta,\xi) &= 0. \end{aligned} \tag{3.5.2}$$

This problem is equivalent to finding critical points of the following functional

$$F_\lambda(\delta,\xi) := J_\lambda\big(W(\delta,\xi)+\phi_0(\delta,\xi)+\phi_1(\delta,\xi)\big), \tag{3.5.3}$$

where J_λ is given by (3.1.2), W, ϕ_0 are defined by (3.2.7) and (3.4.1) respectively, and ϕ_1 is the solution to problem (3.4.5). The following standard result states that critical points of F_λ correspond to solutions of (3.5.2) for small δ, namely, for λ close to $8\pi m$.

Lemma 3.5.1. *There exists δ_0 such that for any $0<\delta<\delta_0$ if (δ,ξ) is a critical point of F_λ, with $\xi\in\Omega^m$ satisfying* (3.3.4) *and $\delta_j>0$, $j=1,\dots,m$ satisfying* (3.2.5)-(3.2.6)*, then $u = W(\delta,\xi)+\phi_0(\delta,\xi)+\phi_1(\delta,\xi)$ is a critical point of J_λ, that is, if $D_\delta F_\lambda(\delta,\xi)=0$ and $D_\xi F_\lambda(\delta,\xi)=0$ then (δ,ξ) satisfies system* (3.5.2)*, i.e., u is a solution to* (3.1.1).

Proof: Let us denote $\phi=\phi_0+\phi_1$, in order to simplify the notation. So, $F_\lambda(\delta,\lambda) = J_\lambda(W+\phi)$. Next, let us differentiate the function F_λ with respect to β for either $\beta=\xi_{kl}$ or $\beta=\delta$. We can differentiate directly $J_\lambda(W+\phi)$ (under the integral sign), so that,

$$\begin{aligned} \partial_\beta F_\lambda(\delta,\xi) &= DJ_\lambda(W+\phi)[\partial_\beta W+\partial_\beta\phi] \\ &= -\int_\Omega \left[\Delta(W+\phi) + \frac{\lambda k^{W+\phi}}{\int_\Omega ke^W}\right][\partial_\beta W+\partial_\beta\phi] \end{aligned}$$

integrating by parts. From (3.5.1) we get that

$$\Delta(W+\phi) + \lambda\left(\frac{k^{W+\phi}}{\int_\Omega ke^W} - \frac{1}{|\Omega|}\right) = \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\Delta PZ_{ij} + c_0\Delta PZ. \tag{3.5.4}$$

Hence, we obtain that

$$\partial_\beta F_\lambda(\delta,\xi) = -\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\int_\Omega \Delta PZ_{ij}[\partial_\beta W+\partial_\beta\phi] - c_0\int_\Omega \Delta PZ[\partial_\beta W+\partial_\beta\phi],$$

since $\int_\Omega[\partial_\beta W+\partial_\beta\phi]=0$. From the results of previous section, this expression defines a continuous function of (δ,ξ). Let us assume that $D_\delta F_\lambda(\delta,\xi)=0$ and $D_\xi F_\lambda(\varepsilon,\xi)=0$. Then, from previous equality for both $\beta=\delta$ and $\beta=\xi_{kl}$

$$\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\int_\Omega \Delta PZ_{ij}[\partial_\beta W+\partial_\beta\phi] + c_0\int_\Omega \Delta PZ[\partial_\beta W+\partial_\beta\phi] = 0.$$

From the estimates of previous section, we get that $\|\partial_\beta \phi\|_\infty \le C$, since $\phi = \phi_0 + \phi_1$. Also, we have that

$$\partial_\delta W(x) = -\frac{1}{\delta} PZ + O(\delta|\log\delta|) \quad \text{and} \quad \partial_{\xi_{kl}} W(x) = \frac{1}{\delta_l} Z_{kl} + O(1) = \frac{1}{\delta_l} PZ_{kl} + O(1)$$

uniformly for $x \in \bar{\Omega}$. Thus, it follows that

$$\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\int_\Omega \Delta PZ_{ij}[PZ + O(\delta)] + c_0 \int_\Omega \Delta PZ[PZ + O(\delta)] = 0$$

$$\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\int_\Omega \Delta PZ_{ij}[PZ_{kl} + O(\delta)] + c_0 \int_\Omega \Delta PZ[PZ_{kl} + O(\delta)] = 0,$$

for all $k = 1, 2$, $j = 1, \dots, m$ with $O(\cdot)$ in the sense of L^∞-norm as $\delta \to 0$. The above system is strictly diagonal dominant and we thus get $c_{ij} = 0$ for all $i = 1, 2$, $j = 1, \dots, m$ and $c_0 = 0$. □

In order to solve for critical points of the function F_λ, a key step is its expected closeness to the function $J_\lambda(W)$, where W is the function defined in (3.2.7), which we analyze in the next section.

Lemma 3.5.2. *The following expansions holds*

$$F_\lambda(\delta, \xi) = J_\lambda(W) + \Theta(\delta, \xi) + \theta_\lambda(\delta, \xi),$$

where

$$\Theta(\delta, \xi) = -\frac{14}{3}\pi\delta^2 \sum_{j=1}^{m} \frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)}$$

and

$$|\theta_\lambda| + \delta|\nabla\theta_\lambda| + \delta^2|D_\delta^2\theta_\lambda| = O(\delta^{3-\sigma}|\log\delta|)$$

uniformly for points $\xi = (\xi_1, \dots, \xi_m)$, $\xi_j \in \Omega$, $j = 1, \dots, m$ *satisfying* (3.3.4), *and with* $\delta_j > 0$, $j = 1, \dots, m$ *satisfying* (3.2.5)-(3.2.6).

Proof: We write $\phi = \phi_0 + \phi_1$ and

$$J_\lambda(W + \phi) - J_\lambda(W) = A + B,$$

where

$$A := J_\lambda(W + \phi) - J_\lambda(W + \phi_0) \quad \text{and} \quad B := J_\lambda(W + \phi_0) - J_\lambda(W).$$

Let us estimate A first. A Taylor expansion give us

$$A = DJ_\lambda(W + \phi)[\phi_1] - \int_0^1 D^2 J_\lambda(W + \phi_0 + t\phi_1)[\phi_1]^2 t\, dt.$$

Testing equation (3.5.1) against ϕ_1 and integrating by parts, we get

$$\int_\Omega \nabla(W + \phi)\nabla\phi_1 - \lambda\frac{\int_\Omega k e^{W+\phi}\phi_1}{\int_\Omega k e^{W+\phi}} = 0$$

i.e., $DJ_\lambda(W + \phi)[\phi_1] = 0$. Thus,

$$A = -\int_0^1 D^2 J_\lambda(W + \phi_0 + t\phi_1)[\phi_1]^2 t\, dt.$$

In fact, we have that

$$A = f'(1) - \int_0^1 f''(t)\,dt, \qquad \text{with} \quad f(t) = J_\lambda(W + \phi_0 + t\phi_1).$$

Hence, we get that

$$\begin{aligned} f'(t) &= DJ_\lambda(W + \phi_0 + t\phi_1)[\phi_1] \\ &= \int_\Omega \left[-\Delta(W + \phi_0 + t\phi_1) - \lambda \frac{ke^{W+\phi_0+t\phi_1}}{\int_\Omega ke^{W+\phi_0+t\phi_1}} \right] \phi_1. \end{aligned}$$

So, by (3.5.1) we get that $f'(1) = DJ_\lambda(W + \phi)[\phi_1] = 0$, since

$$\int_\Omega \phi_1 \Delta PZ_{ij} = 0, \qquad \text{for all } i = 1, 2,\ j = 1, \ldots, m$$

$$\int_\Omega \phi_1 \Delta PZ = 0 \qquad \text{and} \qquad \int_\Omega \phi_1 = 0.$$

On the other hand, we have that

$$\begin{aligned} f''(t) &= D^2 J_\lambda(W + \phi_0 + t\phi_1)[\phi_1]^2 \\ &= -\int_\Omega \left[\Delta\phi_1 + \lambda \frac{ke^{W+\phi_0+t\phi_1}\phi_1}{\int_\Omega ke^{W+\phi_0+t\phi_1}} - \lambda \frac{ke^{W+\phi_0+t\phi_1} \int_\Omega ke^{W+\phi_0+t\phi_1}\phi_1}{\left(\int_\Omega ke^{W+\phi_0+t\phi_1}\right)^2} \right] \phi_1 \\ &= \int_\Omega \left[R_1 + \Lambda(\phi_1) + N_0(\phi_1) + \frac{\lambda ke^W \phi_1}{\int_\Omega ke^W} - \frac{\lambda ke^W \int_\Omega ke^W \phi_1}{\left(\int_\Omega ke^W\right)^2} \right. \\ &\qquad \left. -\lambda \frac{ke^{W+\phi_0+t\phi_1}\phi_1}{\int_\Omega ke^{W+\phi_0+t\phi_1}} + \lambda \frac{k^{W+\phi_0+t\phi_1} \int_\Omega ke^{W+\phi_0+t\phi_1}\phi_1}{\left(\int_\Omega ke^{W+\phi_0+t\phi_1}\right)^2} \right] \phi_1 \\ &= \int_\Omega \left[R_1 + N_0(\phi_1) - \tilde\Lambda_t(\phi_1) \right] \phi_1, \end{aligned} \tag{3.5.5}$$

where

$$\begin{aligned} \tilde\Lambda_t(\phi_1) = &\lambda \left(\frac{ke^{W+\phi_0+t\phi_1}\phi_1}{\int_\Omega ke^{W+\phi_0+t\phi_1}} - \frac{ke^{W+\phi_0}\phi_1}{\int_\Omega ke^{W+\phi_0}} - \frac{k^{W+\phi_0+t\phi_1} \int_\Omega ke^{W+\phi_0+t\phi_1}\phi_1}{\left(\int_\Omega ke^{W+\phi_0+t\phi_1}\right)^2} \right. \\ &\left. + \frac{ke^{W+\phi_0} \int_\Omega ke^{W+\phi_0}\phi_1}{\left(\int_\Omega ke^{W+\phi_0}\right)^2} \right). \end{aligned} \tag{3.5.6}$$

Let us observe that we get $\tilde\Lambda_t(\phi_1)$ from $\Lambda(\phi_1)$ replacing W by $W + \phi_0$ and ϕ_0 by $t\phi_1$. Thus, we obtain

$$A = -\frac{1}{2} \int_\Omega [R_1 + N_0(\phi_1)]\phi_1 + \int_0^1 \left\{ \int_\Omega \tilde\Lambda_t(\phi_1)\phi_1 \right\} t\,dt. \tag{3.5.7}$$

Now, we can estimate

$$\begin{aligned} |A| &\le C\|R_1 + N_0(\phi_1)\|_* \, \|\phi_1\|_\infty + \int_0^1 C\|\tilde\Lambda_t(\phi_1)\|_* \|\phi_1\|_\infty \, |t|\,dt \\ &\le C\,[\|R_1\|_* + \|N_0(\phi_1)\|_*]\,\|\phi_1\|_\infty + \int_0^1 C|t|\|\phi_1\|_\infty^2 \|\phi_1\|_\infty \, |t|\,dt \\ &\le C\left[\delta^{2-\sigma}|\log\delta| + \|\phi_1\|_\infty^2\right] \delta^{2-\sigma}|\log\delta|^2 + C[\delta^{2-\sigma}|\log\delta|]^3 \\ &\le C[\delta^{4-2\sigma}|\log\delta|^3 + \delta^{6-3\sigma}|\log\delta|^6]. \end{aligned}$$

Note that we estimate $\|\tilde{\Lambda}_t(\phi_1)\|_*$, similarly as we have done with $\Lambda(\phi_1)$. Therefore, we get

$$J_\lambda(W+\phi) - J_\lambda(W+\phi_0) = O(\delta^{4-2\sigma}|\log\delta|^3).$$

Let us differentiate with respect to either $\beta = \xi_{kl}$ or $\beta = \delta$. We use the representation (3.5.7) and differentiate directly under the integral sign, thus obtaining for each ξ_{kl}, $k = 1,2$, $l = 1,\dots,m$ or δ,

$$\begin{aligned}\partial_\beta A = & -\frac{1}{2}\int_\Omega \big([\partial_\beta R_1 + \partial_\beta\{N_0(\phi_1)\}]\,\phi_1 + [R_1 + N_0(\phi_1)]\partial_\beta\phi_1\big)\\ & + \int_0^1 \left\{ \int_\Omega \Big[\partial_\beta\{\tilde{\Lambda}_t(\phi_1)\}\phi_1 + \tilde{\Lambda}_t(\phi_1)\partial_\beta\phi_1\Big] \right\} t\,dt.\end{aligned}$$

We use the estimates from previous section and we observe that an estimate for $\|\partial_\beta(\tilde{\Lambda}_t(\phi_1))\|_*$ arises from similar computations for $\|\partial_\beta(\Lambda(\phi_1))\|_*$. So, we find that

$$\|\partial_\beta(\tilde{\Lambda}_t(\phi_1))\|_* \le C\left[|t|\,\|\phi_1\|_\infty^2\|\partial_\beta(W+\phi_0)\|_\infty + |t|\,\|\partial_\beta\phi_1\|_\infty\|\phi_1\|_\infty\right].$$

Hence, by using the estimates of the previous section we obtain

$$\begin{aligned}|\partial_\beta A| \le{}& C\Big([\|\partial_\beta R_1\|_* + \|\partial_\beta\{N_0(\phi_1)\}\|_*]\ \|\phi_1\|_\infty + [\|R_1\|_* + \|N_0(\phi_1)\|_*]\,\|\partial_\beta\phi_1\|_*\\ & + \int_0^1 \Big[\|\partial_\beta\{\tilde{\Lambda}_t(\phi_1)\}\|_*\|\phi_1\|_\infty + \|\tilde{\Lambda}_t(\phi_1)\|_*\|\partial_\beta\phi_1\|_\infty\Big]\ dt\Big)\\ \le{}& C\big(\big[\|\partial_\beta R_1\|_* + \|\partial_\beta(W+\phi_0)\|_\infty\|\phi_1\|_\infty^2 + \|\phi_1\|_\infty\|\partial_\beta\phi_1\|_\infty\big]\ \|\phi_1\|_\infty\\ & + \big[\|R_1\|_* + \|\phi_1\|_\infty^2\big]\,\|\partial_\beta\phi_1\|_* + \big[\|\phi_1\|_\infty^2\|\partial_\beta(W+\phi_0)\|_\infty + \|\partial_\beta\phi_1\|_\infty\|\phi_1\|_\infty\big]\ \|\phi_1\|_\infty\\ & + \|\phi_1\|_\infty^2\|\partial_\beta\phi_1\|_\infty\big)\\ \le{}& C\big(\big[\delta^{1-\sigma} + \delta^{3-2\sigma}|\log\delta|^4 + \delta^{3-2\sigma}|\log\delta|^5\big]\,\delta^{2-\sigma}|\log\delta|^2\\ & + \big[\delta^{2-\sigma}|\log\delta| + \delta^{4-2\sigma}|\log\delta|^2\big]\,\delta^{1-\sigma}|\log\delta|^3\big)\end{aligned}$$

Thus, we conclude that for either $\beta = \xi_{kl}$ or $\beta = \delta$

$$\partial_\beta[J_\lambda(W+\phi) - J_\lambda(W+\phi_0)] = O(\delta^{3-2\sigma}|\log\delta|^4).$$

Let us differentiate $\partial_\delta A$ with respect to δ. Hence, it follows that

$$\begin{aligned}\partial_{\delta\delta}A = & -\frac{1}{2}\int_\Omega \big([\partial_{\delta\delta}R_1 + \partial_{\delta\delta}\{N_0(\phi_1)\}]\,\phi_1 + 2\,[\partial_\delta R_1 + \partial_\delta\{N_0(\phi_1)\}]\,\partial_\delta\phi_1 + [R_1 + N_0(\phi_1)]\partial_{\delta\delta}\phi_1\big)\\ & + \int_0^1 \left\{ \int_\Omega \Big[\partial_{\delta\delta}\{\tilde{\Lambda}_t(\phi_1)\}\phi_1 + 2\partial_\delta\{\tilde{\Lambda}_t(\phi_1)\}\partial_\delta\phi_1 + \tilde{\Lambda}_t(\phi_1)\partial_{\delta\delta}\phi_1\Big] \right\} t\,dt.\end{aligned}$$

From estimates used to obtain (3.4.9) and similarly as above, we get that

$$\partial_{\delta\delta}[J_\lambda(W+\phi) - J_\lambda(W+\phi_0)] = O(\delta^{2-2\sigma}|\log\delta|^5).$$

On the other hand, we have that by a Taylor expansion

$$B = DJ_\lambda(W+\phi_0)[\phi_0] - \int_0^1 D^2J_\lambda(W+t\phi_0)[\phi_0]^2 t\,dt.$$

In fact, it holds

$$B = g'(1) - \int_0^1 g''(t)t\,dt, \qquad \text{with} \quad g(t) = J_\lambda(W + t\phi_0).$$

Hence, we get that

$$\begin{aligned} g'(t) &= DJ_\lambda(W + t\phi_0)[\phi_0] \\ &= \int_\Omega \left[-\Delta(W + t\phi_0) - \lambda \frac{ke^{W+t\phi_0}}{\int_\Omega ke^{W+t\phi_0}} \right] \phi_0. \end{aligned}$$

So, by (3.4.2) we get that

$$\begin{aligned} g'(1) &= -\int_\Omega [R + L(\phi_0) + N(\phi_0)]\phi_0 \\ &= -\int_\Omega \left[R - \tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)} \Delta P Z_{ij} + N(\phi_0) \right] \phi_0, \end{aligned}$$

where we denote $c_{ij}^{(0)} = -\frac{3}{4}\delta_j \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}$. Also, we have that

$$\begin{aligned} g''(t) &= D^2 J_\lambda(W + t\phi_0)[\phi_0]^2 \\ &= -\int_\Omega \left[\Delta\phi_0 + \lambda \frac{ke^{W+t\phi_0}\phi_0}{\int_\Omega ke^{W+t\phi_0}} - \lambda \frac{ke^{W+t\phi_0}\int_\Omega ke^{W+t\phi_0}\phi_0}{\left(\int_\Omega ke^{W+t\phi_0}\right)^2} \right] \phi_0 \\ &= -\int_\Omega \left[-\frac{\lambda ke^W \phi_0}{\int_\Omega ke^W} + \frac{\lambda ke^W \int_\Omega ke^W\phi_0}{\left(\int_\Omega ke^W\right)^2} - \tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)}\Delta P Z_{ij} \right. \\ &\qquad \left. + \lambda \frac{ke^{W+t\phi_0}\phi_0}{\int_\Omega ke^{W+t\phi_0}} - \lambda \frac{k^{W+t\phi_0}\int_\Omega ke^{W+t\phi_0}\phi_0}{\left(\int_\Omega ke^{W+t\phi_0}\right)^2} \right] \phi_0 \\ &= -\int_\Omega \left[-\tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)}\Delta P Z_{ij} + \bar{\Lambda}_t(\phi_0) \right] \phi_0, \end{aligned} \tag{3.5.8}$$

where

$$\bar{\Lambda}_t(\phi_0) = \lambda \left(\frac{ke^{W+t\phi_0}\phi_0}{\int_\Omega ke^{W+t\phi_0}} - \frac{ke^W\phi_0}{\int_\Omega ke^W} - \frac{k^{W+t\phi_0}\int_\Omega ke^{W+t\phi_0}\phi_0}{\left(\int_\Omega ke^{W+t\phi_0}\right)^2} + \frac{ke^W\int_\Omega ke^W\phi_0}{\left(\int_\Omega ke^W\right)^2} \right).$$

Let us observe that $\bar{\Lambda}_t(\phi_0)$ is obtained replacing ϕ_0 by $t\phi_0$ and ϕ_1 by ϕ_0 in $\Lambda(\phi_1)$. Thus, we have that

$$\begin{aligned} B = &-\int_\Omega \left[R - \tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)}\Delta P Z_{ij} + N(\phi_0) \right]\phi_0 \\ &+ \int_0^1 \left\{ \int_\Omega \left[-\tilde{R}_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)}\Delta P Z_{ij} + \bar{\Lambda}_t(\phi_0) \right] \phi_0 \right\} t\,dt \\ = &-\int_\Omega \left[R - \frac{1}{2}\tilde{R}_0 + N(\phi_0) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)}\Delta P Z_{ij} \right]\phi_0 + \int_0^1 \left\{ \int_\Omega \bar{\Lambda}_t(\phi_0)\phi_0 \right\} t\,dt \\ = &\; B_0 + B_1, \end{aligned}$$

where

$$B_0 := -\frac{1}{2}\int_\Omega \left[R_0 + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}^{(0)} \Delta P Z_{ij}\right]\phi_0$$
$$= -\frac{1}{2}\int_\Omega R_0\phi_0 + \frac{3}{8}\sum_{i=1}^{2}\sum_{j=1}^{m}\delta_j \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)}\int_\Omega \Delta P Z_{ij}\phi_0$$

and

$$B_1 := -\int_\Omega \left[R - R_0 + \frac{1}{2}(R_0 - \tilde{R}_0) + N(\phi_0)\right]\phi_0 + \int_0^1 \left\{\int_\Omega \bar{\Lambda}_t(\phi_0)\phi_0\right\} t\,dt.$$

We estimate first B_1. We get that

$$|B_1| \le C\left(\left[\|R - R_0\|_* + \|R_0 - \tilde{R}_0\|_* + \|N(\phi_0)\|_*\right]\|\phi_0\|_\infty + \int_0^1 |t|^2 \|\phi_0\|_\infty^3\, dt\right)$$
$$\le C\left[\delta\left(\delta^{2-\sigma}|\log\delta| + \delta^{2-\sigma} + \delta^2\right) + \delta^3\right]$$
$$\le C\delta^{3-\sigma}|\log\delta|,$$

since $\|\bar{\Lambda}_t(\phi_0)\|_* \le C|t|\,\|\phi_0\|_\infty^2$. Similar as the estimates for the derivatives of A, we differentiate B_1 directly under the integral sign with respect to either $\beta = \xi_{kl}$ or $\beta = \delta$ and we estimate to obtain that

$$|\partial_\beta B_1| \le C\delta^{2-\sigma}|\log\delta|.$$

In the same way, we proceed for the second derivatives $\partial_{\delta\delta}B_1$ and we find that

$$|\partial_{\delta\delta}B_1| \le C\delta^{1-\sigma}|\log\delta|.$$

Now, let us compute B_0. From the definition of R_0 in (3.2.15) and ϕ_0 in (3.4.1) we have that

$$\int_\Omega R_0\phi_0 = \sum_{i=1}^{2}\sum_{j=1}^{m}\sum_{k=1}^{2}\sum_{l=1}^{m}\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\int_\Omega (x-\xi_j)_i e^{U_j} P\psi_{kl}(x).$$

Using (3.4.11), the definition of ψ_{kl}, it follows that if $l = j$, $k = i$ then

$$\int_\Omega (x-\xi_j)_i e^{U_j} P\psi_{ij} = 6\pi\delta_j^2 + O(\delta^3|\log\delta|)$$

and otherwise ($l \ne j$ or $l = j$, $k \ne i$)

$$\int_\Omega (x-\xi_j)_i e^{U_j} P\psi_{kl} = O(\delta^3|\log\delta|).$$

Therefore, we conclude that

$$\int_\Omega R_0\phi_0 = 6\pi\delta^2\sum_{j=1}^{m}\frac{|\nabla\rho_j(\xi_j)|^2}{\rho_j(\xi_j)} + O(\delta^3|\log\delta|),$$

by the choice of δ_j in (3.2.5). On the other hand,

$$\int_\Omega \Delta P Z_{ij}\phi_0 = \int_\Omega \Delta Z_{ij}\phi_0 = -\sum_{k=1}^{2}\sum_{l=1}^{m}\frac{\partial_k\rho_l(\xi_l)}{\rho_l(\xi_l)}\int_\Omega e^{U_j} Z_{ij} P\psi_{kl}.$$

Similarly, it is readily checked that

$$\int_{\Omega} \Delta P Z_{ij} \phi_0 = -\frac{40}{9} \pi \delta_j \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} + O(\delta^2 |\log \delta|).$$

Therefore,

$$B_0 = -\frac{14\pi}{3} \delta^2 \sum_{j=1}^{m} \frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)} + O(\delta^3 |\log \delta|).$$

From similar arguments used in previous estimates and previous sections, this latter expansion is also true for first derivatives $\partial_\beta B_0$ with either $\beta = \xi_{kl}$ or $\beta = \delta$, namely,

$$\partial_{\xi_{kl}} B_0 = -\frac{14\pi}{3} \delta^2 \sum_{j=1}^{m} \partial_{\xi_{kl}} \left(\frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)} \right) + O(\delta^2 |\log \delta|) \qquad \text{and}$$

$$\partial_\delta B_0 = -\frac{28\pi}{3} \delta \sum_{j=1}^{m} \frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)} + O(\delta^2 |\log \delta|).$$

For the second derivative $\partial_{\delta\delta} B_0$ we find that

$$\partial_{\delta\delta} B_0 = -\frac{28\pi}{3} \sum_{j=1}^{m} \frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)} + O(\delta |\log \delta|).$$

Finally, taking $\theta_\lambda(\delta, \xi) = F_\lambda(\delta, \xi) - J_\lambda(W) - \Theta(\delta, \xi)$, where

$$\Theta(\delta, \xi) = -\frac{14}{3} \pi \delta^2 \sum_{j=1}^{m} \frac{|\nabla \rho_j(\xi_j)|^2}{\rho_j(\xi_j)}$$

we have shown that as $\delta \to 0$

$$|\theta_\lambda| + \delta |\nabla \theta_\lambda| + \delta^2 |D_\delta^2 \theta_\lambda| = O(\delta^{3-\sigma} |\log \delta|)$$

uniformly for points $\xi = (\xi_1, \ldots, \xi_m)$, $\xi_j \in \Omega$, $j = 1, \ldots, m$ satisfying (3.3.4), and with $\delta_j > 0$, $j = 1, \ldots, m$ satisfying (3.2.5)-(3.2.6). The continuity in (δ, ξ) of all these expressions is inherited from that of $\phi = \phi_0 + \phi_1$ and its derivatives in (δ, ξ) in the L^∞-norm.

□

3.6 Energy computations

The purpose of this section is to give an asymptotic estimate of $J_\lambda(W)$, where W is the approximate solution defined in (3.2.7) and J_λ is the energy functional (3.1.2) associated to problem (3.1.1).

First, let us see a result which will be useful to get the mentioned expansion.

Lemma 3.6.1. *Given any $f \in C^{2,\gamma}(\bar{\Omega})$, $0 < \gamma < 1$ ($\partial_\delta f \equiv 0$), we have that*

$$\int_{\Omega} e^{U_{\delta,\xi}} f = 8\pi f(\xi) - 4\pi \Delta f(\xi) \delta^2 \log \delta + O(\delta^2),$$

as $\delta \to 0$.

Proof: First, observe that for $r>0$ small we get that

$$\int_\Omega e^{U_{\delta,\xi}} f = \int_{B(\xi,r)} e^{U_{\delta,\xi}} f + O(\delta^2).$$

Also, we get that

$$\begin{aligned}\int_{B(\xi,r)} e^{U_{\delta,\xi}} f = &\int_{B(\xi,r)} e^{U_{\delta,\xi}(x)} \left(f(x) - f(\xi) - \nabla f(\xi)\cdot(x-\xi) - \frac{1}{2}\langle D^2 f(\xi)(x-\xi), x-\xi\rangle \right) dx \\ &+ \int_{B(\xi,r)} e^{U_{\delta,\xi}(x)} \left(f(\xi) + \nabla f(\xi)\cdot(x-\xi) + \frac{1}{2}\langle D^2 f(\xi)(x-\xi), x-\xi\rangle \right) dx.\end{aligned}$$

So, as $\delta \to 0$

$$\int_{B(\xi,r)} e^{U_{\delta,\xi}(x)} \left(f(x) - f(\xi) - \nabla f(\xi)\cdot(x-\xi) - \frac{1}{2}\langle D^2 f(\xi)(x-\xi), x-\xi\rangle \right) dx = O(\delta^2),$$

since

$$f(x) - f(\xi) - \nabla f(\xi)\cdot(x-\xi) - \frac{1}{2}\langle D^2 f(\xi)(x-\xi), x-\xi\rangle = O(|x-\xi|^{2+\gamma})$$

uniformly in $B(\xi,r)$. On the other hand, we get that

$$\begin{aligned}&\int_{B(\xi,r)} e^{U_{\delta,\xi}(x)} \left(f(\xi) + \nabla f(\xi)\cdot(x-\xi) + \frac{1}{2}\langle D^2 f(\xi)(x-\xi), x-\xi\rangle \right) dx \\ &= \int_{B(0,\frac{r}{\delta})} \frac{8}{(1+|y|^2)^2} \left(f(\xi) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} D^2_{ij} f(\xi)\delta^2 y_i y_j \right) dy \\ &= 8f(\xi)\int_{B(0,r/\delta)} \frac{dy}{(1+|y|^2)^2} + 4\delta^2 \sum_{i=1}^{2} D^2_{ii} f(\xi) \int_{B(0,\frac{r}{\delta})} \frac{y_i^2\, dy}{(1+|y|^2)^2} \\ &= 8\pi f(\xi)\left(1 - \frac{\delta^2}{\delta^2+r^2}\right) + 2\pi\delta^2 \Delta f(\xi)\left[\log\left(\frac{\delta^2+r^2}{\delta^2}\right) + \frac{\delta^2}{\delta^2+r^2} - 1\right]\end{aligned}$$

and the conclusion follows. □

Lemma 3.6.2. *Let $m \in \mathbb{Z}^+$ and $d>0$ be a fixed small number and W be the function defined in* (3.2.7). *Under the assumptions* (3.2.5) *and* (3.2.6), *the following expansion holds*

$$J_\lambda(W) = -8\pi - \lambda\log(\pi m) + 4\pi\varphi_m(\xi) + 2(\lambda - 8\pi m)\log\delta + V(\xi)\delta^2\log\delta + O(\delta^2), \qquad (3.6.1)$$

as $\delta \to 0$, uniformly $\xi = (\xi_1,\dots,\xi_m)$ satisfying (3.3.4), *where the function φ_m is defined by*

$$\varphi_m(\xi) = -2\sum_{j=1}^{m} \log k(\xi_j) - \sum_{l\neq j} G(\xi_l,\xi_j) \qquad (3.6.2)$$

and V is the function defined by (3.1.5).

Remark 3.6.1. In the sequel, by $O(\cdot)$ and $o(\cdot)$ we will be uniformly in the region $\xi = (\xi_1,\dots,\xi_m)$, satisfying (3.3.4).

Proof: First, we will evaluate the quadratic part of energy evaluated at W, that is,

$$\frac{1}{2}\int_\Omega |\nabla W|^2 = -\frac{1}{2}\int_\Omega W\Delta W = -\frac{1}{2}\sum_{j=1}^m \int_\Omega W\Delta W_j = \frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m \int_\Omega e^{U_j}W_l.$$

Then, for $j = l$ we have that by Lemma 3.2.1

$$\int_\Omega e^{U_j}W_j = \int_\Omega e^{U_j}\left[U_j - \log(8\delta_j^2) + H(\cdot,\xi_j) + \alpha_{\delta_j,\xi_j} + O(\delta^2)\right].$$

Let us fix a small number $r > 0$. For the first integral in the R.H.S., we get that

$$\int_\Omega e^{U_j}\left[U_j - \log(8\delta_j^2)\right] = \int_{B(0,\frac{r}{\delta_j})} \frac{8}{(1+|y|^2)^2}\log\frac{1}{\delta_j^4(1+|y|^2)^2}\,dy + O(\delta^2).$$

Note that

$$\begin{aligned}\int_{B(0,\frac{r}{\delta_j})} \frac{8}{(1+|y|^2)^2}\log\frac{1}{\delta_j^4(1+|y|^2)^2}\,dy &= 8\int_{B(0,\frac{r}{\delta_j})}\frac{1}{(1+|y|^2)^2}\left[-4\log\delta_j + 2\log\frac{1}{(1+|y|^2)^2}\right]dy\\ &= 8\pi\int_0^{\frac{r}{\delta_j}}\frac{2s}{(1+s^2)^2}\left[-4\log\delta_j + 2\log\frac{1}{(1+s^2)^2}\right]ds\\ &= -32\log\delta_j + \frac{16\pi\delta_j^2}{\delta_j^2+r^2}\log(\delta_j^2+r^2) + \frac{16\pi\delta_j^2}{\delta_j^2+r^2} - 16\pi.\end{aligned}$$

Hence, we obtain that

$$\int_\Omega e^{U_j}\left[U_j - \log(8\delta_j^2)\right] = -16\pi - 32\log\delta_j + O(\delta^2).$$

Next, by the previous Lemma we have that

$$\int_\Omega e^{U_j}H(\cdot,\xi_j) = 8\pi H(\xi_j,\xi_j) - 4\pi\Delta_x H(\xi_j,\xi_j)\delta_j^2\log\delta_j + O(\delta^2).$$

We know that $\Delta_x H(\cdot,\xi_j) = \dfrac{8\pi}{|\Omega|}$, in Ω. Therefore, we conclude that

$$\int_\Omega e^{U_j}H(\cdot,\xi_j) = 8\pi H(\xi_j,\xi_j) - \frac{32\pi^2}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2).$$

Now, using the previous Lemma

$$\int_\Omega e^{U_j}\alpha_{\delta_j,\xi_j} = 8\pi\alpha_{\delta_j,\xi_j} + O(\delta^2),$$

since α_{δ_j,ξ_j} is a constant and $\alpha_{\delta_j,\xi_j} = O(\delta^2|\log\delta|)$. Therefore, we conclude that

$$\int_\Omega e^{U_j}W_j = -16\pi - 32\log\delta_j + 8\pi H(\xi_j,\xi_j) - \frac{64\pi^2}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2).$$

Now, if $l \neq j$ then uniformly for $x \in \bar\Omega \setminus B(\xi_l, r)$ we have that

$$W_l(x) = G(x,\xi_l) + \alpha_{\delta_l,\xi_l} + O(\delta^2).$$

Hence, we get that

$$\int_\Omega e^{U_j} W_l = \int_{B(\xi_j,r)} e^{U_j} W_l + \int_{\Omega\setminus[B(\xi_j,r)\cup B(\xi_l,r)]} e^{U_j} W_l + \int_{B(\xi_l,r)} e^{U_j} W_l,$$

and

$$\begin{aligned}\int_{\Omega\setminus[B(\xi_j,r)\cup B(\xi_l,r)]} e^{U_j} W_l &= \int_{\Omega\setminus[B(\xi_j,r)\cup B(\xi_l,r)]} \frac{8\delta_j^2}{(\delta_j^2+|x-\xi_j|^2)^2}\left[G(x,\xi_l)+O(\delta^2|\log\delta|)\right]dx\\ &= O(\delta^2).\end{aligned}$$

By Lemma 3.2.1, we have that there is a constant $C>0$ independent of δ such that

$$\int_{B(\xi_l,r)} |W_l| \le \int_{B(\xi_l,r)} [|U_l-\log(8\delta_l^2)| + |H(\cdot,\xi_l)| + C'\delta^2|\log\delta|] \le C.$$

Hence, we get that

$$\int_{B(\xi_l,r)} e^{U_j} W_l = O(\delta^2).$$

Also, from similar computations in the proof of Lemma 3.6.1, we find that

$$\begin{aligned}\int_{B(\xi_j,r)} e^{U_j} W_l &= \int_{B(\xi_j,r)} e^{U_j}\left[G(x,\xi_l)+\alpha_{\delta_l,\xi_l}+O(\delta^2)\right]\\ &= 8\pi G(\xi_j,\xi_l) - 4\pi\Delta_x G(\xi_j,\xi_l)\delta_j^2\log\delta_j + 8\pi\alpha_{\delta_l,\xi_l} + O(\delta^2).\end{aligned}$$

Therefore, we obtain that

$$\int_\Omega e^{U_j} W_l = 8\pi G(\xi_j,\xi_l) - \frac{32\pi^2}{|\Omega|}\delta_j^2\log\delta_j - \frac{32\pi^2}{|\Omega|}\delta_l^2\log\delta_l + O(\delta^2).$$

Now, we know that $\delta_j^2 = \delta^2\rho_j(\xi_j)$ and $\log\rho_j(\xi_j) = \log k(\xi_j) + H(\xi_j,\xi_j) + \sum_{l\neq j} G(\xi_j,\xi_l)$ hence, we get that

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla W|^2 &= \frac{1}{2}\sum_{j=1}^m \Bigg[-16\pi - 32\pi\log\delta_j + 8\pi H(\xi_j,\xi_j) - \frac{64\pi^2}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2)\\ &\quad + \sum_{l=1,l\neq j}^m \left(8\pi G(\xi_j,\xi_l) - \frac{32\pi^2}{|\Omega|}\delta_j^2\log\delta_j - \frac{32\pi^2}{|\Omega|}\delta_l^2\log\delta_l + O(\delta^2)\right)\Bigg]\\ &= -8\pi m + \sum_{j=1}^m\Bigg[-16\pi\left(\log\delta + \frac{1}{2}\log\rho_j(\xi_j)\right) + 4\pi H(\xi_j,\xi_j) - \frac{32\pi^2 m}{|\Omega|}\delta_j^2\log\delta_j\\ &\quad + \sum_{l=1,l\neq j}^m 4\pi G(\xi_j,\xi_l)\Bigg] + O(\delta^2)\\ &= -8\pi m - 16\pi m\log\delta + \sum_{j=1}^m\Bigg[-8\pi\log k(\xi_j) - 4\pi H(\xi_j,\xi_j) - 4\pi\sum_{l=1,l\neq j}^m G(\xi_j,\xi_l)\Bigg]\\ &\quad - \frac{32\pi^2 m}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j + O(\delta^2).\end{aligned}$$

Now, let us estimate the second integral term in the energy at W. So, for (the same) $r > 0$ small fixed number, we have that

$$\int_\Omega ke^W = \sum_{j=1}^m \int_{B(\xi_j,r)} ke^W + \int_{\Omega\setminus\cup_{j=1}^m B(\xi_j,r)} ke^W.$$

Given any $j \in \{1,\dots,m\}$ we find that

$$\begin{aligned}\int_{B(\xi_j,r)} ke^W &= \int_{B(\xi_j,r)} k\exp\Bigg(U_j - \log(8\delta_j^2) + H(\cdot,\xi_j) + \alpha_{\delta_j,\xi_j} \\ &\qquad + \sum_{l\neq j}[G(\cdot,\xi_l)+\alpha_{\delta_l,\xi_l}] + O(\delta^2)\Bigg) \\ &= \frac{1}{8\delta_j^2}\int_{B(\xi_j,r)} e^{U_j}\rho_j \exp\Bigg(\sum_{l=1}^m \alpha_{\delta_l,\xi_l}\Bigg)(1+O(\delta^2)).\end{aligned}$$

Hence, using Lemma 3.6.1 we obtain

$$\int_{B(\xi_j,r)} ke^W = \frac{1}{8\delta_j^2} e^{\sum_{l=1}^m \alpha_{\delta_l,\xi_l}}\Big[8\pi\rho_j(\xi_j) - 4\pi\Delta\rho_j(\xi_j)\delta_j^2\log\delta_j + O(\delta^2)\Big].$$

On the other hand, we have that

$$\begin{aligned}\int_{\Omega\setminus\cup_{j=1}^m B(\xi_j,r)} ke^W &= \int_{\Omega\setminus\cup_{j=1}^m B(\xi_j,r)} k\exp\Bigg(\sum_{j=1}^m\big[G(\cdot,\xi_j)+\alpha_{\delta_j,\xi_j}\big]\Bigg)[1+O(\delta^2)] \\ &= \exp\Bigg(\sum_{j=1}^m \alpha_{\delta_j,\xi_j}\Bigg)O(1).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\int_\Omega ke^W &= \exp\Bigg(\sum_{j=1}^m \alpha_{\delta_j,\xi_j}\Bigg)\Bigg\{\sum_{j=1}^m\Big[\frac{\pi}{\delta^2} - \frac{\pi}{2}\Delta\rho_j(\xi_j)\log\delta_j\Big] + O(1)\Bigg\} \\ &= \frac{\pi m}{\delta^2}\exp\Bigg(\sum_{j=1}^m \alpha_{\delta_j,\xi_j}\Bigg)\Bigg\{1 + \frac{1}{m}\sum_{j=1}^m\Big[-\frac{1}{2}\Delta\rho_j(\xi_j)\delta^2\Big(\log\delta + \frac{1}{2}\log\rho_j(\xi_j)\Big)\Big] + O(\delta^2)\Bigg\}\end{aligned}$$

and hence,

$$\begin{aligned}\log\Bigg(\int_\Omega ke^W\Bigg) &= \log(\pi m) - 2\log\delta + \sum_{j=1}^m\Bigg(-\frac{4\pi}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2)\Bigg) \\ &\qquad - \frac{1}{2m}\sum_{j=1}^m \Delta\rho_j(\xi_j)\delta^2\log\delta + O(\delta^2) \\ &= \log(\pi m) - 2\log\delta - \frac{4\pi}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j - \frac{1}{2m}\delta^2\log\delta\sum_{j=1}^m\Delta\rho_j(\xi_j) + O(\delta^2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} J_\lambda(W) =& -8\pi m - 16\pi m\log\delta + 4\pi\varphi_m(\xi) - \frac{32\pi^2 m}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j - \lambda\log(\pi m) \\ &+ 2\lambda\log\delta + \frac{4\pi\lambda}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j + \frac{\lambda}{2m}\delta^2\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta^2) \\ =& -8\pi m - \lambda\log(\pi m) + 4\pi\varphi_m(\xi) + 2(\lambda - 8\pi m)\log\delta + 4\pi\delta^2\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta^2), \end{aligned}$$

since $\lambda = 8\pi m + O(\delta^2|\log\delta|)$. Thus, we get (3.6.1). This completes the proof. □

In order to find critical points of F_λ, we need to know the expansion of the derivatives of F_λ. To this aim, we will show that the expansion (3.6.1) is also true in C^1-sense in ξ and C^2-sense in δ. First, we show the expansion of $\nabla_\xi[J_\lambda(W)]$ in terms of $\nabla\varphi_m$, under the assumptions of Lemma 3.6.2.

Lemma 3.6.3. *The following expansion holds, under the assumptions of Lemma 3.6.2,*

$$\nabla_\xi[J_\lambda(W)] = 4\pi\nabla\varphi_m(\xi) + O(\delta^2|\log\delta|), \tag{3.6.3}$$

uniformly for points $\xi \in \tilde\Omega^m$ satisfying (3.3.4), *as $\delta \to 0$.*

Proof: Let us fix $i \in \{1,2\}$ and $j \in \{1,\dots,m\}$. We have that

$$\partial_{(\xi_j)_i}[J_\lambda(W)] = -\int_\Omega \left[\Delta W + \frac{\lambda k e^W}{\int_\Omega k e^W}\right]\partial_{(\xi_j)_i} W.$$

Hence, we first compute

$$-\int_\Omega \Delta W \partial_{(\xi_j)_i} W = -\sum_{l=1}^m \int_\Omega \Delta W_l \partial_{(\xi_j)_i} W = \sum_{l=1}^m\sum_{q=1}^m \int_\Omega e^{U_l}\partial_{(\xi_j)_i} W_q.$$

Recall that

$$\partial_{(\xi_j)_i} W_j(x) = \frac{4(x-\xi_j)_i}{\delta_j^2 + |x-\xi_j|^2} - \frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\frac{2\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + \partial_{2i}H(x,\xi_j) + O(\delta^2|\log\delta|)$$

and for $q \neq j$

$$\partial_{(\xi_j)_i} W_q(x) = -\partial_{2i}G(\xi_l,\xi_j)\frac{2\delta_l^2}{\delta_l^2 + |x-\xi_l|^2} + O(\delta^2|\log\delta|),$$

uniformly on compact subset of Ω. If $l \neq j$ and $q = l$ then

$$\begin{aligned} \int_\Omega e^{U_l}\partial_{(\xi_j)_i} W_l &= -\int_\Omega e^{U_l}\left[\partial_{2i}G(\xi_l,\xi_j)\frac{2\delta_l^2}{\delta_l^2 + |x-\xi_l|^2} + O(\delta^2|\log\delta|)\right] \\ &= -8\pi\partial_{2i}G(\xi_l,\xi_j) + O(\delta^2|\log\delta|). \end{aligned}$$

If $l = j$ and $q = l$ then

$$\begin{aligned} \int_\Omega e^{U_j}\partial_{(\xi_j)_i} W_j &= \int_\Omega e^{U_j}\left[\frac{4(x-\xi_j)_i}{\delta_j^2 + |x-\xi_j|^2} - \frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\frac{2\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + \partial_{2i}H(x,\xi_j) + O(\delta^2|\log\delta|)\right] \\ &= -8\pi\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)} + 8\pi\partial_{2i}H(\xi_j,\xi_j) + O(\delta^2|\log\delta|). \end{aligned}$$

If $q \neq l$ and $q \neq j$ then

$$\int_\Omega e^{U_l}\partial_{(\xi_j)_i}W_q = \int_{B(\xi_l,r)} e^{U_l}\partial_{(\xi_j)_i}W_q + O(\delta^2) = O(\delta^2).$$

And, if $q \neq l$, $q = j$

$$\begin{aligned}\int_\Omega e^{U_l}\partial_{(\xi_j)_i}W_j &= \int_{B(\xi_l,r)} e^{U_l}\Big[-\frac{\partial_i\rho_j(\xi_j)}{\rho_j(\xi_j)}\frac{2\delta_j^2}{\delta_j^2+|x-\xi_j|^2} + \partial_{2i}G(x,\xi_j) + O(\delta^2|\log\delta|)\Big]\\ &\quad + \int_{B(\xi_j,r)} e^{U_l}\frac{4(x-\xi_j)_i}{\delta_j^2+|x-\xi_j|^2} + O(\delta^2|\log\delta|)\\ &= 8\pi\partial_{2i}G(\xi_l,\xi_j) + O(\delta^2|\log\delta|).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}-\int_\Omega \Delta W\partial_{(\xi_j)_i}W &= \int_\Omega e^{U_j}\partial_{(\xi_j)_i}W_j + \sum_{q=1,q\neq j}^m \int_\Omega e^{U_j}\partial_{(\xi_j)_i}W_q\\ &\quad + \sum_{l=1,l\neq j}^m \Big(\int_\Omega e^{U_l}\partial_{(\xi_j)_i}W_j + \sum_{q=1,q\neq l}^m \int_\Omega e^{U_l}\partial_{(\xi_j)_i}W_q\Big)\\ &= -8\pi\partial_{(\xi_j)_i}(\log\rho_j)(\xi_j) + O(\delta^2|\log\delta|)\\ &= 4\pi\partial_{(\xi_j)_i}\varphi_m(\xi) + O(\delta^2|\log\delta|),\end{aligned}$$

since $-2\partial_{(\xi_j)_i}(\log\rho_j)(\xi_j) = \partial_{(\xi_j)_i}\varphi_m(\xi)$ and $\partial_{2i}H(\xi_j,\xi_j) = 0$.

Now, in order to compute the next term in the R.H.S., first observe that

$$\int_\Omega ke^W = \exp\Big(\sum_{j=1}^m \alpha_{\delta_j,\xi_j}\Big)\left[\frac{\pi m}{\delta^2} - \frac{\pi}{2}\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta^2)\right]$$

and

$$ke^W = \frac{1}{8\delta_j^2}\exp\Big(\sum_{j=1}^m \alpha_{\delta_j,\xi_j}\Big)\rho_j e^{U_j}[1+O(\delta^2)]$$

uniformly in $B(\xi_j,r)$. Hence, we deduce that

$$\frac{\lambda ke^W}{\int_\Omega ke^W} = \frac{\lambda\rho_j e^{U_j}}{8\pi m\rho_j(\xi_j)}\left[1 + \frac{\delta^2\log\delta}{2m}\sum_{l=1}^m \Delta\rho_l(\xi_l) + O(\delta^2)\right]$$

uniformly in $B(\xi_j,r)$ and

$$\frac{\lambda ke^W}{\int_\Omega ke^W} = O(\delta^2) \qquad \text{for all } x \in \Omega\setminus\cup_{j=1}^m B(\xi_j,r).$$

On the other hand, from the definition of W we have that

$$\int_\Omega \frac{\lambda ke^W}{\int_\Omega ke^W}\partial_{(\xi_j)_i}W = \sum_{l=1}^m \int_\Omega \frac{\lambda ke^W}{\int_\Omega ke^W}\partial_{(\xi_j)_i}W_l$$

and for any $l \in \{1, \dots, m\}$

$$\int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l = \sum_{q=1}^m \int_{B(\xi_q, r)} \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l + O(\delta^2).$$

For $l = q = j$, we have

$$\begin{aligned}
\int_{B(\xi_j, r)} \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_j &= \int_{B(\xi_j, r)} \frac{\lambda \rho_j e^{U_j}}{8\pi m \rho_j(\xi_j)} \left[1 + \frac{\delta^2 \log \delta}{2m} \sum_{q=1}^m \Delta \rho_q(\xi_q) + O(\delta^2)\right] \\
&\qquad \times \left[\frac{4(x-\xi_j)_i}{\delta_j^2 + |x-\xi_j|^2} - \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} \frac{2\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + \partial_{2i} H(x, \xi_j) \right. \\
&\qquad \left. + O(\delta^2 |\log \delta|)\right] \\
&= \int_{B(\xi_j, r)} \frac{\lambda \rho_j e^{U_j}}{8\pi m \rho_j(\xi_j)} \left[\left(1 + \frac{\delta^2 \log \delta}{2m} \sum_{q=1}^m \Delta \rho_q(\xi_q)\right) \frac{4(x-\xi_j)_i}{\delta_j^2 + |x-\xi_j|^2} \right. \\
&\qquad \left. + \partial_{2i} H(x, \xi_j) - \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} \frac{2\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + O(\delta^2 |\log \delta|)\right] \\
&= \int_{B(0, \frac{r}{\delta_j})} \frac{\lambda \rho_j(\xi_j + \delta_j y)}{8\pi m \rho_j(\xi_j)} \frac{8}{(1+|y|^2)^2} \left[\left(1 + \frac{\delta^2 \log \delta}{2m} \sum_{q=1}^m \Delta \rho_q(\xi_q)\right) \right. \\
&\qquad \times \frac{4 y_i}{\delta_j (1+|y|^2)} + \partial_{2i} H(\xi_j + \delta_j y, \xi_j) \\
&\qquad \left. - \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} \frac{2}{1+|y|^2}\right] dy + O(\delta^2 |\log \delta|) \\
&= O(\delta^2 |\log \delta|),
\end{aligned}$$

since $\partial_{2i} H(\xi_j, \xi_j) = 0$,

$$\begin{aligned}
\int_{B(0, \frac{r}{\delta_j})} \frac{8}{(1+|y|^2)^2} \frac{4 y_i}{\delta_j (1+|y|^2)} \rho_j(\xi_j + \delta_j y)\, dy &= \frac{32}{\delta_j} \int_{B(0, \frac{r}{\delta_j})} \frac{y_i}{(1+|y|^2)^3} \rho_j(\xi_j + \delta_j y)\, dy \\
= \frac{32}{\delta_j} \int_{B(0, \frac{r}{\delta_j})} \frac{y_i}{(1+|y|^2)^3} &\left[\rho_j(\xi_j) + \nabla \rho_j(\xi_j) \cdot \delta_j y + \frac{1}{2} \langle D^2 \rho_j(\xi_j) \delta_j y, \delta_j y\rangle + O(\delta_j^3 |y|^3)\right] dy \\
= 8\pi \partial_i \rho_j(\xi_j) + O(\delta^2 |\log \delta|)
\end{aligned}$$

and

$$\begin{aligned}
\int_{B(0, \frac{r}{\delta_j})} \frac{8}{(1+|y|^2)^2} &\left[\partial_{2i} H(\xi_j + \delta_j y, \xi_j) - \frac{\partial_i \rho_j(\xi_j)}{\rho_j(\xi_j)} \frac{2}{1+|y|^2}\right] \rho_j(\xi_j + \delta_j y)\, dy \\
&= 8\pi \rho_j(\xi_j) \partial_{2i} H(\xi_j, \xi_j) - 8\pi \partial_i \rho_j(\xi_j) + O(\delta^2 |\log \delta|)
\end{aligned}$$

For $l = j$ and $q \neq j$, we have that

$$\begin{aligned}
\int_{B(\xi_q, r)} \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_j &= \int_{B(\xi_q, r)} \frac{\lambda \rho_q e^{U_q}}{8\pi m \rho_q(\xi_q)} \left[1 + O(\delta^2 \log \delta)\right] \left[\partial_{2i} G(\cdot, \xi_j) + O(\delta^2 |\log \delta|)\right] \\
&= \frac{\lambda}{8\pi m \rho_q(\xi_q)} \left[8\pi \rho_q(\xi_q) \partial_{2i} G(\xi_q, \xi_j) + O(\delta^2 |\log \delta|)\right].
\end{aligned}$$

Therefore, we conclude that

$$\int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_j = \sum_{q=1, q\neq j}^{m} \frac{\lambda}{m}\, \partial_{2i} G(\xi_q, \xi_j) + O(\delta^2 |\log \delta|).$$

Now, for $l \neq j$ we have that if $q \neq l$ then

$$\begin{aligned}\int_{B(\xi_q,r)} \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l &= \int_{B(\xi_q,r)} \frac{\lambda \rho_q e^{U_q}}{8\pi m \rho_q(\xi_q)} \left[1 + O(\delta^2 \log \delta)\right] \\ &\qquad \times \left[-\partial_{2i} G(\xi_l, \xi_j) \frac{2\delta_l^2}{\delta_l^2 + |x - \xi_l|^2} + O(\delta^2 |\log \delta|)\right] \\ &= O(\delta^2 |\log \delta|)\end{aligned}$$

and for $q = l$

$$\begin{aligned}\int_{B(\xi_l,r)} \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l &= \int_{B(\xi_l,r)} \frac{\lambda \rho_l e^{U_l}}{8\pi m \rho_l(\xi_l)} \left[1 + O(\delta^2 \log \delta)\right] \\ &\qquad \times \left[-\partial_{2i} G(\xi_l, \xi_j) \frac{2\delta_l^2}{\delta_l^2 + |x - \xi_l|^2} + O(\delta^2 |\log \delta|)\right] \\ &= -\frac{\lambda}{m}\, \partial_{2i} G(\xi_l, \xi_j) + O(\delta^2 |\log \delta|).\end{aligned}$$

Therefore, we conclude that

$$\int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l = -\frac{\lambda}{m}\, \partial_{2i} G(\xi_l, \xi_j) + O(\delta^2 |\log \delta|),$$

and hence,

$$\int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W = \int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_j + \sum_{l=1, l\neq j}^{m} \int_\Omega \frac{\lambda k e^W}{\int_\Omega k e^W} \partial_{(\xi_j)_i} W_l = O(\delta^2 |\log \delta|).$$

Finally, the conclusion follows. □

Next, we get the expansion of $\partial_\delta[J_\lambda(W)]$ and $\partial_{\delta\delta}[J_\lambda(W)]$ under the assumptions of Lemma 3.6.2.

Lemma 3.6.4. *The following expansions hold*

$$\partial_\delta[J_\lambda(W)] = \frac{2(\lambda - 8\pi m)}{\delta} + 8\pi\delta \log \delta \sum_{j=1}^{m} \Delta \rho_j(\xi_j) + O(\delta) \tag{3.6.4}$$

and

$$\partial_{\delta\delta}[J_\lambda(W)] = -\frac{2(\lambda - 8\pi m)}{\delta^2} + 8\pi \log \delta \sum_{j=1}^{m} \Delta \rho_j(\xi_j) + O(1) \tag{3.6.5}$$

as $\delta \to 0$*, uniformly for* $\xi = (\xi_1, \ldots, \xi_m)$ *satisfying* (3.3.4).

Proof: First, we need an expansion of $\partial_\delta W$ and $\partial_{\delta\delta} W$. We know that

$$W_j(x) = \log \frac{1}{(\delta_j^2 + |x-\xi_j|^2)^2} + H(x,\xi_j) - \frac{4\pi}{|\Omega|}\rho_j(\xi_j)\delta^2 \log\delta + O(\delta^2)$$

uniformly on compact subsets of Ω. Hence, by the integral representation formula (2.2.4) and (3.2.5) we get that

$$\partial_\delta W_j(x) = -\frac{1}{\delta}\left[\frac{4\delta_j^2}{\delta_j^2 + |x-\xi_j|^2} + \frac{8\pi}{|\Omega|}\delta_j^2 \log\delta_j\right] + O(\delta)$$

and

$$\partial_{\delta\delta} W_j(x) = -\frac{1}{\delta^2}\left[\frac{4\delta_j^2(|x-\xi_j|^2 - \delta_j^2)}{(\delta_j^2 + |x-\xi_j|^2)^2} + \frac{8\pi}{|\Omega|}\delta_j^2 \log\delta_j\right] + O(1)$$

uniformly on compact subsets of Ω.

Next, we have that

$$\partial_\delta[J_\lambda(W)] = -\int_\Omega \left[\Delta W + \frac{\lambda k e^W}{\int_\Omega k e^W}\right]\partial_\delta W.$$

Hence, we first compute

$$-\int_\Omega \Delta W \partial_\delta W = \sum_{j=1}^m \sum_{l=1}^m \int_\Omega e^{U_j}\partial_\delta W_l,$$

and

$$\int_\Omega e^{U_j}\partial_\delta W_l = -\frac{1}{\delta}\int_\Omega e^{U_j}\left[\frac{4\delta_l^2}{\delta_l^2 + |x-\xi_l|^2} + \frac{8\pi}{|\Omega|}\delta_l^2\log\delta_l\right] + O(\delta).$$

If $l \neq j$ then

$$\int_\Omega e^{U_j}\frac{4\delta_l^2}{\delta_l^2 + |x-\xi_l|^2}\,dx = O(\delta^2) \quad \text{and if } l=j \text{ then} \quad \int_\Omega e^{U_j}\frac{4\delta_j^2}{\delta_j^2 + |x-\xi_j|^2}\,dx = 16\pi + O(\delta^4).$$

Thus, we obtain that

$$\begin{aligned}
-\int_\Omega \Delta W \partial_\delta W &= \sum_{j=1}^m \left[\int_\Omega e^{U_j}\partial_\delta W_j + \sum_{l=1, l\neq j}^m \int_\Omega e^{U_j}\partial_\delta W_l\right] \\
&= -\frac{16\pi m}{\delta} - \frac{1}{\delta}\sum_{j=1}^m \left[\frac{64\pi^2}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2) + \sum_{l=1,l\neq j}\left(\frac{64\pi^2}{|\Omega|}\delta_l^2\log\delta_l + O(\delta^2)\right)\right] \\
&= -\frac{16\pi m}{\delta} - \frac{1}{\delta}\frac{64\pi^2 m}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j + O(\delta)
\end{aligned}$$

Next, we compute

$$\int_\Omega k e^W \partial_\delta W = \sum_{j=1}^m \int_\Omega k e^W \partial_\delta W_j = \sum_{j=1}^m \left[\sum_{l=1}^m \int_{B(\xi_l,r)} k e^W \partial_\delta W_j + \int_{\Omega\setminus\cup_{l=1}^m B(\xi_l,r)} k e^W \partial_\delta W\right].$$

Hence, we have that

$$\int_{B(\xi_l,r)} ke^W \partial_\delta W_j = -\frac{1}{8\delta\delta_l^2} \exp\left(\sum_{q=1}^m \alpha_{\delta_q,\xi_q}\right) \int_{B(\xi_l,r)} e^{U_l}\rho_l\big(1+O(\delta^2)\big) \times \left(\frac{4\delta_j^2}{\delta_j^2+|x-\xi_j|^2} + \frac{8\pi}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2)\right)$$

Now, if $l \neq j$ then

$$\begin{aligned}\int_{B(\xi_l,r)} ke^W \partial_\delta W_j &= -\frac{1}{8\delta\delta_l^2} \exp\left(\sum_{q=1}^m \alpha_{\delta_q,\xi_q}\right) \int_{B(\xi_l,r)} e^{U_l}\rho_l\left[\frac{8\pi}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2)\right] \\ &= -\frac{1}{\delta}\, e^{\sum_{q=1}^m \alpha_{\delta_q,\xi_q}}\rho_j(\xi_j)\left[\frac{8\pi^2}{|\Omega|}\log\delta_j + O(1)\right].\end{aligned}$$

If $l = j$ then

$$\int_{B(\xi_j,r)} ke^W \partial_\delta W_j = -\frac{1}{8\delta\delta_j^2} \exp\left(\sum_{q=1}^m \alpha_{\delta_q,\xi_q}\right) \int_{B(\xi_j,r)} e^{U_j}\rho_j\left[\frac{4\delta_j^2}{\delta_j^2+|x-\xi_j|^2} + \frac{8\pi}{|\Omega|}\delta_j^2\log\delta_j + O(\delta^2)\right]$$

so, we get that

$$\begin{aligned}\int_{B(\xi_j,r)} e^{U_j}\rho_j \frac{4\delta_j^2}{\delta_j^2+|x-\xi_j|^2}\,dx &= \int_{B(\xi_j,r)} \frac{32}{(1+|y|^2)^3}\rho_j(\xi_j+\delta_j y) \\ &= \int_{B(\xi_j,r)} \frac{32}{(1+|y|^2)^3}\left[\rho_j(\xi_j) + \nabla\rho_j(\xi_j)\cdot\delta_j y + O(\delta^2|y|^2)\right] \\ &= 16\pi\rho_j(\xi_j) + O(\delta^2).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\int_{B(\xi_j,r)} ke^W \partial_\delta W_j &= -\frac{1}{8\delta\delta_j^2}\exp\left(\sum_{q=1}^m \alpha_{\delta_q,\xi_q}\right)\left[16\pi\rho_j(\xi_j) + \frac{64\pi^2}{|\Omega|}\rho_j(\xi_j)\delta_j^2\log\delta_j + O(\delta^2)\right] \\ &= -\frac{1}{\delta}\, e^{\sum_{q=1}^m \alpha_{\delta_q,\xi_q}}\left[\frac{2\pi}{\delta^2} + \frac{8\pi^2}{|\Omega|}\rho_j(\xi_j)\log\delta_j + O(1))\right]\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega\setminus\cup_{l=1}^m B(\xi_l,r)} ke^W \partial_\delta W_j &= \int_{\Omega\setminus\cup_{l=1}^m B(\xi_l,r)} k\exp\left[\sum_{q=1}^m G(\cdot,\xi_q) + \alpha_{\delta_q,\xi_q}\right][1+O(\delta^2)]O(\delta^2|\log\delta|) \\ &= \exp\left(\sum_{q=1}^m \alpha_{\delta_q,\xi_q}\right)O(\delta^2|\log\delta|).\end{aligned}$$

Therefore, we conclude that

$$\int_\Omega ke^W \partial_\delta W = \sum_{j=1}^m \Big[-\frac{1}{\delta}\, e^{\sum_{q=1}^m \alpha_{\delta_q,\xi_q}} \Big(\frac{2\pi}{\delta^2} + \frac{8\pi^2}{|\Omega|}\rho_j(\xi_j)\log\delta_j + O(1)\Big)$$
$$+ \sum_{l=1,l\neq j}^m \Big(-\frac{1}{\delta}\Big)\, e^{\sum_{q=1}^m \alpha_{\delta_q,\xi_q}} \rho_j(\xi_j)\Big(\frac{8\pi^2}{|\Omega|}\log\delta_j + O(1)\Big)\Big]$$
$$= -\frac{1}{\delta}\, e^{\sum_{q=1}^m \alpha_{\delta_q,\xi_q}} \Big[\frac{2\pi m}{\delta^2} + \sum_{j=1}^m \frac{8\pi^2 m}{|\Omega|}\rho_j(\xi_j)\log\delta_j + O(1)\Big].$$

From the expansion of $\int_\Omega ke^W$, we get that

$$-\frac{\lambda}{\int_\Omega ke^W} = -\frac{\lambda\delta^2}{\pi m}\, e^{-\sum_{q=1}^m \alpha_{\delta_q,\xi_q}} \Big\{1 + \frac{1}{2m}\sum_{j=1}^m \Delta\rho_j(\xi_j)\delta^2\log\delta + O(\delta^2)\Big\}$$

and consequently

$$-\int_\Omega \frac{\lambda ke^W}{\int_\Omega ke^W}\partial_\delta W = \Big(\frac{2\lambda}{\delta} + \lambda\delta\sum_{j=1}^m \frac{8\pi}{|\Omega|}\rho_j(\xi_j)\log\delta_j + O(\delta)\Big)$$
$$\times\Big(1 + \frac{1}{m}\delta^2\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta^2)\Big)$$
$$= \frac{2\lambda}{\delta} + \frac{\lambda}{\delta}\frac{8\pi}{|\Omega|}\sum_{j=1}^m \delta_j^2\log\delta_j + \frac{\lambda}{m}\delta\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta).$$

Therefore, we conclude that

$$\partial_\delta[J_\lambda(W)] = \frac{2(\lambda-8\pi m)}{\delta} + \frac{8\pi}{|\Omega|}\frac{(\lambda-8\pi m)}{\delta}\sum_{j=1}^m \delta_j^2\log\delta_j + \frac{\lambda}{m}\delta\log\delta\sum_{j=1}^m \Delta\rho_j(\xi_j) + O(\delta)$$

and (3.6.4) follows since $\lambda = 8\pi m + O(\delta^2|\log\delta|)$.

Finally, (3.6.5) follows from similar computations as above and the expansion of $\partial_{\delta\delta}W$. This, completes the proof. □

3.7 Proofs of the Theorems

In this section, we give a proof of the Theorems.

3.7.1 Proof of Theorem 3.1.2

According to Lemma 3.5.1, we have a solution of Problem (3.1.1) if we adjust (δ,ξ) so that it is a critical point of F_λ defined by (3.5.3). We will assume that $\delta = \mu\varepsilon$ so that $C_0^{-1} \le \mu \le C_0$ for some constant C_0 and $\varepsilon > 0$ such that

$$\varepsilon^2\log\frac{1}{\varepsilon} = |\lambda - 8\pi m|.$$

Note that ε is well defined for all $0 < \varepsilon < \varepsilon_0$ with $0 < \varepsilon_0 < 1$ small enough, since the function given by

$$f : [0, e^{-1/2}] \to [0, (2e)^{-1}], \qquad f(s) = s^2\log\frac{1}{s}$$

is increasing, positive, and hence is invertible. Thus, we take $\varepsilon = f^{-1}(|\lambda - 8\pi m|)$, so that $\varepsilon \to 0$ if and only if $\lambda \to 8\pi m$, and clearly $\varepsilon_0 = f^{-1}(|\lambda_0 - 8\pi m|)$ for λ_0 close enough to $8\pi m$. In this way, we consider $F^*_\lambda(\mu, \xi) = F_\lambda(\mu\varepsilon, \xi)$. It is clear that (μ^*, ξ^*) is a critical point of F^*_λ if and only if $(\mu^*\varepsilon, \xi^*)$ is a critical point of F_λ. Also, we have that

$$\begin{aligned} F^*_\lambda(\mu, \xi) = & -8\pi - \lambda \log(\pi m) + 2(\lambda - 8\pi m)\log\varepsilon + 4\pi\varphi_m(\xi) \\ & + 2(\lambda - 8\pi m)\log\mu + \mu^2\varepsilon^2 \log\varepsilon V(\xi) + O(\varepsilon^2), \end{aligned}$$

uniformly for $\xi = (\xi_1, \ldots, \xi_m)$ satisfying (3.3.4) and $C_0^{-1} \le \mu \le C_0$ as $\lambda \to 8\pi m$, where V is given by (3.1.5). Observe that critical points of F^*_λ are also critical points of $\tilde F_\lambda$ given by

$$\tilde F_\lambda(\mu, \xi) = F^*_\lambda(\mu, \xi) + 8\pi m + \lambda\log(\pi m) + 2(8\pi m - \lambda)\log\varepsilon.$$

By the choice of ε, we have that $\varepsilon^2 \log\varepsilon = -|\lambda - 8\pi m|$, $\varepsilon^2 = o(|\lambda - 8\pi m|)$ and $\varepsilon^{2-\sigma}|\log\varepsilon| = o(|\lambda - 8\pi m|^{1-\sigma})$ as $\lambda \to 8\pi m$. Hence, from Lemmas 3.5.2, 3.6.2, 3.6.3 and 3.6.4 the following expansions follow

$$\tilde F_\lambda(\mu, \xi) = 4\pi\varphi_m(\xi) + 2(\lambda - 8\pi m)\log\mu - \mu^2|\lambda - 8\pi m|\, V(\xi) + o(|\lambda - 8\pi m|)$$

$$D_\xi \tilde F_\lambda(\mu, \xi) = D_\xi F_\lambda(\mu\varepsilon, \xi) = 4\pi\nabla\varphi_m(\xi) + o(|\lambda - 8\pi m|^{1-\sigma})$$

$$D_\mu \tilde F_\lambda(\mu, \xi) = \varepsilon D_\delta F_\lambda(\mu\varepsilon, \xi) = \frac{2(\lambda - 8\pi m)}{\mu} - 2\mu|\lambda - 8\pi m|\, V(\xi) + o(|\lambda - 8\pi m|)$$

and

$$D^2_\mu \tilde F_\lambda(\mu, \xi) = \varepsilon^2 D^2_\delta F_\lambda(\mu\varepsilon, \xi) = -\frac{2(\lambda - 8\pi m)}{\mu^2} - 2|\lambda - 8\pi m|\, V(\xi) + o(|\lambda - 8\pi m|)$$

uniformly for $\xi = (\xi_1, \ldots, \xi_m)$ satisfying (3.3.4) and $C_0^{-1} \le \mu \le C_0$, as $\lambda \to 8\pi m$.

By the assumptions, $V(\xi) \ne 0$ for all $\xi = (\xi_1, \ldots, \xi_m) \in \bar{\mathcal{D}}$. Since $\mathcal{D}$ is connected, $\operatorname{sgn} V(\xi) = \operatorname{sgn} V(\zeta)$ for all $\xi, \zeta \in \mathcal{D}$. Now, let us take $\lambda > 0$ so that $\operatorname{sgn}(\lambda - 8\pi m) = \operatorname{sgn} V(\xi)$, $\xi \in \mathcal{D}$. Thus, we have that $(\lambda - 8\pi m)|\lambda - 8\pi m|^{-1} = V(\xi)|V(\xi)|^{-1}$ for all $\xi \in \mathcal{D}$. Also, let us denote $I = [C_0^{-1}, C_0]$, $i_V = \inf_{\xi\in\mathcal{D}} |V(\xi)|^{-1/2}$ and $s_V = \sup_{\xi\in\mathcal{D}} |V(\xi)|^{-1/2}$. By the assumptions $0 < i_V \le s_V < +\infty$, and hence we choose C_0 so that $\max\{i_V^{-1}, s_V\} < C_0$ and $|V(\xi)|^{-1/2} \in I$ for all $\xi \in \bar{\mathcal{D}}$. For a $\lambda_0 > 0$ define the set

$$I(\lambda_0) = \{\lambda > 0 \mid \operatorname{sgn}(\lambda - 8\pi m) = \operatorname{sgn}(\lambda_0 - 8\pi m) \text{ and } |\lambda - 8\pi m| < |\lambda_0 - 8\pi m|\}.$$

Claim 3.7.1. *There is λ_0 close enough to $8\pi m$ such that for all $\lambda \in I(\lambda_0)$ there exists a C^1 function $\mu(\lambda, \cdot) : \mathcal{D} \to I$ satisfying*

$$D_\mu \tilde F_\lambda(\mu(\lambda, \xi), \xi) = 0, \qquad \textit{for all } \xi \in \mathcal{D}.$$

Proof: First, denote

$$f_\lambda(\mu, \xi) = \frac{1}{2|\lambda - 8\pi m|} D_\mu \tilde F_\lambda(\mu, \xi).$$

Observe that $f_\lambda \to \bar\varphi$ and $D_\mu f_\lambda \to D_\mu \bar\varphi$ uniformly in $I \times \bar{\mathcal{D}}$ as $\lambda \to 8\pi m$, where the function $\bar\varphi$ is given by

$$\bar\varphi(\mu, \xi) = \frac{V(\xi)}{\mu|V(\xi)|} - \mu V(\xi).$$

Note that

$$\partial_\mu \bar\varphi(\mu, \xi) = -\frac{V(\xi)}{\mu^2|V(\xi)|} - V(\xi)$$

and it holds

$$\bar{\varphi}(|V(\xi)|^{-1/2},\xi)=0 \qquad \text{and} \qquad \partial_\mu\bar{\varphi}(|V(\xi)|^{-1/2},\xi)=-2V(\xi)\neq 0$$

for all $\xi\in\mathcal{D}$. Also, note that $\bar{\varphi}(\cdot,\xi)$ is strictly increasing if $V(\xi)<0$ or strictly decreasing if $V(\xi)>0$. Without loss of generality, we shall assume that $V(\xi)>0$, so that we consider $\lambda>8\pi m$ and $I(\lambda_0)=(8\pi m,\lambda_0)$. Let us take $0<\delta<\min\{i_V-C_0^{-1},s_V-C_0\}$, then we get that

$$\bar{\varphi}(\mu,\xi)\geq\bar{\varphi}(|V(\xi)|^{-1/2}-\delta,\xi)=\delta V(\xi)\left[1+\frac{1}{1-\delta|V(\xi)|^{1/2}}\right]>\delta V(\xi)>\delta i_V^2>0$$

for all $\mu\in[C_0^{-1},i_V-\delta]$ and $\xi\in\mathcal{D}$. Similarly,

$$\bar{\varphi}(\mu,\xi)\leq\bar{\varphi}(|V(\xi)|^{-1/2}+\delta,\xi)=-\delta V(\xi)\left[1+\frac{1}{1+\delta|V(\xi)|^{1/2}}\right]<-\delta V(\xi)<-\delta i_V^2<0$$

for all $\mu\in[s_V+\delta,C_0]$ and $\xi\in\mathcal{D}$. Therefore, by uniform convergence there is $\bar{\lambda}>8\pi m$ such that for all $8\pi m<\lambda<\bar{\lambda}$ we have that $f_\lambda(\mu,\xi)>0$ for all $(\mu,\xi)\in[C_0^{-1},i_V-\delta]\times\mathcal{D}$ and $f_\lambda(\mu,\xi)<0$ for all $(\mu,\xi)\in[s_V+\delta,C_0]\times\mathcal{D}$. Then, given $8\pi m<\lambda<\bar{\lambda}$ and using that f_λ is continuous, we obtain that there exists (a unique) $\mu(\lambda,\cdot):\mathcal{D}\to I$ such that

$$f_\lambda(\mu(\lambda,\xi),\xi)=0, \qquad \text{for all } \xi\in\mathcal{D}. \tag{3.7.1}$$

By similar arguments, if $V(\xi)<0$ then there is $\bar{\lambda}<8\pi m$ such that for all $\bar{\lambda}<\lambda<8\pi m$ there exists $\mu(\lambda,\cdot):\mathcal{D}\to I$ satisfying (3.7.1). Furthermore, $\mu(\lambda,\xi)\to|V(\xi)|^{-1/2}$ as $\lambda\to 8\pi m$ for all $\xi\in\mathcal{D}$. Let us show that $\mu(\lambda,\cdot)$ is of class C^1. Define the function $\mathcal{F}:I(\bar{\lambda})\times\bar{\mathcal{D}}\times I\to I(\bar{\lambda})\times\bar{\mathcal{D}}\times\mathbb{R}$, given by

$$\mathcal{F}(\lambda,\xi,\mu)=\big(\lambda,\xi,f_\lambda(\mu,\xi)\big).$$

We will show that there is $\lambda_0\in I(\bar{\lambda})$ such that

$$\mathcal{F}:I(\lambda_0)\times\bar{\mathcal{D}}\times I\to\mathcal{F}(I(\lambda_0)\times\bar{\mathcal{D}}\times I)$$

is invertible. Clearly, $\mathcal{F}$ is onto. Let us see that $\mathcal{F}$ is injective. Suppose the opposite, so, for all $\lambda\in I(\bar{\lambda})$ there exist $(\lambda_i,\xi_i,\mu_i)\in I(\lambda)\times\bar{\mathcal{D}}\times I$, $i=1,2$ such that

$$\mathcal{F}(\lambda_1,\xi_1,\mu_1)=\mathcal{F}(\lambda_2,\xi_2,\mu_2) \qquad \text{and} \qquad (\lambda_1,\xi_1,\mu_1)\neq(\lambda_2,\xi_2,\mu_2).$$

From the definition of $\mathcal{F}$, we get that

$$f_{\lambda_1}(\mu_1,\xi_1)=f_{\lambda_1}(\mu_2,\xi_1) \qquad \text{with} \qquad \mu_1\neq\mu_2.$$

Hence, there are sequences $\{\lambda_n\}_n$, $\{\xi_n\}_n$ and $\{\mu_n^i\}_n$, $i=1,2$ with $\mu_n^1\neq\mu_n^2$ for all n, such that $f_{\lambda_n}(\mu_n^1,\xi_n)=f_{\lambda_n}(\mu_n^2,\xi_n)$ for all n and $\lambda_n\to 8\pi m$ as $n\to+\infty$. Then, up to subsequence there are $\mu^i\in I$, $i=1,2$ and $\xi^*\in\mathcal{D}$ such that $\mu_n^i\to\mu^i$ and $\xi_n\to\xi^*$ as $n\to+\infty$. Suppose that $\mu^1\neq\mu^2$. Letting $n\to+\infty$ we find that $\bar{\varphi}(\mu^1,\xi^*)=\bar{\varphi}(\mu^2,\xi^*)$, and hence $\mu^1\mu^2|V(\xi^*)|=-1$, which is a contradiction. Therefore, $\mu^1=\mu^2$. Without loss of generality, we assume that $\mu_n^1<\mu_n^2$. By the mean value theorem there is $\mu_n^3\in(\mu_n^1,\mu_n^2)$ such that

$$f_{\lambda_n}(\mu_n^1,\xi_n)-f_{\lambda_n}(\mu_n^2,\xi_n)=D_\mu f_{\lambda_n}(\mu_n^3,\xi_n)(\mu_1^2-\mu_n^3)=0.$$

Hence, letting $n\to+\infty$ it follows that $D_\mu\bar{\varphi}(\mu^1,\xi^*)=0$, which implies that $(\mu^1)^2|V(\xi^*)|=-1$. Thus, we conclude that there is $\lambda_0\in I(\bar{\lambda})$ such that

$$\mathcal{F}:I(\lambda_0)\times\bar{\mathcal{D}}\times I\to\mathcal{F}(I(\lambda_0)\times\bar{\mathcal{D}}\times I)$$

is invertible, namely, there exists

$$\mathcal{G}:\mathcal{F}(I(\lambda_0)\times\bar{\mathcal{D}}\times I)\to I(\lambda_0)\times\bar{\mathcal{D}}\times I,\qquad \mathcal{G}=(g_1,g_2,g_3)$$

such that $\mathcal{F}\circ\mathcal{G}=\mathrm{Id}_{\mathcal{F}(I(\lambda_0)\times\bar{\mathcal{D}}\times I)}$. We know that $(\lambda,\xi,0)\in\mathcal{F}(I(\lambda_0)\times\bar{\mathcal{D}}\times I)$. Hence, we get that for all $(\lambda,\xi)\in I(\lambda_0)\times\bar{\mathcal{D}}$

$$\mathcal{F}\big(g_1(\lambda,\xi,0),g_2(\lambda.\xi,0),g_3(\lambda,\xi,0)\big)=(\lambda,\xi,0)$$

which implies that

$$f_\lambda(g_3(\lambda,\xi,0),\xi)=0\qquad\text{for all}\quad(\lambda,\xi)\in I(\lambda_0)\times\bar{\mathcal{D}}.$$

Therefore,

$$g_3(\lambda,\xi,0)=\mu(\lambda,\xi)\qquad\text{for all}\quad(\lambda,\xi)\in I(\lambda_0)\times\bar{\mathcal{D}}.$$

Fixing λ, we have that $\mathcal{F}(\lambda,\cdot,\cdot)$ is C^1 and invertible in $\mathcal{D}\times I$. Hence, it follows that $\mathcal{G}:\mathcal{F}(\{\lambda\}\times\mathcal{D}\times I)\to\{\lambda\}\times\mathcal{D}\times I$ is C^1 and then we conclude that $\mu(\lambda,\cdot)$ is C^1 in $\mathcal{D}$ for all $\lambda\in I(\lambda_0)$. Finally, the conclusion follows from the definition of f_λ. □

Now, let us consider the function $\tilde{\varphi}_\lambda$ given by

$$\tilde{\varphi}(\xi)=\tilde{F}_\lambda(\mu(\lambda,\xi),\xi),\qquad\xi\in\mathcal{D}\qquad\text{with}\quad\lambda\in I(\lambda_0).$$

Since, $C_0^{-1}\le\mu(\lambda,\xi)\le C_0$ for all $(\lambda,\xi)\in I(\lambda_0)\times\bar{\mathcal{D}}$, it follows that

$$\tilde{\varphi}_\lambda(\xi)=4\pi\varphi_m(\xi)+O(|\lambda-8\pi m|)\qquad\text{and}\qquad\nabla\tilde{\varphi}_\lambda(\xi)=4\pi\varphi_m(\xi)+o(|\lambda-8\pi m|^{1-\sigma})$$

uniformly for $\xi\in\mathcal{D}$. Hence, $\tilde{\varphi}_\lambda$ is a C^1 small perturbation of $4\pi\varphi_m$. Since $\mathcal{C}$ is a stable critical value of φ_m in $\mathcal{D}$, it follows that there is a critical point ξ_λ of $\tilde{\varphi}_\lambda$ in $\mathcal{D}$ for λ close enough to $8\pi m$, and

$$\nabla\tilde{\varphi}_\lambda(\xi_\lambda)=4\pi\nabla\varphi_m(\xi_\lambda)+o(1)\qquad\text{as}\quad\lambda\to8\pi m.$$

Also, we have that

$$0=\nabla\tilde{\varphi}_\lambda(\xi_\lambda)=D_\mu\tilde{F}_\lambda\big(\mu(\lambda,\xi_\lambda),\xi_\lambda\big)D_\xi\mu(\lambda,\xi_\lambda)+D_\xi\tilde{F}_\lambda\big(\mu(\lambda,\xi_\lambda)\big).$$

Finally, $(\mu(\lambda,\xi_\lambda),\xi_\lambda)$ turns out to be a critical point of $\tilde{F}_\lambda$. Note that $\xi_\lambda\to\xi^*$ as $\lambda\to8\pi m$, where ξ^* is a critical point of φ_m. The verification of (3.1.4) follows by construction of the approximating solutions $W=\sum_{j=1}^m PU_{\delta_j,\xi_j}$. □

3.7.2 Proof of Theorem 3.1.3

Here, we assume a different kind of condition on critical values of φ_m. It turns out that the previous proof works out in this situation, since $\tilde{\varphi}_\lambda$ is C^1-close to a function with a non-trivial critical level in $\mathcal{D}$. Indeed, by the assumptions, $V(\xi)\ne0$ for all $\xi=(\xi_1,\dots,\xi_m)\in\bar{\mathcal{D}}$. Hence, it follows that there is λ_0 close enough to $8\pi m$ such that for all $\lambda\in I(\lambda_0)$ there exists a C^1 function $\mu(\lambda,\cdot):\mathcal{D}\to I$ satisfying

$$D_\mu\tilde{F}_\lambda(\mu(\lambda,\xi),\xi)=0,\qquad\text{for all }\xi\in\mathcal{D}.$$

The function $\tilde{\varphi}_\lambda$ given by

$$\tilde{\varphi}(\xi)=\tilde{F}_\lambda(\mu(\lambda,\xi),\xi),\qquad\xi\in\mathcal{D}\qquad\text{with}\quad\lambda\in I(\lambda_0)$$

is a C^1 small perturbation of $4\pi\varphi_m$. Since $\mathcal{C}$ is a non-trivial critical level of φ_m in $\mathcal{D}$, it follows that there is a critical point ξ_λ of $\tilde{\varphi}_\lambda$ in $\mathcal{D}$ for λ close enough to $8\pi m$. Finally, $(\mu(\lambda,\xi_\lambda),\xi_\lambda)$ turns out to be a critical point of $\tilde{F}_\lambda$. Note that $\xi_\lambda\to\xi^*$ as $\lambda\to8\pi m$, where ξ^* is a critical point of φ_m. The verification of (3.1.4) follows by construction of the approximating solutions $W=\sum_{j=1}^m PU_{\delta_j,\xi_j}$. □

3.7.3 Proof of Theorem 3.1.1: Ω is a flat torus

Note that the Gaussian curvature when Ω is a torus is $K \equiv 1$. Assume that $\xi^* = (\xi_1^*, \dots, \xi_m^*) \in \bar{\Omega}^m \setminus \mathcal{E}_m$ is a non degenerate critical point of φ_m and $\mathcal{L}(\xi^*) \neq 0$. Then, we have that $V(\xi^*) = 4\pi\mathcal{L}(\xi^*)$, since it is readily checked that

$$\Delta\rho_j(x) = \rho_j(x)\left[|\nabla(\log\rho_j)(x)|^2 + \Delta(\log k)(x) + \frac{8\pi m}{|\Omega|}\right]. \qquad (3.7.2)$$

and $-2\nabla(\log\rho_j)(\xi_j^*) = \nabla_{\xi_j}\varphi_m(\xi^*) = 0$. Therefore, there exists a connected neighborhood $\mathcal{D}$ of ξ^* compactly contained in $\bar{\Omega}^m \setminus \mathcal{E}_m$ such that $V(\xi) \neq 0$ for all $\xi \in \mathcal{D}$. Finally, in this context the same proof for Theorem 3.1.2 works out here. □

3.7.4 Proof of Theorem 3.1.4

Assume that $k \equiv 1$ and $m \geq 2$. We know that $H(x,x) = H(y,y)$ for all $x, y \in \bar{\Omega}$, since $\bar{\Omega}$ is a torus. Hence, it is enough considering that

$$\varphi_m(\xi) = -\sum_{l\neq j} G(\xi_l, \xi_j)$$

Also, we get that

$$\rho_j(x) = \exp\left(H(x,\xi_j) + \sum_{l\neq j} G(x,\xi_l)\right)$$

and using (3.7.2) we find that $\Delta\rho_j(x) \geq 0$ for all $x \in \bar{\Omega}^m$. Thus, we conclude that $V(\xi) > 0$ for all $\xi \in \bar{\Omega}^m \setminus \mathcal{E}_m$. We know that G is bounded from below in $\bar{\Omega} \times \bar{\Omega}$ and hence, φ_m has a global maximum in $\bar{\Omega}^m$. Therefore, 3.1.3 is applicable and the conclusion follows. For $m = 1$, we have that the functional $\tilde{F}_\lambda(\mu(\lambda,\xi),\xi)$ is bounded and the conclusion follows. □

Chapter 4

Liouville Equation on the Torus

4.1 Introduction and statements of main results

In this chapter, we study the elliptic partial differential equation on the torus with exponential nonlinearity and a singular source

$$\begin{cases} -\Delta u = \varepsilon^2 e^u - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 e^u + \frac{4\pi N}{|\Omega|} - 4\pi N \delta_p, & \text{in } \Omega, \\ u \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega u = 0, \end{cases} \tag{4.1.1}$$

where $\varepsilon > 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(\beta/\alpha) > 0$,

$$\Omega = \{z = s\alpha + t\beta \in \mathbb{C} \mid 0 < s, t < 1\},$$

$p \in \Omega$ and $N \geq 0$. This equation, which is the corresponding Liouville equation on the torus, and similar ones have been extensively studied over last decades. For the regular case $N = 0$, as we have already mention, due to the presence of an exponential nonlinearity, this type of equation arises in various context such as astrophysics and combustion theory, see [11, 41, 58] and references therein, the prescribed Gaussian curvature problem in a compact manifold with its related mean field version [13, 14, 47] and in statistical mechanics [7, 8, 24]. Recently, motivated by finding vortex solutions of Maxwell-Chern-Simons-Higgs theory, this type of equation with singular data, namely $N \neq 0$, has drawn a lot of attentions. For recent developments of these subjects, we refer the readers to [6, 9, 10, 24, 55, 56, 60, 61, 62, 65, 68].

Observe that (4.1.1) is equivalent to

$$\begin{cases} -\Delta u = \varepsilon^2 k(x) e^u - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(z) e^{u(z)}\, dz, & \text{in } \Omega, \\ u \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega u = 0 \end{cases} \tag{4.1.2}$$

where $k = e^{u_0}$ and u_0 is the unique solution of the problem

$$\begin{cases} -\Delta u_0 = \frac{4\pi N}{|\Omega|} - 4\pi N \delta_p, & \text{in } \Omega, \\ u_0 \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega u_0 = 0, \end{cases}$$

so that k is positive everywhere except at $x = p$ and $k(x) \sim |x-p|^{2N}$, as $x \to p$. Furthermore, problem (4.1.2) admits a variational structure, in the sense that weak solutions for (4.1.2) are the critical points of the following energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \varepsilon^2 \int_\Omega k(x)e^u, \qquad u \in E(\Omega), \tag{4.1.3}$$

where

$$\mathcal{H}(\Omega) = \{u \in H^1_{\rm loc}(\mathbb{R}^2) \mid u \text{ is doubly periodic with periodic cell domain } \Omega\}$$

and

$$E(\Omega) = \left\{ u \in \mathcal{H}(\Omega) : \int_\Omega u = 0 \right\}.$$

In fact, a critical point of J_ε on $E(\Omega)$ yields to a doubly periodic function on $\partial\Omega$ with zero average and satisfying

$$-\Delta u = \varepsilon^2\, k(x)e^u - \lambda, \qquad \text{in } \Omega,$$

for some Lagrange multiplier λ. Integrating on Ω, we get that

$$\lambda = \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(z)e^{u(z)}\,dz$$

and we recover a solution to (4.1.2). For any $\varepsilon > 0$ sufficiently small, the functional given by (4.1.3) has a local minimum which is a solution to (4.1.2) close to 0. Furthermore, the Moser-Trudinger inequality assures the existence of a second solution, which can be obtained as a mountain pass critical point for J_ε, and this second solution turns out to be unbounded as $\varepsilon \to 0$.

Our purpose is to study the existence of solution to (4.1.1), for ε positive and small, under some assumption on the weight N of the source, and to describe the asymptotic behavior of such solutions as $\varepsilon \to 0$. Indeed, we prove that, if $1 \le m < N+1$, then we can construct solutions to (4.1.1) which concentrate and blow-up, as $\varepsilon \to 0$, around some given m points of the torus Ω. Moreover, we find conditions under which there is a family of solutions of (4.1.2) exhibiting m concentration points. These are the singular limits.

Let us mention that concentration phenomena of this type has been addressed also for the problem

$$\begin{cases} -\Delta u = \varepsilon^2 e^u - 4\pi N\delta_p, & \text{in } \Omega, \\ \qquad u = 0 \quad \text{on} & \partial\Omega, \end{cases} \tag{4.1.4}$$

where Ω is now a bounded smooth domain. The regular case $N = 0$, sometimes referred to as the Gelfand problem [41], has been broadly studied. When $\varepsilon > 0$ is sufficiently small, it has long been known the existence of both a small and large solution as $\varepsilon \to 0$. This large solution of (4.1.4) was found in simply connected domains in [70], see also [26] for earlier work on existence. In general, the analysis of the blowing-up behavior for the large solution, after works [5, 51, 57, 59, 64] yields that, if u_ε is a family of solutions to (4.1.4) (with $N = 0$) for which $\varepsilon^2 \int_\Omega e^{u_\varepsilon}$ remains uniformly bounded, then necessarily there is an integer $m \ge 1$ such that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega e^{u_\varepsilon} = 8\pi m.$$

Moreover, there are points $\xi_1^\varepsilon, \dots, \xi_m^\varepsilon \in \Omega$ which remain away one from each other and away from $\partial\Omega$, such that u_ε is uniformly bounded on $\Omega \setminus \cup_{j=1}^m B(\xi_j^\varepsilon, \delta)$ and $\sup_{B(\xi_j^\varepsilon,\delta)} u_\varepsilon \to +\infty$ for any small $\delta > 0$. Existence results of solutions with the above properties has been addressed in [1, 31, 37].

In particular, in [31] it was shown that u_ε as above exists for any $m \geq 1$ provided that the domain is not simple connected.

For the problem (4.1.4) with $N > 0$, in the works [4, 2, 66] important progress have been achieved in the understanding of blowing-up solutions from the point of view of profile of blowing-up solutions, quantization of blow-up levels and Harnack-type estimates. See [67] for a complete account on the topic. Also, in [19], the authors consider the mean field equation with singular data, and get some estimates for blowing up solutions.

Several existence results of blowing-up solutions to (4.1.4) with $N > 0$ were shown in [31, 32, 35, 36]. In particular, the author in [35, 36] shows a construction of blowing-up solutions which concentrate around p provided that $\alpha \notin \mathbb{N}$. In case $N \in \mathbb{N}$ and Ω is a simply connected domain, the authors [32] present a construction of a blowing-up solutions concentrating at $N+1$ vertices of any sufficiently tiny regular polygon with a suitable center, as $\varepsilon \to 0$. On the other hand, a family of solutions which concentrate away from p have been built in [31] whenever $1 \leq m < N+1$, regardless whether or not N is an integer. An interesting question is whether this latter situation may be for the problem (4.1.1). In this chapter, we prove that such a family of solutions u_ε does actually exists.

Our main result states as follows.

Theorem 4.1.1. *Assume that $N > 0$ and $1 \leq m < N+1$. Then there exists a family of solutions $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ to (4.1.1) such that*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega e^{u_\varepsilon} = 8\pi m.$$

Moreover, there are points $\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon \in \Omega$, which remain uniformly away from p and for all $i \neq j$, $dist(\xi_i^\varepsilon - \xi_j^\varepsilon, \alpha\mathbb{Z} + \beta\mathbb{Z})$ remain uniformly away from zero, for which u_ε remains uniformly bounded on $\Omega \setminus \cup_{i=1}^m B(\xi_i^\varepsilon, \delta)$ and $\sup_{B(\xi_j^\varepsilon, \delta)} u_\varepsilon \to +\infty$ for any small $\delta > 0$.

Let stress that the solutions found in the above result have concentration at points different from p. The problem of finding solutions with an additional concentration around the source is of different nature. In case they exist, they provide an extra contribution $8\pi(1+N)$ to the first above limit. See [2, 32, 66] for some related topics.

The location of concentration points ξ_i^ε, $i = 1, \ldots, m$ is characterized in terms of a functional φ_m defined explicitly in terms of the Green's function $G = G(x,y)$ of $-\Delta$ with respect to doubly periodic boundary conditions on $\partial\Omega$, which satisfy

$$\begin{cases} -\Delta_x G(\cdot, y) = 8\pi\delta_y - \frac{8\pi}{|\Omega|}, & \text{in } \Omega, \\ G(\cdot, y) \quad \text{is doubly periodic on } \partial\Omega, \\ \int_\Omega G(x,y)\, dx = 0. \end{cases}$$

In fact, taking $\xi = (\xi_1, \ldots, \xi_m)$ we have that

$$\nabla \varphi_m(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

where

$$\varphi_m(\xi) = N \sum_{j=1}^m G(\xi_j, p) - \sum_{l \neq j} G(\xi_l, \xi_j),$$

that is to say, up to subsequences, the m- tuple $(\xi_1^\varepsilon, \ldots, \xi_m^\varepsilon)$ converge to a critical point of the functional φ_m.

We shall deduce Theorem 4.1.1 by applying a more general result, due to the equivalence with problem (4.1.2). Indeed, we construct a family of solutions to (4.1.2) which blowing up at m different points of Ω. These points will be characterized by a functional which involves the function k and the Green's function. Given a non-negative doubly periodic function k on $\partial\Omega$, define

$$\varphi_m(\xi) = -2\sum_{j=1}^{m} \log k(\xi_j) - \sum_{l\neq j} G(\xi_l, \xi_j)$$

and denote

$$\tilde{\Omega} = \{x \in \bar{\Omega} \mid k(x) > 0\ \}, \tag{4.1.5}$$

set we always assume non-empty. An observation we make is that in any compact subset of $\tilde{\Omega}^m$, we may define, without ambiguity,

$$\varphi_m(x_1, \dots, x_m) = -\infty \quad \text{if } \operatorname{dist}(x_i - x_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) = 0 \text{ for some } i \neq j.$$

Thus, the level of blowing up points will be near a *nontrivial critical value* of φ_m.

In the next result we assume $k \geq 0$, $k \not\equiv 0$, k is doubly periodic on $\partial\Omega$ and $k \in C(\bar{\Omega}) \cap C^2(\tilde{\Omega})$ where $\tilde{\Omega}$ is given by (4.1.5).

Theorem 4.1.2. *Let $m \geq 1$ and assume that there is an open set $\mathcal{D}$ compactly contained in $\tilde{\Omega}^m$ where φ_m has a non-trivial critical level $\mathcal{C}$. Then, there exists a solution u_ε to* (4.1.2)*, with*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8m\pi\ .$$

Moreover, there is an m-tuple $(x_1^\varepsilon, \dots, x_m^\varepsilon) \in \mathcal{D}$, such that as $\varepsilon \to 0$

$$\nabla\varphi_m(x_1^\varepsilon, \dots, x_m^\varepsilon) \to 0, \quad \varphi_m(x_1^\varepsilon, \dots, x_m^\varepsilon) \to \mathcal{C},$$

for which u_ε remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B(x_i^\varepsilon, \delta)$, for any $\delta > 0$

$$\sup_{B(x_i^\varepsilon,\delta)} u_\varepsilon \to +\infty, \qquad \text{and} \qquad \varepsilon^2 k(x) e^{u_\varepsilon} - 8\pi \sum_{i=1}^{m} \delta_{x_i^\varepsilon} \rightharpoonup 0$$

as $\varepsilon \to 0$ in measure sense.

We will see that for the functional φ_m corresponding to problem (4.1.1), such a set $\mathcal{D}$ actually exists under the assumption $1 \leq m < N + 1$. Thus, we conclude the result of Theorem 4.1.1.

In [1] the authors prove that for each non-degenerate critical point of the corresponding analogue φ_m for the problem (4.1.4) with $N = 0$, there exist a family of solutions u_ε concentrating at this point as $\varepsilon \to 0$. Moreover, they construct a very precise approximation of the actual solution and an application of Banach fixed point theorem, uses non-degeneracy in essential way. For the related mean field version of problem (4.1.2) in a compact two-dimensional Riemannian manifold, Chen and Lin construct blowing up solutions as a major step in their program for computation of degrees in [18]. This construction also seems to rely in essential way on the assumption of non-degenerate critical points.

On the other hand, for the problem (4.1.4) with $N = 0$, in [31, 37] have been built a solution with concentration points near topologically nontrivial critical point. However, the case $N > 0$ was considered in [31] and also, the notion of nontrivial critical value allow them to get the result.

Another consequence of our procedure is the existence of blowing-up solutions in case $\inf_\Omega k > 0$. In particular, we get the existence of blowing-up solutions when $N = 0$ in problem (4.1.1).

Theorem 4.1.3. *Assume that* $\inf_\Omega k > 0$*. Then given* $m \in I\!N$ *there exist a family of solutions* $\{u_\varepsilon\}$ *to equation* (4.1.2) *such that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon(x)}\, dx = 8\pi m.$$

The proof of Theorem 4.1.2 relies on the construction of an approximate solution, different from those in [1, 18], and is rather close to those present in [31, 37], which turns out to be precise enough. In fact, we use a family of solutions of the Liouville equation in $\mathbb{R}^2$ to construct an approximate solution, suitable scaled and projected to make it up to a good order for the boundary conditions. Solutions are found as a small additive perturbation of these initial approximation. A linearization procedure leads to a finite dimensional reduction, where the reduced problem corresponds to that of adjusting variationally the location of the concentration point. An important element in the reduction procedure, of independent interest, is the bounded invertibility of the linearized operator in suitable L^∞-weighted spaces. This functional analytic setting have been used in several works [29, 30, 31, 27, 38, 39, 40] to detect bubbling from above the critical exponent in higher dimensional problems and in Liouville type equations, and non-degeneracy of critical points of the analogue of φ_m in that context. The local notion of nontrivial critical value in (2.1.1)-(2.1.2) was introduced in [28] in the analysis of concentration phenomena in nonlinear Schrödinger equations. And it was also used in [31, 38].

4.2 Preliminaries and a first approximation of the solution

The main idea to construct an approximation of a solution is to use the functions $U_{\delta,\xi}$ defined by (2.3.3), with a suitable choice of δ. Let m be a positive integer and choose $\xi_1, \dots, \xi_m \in \Omega$ with $k(\xi_j) > 0$, $j = 1, \dots, m$ and $\xi_j \neq \xi_j$ if $i \neq j$. Let μ_j, $j = 1, \dots, m$ be positive numbers, and consider the function

$$u_j(x) = \log \frac{8\mu_j^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2 k(\xi_j)}, \tag{4.2.1}$$

where μ_j, $j = 1, \dots, m$ are parameters to be determined. Note that

$$u_j = U_{\mu_j\varepsilon,\xi_j} - \log[\varepsilon^2 k(\xi_j)].$$

In order to satisfy the boundary conditions, consider the functions U_j, $j = 1, \dots, m$ given by

$$\begin{cases} -\Delta U_j = \varepsilon^2 k(\xi_j) e^{u_j} - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(\xi_j) e^{u_j(x)}\, dx, & \text{in } \Omega, \\ U_j \quad \text{doubly periodic on} \quad \partial\Omega, & \\ \int_\Omega U_j = 0, & \end{cases} \tag{4.2.2}$$

namely, $U_j = Pu_j$, where P is the projection operator introduced in (2.2.5). Let us denote $U_j = u_j + H_j$. Observe that

$$\frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(\xi_j) e^{u_j(x)}\, dx = \frac{8\pi}{|\Omega|} + O(\varepsilon^2). \tag{4.2.3}$$

Indeed, choosing $\delta > 0$ small enough, we have

$$\int_\Omega \varepsilon^2 k(\xi_j) e^{u_j(x)}\, dx = \int_{B(\xi_j,\delta)} \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |y-\xi_j|^2)^2}\, dx + O(\varepsilon^2)$$

and taking $\mu_j \varepsilon y = x - \xi_j$ (4.2.3) follows. We want to know the behavior of U_j away from ξ_j and around ξ_j. We obtain the following characterization, assuming that for all $j = 1, \dots, m$, $C_0^{-1} \le \mu_j \le C_0$ for some constant C_0. Using the integral representation formula (2.2.4) and similarly to Lemma 3.2.1 we get the following fact.

Lemma 4.2.1. *The function U_j, which is the solution of (4.2.2), satisfies*

$$U_j(x) = u_j(x) + H(x, \xi_j) - \log \frac{8\mu_j^2}{k(\xi_j)} + O(\varepsilon^2 |\log \varepsilon|) \tag{4.2.4}$$

where the term $O(\cdot)$ is uniform in C^2-sense on compact subsets of Ω. In particular,

$$U_j(x) = G(x, \xi_j) + O(\varepsilon^2 |\log \varepsilon|), \tag{4.2.5}$$

where the term $O(\cdot)$ is uniform in C^2-sense on compact subsets of $\Omega \setminus \{\xi_j\}$.

Using the previous result we get the behavior of the function H_j on compact subsets of Ω

$$H_j(x) = H(x, \xi_j) - \log \frac{8\mu_j^2}{k(\xi_j)} + O(\varepsilon^2 |\log \varepsilon|) \tag{4.2.6}$$

uniformly in C^2-sense for x on compact subset of Ω.

Our first approximation is

$$U(x) = U_1(x) + \cdots + U_m(x), \qquad x \in \Omega. \tag{4.2.7}$$

where U_j are given by (4.2.2) with the numbers μ_j, $j = 1, \dots, m$ defined by

$$\log(8\mu_j^2) = \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j), \qquad j = 1, \dots, m. \tag{4.2.8}$$

In order to have a good approximation, we need to verify $H_j(\xi_j) + \sum_{i=1, i\neq j}^m U_i(\xi_j) \to 0$ as $\varepsilon \to 0$. In fact, by (4.2.4) and (4.2.5) we get readily the following result.

Remark 4.2.1. If we choose μ_j, $j = 1, \dots, m$ given by (4.2.8) then $U(\xi_j) - u_j(\xi_j) \to 0$ as $\varepsilon \to 0$. Also, we have that $U(\xi_j) = -4 \log \mu_j \varepsilon + H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j) + \varepsilon^2 |\log \varepsilon| \Theta_\varepsilon(\xi)$, where Θ_ε is a bounded function of $\xi = (\xi_1, \dots, \xi_m)$.

On the other hand, it is possible to show that u satisfies (4.1.2) if and only if $v(y) = u(\varepsilon y)$ satisfies

$$\begin{cases} -\Delta v = \varepsilon^4 k(\varepsilon y) e^v - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon z) e^v(z)\, dz, & \text{in } \Omega_\varepsilon, \\ v \quad \text{doubly periodic on} & \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} v = 0 \end{cases} \tag{4.2.9}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and $|\Omega_\varepsilon| = \varepsilon^{-2}|\Omega|$. Taking, the initial approximation in expanded variables as

$$V(y) = U(\varepsilon y), \tag{4.2.10}$$

we look for a solution v of (4.2.9) of the form $v = V + \phi$. We also write $\xi_j' = \varepsilon^{-1}\xi_j$. Now, in terms of ϕ, the problem (4.1.2) becomes

$$\begin{cases} L(\phi) = -[R + N(\phi)], & \text{in } \Omega_\varepsilon, \\ \phi \quad \text{doubly periodic on} & \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi = 0 \end{cases} \tag{4.2.11}$$

where

$$L(\phi) = \Delta\phi + K(y)\phi - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} K(z)\phi(z)\,dz, \qquad K(y) := \varepsilon^4 k(\varepsilon y)\, e^{V(y)},$$

$$R(y) = \Delta V(y) + \varepsilon^4 k(\varepsilon y)\, e^{V(y)} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon z)\, e^{V(z)}\,dz, \tag{4.2.12}$$

and

$$N(\phi) = K(y)(e^{\phi} - \phi - 1) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} K(z)(e^{\phi(z)} - \phi(z) - 1)\,dz. \tag{4.2.13}$$

Let us stress that R, $L(\phi)$ and $N(\phi)$ satisfy

$$\int_{\Omega_\varepsilon} R = \int_{\Omega_\varepsilon} L(\phi) = \int_{\Omega_\varepsilon} N(\phi) = 0.$$

Let us see how V behaves, namely, we want to measure how well V solves the above problem.

Lemma 4.2.2. *Assume* (4.2.8) *holds true. Then there exists a constant $C > 0$ independent of ε such that for any $y \in \Omega_\varepsilon$,*

$$|R(y)| \le C\varepsilon \sum_{j=1}^{m} \frac{1}{1 + |y - \xi_j'|^{2+\sigma}}, \tag{4.2.14}$$

where $0 < \sigma < 1$ is a small fixed constant and

$$K(y) = \sum_{j=1}^{m} \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}[1 + \theta_\varepsilon(y)], \tag{4.2.15}$$

with

$$|\theta_\varepsilon(y)| \le C\varepsilon \sum_{j=1}^{m} [|y - \xi_j'| + 1]. \tag{4.2.16}$$

Proof: Let us fix a small number $\delta > 0$ and observe that $\varepsilon^4 k(\varepsilon y) e^{V(y)} = \varepsilon^4 k(x) e^{U(x)}$ with $x = \varepsilon y$. Note that, $|y - \xi_j'| > \frac{\delta}{\varepsilon}$ if and only if $|x - \xi_j| > \delta$. Hence, we see that if $|y - \xi_j'| > \frac{\delta}{\varepsilon}$ for all $j = 1, \ldots, m$ then

$$\varepsilon^4 k(\varepsilon y) e^{V(y)} = O(\varepsilon^4). \tag{4.2.17}$$

Similarly, we have

$$\Delta V(y) = \varepsilon^2 \Delta U(x) = -\varepsilon^2 \sum_{j=1}^{m} \left(\varepsilon^2 k(\xi_j) e^{u_j(x)} - \frac{\varepsilon^2 k(\xi_j)}{|\Omega|} \int_{\Omega} e^{u_j} \right)$$

and hence

$$\Delta V(y) = \frac{8\pi m}{|\Omega_\varepsilon|} + O(\varepsilon^4) \qquad \text{if} \quad |y - \xi_j'| > \frac{\delta}{\varepsilon} \quad \text{ for all } j = 1, \ldots, m. \tag{4.2.18}$$

On the other hand, assume that for certain j, $|y - \xi_j'| < \delta/\varepsilon$. Then setting $y = \xi_j' + z$ we get

$$K(y) = \varepsilon^4 k(\xi_j + \varepsilon z) \frac{8\mu_j^2}{(\mu_j^2 \varepsilon^2 + |\xi_j - \varepsilon z - \xi_j|^2)^2 k(\xi_j)} \exp\left(H_j(\xi_j + \varepsilon z) + \sum_{l \ne j} U_l(\xi_j + \varepsilon z) \right)$$

$$= k(\xi_j + \varepsilon z) \frac{8\mu_j^2}{k(\xi_j)(\mu_j^2 + |z|^2)^2} \exp\left(H_j(\xi_j + \varepsilon z) + \sum_{l \ne j} U_l(\xi_j + \varepsilon z) \right).$$

Now, we know that by the choice of μ_j, $j = 1, \ldots, m$ in (4.2.8) and (4.2.6)

$$\begin{aligned} H_j(\xi_j + \varepsilon z) &= H(\xi_j + \varepsilon z, \xi_j) - \Big[H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j)\Big] + O(\varepsilon^2 |\log \varepsilon|) \\ &= -\sum_{l \neq j} G(\xi_l, \xi_j) + O(\varepsilon |z|) + O(\varepsilon^2 |\log \varepsilon|). \end{aligned}$$

From Lemma 4.2.1, we deduce that for $l \neq j$

$$U_l(\xi_j + \varepsilon z) - G(\xi_l, \xi_j) = G(\xi_j + \varepsilon z, \xi_l) - G(\xi_l, \xi_j) + O(\varepsilon^2 |\log \varepsilon|) = O(\varepsilon |z|) + O(\varepsilon^2 |\log \varepsilon|)$$

in the considered region. Taking into account these relations we get then that

$$\begin{aligned} K(y) &= \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2} \cdot \frac{k(\xi_j + \varepsilon z)}{k(\xi_j)} \exp\Big(H(\xi_j + \varepsilon z, \xi_j) - H(\xi_j, \xi_j) \\ &\qquad + \sum_{l \neq j} [U_l(\xi_j + \varepsilon z) - G(\xi_l, \xi_j)] + O(\varepsilon^2 |\log \varepsilon|) \Big) \\ &= \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} [1 + O(\varepsilon |z|) + O(\varepsilon^2 |\log \varepsilon|)], \qquad |y - \xi_j'| < \frac{\delta}{\varepsilon}. \end{aligned} \tag{4.2.19}$$

We also have

$$\Delta V(y) = \varepsilon^2 \left(-\varepsilon^2 k(\xi_j) e^{u_j(x)} + \frac{\varepsilon^2 k(\xi_j)}{|\Omega|} \int_\Omega e^{u_j} - \sum_{l \neq j} \left[\varepsilon^2 k(\xi_l) e^{u_l(x)} - \frac{\varepsilon^2 k(\xi_l)}{|\Omega|} \int_\Omega e^{u_l} \right] \right).$$

So, by (4.2.3) we conclude that in this region

$$\begin{aligned} \Delta V(y) &= -\varepsilon^4 \frac{8\mu_j^2}{(\mu_j^2 \varepsilon^2 + \varepsilon^2 |y - \xi_j'|^2)^2} + \frac{8\pi m}{|\Omega|} \varepsilon^2 + O(\varepsilon^4) \\ &= -\frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} + \frac{8\pi m}{|\Omega_\varepsilon|} + O(\varepsilon^4). \end{aligned} \tag{4.2.20}$$

Also, we have that

$$\int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon z) e^{V(z)}\, dz = \varepsilon^2 \int_\Omega k(x) e^{U(x)}\, dx = \sum_{j=1}^m \varepsilon^2 \int_{B(\xi_j, \delta)} k(x) e^{U(x)}\, dx + A_\varepsilon.$$

Observe that $A_\varepsilon = \varepsilon^2 \Theta_\varepsilon(\xi)$ with Θ_ε a uniformly bounded function as $\varepsilon \to 0$. Now, by Lemma 4.2.1

$$\begin{aligned} \varepsilon^2 \int_{B(\xi_j,\delta)} k(x) e^{U(x)}\, dx &= \varepsilon^2 \int_{B(\xi_j,\delta)} k(x) \exp\Big(u_j(x) + H_j(x) + \sum_{l \neq j} U_l(x) \Big)\, dx \\ &= \varepsilon^2 \int_{B(\xi_j,\delta)} \frac{8\mu_j^2 k(x)}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2 k(\xi_j)} \exp\Big(H_j(x) + \sum_{l \neq j} U_l(x) \Big)\, dx \\ &= \frac{1}{\mu_j^4 \varepsilon^2} \int_{B(\xi_j,\delta)} \frac{k(x) e^{H(x,\xi_j) + \sum_{l \neq j} G(x,\xi_l) + O(\varepsilon^2 |\log \varepsilon|)}}{\Big(1 + \big(\frac{|x - \xi_j|}{\mu_j \varepsilon}\big)^2\Big)^2}\, dx \quad (x - \xi_j = \mu_j \varepsilon y) \\ &= \frac{1}{\mu_j^2} \int_{B(0, \frac{\delta}{\mu_j \varepsilon})} \frac{k(\xi_j + \mu_j \varepsilon y) e^{H(\xi_j + \mu_j \varepsilon y, \xi) + \sum_{l \neq j} G(\xi_j + \mu_j \varepsilon y, \xi_l)}}{(1 + |y|^2)^2}\, dy + O(\varepsilon^2 |\log \varepsilon|) \\ &= \pi \frac{k(\xi_j)}{\mu_j^2} e^{H(\xi_j,\xi_j) + \sum_{l \neq j} G(\xi_j, \xi_l)} + \varepsilon^2 |\log \varepsilon| \Theta_\varepsilon(\xi). \end{aligned}$$

And using the choice of μ_j, $j = 1, \ldots, m$ in (4.2.8), we get that

$$\varepsilon^2 \int_\Omega k(x) e^{U(x)}\, dx = 8\pi m + \varepsilon^2 |\log \varepsilon| \Theta_\varepsilon(\xi). \tag{4.2.21}$$

In summary, combining (4.2.17)-(4.2.21) we have established the following fact

$$R(y) = O(\varepsilon^4 |\log \varepsilon|), \qquad \text{if} \quad |y - \xi_j'| > \frac{\delta}{\varepsilon} \text{ for all } j = 1, \ldots, m$$

and if $|y - \xi_j'| < \frac{\delta}{\varepsilon}$ for some j

$$R(y) = \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} [O(\varepsilon |y - \xi_j'|) + O(\varepsilon^2 |\log \varepsilon|)] + O(\varepsilon^4 |\log \varepsilon|).$$

Therefore, from the definition of $*$-norm we conclude (4.2.14).

The estimates (4.2.15) and (4.2.16) follows from (4.2.17), (4.2.19) and similar arguments used to obtain (4.2.14). Indeed, note that if $|y - \xi_j'| < \frac{\delta}{\varepsilon}$ for some j then

$$K(y) = \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} [1 + O(\varepsilon |y - \xi_j'|) + O(\varepsilon^2 |\log \varepsilon|)]$$

and if $|y - \xi_j'| > \frac{\delta}{\varepsilon}$ for all $j = 1, \ldots, m$ then

$$K(y) = O(\varepsilon^4).$$

Therefore, (4.2.15) and (4.2.16) follows. This completes the proof. □

4.3 The associated linear problem

In this section, we will study the linearized operator under suitable orthogonality conditions. Thus we set

$$L(\phi) = \Delta \phi + K\phi - \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} K\phi, \tag{4.3.1}$$

for functions ϕ defined on Ω_ε, where K is a function that satisfies (4.2.15) and (4.2.16). Throughout the main part of this section, we only assume that the numbers μ_j, $j = 1, \ldots, m$ appearing in (4.2.15) satisfy $C_0^{-1} \le \mu_j \le C_0$ for all $j = 1, \ldots, m$ independently of ε and that the points $\xi_j \in \Omega$, $j = 1, \ldots, m$ are uniformly separated from each other, namely,

$$\operatorname{dist}(\xi_l - \xi_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) \ge \delta \qquad \text{for} \quad l \ne j \ , \tag{4.3.2}$$

where $\delta > 0$ is fixed. Recall that, from (4.2.8), we have that $\mu_j = \mu_j(\xi_1', \ldots, \xi_m')$, which will be considered at the end of this section.

Let us observe that $L(\phi) = \tilde{L}(\phi) + c(\phi)$, where

$$\tilde{L}(\phi) = \Delta \phi + K\phi \tag{4.3.3}$$

and $c(\phi) := -\frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} K\phi$. Formally, if we center the system of coordinates at ξ_j', by setting $z = y - \xi_j'$, then the operator $\tilde{L}$ approaches the linear operator in $\mathbb{R}^2$,

$$L_j(\phi) = \Delta \phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2} \phi,$$

namely, equation $\Delta v + e^v = 0$ linearized around the radial solution $v_j(z) = \log \frac{8\mu_j^2}{(\mu_j^2+|z|^2)^2}$. An important fact to develop the desired solvability theory is the non-degeneracy of v_j modulo the natural invariance of the equations under translations and dilations, $\zeta \mapsto v_j(z-\zeta)$ and $s \mapsto v_j(sz) - 2\log s$. Thus we set,

$$Y_{ij}(z) = -\frac{1}{4}\frac{\partial}{\partial \zeta_i} v_j(z+\zeta)\bigg|_{\zeta=0} = \frac{z_i}{\mu_j^2 + |z|^2}, \qquad i = 1,2, \qquad \text{and}$$

$$Y_{0j}(z) = -\frac{1}{2}\frac{\partial}{\partial s}[v_j(sz) + 2\log s]\bigg|_{s=1} = 1 - \frac{2\mu_j^2}{\mu_j^2 + |z|^2}.$$

As it is well know, it turns out that the only bounded solutions of $L_j(\phi) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of the Y_{ij}, $i = 0,1,2$, see [1] for a proof. Let us denote also $Z_{ij}(y) := Y_{ij}(y - \xi_j')$, $i = 0,1,2$ and $j = 1,\dots,m$. Also, an important goal in the study of operator L is to get rid of the presence of the term $c(\phi)$.

Additionally, let us consider a large but fixed number $R_0 > 0$ and a non-negative cut-off function $\chi = \chi(\rho)$ with $\chi(\rho) = 1$ if $\rho < R_0$ and $\chi(\rho) = 0$ if $\rho > R_0 + 1$. We denote

$$\chi_j(y) = \chi(|y - \xi_j'|).$$

Given h of class $C^{0,\alpha}(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} h = 0$, we prove first a priori estimates for the problem

$$\begin{cases} L(\phi) \;=\; h, & \text{in } \Omega_\varepsilon, \\ \phi \quad \text{doubly periodic on} & \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \quad \text{for all } i = 1,2, j = 1,\dots,m & \int_{\Omega_\varepsilon} \phi = 0, \end{cases} \tag{4.3.4}$$

where points $\xi_j \in \Omega$, $j = 1,\dots,m$ satisfy (4.3.2). Thus, we consider the norms

$$\|\psi\|_\infty = \sup_{y\in\Omega_\varepsilon} |\psi(y)|, \quad \|\psi\|_* = \sup_{y\in\Omega_\varepsilon} \left[\sum_{j=1}^m (1 + |y - \xi_j'|)^{-2-\sigma}\right]^{-1} |\psi(y)|,$$

where $0 < \sigma < 1$ is a small fixed constant.

Proposition 4.3.1. *Let $\delta > 0$ be fixed. There exist positive numbers ε_0 and C, such that for any points $\xi_j \in \tilde{\Omega}$, $j = 1,\dots,m$, which satisfy* (4.3.2)*, and any solution ϕ to problem* (4.3.4)*, one has*

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*, \tag{4.3.5}$$

for all $\varepsilon < \varepsilon_0$.

We observe that the orthogonality conditions in the problem above are only taken with respect to the elements of the approximate kernel due to translations. Our functional frame is $E(\Omega_\varepsilon)$.

Proof: The proof of estimate (4.3.5) consists of several steps. Let assume the opposite, namely, the existence of sequences $\varepsilon_n \to 0$, points $\xi_j^n \in \Omega$, which satisfy (4.3.2), functions h_n with $\log\frac{1}{\varepsilon_n}\|h_n\|_* \to 0$ as $n \to +\infty$, ϕ_n with $\|\phi_n\|_\infty = 1$, and

$$\begin{cases} L(\phi_n) \;=\; h_n, & \text{in } \Omega_{\varepsilon_n}, \\ \phi_n \quad \text{doubly periodic on} & \partial\Omega_{\varepsilon_n}, \\ \int_{\Omega_{\varepsilon_n}} \chi_j Z_{ij}\phi_n = 0, \quad \text{for all } i = 1,2, j = 1,\dots,m & \int_{\Omega_{\varepsilon_n}} \phi_n = 0, \end{cases} \tag{4.3.6}$$

Without loss of generality, we assume that $\xi_j^n \to \xi_j^*$ as $n \to +\infty$ and $\xi_j^* \in \Omega$ for all $j = 1, \dots, m$, by the doubly periodic boundary conditions. Let us define $\hat{\phi}_n(x) := \phi_n(x/\varepsilon_n)$ for $x \in \Omega$. We have the following fact.

Claim 4.3.1. *There holds $\hat{\phi}_n \to 0$ as $n \to +\infty$ in $C^{2,\alpha}$ uniformly over compact subsets of $\bar{\Omega} \setminus \{\xi_1^*, \dots, \xi_m^*\}$. In particular, given any $\delta_0 > 0$ we have*

$$\|\hat{\phi}_n\|_{L^\infty(\Omega \setminus \cup_{j=1}^m B(\xi_j^n, \delta_0))} \to 0 \qquad \text{as } n \to +\infty. \tag{4.3.7}$$

Proof: Note that as $n \to +\infty$

$$c(\phi_n) = -\frac{1}{|\Omega_{\varepsilon_n}|} \int_{\Omega_{\varepsilon_n}} \varepsilon_n^4 k(\varepsilon_n z) e^{V(z)} \phi_n(z)\, dz = -\frac{\varepsilon_n^2}{|\Omega|} \int_\Omega \varepsilon_n^2 k(x) e^{U(x)} \hat{\phi}_n(x)\, dx = O(\varepsilon_n^2),$$

since ϕ_n is uniformly bounded and from the definition of U

$$\frac{1}{|\Omega|} \int_\Omega \varepsilon_n^2 k(x) e^{U(x)} \hat{\phi}_n(x)\, dx = O(1).$$

Hence, up to a subsequence, we get that as $n \to +\infty$

$$\frac{1}{|\Omega|} \int_\Omega \varepsilon_n^2 k(x) e^{U(x)} \hat{\phi}_n(x)\, dx = c_0 + o(1).$$

Furthermore, we have that $\hat{\phi}$ is doubly periodic on on $\partial\Omega$,

$$\|\hat{\phi}_n\|_{L^\infty(\Omega)} = 1 \qquad \text{and} \qquad \int_\Omega \hat{\phi}_n = 0.$$

Then

$$\Delta\hat{\phi}_n(x) = \frac{1}{\varepsilon_n^2} \Delta\phi_n\left(\frac{x}{\varepsilon_n}\right) = -\varepsilon_n^2 k(x) e^{U(x)} \hat{\phi}_n(x) + \frac{1}{|\Omega|} \int_\Omega \varepsilon_n^2 k e^U \hat{\phi}_n + \hat{h}_n(x),$$

where $\hat{h}_n(x) = \frac{1}{\varepsilon_n^2} h_n\left(\frac{x}{\varepsilon_n}\right)$. Hence, given $\delta' > 0$ we get

$$\Delta\hat{\phi}_n(x) = O(\varepsilon_n^2) + c_0 + o(1) \qquad \text{uniformly for} \quad x \in \Omega \setminus \cup_{j=1}^m B(\xi_j^n, \delta')$$

since if $|y - (\xi_j^n)'| > \frac{\delta'}{\varepsilon_n}$ for all $j = 1, \dots, m$ then $K(y) = O(\varepsilon^4)$ and if $|x - \xi_j^n| > \delta'$ for all $j = 1, \dots, m$ then

$$\frac{1}{\varepsilon_n^2} \left| h_n\left(\frac{x}{\varepsilon_n}\right) \right| \le \|h_n\|_* \left[\sum_{j=1}^m \frac{\varepsilon_n^{1-\sigma}}{\left(\varepsilon_n + |x - \xi_j^n|\right)^{2+\sigma}} \right] \le C\|h_n\|_*.$$

Therefore, passing to a subsequence $\hat{\phi}_n \to \hat{\phi}$ as $n \to +\infty$ in $C^{2,\alpha}$ sense over compact subsets of $\Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$. Also, we have that

$$\begin{cases} \Delta\hat{\phi} = c_0, & \text{in } \Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}, \\ \hat{\phi} \quad \text{doubly periodic on} & \partial\Omega \end{cases}$$

Since $|\hat{\phi}(x)| \le 1$ for all $x \in \Omega \setminus \{\xi_1^*, \dots, \xi_m^*\}$, it follows that $\hat{\phi}$ can be extended continuously to Ω and satisfies

$$\begin{cases} \Delta\hat{\phi} = c_0, & \text{in } \Omega, \\ \hat{\phi} \quad \text{doubly periodic on} & \partial\Omega, \\ \int_\Omega \hat{\phi} = 0, \end{cases} \tag{4.3.8}$$

using dominated convergence. By, $\int_\Omega \Delta\hat{\phi} = 0$ we get that $c_0 = 0$. Therefore, $\hat{\phi} \equiv 0$, and the claim follows. □

We follow ideas shows in [31, 40] to prove an estimate for ϕ_n. We use that $\tilde{L}$ is given by (4.3.3), where K is a function that satisfies (4.2.15) and (4.2.16).

Claim 4.3.2. *The operator $\tilde{L}$ satisfies maximum principle in $\Omega_{R,\delta_0} := \cup_{j=1}^m [B(\xi_j', \frac{\delta_0}{\varepsilon}) \setminus \bar{B}(\xi_j', R)]$ for $R > 0$ large enough and $\delta_0 > 0$ small.*

Proof: First, observe that from (4.2.15) and (4.2.16) we have that there is a constant D_0 such that for all $y \in \Omega_\varepsilon$

$$K(y) \leq D_0 \sum_{j=1}^m \frac{1}{(1 + |y - \xi_j'|^2)^2}.$$

Now, consider the increasing function $Y_0(r) = \dfrac{r^2 - 1}{1 + r^2}$, radial solution in $\mathbb{R}^2$ of

$$\Delta Y_0 + \frac{8}{(1 + r^2)^2} Y_0 = 0.$$

Define a comparison function in Ω_ε,

$$g(y) = \sum_{j=1}^m Y_0(a|y - \xi_j'|) = \sum_{j=1}^m \frac{a^2|y - \xi_j'|^2 - 1}{1 + a^2|y - \xi_j'|^2}, \quad y \in \Omega_\varepsilon.$$

Let us observe that

$$-\Delta g = \sum_{j=1}^m \frac{8a^2(a^2|y - \xi_j'|^2 - 1)}{(1 + a^2|y - \xi_j'|^2)^3}.$$

So, that for $|y - \xi_j'| > R$ for all j,

$$\begin{aligned} -\Delta g &\geq 8 \sum_{j=1}^m \frac{a^2}{(1 + a^2|y - \xi_j'|^2)^2} \frac{a^2R^2 - 1}{1 + a^2R^2} \geq 2 \sum_{j=1}^m \frac{a^2}{(1 + a^2|y - \xi_j'|^2)^2} \\ &\geq 2 \sum_{j=1}^m \frac{a^2R^4}{(1 + a^2R^2)^2} \frac{1}{|y - \xi_j'|^4} \geq \frac{1}{2a^2} \sum_{j=1}^m \frac{1}{|y - \xi_j'|^4}, \end{aligned}$$

if we choose $a^2R^2 > \frac{5}{3} > 1$. On the other hand, it is readily checked that $g(y) \leq m$ so, in the same region,

$$Kg \leq D_0 m \sum_{j=1}^m \frac{1}{(1 + |y - \xi_j'|^2)^2} \leq D_0 m \sum_{j=1}^m \frac{1}{|y - \xi_j'|^4}.$$

Hence, if a is taken so that $0 < a < \frac{1}{\sqrt{2D_0 m}}$ and fixed, and $R > 0$ is chosen such that $a^2R^2 > \frac{5}{3}$, then we have that

$$\tilde{L}(g) = \Delta g + Kg \leq \left(-\frac{1}{2a^2} + D_0 m \right) \sum_{j=1}^m \frac{1}{|y - \xi_j'|^4} < 0 \qquad \text{in } \Omega_{R,\delta_0}.$$

Since, for all $y \in \Omega_{R,\delta_0}$

$$g(y) \geq m \frac{a^2R^2 - 1}{1 + a^2R^2} \geq \frac{m}{4} > 0,$$

we then conclude that $\tilde{L}$ satisfies Maximum principle, namely if $\tilde{L}(\psi) \leq 0$ in Ω_{R,δ_0} and $\psi \geq 0$ on $\partial\Omega_{R,\delta_0}$ then $\psi \geq 0$ in Ω_{R,δ_0}. □

Let us fix such a number $R > 0$ which we may take larger whenever it is needed and a small $\delta_0 > 0$. Now, let us consider the "annulus norm" and "boundary annulus norm"

$$\|\phi\|_a = \|\phi\|_{L^\infty(\Omega_{R,\delta_0})} \qquad \text{and} \qquad \|\phi\|_b = \|\phi\|_{L^\infty(\partial\Omega_{R,\delta_0})}.$$

Note that $\partial\Omega_{R,\delta_0} = \cup_{j=1}^m [\partial B(\xi'_j, R) \cup \partial B(\xi'_j, \frac{\delta_0}{\varepsilon})]$. We have the following estimate.

Claim 4.3.3. *There is a constant $C > 0$ such that if $\tilde{L}(\phi) = h$ in Ω_ε then*

$$\|\phi\|_a \le C[\|\phi\|_b + \|h\|_*]. \tag{4.3.9}$$

Proof: We will establish this inequality with the use of suitable barriers. Let M be a large number such that for all j, $\Omega_\varepsilon \subset B(\xi'_j, \frac{M}{\varepsilon})$. Consider now the solution of the problem

$$-\Delta\psi_j = \frac{2}{|y-\xi'_j|^{2+\sigma}}, \quad R < |y-\xi'_j| < \frac{M}{\varepsilon},$$

$$\psi_j(y) = 0 \text{ for } |y-\xi'_j| = R, \quad |y-\xi'_j| = \tfrac{M}{\varepsilon}.$$

A direct computation shows that

$$\psi_j(r) = \frac{2}{\sigma^2 R^\sigma} - \frac{2}{\sigma^2 r^\sigma} + \left[\frac{2\varepsilon^\sigma}{\sigma^2 M^\sigma} - \frac{2}{\sigma^2 R^\sigma}\right] \frac{\log\frac{r}{R}}{\log\frac{M}{\varepsilon R}}, \qquad r = |y-\xi_j|.$$

Note that

$$\frac{2\varepsilon^\sigma}{\sigma^2 M^\sigma} - \frac{2}{\sigma^2 R^\sigma} < 0 \qquad \text{and} \qquad 0 \le \psi_j \le \frac{2}{\sigma^2 R^\sigma} - \frac{2\varepsilon^\sigma}{\sigma^2 M^\sigma} \le \frac{2}{\sigma^2 R^\sigma}.$$

hence these functions ψ_j have a uniform bound independent of ε. On the other hand, let us consider the function g defined above, and let us set

$$\psi(y) = 4\|\phi\|_b\, g(y) + \|h\|_* \sum_{j=1}^m \psi_j(y).$$

Then, it is easily checked that, choosing R larger if necessary, $\tilde{L}(\psi) \le h$ and $\psi \ge |\phi|$ on $\partial\Omega_{R,\delta_0}$. Hence $|\phi| \le \psi$ in Ω_{R,δ_0}. In fact, we have that for all $y \in \partial\Omega_{R,\delta_0}$

$$\psi(y) \ge 4\|\phi\|_b\, g(y) \ge \|\phi\|_b \ge |\phi(y)|.$$

Also, we have that choosing $2D_0 m \le \sigma^2 R^\sigma$ (for R large enough)

$$\begin{aligned}
\tilde{L}(\psi) &= 4\|\phi\|_b\, \tilde{L}(g) + \|h\|_* \sum_{j=1}^m \tilde{L}(\psi_j) < \|h\|_* \sum_{j=1}^m (\Delta\psi_j + K\psi_j) \\
&\le \|h\|_* \sum_{j=1}^m \left(-\frac{2}{|y-\xi'_j|^{2+\sigma}} + D_0 \sum_{l=1}^m \frac{1}{(1+|y-\xi'_l|^2)^2} \frac{2}{\sigma^2 R^\sigma} \right) \\
&\le \|h\|_* \sum_{j=1}^m \left(-\frac{2}{|y-\xi'_j|^{2+\sigma}} + \frac{2D_0 m}{\sigma^2 R^\sigma} \frac{1}{(1+|y-\xi'_j|^2)^2} \right) \\
&\le -\|h\|_* \sum_{j=1}^m \frac{1}{(1+|y-\xi'_j|)^{2+\sigma}} \le h,
\end{aligned}$$

since
$$-\frac{2}{|y-\xi_j'|^{2+\sigma}}+\frac{2D_0m}{\sigma^2R^\sigma}\,\frac{1}{(1+|y-\xi_j'|^2)^2}\le-\frac{1}{(1+|y-\xi_j'|)^{2+\sigma}}.$$
Hence, we conclude that $|\phi(y)|\le\psi(y)$ for all $R<|y-\xi_j'|<\frac{\delta_0}{\varepsilon}$, $j=1,\dots,m$ and the claim follows. □

The following intermediate result provides another estimate. Again, for notational simplicity *we omit* the subscript n in the quantities involved.

Lemma 4.3.1. *There exist constants $C>0$ such that for large n*
$$\|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',\frac{\delta_0}{\varepsilon}))}\le C\left\{\|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',R))}+o(1)\right\}. \tag{4.3.10}$$

Proof: First, note that from estimate (4.3.9) we deduce that there is a constant $C>0$ such that if $L(\phi)=h$ in Ω_ε then
$$\|\phi\|_a\le C\left[\|\phi\|_b+\|h\|_*+\frac{|c(\phi)|}{\varepsilon^2}\right]. \tag{4.3.11}$$
Indeed, let us consider the function
$$\tilde\phi(y)=\phi(y)+c(\phi)\,\frac{|y-\xi_j'|^2}{4},\qquad y\in\Omega_\varepsilon.$$
Then,
$$\tilde L(\tilde\phi)=\tilde L(\phi)+c(\phi)+c(\phi)\,\frac{|y-\xi_j'|^2}{4}K=h+c(\phi)\,\frac{|y-\xi_j'|^2}{4}K.$$
From (4.2.15) and (4.2.16), it readily follows that $\|K\|_*\le C$. Thus, by estimate (4.3.9) we get that there is a constant $C>0$ such that
$$\|\tilde\phi\|_a\le C\left[\|\tilde\phi\|_b+\|h\|_*+\frac{|c(\phi)|}{\varepsilon^2}\right],$$
since
$$\left\|h+\frac{|c(\phi)|}{4}|\cdot-\xi_j'|^2K\right\|_*\le\|h\|_*+\frac{|c(\phi)|}{4}\|K\|_*\sup_{y\in\Omega_\varepsilon}|y-\xi_j'|^2.$$
Also, we have that
$$\begin{aligned}\|\phi\|_a&\le\|\tilde\phi\|_a+\left\|\frac{c(\phi)}{4}|\cdot-\xi_j'|^2\right\|_a\\&\le C\left[\|\tilde\phi\|_b+\|h\|_*+\frac{|c(\phi)|}{\varepsilon^2}\right]+\frac{|c(\phi)|}{4}\sup_{y\in\Omega_\varepsilon}|y-\xi_j'|^2\\&\le C\left[\|\phi\|_b+\|h\|_*+\frac{|c(\phi)|}{\varepsilon^2}\right],\end{aligned}$$
since
$$\|\tilde\phi\|_b\le\|\phi\|_b+\frac{|c(\phi)|}{\varepsilon^2}.$$
From (4.3.7) we find that for large n
$$\|\phi\|_{L^\infty(\Omega_\varepsilon\setminus\cup_{l=1}^m B(\xi_j',\frac{\delta_0}{\varepsilon}))}=o(1). \tag{4.3.12}$$

Furthermore, we have that $c(\phi) = o(\varepsilon^2)$, since $c_0 = 0$. By the assumption, we know that $\|h\|_* = o(1)$. Now, from (4.3.11) it is clear that

$$\begin{aligned}\|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j', \frac{\delta_0}{\varepsilon}))} &\le \max\{\|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',R))}, \|\phi\|_a\} \\ &\le \|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',R))} + C\left[\|\phi\|_b + \|h\|_* + \frac{|c(\phi)|}{\varepsilon^2}\right] \\ &\le C\|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',R))} + o(1),\end{aligned}$$

since by (4.3.12) we get that

$$\|\phi\|_b \le \|\phi\|_{L^\infty(\cup_{j=1}^m \partial B(\xi_j',R))} + o(1) \le \|\phi\|_{L^\infty(\cup_{j=1}^m B(\xi_j',R))} + o(1).$$

Therefore, we conclude (4.3.10) and this completes the proof. □

We continue with the proof of Proposition 4.3.1 and we get the following fact.

Claim 4.3.4. *There exists an index $j \in \{1, \dots, m\}$ such that passing to a subsequence if necessary,*

$$\liminf_{n\to\infty} \|\phi_n\|_{L^\infty(B((\xi_j^n)',R))} \ge \kappa > 0. \tag{4.3.13}$$

Proof: Arguing by contradiction, if for all $j = 1, \dots, m$

$$\liminf_{n\to\infty} \|\phi_n\|_{L^\infty(B((\xi_j^n)',R))} = 0,$$

then (4.3.10) and (4.3.12) implies that, passing to a subsequence if necessary, $\|\phi_n\|_\infty \to 0$ as $n \to +\infty$. On the other hand, we know that $\|\phi\|_\infty = 1$ for all $n \in \mathbb{N}$. This conclude (4.3.13). □

Let us set $\psi_{n,j}(z) = \phi_n((\xi_j^n)' + z)$ for any j. We notice that $\psi_{n,j}$ satisfies

$$\Delta\psi_{n,j} + K((\xi_j^n)' + z)\,\psi_{n,j} = h_n((\xi_j^n)' + z) - c(\phi_n), \quad \text{in } \Omega_{n,j} \equiv \Omega_{\varepsilon_n} - (\xi_j^n)'\ .$$

Elliptic estimates and (4.3.13) readily imply that ψ_n converges uniformly over compact subsets of $\mathbb{R}^2$ to a bounded, non-zero solution ψ_j^* of

$$\Delta\psi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\psi = 0.$$

This implies that ψ_j^* is a linear combination of the functions Y_{ij}, $i = 0, 1, 2$. Thus, we have that for some constants a_{ij}, $i = 0, 1, 2$

$$\psi_j^* = a_{0j}Y_{0j} + a_{1j}Y_{1j} + a_{2j}Y_{2j}.$$

But, from (4.3.6), orthogonality conditions over $\psi_{n,j}$ pass to the limit thanks to $\|\psi_{n,j}\|_\infty \le C$ and dominated convergence, namely,

$$\int_{\mathbb{R}^2} \chi(|z|)Y_{ij}\psi_j^* = 0, \qquad \text{for} \quad i = 1, 2.$$

This implies that $a_{1j} = a_{2j} = 0$ and $\psi_j^* = a_{0j}\,Y_{0j}$. A contradiction with (4.3.13) arises if we are able to show that $a_{0j} = 0$. The assumption on h_n, $|\log\varepsilon_n|\,\|h_n\|_* = o(1)$ allows us to get $a_{0j} = 0$.

Claim 4.3.5. *There holds $a_{0j} = 0$.*

Proof: Let us construct a suitable test function in order to get the additional orthogonality relation

$$\int_{\mathbb{R}^2} \frac{8\mu_j^2(\mu_j^2 - |z|^2)}{(\mu_j^2 + |z|^2)^3}\, \psi_j^*(z)\, dz = 0, \tag{4.3.14}$$

which implies $a_{0j} = 0$ as claimed. We will use an idea developed first in [37] and then exploited in [38, 39, 40]. This idea has been used also to prove claim 3.3.4.

Define the functions $w_{n,j}$ and $\tilde{w}_{n,j}$ for $x \in \Omega$ given by

$$w_{n,j}(x) = \frac{4}{3}\log(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)\frac{\mu_j^2\varepsilon^2 - |x-\xi_j|^2}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2} + \frac{8}{3}\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2}$$

and

$$\tilde{w}_{n,j}(x) = -\frac{2\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2}.$$

Let us note that $w_{n,j}$ and $\tilde{w}_{n,j}$ satisfy

$$\Delta w_{n,j} + \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} w_{n,j} = 2\frac{8\mu_j^2\varepsilon^2(\mu_j^2\varepsilon^2 - |x-\xi_j|^2)}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^3}$$

and

$$\Delta \tilde{w}_{n,j} + \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \tilde{w}_{n,j} = -\frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2}.$$

Consider the test function $Z_{n,j}$ satisfying

$$\begin{cases} \Delta Z_{n,j} = \Delta z_{n,j} - \frac{1}{|\Omega|}\int_\Omega \Delta z_{n,j}(x)\, dx, & \text{in } \Omega, \\ Z_{n,j} \quad \text{doubly periodic on} \quad \partial\Omega, \\ \int_\Omega Z_{n,j} = 0, \end{cases}$$

where $z_{n,j} = w_{n,j} + \frac{2}{3}H(\xi_j,\xi_j)\tilde{w}_{n,j}$. Observe that from the representation formula (2.2.4) we get that

$$Z_{n,j} - z_{n,j} - \frac{2}{3}H(\cdot,\xi_j) = O(\varepsilon^2|\log\varepsilon|), \tag{4.3.15}$$

in C^2-sense over compact subset of Ω. Recall that $\hat{\phi}_n$ satisfies

$$\Delta\hat{\phi}_n + \varepsilon_n^2 k(x)e^U\hat{\phi}_n - \frac{1}{|\Omega|}\int_\Omega \varepsilon_n^2 k e^U \hat{\phi}_n = \hat{h}, \qquad \text{in} \quad \Omega.$$

Multiply this equation by $Z_{n,j}$ and integrate on Ω, since $\int_\Omega Z_{n,j} = \int_\Omega \hat{\phi}_n = 0$ we get that

$$\begin{aligned} \int_\Omega \hat{h} Z_{n,j} &= \int_\Omega \hat{\phi}_n\left(\Delta Z_{n,j} + \varepsilon^2 k(x)e^U Z_{n,j}\right) = \int_\Omega \hat{\phi}_n \Delta z_{n,j} + \int_\Omega \varepsilon^2 k(x)e^U\hat{\phi}_n Z_{n,j} \\ &= 2\int_\Omega \hat{\phi}_n \frac{8\mu_j^2\varepsilon^2(\mu_j^2\varepsilon^2 - |x-\xi_j|^2)}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^3} + \int_\Omega \hat{\phi}_n \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2}\left(Z_{n,j} - z_{n,j} - \frac{2}{3}H(\xi_j,\xi_j)\right) \\ &\quad + \int_\Omega \left(\varepsilon^2 k(x)e^U - \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2}\right)\hat{\phi}_n Z_{n,j} \end{aligned}$$

From (4.3.15) we get that $Z_{n,j} = z_{n,j} + O(1) = O(|\log \varepsilon_n|)$, then

$$\int_\Omega \hat{h} Z_{n,j} = \int_{\Omega_{\varepsilon_n}} h_n(y) Z_{n,j}(\varepsilon_n y)\, dy = O(|\log \varepsilon_n|\, \|h_n\|_*) = o(1)$$

as $n \to +\infty$. On the other hand, we have that $\psi_{n,j}(z) = \hat{\phi}_n(\xi_j^n + \varepsilon_n z)$ then as $n \to +\infty$

$$\int_\Omega \hat{\phi}_n(x) \frac{8\mu_j^2\varepsilon^2(\mu_j^2\varepsilon^2 - |x-\xi_j|^2)}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^3}\, dx = \int_{B\left(0,\frac{\delta}{\varepsilon_n}\right)} \psi_{n,j}(z) \frac{8\mu_j^2(\mu_j^2 - |z|^2)}{(\mu_j^2 + |z|^2)^3}\, dx + O(\varepsilon_n^2)$$

$$\begin{aligned}
\int_\Omega \hat{\phi}_n \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} &\left(Z_{n,j} - z_{n,j} - \frac{2}{3} H(\xi_j,\xi_j) \right) \\
&= \int_\Omega \hat{\phi}_n \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \left(Z_{n,j}(x) - z_{n,j}(x) - \frac{2}{3} H(x,\xi_j) \right) dx \\
&\quad + \frac{2}{3} \int_\Omega \hat{\phi}_n \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \left(H(x,\xi_j) - H(\xi_j,\xi_j) \right) dx \\
&= O(\varepsilon_n)
\end{aligned}$$

$$\begin{aligned}
\int_\Omega &\left(\varepsilon^2 k(x) e^U - \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \right) \hat{\phi}_n Z_n \\
&= \int_{B(\xi_j,\delta)} \left(\varepsilon^2 k(x) e^U - \frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2 + |x-\xi_j|^2)^2} \right) \hat{\phi}_n Z_{n,j} + O(\varepsilon_n^2 |\log \varepsilon_n|) \\
&\quad + \sum_{l=1, l\neq j}^m \int_{B(\xi_l,\delta)} \varepsilon^2 k(x) e^U \hat{\phi}_n Z_{n,j} \\
&= \int_{B(\xi_j', \frac{\delta}{\varepsilon_n})} \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2} \theta_\varepsilon(y) \phi_n(y) Z_{n,j}(\varepsilon_n y)\, dy + O(\varepsilon_n^2 |\log \varepsilon_n|) \\
&\quad + \sum_{l=1, l\neq j}^m \int_{B(0,\frac{\delta}{\varepsilon_n})} \frac{8\mu_l^2}{(\mu_l^2 + |z|^2)^2} [1 + O(\varepsilon|z|) + O(\varepsilon^2 |\log\varepsilon|)] \psi_{n,l}(z) Z_{n,j}(\xi_l + \varepsilon z)\, dz \\
&= o(1),
\end{aligned}$$

since if $l \neq j$ then we find that $Z_{n,j}(\xi_l + \varepsilon z) = \frac{2}{3} G(\xi_l, \xi_j) + O(\varepsilon)$ for all $|z| < \frac{\delta}{\varepsilon}$ and

$$\int_{B(0,\frac{\delta}{\varepsilon_n})} \frac{8\mu_l^2}{(\mu_l^2 + |z|^2)^2} [1 + O(\varepsilon|z|) + O(\varepsilon^2 |\log\varepsilon|)] \psi_{n,l}(z) Z_{n,j}(\xi_l + \varepsilon z)\, dz = o(1)$$

thanks to dominated convergence. Therefore, we conclude (4.3.14) and hence, $a_{0j} = 0$.

□

This conclude the proof of proposition 4.3.1. □

Consider the linear problem of finding a function ϕ and scalars c_{ij}, $i = 1, 2$, $j = 1, \ldots, m$ and such that

$$\begin{cases} L(\phi) = h + \sum_{i=1}^2 \sum_{j=1}^m c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \phi \quad \text{doubly periodic on} & \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \phi = 0, \quad \text{for all } i = 1, 2,\ j = 1, \ldots, m & \int_{\Omega_\varepsilon} \phi = 0, \end{cases} \tag{4.3.16}$$

where $h \in L^\infty(\Omega_\varepsilon)$, $\|h\|_* < +\infty$ and $\int_{\Omega_\varepsilon} h = 0$. Our main result for the problem (4.3.16) states its solvability, for any points $\xi_j \in \Omega$ uniformly separated from each other. Let us stress that the right hand side of the equation of $L(\phi)$ integrates zero.

We are now ready for the proof of our main result of this section.

Proposition 4.3.2. *Let $\delta > 0$ be fixed. There exist positive numbers ε_0 and C, such that for any points $\xi_j \in \tilde{\Omega}$, $j = 1, \ldots, m$, satisfying (4.3.2), there is a unique solution to problem (4.3.16) for all $\varepsilon < \varepsilon_0$. Moreover,*

$$\|\phi\|_\infty \le C \left(\log \frac{1}{\varepsilon}\right) \|h\|_*, \qquad |c_{ij}| \le C\|h\|_*, \quad i = 1, 2,\ j = 1, \ldots, m. \tag{4.3.17}$$

Proof: We begin by establishing the validity of the a priori estimate (4.3.17). We have the equation

$$L(\phi) = h + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\chi_j Z_{ij}. \tag{4.3.18}$$

So, by a priori estimates

$$\|\phi\|_\infty \le C\left(\log \frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\right]. \tag{4.3.19}$$

since for all $i = 1, 2$, $j = 1, \ldots, m$ we have that $\|\chi_j Z_{ij}\|_* \le C$. Hence, it suffices to estimate the values of the constants $|c_{ij}|$. Next, we consider a smooth cut-off function $\eta = \eta(r)$ with the following properties: $\eta(r) = 1$ for $r < \frac{\delta}{4\varepsilon}$, $\eta(r) = 0$ for $r > \frac{\delta}{3\varepsilon}$, $|\eta'(r)| \le C\varepsilon$, $|\eta''(r)| \le C\varepsilon^2$. Then we set

$$\eta_j(y) = \eta(|y - \xi_j'|) \tag{4.3.20}$$

We test equation (4.3.18) against $\eta_j Z_{ij}$ to find for $i = 1, 2$

$$\langle L(\phi), \eta_j Z_{ij}\rangle = \langle h, \eta_j Z_{ij}\rangle + c_{ij}\int_{\Omega_\varepsilon} \chi_j |Z_{ij}|^2, \tag{4.3.21}$$

where $\langle f, g\rangle = \int_{\Omega_\varepsilon} fg$. Now, we find that

$$\langle L(\phi), \eta_j Z_{ij}\rangle = \langle \Delta\phi + K\phi + c(\phi), \eta_j Z_{ij}\rangle = \int_{\Omega_\varepsilon} [\Delta\eta_j Z_{ij} + K\eta_j Z_{ij}]\,\phi + c(\phi)\int_{\Omega_\varepsilon} \eta_j Z_{ij}.$$

Thus, we get that

$$\langle L(\phi), \eta_j Z_{ij}\rangle = \langle \phi, \tilde{L}(\eta_j Z_{ij})\rangle, \qquad \text{for all}, \quad i = 1, 2,\ j = 1, \ldots, m.$$

And we have

$$\begin{aligned}\tilde{L}(\eta_j Z_{ij}) &= \Delta\eta_j Z_{ij} + 2\nabla\eta_j\nabla Z_{ij} + \eta_j(\Delta Z_{ij} + KZ_{ij})\\ &= \Delta\eta_j Z_{ij} + 2\nabla\eta_j\nabla Z_{ij} + e^{v_j(\cdot - \xi_j')}\theta_\varepsilon\eta_j Z_{ij} + O(\varepsilon^4).\end{aligned}$$

Furthermore, we find for $i = 1, 2$

$$\begin{aligned}\int_{\Omega_\varepsilon} |\Delta\eta_j Z_{ij}| &\le \int_{B(\xi_j', \frac{\delta}{3\varepsilon})\setminus B(\xi_j', \frac{\delta}{4\varepsilon})} \left[|\eta''(|y - \xi_j'|)| + \frac{1}{|y - \xi_j'|}|\eta'(|y - \xi_j'|)|\right]\frac{|y - \xi_j'|}{\mu_j^2 + |y - \xi_j'|^2}\,dy\\ &\le C\varepsilon^2\int_{B(\xi_j', \frac{\delta}{3\varepsilon})\setminus B(\xi_j', \frac{\delta}{4\varepsilon})} \frac{|y - \xi_j'|}{\mu_j^2 + |y - \xi_j'|^2}\,dy + C\varepsilon\int_{B(\xi_j', \frac{\delta}{3\varepsilon})\setminus B(\xi_j', \frac{\delta}{4\varepsilon})} \frac{1}{\mu_j^2 + |y - \xi_j'|^2}\,dy,\\ &\le C\varepsilon\end{aligned}$$

$$\int_{\Omega_\varepsilon} |\nabla\eta_j \nabla Z_{ij}| \le \int_{B(\xi_j',\frac{\delta}{3\varepsilon})\setminus B(\xi_j',\frac{\delta}{4\varepsilon})} |\eta'(|y-\xi_j'|)|\,|\nabla Z_{ij}(y)|\,dy \le C\varepsilon$$

and similarly

$$\int_{\Omega_\varepsilon} |e^{v_j(\cdot-\xi_j')}\theta_\varepsilon \eta_j Z_{ij}| \le C\varepsilon.$$

Thus, for $i=1,2$

$$\left|\langle \phi, \tilde{L}(\eta_j Z_{ij})\rangle\right| \le C\varepsilon\|\phi\|_\infty.$$

From (4.3.19) and (4.3.21), we get the following inequality for $i=1,2$

$$\begin{aligned}|c_{ij}|\left|\int_{\Omega_\varepsilon} \chi_j Z_{ij}^2\right| &\le \left|\int_{\Omega_\varepsilon} \phi\,\tilde{L}(\eta_j Z_{ij})\right| + \int_{\Omega_\varepsilon} |h|\,|\eta_j Z_{ij}| \le C\varepsilon\|\phi\|_\infty + C\|h\|_* \\ &\le C\varepsilon \log\frac{1}{\varepsilon}\left[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\right] + C\|h\|_*\end{aligned}$$

and the estimate

$$|c_{ij}|\left|\int_{\Omega_\varepsilon} \chi_j Z_{ij}^2\right| \le C\left[\|h\|_* + \varepsilon\log\frac{1}{\varepsilon}\Big(\sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\Big)\right]. \tag{4.3.22}$$

Also, we have that there is a constant $C = C(R_0)$ independent of ε such that

$$\left|\int_{\Omega_\varepsilon} \chi_j Z_{ij}^2\right| \ge C.$$

Combining this estimate with (4.3.22) we obtain for $i=1,2$

$$|c_{ij}| \le C\left[\|h\|_* + \varepsilon\log\frac{1}{\varepsilon}\Big(\sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\Big)\right]$$

which implies $|c_{ij}| \le C\|h\|_*$ for all $i=1,2,\ j=1,\dots,m$. It follows finally from (4.3.19) that $\|\phi\|_\infty \le C(\log\frac{1}{\varepsilon})\|h\|_*$ and the a priori estimate has been thus proven. It only remains to prove the solvability assertion. To this purpose we consider the space

$$H = \left\{\phi \in E(\Omega_\varepsilon)\ :\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\,\phi \,=\, 0 \quad \text{for } \ i=1,2,\ j=1,\dots,m\right\},$$

endowed with the usual inner product $[\phi,\psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (4.3.16) expressed in weak form is equivalent to that of finding a $\phi \in H$, such that

$$[\phi,\psi] = \int_{\Omega_\varepsilon} [K\phi - h]\,\psi\,dx, \qquad \text{for all } \ \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = \mathcal{K}(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where $\mathcal{K}$ is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = \mathcal{K}(\phi)$ has only the zero solution in H. This last equation is equivalent to (4.3.16) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (4.3.17). This finishes the proof. □

The result of Proposition 4.3.2 implies that the unique solution $\phi = T(h)$ of (4.3.16) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ and $\int_{\Omega_\varepsilon} h = 0$, into L^∞, with norm bounded by $C|\log \varepsilon|$.

It is important for later purposes to understand the differentiability of the operator T with respect to the variable ξ_i'. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. Let us recall that ϕ satisfies the equation

$$L(\phi) = h + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\, \chi_j\, Z_{ij},$$

and the doubly periodic and orthogonality conditions, for some (uniquely determined) constants $c_{ij} = c_{ij}(\xi')$, $i = 1, 2$, $j = 1, \ldots, m$. We want to compute derivatives of ϕ with respect to the parameters ξ_{kl}'. Formally

$$\Delta(\partial_{\xi_{kl}'}\phi) + K\partial_{\xi_{kl}'}\phi - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} K\partial_{\xi_{kl}'}\phi = -\partial_{\xi_{kl}'}K\,\phi + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\partial_{\xi_{kl}'}K\phi + \sum_{i=1}^{2}\sum_{j=1}^{m}\partial_{\xi_{kl}'}(c_{ij}\chi_j Z_{ij})$$

so, $X = \partial_{\xi_{kl}'}\phi$ should satisfy

$$L(X) = -\partial_{\xi_{kl}'}K\,\phi + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\partial_{\xi_{kl}'}K\phi + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\,\partial_{\xi_{kl}'}(\chi_j Z_{ij}) + \sum_{i=1}^{2}\sum_{j=1}^{m} d_{ij}\chi_j Z_{ij}\,,$$

where (still formally) $d_{ij} = \partial_{\xi_{kl}'}(c_{ij})$, $i = 1, 2$, $j = 1, \ldots, m$. The orthogonality conditions now become

$$\int_{\Omega_\varepsilon}\chi_j Z_{ij} X = -\int_{\Omega_\varepsilon}\partial_{\xi_{kl}'}(\chi_j Z_{ij})\phi, \qquad i = 1, 2, j = 1, \ldots, m.$$

Observe that $\partial_{\xi_{kl}'}(\chi_j Z_{ij})$ is not necessarily identically zero, since $\mu_j = \mu_j(\xi_1', \ldots, \xi_m')$ by (4.2.8).

We will recast X as follows. Let us consider η_j, a smooth cut-off function as in (4.3.20). We consider the constants b_{ij} defined as

$$b_{ij}\int_{\Omega_\varepsilon}\chi_j|Z_{ij}|^2 := \int_{\Omega_\varepsilon}\phi\;\partial_{\xi_{kl}'}(\chi_j Z_{ij}), \qquad i = 1, 2, j = 1, \ldots, m$$

and the function

$$f := \sum_{i=1}^{2}\sum_{j=1}^{m}\Big[b_{ij}\,L(\eta_j\,Z_{ij}) + c_{ij}\,\partial_{\xi_{kl}'}(\chi_j Z_{ij})\Big] - \partial_{\xi_{kl}'}K\,\phi + \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\partial_{\xi_{kl}'}K\phi.$$

Then the function X above can be uniquely expressed as

$$X = T(f) - \sum_{i=1}^{2}\sum_{j=1}^{m} b_{ij}\,\eta_j\,Z_{ij}.$$

This computation is not just formal. Arguing directly by definition it shows that indeed $\partial_{\xi_{kl}'}\phi = X$. Also, we find that $\|f\|_* \le C\varepsilon^{-\sigma}(\log\frac{1}{\varepsilon})\|h\|_*$. In fact, we get that

$$\|f\|_* \le \sum_{i=1}^{2}\sum_{j=1}^{m}\Big[|b_{ij}|\,\|L(\eta_j Z_{ij})\|_* + |c_{ij}|\,\|\partial_{\xi_{kl}'}(\chi_j Z_{ij})\|_*\Big] + \|\partial_{\xi_{kl}'}K\phi\|_* + \left\|\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\partial_{\xi_{kl}'}K\phi\right\|_*.$$

First, note that $|b_{ij}| \leq C\|\phi\|_\infty \|\partial_{\xi'_{kl}}(\chi_j Z_{ij})\|_*$. Since, $\operatorname{supp}\chi_j \subseteq B(\xi'_j, R_0+1)$, we get that $\|\partial_{\xi'_{kl}}(\chi_j Z_{ij})\|_* \leq C$. Thus, we obtain that

$$|b_{ij}| \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*.$$

Next, we estimate $\|L(\eta_j Z_{ij})\|_*$. So, we have that

$$\|L(\eta_j Z_{ij})\|_* = \|\tilde{L}(\eta_j Z_{ij}) + c(\eta_j Z_{ij})\|_* \leq \|\tilde{L}(\eta_j Z_{ij})\|_* + \|c(\eta_j Z_{ij})\|_*.$$

We know that in $B(\xi'_j, \frac{\delta}{3\varepsilon})$

$$\tilde{L}(\eta_j Z_{ij}) = \Delta\eta_j Z_{ij} + 2\nabla\eta_j\nabla Z_{ij} + e^{v_j(\cdot - \xi'_j)}\theta_\varepsilon\eta_j Z_{ij} + O(\varepsilon^4)$$

and $\tilde{L}(\eta_j Z_{ij}) = 0$ in $\Omega_\varepsilon \setminus B(\xi'_j, \frac{\delta}{3\varepsilon})$. Therefore, from the definition of $*$-norm we obtain that $\|\tilde{L}(\eta_j Z_{ij})\|_* \leq C\varepsilon^{1-\sigma}$.

On the other hand, we know that

$$c(\eta_j Z_{ij}) = -\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} K\eta_j Z_{ij}$$

and hence, we estimate

$$\|c(\eta_j Z_{ij})\|_* \leq \frac{\varepsilon^2}{|\Omega|}\left|\int_{\Omega_\varepsilon} K\eta_j Z_{ij}\right| \sup_{y\in\Omega_\varepsilon}(1+|y-\xi'_j|)^{2+\sigma} \leq C\varepsilon^{1-\sigma},$$

since

$$\int_{\Omega_\varepsilon}\Delta(\eta_j Z_{ij}) = 0 \qquad \text{and} \qquad \int_{\Omega_\varepsilon}\left|\tilde{L}(\eta_j Z_{ij})\right| \leq C\varepsilon.$$

Now, we estimate $\partial_{\xi'_{kl}}K$. We know that $K(y) = \varepsilon^4 k(\varepsilon y)e^{V(y)}$ and hence we find that $\partial_{\xi'_{kl}}K(y) = \varepsilon^4 k(\varepsilon y)e^{V(y)}\partial_{\xi'_{kl}}V(y)$. From the definition of V, we have that

$$\partial_{\xi'_{kl}}V(y) = \partial_{\xi'_{kl}}[U(\varepsilon y)] = \sum_{j=1}^m \partial_{\xi'_{kl}}[U_j(\varepsilon y)] = \varepsilon\sum_{j=1}^m \partial_{\xi_{kl}}U_j(\varepsilon y).$$

Using the integral representation (2.2.4) we deduce that for all $j = 1,\dots,m$

$$\partial_{\xi_{kl}}U_j(x) = \partial_{\xi_{kl}}\left(u_j(x) - \log\frac{8\mu_j^2}{k(\xi_j)} + H(x,\xi_j)\right) + O(\varepsilon^2|\log\varepsilon|)$$

uniform in C^2-sense on compact subsets of Ω. In particular, for $j \neq l$

$$\partial_{\xi_{kl}}U_j(x) = \frac{-2\partial_{\xi_{kl}}(\mu_j^2)\varepsilon^2}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2} + O(\varepsilon^2|\log\varepsilon|)$$

and

$$\partial_{\xi_{kl}}U_l(x) = \frac{4(x-\xi_l)_k}{\mu_l^2\varepsilon^2 + |x-\xi_l|^2} - \frac{2\partial_{\xi_{kl}}(\mu_l^2)\varepsilon^2}{\mu_l^2\varepsilon^2 + |x-\xi_l|^2} + \partial_{2k}H(x,\xi_l) + O(\varepsilon^2|\log\varepsilon|).$$

Hence, we have that $\|\partial_{\xi_{kl}}U_l\|_\infty \leq \frac{C}{\varepsilon}$ and for all $j \neq l$ $\|\partial_{\xi_{kl}}U_j\|_\infty \leq C$. Thus, we get that $\|\partial_{\xi'_{kl}}V\|_\infty \leq C$ and $\|\partial_{\xi'_{kl}}K\|_\infty \leq \|K\|_*\|\partial_{\xi'_{kl}}V\|_\infty \leq C$. With this estimate, we get that

$$\|\partial_{\xi'_{kl}}K\phi\|_* \leq \|\partial_{\xi'_{kl}}K\|_*\|\phi\|_\infty \leq C\left(\log\frac{1}{\varepsilon}\right)\|h\|_*$$

and

$$\left\| \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \partial_{\xi'_{kl}} K\phi \right\|_* \leq C \|\partial_{\xi'_{kl}} K\|_* \|\phi\|_\infty \sup_{y\in\Omega_\varepsilon} (1+|y-\xi'_j|)^\sigma \leq \frac{C}{\varepsilon^\sigma} \left(\log \frac{1}{\varepsilon}\right) \|h\|_*.$$

Therefore, we conclude that

$$\begin{aligned}\|f\|_* &\leq C \left[\varepsilon^{1-\sigma} \left(\log \frac{1}{\varepsilon}\right) \|h\|_* + \|h\|_* + \left(\log \frac{1}{\varepsilon}\right) \|h\|_* + \varepsilon^{-\sigma} \left(\log \frac{1}{\varepsilon}\right) \|h\|_*\right] \\ &\leq \frac{C}{\varepsilon^\sigma} \left(\log \frac{1}{\varepsilon}\right) \|h\|_*.\end{aligned}$$

Moreover, using Proposition 4.3.2, we find that

$$\|\partial_{\xi'_{kl}} \phi\|_* \leq \|T(f)\|_\infty + \sum_{i=1}^{2} \sum_{j=1}^{m} |b_{ij}| \, \|\eta_j Z_{ij}\|_\infty \leq C \left(\log \frac{1}{\varepsilon}\right) \|f\|_* + C \left(\log \frac{1}{\varepsilon}\right) \|h\|_*.$$

Finally, we conclude that

$$\|\partial_{\xi'_{kl}} T(h)\|_\infty \leq \frac{C}{\varepsilon^\sigma} \left(\log \frac{1}{\varepsilon}\right)^2 \|h\|_* \quad \text{for all } k=1,2, l=1,\dots,m. \tag{4.3.23}$$

This estimate is of crucial importance in the arguments to come. Remark that $0 < \sigma < 1$.

4.4 The nonlinear problem

In this section, instead of solve directly the problem (4.2.11) we shall solve an intermediate problem. First, we consider the following auxiliary non linear problem

$$\begin{cases} L(\phi) = -[R + N(\phi)] + \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} \chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \quad \phi \quad \text{doubly periodic on} \quad \partial\Omega_\varepsilon, & \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij} \phi = 0, \quad \text{for all } i=1,2, \ j=1,\dots,m & \int_{\Omega_\varepsilon} \phi = 0. \end{cases} \tag{4.4.1}$$

where K, R and $N(\phi)$ are given by (4.2.15), (4.2.12) and (4.2.13) respectively.

Lemma 4.4.1. *Let $\delta > 0$. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and for any $\xi_1, \dots, \xi_m \in \tilde{\Omega}$ satisfying (4.3.2), problem (4.4.1) admits a unique solution ϕ, c_{ij}, $i = 1, 2$, $j = 1, \dots, m$ such that*

$$\|\phi\|_\infty \leq C\varepsilon |\log \varepsilon|. \tag{4.4.2}$$

Furthermore, denoting $\xi' = (\xi'_1, \dots, \xi'_m)$, the function $\xi' \mapsto \phi(\xi') \in C(\bar{\Omega}_\varepsilon)$ is C^1 and

$$\|D_{\xi'} \phi\|_\infty \leq C\varepsilon^{1-\sigma} |\log \varepsilon|^2. \tag{4.4.3}$$

Proof: First, note that $R \in L^\infty(\Omega_\varepsilon)$, $\|R\|_* < +\infty$, $\int_{\Omega_\varepsilon} R = 0$ and $\int_{\Omega_\varepsilon} N(\phi) = 0$ for any $\phi \in C(\bar{\Omega}_\varepsilon)$. Next, we observe that in terms of the operator T defined in Proposition 4.3.2, the latter problem becomes

$$\phi = -T\left(R + N(\phi)\right) := \mathcal{A}(\phi). \tag{4.4.4}$$

For a given number $\nu > 0$, let us consider

$$\mathcal{F}_\nu = \{\phi \in C(\bar{\Omega}_\varepsilon) : \|\phi\|_\infty \leq \nu\varepsilon |\log \varepsilon|\}$$

From the Proposition 4.3.2, we get

$$\|\mathcal{A}(\phi)\|_\infty \leq C|\log\varepsilon|\|R+N(\phi)\|_* \leq C|\log\varepsilon|\left[\|R\|_* + \|N(\phi)\|_*\right].$$

From (4.2.14) it follows the estimate $\|R\|_* \leq C\varepsilon$. Furthermore,

$$\begin{aligned}\|N(\phi)\|_* &\leq \|K\|_*\,\|e^\phi - \phi - 1\|_\infty + \frac{\varepsilon^2}{|\Omega|}\left\|\int_{\Omega_\varepsilon} K(z)(e^{\phi(z)} - \phi(z) - 1)\,dz\right\|_* \\ &\leq C\|\phi\|_\infty^2 + \frac{C}{\varepsilon^\sigma}\|K\|_*\,\|e^\phi - \phi - 1\|_\infty \\ &\leq \frac{C}{\varepsilon^\sigma}\|\phi\|_\infty^2.\end{aligned}$$

Hence, we get for any $\phi \in \mathcal{F}_\nu$,

$$\begin{aligned}\|\mathcal{A}(\phi)\|_\infty &\leq C|\log\varepsilon|\left[\varepsilon + \frac{1}{\varepsilon^\sigma}\|\phi\|_\infty^2\right] \leq C|\log\varepsilon|\left[\varepsilon + \nu^2\varepsilon^{2-\sigma}|\log\varepsilon|^2\right] \\ &\leq C\varepsilon|\log\varepsilon|\left[1 + \nu^2\varepsilon^{1-\sigma}|\log\varepsilon|^2\right].\end{aligned}$$

Given any $\phi_1, \phi_2 \in \mathcal{F}_\nu$, we have $\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) = -T\left(N(\phi_1) - N(\phi_2)\right)$,

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_\infty \leq C|\log\varepsilon|\,\|N(\phi_1) - N(\phi_2)\|_*\,,$$

$$N(\phi_1) - N(\phi_2) = K(e^{\phi_1} - \phi_1 - [e^{\phi_2} - \phi_2]) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} K(e^{\phi_1} - \phi_1 - [e^{\phi_2} - \phi_2])$$

and

$$\begin{aligned}\|N(\phi_1) - N(\phi_2)\|_* &\leq C(\|\phi_1\|_\infty + \|\phi_2\|_\infty)\|\phi_1 - \phi_2\|_\infty + \frac{C}{\varepsilon^\sigma}\|K\|_*\|e^{\phi_1} - \phi_1 - [e^{\phi_2} - \phi_2]\|_\infty \\ &\leq C\nu\varepsilon^{1-\sigma}|\log\varepsilon|\,\|\phi_1 - \phi_2\|_\infty\end{aligned}$$

with C independent of ν. Therefore,

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_\infty \leq C\nu\varepsilon^{1-\sigma}|\log\varepsilon|^2\|\phi_1 - \phi_2\|_\infty$$

It follows that for all ε sufficiently small $\mathcal{A}$ is a contraction mapping of $\mathcal{F}_\nu$ (for ν large enough), and therefore a unique fixed point of $\mathcal{A}$ exists in $\mathcal{F}_\nu$.

Let us now discuss the differentiability of ϕ depending on ξ', i.e., $\xi' \mapsto \phi(\xi') \in C(\bar{\Omega}_\varepsilon)$ is C^1. Since R depends continuously (in the $*$-norm) on ξ', using the fixed point characterization (4.4.4), we deduce that the mapping $\xi' \mapsto \phi$ is also continuous. Then, formally

$$\begin{aligned}\partial_{\xi'_{kl}} N(\phi) = &\,\partial_{\xi'_{kl}} K(e^\phi - \phi - 1) + K[e^\phi - 1]\partial_{\xi'_{kl}}\phi \\ &- \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\left(\partial_{\xi'_{kl}} K(e^\phi - \phi - 1) + K[e^\phi - 1]\partial_{\xi'_{kl}}\phi\right).\end{aligned}$$

It can be checked that $\|\partial_{\xi'_{kl}} K\|_*$ and $\int_{\Omega_\varepsilon}|\partial_{\xi'_{kl}} K|$ are uniformly bounded, so we conclude

$$\|\partial_{\xi'_{kl}} N(\phi)\|_* \leq \frac{C}{\varepsilon^\sigma}\|\phi\|_\infty^2 + \frac{C}{\varepsilon^\sigma}\|\phi\|_\infty\,\|\partial_{\xi'_{kl}}\phi\|_\infty \leq C\varepsilon^{1-\sigma}|\log\varepsilon|\left[\varepsilon|\log\varepsilon| + \|\partial_{\xi'_{kl}}\phi\|_\infty\right].$$

Also, observe that we have

$$\partial_{\xi'_{kl}}\phi = -(\partial_{\xi'_{kl}} T)\left(R + N(\phi)\right) - T\left(\partial_{\xi'_{kl}}\left[R + N(\phi)\right]\right).$$

So, using (4.3.23), we get

$$\begin{aligned}\|\partial_{\xi'_{kl}}\phi\|_\infty &\le \frac{C}{\varepsilon^\sigma}|\log\varepsilon|^2\,\|R+N(\phi)\|_* + C|\log\varepsilon|\,\|\partial_{\xi'_{kl}}(R+N(\phi))\|_*\\ &\le C|\log\varepsilon|\left[\frac{|\log\varepsilon|}{\varepsilon^\sigma}(\|R\|_*+\|N(\phi)\|_*)+\|\partial_{\xi'_{kl}}R\|_*+\|\partial_{\xi'_{kl}}N(\phi)\|_*\right].\end{aligned}$$

Let us estimate $\|\partial_{\xi'_{kl}}R\|_*$. We know that

$$\partial_{\xi'_{kl}}R(y)=\Delta\partial_{\xi'_{kl}}V(y)+\varepsilon^4k(\varepsilon y)e^V\partial_{\xi'_{kl}}V-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\varepsilon^4k(\varepsilon z)e^V(z)\partial_{\xi'_{kl}}V(z)\,dz.$$

Thus, we have that

$$\begin{aligned}\Delta\partial_{\xi'_{kl}}V(y)&=-\varepsilon^2\sum_{j=1}^m\partial_{\xi'_{kl}}\left(\varepsilon^2k(\xi_j)e^{u_j(\varepsilon y)}-\frac{1}{|\Omega|}\int_\Omega\varepsilon^2k(\xi_j)e^{u_j}\right)\\ &=-\varepsilon^2\sum_{j=1}^m\left(\varepsilon^3\partial_{\xi_{kl}}[k(\xi_j)e^{u_j(\varepsilon y)}]-\frac{\varepsilon}{|\Omega|}\int_\Omega\varepsilon^2\partial_{\xi_{kl}}[k(\xi_j)e^{u_j}]\right).\end{aligned}$$

Observe that

$$\partial_{\xi_{kl}}[k(\xi_l)e^{u_l(x)}]=\frac{32\mu_l^2(x-\xi_l)_k}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^3}+\frac{8\partial_{\xi_{kl}}(\mu_l^2)(|x-\xi_l|^2-\mu_l^2\varepsilon^2)}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^3}$$

and for $j\neq l$

$$\partial_{\xi_{kl}}[k(\xi_j)e^{u_j(x)}]=\frac{8\partial_{\xi_{kl}}(\mu_j^2)}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}\,\frac{|x-\xi_j|^2-\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+|x-\xi_j|^2}.$$

Hence, we get that

$$\begin{aligned}\varepsilon^2\int_\Omega\partial_{\xi_{kl}}[k(\xi_l)e^{u_l(x)}]\,dx&=\frac{\partial_{\xi_{kl}}(\mu_l^2)}{\mu_l^2}\int_{B(\xi_l,\delta)}\frac{8\mu_l^2\varepsilon^2(|x-\xi_l|^2-\mu_l^2\varepsilon^2)}{(\mu_l^2\varepsilon^2+|x-\xi_l|^2)^3}\,dx+O(\varepsilon^2)\\ &=O(\varepsilon^2)\end{aligned}$$

and similarly for $j\neq l$

$$\begin{aligned}\varepsilon^2\int_\Omega\partial_{\xi_{kl}}[k(\xi_j)e^{u_j(x)}]\,dx&=\frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2}\int_{B(\xi_j,\delta)}\frac{8\mu_j^2\varepsilon^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}\,\frac{|x-\xi_j|^2-\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+|x-\xi_j|^2}\,dx+O(\varepsilon^2)\\ &=O(\varepsilon^2).\end{aligned}$$

Also, we have that

$$\Delta\partial_{\xi'_{kl}}V(y)=-\frac{32\mu_l^2(y-\xi'_l)_k}{(\mu_l^2+|y-\xi'_l|^2)^3}-\varepsilon\sum_{j=1}^m\frac{8\partial_{\xi_{kl}}(\mu_j^2)(|y-\xi'_j|^2-\mu_j^2)}{(\mu_j^2+|y-\xi'_j|^2)^3}+O(\varepsilon^5).$$

On the other hand, we have that

$$\begin{aligned}\int_{\Omega_\varepsilon}\varepsilon^4k(\varepsilon z)e^{V(z)}\partial_{\xi'_{kl}}V(z)\,dz&=\int_\Omega\varepsilon^2k(x)e^{U(x)}\varepsilon\partial_{\xi_{kl}}U(x)\,dx\\ &=\sum_{j=1}^m\varepsilon^3\int_{B(\xi_j,\delta)}k(x)e^{U(x)}\partial_{\xi_{kl}}U(x)\,dx+O(\varepsilon^3).\end{aligned}$$

For any j, denote $\rho_j(x) = k(x)e^{H(x,\xi_j)+\sum_{q\neq l} G(x,\xi_q)}$. Now, by similar computations to get (4.2.21), if $j \neq l$ then

$$
\begin{aligned}
&\varepsilon^3 \int_{B(\xi_j,\delta)} k(x)e^{U(x)}\partial_{\xi_{kl}}U(x)\,dx \\
&\quad= \varepsilon^3 \int_{B(\xi_j,\delta)} \frac{\rho_j(x)}{\mu_j^4\varepsilon^4\left(1+\left(\frac{|x-\xi_j|}{\mu_j\varepsilon}\right)^2\right)^2}(1+O(\varepsilon^2|\log\varepsilon|)) \\
&\qquad\quad \times\left(\frac{-2\partial_{\xi_{kl}}(\mu_j^2)\varepsilon^2}{\mu_j^2\varepsilon^2+|x-\xi_j|^2} + \partial_{2k}G(x,\xi_l) + O(\varepsilon^2|\log\varepsilon|)\right)dx \\
&\quad= \frac{\varepsilon}{\mu_j^2}\int_{B(0,\frac{\delta}{\mu_j\varepsilon})} \frac{\rho_j(\xi_j+\mu_j\varepsilon y)}{(1+|y|^2)^2}\left(\frac{-2\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2(1+|y|^2)} + \partial_{2k}G(\xi_j+\mu_j\varepsilon y,\xi_l)\right)dy \\
&\qquad + O(\varepsilon^3|\log\varepsilon|) \\
&\quad= -\frac{2\partial_{\xi_{kl}}(\mu_j^2)\varepsilon}{\mu_j^4}\left[\frac{\pi}{2}\rho_j(\xi_j) + O(\varepsilon^2)\right] + \frac{\varepsilon}{\mu_j^2}\left[\pi\rho_j(\xi_j)\partial_{2k}G(\xi_j,\xi_l) + O(\varepsilon^2|\log\varepsilon|)\right] \\
&\qquad + O(\varepsilon^3|\log\varepsilon|) \\
&\quad= O(\varepsilon^3|\log\varepsilon|),
\end{aligned}
$$

since $8\mu_j^2 = \rho_j(\xi_j)$, $\partial_{\xi_{kl}}(\mu_j^2) = \mu_j^2\partial_{2k}G(\xi_j,\xi_l)$ for $j \neq l$. Also, we have that

$$
\begin{aligned}
&\varepsilon^3 \int_{B(\xi_l,\delta)} k(x)e^{U(x)}\partial_{\xi_{kl}}U(x)\,dx \\
&\quad= \varepsilon^3 \int_{B(\xi_l,\delta)} \frac{\rho_l(x)}{\mu_l^4\varepsilon^4\left(1+\left(\frac{|x-\xi_l|}{\mu_l\varepsilon}\right)^2\right)^2}(1+O(\varepsilon^2|\log\varepsilon|)) \\
&\qquad\quad \times\left(\frac{4(x-\xi_l)_k - 2\partial_{\xi_{kl}}(\mu_l^2)\varepsilon^2}{\mu_l^2\varepsilon^2+|x-\xi_l|^2} + \partial_{2k}H(x,\xi_l) + O(\varepsilon^2|\log\varepsilon|)\right)dx \\
&\quad= \int_{B(0,\frac{\delta}{\mu_l\varepsilon})}\left[\frac{4}{\mu_l^3}\frac{y_k\rho_l(\xi_l+\mu_l\varepsilon y)}{(1+|y|^2)^3} - \frac{2\partial_{\xi_{kl}}(\mu_l^2)\varepsilon}{\mu_l^4}\frac{\rho_l(\xi_l+\mu_l\varepsilon y)}{(1+|y|^2)^3}\right. \\
&\qquad\quad \left. + \frac{\varepsilon}{\mu_l^2}\frac{\rho_l(\xi_l+\mu_l\varepsilon y)}{(1+|y|^2)^2}\partial_{2k}H(\xi_l+\mu_l\varepsilon y,\xi_l)\right]dy + O(\varepsilon^3|\log\varepsilon|) \\
&\quad= O(\varepsilon^3|\log\varepsilon|),
\end{aligned}
$$

since $8\mu_l^2 = \rho_l(\xi_l)$, $\partial_k\rho_l(\xi_l) = 8\partial_{\xi_{kl}}(\mu_l^2)$, $\partial_{2k}H(\xi_l,\xi_l) = 0$ and

$$
\begin{aligned}
&\int_{B(0,\frac{\delta}{\mu_l\varepsilon})} \frac{y_k\rho_l(\xi_l+\mu_l\varepsilon y)}{(1+|y|^2)^3}\,dy \\
&\quad= \int_{B(0,\frac{\delta}{\mu_l\varepsilon})} \frac{y_k}{(1+|y|^2)^3}\left[\rho_l(\xi_l) + \nabla\rho_l(\xi_l)\cdot\mu_l\varepsilon y + \mu_l^2\varepsilon^2\langle D^2\rho_l(\xi_l)y,y\rangle + O(\mu_l^3\varepsilon^3|y|^3)\right]dy \\
&\quad= \frac{\pi}{4}\partial_k\rho_l(\xi_l)\mu_l\varepsilon + O(\varepsilon^3|\log\varepsilon|).
\end{aligned}
$$

Therefore, we conclude that

$$
\int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon z)e^{V(z)}\partial_{\xi'_{kl}}V(z)\,dz = O(\varepsilon^3|\log\varepsilon|).
$$

On the other hand, we know that

$$\partial_{\xi'_{kl}} V(y) = \frac{4(y-\xi'_l)_k}{\mu_l^2 + |y-\xi'_l|^2} + \varepsilon \partial_{2k} H(\varepsilon y, \xi_l) - \sum_{j=1}^m \frac{2\partial_{\xi_{kl}}(\mu_j^2)\varepsilon}{\mu_j^2 + |y-\xi'_j|^2} + O(\varepsilon^3 |\log \varepsilon|).$$

If $|y-\xi'_j| > \frac{\delta}{\varepsilon}$ for all $j = 1, \ldots, m$ then $\varepsilon^4 k(\varepsilon y) e^{V(y)} = O(\varepsilon^4)$ and $\partial_{\xi'_{kl}} V(y) = O(\varepsilon)$. Similarly, in the same region $\Delta \partial_{\xi'_{kl}} V(y) = O(\varepsilon^5)$. Hence, we get that in the considered region $\partial_{\xi'_{kl}} R(y) = O(\varepsilon^5 |\log \varepsilon|)$, for all $j = 1, \ldots, m$. Now, if $|y - \xi'_j| < \frac{\delta}{\varepsilon}$ for some $j \in \{1, \ldots, m\}$ then

$$\varepsilon^4 k(\varepsilon y) e^{V(y)} = \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi'_j|^2)^2} [1 + O(\varepsilon|y-\xi'_j|) + O(\varepsilon^2 |\log \varepsilon|)].$$

We have that for $j \neq l$

$$\partial_{\xi'_{kl}} V(y) = -\frac{2\partial_{\xi_{kl}}(\mu_j^2)\varepsilon}{\mu_j^2 + |y-\xi'_j|^2} + O(\varepsilon) = O(\varepsilon)$$

and

$$\begin{aligned}
\partial_{\xi'_{kl}} R(y) &= -\varepsilon \frac{8\partial_{\xi'_{kl}}(\mu_j^2)(|y-\xi'_j|^2 - \mu_j^2)}{(\mu_j^2 + |y-\xi'_j|^2)^3} \\
&\quad + \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi'_j|^2)^2} \left[1 + O(\varepsilon|y-\xi'_j|) + O(\varepsilon^2|\log\varepsilon|)\right] O(\varepsilon) + O(\varepsilon^5 |\log \varepsilon|) \\
&= \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi'_j|^2)^2} \left[O(\varepsilon) + O(\varepsilon^2 |y-\xi'_j|)\right] + O(\varepsilon^5 |\log\varepsilon|).
\end{aligned}$$

For $j = l$, we find that

$$\begin{aligned}
\partial_{\xi'_{kl}} R(y) &= -\frac{32\mu_l^2 (y-\xi'_l)_k}{(\mu_l^2 + |y-\xi'_l|^2)^3} - \varepsilon \frac{8\partial_{\xi'_{kl}}(\mu_l^2)(|y-\xi'_l|^2 - \mu_l^2)}{(\mu_j^2 + |y-\xi'_j|^2)^3} \\
&\quad + \frac{8\mu_l^2}{(\mu_l^2 + |y-\xi'_l|^2)^2} \left[1 + O(\varepsilon|y-\xi'_l|) + O(\varepsilon^2|\log\varepsilon|)\right] \left[\frac{4(y-\xi_l)_k}{\mu_l^2 + |y-\xi'_l|^2} + O(\varepsilon)\right] \\
&\quad + O(\varepsilon^5 |\log\varepsilon|) \\
&= \frac{8\mu_l^2}{(\mu_l^2 + |y-\xi'_l|^2)^2} \left[O(\varepsilon) + O(\varepsilon|y-\xi'_l|)\right] + O(\varepsilon^5|\log\varepsilon|).
\end{aligned}$$

Therefore, from the definition of *-norm we conclude that$\|\partial_{\xi'_{kl}} R\|_* \leq C\varepsilon$. Hence, we find the following estimate

$$\begin{aligned}
\|\partial_{\xi'_{kl}} \phi\|_\infty &\leq C|\log\varepsilon| \left[\frac{|\log\varepsilon|}{\varepsilon^\sigma} \left(\varepsilon + \frac{1}{\varepsilon^\sigma}\|\phi\|_\infty^2\right) + \varepsilon + \varepsilon^{1-\sigma}|\log\varepsilon| \left(\varepsilon|\log\varepsilon| + \|\partial_{\xi'_{kl}}\phi\|_\infty\right)\right] \\
&\leq C|\log\varepsilon| \left[\varepsilon^{1-\sigma}|\log\varepsilon| + \varepsilon^{2-2\sigma}|\log\varepsilon|^3 + \varepsilon + \varepsilon^{2-\sigma}|\log\varepsilon|^2 + \varepsilon^{1-\sigma}|\log\varepsilon|\, \|\partial_{\xi'_{kl}}\phi\|_\infty\right].
\end{aligned}$$

Thus, we conclude

$$\|\partial_{\xi'_{kl}}\phi\|_\infty \leq C\varepsilon^{1-\sigma}|\log\varepsilon|^2.$$

The above computations can be made rigorous by using the implicit function theorem and the fixed point representation (4.4.4) which guarantees C^1 regularity in ξ'.

□

4.5 Variational reduction

In view of Lemma 4.4.1, given $\delta > 0$ and any points $\xi_1, \dots, \xi_m \in \Omega$ satisfying $\mathrm{dist}(\xi_i - \xi_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) > \delta$ for all $i \neq j$, we consider $\phi(\xi')$, $c_{ij}(\xi')$, $i = 1, 2$, $j = 1, \dots, m$, where $\xi = (\xi_1, \dots, \xi_m)$ and $\xi = \varepsilon\xi'$, to be the unique solution to (4.4.1) satisfying (4.4.2) and (4.4.3).

After problem (4.4.1) has been solved, we find a solution to problem (4.2.9) and hence to the original problem if ξ' is such that

$$c_{ij}(\xi') = 0, \qquad i = 1, 2, \quad j = 1, \dots, m. \tag{4.5.1}$$

This problem is equivalent to finding critical points of a functional of $\xi = \varepsilon\xi'$. Let us consider J_ε, given by (4.1.3) and define

$$F_\varepsilon(\xi) := J_\varepsilon\big(U(\xi) + \tilde{\phi}(\xi)\big), \tag{4.5.2}$$

where (with slight abuse of notation) $U = U(\xi) = U(x, \xi)$ and $\tilde{\phi} = \tilde{\phi}(\xi) = \tilde{\phi}(x, \xi)$ are the functions defined on Ω from the relations

$$U(x, \xi) = V\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right) \qquad \text{and} \qquad \tilde{\phi}(x, \xi) = \phi\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right),$$

The following result states that critical points of F_ε correspond to solutions of (4.5.1) for small ε.

Lemma 4.5.1. *There exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$, if $\xi \in \Omega^m$ satisfying (4.3.2) is a critical point of F_ε then $u = U(\xi) + \tilde{\phi}(\xi)$ is a critical point of J_ε, that is, if $D_\xi F_\varepsilon(\xi) = 0$ then ξ satisfies system (4.5.1), i.e., u is a solution to (4.1.2).*

Proof: Define the energy functional I_ε associated to problem (4.2.9), namely,

$$I_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^v.$$

It is easy to see that

$$I_\varepsilon(V(\xi') + \phi(\xi')) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)). \tag{4.5.3}$$

Let us differentiate the function $F_\varepsilon(\xi)$ with respect to ξ. Since (4.5.3), we can differentiate directly $I_\varepsilon(V + \phi)$ (under the integral sign), so that

$$\begin{aligned} \partial_{\xi_{kl}} F_\varepsilon(\xi) &= \frac{1}{\varepsilon}\partial_{\xi'_{kl}}\left[I_\varepsilon(V + \phi)\right] = \frac{1}{\varepsilon} DI_\varepsilon(V + \phi)\left[\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right] \\ &= \frac{1}{\varepsilon}\left[\int_{\Omega_\varepsilon} \nabla(V + \phi)\,\nabla\left(\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right) - \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^{V+\phi}\left(\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right)\right]. \end{aligned}$$

We know that

$$\begin{cases} \Delta(V + \phi) + \varepsilon^4 k(\varepsilon y)\, e^{V+\phi} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon z)\, e^{V+\phi}\, dz = \sum_{i=1}^2 \sum_{j=1}^m c_{ij}\chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \phi \quad \text{doubly periodic on} \quad \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} (V + \phi) = 0. \end{cases} \tag{4.5.4}$$

So, integrating by parts, we get

$$\begin{aligned} \partial_{\xi_{kl}} F_\varepsilon(\xi) &= \frac{1}{\varepsilon}\left[-\int_{\Omega_\varepsilon} \Delta(V + \phi)\,\left(\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right) - \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^{V+\phi}\left(\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right)\right] \\ &= -\frac{1}{\varepsilon}\sum_{i=1}^2\sum_{j=1}^m c_{ij}\int_{\Omega_\varepsilon} \chi_j Z_{ij}\,\left[\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right], \end{aligned}$$

since $\int_{\Omega_\varepsilon}\left(\partial_{\xi'_{kl}}V+\partial_{\xi'_{kl}}\phi\right)=0$. From the results of the previous section, this expression defines a continuous function of ξ', and hence of ξ. Let us assume that $D_\xi F_\varepsilon(\xi)=0$. Then, from the latter equality

$$\sum_{i=1}^{2}\sum_{j=1}^{m}c_{ij}\int_{\Omega_\varepsilon}\chi_j Z_{ij}\left[\partial_{\xi'_{kl}}V+\partial_{\xi'_{kl}}\phi\right]=0,\qquad k=1,2,\ l=1,\dots,m.$$

Using (4.4.3) and $\partial_{\xi'_{kl}}V=4Z_{kl}+O(\varepsilon)$, where $O(\varepsilon)$ is in the L^∞ norm, it follows

$$\sum_{i=1}^{2}\sum_{j=1}^{m}c_{ij}\int_{\Omega_\varepsilon}\chi_j Z_{ij}\ [Z_{kl}+o(1)]=0,\qquad k=1,2,\ l=1,\dots,m.$$

with $o(1)$ small in the sense of the L^∞ norm as $\varepsilon\to 0$. The above system is diagonal dominant and we thus get $c_{ij}=0$ for $i=1,2,\ j=1,\dots,m$.

□

In order to solve for critical points of the function F_ε, a key step is its expected closeness to the function $J_\varepsilon(U)$, where U is the function defined in (4.2.7), which we will analyze in the next section.

Lemma 4.5.2. *The following expansions holds*

$$F_\varepsilon(\xi)=J_\varepsilon(U)+\theta_\varepsilon(\xi),$$

where

$$|\theta_\varepsilon|=O(\varepsilon^2|\log\varepsilon|)\qquad\textit{and}\qquad|\nabla\theta_\varepsilon|=O(\varepsilon^{1-\sigma}|\log\varepsilon|^2),\qquad\textit{as}\quad\varepsilon\to 0,$$

uniformly on points $\xi=(\xi_1,\dots,\xi_m)\in\Omega^m$ satisfying the constraints (4.3.2).

Proof: Since we have, $I_\varepsilon(V)=J_\varepsilon(U)$ and (4.5.3), we write

$$J_\varepsilon(U+\tilde\phi)-J_\varepsilon(U)=I_\varepsilon(V+\phi)-I_\varepsilon(V):=A.$$

Let us estimate A first. A Taylor expansion gives us

$$A=DI_\varepsilon(V+\phi)[\phi]-\int_0^1 D^2I_\varepsilon(V+t\phi)[\phi]^2\,t\,dt.$$

Testing equation (4.5.4) against ϕ and integrating by parts, we get

$$\int_{\Omega_\varepsilon}\nabla(V+\phi)\,\nabla\phi-\int_{\Omega_\varepsilon}\varepsilon^4k(\varepsilon y)e^{V+\phi}\phi=0,$$

i.e., $DI_\varepsilon(V+\phi)[\phi]=0$. Thus,

$$A=-\int_0^1 D^2I_\varepsilon(V+t\phi)[\phi]^2\,t\,dt.$$

We know that $K(y)=\varepsilon^4k(\varepsilon y)e^{V(y)}$ and ϕ satisfies

$$-\Delta\phi=R+K\phi+N(\phi)-\sum_{i=1}^{2}\sum_{j=1}^{m}c_{ij}\chi_jZ_{ij}-\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}K\phi$$

using (4.4.1). Also, we have that

$$
\begin{aligned}
D^2 I_\varepsilon(V+t\phi)[\phi]^2 &= -\int_{\Omega_\varepsilon} \phi\,\Delta\phi - \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^{V+t\phi}\phi^2 \\
&= \int_{\Omega_\varepsilon} [R+N(\phi)]\,\phi + \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^{V}(1-e^{t\phi})\phi^2.
\end{aligned}
$$

Thus,

$$
A = -\int_0^1 \left(\int_{\Omega_\varepsilon} [R+N(\phi)]\,\phi + \int_{\Omega_\varepsilon} \varepsilon^4 k(\varepsilon y) e^{V}(1-e^{t\phi})\phi^2 \right) t\,dt. \tag{4.5.5}
$$

Now, we can estimate

$$
\begin{aligned}
|A| &\le \int_0^1 C\big\| [R+N(\phi)]\,\phi + \varepsilon^4 k(\varepsilon y) e^{V}(1-e^{t\phi})\phi^2 \big\|_* \,dt \\
&\le \int_0^1 C\left[\|\phi\|_\infty (\|R\|_* + \|N(\phi)\|_*) + \|K\|_*\,|1-e^{t\phi}|\|\phi\|_\infty^2 \right] dt \\
&\le C\left[\varepsilon^2|\log\varepsilon| + \varepsilon^{3-\sigma}|\log\varepsilon|^2 + \varepsilon^3|\log\varepsilon|^3 \right],
\end{aligned}
$$

since $\|R\|_* \le C\varepsilon$, $\|N(\phi)\|_* \le C\varepsilon^{-\sigma}\|\phi\|_\infty^2$ and $\|\phi\|_\infty \le C\varepsilon|\log\varepsilon|$. Therefore, we get

$$
I_\varepsilon(V+\phi) - I_\varepsilon(V) = O(\varepsilon^2|\log\varepsilon|).
$$

Let us differentiate with respect to ξ'. We use representation (4.5.5) and differentiate directly under the integral sign, thus obtaining, for each $k=1,2,\ l=1,\dots,m$

$$
-\partial_{\xi'_{kl}}[A] = \int_0^1 \left(\int_{\Omega_\varepsilon} \partial_{\xi'_{kl}}\left[\{R+N(\phi)\}\,\phi\right] + \int_{\Omega_\varepsilon} \varepsilon^4 \partial_{\xi'_{kl}}\left[k(\varepsilon y) e^{V}(1-e^{t\phi})\phi^2 \right] \right) t\,dt
$$

We analyze and estimate each term, so,

$$
\begin{aligned}
&\left| \int_0^1 \left(\int_{\Omega_\varepsilon} \partial_{\xi'_{kl}}\left[\{R+N(\phi)\}\,\phi\right] \right) t\,dt \right| \\
&\quad \le \left| \int_{\Omega_\varepsilon} \partial_{\xi'_{kl}}\left[\{R+N(\phi)\}\,\phi\right] \right| \\
&\quad \le C\left[\left(\|\partial_{\xi'_{kl}} R\|_* + \|\partial_{\xi'_{kl}} N(\phi)\|_* \right) \|\phi\|_\infty + (\|R\|_* + \|N(\phi)\|_*)\, \|\partial_{\xi'_{kl}}\phi\|_\infty \right] \\
&\quad \le C\left[\varepsilon|\log\varepsilon|(\varepsilon + \varepsilon^{1-\sigma}|\log\varepsilon|\{\varepsilon|\log\varepsilon| + \varepsilon^{1-\sigma}|\log\varepsilon|^2\}) + \varepsilon^{1-\sigma}|\log\varepsilon|^2(\varepsilon + \varepsilon^{2-\sigma}|\log\varepsilon|^2) \right] \\
&\quad \le C\varepsilon^{2-\sigma}|\log\varepsilon|^2
\end{aligned}
$$

using Lemma 4.4.1 and the computations in the proof. Now, similarly as above

$$
\begin{aligned}
&\left| \int_0^1 \left(\int_{\Omega_\varepsilon} \varepsilon^4 \partial_{\xi'_{kl}}\left[k(\varepsilon y) e^{V}(1-e^{t\phi})\,\phi^2 \right] \right) t\,dt \right| \\
&\quad \le \int_0^1 C\left\| \varepsilon^4 k(\varepsilon y) e^{V}\left[\partial_{\xi'_{kl}} V(1-e^{t\phi})\,\phi^2 - e^{t\phi}\,t\,\partial_{\xi'_{kl}}\phi\,\phi^2 + 2(1-e^{t\phi})\,\phi\,\partial_{\xi'_{kl}}\phi \right] \right\|_* |t|\,dt \\
&\quad \le C\left[\|K\|_*\,\|\partial_{\xi'_{kl}} V\|_\infty |t|\,\|\phi\|_\infty^3 + \|K\|_*\,|t|\,\|\phi\|_\infty^2\,\|\partial_{\xi'_{kl}}\phi\|_\infty \right] \\
&\quad \le C\varepsilon^{3-\sigma}|\log\varepsilon|^4.
\end{aligned}
$$

Thus, we conclude

$$
\partial_{\xi'_{kl}}\left[I_\varepsilon(V+\phi) - I_\varepsilon(V) \right] = O(\varepsilon^{2-\sigma}|\log\varepsilon|^2), \qquad k=1,2,\ l=1,\dots,m.
$$

Now, taking $\tilde{\theta}_\varepsilon(\xi') = \theta_\varepsilon(\varepsilon\xi')$ with $\theta_\varepsilon(\xi) = F_\varepsilon(\xi) - J_\varepsilon(U)$, we have shown that

$$|\tilde{\theta}_\varepsilon| + \frac{\varepsilon^\sigma}{|\log\varepsilon|}|\nabla_{\xi'}\tilde{\theta}_\varepsilon| = O(\varepsilon^2|\log\varepsilon|), \qquad \text{as} \quad \varepsilon \to 0.$$

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. $\square$

4.6 Energy computations of approximate solution

The purpose of this section is to give an asymptotic estimate of $J_\varepsilon(U)$ where U is the approximate solution defined in (4.2.7) and J_ε is the energy functional (4.1.3) associated to Problem (4.1.2).

We have the following result.

Lemma 4.6.1. *Let $m \in \mathbb{Z}^+$ and $\delta > 0$ be a fixed small number and U be the function defined in* (4.2.7). *With the choice* (4.2.8) *for the parameter μ_j, $j = 1, \ldots, m$, the following expansion holds*

$$J_\varepsilon(U) = -16\pi m + 8\pi m\log 8 - 16\pi m\log\varepsilon - 32\pi^2 m H^*(0) + 4\pi\varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi) \tag{4.6.1}$$

where the function φ_m is defined by

$$\varphi_m(\xi) = -2\sum_{j=1}^m \log k(\xi_j) - \sum_{l\neq j} G(\xi_l, \xi_j). \tag{4.6.2}$$

In (4.6.1), *Θ_ε is a smooth function of $\xi = (\xi_1, \ldots, \xi_m)$, bounded together with its derivatives, as $\varepsilon \to 0$ uniformly on points $\xi_1, \ldots, \xi_m \in \Omega$ that satisfy dist$(\xi_i - \xi_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) > \delta$ for all $i \neq j$.*

Remark 4.6.1. In the sequel, by $\theta_\varepsilon, \Theta_\varepsilon$ we will denote generic functions of ξ that are bounded in the region $\text{dist}(\xi_i - \xi_j, \alpha\mathbb{Z} + \beta\mathbb{Z}) > \delta$ for all $i \neq j$.

Proof: First, we will evaluate the quadratic part of energy evaluated at U, that is,

$$\frac{1}{2}\int_\Omega |\nabla U|^2\,dx = -\frac{1}{2}\int_\Omega U\Delta U\,dx = -\frac{1}{2}\sum_{j=1}^m\int_\Omega U\Delta U_j\,dx.$$

Using the equation (4.2.2) of U_j, we have

$$\begin{aligned}\int_\Omega U(-\Delta U_j)\,dx &= \int_\Omega U(x)\left[\varepsilon^2 k(\xi_j)e^{u_j(x)} - \frac{1}{|\Omega|}\int_\Omega \varepsilon^2 k(\xi_j)e^{u_j}\right]dx\\ &= \int_\Omega \varepsilon^2 k(\xi_j)e^{u_j(x)}U(x)\,dx\end{aligned}$$

since $\int_\Omega U = 0$. Given $0 < \delta_0 < \delta$ we have

$$\begin{aligned}\int_\Omega \varepsilon^2 k(\xi_j)e^{u_j(x)}U(x)\,dx &= \int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j)e^{u_j(x)}U(x)\,dx + \int_{B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j)e^{u_j(x)}U(x)\,dx\\ &= \int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j)e^{u_j(x)}U(x)\,dx + 8\pi U(\xi_j)\\ &\quad + 8\int_{B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{U(\xi_j+\mu_j\varepsilon y) - U(\xi_j)}{(1+|y|^2)^2}\,dy - 8U(\xi_j)\int_{\mathbb{R}^2\setminus B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{dy}{(1+|y|^2)^2}.\end{aligned}$$

From the definition of U, we find

$$\int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U(x)\,dx = \sum_{l=1}^{m} \int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx$$

and for all $l \neq j$ we decompose

$$\int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx = \int_{B(\xi_l,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx + \int_{\Omega\setminus(B(\xi_j,\delta_0)\cup B(\xi_l,\delta_0))} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx.$$

Since, for any $l \in \{1,\dots,m\}$ we have $U_l(x) = G(x,\xi_l) + O(\varepsilon^2|\log\varepsilon|)$ uniformly in $\bar\Omega\setminus B(\xi_l,\delta_0)$, U_l is uniformly bounded in $\bar\Omega\setminus B(\xi_l,\delta_0)$ by a constant independent of ε. Besides $\varepsilon^2 k(\xi_j) e^{u_j} = O(\varepsilon^2)$ uniformly in $\bar\Omega\setminus B(\xi_j,\delta_0)$. Hence we get

$$\int_{\Omega\setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_j(x)\,dx = O(\varepsilon^2)$$

and for all $l \neq j$

$$\int_{\Omega\setminus(B(\xi_j,\delta_0)\cup B(\xi_l,\delta_0))} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx = O(\varepsilon^2).$$

Note that by Lemma 4.2.1 we have that uniformly for all $x \in B(\xi_l,\delta_0)$

$$U_l(x) = \log\frac{1}{(\mu_l\varepsilon^2 + |x-\xi_l|^2)^2} + H(x,\xi_l) + O(\varepsilon^2|\log\varepsilon|).$$

So, we have that

$$\int_{B(\xi_l,\delta_0)} |U_l(x)|\,dx \le \int_{B(\xi_l,\delta_0)} 2\left|\log(\mu_l^2\varepsilon^2 + |x-\xi_l|^2)\right|\,dy + \tilde C \le C.$$

Hence, we obtain that for all $l \neq j$

$$\int_{B(\xi_l,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U_l(x)\,dx = O(\varepsilon^2).$$

And we conclude

$$\int_{\Omega\setminus(B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} U(x)\,dx = O(\varepsilon^2).$$

Also, we know that $U(\xi_j) = -4\log\mu_j\varepsilon + H(\xi_j,\xi_j) + \sum_{l\neq j} G(\xi_l,\xi_j) + \varepsilon^2|\log\varepsilon|\Theta_\varepsilon(\xi)$, so

$$U(\xi_j)\int_{\mathbb{R}^2\setminus B(0,\frac{\delta_0}{\mu_j\varepsilon})} \frac{dy}{(1+|y|^2)^2} = O(\varepsilon^2|\log\varepsilon|).$$

On the other hand, observe that

$$U(\xi_j+\mu_j\varepsilon y) - U(\xi_j) = U_j(\xi_j+\mu_j\varepsilon y) - U_j(\xi_j) + \sum_{l\neq j}[U_l(\xi_j+\mu_j\varepsilon y) - U_l(\xi_j)].$$

Hence, we find that

$$U_j(\xi_j+\mu_j\varepsilon y) - U_j(\xi_j) = \log\frac{1}{(1+|y|^2)^2} + H_j(\xi_j+\mu_j\varepsilon y) - H_j(\xi_j),$$

since

$$u_j(\xi_j+\mu_j\varepsilon y)-u_j(\xi_j)=\log\frac{\mu_j^4\varepsilon^4}{(\mu_j^2\varepsilon^2+\mu_j^2\varepsilon^2|y|^2)^2}.$$

Now, from Lemma 4.2.1 we can deduce that for $y\in B\big(0,\frac{\delta_0}{\mu_j\varepsilon}\big)$

$$H_j(\xi_j+\mu_j\varepsilon y)-H_j(\xi_j)=H(\xi_j+\mu_j\varepsilon y,\xi_j)-H(\xi_j,\xi_j)+O(\varepsilon^2|\log\varepsilon|).$$

By the choice of δ_0 we get that $B(\xi_j,\delta_0)\cap B(\xi_l,\delta_0)=\emptyset$ for all $l\neq j$, so from Lemma 4.2.1 we have that for $y\in B\big(0,\frac{\delta_0}{\mu_j\varepsilon}\big)$

$$U_l(\xi_j+\mu_j\varepsilon y)-U_l(\xi_j)=G(\xi_j+\mu_j\varepsilon y,\xi_l)-G(\xi_j,\xi_l)+O(\varepsilon^2|\log\varepsilon|).$$

Then, for $y\in B(0,\frac{\delta_0}{\mu_j\varepsilon})$

$$\begin{aligned}U(\xi_j+\mu_j\varepsilon y)-U(\xi_j)=\;&\log\frac{1}{(1+|y|^2)^2}+H(\xi_j+\mu_j\varepsilon y,\xi_j)-H(\xi_j,\xi_j)\\&+\sum_{l\neq j}[G(\xi_j+\mu_j\varepsilon y,\xi_l)-G(\xi_j,\xi_l)]+O(\varepsilon^2|\log\varepsilon|).\end{aligned}$$

Also, we obtain that

$$\begin{aligned}\int_{B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{1}{(1+|y|^2)^2}\log\frac{1}{(1+|y|^2)^2}\,dy&=-2\pi\int_0^{\delta_0/\mu_j\varepsilon}\frac{-2r}{(1+r^2)^2}\log\frac{1}{(1+r^2)^2}\,dr\\&=-2\pi\int_1^{\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}}\log t\,dt\\&=-2\pi\left[\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}\log\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}-\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}+1\right].\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\int_{B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{U(\xi_j+\mu_j\varepsilon y)-U(\xi_j)}{(1+|y|^2)^2}\,dy=&-2\pi\left[\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}\log\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}-\frac{\mu_j^2\varepsilon^2}{\mu_j^2\varepsilon^2+\delta_0^2}+1\right]\\&+\int_{B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{H(\xi_j+\mu_j\varepsilon y,\xi_j)-H(\xi_j,\xi_j)}{(1+|y|^2)^2}\,dy\\&+\sum_{l\neq j}\int_{B(0,\frac{\delta_0}{\mu_j\varepsilon})}\frac{G(\xi_j+\mu_j\varepsilon y,\xi_l)-G(\xi_j,\xi_l)}{(1+|y|^2)^2}\,dy+O(\varepsilon^2|\log\varepsilon|)\\=&-2\pi+\varepsilon^2|\log\varepsilon|\Theta_\varepsilon(\xi).\end{aligned}$$

And again using $U(\xi_j)=-4\log\mu_j\varepsilon+H(\xi_j,\xi_j)+\sum_{l\neq j}G(\xi_j,\xi_l)+\varepsilon^2|\log\varepsilon|\Theta_\varepsilon(\xi)$ we conclude

$$\int_\Omega\varepsilon^2k(\xi_j)e^{u_j(x)}U(x)\,dx=-16\pi-32\pi\log\mu_j\varepsilon+8\pi\Big(H(\xi_j,\xi_j)+\sum_{l\neq j}G(\xi_j,\xi_l)\Big)+\varepsilon\Theta_\varepsilon(\xi).$$

Therefore,

$$\frac12\int_\Omega|\nabla U|^2\,dx=-8\pi m+\sum_{j=1}^m\left[-16\pi\log\mu_j\varepsilon+4\pi\Big(H(\xi_j,\xi_j)+\sum_{l=1,l\neq j}^m G(\xi_j,\xi_l)\Big)\right]+\varepsilon\Theta_\varepsilon(\xi).\quad(4.6.3)$$

On the other hand, from (4.2.21) we know that the second term in the energy functional satisfies

$$\varepsilon^2 \int_\Omega k(x) e^{U(x)}\, dx = 8\pi m + \varepsilon^2 |\log \varepsilon| \Theta_\varepsilon(\xi).$$

Using (4.6.3) and (4.2.21), we conclude

$$\begin{aligned} J_\varepsilon(U) &= -16\pi m + \sum_{j=1}^m \left[-16\pi \log \mu_j \varepsilon + 4\pi \left(H(\xi_j, \xi_j) + \sum_{l=1, l\neq j}^m G(\xi_j, \xi_l) \right) \right] + \varepsilon \Theta_\varepsilon(\xi) \\ &= -16\pi m + 8\pi m \log 8 - 16\pi m \log \varepsilon - 4\pi \sum_{j=1}^m \left[2 \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{l=1, l\neq j}^m G(\xi_l, \xi_j) \right] \\ &\quad + \varepsilon \Theta_\varepsilon(\xi), \end{aligned} \tag{4.6.4}$$

since by the choice of μ_j

$$16\pi \log \mu_j \varepsilon = 8\pi \left[\log k(\xi_j) - \log 8 + H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j) \right] + 16\pi \log \varepsilon.$$

Recall that $H(\xi_j, \xi_j) = 8\pi H^*(0)$ and $H(\xi_j + \mu_j \varepsilon y, \xi_j) - H(\xi_j, \xi_j) = \frac{2\pi}{|\Omega|} \mu_j^2 \varepsilon^2 |y|^2 + 8\pi H^*(\mu_j \varepsilon y) - 8\pi H^*(0)$ for all $j = 1, \ldots, m$. Hence, we conclude (4.6.1). The C^1-closeness is a direct consequence of the fact that Θ_ε is bounded together with its derivatives in the considered region. In fact, we will show that

$$\partial_{\xi_{kl}} [J_\varepsilon(U)] = 4\pi \partial_{\xi_{kl}} \varphi_m(\xi_1, \ldots, \xi_m) + O(\varepsilon^2 |\log \varepsilon|) \tag{4.6.5}$$

in the considered region. First, observe that

$$\partial_{\xi_{kl}} [J_\varepsilon(U)] = D J_\varepsilon(U)[\partial_{\xi_{kl}} U] = -\int_\Omega \left[\Delta U + \varepsilon^2 k e^U \right] \partial_{\xi_{kl}} U.$$

Now, we have that

$$\int_\Omega \partial_{\xi_{kl}} U (-\Delta U) = \sum_{j=1}^m \int_\Omega \partial_{\xi_{kl}} U (-\Delta U_j) = \sum_{j=1}^m \int_\Omega \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U(x)\, dx$$

using the equation (4.2.2) of U_j and $\int_\Omega \partial_{\xi_{kl}} U = 0$. Again, we consider $0 < \delta_0 < \delta$. So, we have

$$\begin{aligned} \int_\Omega \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U(x)\, dx &= \int_{\Omega \setminus B(\xi_j, \delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U(x)\, dx + 8\pi \partial_{\xi_{kl}} U(\xi_j) \\ &\quad + 8 \int_{B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{\partial_{\xi_{kl}} U(\xi_j + \mu_j \varepsilon y) - \partial_{\xi_{kl}} U(\xi_j)}{(1 + |y|^2)^2}\, dy \\ &\quad - 8 \partial_{\xi_{kl}} U(\xi_j) \int_{\mathbb{R}^2 \setminus B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{dy}{(1 + |y|^2)^2}. \end{aligned}$$

From the definition of U, we find

$$\int_{\Omega \setminus B(\xi_j, \delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U(x)\, dx = \sum_{q=1}^m \int_{\Omega \setminus B(\xi_j, \delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx$$

and for all $q \neq j$ we decompose

$$\int_{\Omega \setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx = \int_{B(\xi_q,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx + \int_{\Omega \setminus (B(\xi_j,\delta_0) \cup B(\xi_q,\delta_0))} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx.$$

We know that for any q, $\partial_{\xi_{kl}} U_q$ is uniformly bounded in $\bar{\Omega} \setminus B(\xi_q, \delta_0)$ by a constant independent of ε. Hence we get

$$\int_{\Omega \setminus B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_j(x)\, dx = O(\varepsilon^2)$$

and for all $q \neq j$

$$\int_{\Omega \setminus (B(\xi_j,\delta_0) \cup B(\xi_q,\delta_0))} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx = O(\varepsilon^2).$$

Furthermore, observe that for $q \neq l$, $\partial_{\xi_{kl}} U_q$ is uniformly bounded in Ω and

$$\partial_{\xi_{kl}} U_l(x) = \frac{4(x-\xi_l)_k}{\mu_l^2 \varepsilon^2 + |x-\xi_l|^2} + O(1)$$

in Ω. Then we have that

$$\int_{B(\xi_l,\delta_0)} |\partial_{\xi_{kl}} U_l(x)|\, dx \leq C_1 \int_{B(\xi_l,\delta_0)} \frac{|x-\xi_l|}{\mu_l^2 \varepsilon^2 + |x-\xi_l|^2}\, dx + C_2 \leq C$$

Hence, we obtain that for all $q \neq j$

$$\int_{B(\xi_q,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U_q(x)\, dx = O(\varepsilon^2).$$

Thus, we conclude that

$$\int_{\Omega \setminus (B(\xi_j,\delta_0)} \varepsilon^2 k(\xi_j) e^{u_j(x)} \partial_{\xi_{kl}} U(x)\, dx = O(\varepsilon^2).$$

On the other hand, we know that

$$\partial_{\xi_{kl}} U(x) = \frac{4(x-\xi_l)_k}{\mu_l^2 \varepsilon^2 + |x-\xi_l|^2} + \partial_{2k} H(x,\xi_l) - 2 \sum_{q=1}^{m} \frac{\partial_{\xi_{kl}} (\mu_q)^2 \varepsilon^2}{\mu_q^2 \varepsilon^2 + |x-\xi_q|^2} + O(\varepsilon^2 |\log \varepsilon|),$$

hence, we get that there is a constant $C > 0$ independent of ε such that for all $j = 1, \ldots, m$, $|\partial_{\xi_{kl}} U(\xi_j)| \leq C$. So, we get that

$$\partial_{\xi_{kl}} U(\xi_j) \int_{\mathbb{R}^2 \setminus B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{dy}{(1+|y|^2)^2} = O(\varepsilon^2).$$

Now, observe that if $j \neq l$ then

$$\partial_{\xi_{kl}} U(\xi_j + \mu_j \varepsilon y) - \partial_{\xi_{kl}} U(\xi_j) = \partial_{2k} G(\xi_j + \mu_j \varepsilon y, \xi_l) - 2 \frac{\partial_{\xi_{kl}} (\mu_j)^2}{\mu_j^2 (1+|y|^2)} - \partial_{2k} G(\xi_j, \xi_l) + 2 \frac{\partial_{\xi_{kl}} (\mu_j)^2}{\mu_j^2} + O(\varepsilon^2 |\log \varepsilon|)$$

and

$$\partial_{\xi_{kl}} U(\xi_l + \mu_l \varepsilon y) - \partial_{\xi_{kl}} U(\xi_l) = \frac{4y_k}{\mu_l \varepsilon (1+|y|^2)} + \partial_{2k} H(\xi_l + \mu_l \varepsilon y, \xi_l) - 2\frac{\partial_{\xi_{kl}}(\mu_l)^2}{\mu_l^2(1+|y|^2)} - \partial_{2k} H(\xi_l, \xi_l) + 2\frac{\partial_{\xi_{kl}}(\mu_l)^2}{\mu_l^2} + O(\varepsilon^2 |\log \varepsilon|).$$

Also, we obtain that

$$\int_{B(0,\frac{\delta_0}{\mu_j \varepsilon})} \frac{2}{(1+|y|^2)^3}\, dy = \pi \left[1 - \frac{\mu_j^4 \varepsilon^4}{(\mu_j^2 \varepsilon^2 + \delta_0^2)^2} \right].$$

Thus, we have that for any j

$$\int_{B(0,\frac{\delta_0}{\mu_j \varepsilon})} \frac{\partial_{\xi_{kl}} U(\xi_j + \mu_j \varepsilon y) - \partial_{\xi_{kl}} U(\xi_j)}{(1+|y|^2)^2}\, dy = \pi \frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2} + O(\varepsilon^2 |\log \varepsilon|).$$

Also, we know that for $j \neq l$

$$\partial_{\xi_{kl}} U(\xi_j) = \partial_{2k} G(\xi_j, \xi_l) - 2\frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2} + O(\varepsilon^2 |\log \varepsilon|)$$

and

$$\partial_{\xi_{kl}} U(\xi_l) = -2\frac{\partial_{\xi_{kl}}(\mu_l^2)}{\mu_l^2} + O(\varepsilon^2 |\log \varepsilon|).$$

Thus, we conclude that

$$\begin{aligned}\int_\Omega \partial_{\xi_{kl}} U(-\Delta U) =& 8\pi \left(-2\frac{\partial_{\xi_{kl}}(\mu_l^2)}{\mu_l^2} + \sum_{j=1, j\neq l}^m \left[\partial_{2k} G(\xi_j, \xi_l) - 2\frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2} \right] \right) \\ &+ 8\pi \sum_{j=1}^m \frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2} + O(\varepsilon^2 |\log \varepsilon|) \\ =& - 8\pi \sum_{j=1}^m \frac{\partial_{\xi_{kl}}(\mu_j^2)}{\mu_j^2} + 8\pi \sum_{j=1, j\neq l}^m \partial_{2k} G(\xi_j, \xi_l) + O(\varepsilon^2 |\log \varepsilon|).\end{aligned}$$

On the other hand, we know that

$$\varepsilon^2 \int_\Omega k(x) e^{U(x)} \partial_{\xi_{kl}} U(x)\, dx = O(\varepsilon^2 |\log \varepsilon|).$$

Therefore, using the choice of μ_j, we conclude

$$\begin{aligned}\partial_{\xi_{kl}}[J_\varepsilon(U)] &= -8\pi \sum_{j=1}^m \partial_{\xi_{kl}}(\log \mu_j^2) + 8\pi \sum_{j=1, j\neq l}^m \partial_{2k} G(\xi_j, \xi_l) + O(\varepsilon^2 |\log \varepsilon|) \\ &= -8\pi \partial_{\xi_{kl}} \left(\sum_{j=1}^m \left[\log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{q=1, q\neq j}^m G(\xi_q, \xi_j) \right] \right) \\ &\quad + 8\pi \partial_{\xi_{kl}} \left(\sum_{j=1, j\neq l}^m G(\xi_j, \xi_l) \right) + O(\varepsilon^2 |\log \varepsilon|) \\ &= -4\pi \partial_{\xi_{kl}} \left(\sum_{j=1}^m \left[2\log k(\xi_j) + \sum_{q=1, q\neq j}^m G(\xi_q, \xi_j) \right] \right) + O(\varepsilon^2 |\log \varepsilon|)\end{aligned} \tag{4.6.6}$$

since $H(\xi_j, \xi_j)$ is constant, $G(\xi_q, \xi_j) = G(\xi_j, \xi_q)$ for all $j \neq q$, $\partial_{1k} G(\xi_l, \xi_q) = \partial_{2k} G(\xi_q, \xi_l)$,

$$\partial_{\xi_{kl}} \left(\sum_{j=1}^{m} \sum_{q=1, q\neq j}^{m} G(\xi_q, \xi_j) \right) = 2\partial_{\xi_{kl}} \left(\sum_{j=1, j\neq j}^{m} G(\xi_j, \xi_j) \right)$$

and

$$\sum_{j=1}^{m} \sum_{q=1, q\neq j}^{m} G(\xi_q, \xi_j) = \sum_{q=1, q\neq l}^{m} G(\xi_q, \xi_l) \sum_{j=1, j\neq l}^{m} \sum_{q=1, q\neq j}^{m} G(\xi_q, \xi_j).$$

Thus, we conclude (4.6.5). This completes the proof. □

4.7 Proof of Theorems

4.7.1 Proof of Theorem 4.1.2.

Let us consider the set $\mathcal{D}$ as in the statement of the theorem, $\mathcal{C}$ the associated critical value and $\xi \in \mathcal{D}$. According to Lemma 4.5.1, we have a solution of Problem (4.1.2) if we adjust ξ so that it is a critical point of F_ε defined by (4.5.2). This is equivalent to finding a critical point of

$$\tilde{F}_\varepsilon(\xi) = \frac{1}{4\pi} \left[F_\varepsilon(\xi) + 16m\pi \log \varepsilon + 16\pi m - 8\pi m \log 8 + 32\pi^2 m H^*(0) \right].$$

On the other hand, from Lemmas 4.5.2 and 4.6.1, we have that for $\xi \in \mathcal{D}$, such that its components satisfy $|\xi_i - \xi_j| \geq \delta$,

$$\tilde{F}_\varepsilon(\xi) = \varphi_m(\xi) + \varepsilon^\gamma \Theta_\varepsilon(\xi), \qquad \text{with} \quad 0 < \gamma < 1 - \sigma$$

where Θ_ε and $\nabla_\xi \Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$.

Let us observe that if $M > \mathcal{C}$, then assumptions (2.1.1), (2.1.2) still hold for the function $\min\{M, \varphi_m(\xi)\}$ as well as for $\min\{M, \varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi)\}$. It follows that the function $\min\{M, \tilde{F}_\varepsilon(\xi)\}$ satisfies for all ε small assumptions (2.1.1),(2.1.2) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_\varepsilon < M$ which is close to $\mathcal{C}$ in this region. If $\xi_\varepsilon \in \mathcal{D}$ is a critical point at this level for $\tilde{F}_\varepsilon(\xi)$, then since

$$\tilde{F}_\varepsilon(\xi_\varepsilon) \leq \mathcal{C}_\varepsilon < M$$

we have that there exists a $\delta > 0$ such that $|\xi_{\varepsilon,j} - \xi_{\varepsilon,i}| > \delta$. This implies C^1-closeness of $\tilde{F}(\xi)$ and $\varphi_m(\xi)$ at this level, hence $\nabla \varphi_m(\xi_\varepsilon) \to 0$. The function $u_\varepsilon = U(\xi_\varepsilon) + \tilde{\phi}(\xi_\varepsilon)$ is therefore a solution as predicted by the theorem. □

4.7.2 Proof of Theorem 4.1.1.

According to the result of Theorem 4.1.2, it is sufficient to establish that given $m \geq 1$, φ_m has a nontrivial critical value in some open set $\mathcal{D}$, compactly contained in $\bar{\Omega}^m$. We will use an idea developed in [31]. Let us observe that the function φ_m becomes

$$\varphi_m(y_1, \ldots, y_m) = \sum_{j=1}^{m} NG(y_j, p) - \sum_{i\neq j} G(y_i, y_j).$$

The domain $\mathcal{D}$ is chosen as $\mathcal{D} = \Omega_\delta^m$, where

$$\Omega_\delta = \{ y \in \bar{\Omega} \ : \ |y - p| > \delta \ \}$$

where δ is a small positive number and $p \in \Omega$ (open cell). Consider a closed, smooth Jordan curve γ contained in Ω which encloses the point p. We let S to be the image of γ, $B_0 = \emptyset$ and $B = S \times \cdots \times S = S^m$.

Let us recall that

$$\Omega = \{z = s\alpha + t\beta \in \mathbb{C} \mid 0 < s, t < 1\}$$

with $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Im}(\beta/\alpha) > 0$. By the doubly periodic conditions it will be useful to consider the distance

$$d(x, y) = \mathrm{dist}(x - y, \alpha\mathbb{Z} + \beta\mathbb{Z}) := \inf\{|x - y + n_1\alpha + n_2\beta| \, : \text{ for all } n_1, n_2 \in \mathbb{Z}\}.$$

Note that for a small $\delta > 0$ we have that $d(y, p) \le \delta$ implies $d(x, y) = |y - p|$. Furthermore, for any $x, y \in \mathbb{R}^2$ there is $z \in \alpha\mathbb{Z} + \beta\mathbb{Z}$ such that $d(x, y) = |x - y + z|$ and by the periodicity of the Green's function $G(x, y) = G(x + z, y) = G(x, y - z)$.

Then define

$$\mathcal{C} = \inf_{\Phi \in \Gamma} \sup_{z \in B} \varphi_m(\Phi(z)), \tag{4.7.1}$$

where $\Phi \in \Gamma$ if and only if $\Phi(z) = \Psi(1, z)$ with $\Psi : [0, 1] \times B \to \mathcal{D}$ continuous and $\Psi(0, z) = z$.

Lemma 4.7.1. *There exists $K > 0$, independent of the small number δ used to define $\mathcal{D}$ such that $\mathcal{C} \ge -K$.*

Proof. We need to prove the existence of $K > 0$ independent of small δ such that if $\Phi \in \Gamma$, then there exists a $\bar{z} \in B$ with

$$\varphi_m(\Phi(\bar{z})) \ge -K. \tag{4.7.2}$$

Let us write

$$\Phi(z) = (\Phi_1(z), \ldots, \Phi_m(z)).$$

Identifying the components of the above m-tuple with complex numbers and given m points $\zeta_1, \ldots, \zeta_m \in S^1$, we shall establish the existence of $\bar{z} \in B$ such that

$$\frac{\Phi_j(\bar{z}) - p}{|\Phi_j(\bar{z}) - p|} = \zeta_j \quad \text{for all } j = 1, \ldots, m. \tag{4.7.3}$$

This fact was shown in [31]. For the sake of self-containment we shall present a proof here. To prove (4.7.3), we consider an orientation-preserving homeomorphism $h : S^1 \to S$ and the map $f : T^m \to T^m$ defined as $f(\zeta) = (f_1(\zeta), \ldots, f_m(\zeta))$ with

$$T^m = \underbrace{S^1 \times \cdots \times S^1}_{m},$$

and

$$f_j(\zeta_1, \ldots, \zeta_m) = \frac{\Phi_j(h(\zeta_1), \ldots, h(\zeta_m)) - p}{|\Phi_j(h(\zeta_1), \ldots, h(\zeta_m)) - p|}.$$

We define a homotopy $F : [0, 1] \times T^m \to T^m$ by

$$F_j(t, \zeta) = \frac{\Psi_j(t, h(\zeta_1), \ldots, h(\zeta_m)) - p}{|\Psi_j(t, h(\zeta_1), \ldots, h(\zeta_m)) - p|}.$$

Notice that $F(1, \zeta) = f(\zeta)$ and

$$F(0, \zeta) = \left(\frac{h(\zeta_1) - p}{|h(\zeta_1) - p|}, \ldots, \frac{h(\zeta_m) - p}{|h(\zeta_m) - p|} \right),$$

which is a homeomorphism of T^m. The existence of $\bar{z}$ such that relation (4.7.3) holds follows from establishing that f is onto, which we show next.

The torus T^m can be identified with the closed manifold embedded in $\mathbb{R}^{m+1}$ parameterized as

$$\zeta : (\theta_1, \dots, \theta_m) \in [0, 2\pi)^m \mapsto (\rho_1 e^{i\theta_1}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2}, 0_{m-2}) + \cdots + (0_{m-1}, \rho_m e^{i\theta_m}),$$

where $0 < \rho_m < \cdots < \rho_1$ and we have denoted $0_k = \underbrace{(0, \dots, 0)}_{k}$, $e^{i\theta_j} = (\cos\theta_j, \sin\theta_j)$. We consider as well the solid torus $\hat{T}^m$ parameterized as

$$(\theta_1, \dots, \theta_m, \rho) \in [0, 2\pi)^m \times [0, \rho_m] \mapsto (\rho_1 e^{i\theta_1}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2}, 0_{m-2}) + \cdots + (0_{m-1}, \rho e^{i\theta_m}).$$

Obviously $\partial \hat{T}^m = T^m$ in $\mathbb{R}^{m+1}$.

With slight abuse of notation, we consider the map $f : T^m \to T^m$, induced from the original f under the above identification, namely

$$f(\zeta) = (\rho_1 f_1(\zeta), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta), 0_{m-2}) + \cdots + (0_{m-1}, \rho_m f_m(\zeta)).$$

f then can be extended continuously to the whole solid torus as $\tilde{f} : \hat{T}^m \to \mathbb{R}^{m+1}$ defined simply as

$$f(\zeta, \rho) = (\rho_1 f_1(\zeta), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta), 0_{m-2}) + \cdots + (0_{m-1}, \rho f_m(\zeta)).$$

$\tilde{f}$ is homotopic to a homeomorphism of $\hat{T}^m$, along a deformation which applies $\partial \hat{T}^m$ into itself. Thus if $P \in \operatorname{int}(\hat{T}^m)$ then $\deg(\tilde{f}, \hat{T}^m, P) \neq 0$ and hence there exists $Q \in \hat{T}^m$ such that $\tilde{f}(Q) = P$. Thus if we fix angles $(\theta_1^*, \dots, \theta_m^*) \in [0, 2\pi)^m$ and $\rho^* \in (0, \rho_m)$ then there exist $\zeta^{**} \in T^m$ and $\rho^{**} \in (0, \rho_m)$ such that

$$(\rho_1 f_1(\zeta^{**}), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta^{**}), 0_{m-2}) + \cdots + (0_{m-1}, \rho^{**} f_m(\zeta^{**})) =$$

$$(\rho_1 e^{i\theta_1^*}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2^*}, 0_{m-2}) + \cdots + (0_{m-1}, \rho^* e^{i\theta_m^*}).$$

A direct computation shows then that $f_j(\zeta^{**}) = e^{i\theta_j^*}$ for all j and also $\rho^* = \rho^{**}$. It then follows that f is onto. This concludes the proof of (4.7.3).

Now, we will choose $\zeta_1, \dots, \zeta_m \in S^1$ as follows: denote $\zeta_j = e^{2\pi i\theta_j}$, with $\theta_j \in [0, 1)$ and define $t_j^* := \max\{t > 0 \mid p + t\zeta_j \in \bar{\Omega}\}$. Thus, we get that $p + t_j^*\zeta_j \in \partial\Omega$. Also, for $j = 1, \dots, m$ define the sets

$$A_j = \left\{ \frac{t_j^*\zeta_j + n_1\alpha + n_2\beta}{|t_j^*\zeta_j + n_1\alpha + n_2\beta|} \ : \ n_1, n_2 = -1, 0, 1,\ n_1^2 + n_2^2 \neq 0 \text{ and } p + t_j^*\zeta_j + n_1\alpha + n_2\beta \in \partial\Omega \right\}.$$

Then, we choose any $\theta_1 \in [0, \frac{1}{m})$ and θ_j for $j > 1$ such that $\theta_j \in [\frac{j-1}{m}, \frac{j}{m})$ and $\zeta_j = e^{2\pi i\theta_j} \notin \cup_{l=1}^{j-1} A_l$. Hence, we have that there is a constant $\sigma_0 > 0$ such that if $y \in \mathcal{D}$ satisfies

$$\frac{y_j - p}{|y_j - p|} = \zeta_j \quad \text{for all } j = 1, \dots, m$$

then

$$|y_j - y_l + n_1\alpha + n_2\beta| > \sigma_0 \qquad \text{for all } j \neq l \text{ and } n_1, n_2 = -1, 0, 1,\ n_1^2 + n_2^2 \neq 0,$$

namely, $d(y_j, y_l) \geq \sigma_0$ and $H(y_j, y_l) = O(1)$ uniformly for all $j \neq l$. Furthermore, $\zeta_j \neq \zeta_l$ if $j \neq l$ and there exist a constant $C > 0$ such that $|y_l - y_j| \geq C|y_l - p|$ for all $l \neq j$.

Now, from (2.2.2), it is clearly that

$$\varphi_m(y_1,\ldots,y_m) = 4N\sum_{j=1}^{m}\log\frac{1}{|y_j-p|} + 4\sum_{l\neq j}\log|y_j-y_l| + O(1).$$

Fix j, then we have

$$4N\log\frac{1}{|y_j-p|} + 4\sum_{l=1,l\neq j}^{m}\log|y_j-y_i| \geq 4\left(N\log\frac{1}{|y_j-p|} - (m-1)\log\frac{1}{|y_j-p|}\right) + O(1).$$

Since $N > m-1$ by assumption, the above quantity is uniformly bounded below, hence the value $\mathcal{C}$ is bounded below independently of δ, as desired. □

The second step we have to carry out to make Theorem 4.1.2 applicable is to establish the validity of assumption (2.1.2). To this end we need to establish a couple of preliminary facts on the half plane

$$\mathcal{H} = \{(x^1,x^2)\ :\ x^1 \geq 0\}.$$

Lemma 4.7.2. *Consider the function of k distinct points on $\mathcal{H}$*

$$\Psi_k(x_1,\ldots,x_k) = -4\sum_{i\neq j}\log|x_i-x_j|.$$

Let I_+ denote the set of indices i for which $x_i^1 > 0$ and I_0 that for which $x_i^1 = 0$. Then, either

$$\nabla_{x_i}\Psi_k(x_1,\ldots,x_k) \neq 0, \quad \textit{for some } i\in I_+,$$

or

$$\frac{\partial}{\partial x_{i2}}\Psi_k(x_1,\ldots,x_k) \neq 0, \quad \textit{for some } i\in I_0.$$

Proof. We have that

$$\frac{\partial}{\partial\lambda}\Psi_k(\lambda x_1,\ldots,\lambda x_k)|_{\lambda=1} =$$
$$\sum_{i\in I_+}\nabla_{x_i}\Psi_k(x_1,\ldots,x_k)\cdot x_i + \sum_{i\in I_0}\partial x_{i2}\Psi_k(x_1,\ldots,x_k)x_{i2}.$$

On the other hand,

$$\frac{\partial}{\partial\lambda}\Psi_k(\lambda x_1,\ldots,\lambda x_k)|_{\lambda=1} = -4\frac{\partial}{\partial\lambda}[k(k-1)\log\lambda]|_{\lambda=1} \neq 0,$$

and the result follows. □

Now, we are ready to prove the validity of assumption (2.1.2) which in this case reads as follows:

Lemma 4.7.3. *Given $K>0$ there exists a $\delta>0$ such that if $\xi=(\xi_1,\ldots,\xi_m)\in\mathcal{D}_\delta$ and $|\varphi_m(\xi)|\leq K$, then there is a vector τ, tangent to $\partial\mathcal{D}_\delta$ at ξ such that*

$$\nabla\varphi_m(\xi)\cdot\tau \neq 0$$

Proof: Let us assume the opposite, namely the existence of a sequence $\delta_n > 0$, $\delta_n \to 0$, and points $\xi^n \in \partial\mathcal{D}_{\delta_n}$ such that $\xi^n = (\xi_1^n, \dots, \xi_m^n)$,

$$\nabla_{\xi_i}\varphi_m(\xi_1^n, \dots, \xi_m^n) = 0 \qquad \text{if} \quad \xi_i^n \in \Omega_{\delta_n}$$

and

$$\nabla_{\xi_i}\varphi_m(\xi_1^n, \dots, \xi_m^n) \cdot \tau_i = 0 \qquad \text{if} \quad \xi_i^n \in \partial\Omega_{\delta_n}, \qquad \text{for all} \quad \tau_i \in T_{\xi_i^n}(\partial\Omega_{\delta_n}),$$

where $T_{\xi_i}(\partial\Omega_{\delta_n})$ is the set of all vectors τ tangent to $\partial\Omega_{\delta_n}$ at ξ_i. Note that $\partial\Omega_{\delta_n} = \{y \in \Omega : |y-p| = \delta_n\}$. From the assumptions of the Lemma follows that there is a point $\xi_i^n \in \partial\Omega_{\delta_n}$ for all n (up to subsequence). Hence, $\xi_i^n \to p$ as $n \to +\infty$ and $G(\xi_i^n, p) \to +\infty$ as $n \to +\infty$. Since the values of φ_m remains uniformly bounded, necessarily we must have that there are two different points ξ_j^n, ξ_l^n such that ξ_j^n and ξ_l^n are becoming close, namely $d(\xi_j^n, \xi_l^n) \to 0$ as $n \to +\infty$. Let us set $\rho_n = \inf_{i\neq j} d(\xi_i^n, \xi_j^n)$, so $\rho_n \to 0$ as $n \to +\infty$. Without loss of generality, we can assume $\rho_n = d(\xi_1^n, \xi_2^n)$. Let $\zeta_j^n \in \alpha\mathbb{Z} + \beta\mathbb{Z}$ such that $d(\xi_j^n, \xi_1^n) = |\xi_j^n - \xi_1^n - \zeta_j^n|$. We define

$$x_j^n = \frac{\xi_j^n - \xi_1^n - \zeta_j^n}{\rho_n}.$$

Clearly there exists a k, $2 \le k \le m$ such that

$$\lim_{n\to+\infty} |x_j^n| < +\infty, \quad j = 1, \dots, k \quad \text{and} \quad \lim_{n\to+\infty} |x_j^n| = +\infty, \quad j > k$$

For $j \le k$ we set $\tilde{x}_j = \lim_{n\to+\infty} x_j^n$. Note that $\tilde{x}_1 = 0$ and $|\tilde{x}_2| = 1$. Define

$$\tilde{\varphi}_m(x_1, \dots, x_n) = \varphi_m(\xi_1^n + \rho_n x_1, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m).$$

We have

$$\partial_{(x_j)_l}\tilde{\varphi}_m(x_1, \dots, x_n) = \rho_n \partial_{(\xi_j)_l}\varphi_m(\xi_1^n + \rho_n x_1, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m).$$

for all $l = 1, 2$, $j = 1, \dots, m$. Observe that

$$\begin{aligned}\partial_{(\xi_j)_l}\varphi_m(y_1, \dots, y_n) = {} & N\left(-4\frac{(y_j - p)_l}{|y_j - p|^2} + \partial_{1l}H(y_j, p)\right) \\ & - 2\sum_{i=1, i\neq j}^{m}\left(-4\frac{(y_j - y_i)_l}{|y_j - y_i|^2} + \partial_{2l}H(y_i, y_j)\right)\end{aligned}$$

and

$$\begin{aligned}&\partial_{(x_j)_l}\tilde{\varphi}_m(x_1^n, \dots, x_k^n, 0, \dots, 0) \\ &\quad = -4N\rho_n\frac{(\xi_j^n - p)_l}{|\xi_j^n - p|^2} + \rho_n N\partial_{1l}H(\xi_j^n, p) - 2\sum_{i=1, i\neq j}^{m}\Bigg(-4\rho_n\frac{(\xi_j^n - \xi_i^n + \zeta_i^n - \zeta_j^n)_l}{|\xi_j^n - \xi_i^n + \zeta_i^n - \zeta_j^n|^2} \\ &\qquad + \rho_n\partial_{2l}H(\xi_i^n - \zeta_i^n, \xi_j^n - \zeta_j^n)\Bigg) \\ &\quad = -4N\rho_n\frac{(\xi_j^n - p)_l}{|\xi_j^n - p|^2} + \rho_n N\partial_{1l}H(\xi_j^n, p) + 8\sum_{i=1, i\neq j}^{m}\frac{(x_j^n - x_i^n)_l}{|x_j^n - x_i^n|^2} \\ &\qquad - 2\sum_{i=1, i\neq j}^{m}\rho_n\partial_{2l}H(\xi_i^n - \zeta_i^n, \xi_j^n - \zeta_j^n),\end{aligned}$$

since, $\xi_j^n - \xi_i^n + \zeta_i^n - \zeta_j^n = \rho_n(x_j^n - x_i^n)$.

We consider two cases:

(1) either

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} = +\infty;$$

(2) or there exists $C_0 > 0$ such that for all n

$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \le C_0,$$

where $\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n}) = \inf\{d(\xi_1^n, y) \mid y \in \partial\Omega_{\delta_n}\}$.

In case 1, it is easy to see that actually

$$\lim_{n\to+\infty} \frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} = +\infty, \qquad \text{for all} \quad j = 1, \dots, k.$$

Indeed, we have that $\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n}) \le d(\xi_j^n, \xi_1^n) + \operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})$. Furthermore, points ξ_1^n, ..., ξ_k^n are all interior to Ω_{δ_n}, hence

$$\nabla_{\xi_j} \varphi_m(\xi_1^n, \dots, \xi_m^n) = 0 \quad \text{for all} \quad n, \quad \text{for} \quad j = 1, \dots, k.$$

Then, from the inequality $\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \le d(\xi_j^n, p)$, we deduce $\lim_{n\to+\infty} \rho_n \partial_{1l} G(\xi_j^n, p) = 0$. Now, note that for any $1 \le j \le k$ and $i \ge k+1$ we get that

$$\lim_{n\to+\infty} \frac{d(\xi_i^n, \xi_j^n)}{\rho_n} = +\infty.$$

Also, if $d(\xi_i^n, \xi_j^n) \ge \sigma_0$ for all n (up to subsequence) and for some $\sigma_0 > 0$, then there exists $x \in \mathbb{R}^2 \setminus (\alpha\mathbb{Z} + \beta\mathbb{Z})$ such that $|\xi_i^n - \xi_j^n - x| \to 0$ as $n \to +\infty$. Hence, we find that

$$\rho_n \partial_{2l} G(\xi_i^n, \xi_j^n) = -4\rho_n \frac{(\xi_i^n - \xi_j^n)_l}{|\xi_i^n - \xi_j^n|^2} + \rho_n \partial_{2l} H(\xi_i^n, \xi_j^n) = O(\rho_n),$$

as $n \to +\infty$. If $d(\xi_i^n, \xi_j^n) = o(1)$ as $n \to +\infty$, then there exists $\zeta_{ij}^n \in \alpha\mathbb{Z} + \beta\mathbb{Z}$ such that $d(\xi_i^n, \xi_j^n) = |\xi_i^n - \xi_j^n + \zeta_{ij}^n|$ and we have that as $n \to +\infty$

$$\begin{aligned} \rho_n \partial_{2l} G(\xi_i^n, \xi_j^n) &= -4\rho_n \frac{(\xi_i^n - \xi_j^n + \zeta_{ij}^n)_l}{|\xi_i. - \xi_j^n + \zeta_{ij}^n|^2} + \rho_n \partial_{2l} H(\xi_i^n + \zeta_{ij}^n, \xi_j^n) \\ &= O\left(\frac{\rho_n}{d(\xi_i^n, \xi_j^n)}\right) + O(\rho_n), \end{aligned}$$

since $|\xi_i^n - \xi_j^n + \zeta_{ij}^n| \to 0$. Therefore, we deduce that

$$\lim_{n\to+\infty} \partial_{(x_j)_l} \tilde\varphi_m(x_1^n, \dots, x_k^n, 0, \dots, 0) = -4\partial_{(x_j)_l} \left(\sum_{i\ne q, i,q\le k} \log \frac{1}{|\tilde x_q - \tilde x_i|} \right) = 0.$$

Note that $\rho_n |x_j^n - x_i^n| \ge |\xi_i^n - \xi_j^n + \zeta_{ij}^n| = d(\xi_j^n, \xi_i^n) \ge \rho_n$ implies $\tilde x_j \ne \tilde x_i$ for all $i \ne j$, $i, j \le k$. Hence, we deduce that this last equality is true for any $j \le k$, $l = 1, 2$. Thus, we arrive at a contradiction with Lemma 4.7.2, which proves impossibility of this case.

On the other hand, in case 2 there exist a constant $C_1 > 0$ such that

$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \le C_1, \qquad \text{for all} \quad j = 1, \dots, k.$$

In fact, it easily follows from the inequality $\text{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \leq d(\xi_j^n, \xi_1^n) + \text{dist}(\xi_1^n, \partial\Omega_{\delta_n})$. Also, note that we have

$$|\xi_j^n - p| \leq \delta_n + \text{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \leq \delta_n + C_1\rho_n, \qquad \text{for all } j = 1, \dots, k.$$

Hence, we get that $\xi_j^n \to p$ for all $j = 1, \dots, k$. Let us stress that $|\xi_j^n - p| \to 0$ if and only if $d(\xi_j^n, p) \to 0$ as $n \to +\infty$, since p is an interior point of the open cell Ω.

Assume first that there exists a constant $C > 0$ such that $\delta_n < C\rho_n$. Hence, we get that $|\xi_j^n - p| \leq (C + C_1)\rho_n$. Observe that

$$\begin{aligned}\varphi_m(\xi_1^n, \dots, \xi_m^n) &= N \sum_{j=1}^m \left[-4\log|\xi_j^n - p| + H(\xi_j^n, p)\right] - \sum_{i\neq j} G(\xi_i^n + \zeta_{ij}^n, \xi_j^n)\\ &= \sum_{j=1}^m s_j^n + O(1),\end{aligned}$$

where we denote

$$s_j^n := 4N\log\frac{1}{|\xi_j^n - p|} + \sum_{i=1, i\neq j}^m 4\log|\xi_i^n - \xi_j^n + \zeta_{ij}^n|.$$

Then, we get that for all $j = 1, \dots, k$

$$\begin{aligned}s_j^n &\geq 4N\log\frac{1}{(C + C_1)\rho_n} + \sum_{i=1, i\neq j}^m 4\log\rho_n\\ &\geq 4N\log\frac{1}{\rho_n} + 4(m-1)\log\rho_n + 4N\log\frac{1}{C + C_1}\\ &\geq 4\log\frac{1}{\rho_n^{N-(m-1)}} + \tilde{C}_1.\end{aligned}$$

Under the assumption $N + 1 > m$, we obtain that $\sum_{j=1}^k s_j^n \to +\infty$ as $n \to +\infty$. If $k = m$ then we conclude that $\varphi_m(\xi_1^n, \dots, \xi_m^n) \to +\infty$, as $n \to +\infty$, which is contradiction, since $\varphi_m(\xi^n)$ is bounded uniformly in n. Therefore, it holds that $k \leq m - 1$. Next, let us estimate the sum $\sum_{j=k+1}^m s_j^n$, with $k \leq m-2$. We can isolate groups of those points according to the asymptotic form of their mutual distances. For example, we can define:

$$\rho_n^1 = \inf_{i\neq j, i,j>k} d(\xi_j^n, \xi_i^n),$$

and consider those points whose mutual distances are $O(\rho_n^1)$, and so on. For each group of those points (also those with indices higher than k) the argument given above in the Case 1 applies. This means that not only those points become close to one another but also that their distance to the boundary $\partial\Omega_{\delta_n}$ is comparable with their mutual distance. Observe that for any $j \in \{k+1, \dots, m\}$ we have

$$\sum_{i=1, i\neq j}^m 4\log|\xi_i^n - \xi_j^n + \zeta_{ij}^n| = \sum_{i=1}^k 4\log|\xi_i^n - \xi_j^n + \zeta_{ij}^n| + \sum_{i=k+1, i\neq j}^m 4\log|\xi_i^n - \xi_j^n + \zeta_{ij}^n|.$$

First, assume that there is a constant $d_0 > 0$ such that $\rho_n^1 \geq d_0$ for all n. Then, at most there is certain $j_0 \in \{k+1,\dots,m\}$ such that $\xi_{j_0}^n \to p$ as $n \to +\infty$, since $\xi_i^n \to p$ as $n \to +\infty$ for all $i = 1,\dots,k$. Thus, for those $j \in \{k+1,\dots,m\}$ such that there is a constant $r_0 > 0$ satisfying $|\xi_j^n - p| \geq r_0$ for all n and we have that $d(\xi_i^n, \xi_j^n) = |\xi_i^n - \xi_j^n + \zeta_{ij}^n| \geq r_1$ for all n, $i = 1,\dots,m$ for some constant $r_1 > 0$. Hence, we get that $s_j^n = O(1)$ as $n \to +\infty$. Now, for that $j_0 \in \{k+1,\dots,m\}$ such that $\xi_{j_0}^n \to p$ as $n \to +\infty$, we have that $\zeta_{ij_0}^n = 0$ and $|\xi_{j_0}^n - \xi_i^n| \to 0$ as $n \to +\infty$ for all $i = 1,\dots,k$. Thus, taking $\tilde\rho_n = \inf_{i=1,\dots,k} |\xi_i^n - \xi_{j_0}^n|$, we consider two cases:

(a) either

$$\lim_{n\to+\infty} \frac{\mathrm{dist}(\xi_{j_0}^n, \partial\Omega_{\delta_n})}{\tilde\rho_n} = +\infty;$$

(b) or there exists $C_2 > 0$ such that for all n

$$\frac{\mathrm{dist}(\xi_{j_0}^n, \partial\Omega_{\delta_n})}{\tilde\rho_n} \leq C_2,$$

since $\tilde\rho_n \to 0$ as $n \to +\infty$. In case (a), after scaling with $\tilde\rho_n$ around $\xi_{j_0}^n$ and arguing similarly as in the Case 1 we get a contradiction with Lemma 4.7.2. Thus, case (a) cannot hold and it does hold case (b). In case (b) we have that

$$\begin{aligned} s_{j_0}^n &\geq 4N \log \frac{1}{(C+C_2)\tilde\rho_n} + \sum_{i=1}^{k} 4\log\tilde\rho_n + \sum_{i=k+1, i\neq j_0}^{m} 4\log d_0 \\ &\geq 4N\log\frac{1}{\tilde\rho_n} + 4k\log\tilde\rho_n + 4N\log\frac{1}{C+C_2} + 4(m-k-1)\log d_0 \\ &\geq 4\log\frac{1}{\tilde\rho_n^{N-k}} + \tilde C_2. \end{aligned}$$

Since $N > m-1 > k$, we conclude that as $n \to +\infty$, $s_{j_0}^n \to +\infty$. If $k = m-1$ then by similar arguments as above, depending on whether or not $\xi_m^n \to p$ as $n \to +\infty$, we get that either $s_m^n = O(1)$ or $s_m^n \to +\infty$ as $n \to +\infty$. Therefore, in any case $k \leq m-2$ with $\rho_n^1 \geq d_0$ or $k = m-1$, we conclude that $\varphi_m(\xi_1^n,\dots,\xi_m^n) \to +\infty$ as $n \to +\infty$, which is contradiction, since $\varphi_m(\xi^n)$ is bounded uniformly in n. Thus, it holds that $\rho_n^1 \to 0$ as $n \to +\infty$ when $k \leq m-2$. Since $\rho_n \leq \rho_n^1$, we get that $\delta_n < C\rho_n^1$. Similarly as above, without loss of generality, we can assume that $\rho_n^1 = |\xi_{k+1}^n - \xi_{k+2}^n|$. If

$$\lim_{n\to+\infty} \frac{\mathrm{dist}(\xi_{k+1}^n, \partial\Omega_{\delta_n})}{\rho_n^1} = +\infty$$

then the argument given above in the Case 1 applies. Thus, it holds that there is a constant $C_3 > 0$ such that

$$\frac{\mathrm{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n^1} \leq C_3, \qquad \text{for all} \quad j = k+1,\dots,k',$$

where $k+2 \leq k' \leq m$. In this case, we get that $\xi_j^n \to p$ for all $j = k+1,\dots,k'$ as $n \to +\infty$, since $\rho_n^1 \to 0$ as $n \to +\infty$ by the assumption and for all $j = k+1,\dots,k'$,

$$|\xi_j^n - p| \leq \delta_n + \mathrm{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \leq \delta_n + C_3\rho_n^1.$$

Thus, for each $j \in \{k+1,\dots,k'\}$ we have that $|\xi_i^n - \xi_j^n| \to 0$ as $n \to +\infty$ for all $i = 1,\dots,k$. Let us consider $\tilde\rho_n^j = \inf_{i=1,\dots,k} |\xi_j^n - \xi_i^n|$. Note that $\rho_n \leq \rho_n^1$, $\rho_n \leq \tilde\rho_n^j$ for all $j = k+1,\dots,k'$ and ρ_n,

ρ_n^1 and $\tilde{\rho}_n^j \to 0$ as $n \to +\infty$. For any $j \in \{k+1, \dots, m\}$, if

$$\lim_{n\to+\infty} \frac{\text{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\tilde{\rho}_n^j} = +\infty,$$

then the Case 1 applies. Therefore, there is a constant $\tilde{C}_3 > 0$ such that $\text{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \leq \tilde{C}_3\tilde{\rho}_n^j$. Hence, we get that

$$|\xi_j^n - p| \leq \delta_n + C_3\rho_n^1 \leq C\rho_n + C_3\rho_n^1 \leq (C + C_3)\rho_n^1$$

and

$$|\xi_j^n - p| \leq \delta_n + \text{dist}(\xi_j^n, \partial\Omega_{\delta_n}) \leq C\rho_n + \tilde{C}_3\tilde{\rho}_n^j \leq (C + \tilde{C}_3)\tilde{\rho}_n^j$$

for all $j = k+1, \dots, k'$. And we find the estimate

$$\begin{aligned} s_j^n &\geq 4(N - m + 1)\log\frac{1}{|\xi_j^n - p|} + 4k\log\frac{1}{(C+\tilde{C}_3)\tilde{\rho}_n^j} + 4(m-k-1)\log\frac{1}{(C+C_3)\rho_n^1} \\ &\quad + \sum_{i=1}^{k} 4\log\tilde{\rho}_n^j + \sum_{i=k+1, i\neq j}^{m} 4\log\rho_n^1 \\ &\geq 4(N-m+1)\log\frac{1}{\rho_n^1} + C_3' \end{aligned}$$

for all $j = k+1, \dots, k'$. On the other hand, if $k' \leq m-1$ then the sum

$$\sum_{j=k'+1}^{m}\left[4N\log\frac{1}{|\xi_j^n - p|} + \sum_{i=1, i\neq j}^{m} 4\log|\xi_i^n - \xi_j^n + \zeta_{ij}^n|\right]$$

could be estimated similarly as above. Therefore, in any case, we conclude

$$\varphi_m(\xi_1^n, \dots, \xi_m^n) \to +\infty \qquad \text{as} \quad n \to +\infty,$$

which is contradiction, since $\varphi_m(\xi^n)$ is bounded uniformly in n.

Finally, it remains to consider that $\rho_n = o(\delta_n)$ as $n \to +\infty$. Observe that $|\xi_j^n - p| \geq \delta_n$ for all $j = 1, \dots, k$, and hence

$$\lim_{n\to+\infty} \rho_n \frac{\xi_j^n - p}{|\xi_j^n - p|^2} = 0$$

If all points $\xi_1^n, \dots, \xi_k^n$ are interior to Ω_{δ_n} then after scaling with ρ_n we argue as in case 1 above to reach a contradiction with Lemma 4.7.2. Suppose that $\{1, \dots, k\} = I_1 \cup I_2$, where I_1 is the set of indices j for which $\xi_j^n \in \partial\Omega_{\delta_n}$ and I_2 that for $\xi_j^n \in \Omega_{\delta_n}$. So, $I_1, I_2 \neq \emptyset$ and $I \cap I_2 = \emptyset$. Then, we have that

$$\nabla_{\xi_j}\varphi_m(\xi_1^n, \dots, \xi_m^n)\cdot\tau = 0 \qquad \text{for all} \quad \tau \in T_{\xi_j^n}(\partial\Omega_{\delta_n}), \qquad \text{for all } j \in I_1$$

and

$$\nabla_{\xi_i}\varphi_m(\xi_1^n, \dots, \xi_m^n) = 0 \qquad \text{for all } i \in I_2.$$

Let $j \in I_1$, by the definition of $\partial\Omega_{\delta_n}$, we can take $\tau_j^n \in T_{\xi_j^n}(\partial\Omega_{\delta_n})$, $|\tau_j^n| = 1$ and $(\xi_j^n - p)\cdot\tau_j^n = 0$ for all n. Then, for $j \in I_1$ we get that

$$\lim_{n\to+\infty} \nabla_{x_j}\tilde{\varphi}_m(x_1^n, \dots, x_k^n, 0, \dots, 0)\cdot\tau_j^n = 8\sum_{l=1, l\neq j}^{k} \frac{\tilde{x}_j - \tilde{x}_l}{|\tilde{x}_j - \tilde{x}_l|^2}\cdot\tau_j = 0, \tag{4.7.4}$$

where $|\tau_j| = 1$ and $\tau_j^n \to \tau_j$ as $n \to +\infty$ (up to subsequence) and for $i \in I_2$

$$\lim_{n\to+\infty} \nabla_{x_i}\tilde{\varphi}_m(x_1^n, \dots, x_k^n, 0, \dots, 0) = 8 \sum_{l=1, l\neq i}^{k} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2} = 0. \tag{4.7.5}$$

In order to get a contradiction, we will use the following fact.

Claim 4.7.1. *Let $a_n, b_n \in I\!R^2$ be, such that $a_n \neq b_n$ for all n, $\delta_n = |a_n| = |b_n|$, $\delta_n \to 0$, $C_0\rho_n \leq |a_n - b_n| \leq C_1\rho_n$, for some constants $C_0, C_1 > 0$ and $\rho_n = o(\delta_n)$ as $n \to +\infty$. Then, up to subsequence*

$$\lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{a_n}{\delta_n} = \lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{b_n}{\delta_n} = 0.$$

Proof: First, it is clear that the limits are finite. By the definition of $|a_n - b_n|$, we have

$$C_0^2\rho_n^2 \leq (a_n - b_n) \cdot a_n - (a_n - b_n) \cdot b_n \leq C_1^2\rho_n^2$$

and

$$C_0^2\frac{\rho_n}{\delta_n} \leq \frac{a_n - b_n}{\rho_n} \cdot \frac{a_n}{\delta_n} - \frac{a_n - b_n}{\rho_n} \cdot \frac{b_n}{\delta_n} \leq C_1^2\frac{\rho_n}{\delta_n}.$$

Thus, we get

$$\lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{a_n}{\delta_n} = \lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{b_n}{\delta_n}.$$

On the other hand, $|a_n|^2 = (a_n - b_n) \cdot a_n + a_n \cdot b_n$ and $|b_n|^2 = -(a_n - b_n) \cdot b_n + a_n \cdot b_n$. Hence, we get that

$$(a_n - b_n) \cdot a_n + (a_n - b_n) \cdot b_n = 0 \qquad \text{and} \qquad \frac{a_n - b_n}{\rho_n} \cdot \frac{a_n}{\delta_n} + \frac{a_n - b_n}{\rho_n} \cdot \frac{b_n}{\delta_n} = 0.$$

Then, letting $n \to +\infty$ the claim follows. □

Similarly, we have a variant of the above conclusion.

Claim 4.7.2. *Let $a_n, b_n \in I\!R^2$ be, such that $a_n \neq b_n$ for all n, $\delta_n = |a_n| < |b_n|$, $\delta_n \to 0$, $C_0\rho_n \leq |a_n - b_n| \leq C_1\rho_n$, for some constants $C_0, C_1 > 0$ and $\rho_n = o(\delta_n)$ as $n \to +\infty$. Then, up to subsequence*

$$\lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{a_n}{\delta_n} = \lim_{n\to+\infty} \frac{a_n - b_n}{\rho_n} \cdot \frac{b_n}{\delta_n} \leq 0.$$

Now, taking $a_n = \xi_j^n - p$, $b_n = \xi_i^n - p$ and $j \in I_1$, we have that for $i \in I_1$

$$\lim_{n\to+\infty} \frac{\xi_j^n - \xi_i^n}{\rho_n} \cdot \frac{\xi_j^n - p}{\delta_n} = (\tilde{x}_j - \tilde{x}_i) \cdot \tilde{x} = 0, \tag{4.7.6}$$

where

$$\tilde{x} = \lim_{n\to+\infty} \frac{\xi_j^n - p}{\delta_n}$$

and for $i \in I_2$

$$\lim_{n\to+\infty} \frac{\xi_j^n - \xi_i^n}{\rho_n} \cdot \frac{\xi_j^n - p}{\delta_n} = (\tilde{x}_j - \tilde{x}_i) \cdot \tilde{x} \leq 0. \tag{4.7.7}$$

Note that from the equality

$$\frac{\xi_j^n - p}{\delta_n} = \frac{\xi_i^n - p}{\delta_n} + \frac{\rho_n}{\delta_n}\frac{\xi_j^n - \xi_i^n}{\rho_n}$$

we get that $\tilde{x}$ is independent of $j = 1, \ldots, k$. Also, we have that

$$\lim_{n\to+\infty} \frac{\xi_j^n - p}{\delta_n} \cdot \tau_j^n = \tilde{x} \cdot \tau_j = 0$$

since $(\xi_j^n - p) \cdot \tau_j^n = 0$ for all n and $j \in I_1$. Hence, we can assume that $\tau = \tau_j$ for all $j \in I_1$.

Now, let us take $j \in I_1$ and observe that from (4.7.6) and (4.7.7) we find that

$$\sum_{i=1, i\neq j}^{k} \frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} \cdot \tilde{x} = \sum_{i\in I_2} \frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} \cdot \tilde{x} \leq 0.$$

On the other hand, for $i \in I_2$, we get that from (4.7.5)

$$\frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} = \sum_{l=1, l\neq i, l\neq j}^{k} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2} = \sum_{l\in I_1, l\neq j} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2} + \sum_{l\in I_2, l\neq i} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2}$$

and hence,

$$\sum_{i\in I_2} \frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} = \sum_{i\in I_2} \sum_{l\in I_1, l\neq j} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2}.$$

Therefore, using (4.7.7) we have that

$$\sum_{i\in I_2} \frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} \cdot \tilde{x} = \sum_{i\in I_2} \sum_{l\in I_1, l\neq j} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2} \cdot \tilde{x} \geq 0.$$

Thus, we conclude that for all $j \in I_1$

$$\sum_{i=1, i\neq j}^{k} \frac{\tilde{x}_j - \tilde{x}_i}{|\tilde{x}_j - \tilde{x}_i|^2} \cdot \tilde{x} = 0.$$

Therefore, from (4.7.4) and $\tilde{x} \cdot \tau = 0$, it follows that

$$\sum_{l=1, l\neq i}^{k} \frac{\tilde{x}_i - \tilde{x}_l}{|\tilde{x}_i - \tilde{x}_l|^2} = 0, \qquad \text{for all } i = 1, \ldots, k.$$

Thus, we get a contradiction with Lemma 4.7.2 and Case 2 cannot hold.

In summary we reached now a contradiction with the assumptions of the Lemma. The proof is complete.

□

4.7.3 Proof of Theorem 4.1.3.

According to Lemma 4.5.1, given $m \geq 1$, we have a solution of Problem (4.1.2) if we adjust ξ so that it is a critical point of F_ε defined by (4.5.2). This is equivalent to finding a critical point of

$$\tilde{F}_\varepsilon(\xi) = \frac{1}{4\pi}\left[F_\varepsilon(\xi) + 16m\pi \log\varepsilon + 16\pi m - 8\pi m \log 8 + 32\pi^2 H^*(0)\right].$$

On the other hand, from Lemmas 4.5.2 and 4.6.1, we have that for $\xi \in \Omega^m$, such that $k(\xi_j) > 0$ and its components satisfy $|\xi_i - \xi_j| \geq \delta$,

$$\tilde{F}_\varepsilon(\xi) = \varphi_m(\xi) + \varepsilon^\gamma \Theta_\varepsilon(\xi), \qquad \text{with} \quad 0 < \gamma < 1 - \sigma$$

where Θ_ε and $\nabla_\xi \Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$, and $\alpha \neq 0$ and β are universal constants. By the assumption $\inf_\Omega k > 0$, we get that $-2 \log k$ is bounded from above. Since the Green's function G is bounded from below in $\Omega \times \Omega$, it holds φ_m is bounded from above and there is a global maximum, that is to say, a critical value $\mathcal{C}$ such that

$$\mathcal{C} = \max_{\Omega^m} \varphi_m.$$

Now, taking $\mathcal{D} = \{x \in \Omega^m \mid \varphi_m(x) > \mathcal{C}/2\}$ we have that there exists a $\delta > 0$ such that $|\xi_j - \xi_i| \geq \delta$ for any $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{D}$. Hence, $\alpha \tilde{F}_\varepsilon + \beta$ is uniformly bounded from above in $\mathcal{D}$ and there is a critical value

$$\mathcal{C}_\varepsilon = \max_{\mathcal{D}} [\alpha \tilde{F}_\varepsilon + \beta]$$

which is close to $\mathcal{C}$ in this region. From C^1-closeness of $\alpha \tilde{F} + \beta$ and φ_m in the region $\mathcal{D}$ it follows that, if $\xi_\varepsilon \in \mathcal{D}$ is a critical point at level $\mathcal{C}_\varepsilon$ then $\nabla \varphi_m(\xi_\varepsilon) \to 0$. The function $u_\varepsilon = U(\xi_\varepsilon) + \tilde{\phi}(\xi_\varepsilon)$ is therefore a solution as predicted by the theorem. $\square$

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