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RACIONALIZABILIDAD EN JUEGOS Y COORDINACIÓN DE ANTICIPACIONES

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RESUMEN

En este trabajo evaluamos la estabilidad eductiva de los equilibrios de una clase de modelos económicos compuestos de un continuo de agentes y de un estado agregado del mundo sobre el cual los agentes tienen una influencia infinitesimal.

Usando como cuadro general una clase de juegos no atómicos con un continuo de agentes, introducimos primero el concepto de racionalizabilidad. Cuando el pago de los jugadores depende de las estrategias de los rivales sólo a través del valor de la integral del perfil de estrategias, proponemos una definición del conjunto de estados (puntualmente) racionalizables y entregamos una caracterización de estos conjuntos, a través de la eliminación iterativa de puntos del conjunto de estados, para el caso en que el juego satisface hipótesis adecuadas de continuidad y medibilidad de la función de pagos.

Definimos entonces la racionalidad fuerte (o estabilidad eductiva) como la unicidad de la solución racionalizable del sistema económico y estudiamos la relación entre este concepto de estabilidad y la estabilidad iterativa en anticipaciones. La caracterización obtenida para la racionalizabilidad, nos permite explorar el enfoque local de la estabilidad de anticipaciones.

Demostramos que en presencia de complementariedad estratégica, la unicidad del equilibrio es equivalente a su estabilidad eductiva. La heterogeneidad de creencias no juega ningún rol en la coordinación de anticipaciones, pues la estabilidad eductiva resulta ser equivalente la estabilidad iterativa en anticipaciones.

Por otro lado, en presencia de sustitutabilidad estratégica, si bien la estabilidad eductiva es también equivalente la estabilidad iterativa en anticipaciones, la unicidad del equilibrio no asegura su estabilidad global.

Estudiamos también un duopolio donde las firmas deciden en una primera etapa su capacidad de producción y compiten secuencialmente en precios en una segunda etapa, en la cual el rol de líder es determinado aleatoriamente. Obtenemos en este contexto que el resultado de equilibrio de Cournot puede ser sostenido como equilibrio perfecto en sub-juegos en estrategias puras del juego completo. Obtenemos también que existe la posibilidad de encontrar equilibrios diferentes al de Cournot, como consecuencia del orden aleatorio del juego y de lo atractivo que resulta el rol de seguidor en el sub-juego en precios. Finalmente, damos una condición suficiente para la existencia de tales equilibrios.

ABSTRACT

In this work we evaluate the eductive stability of equilibria in a class of models that feature a continuum of agents and an aggregate state of the world over which agents have an infinitesimal influence.

Set in the framework of a class of games with a continuum of players, we first introduce the concept of rationalizability. When the payoff of a player depends on his opponents strategies only through the value of the integral of the strategy profile, we propose a definition for the set of (point-) rationalizable states; we provide a characterization of such sets, through the iterative elimination of points in the set of states, for the case where the game's payoffs function satisfies suitable continuity and measurability hypothesis.

We define strong rationality (or eductive stability) as the uniqueness of the rationalizable solution of the economic system and we study the relation between this stability concept and iterative expectational stability. The characterization obtained for rationalizability allows us to explore the local viewpoint of expectational stability.

We prove that in the presence of strategic complementarities, uniqueness of equilibrium is equivalent to its stability. Heterogeneity of beliefs plays no role in expectational coordination, since eductive stability is equivalent to iterative expectational stability.

On the other hand in the presence of strategic substitutabilities, although eductive stability is as well equivalent to iterative expectational stability, uniqueness of equilibrium does not assure its global stability.

We study in the last chapter a duopolistic model in which, first, firms engage simultaneously in capacity and compete sequentially in prices in a second stage, in which leadership is determined randomly. We obtain in this context that the Cournot outcome can be sustained as a pure strategy sub-game perfect Nash equilibrium of the whole game. We obtain as well that it is possible to find equilibria in which firms produce strictly more than the Cournot outcome, as a consequence of the random timing of the game and the attractiveness of being follower in the price sub-game. We give a sufficient condition to have existence of such equilibria.

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Though I know I'll never lose affection
For people and things that went before,
I know I'll often stop and think about them
In my life I love you more.

de *In my life*

Try to see it my way,
Only time will tell if I am right or I am wrong,

de *We can work it out*

The Beatles

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CHAPTER 1

Introduction

In recent years, significant effort is being directed to give formal justifications for the Rational Expectations Hypothesis in economics. In this line of research the "eductive" and "learning" viewpoints of rational expectations put emphasis into the idea that Rational Expectations Equilibria should be explained rather than assumed. Our interest is on the eductive explanations, which rely on the analysis of mental processes of the economic system's participating agents that seek to forecast, not directly the outcome itself of the system, but the forecasts of forecasts of the other agents, in order to anticipate such an outcome. This is, even in the case in which there is no uncertainty about the situation in which economic agents are immersed (what we would call structural uncertainty) there is always *strategic uncertainty*, agents have to rely on forecasts to justify their actions. One important attempt to set a clear set of assumptions that justify this concept is that of *Strongly Rational Expectations Equilibrium* proposed by Guesnerie (1992, 2002). Guesnerie transposes in economic contexts the ideas that in game-theoretical frameworks are behind the concepts of rationalizability, in order to ask whether rational economic agents may "educe" a so-called Rational Expectations Equilibrium. These ideas rely on two basic hypothesis: individual rationality and common knowledge. The emphasis is put on the formation of beliefs through a process that rules out unreasonable outcomes following these two hypothesis.

Bernheim (1984) and Pearce (1984) introduced the concept of Rationalizable Strategies, in the context of games with a finite number of players, as the adequate solution concept when players are modeled as rational agents that take decisions independently and in ignorance of the strategies adopted by the other players [1]. These agents must rely in forecasts and an internally consistent system of beliefs that justify such forecasts. Rationality of players

[1] See as well the papers by Tan and da Costa Werlang (1988) and Basu and Weibull (1991) for characterizations and justifications of Rationalizable Solutions

means that players maximize expected utility subject to some prior regarding the choice of their opponents, this prior must not contradict any information that they might have and players know that their opponents maximize expected utility and rely on such priors. We may say that players are rational and that rationality is common knowledge, they optimize and it is in their interest to use the correct forecast.

In Bernheim (1984), Rationalizability is introduced through the definition of a *consistent system of beliefs* in the context of general normal form games, while Pearce (1984) focuses in an iterative procedure of elimination of non-optimal strategies in the special case of games where the sets of actions are finite. The two papers can be merged using Proposition 3.2 in Bernheim (1984) (see page 8 of this report), which states that for games where the strategy sets are compact subsets of $\mathbb{R}^n$ and the payoff functions are continuous, then the set of Rationalizable Strategies is characterized by two equivalent definitions:

1. It is the result of the iterative elimination of non-best-reply strategies[2].
2. It is the largest set of strategy profiles that satisfies being a fixed point of the process of elimination of non-best-replies.

Under these premises, the formation of beliefs and the plausibility of equilibrium outcome has been analyzed in a series of works (Desgranges and Heinemann 2005, Guesnerie 2005, Desgranges and Gauthier 2004, Guesnerie 2002, Guesnerie 1992, among others) by setting an economic situation into a game-theoretical framework in order to use the concept of Rationalizability as a stability test for equilibria, regarding forecasts. In an economic system, following Guesnerie (1992), an equilibrium is said to be *Strongly Rational* or *Eductively Stable* if it is the unique outcome associated to the set of *Rationalizable Solutions* of the game form of such a system. From the expectations formation point of view, the economic agents should conclude that the only possible outcome of the system must be it's unique equilibrium, thus justifying the rational expectations hypothesis. It is in these good cases, as stated by Guesnerie, that "the rational expectations forecast is the necessary outcome of agents' mental activities which have clear and appealing grounds". We may then say that agents will *coordinate* in the equilibrium.

Eductive Stability, however, has been studied mainly in models that feature a continuum of agents, many of which have macroeconomic inspirations, to model situations in which agents are "small" with respect to the economy, in the sense that they have no individual influence over the economic system in which they are immersed. Examples of these models can be found in the articles mentioned above as well as in Evans and Guesnerie (1993), where the authors study Eductive Stability in a general linear model of Rational Expectations or

[2] Strategies that are never the result of individual payoff maximization.

Evans and Guesnerie (2003, 2005) for dynamics in macroeconomics. See as well the book by Chamley (2004), whose second part treats Coordination based on eductive stability. These and other works that feature a continuum of agents, including the seminal work by Guesnerie (1992), feature intuitive and/or context-specific, for the considered model, definitions of Rationalizability. It is then of primary interest to present a model of game that first, captures the features of these economic models and second, allows us to provide a general theoretical framework in which we can study Rationalizability as a natural extension of the finite player games described above.

In the first part of this presentation, Chapter 2, we aim to link the established game-theoretical concepts to its' economic applications in macroeconomics and economic models. We present a general framework of a game with a continuum of players over which the ideas of Bernheim and Pearce can be formally described and characterized. We define there the concepts of Rationalizable and Point-Rationalizable States, and Point-Rationalizable Strategies. The definition of (*standard*) Rationalizable Strategies for games with a continuum of players is nevertheless a more delicate task. We give a formal definition and characterization for this last concept in the particular case where the set of actions is finite. This framework allows then to work with the concept of Rationalizability in a very general setting. It is as well a framework that encompasses a large literature of specific economic and macroeconomic models including the ones already mentioned (see also Townsend (1978)). With this we aim as well to put together together, explicitly, lines of economic research that seemed to be unrelated in the literature until know, as are macroeconomics and games with a continuum of agents in a general setting. Then, following Guesnerie (1992) and using Rationalizability as a test for stability of equilibria, we can formally define and study Eductive Stability, in a general framework.

In Chapter 3 we make use of the framework, definitions and characterization of Rationalizable Solutions presented in Chapter 2, to address stability of equilibria (local and global). Part of our interest is the comparison between the already mentioned stability concept of Eductive Stability and Expectational Stability (Lucas 1978; DeCanio 1979). Expectational stability is related to the iterations of a best response mapping and can therefore be interpreted as a process that tends to seek outcomes where agents have homogeneous expectations as opposed to Strong Rationality in which we allow agents to have heterogeneous expectations. The characterizations of rationalizability obtained in Chapter 2 allow us to assess local eductive stability of equilibria as the result of the eductive process described in that Chapter, started from a neighborhood of an equilibrium. The comparison between these two stability concepts is stressed then as well from this local viewpoint in Chapter 3.

Further on we will explore the implications over stability of equilibria of the presence of a specific structure in the underlying game form of the economic model. We endow the game-theoretical model of Chapter 2 with a lattice structure in order to introduce strategic complementarity or substitutability and study how the economic setting inherits this ordered structure. We show that it is possible to go from a strategic point of view to a model in which there is an aggregate value that summarizes all the information that agents need in order to take a decision. This is, the ordered structure of a strategy profile set (a set of functions) is passed on to the set of aggregate values obtained from such profiles.

Once these tools are established we argue that the presence of Strategic Complementarities helps expectational coordination. The results in the literature of strategic complementarities in games with a finite number of players (Milgrom and Roberts 1990; Vives 1990) are extended, as expected, for the case of a continuum of players. Indeed, we will see in Chapter 3 that under such a setting, uniqueness of equilibrium is a sufficient condition for it to be Eductively Stable. We will show as well that this equilibrium is Eductively Stable if and only if it is Iterative Expectationaly Stable, simplifying the stability test for the former concept. On the other hand, although the presence of Strategic Substitutabilties gives some more structure to the stability tests, we can not say as much of stability of equilibria in this second case. Uniqueness of equilibrium does not give its' stability, nor eductvie nor iteratively expectational, but we show that the study of the second iterate of a best response type mapping may give some light. In this case, if the second iterate of the mapping has a unique fixed point, then we can say that the there is a unique rationalizable solution and consequently it will be a strongly rational equilibrium. We also have that iterative expectational stability and eductive stability are equivalent (we have one if and only if we have the other) and so heterogeneity of expectations is not relevant under the strategic substitutabilities setting.

Chapter 4 constitutes a short essay on duopolistic competition. Inspired by the work of Kreps and Scheinkman (1983), an exercise is proposed in order to reestablish the existence of pure strategy equilibrium in a price competition capacity constrained duopoly. The main issue in the original paper of Kreps and Scheinkman was to clarify the difference between *Cournot* and *Bertrand* competition, understood as competition on quantities opposed to competition on prices. The authors argue that the significant differences between these two approaches is not only the strategy space of the game (quantities vs. prices), but that timing of decision was relevant. To illustrate this, they present a duopoly in which price competition comes after the decision on production [3], thus reversing the order of decision from Bertrand competition where production is decided after prices are revealed, concluding that the whole game had as outcome the Cournot quantities and prices, implying that both: competition

[3] Although in the literature that has followed this article, the emphasis has been made in the interpretation that says that in the quantity setting game the competition is in *capacity* and not *production*, we use the terminology of the authors (production). Even more, using the interpretation of capacity setting in their model would imply that: 1. there is no cost of production in the second stage when the decision on quantities is made first, and 2. the interpretation of *quantity* changes from one game to the other, production in the Bertrand game, capacity in the reversed game. We leave the interpretation to the reader.

on prices and production following the realization of the demand, are required for perfectly competitive outcomes as in the Bertrand approach.

However, for a non negligible set of pairs of productions quantities, the capacity constrained price subgame had no pure strategy equilibrium. This inspires the exercise presented in this Chapter. We seek for pure strategy outcomes since they are economically more pertinent. The second stage subgame is modified and price competition is turned to be sequential in order to restore pure strategies at equilibrium and hopefully in the whole game-tree. Since in the related literature there has been as well some emphasis on the competitive outcome as the result of capacity constrained price competition, this issue is also partially assessed; we give a condition that allows to find equilibria that deliver quantities strictly greater than those that emerge at the Cournot equilibrium (see Section 1 and Subsection 4.2 of Chapter 4). In previous works, non Cournot outcomes are found by departing from the setting of Kreps and Scheinkman (1983). In the present work there are no changes in the assumptions of demand rationing, nor on properties of the costs functions.

Structure of the Thesis

The report consist of three chapters, each one of which contains an article that has been submitted to international journals and/or conferences. The results of Chapter 2 are available as a working paper at the PARIS SCHOOL OF ECONOMICS, PARIS, FRANCE and as a technical report at the CENTER OF MATHEMATICAL MODELING, SANTIAGO, CHILE; they have been presented in the THIRD WORLD CONGRESS OF THE GAME THEORY SOCIETY and the 2008 EUROPEAN MEETING OF THE ECONOMETRIC SOCIETY and at several seminars. A preliminary version of an article with the results of Chapter 3 has been presented in the IESE CONFERENCE ON COMPLEMENTARITIES AND INFORMATION and at several seminars. An abridged version of the results of Chapter 4 are available as a working paper at the PARIS SCHOOL OF ECONOMICS, PARIS, FRANCE and as a technical report at the CENTER OF MATHEMATICAL MODELING, SANTIAGO, CHILE; they have been presented at the 2007 CONFERENCE OF THE SOCIETY FOR THE ADVANCEMENT OF ECONOMIC THEORY. Chapters 2 and 3 include the full text of each publication, preceded by an introductory section. Al chapters include a reference to the most recent version of each article. We close the presentation with conclusions and the bibliography of the thesis.

CHAPTER 2

Rationalizability in Games with a continuum of players

Introductory Notes

In this chapter we present a framework in which it is possible to define the concept of rationalizability in the context of economic models that feature a continuum of agents and to obtain characterizations of the sets of rationalizable solutions. We introduce a suitable class of non-atomic games and explore this concept. Rationalizability has been defined in finite player games by Bernheim (1984) and Pearce (1984). For the sake of completeness, we summarize below the presentation in Bernheim.

A *finite player game* is a triplet $\left\langle J,(S_i)_{i\in J},(\pi_i)_{i\in J}\right\rangle$ where J is a finite set of players, for each $i \in J$, $S_i \subseteq \mathbb{R}^n$ are sets of strategies or actions available for the players and the functions $\pi_i : \prod_{i\in J} S_i \to \mathbb{R}$ give the payoff for each player and each profile of strategies.

The key concept of a game is that of Nash Equilibrium. A strategy profile $s^* \in \prod_{i\in J} S_i$ is a Nash Equilibrium of a game if for each player, his strategy s_i^* is a maximizer of the payoff function, given the strategies of his opponents:

$$\forall\, i \in J, \quad \pi_i(s^*) \geq \pi_i\big(y, s_{-i}^*\big) \quad \forall y \in S_i$$

To define Rationalizable Strategies, Bernheim formalizes the idea of *system of beliefs* and defines a *consistent system of beliefs*. A system of beliefs for a player $i \in J$ represents the possible forecasts of the player concerning the forecasts over forecasts of his opponents, concerning what any player would do. These forecasts take the form of borel measurable subsets of the players' strategy sets. If a system of beliefs gives only singletons, then he calls it a *system of point beliefs*. A consistent system of beliefs simply emphasizes the idea that players should consider in their forecast that the opponents are rational and so are optimizing with respect to some forecasts of their own. Rationality and common knowledge of rationality then imply that a system of beliefs must satisfy a consistent condition stressed as follows:

> If player i thinks it is possible that player i_1 thinks it is possible that,. . . player i_{n-1} thinks it is possible that player i_n might take an action s_{i_n}, then s_{i_n} must be a best response to some subjective distribution over i_n's opponents' strategies, where anything receiving nonzero probability in this distribution must be something which i thinks it is possible that i_1 thinks it is possible that,. . . i_{n-1} thinks it is possible that i_n thinks his opponents might possibly do.[1]

With these tools, Bernheim defines Rationalizability as follows.

Definition 2.1 (Bernheim (1984))**.** s_i is a *Rationalizable Strategy* for player i iff there exists some consistent system of beliefs for player i and some probability measure $\mu_{-i} \in \prod_{j \neq i} \mathcal{P}(S_j)$, such that $s_i \in \mathbb{B}\text{r}_i(\mu_{-i})$, and μ_{-i} is a subjective probability distribution that gives zero probability to actions of the opponents of i that are ruled out by this system of beliefs. In the particular case where the system is of point beliefs, we say that s_i is a point rationalizable strategy.

If the players of a game take actions independently and in ignorance of the actions taken by their opponents, it is pertinent to consider the mapping $\mathbb{B}\text{r}_i : \prod_{j \neq i} \mathcal{P}(S_j) \rightrightarrows S_i$ that assigns to each product probability measure μ_{-i} over the product borel field, the set of expected utility maximizers:

$$\mathbb{B}\text{r}_i(\mu_{-i}) := \text{argmax}_{y \in S_i} \mathbb{E}_{\mu_{-i}} \left[\pi_i(y, s_{-i}) \right]$$

For $\overline{\mathcal{B}} = \prod_{j \in J} \mathcal{B}(S_j)$, we define a mapping $R : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ that eliminates strategies that are not expected utility maximizers with respect to (subjective) probability measures whose supports are contained in certain corresponding subsets $\prod_{j \neq i} \text{Proj}_j(H) \subseteq S_{-i}$ for each player $i \in J$:

$$R(H) := \prod_{i \in J} \bigcup_{\mu_{-i} \in \prod_{j \neq i} \mathcal{P}\left(\text{Proj}_j(H)\right)} \mathbb{B}\text{r}_i(\mu_{-i})$$

The set $R(H)$ contains all and only strategy profiles that can be obtained from the independent actions of players that react optimally to some prior with support on the projection of the set H over the product set of strategy profiles of the opponents.

[1] Taken form Bernheim (1984) p. 1014.

Call $\overline{S} \equiv \prod_{j \in J} S_j$. The following is the characterization, using the mapping R, of Rationalizable Strategies for a game as the one described above.

Proposition 2.2 (Bernheim, 1984)**.** *In a game with continuous payoff functions and compact strategy sets, the set of Rationalizable Strategies, $\mathbb{R}_{\overline{S}}$, is characterized as follows:*

$$\mathbb{R}_{\overline{S}} \textit{ is the maximal subset } H \subseteq \overline{S} \textit{ such that } R(H) = H$$

$$\mathbb{R}_{\overline{S}} \equiv \bigcap_{t \geq 0} R^t\left(\overline{S}\right)$$

The proposition states that the set of Rationalizable Strategies is the result of the iterative elimination of strategies that are not best-replies to forecasts considering all of the remaining strategy profiles. There is another definition on the paper of Bernheim, that of Point-Rationalizable Strategies, that considers only forecasts as points in the strategy sets instead of probability assessments. The characterizations are analogous to the ones presented in Proposition 2.2 but considering only strategy profiles instead of probability measures at the best reply correspondence level.

Care is needed when we pass from this setting to a setting in which the player set is no longer finite, but an interval of $\mathbb{R}$, this is the principal issue to be treated in this Chapter.

Full Paper. Current version: Jara-Moroni (2008b)

Rationalizability in Games with a continuum of players

Abstract

The concept of Rationalizability has been used in the last fifteen years to study stability of equilibria on models with continuum of agents such as standard competitive markets, macroeconomic dynamics and currency attacks. However, Rationalizability has been formally defined in a general setting only for games with a finite number of players and there is no general definition for Rationalizability in the case of games with continuum of players. In this work, we propose a definition for Point-Rationalizable Strategies in the context of *Non-Atomic Noncooperative Games with a Continuum of Players*. In a special class of these games where the payoff of a player depends only on his own strategy and an aggregate value that represents the *state* of the game, state that is obtained from the actions of all the players, we define the sets of *Point-Rationalizable States* and *Rationalizable States*. These last sets are characterized and some of their properties are explored. We study as well *standard* Rationalizability in a subclass of these games. We present an exploratory framework that encompasses the previously mentioned models, over which we can link the established theory and its' macroeconomic applications on stability properties of equilibria.

Keywords: Rationalizable Strategies, Non-atomic Games, Expectational Coordination, Rational Expectations, Eductive Stability, Strong Rationality.

JEL Classification: D84, C72, C62.

1 Introduction

The concept of *Strong Rationality* was first introduced by Guesnerie (1992) in a model of a standard market with a continuum of producers. An equilibrium of the market is there said to be *Strongly Rational*, or *Eductively Stable*, if it is the only *Rationalizable Solution* of the economic system. Inspired in the work of Muth (1961), the purpose of such an exercise was to give a rationale for the Rational Expectations Hypothesis. Strong rationality has been studied as well in macroeconomic models in terms of stability of equilibria. See for instance Evans and Guesnerie (1993), where they study Eductive Stability in a general linear model of Rational Expectations or Evans and Guesnerie (2003, 2005) for dynamics in macroeconomics. More examples of applications of Strong Rationality can be found in the recent book by Chamley (2004) where he presents models of Stag Hunts in the context of coordination in games with strategic complementarities.

The *Rationalizable Solution* of the economic system assessed by Guesnerie in the definition of *Strong Rationality*, refers to the concept of *Rationalizable Strategies* as defined by Pearce (1984) in the context of games with a finite number of players and finite sets of strategies. *Rationalizable Strategies* were formally introduced by Bernheim (1984) and Pearce (1984) as the "adequate" solution concept under the premises that players are rational utility maximizers that take decisions independently and that rationality is common knowledge. Adequate because Rationalizable Strategy Profiles are outcomes of a game that cannot be discarded based only on agents' rationality and common knowledge. The work of Pearce focused mainly in refinements of equilibria of extensive form finite games, while Bernheim gave a definition and characterization in the context of general normal form games, along with comparison between the set of Nash Equilibria and the set of Rationalizable Strategy Profiles. In both papers and later treatments, however, the definition and characterization of rationalizable "solutions" were developed for games with a finite number of players.

On the other hand, each one of the works that are mentioned in the first paragraph of this introduction including the seminal work by Guesnerie (1992), feature intuitive and/or context-specific definitions of the concept of *Rationalizable Solution*, adapting the original definitions and characterizations of Rationalizable Strategies, based on the intuitions behind them, to models with a continuum of agents. It is this gap between the established theory and its' economic applications that motivate this work. Since there is no established definition for Rationalizable Strategies, or Rationalizability for what matters, in a general framework with a continuum of agents, in this paper then we link the game-theoretical concept of Rationalizability to its' applications in macroeconomics and economic models, proposing a general definition in the context of games with a continuum of players.

To motivate this presentation let us describe the model and illustrate how the Rationalizability concept is presented in Guesnerie's (1992) work.

Example 2.3. Consider that we have a group of farmers, represented by the $[0,1] \equiv I$ interval, that participate in a market in which production decisions are taken one period before production is sold. Each farmer $i \in I$ has a cost function $c_i : \mathbb{R}_+ \to \mathbb{R}$. The price p at which the good is sold is obtained from the (given) inverse demand function $P : \mathbb{R}_+ \to [0, p_{\max}]$ evaluated in total aggregate production $p = P\big(\int q(i)\,\mathrm{di}\big)$ where $q(i)$ is farmer i's production. Since an individual change in production does not change the value of the price, the product is sold at price-taking behavior, so each farmer $i \in I$ maximizes his payoff function $u(i, \cdot\,, \cdot\,) : \mathbb{R}_+ \times [0, p_{\max}] \to \mathbb{R}$ defined by $u(i, q(i)\,, p) \equiv pq(i) - c_i(q(i))$. An equilibrium of this system is a price p^* such that $p^* = P\big(\int q^*(i)\,\mathrm{di}\big)$ and $u(i, q^*(i)\,, p^*) \geq u(i, q, p^*)\ \forall\ q \in \mathbb{R},\ \forall\ i \in I$.

At the moment of taking the production decision, farmers do not actually know the value of the price at which their production will be sold. Consequently they have to rely on forecasts of the price or of the production decision of the other farmers. The concept in scrutiny in our work is related to *how this (these) forecast(s) is (are) generated.*

Forecasts of farmers should be rational in the sense that no *unreasonable* price should be given positive probability of being achieved. It is in this setting that Guesnerie introduces the concept of *strong rationality* or *eductive stability* [2] as the uniqueness of *rationalizable prices* which are obtained from the elimination of the unreasonable forecasts of possible outcomes. To obtain these rationalizable prices, Guesnerie describes, in what he calls the *eductive procedure*, how the unreasonable prices can be eliminated using an iterative process of elimination of non-best-response strategies.

Now let us illustrate how the eductive process works in this setting. From the farmers problem we can obtain for each farmer his supply function $s(i, \cdot\,) : [0, p_{max}] \to \mathbb{R}_+$. The structure of the payoff function implies that for a given forecast μ of a farmer i over the value of the price, his optimal production is obtained evaluating his supply function in the expectation under μ of the price, $\mathbb{E}_\mu\,[p] : s(i, \mathbb{E}_\mu\,[p])$. Farmers know that a price higher than p_{max} gives no demand and so prices higher than those are unreasonable. Since all farmers can obtain this conclusion, all farmers know that the other farmers should not have forecasts that give positive weight to prices that are greater than p_{max}. The expectation of each of the farmers' forecasts then cannot be greater than p_{max} and so under necessary measurability hypothesis we can claim that aggregate supply can not be greater than $S(p_{max}) = \int_0^1 s(i, p_{max})\,\mathrm{di}$. Since all farmers know that aggregate supply can not be greater than $S(p_{max})$, they know

[2] An equilibrium of an economic system is said to be *strongly rational* or *eductively stable* if it is the only Rationalizable outcome of the system. We will refer equivalently to outcomes as begin strongly rational or eductively stable.

then that the price, obtained through the inverse demand function, can not be smaller than $p^1_{min} = P(S(p_{max}))$. All farmers know then that forecasts are constrained by the interval $[p^1_{min}, p_{max}]$. They have discarded all the prices above p_{max} and below p^1_{min}. This same reasoning can be made now starting from this new interval.

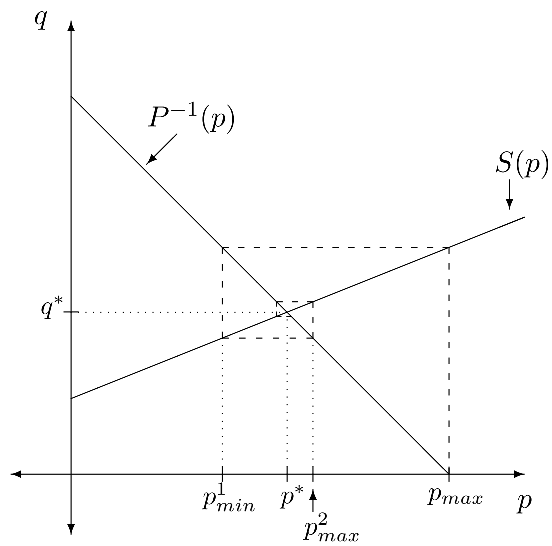


Figure 2.1: The *eductive process*

In Figure 2.1 we can see the aggregate supply function depicted along with the demand function. We have seen that to eliminate unreasonable prices in this model we only need these two functions. The process, as described in the Figure, continues until the farmers eliminate all the prices except the unique equilibrium price p^*. We say then that this price is *(globally) eductively stable*. Note that the eductive process could "fail", in the sense that it could give more than only the equilibrium point. This could happen for instance if $S(p_{max}) \geq P^{-1}(0)$. In this situation the *rationalizable set* would be the whole interval $[0, p_{max}]$, since farmers would not be able to eliminate prices belonging to this interval.

For more details on the example the reader is referred to the paper of Guesnerie (1992). The iterative process of elimination of unreasonable prices is inspired by the work of Pearce. However, Pearce's definition of Rationalizability is stressed in the particular framework of a game with a finite number of players where the sets of actions are finite. The approach followed by Pearce assesses rationality and common knowledge of rationality, by considering an iterative process of elimination of non-best-responses (or non-expected-utility-maximizers). This process is overtaken on the set of mixed strategies of the players. Starting with the whole set of mixed strategy profiles, players eliminate at each step of the process the mixed strategies that are not best response to some product probability measure over the set of remaining profiles of mixed strategies of the opponents. This process ends in a finite number of iterations and delivers a set that Pearce defines to be the set of Rationalizable Strategies.

Still, this argument may not be valid in more general contexts. Bernheim's approach to Rationalizable Strategies relies the formalization of the ideas of *system of beliefs* and *consistent system of beliefs*. A system of beliefs for a player represents the possible forecasts of the player concerning the forecasts over forecasts of his opponents, concerning what any player would do. These forecasts take the form of borel measurable subsets of the players' strategy sets. If a system of beliefs gives only singletons, then it is called a *system of point beliefs*. Rationality and common knowledge of rationality then imply that a system of beliefs must satisfy a consistency condition. A consistent system of beliefs simply emphasizes the idea that players should consider in their forecast that the opponents are rational and so are optimizing with respect to some forecasts of their own.

According to Bernheim, a strategy s_i is a *Rationalizable Strategy* for player i if there exists some consistent system of beliefs for this player and some subjective product probability measure over the set of strategy profiles of the opponents, that gives zero probability to actions of the opponents of i that are ruled out by this system of beliefs and such that the strategy s_i maximizes expected payoff with respect to this probability measure. In the particular case where the system is of point beliefs, Bernheim calls s_i a *Point-Rationalizable Strategy*.

In this context, the Rationalizable Set as defined by Bernheim may fail to be the result of the iterated elimination of non-best-responses as described by Pearce. Bernheim proves that in a game with a finite number of players, compact strategy sets and continuous payoff functions, the set of Rationalizable Strategy Profiles is in fact the result of the iterative elimination of strategies that are not best-replies to forecasts considering all of the remaining strategy profiles [3]. This result proves as well, as Bernheim and Pearce claim, that their definitions are indeed equivalent [4]. The characterizations of rationalizability presented by Bernheim are actually related to two properties that rationalizable sets should be asked to fulfill. This is, the rationalizable set must (i) be a subset (hopefully equal) of the set that results from the iterated elimination process, but above all it should (ii) be a fixed point of this process, or, at least, it should be contained in its image through the process [5].

Recent papers address the issue of the set obtained as the limit of processes of iterated elimination of non-best-response-strategies, not being a fixed point of the iterated process in

[3]See Propositions 3.1 and 3.2 in Bernheim (1984). Proposition 3.2 states that the set of Rationalizable Strategies is as well the largest set that satisfies being a fixed point of the process of elimination of strategies. Proposition 3.1 gives an analogous characterization for Point-Rationalizable Strategies considering in the definition of the process of elimination of non-best-response-strategies only the Dirac measures over the remaining sets, instead of all the measures.

[4]Proposition 3.2 in Bernheim (1984). Then, what Pearce defines as the rationalizable set, is named by Bernheim the set of *rationalizable mixed strategies*.

[5]This pertains to some type of *best response property* that the rationalizable set must satisfy.

general normal form games with finite number of players; and go beyond to explore more complex iterated processes of elimination of strategies (see for instance Dufwenberg and Stegeman (2002), Apt (2007), Chen, Long, and Luo (2007) [6]). The problem rises then only if the assumptions on utility functions and strategy sets are relaxed (namely the cases for unbounded strategy sets and/or discontinuous utility functions).

The question surfaces on how should this process be defined in the context of a continuum of players? When can we claim that the result of the iterative elimination process gives a set that we may call *Rationalizable*?

Example 2.3 gives clear insight on how to face these questions. The particular structure of this example allows us to look at outcomes on the set of prices (or aggregate production), instead of the set of strategy profiles (production profiles), as is done in Pearce or Bernheim. This allows for a special characterization of the *rationalizable set* as the limit of an iterative process of elimination of unreasonable prices, and not necessarily production profiles. The eductive procedure consists in eliminating the prices that do not emerge as a consequence of farmers taking productions decisions that are best responses to the remaining strategy profiles, or equivalently, remaining values of aggregate production or prices [7].

There are three main issues to take into account when we pass from the finite to the continuous player sets. The first one is how to address forecasts. In the finite player case it is direct to use product measures as forecasts and take expectation over payoff functions to make decisions. This is not evident in the continuum case. The second issue stems from the first one and is related to the space in which one should seek the *rationalizable set*. The set of strategy profiles may not be appealing in contexts where the set of players is a continuum. The third one relates to give conditions to have a well defined process of iterated elimination of *outcomes*. As we have already said, Guesnerie's approach is Pearce's approach in a situation with a continuum of players. This approach is a reasonable and natural way to overtake the rationalizability argument. Nevertheless, and in the light of Bernheim's Proposition 3.2, we see that care is needed to claim that the limit of the process of iterated elimination is in fact a set that we could call of *rationalizable outcomes*. Moreover, the process itself could well be undefined without proper assumptions. Of course, as we prove below in Theorems 2.10 and 2.22, this is not an issue in Guesnerie's setting.

[6] Dufwenberg and Stegeman (2002) and Chen, Long, and Luo (2007) put emphasis in Reny's (1999) better-reply secure games.

[7] A second characteristic of this setting is that the eductive procedure can be done by simply eliminating prices that are beyond the upper and lower bounds that are obtained in each iteration. However, this comes from the monotonicity properties of the aggregation operator (the integral) and the supply function of the farmers. It is not always the case that the eductive process works this way. This second feature of example 2.3 is more related to the ordered structure of the games studied in Milgrom and Roberts (1990) and so is left for a further treatment (Guesnerie and Jara-Moroni 2007).

We make the emphasis then in two features of this example [8]: (i) there is a continuum of producers that interact and (ii) payoffs of producers depend on an aggregate value that cannot be affected unilaterally by any agent, this aggregate variable has all the relevant information that producers need to take a decision. We are interested in defining *Rationalizability* in a general setting considering these features. We will adapt the concept of Rationalizable Strategy from the finite game-theoretical world to the context of a class of non-atomic non-cooperative games with a continuum of players. One part of the task then is to find a suitable model of game with a continuum of players, in which one could be able to define and characterize *Rationalizable Outcomes*.

In what follows, we will present a framework of a general class of non-cooperative games with a continuum of players, in which we explore the ideas of rationalizability. We will begin by loosely defining the concept of *Point-Rationalizable Strategies* in a general setting. Then we will turn to the special case where payoffs depend on players' own actions and the average of the actions taken by all the players. We will call this average the *state* of the game, and we will define the sets of *Point-Rationalizable States* and *Rationalizable States*. This last approach is not evident nor a generalization of finite player games, since in "small" games, and as opposed to what we do here, players can actually affect directly and unilaterally the payoff of other players. Our main results are Theorems 2.10 and 2.22 where we characterize these sets as the results of iterated elimination of states. More precisely, we extend Propositions 3.1 and 3.2 in Bernheim (1984) to (Point-)Rationalizable States in the context of games with compact strategy sets, continuous utility functions and a continuum of players. The need for these two Theorems comes from the proof of Proposition 3.2, where a convergent subsequence extraction argument is used, argument that is no longer valid in the context of a continuum of players. A different limit concept is needed to conclude. Moreover, certain measurability properties must be required to have a well defined process of iterated elimination. Consequently, we will get a setting with a continuum of players in which it is possible to study rationalizability and general properties of (locally) strongly rational equilibria as in the economic applications.

The remainder of the paper is as follows: in section 2 we introduce *games with a continuum of players* and some notation; in section 3 we define Point-Rationalizable Strategies in the context of these games and, for the particular class of games with an aggregate state, we define as well Point-Rationalizable States. The main result of this section is the study of the set of Point-Rationalizable States, for which we give a characterization and show its' convexity and compactness. In 3.4 we introduce the concept of *Strongly Point Rational Equilibrium* and explore the relation between Point-Rationalizable Strategies and States. We argue in

[8] Similar features and structure can be found as well in Evans and Guesnerie (1993), in Chapter 11 of Chamley (2004), in Stag Hunt models (see also Morris and Shin (1998)), Chatterji and Ghosal (2004) and in Guesnerie (2005), among others.

favor of the use of this last approach, states instead of strategies, in the context of these games. In section 4 we define and characterize Rationalizable States. Before concluding, we explore the concept of Rationalizability in terms of strategy profiles, in the particular setting in which (pure) strategies are chosen from finite sets and payoffs depend on the integral of the profile of mixed strategies (Schmeidler 1973). We close the presentation with comments and conclusions in section 5.

2 Games With a Continuum of Players

Since the concept of Strong Rationality introduced by Guesnerie in his paper, relies on a concept that comes from the game-theory literature, our interest is to look at the setting described in the example as a strategic interaction situation. This idea of strategic interaction is then: payoffs of agents depend on the actions of other agents. This interaction would occur through the aggregation of the production and the evaluation in the price function. The payoff of a single farmer depends on the production of all the farmers through P and $u(i, \cdot, \cdot)$, as follows: each farmer $i \in I$ chooses a production $q(i)$ in the positive interval. The price is determined by evaluating the price function in the value of total production, that is on the integral of the production profile. Each agent $i \in I$ obtains payoff $u(i, q(i), p)$. The second feature we need in the mathematical formulation, is that it allows to model the inability of single agents to influence the state of the system, in this case the price, or for what matters, total production, which calls for a mathematical formulation where the weight of single agent is small compared to the whole set or the remaining agents. These two features are captured in the mathematical model presented below.

We consider then games with a continuum of players. Schmeidler (1973) introduced a concept of equilibrium and gave existence results in games where a strategy profile is an equivalence class of measurable functions from the set of players into a strategy set, and the payoff function of a player depends on his own strategy and the strategy profile played. A different approach was presented later by Mas-Colell (1984) [9] and more general frameworks can be found in Khan and Papageorgiou (1987) and Khan, Rath, and Sun (1997) as well. For a comprehensive review of games with many players see Khan and Sun (2002). We will focus mainly in Schmeidler's general setting and specially in games where payoffs depend on an "average" of the actions taken by all the players (Rath 1992).

In a Non-Atomic Game the set of players is a non-atomic measure space $(I, \mathcal{I}, \lambda)$ where

[9] In Mas-Colell (1984) what matters is not strategy profiles but a distribution on the product set of payoff functions and strategies.

I is the set of interacting agents $i \in I$ and λ is a non atomic measure on $\mathcal{I}$. This is, $\forall E \in \mathcal{I}$ such that $\lambda(E) > 0$, $\exists F \in \mathcal{I}$ such that $0 < \lambda(F) < \lambda(E)$. We will consider the set of players I as the unit interval in $\mathbb{R}$ and the non-atomic measure λ to be the Lebesgue measure.

Given a set $X \subseteq \mathbb{R}^n$ we will denote the set of equivalence classes of measurable functions from I to X as X^I. We identify then, for a general set X of available actions, X^I with the set of strategy profiles. So a strategy profile is a measurable function from I to X, the set of strategies. By doing this we are assuming that all players have the same strategy set. We will denote S the set of strategies and we will not make a difference a priori between pure or mixed strategies. However, since we assume that S is in $\mathbb{R}^n$ it is better to think of this set as a set of pure strategies. We will come back to this issue on Section 4.

For each player $i \in I$, we will denote by $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$ the general payoff functions of a game, that depend on the action of each player as an element of the set S and the profile of strategies as an element of the set S^I described as above. To specify how the functions $\pi(i, \cdot, \cdot)$ depend on these variables, we will use auxiliary functions that depend on the action taken by the player in his strategy set S and some vector taken from a set $X \subset \mathbb{R}^K$, that is obtained from the strategy profile $\mathbf{s}$. The functions $\pi(i, \cdot, \cdot)$ will be obtained then by an operation between these auxiliary functions and some other mathematical objects[10].

2.1 Payoff Functions that Depend on the Integral of the Strategy Profile

Our aim is to capture the relevant features of a wide variety of models that are similar to the one described in Example 2.3, in the Introduction. Consider then the class of models where there is a set $\mathcal{A} \subseteq \mathbb{R}^K$ and a variable $a \in \mathcal{A}$ that represent, respectively, the *set of states* and the *state* of an economic system. For each agent $i \in I$, the payoff function is now defined on the product of S and $\mathcal{A}$, $u(i, \cdot, \cdot) : S \times \mathcal{A} \to \mathbb{R}$ and depends on his own action $s(i) \in S$ and the state of the system $a \in \mathcal{A}$. Finally, we have an aggregation operator: $A : S^I \to \mathcal{A}$ that gives the state of the system $a = A(\mathbf{s})$ when agents take the action profile $\mathbf{s}$.

In the example, the state of the system could have been identified with aggregate production or the price, and the aggregation operator would have been the integral of the production profile or the evaluation of the price function on such a quantity (respectively).

Agents' impossibility of affecting unilaterally the state of the system is formalized by the

[10]See equations 2.1 and 2.17.

following property of A:

$$A(\mathbf{s}) = A(\mathbf{s}') \;\; \forall\, \mathbf{s},\, \mathbf{s}' \in \operatorname{dom} A \text{ such that } \lambda(\{l \in I \;:\; s(l) \neq s'(l)\}) = 0$$

That is, since A is defined on S^I, for all strategy profiles that are in the same equivalence class of S^I, the value of the mapping A is the same.

To capture this setting, let S be now a compact subset of $\mathbb{R}^n$. The aggregation operator is chosen for convenience to be the integral with respect to the Lebesgue measure:

$$A(\mathbf{s}) \equiv \int_I \mathbf{s}(i)\,\mathrm{di}$$

so that S^I, the set of measurable functions from I to S, is contained in $\operatorname{dom} A$, the set of integrable functions from I to $\mathbb{R}^n$, and the set $\mathcal{A}$ is $\mathcal{A} \equiv \operatorname{co}\{S\}$ [11].

The payoff functions $\pi(i,\,\cdot\,,\,\cdot\,)$ mentioned above in the description of a game are calculated by composing the functions $u(i,\,\cdot\,,\,\cdot\,)$ and A of the economic system, that is

$$\begin{aligned} \pi(i, s(i)\,, \mathbf{s}) &:= u(i, s(i)\,, A(\mathbf{s})) \\ &\equiv u\left(i, s(i)\,, \int_I s(i)\;\;\mathrm{di}\right). \end{aligned} \tag{2.1}$$

In this way we are in Rath's extension of Schmeidler's formulation of games with a continuum of players, where, in a particular class of these games, agents' utility functions depend on their own actions, that are elements of a general compact set, and an "average" of all agents' actions. The description of a game will be given then by a mapping that associates each player $i \in I$ with a real valued continuous function $u(i,\,\cdot\,,\,\cdot\,)$ defined on $S \times \mathcal{A}$.

We denote the set of real valued bounded continuous functions defined on a space X by $C_{\mathrm{b}}(X)$. Let $\mathcal{U}_{S\times\mathcal{A}} := C_{\mathrm{b}}(S \times \mathcal{A})$ denote the set of real valued continuous functions defined on $S \times \mathcal{A}$ endowed with the sup norm topology.

To denote *games with a continuum of players* that have an *aggregate state* as above, we will use the notation $\mathbf{u}$. Throughout the document when we refer to such games, we will be using the assumption that the function $\mathbf{u} : I \to \mathcal{U}_{S\times\mathcal{A}}$ that associates players with their

[11] The aggregation operator can as well be the integral of the strategy profile with respect to any measure that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\left(\int_I s(i)\;\;\mathrm{d}\bar{\lambda}(i)\right)$$

where $\bar{\lambda}$ is absolutely continuous with respect to the lebesgue measure and $G : \operatorname{co}\{S\} \to \mathcal{A}$ is a continuous function; the results in this work could well be extended to this setting. For instance Theorem 2.10 holds and if G is affine, Corollary 2.12 holds.

payoff functions is measurable (Rath 1992).

This is in opposition to when we refer to more general games related to the function π that to each player $i \in I$ associates a payoff function $\pi(i) : S \times S^I \to \mathbb{R}$ over which we make no general assumptions. We will note then equivalently $\mathbf{u}(i)$ and $u(i, \cdot, \cdot)$. Since the set S is compact, so is $\mathcal{A}$ and so the payoff functions $\mathbf{u}(i)$ are as well bounded. We will call *states* the elements of the set $\mathcal{A}$. Under this description of the game, the fact that payoffs depend on the strategy profiles is given by the rules of the game, and not the payoff function, i.e. the fact that the state of the game is calculated with the integral of the strategy profile.

A *Nash Equilibrium* of a game π is a strategy profile $\mathbf{s}^* \in S^I$ such that λ-almost-everywhere in I:

$$\pi(i, s^*(i), \mathbf{s}^*) \geq \pi(i, y, \mathbf{s}^*) \qquad \forall\, y \in S,$$

This is simply re-stated for a game $\mathbf{u}$ as a strategy profile $\mathbf{s}^* \in S^I$ such that λ-almost-everywhere in I:

$$u\left(i, s^*(i), \int_I \mathbf{s}^* \,\mathrm{di}\right) \geq u\left(i, y, \int_I \mathbf{s}^* \,\mathrm{di}\right) \qquad \forall\, y \in S,$$

In this framework Rath shows that for every game there exists a Nash Equilibrium.

Theorem 2.4 (Rath 1992)**.** *Every game* $\mathbf{u}$ *has a (pure strategy) Nash Equilibrium.*

We present a proof for this Theorem below. The proof in Rath's paper uses Kakutani's fixed point theorem on the mapping Γ that maps a state $a \in \mathcal{A}$ into all the possible states that rise as the consequence of agents taking best response actions to this state. This mapping goes from the convex and compact set $\mathcal{A} \subset \mathbb{R}^n$ into itself and is proved to have a closed graph with non-empty, convex values. The only step where one should be careful is on the proof for non-emptiness of $\Gamma(a)$ in which a measurable selection argument is needed. This is a consequence of the assumption on measurability of the mapping that defines the game. The proof presented herein makes use of Lemma 2.6 stated below. As Rath mentions in his paper, the assumptions on continuity and measurability of the payoff functions are both hidden in the definition of the function $\mathbf{u}$ that represents a game.

3 Point-Rationalizability

Recall that we are interested in situations where players act in ignorance of the actions taken by their opponents. Thus, they must rely on forecasts or subjective priors over the possible outcomes. We assume that agents are rational not only in the sense that they act by maximizing their payoff, but also considering that the subjective priors that they form do not contradict any information that they may have.

The two main assumptions on player's behavior that justify *Rationalizable Strategies* as a solution concept can be summed up to two basic principles: rationality of agents and common knowledge (structural and of rationality of agents) (Pearce 1984; Bernheim 1984; Tan and da Costa Werlang 1988). The implications of these assumptions can be exhausted, as is done in Pearce (1984) and Guesnerie (1992), by considering sequential and independent reasoning by the agents, where they rule out certain outcomes of the system as impossible.

Since agents are rational, they only use strategies that are best responses to some forecast over the possible strategy profiles that can actually be played by the others. Hence, the assumption of rationality implies that strategies that are not best responses will never be played. Following the assumption of common knowledge, each agent knows that all other agents are rational. They can then reach the same conclusion: that only best responses can be played; and taking that into account, each agent may discover that some of his (remaining) strategies are no longer best responses and so he will eliminate them. Then rationality implies that forecasts will be restricted to strategy profiles that are not eliminated. Since all agents are rational and know this second conclusion, they can continue this process of elimination of strategies. This generates a sequence of elimination of non-best-responses that under suitable hypothesis will converge in a sense to be formalized to some (hopefully strict) subset of the original strategy profile set. Guesnerie names this procedure the *eductive process* and we will use this terminology.

Following the terminology of Bernheim we will make a difference between *Rationalizability*, understood as forecasts being general probability measures on the sets of outcomes, and *Point-Rationalizability*, understood as forecasts being points or dirac probability measures on the sets of outcomes. We will continue now by giving a formal definition of the concept of Point-Rationalizability for the case of games with a continuum of agents. Further-on we will address the issue of standard Rationalizability.

3.1 Point-Rationalizable Strategies

The first and natural attempt is to go directly from the finite player case into the continuous case. In this approach, players have forecasts over the set of strategies of each of their opponents. These forecasts are in the form of points in these sets and are so represented by functions from I into S.

Consider the following line of reasoning. Given the strategy profile set S^I, all players know that each player will only play a strategy that is a best response to some strategy profile $\mathbf{s} \in S^I$. For each player then we may define the best response mapping $\mathrm{Br}(i, \cdot) : S^I \rightrightarrows S$:

$$\mathrm{Br}(i, \mathbf{s}) := \operatorname{argmax} \{\pi(i, y, \mathbf{s}) \ : \ y \in S\} . \tag{2.2}$$

The mapping $\mathrm{Br}(i, \cdot)$ gives the optimal set for player $i \in I$ facing a strategy profile $\mathbf{s}$. We use the function $\pi(i, \cdot, \cdot)$ that associates strategy profiles to payoffs in a general way. As we said before, rationality of players implies that they will only use strategies that are optimal to some forecast. So players can discard for each player $i \in I$ strategies that are outside the sets

$$\mathrm{Br}\big(i, S^I\big) \equiv \bigcup_{\mathbf{s} \in S^I} \mathrm{Br}(i, \mathbf{s}) ,$$

so strategy profiles can be actually secluded into the set:

$$S_1^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of the correspondence} \\ i \rightrightarrows \mathrm{Br}\big(i, S^I\big) \end{array} \right\} .$$

That is, players will not play a strategy that is not a best response to some strategy profile. This is captured by selections of the mapping $i \rightrightarrows \mathrm{Br}\big(i, S^I\big)$. Taking this into account, agents can deduce, at a step t of this process, that strategy profiles must actually be in the set S_t^I,

$$S_t^I \equiv \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of the correspondence} \\ i \rightrightarrows \mathrm{Br}\big(i, S_{t-1}^I\big) \end{array} \right\} .$$

This exercise motivates the definition of a recursive process of elimination of non best responses. For this, denoting by $\mathcal{P}(X)$ the set of subsets of a certain set X, we define the mapping $Pr : \mathcal{P}\big(S^I\big) \to \mathcal{P}\big(S^I\big)$ that to each subset $H \subseteq S^I$ associates the set $Pr(H)$ defined by:

$$Pr(H) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a (measurable) selection of} \\ \text{the correspondence } i \rightrightarrows \mathrm{Br}(i, H) \end{array} \right\} . \tag{2.3}$$

This definition is analogous to the one given by Pearce and by Bernheim [12]. In the context of a continuum of players, however, the set $Pr(H)$ could well be empty if we do not make appropriate assumptions about the payoff function π. A sufficient condition for non-emptiness of $Pr(H)$ is non-emptyness of the sets $\text{Br}(i,H)$ λ-almost-everywhere in I along with measurability of the correspondence $i \rightrightarrows \text{Br}(i,H)$. The mapping Pr represents strategy profiles that are obtained as the reactions of players to strategy profiles contained in the set $H \subseteq S^I$. It has to be kept in mind that strategies of different players in a strategy profile in $Pr(H)$ can be the reactions to (possibly) different strategy profiles in H.

The line of reasoning developed above implies that a strategy profile that is point rationalizable should never be eliminated during the process generated by the iterations of Pr. Let us note $Pr^t\big(S^I\big) \equiv Pr\big(Pr^{t-1}\big(S^I\big)\big)$ and $Pr^0\big(S^I\big) \equiv S^I$. The set $Pr^t\big(S^I\big)$ is the one obtained in the t^{th} step of the process of elimination of non-best-response strategy profiles. It is direct to see that $Pr^1\big(S^I\big) \equiv S^I_1$ and $Pr^t\big(S^I\big) \equiv S^I_t$. Note that the process $\big\{Pr^t\big(S^I\big)\big\}_{t=0}^{+\infty}$ gives a nested family of subsets of S^I and so a point that is never eliminated should be in the intersection of all of them. This means that the set of point-rationalizable strategies, from now on denoted $\mathbb{P}_S$, must satisfy:

$$\mathbb{P}_S \subseteq \bigcap_{t=0}^{+\infty} Pr^t\big(S^I\big)\,. \tag{2.4}$$

However, it is not enough to ask for this property, since rationality of players implies that a strategy should only be played if it is justified by a rationalizable strategy profile. The point-rationalizable set must have the *best response property*: each strategy that participates in a strategy profile in $\mathbb{P}_S$ must be a best response to some (possibly different) strategy profile in $\mathbb{P}_S$.We capture this second feature by asking condition (2.5),

$$\mathbb{P}_S \subseteq Pr(\mathbb{P}_S)\,. \tag{2.5}$$

Note that condition (2.5) implies (2.4), since a set that satisfies (2.5) would never be eliminated. The ideal situation would be that the result of the eductive process gave the set of point-rationalizable strategies. This would be the case only if $Pr\big(\bigcap_{t=0}^{+\infty} Pr^t\big(S^I\big)\big) = \bigcap_{t=0}^{+\infty} Pr^t\big(S^I\big)$, which as we mentioned in the introduction is not necessarily true in all generality, we give an example in the next subsection.

Nevertheless, with the concepts displayed so far, we are able to give a definition for the Point-Rationalizable Strategy Profiles set.

Definition 2.5. The set of *Point-Rationalizable Strategy Profiles* is the maximal subset $H \subseteq S^I$ that satisfies condition (2.5) and we note it $\mathbb{P}_S$.

[12] See Definition 1 in Pearce (1984) and Section 3(b) in Bernheim (1984).

For each player, $i \in I$, there will be a set of Point-Rationalizable Strategies, namely the union, over all the Point-Rationalizable Strategy Profiles in $\mathbb{P}_S$, of the best response set of the considered player. That is, the set of Point-Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{P}_S(i) := \bigcup_{\mathbf{s} \in \mathbb{P}_S} \mathrm{Br}(i, \mathbf{s})$$

A well known result for the case of games with a finite number of players is that all Nash Equilibria of the game are elements of the Point-Rationalizable Strategies set (Bernheim 1984). The same is true for our definition, since if $\mathbf{s}^*$ is a Nash Equilibrium, then it is a selection taken from $i \rightrightarrows \mathrm{Br}(i, \mathbf{s}^*)$ and so it satisfies $\{\mathbf{s}^*\} \subseteq Pr(\{\mathbf{s}^*\})$ which implies the property.

We now turn to a different approach to Rationalizability. In the context that interests us, players form expectations not on the space of strategy profiles, but on the set of states of the game. Thus Rationalizability should also be stated in terms of forecasts on this set of states. This is what we present in the next subsection.

3.2 Point-Rationalizable States

We turn to the particular class of games with a continuum of players where payoffs depend explicitly on the average of the actions of all the players, which we call the state of the game. In this framework it is natural to model agents as having forecasts on the set of states, rather than on the set of strategy profiles, since the relevant information that agents need to take a decision is the value of the state a [13].

In what follows, we will define Point-Rationalizability on the set of states. So now instead of using the correspondence $\mathrm{Br}(i, \cdot)$ defined in (2.2), we use the mapping $B(i, \cdot) : \mathcal{A} \rightrightarrows S$ that gives the optimal strategy set given a state of the system,

$$B(i, a) := \mathrm{argmax}\,\{u(i, y, a) \;\;:\;\; y \in S\}\,.$$

[13] See as well Guesnerie (2002) for a discussion on this matter.

There are two main differences between this approach an the one presented in the previous subsection. First, here we use the specific function $\mathbf{u}$ that defines a game with an aggregate state instead of the general function π as in (2.2), and second, the mapping $B(i, \cdot)$ goes from $\mathcal{A} \subset \mathbb{R}^n$, instead of S^I, to $S \subset \mathbb{R}^n$. It is direct to see, however, that for a given strategy profile $\mathbf{s}$, in the context of a game $\mathbf{u}$, $\text{Br}(i, \mathbf{s}) \equiv B\big(i, \int \mathbf{s}\big)$. For each $i \in I$ and a set $X \subseteq \mathcal{A}$, consider the image through $B(i, \cdot)$ of the set X

$$B(i, X) := \bigcup_{a \in X} B(i, a)\,.$$

Let us now look at the process of elimination of non reachable or non generated states. Suppose that initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$. Then, in a first order basis, an agent can assume that any of the states $a \in X$ can be the actual state, but point expectations are actually constrained by X, so the possible actions of a player $i \in I$ are constrained to the set $B(i, X)$. Since all players know this, each one of them can discard all strategy profiles $\mathbf{s} \in S^I$ that are not selections of the set valued mapping $i \rightrightarrows B(i, X)$. Then, if the players know that forecasts are restricted to $X \subseteq \mathcal{A}$, they will know that the actual outcome has to be a state associated through the aggregation operator to some measurable selection of that mapping.

Therefore, given $X \subseteq \mathcal{A}$ consider the set of all the measurable selections taken from the correspondence $i \rightrightarrows B(i, X)$ that to each agent $i \in I$ associates the set $B(i, X)$. Then, take all the possible images through the aggregation mapping of such functions. We define then the mapping $\tilde{Pr} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ that to each set $X \subseteq \mathcal{A}$ associates the set $\tilde{Pr}(X) \subseteq \mathcal{A}$ defined by:

$$\tilde{Pr}(X) := \left\{ a \in \mathcal{A} : \ a = A(\mathbf{s}), \ \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the} \\ \text{correspondence } i \rightrightarrows B(i, X) \end{array} \right\}. \tag{2.6}$$

Our assumptions on the aggregation operator A allow us to re-write definition (2.6) as the integral of a set valued mapping [14]:

$$\tilde{Pr}(X) \equiv \int_I B(i, X)\, \text{di}.$$

Before continuing, we state a relevant property associated to the mapping B.

Lemma 2.6. *In a game $\mathbf{u}$, for a non-empty closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$*

[14]The integral of a correspondence $F : I \rightrightarrows \mathbb{R}^n$ is calculated, following Aumann (1965), as the set of integrals of all the integrable selections of F. This is,

$$\int_I F(i)\, \text{di} \equiv \left\{ \int_I f(i)\, \text{di} \ : \ f \text{ is an integrable selection of } F \right\}$$

where $\int f \text{di} := \big(\int f_1(i)\, \text{di}, \ldots, \int f_n(i)\, \text{di}\big)$.

is measurable and has non-empty compact values.

Proof of Lemma 2.6.

We show first that the mapping $\mathrm{G} : I \rightrightarrows \mathcal{A} \times S$, that associates with each agent $i \in I$ the graph of the best response mapping $B(i, \cdot)$, $\mathrm{G}(i) := \operatorname{gph} B(i, \cdot)$, is measurable.

Take a closed set $C \subseteq \mathcal{A} \times S$. We need to prove that the set

$$\mathrm{G}^{-1}(C) \equiv \{i \in I \ : \ C \cap \operatorname{gph} B(i, \cdot) \neq \emptyset\}$$

is measurable. Consider the subset $U \subseteq \mathcal{U}_{S\times\mathcal{A}}$ defined by:

$$U := \{\ g \in \mathcal{U}_{S\times\mathcal{A}} \ : \ \exists\,(a,s) \in C \text{ such that } g(s,a) \geq g(y,a) \ \forall\ y \in S\}$$

note that $\mathbf{u}^{-1}(U) \equiv \mathrm{G}^{-1}(C)$ and so, from the measurability assumption over $\mathbf{u}$, it suffices to prove that U is closed. That is, we have to show that for any sequence $\{g^\nu\}_{\nu\in\mathbb{N}} \subset U$, such that $g^\nu \to g^*$ uniformly $g^* \in U$.

Since the functions g^ν are finite and continuous in $S \times \mathcal{A}$, from Weierstrass' Theorem g^* is continuous and so it belongs to $\mathcal{U}_{S\times\mathcal{A}}$. Moreover, g^ν converges continuously to g^*, that is, for any convergent sequence (a^ν, s^ν) with limit (a^*, s^*), the sequence $g^\nu(s^\nu, a^\nu)$ converges to $g^*(s^*, a^*)$. Indeed, consider any $\varepsilon > 0$. By the continuity of g^* there exists $\bar{\nu}_1 \in \mathbb{N}$ such that $\forall\ \nu > \bar{\nu}_1$,

$$2|g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \frac{\varepsilon}{2}.$$

From the uniform convergence of g^ν we get that there exists $\bar{\nu}_2 \in \mathbb{N}$ such that,

$$|g^\nu(s,a) - g^*(s,a)| < \frac{\varepsilon}{2} \qquad \text{for all } \nu \geq \bar{\nu}_2 \text{ and } \forall\,(s,a) \in S \times \mathcal{A},$$

in particular this is true for all the elements of the sequence of points. We get then that $\forall$ $\nu \geq \max\{\bar{\nu}_1, \bar{\nu}_2\}$,

$$\begin{aligned} |g^\nu(s^\nu, a^\nu) - g^*(s^*, a^*)| \leq\ & |g^\nu(s^\nu, a^\nu) - g^*(s^\nu, a^\nu)| \\ & + |g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \varepsilon. \end{aligned}$$

We have to show then that there exists a point $(a, s) \in C$ such that $g^*(s, a) \geq g^*(y, a)$ $\forall\ y \in S$. Since $g^\nu \in U$, we have for each $\nu \in \mathbb{N}$, points $(a^\nu, s^\nu) \in C$ such that $g^\nu(s^\nu, a^\nu) \geq g^\nu(y, a^\nu)$ $\forall\ y \in S$. Let $(a^*, s^*) \in C$ be the limit of a convergent subsequence of $\{(a^\nu, s^\nu)\}_{\nu\in\mathbb{N}}$, which without loss of generality we can take to be the same sequence. We see that (a^*, s^*) is the point we are looking for since for a fixed $y \in S$, continuous convergence implies that in the limit

$$g^*(s^*, a^*) \geq g^*(y, a^*)\,.$$

We conclude then that $g^* \in U$. Thus, U is closed and since $\mathbf{u}$ is a measurable mapping, $\mathbf{u}^{-1}(U)$ is measurable.

With this in mind, consider a closed set $X \subseteq \mathcal{A}$ and the mapping $i \rightrightarrows B(i, X)$. Applying Theorem 14.13 in Rockafellar and Wets (1998) to the constant mapping $i \rightrightarrows X$ along with G above, we get that the correspondence $i \rightrightarrows B(i, X)$ is measurable and has closed values (hence compact since S is compact).

■

With Lemma 2.6 we can now prove Theorem 2.4.

Proof.

Consider the correspondence $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$ defined by [15]

$$\Gamma(a) := \int_I B(i, a)\, \mathrm{di}.$$

Note that a fixed point of Γ defines an equilibrium of the game $\mathbf{u}$. Lemma 2.6 implies that for all $a \in \mathcal{A}$, $\Gamma(a) \neq \emptyset$. By definition, for all $a \in \mathcal{A}$, $\Gamma(a)$ is convex. Under our assumptions, the correspondences $B(i, \cdot\,) : \mathcal{A} \rightrightarrows S$ are u.s.c. and from Aumann (1976) so is Γ. This last assertion implies as well that $\Gamma(a)$ is compact $\forall\, a \in \mathcal{A}$. Applying Kakutani's fixed point Theorem we get that there exists $a^* \in \mathcal{A}$ such that $a^* \in \Gamma(a^*)$.

■

Lemma 2.6 above and Theorem 2 in Aumann (1965) assure that $\tilde{Pr}(X)$ is non empty and closed whenever X is non empty and closed. With this set to set mapping we can define a set of point rationalizable states.

[15] The integral of a correspondence $F : I \rightrightarrows \mathbb{R}^n$ is calculated, following Aumann (1965), as the set of integrals of all the integrable selections of F. This is,

$$\int_I F(i)\, \mathrm{di} \equiv \left\{ \int_I f(i)\, \mathrm{di} \;\; : \;\; f \text{ is an integrable selection of } F \right\}$$

where $\int f \mathrm{di} := \left(\int f_1(i)\, \mathrm{di}, \ldots, \int f_n(i)\, \mathrm{di} \right)$.

As we did in the previous subsection, consider the process given by iterations of $\tilde{Pr}$. That is,

$$\begin{aligned} \tilde{Pr}^0(\mathcal{A}) &:= \mathcal{A} \\ \tilde{Pr}^{t+1}(\mathcal{A}) &:= \tilde{Pr}\left(\tilde{Pr}^t(\mathcal{A})\right) \quad \text{for } t \geq 1. \end{aligned}$$

Observe that $\tilde{Pr}^{t+1}(\mathcal{A}) \subseteq \tilde{Pr}^t(\mathcal{A})$, this is not necessarily true for any subset $X \subseteq \mathcal{A}$. The set of Point-Rationalizable States, $\mathbb{P}_{\mathcal{A}}$, must then satisfy:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A})\,. \tag{2.7}$$

The right hand side of (2.7) represents the iterative elimination of non reachable states. At each step of this process, players only keep in mind the states that could be reached following rational actions based on point expectations given by the set of the previous step. If a state is not reached by actions following forecasts constrained at a certain step of the process, then it is not rationalizable. Since the family of sets $\left\{\tilde{Pr}^t(\mathcal{A})\right\}_{t=0}^{+\infty}$ is a nested (decreasing) family of closed subsets of $\mathbb{R}^n$, the infinite intersection in expression (2.7) turns out to be the exact *Painlevé-Kuratowski* limit of the sequence of sets.

The second condition that the set of Point-Rationalizable States must satisfy is:

$$\mathbb{P}_{\mathcal{A}} \subseteq \tilde{Pr}(\mathbb{P}_{\mathcal{A}})\,. \tag{2.8}$$

Condition (2.8) stands for the fact that Point-Rationalizable States should be justified by Point-Rationalizable States. This means that if a state is Point-Rationalizable, it should rise as the consequences of players taking actions as reactions to point forecasts in the set of Point-Rationalizable States. Analogously to the case where point forecasts are taken over strategy profiles, it is direct to see that condition (2.8) implies (2.7). That is, if a set $X \subseteq \mathcal{A}$ satisfies condition (2.8) then $X \subseteq \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A})$. So we define the set of Point-Rationalizable States as follows:

Definition 2.7. The set of Point-Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies condition (2.8) and we note it $\mathbb{P}_{\mathcal{A}}$.

Remark 2.8. Note that for the case of forecasts over the set of states, defining player-specific rationalizable states set makes no sense. This approach calls for different mathematical tools since now we are dealing with a set in a finite dimensional space as opposed to Definition 2.5. Moreover, the exercise of obtaining Point-Rationalizable States gives clear insights on properties of the Point-Rationalizable Strategy Profiles set, particularly for strongly rational equilibria, as can be seen in Proposition 2.17 below.

Remark 2.9. Conditions (2.5) and (2.8) are related to the definition of *Tight Sets Closed Under Rational Behavior* (Tight CURB Sets) given in Basu and Weibull (1991). Indeed Basu and Weibull make the observation that the set of rationalizable strategy profiles in a finite game with compact strategy sets and continuous payoff functions, is in fact the maximal tight curb set, which is analogous to our definitions of Point-Rationalizability.

We know give an answer to the question of whether we can obtain the same conclusion as in Bernheim's Proposition 3.2 in our context. Our main result, Theorem 2.10, states that under the hypothesis of Rath's setting we have that the set of Point-Rationalizable States, is actually the one obtained from the eductive process, and so we obtain a first characterization of this set.

Theorem 2.10. *Let us write* $\mathbb{P}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{Pr}^{t}(\mathcal{A})$*. The set of Point-Rationalizable States of a game* $\boldsymbol{u}$ *can be calculated as*

$$\begin{aligned} \mathbb{P}_{\mathcal{A}} &\equiv \mathbb{P}'_{\mathcal{A}} \\ &\equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^{t}(\mathcal{A}) \end{aligned}$$

Proof.

We will show that:

$$\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}}$$

Let us begin by showing that $\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \subseteq \mathbb{P}'_{\mathcal{A}}$. Indeed, if $a \in \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})$ then, by the definition of $\tilde{Pr}$, there exists a measurable selection $\mathbf{s} : I \to S$ of $i \rightrightarrows B(i, \mathbb{P}'_{\mathcal{A}})$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{Pr}^{t}(\mathcal{A})\ \forall\, t \geq 0$, we have that $B(i, \mathbb{P}'_{\mathcal{A}}) \subseteq B\Big(i, \tilde{Pr}^{t}(\mathcal{A})\Big)\ \forall\, t \geq 0\ \forall\, i \in I$. So $\mathbf{s}$ is a selection of $i \rightrightarrows B\Big(i, \tilde{Pr}^{t}(\mathcal{A})\Big)$ and then $a \in \tilde{Pr}^{t+1}(\mathcal{A})\ \forall\, t \geq 0$, which means that $a \in \mathbb{P}'_{\mathcal{A}}$.

Now we show that $\mathbb{P}'_{\mathcal{A}} \subseteq \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})$. For this inclusion, consider the following sequence $F^t : I \rightrightarrows S$, $t \geq 0$, of set valued mappings:

$$\begin{aligned} & F^0(i) := S & \forall\, i \in I \\ \forall\, i \in I \quad & F^t(i) := B\Big(i, \tilde{Pr}^{t-1}(\mathcal{A})\Big) & t \geq 1 \end{aligned}$$

As we said before, we have that

$$\tilde{Pr}^{t}(\mathcal{A}) \equiv \int_I F^t(i)\ \text{di}.$$

Since $\mathbf{u}(i) \in \mathcal{U}_{S\times\mathcal{A}}$, then $\forall\, i \in I$ the mapping $B(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S$ is u.s.c. and, as a consequence, the set $B(i, X)$ is compact for any compact subset $X \subseteq \mathcal{A}$ (Berge 1997). Since $\mathcal{A} \equiv \int_I F^0$, Aumann (1965) gives that $\mathcal{A}$ is non empty and compact[16]. From Lemma 2.6 we get that F^1 is measurable and compact valued and by induction over t, we get that for all $t \geq 1$, $\tilde{Pr}^{t-1}(\mathcal{A})$ is non empty, convex and compact, and F^t is measurable and compact valued.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t, noted p-lim $\sup_t F^t$, obtained as:

$$F(i) := \big(\text{p-lim}\sup\nolimits_t F^t\big)(i) \equiv \limsup_t F^t(i)$$

where the right hand side is the set of all cluster points of sequences $\{y^t\}_{t\in\mathbb{N}}$ such that $y^t \in F^t(i)$. From Rockafellar and Wets[17] we get that F is measurable and compact valued.

So now let us take a point $a \in \mathbb{P}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t\in\mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976) we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t\in\mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int \mathbf{s}^t$.

Now it suffices to check that $F(i) \subseteq B(i, \mathbb{P}'_{\mathcal{A}})$, since then we would have

$$a \in \int_I F\mathrm{di} \subseteq \int_I B(i, \mathbb{P}'_{\mathcal{A}})\,\mathrm{di} \equiv \tilde{Pr}(\mathbb{P}'_{\mathcal{A}})\,.$$

This comes from the upper semi continuity of $B(i,\,\cdot\,)$. Take $y \in F(i)$. From the definition of F, y is a cluster point of a sequence $\{y^t\}_{t\in\mathbb{N}}$ such that $y^t \in F^t(i)$. That is, there is a sequence of elements of $\mathcal{A}$, $\{a^k\}_{k\in\mathbb{N}}$ such that $a^k \in \tilde{Pr}^{t_k-1}(\mathcal{A})$, $y^{t_k} \in B\big(i, a^k\big)$ and $\lim_k y^{t_k} = y$. Through a subsequence of $\{a^k\}_{k\in\mathbb{N}}$ we get that the limit of $\{y^{t_k}\}_{k\in\mathbb{N}}$ must belong to $B(i, \mathbb{P}'_{\mathcal{A}})$, since all cluster points of $\{a^k\}_{k\in\mathbb{N}}$ are in $\mathbb{P}'_{\mathcal{A}}$, being the intersection of a nested family of compact sets.

■

The previous theorem gives a characterization of Point-Rationalizable States that is analogous to the one given for Point-Rationalizable Strategies in Bernheim, in the case of finite player games with compact strategy sets and continuous utility functions. The difficulty of Theorem 2.10 is to identify the adequate convergence concept for the eductive process. In the case of finite player games there is no such a question, since in that case the technique is simply to take a convergent subsequence of points (in the finite dimensional strategy profile

[16] We have already noted that in fact $\mathcal{A} \equiv \mathrm{co}\,\{S\}$ and so in particular it is also convex, which is of no relevance in this proof, but is the key property in Corollary 2.12.

[17] See Rockafellar and Wets (1998) Ch. 4 and 5 and Theorem 14.20 .

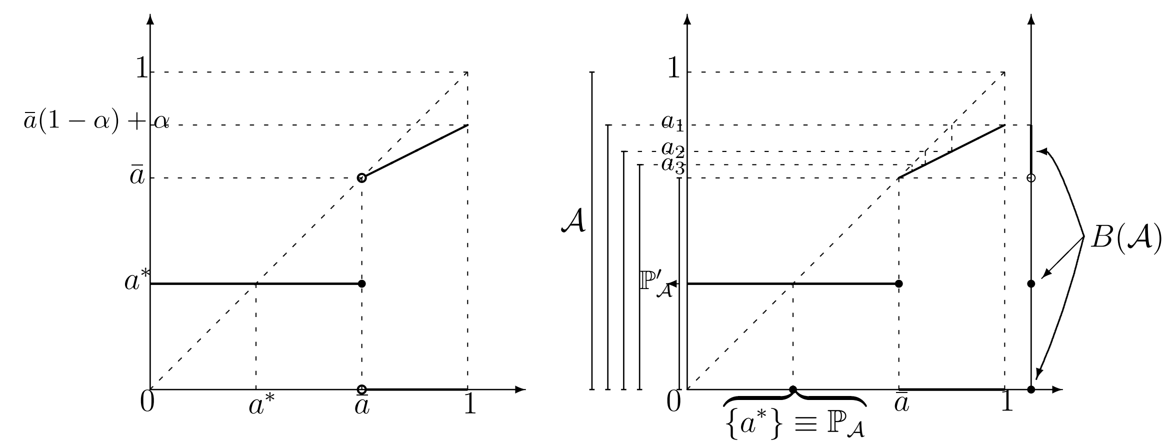
set) from the sequence of sets that participate in the eductive procedure and using (semi) continuity arguments of the best response mappings obtain the result [18]. However, in our setting these arguments fail to prevail. From the proof of the Theorem, we see that the set of Point-Rationalizable States is obtained as the integral of the point-wise upper limit of a sequence of set valued mappings. So the relevant improvement in the proof (besides measurability requirements) is to give the adequate limit concept.

To see that the Theorem is not vacuous consider the following example.

Example 2.11. Consider the game where $I \equiv [0,1]$, $S \equiv [0,1]$ and $\mathbf{u}(i) \equiv u : [0,1]^2 \to \mathbb{R}$ for all $i \in I$, such that it generates the following best reply correspondence, depicted in Figure 2.2a:

$$B(a) = \begin{cases} a^* & \text{if } a \leq \bar{a}, \\ \{0, \bar{a}(1-\alpha) + a\alpha\} & \text{if } a > \bar{a}, \end{cases}$$

where a^*, $\bar{a}$ and α are in $]0,1[$ and $a^* < \bar{a}$. It is clear that this game does not satisfy the hypothesis of Theorem 2.10 since no continuous utility function may give rise to such a best response correspondence.



(a) The best reply correspondence $B : [0,1] \rightrightarrows [0,1]$.

(b) The elimination of non-generated-states with $\tilde{Pr}(X) \equiv \text{co}\,\{B(X)\}$ and the Point-Rationalizable State set.

Figure 2.2: Example 2.11: The set of Point-Rationalizable States is not the set $\mathbb{P}'_{\mathcal{A}}$.

The only equilibrium of the game is a^*.

Since all the players have the same best reply correspondence, the process of elimination of non-generated-states is obtained by:

$$\tilde{Pr}(X) \equiv \text{co}\,\{B(X)\}$$

[18] See the proof of Proposition 3.1 in Bernheim (1984).

The image through the best reply correspondence of the state set is

$$B(\mathcal{A}) \equiv \{0, a^*\} \cup \left]\bar{a}, \bar{a}(1-\alpha)+\alpha\right],$$

then $\tilde{Pr}(\mathcal{A}) \equiv \mathrm{co}\,\{B(\mathcal{A})\} \equiv [0, \bar{a}(1-\alpha)+\alpha]$.

Then the second iteration is obtained by

$$\begin{aligned}\tilde{Pr}^2(\mathcal{A}) \equiv \mathrm{co}\left\{B\Big(\tilde{Pr}(\mathcal{A})\Big)\right\} &\equiv \mathrm{co}\,\{B([0, \bar{a}(1-\alpha)+\alpha])\} \\ &\equiv \mathrm{co}\,\{\{0, a^*\} \cup \left]\bar{a}, \bar{a}(1-\alpha)+\alpha(\bar{a}(1-\alpha)+\alpha)\right]\} \\ &\equiv \left[0, \bar{a}\big(1-\alpha^2\big)+\alpha^2\right].\end{aligned}$$

We see from the form of the best reply correspondence that on each iteration of $\tilde{Pr}$ we get an interval of the form $[0, a_t]$ where the sequence $\{a_t\}_{t=0}^{+\infty}$ satisfies:

$$a_{t+1} = \bar{a}(1-\alpha) + a_t\alpha,$$

which gives, for $t \geq 1$,

$$a_t = \bar{a}\big(1-\alpha^t\big) + \alpha^t,$$

with $a_0 = 1$, and so the sequence is decreasing and converges to $\bar{a}$ (see Figure 2.2b). This allows for us to see that $\mathbb{P}'_{\mathcal{A}} \equiv [0, \bar{a}]$. However,

$$\begin{aligned}\tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) &\equiv \mathrm{co}\,\{B(\mathbb{P}'_{\mathcal{A}})\} \equiv \mathrm{co}\,\{B([0,\bar{a}])\} \equiv \mathrm{co}\,\{\{a^*\}\} \equiv \{a^*\} \\ &\subsetneq \mathbb{P}'_{\mathcal{A}}.\end{aligned}$$

and so $\mathbb{P}'_{\mathcal{A}}$ is not equal to $\mathbb{P}_{\mathcal{A}}$, which is in fact equal to the set of equilibria: the singleton $\{a^*\}$.

Theorem 2.4 implies that if the set of Point-Rationalizable Strategies (or States for what it matters) is well defined, then it is not empty, since as we already said, all the equilibria belong to this set. From Theorem 2.10 we get as a Corollary that in the games that we are considering, the set of Point-Rationalizable States is well behaved.

Corollary 2.12. *The set of Point-Rationalizable States of a game $\boldsymbol{u}$ is well defined, non-empty, convex and compact.*

Proof.

From Theorem 2.10, $\mathbb{P}_{\mathcal{A}}$ is the intersection of a nested family of non-empty compact convex sets. From Theorem 2.4 we get that there is point $a^* \in \mathcal{A}$ such that $a^* \in \tilde{Pr}^t(\mathcal{A})\ \forall\ t$. These two facts lead us to conclude that this intersection is compact, convex and non empty.

■

The properties stated in this Corollary are not trivial. In games with finite number of players we can find examples where the outcome of the iterative elimination of non-best-replies is an empty set. The same can be true in our context when we withdraw the continuity hypothesis of utility functions, we present below an example of a game with non-continuous payoffs:

Example 2.13 (Based on Dufwenberg and Stegeman (2002))**.** Consider the game where $I \equiv [0,1]$, $S \equiv [0,1]$ and $\mathbf{u}(i) \equiv u : [0,1]^2 \to \mathbb{R}$ for all $i \in I$, such that:

$$u(y,a) = \begin{cases} 1 - y & \text{if } 0 < \frac{a}{2} \leq y, \\ y & \text{if not.} \end{cases}$$

Then, the best reply correspondence is the same for all players:

$$B(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

The mapping Γ turns out to be equal to the best reply correspondence:

$$\Gamma(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{a}{2} & \text{if } a > 0. \end{cases}$$

This mapping has no fixed point and so in this game there is no equilibrium.

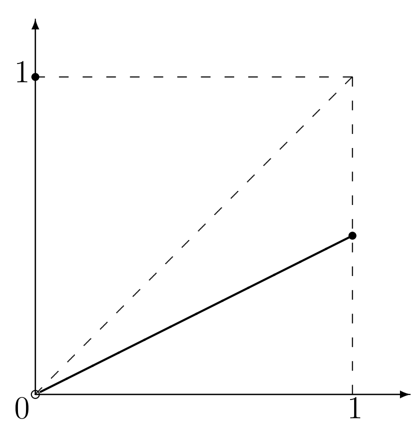


Figure 2.3: The best reply correspondence $B : [0,1] \rightrightarrows [0,1]$.

Let us study the Point-Rationalizable States set. The image through the best reply correspondence of the state set is $B(\mathcal{A}) \equiv \{1\} \cup]0, 1/2]$, then

$$\tilde{Pr}(\mathcal{A}) \equiv \text{co}\{B(\mathcal{A})\} \equiv]0, 1],$$

the second iteration gives

$$\tilde{Pr}^2(\mathcal{A}) \equiv \text{co}\left\{B\left(\tilde{Pr}(\mathcal{A})\right)\right\} \equiv \text{co}\{B(]0,1])\} \equiv]0, 1/2]$$

and the third

$$\tilde{Pr}^3(\mathcal{A}) \equiv \text{co}\left\{B\left(\tilde{Pr}^2(\mathcal{A})\right)\right\} \equiv \text{co}\{B(]0,1/2])\} \equiv]0, 1/4], \ \ldots, \text{etc..}$$

Then the set $\mathbb{P}'_{\mathcal{A}} \equiv \emptyset$ and so $\mathbb{P}_{\mathcal{A}} \equiv \emptyset$, that is, in this example there is no set $X \subseteq \mathcal{A}$ that satisfies $X \subseteq \tilde{Pr}(X)$.

Although Theorem 2.10 and Corollary 2.12 assure that the eductive procedure achieves a non-empty Point-Rationalizable set of states, we see from Examples 2.11 and 2.13 that even in cases where the eductive procedure fails, in the sense that it does not deliver the Point-Rationalizable set, we may still identify a set as the *correct* Point-Rationalizable State set following our Definition 2.7 (in Example 2.11 the set $\mathbb{P}_{\mathcal{A}}$ is the singleton that contains the equilibrium and in Example 2.13 it is the empty set). Even more, the eductive procedure can (obviously) help to locate such set even in the case of failure and emptiness. The original motivation to introduce rationalizability in economic contexts is the plausibility of the Rational Expectations Hypothesis. In consequence we allow an empty (Point-)Rationalizable set, under the definition of rationalizability, interpreting such as a pessimistic answer to the possibility of the coordination of expectations.

Another property stated in Corollary 2.12 and an important consequence of Theorem 2.10, is the convexity of the Point-Rationalizable States set, since in the case where we have multiple equilibria in the set of states, we know that the convex hull of this set of equilibria is also contained in the set of Point-Rationalizable States. This inclusion may be strict since if we have multiple equilibria in the set of strategies, even with uniqueness in the set of states, we may have multiple Point-Rationalizable States [19]. Convexity of the Point-Rationalizable States set is a relevant property since it has not been obtained (to our knowledge) for any other concept related to Rationalizability.

[19] See Proposition 2.17.

3.3 Point-Rationalizable Strategies vs. Point-Rationalizable States

It is straightforward to ask how the concepts that we just defined are related. To address this issue, note that for the class of non-atomic games that we are considering, the iterations of the strategy-elimination and state-elimination mappings (Pr and $\tilde{Pr}$) are equivalent in the following sense. Consider the set to set mappings $\bar{A}$ and $\bar{B}$ defined below. Let $\bar{A} : \mathcal{P}(S^I) \to \mathcal{P}(\mathcal{A})$ be defined by:

$$\begin{aligned}\bar{A}(H) &:= \left\{ a \in \mathcal{A} : \begin{array}{l} a = \int_I s(i)\,\mathrm{di} \text{ and } \mathbf{s} \text{ is a measur-} \\ \text{able function in } H \end{array} \right\} \\ &\equiv A(H)\end{aligned}$$

and let $\bar{B} : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(S^I)$ be:

$$\bar{B}(X) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of} \\ i \rightrightarrows B(i, X) \end{array} \right\}.$$

These mappings satisfy

$$\begin{aligned} X_1 \subseteq X_2 \subseteq \mathcal{A} &\implies \bar{B}(X_1) \subseteq \bar{B}(X_2) \\ H_1 \subseteq H_2 \subseteq S^I &\implies \bar{A}(H_1) \subseteq \bar{A}(H_2)\,. \end{aligned} \tag{2.9}$$

Then, in the context of a game $\mathbf{u}$, the mappings Pr and $\tilde{Pr}$ satisfy:

$$\begin{aligned} Pr(H) &\equiv \bar{B}\big(\bar{A}(H)\big) \\ \tilde{Pr}(X) &\equiv \bar{A}\big(\bar{B}(X)\big) \end{aligned} \tag{2.10}$$

In particular, we get,

$$\tilde{Pr}^0(\mathcal{A}) \equiv \mathcal{A} \equiv \mathrm{co}\,\{S\} \equiv \bar{A}\big(S^I\big)$$

which implies by induction that:

$$\begin{aligned} \tilde{Pr}^t(\mathcal{A}) &\equiv \bar{A}\big(Pr^t\big(S^I\big)\big) \\ Pr^t\big(S^I\big) &\equiv \bar{B}\Big(\tilde{Pr}^{t-1}(\mathcal{A})\Big) \end{aligned}$$

Theorem 2.14. *In a game* $\mathbf{u}$ *we have:*

$$\mathbb{P}_S \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \quad \text{and} \quad \mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_S)\,.$$

Proof.

Note that from (2.10) we have

$$\begin{aligned} Pr\big(\bar{B}(\mathbb{P}_{\mathcal{A}})\big) &\equiv \bar{B}\Big(\tilde{Pr}(\mathbb{P}_{\mathcal{A}})\Big) \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \\ \tilde{Pr}\big(\bar{A}(\mathbb{P}_S)\big) &\equiv \bar{A}(Pr(\mathbb{P}_S)) \ \ \equiv \bar{A}(\mathbb{P}_S) \end{aligned}$$

That is, the sets $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq S^I$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathcal{A}$ satisfy conditions (2.5) and (2.13) respectively, which implies that $\bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S$ and $\bar{A}(\mathbb{P}_S) \subseteq \mathbb{P}_{\mathcal{A}}$. Then

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S) \equiv \bar{B}\big(\bar{A}(\mathbb{P}_S)\big) \subseteq \bar{B}(\mathbb{P}_{\mathcal{A}}) \subseteq \mathbb{P}_S.$$

The second equality comes from (2.10) while the first inclusion comes from (2.9) and the previous observation. The proof for the second statement is analogous.

■

We see from Theorem 2.14 that in the context that we are considering, the set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. This implies that in the models that interest us, it is equivalent to study Point-Rationalizability in terms of strategies or states, an intuition claimed by Guesnerie and, of course, present in Example 2.3.

Theorems 2.14 and 2.10 together imply that the set of Point-Rationalizable Strategies $\mathbb{P}_S$ can be actually computed, in this setting, as the limit of the strategy elimination process governed by Pr (see condition (2.4)) answering a question that remained unanswered. In consequence, we have that in a game $\mathbf{u}$ we can obtain the sets of Point-Rationalizable States and Strategies through the eductive process in the respective set ($\mathcal{A}$ or S^I).

Corollary 2.15. *Let us write* $\mathbb{P}'_S := \bigcap_{t=0}^{\infty} Pr^t\big(S^I\big)$*. The set of Point-Rationalizable Strategy Profiles of a game* $\mathbf{u}$ *can be calculated as*

$$\begin{aligned} \mathbb{P}_S &\equiv \mathbb{P}'_S \\ &\equiv \bigcap_{t=0}^{\infty} Pr^t\big(S^I\big) \end{aligned}$$

3.4 Strongly Point Rational Equilibrium

As we have already said, Guesnerie defines the concept of (local) Strongly Rational Equilibrium as an equilibrium that is the only Rationalizable State of an economic system. A particular feature of the work therein developed is that although the definition of Rationalizability is stressed in terms of strategy profiles, that is, on the profile of individual production quantities, the study of the (local) stability of the (unique) equilibrium can be developed in terms of aggregate production or even prices (see note 13). In our context, Strong Rationality would be defined in terms of the aggregate variable $a \in \mathcal{A}$.

Our purpose in this section then is to explore the relation between strategy profiles and aggregate states when we are interested in Strong Rationality and Point-Rationalizability.

Definition 2.16. An equilibrium state $a^* \in \mathcal{A}$ is a *Strongly Point Rational Equilibrium* if $\mathbb{P}_{\mathcal{A}} = \{a^*\}$.

Note that if a state a^* satisfies $\mathbb{P}_{\mathcal{A}} = \{a^*\}$, then it is the unique equilibrium of the system since (i) all equilibria are in $\mathbb{P}_{\mathcal{A}}$ and (ii) $\tilde{Pr}(\{a^*\}) \equiv \{a^*\}$ implies that a^* is the unique value obtained from the integral of the best response mapping $i \rightrightarrows B(i, a^*)$ and so is an equilibrium. Analogously if $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$, then $\mathbf{s}^*$ is the unique Nash Equilibrium of the game, since all Nash Equilibria are in $\mathbb{P}_S$ and $Pr(\{\mathbf{s}^*\}) \equiv \{\mathbf{s}^*\}$ implies that $\mathbf{s}^*$ is the (unique) measurable selection of $i \rightrightarrows \mathrm{Br}(i, \mathbf{s}^*)$ and so it is a Nash Equilibrium. In particular this says that $\mathrm{Br}(i, \mathbf{s}^*)$ is λ-a.e. single valued and hence can be associated to the concept of strict Nash Equilibrium.[20]

Proposition 2.17. *If $\boldsymbol{s}^*$ is a Nash Equilibrium of $\boldsymbol{u}$ and $\int_I \boldsymbol{s}^* = a^*$, then:*

$$a^* \textit{ is Strongly Point Rational} \implies \mathbb{P}_S \equiv \left\{ \boldsymbol{s} \in S^I \;:\; \begin{array}{c} s \textit{ is a measur-} \\ \textit{able selection of} \\ i \rightrightarrows B(i, a^*) \end{array} \right\}$$

$$\mathbb{P}_S \equiv \{\boldsymbol{s}^*\} \implies a^* \textit{ is Strongly Point Rational}$$

In particular, if $B(i, \cdot)$ is single valued at a^* λ-a.e. on I, then,

$$a^* \textit{ is Strongly Point Rational} \iff \mathbb{P}_S \equiv \{\boldsymbol{s}^*\}$$

Proof.

[20] A situation in which any unilateral deviation incurs in a loss.

By the definition of $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{P}_S$ and the property of $\mathbb{P}_{\mathcal{A}}$ stated in Theorem 2.14 we have,

$$\bar{B}(\mathbb{P}_{\mathcal{A}}) \equiv \mathbb{P}_S \tag{2.11}$$

Suppose that $\mathbb{P}_S \equiv \{\mathbf{s}^*\}$. Then, a^* is an equilibrium, so it satisfies $a^* \in \mathbb{P}_{\mathcal{A}}$ which in turn implies that $\mathbb{P}_{\mathcal{A}} \neq \emptyset$. Property (2.11) gives

$$\mathbb{P}_S \equiv \{\mathbf{s}^*\} \quad \Longrightarrow \quad \tilde{Pr}(\mathbb{P}_{\mathcal{A}}) \equiv \{a^*\}$$

and by the definition of $\mathbb{P}_{\mathcal{A}}$,

$$\mathbb{P}_{\mathcal{A}} \equiv \tilde{Pr}(\mathbb{P}_{\mathcal{A}})\,.$$

For the proof in the opposite sense, analogously we get:

$$\bar{A}(\mathbb{P}_S) \equiv \mathbb{P}_{\mathcal{A}} \tag{2.12}$$

Since $\mathbf{s}^*$ is a Nash Equilibrium, $\mathbb{P}_S \neq \emptyset$ and then from (2.12) we get:

$$\mathbb{P}_{\mathcal{A}} = \{a^*\} \quad \Longrightarrow \quad Pr(\mathbb{P}_S) \equiv \left\{ \mathbf{s} \in S^I \; : \; \begin{array}{l} \mathbf{s} \text{ is a measur-} \\ \text{able selection of} \\ i \rightrightarrows B(i, a^*) \end{array} \right\}$$

And from the definition of $\mathbb{P}_S$,

$$\mathbb{P}_S \equiv Pr(\mathbb{P}_S)$$

■

Proposition 2.17 shows that it is possible to study Eductive Stability of models with continuum of agents that fit our framework using the set of states as well as the set of strategies. Moreover, it can be even desirable to use the former approach since (local) uniqueness and stability are more pertinent in terms of the state of the system rather than in terms of strategy profiles, as is discussed in the previous section. For instance, the study of Strategic Complementarities in coordination games or Strategic Substituability in general models as well, can be undertaken by looking at states of the system rather than strategy profiles (see the books by Cooper (1999) and Chamley (2004). See as well Guesnerie (2005) and Guesnerie and Jara-Moroni (2007)).

4 Rationalizability vs Point-Rationalizability

Rationalizability differs from Point-Rationalizability on the way we address forecasts. For Rationalizability, forecasts are no longer points in the corresponding sets but probability distributions whose supports are contained in these sets. Then, when a player has a subjective probability forecast over what may occur with the rest of the economic system, he maximizes his expected utility with respect to such a probability distribution to make a decision. Rationality implies that players should not give positive weight in their forecasts to strategies that are not best response to some rationally generated forecast.

Rationalizable Strategy Profiles, for instance, should be obtained from a similar exercise as done in Subsection 3.1, but considering forecasts as probability measures over the set of strategies of the opponents. Loosely speaking, each player should consider a profile of probability measures (his subjective forecasts over each of his opponents' play) and maximize some expected utility, expectation taken over an induced probability measure over the set of strategy profiles.

A difficulty in a context with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the player's name. There is no straightforward solution in any case. However, in our framework it is possible to bypass this difficulty. Using the intuition just described for the strategy profiles case, we will reformulate the processes of elimination of strategies and states described by equations (2.3) and (2.6) by considering procedures where players eliminate strategies that are not best response to any possible (subjective probability) forecast (profile) over a given set of states or strategy profiles.

We present first, in the next Subsection, the concept of Rationalizable States, where forecasts and the process of elimination are taken over the set of states $\mathcal{A}$. Then, in Subsection 4.4 we will make use of Schmeidler's original framework of games with continuum of players, to formalize the idea of Rationalizable Strategy Profiles in that context.

4.1 Forecasts over the set of states

Before we enter into context we need to introduce some concepts and some notation. for a Borel set $X \subseteq \mathbb{R}^n$ we denote by $\mathcal{P}(X)$ the set of probability measures on the Borel subsets of X. Equivalently this is the set of probability measures on the Borel sets of $\mathbb{R}^n$ whose support is in X. We will endow the set $\mathcal{P}(X)$ with the weak* topology $w^* = \sigma(\mathcal{P}(X), C_b(X))$ [21]. For a Borel subset Z of X in $\mathbb{R}^n$, $\mathcal{P}(Z)$ can be considered to be a subset of $\mathcal{P}(X)$ and the weak* topology on $\mathcal{P}(Z)$ is the relativization of the weak* topology on $\mathcal{P}(X)$ to $\mathcal{P}(Z)$. The set X can be topologically identified with a subspace of $\mathcal{P}(X)$ by the embedding $x \mapsto \delta_x$. An important property that we will use is that X is compact if and only if $\mathcal{P}(X)$ is compact (and metrizable, since we use the norm in $\mathbb{R}^n$) [22].

As we said before, in the setting of Rath there is a simple way to get through the inconvenience of defining an induced probability measure over the set of strategy profiles, using the presence of the state variable of the game over which players have an infinitesimal influence.

We consider then games with an aggregate state. In this setting, we again consider players as having forecasts over the set of states rather than over each of the individual strategy sets. That is, forecasts are probability measures over the set of states rather than profiles of probabilities over the set of strategies. We define then for each player the set valued mapping that gives the actions that maximize expected utility given a probability measure μ over the set of states $\mathcal{A}$, $\mu \in \mathcal{P}(\mathcal{A})$, $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$:

$$\begin{aligned} \mathbb{B}(i, \mu) : &= \operatorname{argmax}_{y \in S} \mathbb{E}_\mu \left[u(i, y, a) \right] \\ &\equiv \operatorname{argmax}_{y \in S} \int_{\mathcal{A}} u(i, y, a) \, \mathrm{d}\mu(a) \, . \end{aligned}$$

As it has been along all this document, we can describe then, using $\mathbb{B}(i, \cdot)$, the process of elimination of unreasonable states, considering that players could now use probability forecasts over the set of states. If it is common knowledge that the actual state is restricted to a subset $X \subseteq \mathcal{A}$ then players will use strategies only in the set $\mathbb{B}(i, \mathcal{P}(X)) := \cup_{\mu \in \mathcal{P}(X)} \mathbb{B}(i, \mu)$. This is, each player $i \in I$ will behave optimally with respect to some subjective belief about the outcome of the game, whose support is contained in X. This means that rational strategy profiles have to be selections of the correspondence $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$ that maps the set of players on their set of optimal responses with respect to any possible forecast over X. The state of the game will then be the integral of one of these selections. This process is described with the mapping $\tilde{R} : \mathcal{B}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$:

$$\begin{aligned} \tilde{R}(X) :&= \left\{ \int_I s(i) \, \mathrm{di} : \begin{array}{l} \mathbf{s} \in S^I, \mathbf{s} \text{ is a measurable selection} \\ \text{of } i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X)) \end{array} \right\} . \\ &\equiv \int_I \mathbb{B}(i, \mathcal{P}(X)) \, \mathrm{di} \end{aligned}$$

The set $\tilde{R}(X)$ gives the set of states that are obtained as consequence of optimal behavior

[21] Recall that $C_b(X)$ is the space of real valued bounded continuous functions on X.
[22] See Aliprantis and Border (1999) for detailed treatments of these and other results.

when common knowledge about the outcome of the game is represented by X. As we said before, the difference between $\tilde{Pr}$ and $\tilde{R}$ is that the second process considers expected utility maximizers. For a given Borel set $X \subseteq \mathcal{A}$, X can be embedded into $\mathcal{P}(X)$. This means that $B(i, X) \subseteq \mathbb{B}(i, \mathcal{P}(X))$ and so we have that $\tilde{Pr}(X) \subseteq \tilde{R}(X)$.

Proposition 2.18. *In a game $\boldsymbol{u}$, if $X \subseteq \mathcal{A}$ is nonempty and closed, then $\tilde{R}(X)$ is nonempty, convex and closed.*

For the proof we will make use of the following Lemma:

Lemma 2.19. *Let Y and X be compact subsets of $\mathbb{R}^n$. Given a function $u \in C_{\mathrm{b}}(Y \times X)$, the function $U : Y \times \mathcal{P}(X) \to \mathbb{R}$, which to each $(y, \mu) \in Y \times \mathcal{P}(X)$ associates the expectation*

$$U(y, \mu) \equiv \int_X u(y, x)\, \mathrm{d}\mu(x)\,,$$

is continuous when $\mathcal{P}(X)$ is endowed with the weak topology.*

Proof.

We write U as the composition of two functions:

$$\begin{aligned} (y, \mu) \in Y \times \mathcal{P}(X) \quad & \to (u(y, \cdot\,), \mu) \in C_{\mathrm{b}}(X) \times \mathcal{P}(X) \\ \text{and } (f, \mu) \in C_{\mathrm{b}}(X) \times \mathcal{P}(X) \quad & \to \int_X f(x)\, \mathrm{d}\mu(x) \end{aligned}$$

If we endow $C_{\mathrm{b}}(X)$ with the sup norm topology and $\mathcal{P}(X)$ with the weak* topology, from Corollary 15.7 in Aliprantis and Border (1999) we get that $(f, \mu) \to \int f \mathrm{d}\mu$ is continuous on $C_{\mathrm{b}}(X) \times \mathcal{P}(X)$.

Therefore, the result will follow from the continuity of the function

$$(y, \mu) \to \mathcal{F}(y, \mu) = (u(y, \cdot\,), \mu)\,.$$

Note first that this function is defined component to component by functions that depend each only on one variable, this is $\mathcal{F}(y, \mu) = (\mathcal{F}_1(y)\,, \mathcal{F}_2(\mu))$, and second that $\mathcal{F}_2$ is the identity. Thus, we only need to prove that $\mathcal{F}_1 : Y \to C_{\mathrm{b}}(X)$ is continuous for the sup norm topology in $C_{\mathrm{b}}(X)$.

Let $y^\nu \to y$ and take $\varepsilon > 0$.

Since $Y \times X$ is compact and u is in $C_{\mathrm{b}}(Y \times X)$, this function is as well uniformly continuous. Thus, $\exists\ \delta > 0$ (that depends only on ε) such that

$$\|\ (y,x) - (y',x')\| < \delta \quad \Longrightarrow \quad |u(y,x) - u(y',x')| < \frac{\varepsilon}{3}$$

Since X is compact $\exists\ \{x_1,\ldots,x_N\} \subset X$ such that $X \subseteq \cup_{i=1}^{N} \mathrm{B}(x_i,\delta)$. This is, for any $x \in X$ there exists x_i in the previous set such that $x \in \mathrm{B}(x_i,\delta)$.

Finally, since $y^\nu \to y$, there exist for each x_i numbers $\bar{\nu}_i$ such that

$$\|\ (y^\nu, x_i) - (y, x_i)\| < \delta$$

for all $\nu \geq \bar{\nu}_i$.

All together gives, for $\nu \geq \max\{\bar{\nu}_i\ :\ i \in \{1,\ldots,N\}\}$ and $x \in X$:

$$\begin{aligned} |u(y^\nu,x) - u(y,x)| &< |u(y^\nu,x) - u(y^\nu,x_i)| + |u(y^\nu,x_i) - u(y,x_i)| \\ &\qquad + |u(y,x_i) - u(y,x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

We conclude that $u(y^\nu,\cdot)$ converges to $u(y,\cdot)$ for the sup norm topology, which completes the proof.

∎

Proof of Proposition 2.18.

From our assumptions, $\mathbf{u}(i)$ belongs to $C_{\mathrm{b}}(S \times \mathcal{A})$, and S and $\mathcal{A}$ are compact sets in $\mathbb{R}^n$, so Lemma 2.19, along with Berges' Theorem, imply that for each $i \in I$ the correspondence $\mathbb{B}(i,\cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ is upper semi continuous. Since X is closed, it is compact, which gives that $\mathcal{P}(X)$ is a compact subset of $\mathcal{P}(\mathcal{A})$ and so the set $\mathbb{B}(i,\mathcal{P}(X))$ is closed for every $i \in I$. From Theorem 4 in Aumann (1965) we get that $\int_I \mathbb{B}(i,\mathcal{P}(X))\,\mathrm{d}i$ is closed.

On the other hand, Lemma 2.6 states that the correspondence $i \rightrightarrows B(i,X)$ is measurable, which means that it has a measurable selection $\mathbf{s}$. Since $B(i,X) \subseteq \mathbb{B}(i,\mathcal{P}(X))$, $\mathbf{s}$ is also a selection of $i \rightrightarrows \mathbb{B}(i,\mathcal{P}(X))$. This implies that $\int_I \mathbb{B}(i,\mathcal{P}(X))\,\mathrm{d}i$ is nonempty.

Convexity comes from the fact that $\tilde{R}(X)$ is obtained as an integral of a set valued mapping.

∎

The previous Proposition allows us to define the Eductive Process in this case. As usual then we consider the iterative elimination of non generated states, but now allowing for probability forecasts of players. The iterative process begins with the whole set of outcomes, in this case $\mathcal{A}$.

$$\tilde{R}^0(\mathcal{A}) := \mathcal{A}$$

Then, on each iteration, the states that are not reached by the process $\tilde{R}$ are eliminated:

$$\tilde{R}^{t+1}(\mathcal{A}) := \tilde{R}\Big(\tilde{R}^t(\mathcal{A})\Big) .$$

Recall that since we start the process at $\mathcal{A}$, what we get is a nested family of sets that, following Proposition 2.18, are nonempty convex and closed. The Eductive Process gives then the set,

$$\mathbb{R}'_{\mathcal{A}} := \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A}) .$$

Theorem 2.20. *In a game* $\boldsymbol{u}$*, the set* $\mathbb{R}'_{\mathcal{A}}$ *is non empty, convex and closed.*

Proof.

$\mathbb{R}'_{\mathcal{A}}$ is the intersections of closed convex sets, so it is convex and closed. Theorem 2.4 assures that $\mathbb{R}'_{\mathcal{A}}$ is nonempty, since equilibria belong to every set $\tilde{R}^t(\mathcal{A})$.

■

As it was the case before, the assumptions of rationality and common knowledge of rationality imply that players must take into account that all their opponents construct their subjective forecasts rationally. This is formalized by asking that the set of Rationalizable States must be a subset of $\mathbb{R}'_{\mathcal{A}}$ (analogously to (2.7)), in the sense that states that are eliminated can not rise with positive probability and hence are not rationalizable. On the other hand, if a state is rationalizable then it must be an outcome associated to optimal reactions to forecasts with support in the set of Rationalizable States, this means that the set of Rationalizable States must satisfy an analogous condition to (2.8). This is, the set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$ must satisfy

$$\mathbb{R}_{\mathcal{A}} \subseteq \tilde{R}(\mathbb{R}_{\mathcal{A}}) . \tag{2.13}$$

Note that if a set satisfies condition (2.13), then it is a subset of the set $\mathbb{R}'_{\mathcal{A}}$.

Definition 2.21. The set of Rationalizable States is the maximal subset $X \subseteq \mathcal{A}$ that satisfies:

$$X \subseteq \tilde{R}(X)$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

As we have already said, from the definition of the set of Rationalizable States, we have $\mathbb{R}_{\mathcal{A}} \subseteq \mathbb{R}'_{\mathcal{A}}$. The following Theorem, analogous of Theorem 2.10, shows that these two sets are actually the same.

Theorem 2.22. *The set of Rationalizable States of a game* $\boldsymbol{u}$ *can be calculated as*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

Proof.

The proof follows the proof of Theorem 2.10. There are some aspects in which attention must be put.

We will show that:

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \mathbb{R}'_{\mathcal{A}}$$

Theorem 2.20 assures that $\tilde{R}(\mathbb{R}'_{\mathcal{A}})$ is correctly defined.

Suppose that $a \in \tilde{R}(\mathbb{R}'_{\mathcal{A}})$. By the definition of $\tilde{R}$, there exists a measurable selection $\mathbf{s} : I \to S$ of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, such that $a = \int_I \mathbf{s}$. Since $\mathbb{R}'_{\mathcal{A}}$ is a Borel set and $\mathbb{R}'_{\mathcal{A}} \subseteq \tilde{R}^t(\mathcal{A})$, which are as well Borel sets $\forall\, t \geq 0$, we have that $\mathcal{P}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)$ are well defined and, $\forall$ $t \geq 0$, $\forall\, i \in I$, $\mathbb{B}(i, \mathcal{P}(\mathbb{P}'_{\mathcal{A}})) \subseteq \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)\Big)$. So $\mathbf{s}$ is a selection of $i \rightrightarrows \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^t(\mathcal{A})\Big)\Big)$ and then $a \in \tilde{R}^{t+1}(\mathcal{A})$ $\forall\, t \geq 0$, which means that $a \in \mathbb{R}'_{\mathcal{A}}$. This proves that $\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \subseteq \mathbb{R}'_{\mathcal{A}}$.

For the other inclusion, we consider again a sequence of set valued mappings $F^t : I \rightrightarrows S$, $t \geq 0$, whose p-lim sup limit will be again very helpful. Consider then $\forall\, i \in I$,

$$\begin{aligned} F^0(i) &:= S \\ F^t(i) &:= \mathbb{B}\Big(i, \mathcal{P}\Big(\tilde{R}^{t-1}(\mathcal{A})\Big)\Big) \quad t \geq 1 \end{aligned}$$

We have now $\forall\, t \geq 0$,

$$\tilde{R}^t(\mathcal{A}) \equiv \int_I F^t(i) \;\, \mathrm{di}.$$

We know that $\mathcal{A} \equiv \int_I F^0$ is non empty and compact. From Proposition 2.18 we get that so are the sets $\tilde{R}^t(\mathcal{A})$ for all $t \geq 1$.

From Lemma 2.19, we get that $\forall\, i \in I$ the mapping $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ is u.s.c. and, as a consequence, the set $\mathbb{B}(i, \mathcal{P}(X))$ is compact for any compact subset $X \subseteq \mathcal{A}$. So the correspondences F^t are compact valued. From the proof of Theorem 2.10 we get that they all have a measurable selection, since for a closed set X, $B(i,X) \subseteq \mathbb{B}(i,\mathcal{P}(X))$ and from Lemma 2.6 the mappings $i \rightrightarrows B(i,X)$ are measurable and hence have a measurable selection.

Consider then the set valued mapping $F : I \rightrightarrows S$ defined as the point-wise lim sup of the sequence F^t:

$$F(i) := \left(\text{p-}\limsup_t F^t\right)(i) \equiv \limsup_t F^t(i)\,.$$

So now let us take a point $a \in \mathbb{R}'_{\mathcal{A}}$. That is, $a \in \int_I F^t$ for all $t \geq 0$. This gives a sequence of measurable selections $\{\mathbf{s}^t\}_{t\in\mathbb{N}}$, such that $a = \int_I \mathbf{s}^t$. From the Lemma proved in Aumann (1976) we get that $a \in \int_I F$, since for each $i \in I$ the cluster points of $\{s^t(i)\}_{t\in\mathbb{N}}$ belong to $F(i)$ and a is the trivial limit of the constant sequence $\int \mathbf{s}^t$.

To show that $F(i) \subseteq \mathbb{B}(i,\mathcal{P}(\mathbb{R}'_{\mathcal{A}}))$, we use that the weak* topology in $\mathcal{P}(\mathcal{A})$ is metrizable and the upper semi continuity of $\mathbb{B}(i,\,\cdot\,) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ to give an argument that follows the one at the end of the proof of Theorem 2.10.

■

We get directly that,

Proposition 2.23. *In a game* $\boldsymbol{u}$ *we have:*

$$\mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

In Proposition 2.23 it is not possible to obtain the equality in a general context. We give a sufficient condition in our setting to have it.

Proposition 2.24. *If in a game* $\boldsymbol{u}$*, we have* $\forall\ \mu \in \mathcal{P}(\mathcal{A})$*:*

$$\mathbb{E}_\mu\left[u(i,y,a)\right] \equiv u(i,y,\mathbb{E}_\mu\left[a\right])$$

then

$$\mathbb{P}_{\mathcal{A}} \equiv \mathbb{R}_{\mathcal{A}}$$

Proposition 2.24 says that if the utility functions are affine in the state variable, then we have that the Point-Rationalizable States set is equal to the set of Rationalizable States. Below we will see that we do have a general setting where the set of Rationalizable Strategies is well defined and in which we can get a result of equivalence between Point and standard Rationalizability in the strategy sets, improving the statement of Proposition 2.23. We address this issue in Subsection 4.4.

Proof.

If $\mathbb{E}_\mu[u(i,y,a)] \equiv u(i,y,\mathbb{E}_\mu[a])$ then

$$\mathbb{B}(i,\mu) \equiv B(i,\mathbb{E}_\mu[a]),$$

which implies that

$$\mathbb{B}(i,\mathcal{P}(X)) \equiv \bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu[a]).$$

For a convex set $X \subseteq \mathcal{A}$ we have $\mathbb{E}_\mu[a] \in X$, $\forall\ \mu \in \mathcal{P}(X)$.

This implies that under the hypothesis of the Proposition if X is convex,

$$\bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu[a]) \subseteq \bigcup_{a\in X} B(i,a)$$

and consequently

$$B(i,X) \subseteq \mathbb{B}(i,\mathcal{P}(X)) \equiv \bigcup_{\mu\in\mathcal{P}(X)} B(i,\mathbb{E}_\mu[a]) \subseteq \bigcup_{a\in X} B(i,a) \equiv B(i,X).$$

This is, $\mathbb{B}(i,\mathcal{P}(X)) \equiv B(i,X)$ and so $\tilde{R}(X) \equiv \tilde{Pr}(X)$.

Noting that $\mathcal{A}$ is convex, we get $\tilde{R}(\mathcal{A}) \equiv \tilde{Pr}(\mathcal{A})$ which are as well convex. By induction we get that $\tilde{Pr}^t(\mathcal{A}) \equiv \tilde{R}^t(\mathcal{A})\ \forall\ t$ which gives (using the previous notation)

$$\mathbb{R}'_{\mathcal{A}} \equiv \bigcap_{t=0}^{\infty} \tilde{R}^t(\mathcal{A}) \equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^t(\mathcal{A}) \equiv \mathbb{P}'_{\mathcal{A}} \tag{2.14}$$

where these intersections give closed convex sets.

Finally we get,

$$\tilde{R}(\mathbb{R}'_{\mathcal{A}}) \equiv \tilde{R}(\mathbb{P}'_{\mathcal{A}}) \equiv \tilde{Pr}(\mathbb{P}'_{\mathcal{A}}) \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}}$$

which implies that $\mathbb{R}'_{\mathcal{A}} \equiv \mathbb{R}_{\mathcal{A}}$. The first inequality comes from (2.14), the second one is true because $\mathbb{P}'_{\mathcal{A}}$ is convex, the third one comes from Theorem 2.10 which states that $\mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$ and the last one comes again from (2.14).

We conclude that $\mathbb{R}_{\mathcal{A}} \equiv \mathbb{R}'_{\mathcal{A}} \equiv \mathbb{P}'_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}}$.

■

4.2 More of Example 2.3

Let us now illustrate our results and definitions with the example that motivates this presentation. In this example the strategy set was $\mathbb{R}_+$. Without loss of generality we can assume it to be a compact interval $S \equiv [0, q_{\max}]$, where $q_{\max}$ could be the quantity that makes the price equal to $0 : \; q_{\max} := \inf \{P^{-1}(0)\}$.

Now we identify the state set. As we have already said, we could choose the state set to be the set of aggregate production quantities or the set of prices. This depends on the aggregation operator that we are considering.

- Let us consider first A to be the operator that gives aggregate production quantities. This is, $A : S^I \to \mathbb{R}_+$

  $$A(\mathbf{s}) \equiv \int_I s(i)\, \mathrm{di}.$$

  The set $\mathcal{A}$ is the interval

  $$\begin{aligned} \mathcal{A} &\equiv \left\{ q \in \mathbb{R} \; : \; \exists\, \mathbf{s} \in S^I, \; q = \int_I s(i)\, \mathrm{di} \right\} \\ &\equiv \mathrm{co}\left\{ [0, q_{\max}] \right\} \\ &\equiv [0, q_{\max}]\,. \end{aligned}$$

  The payoff function $u(i, \cdot\,, \cdot\,) : [0, q_{\max}] \times [0, q_{\max}] \to \mathbb{R}$ is then:

  $$u(i, s(i)\,, Q) \equiv P(Q)\, s(i) - c_i(s(i))\,.$$

  If we assume P and c_i to be continuous and that the measurability requirement over the function $i \to c_i(\,\cdot\,)$ is met (for instance if all the producers have the same cost function), Theorem 2.10 holds and so we can compute the Point-Rationalizable State set using the eductive procedure described by the right hand side of (2.7). We get as well the result of Corollary 2.12 and we know then that this set is a compact interval.

- Now suppose that we use a variation, as in footnote 11, of the aggregation operator. In the same setting we will consider the state set to be the set of prices. This is, $A : S^I \to \mathbb{R}_+$

  $$A(\mathbf{s}) \equiv P\left(\int_I s(i)\, \mathrm{di} \right).$$

This is not the aggregation operator for which we obtained the results. However, we will see below that they still hold. The set of states $\mathcal{A}$ will be identified with the set:

$$\begin{aligned}\mathcal{A} &\equiv \{p \in \mathbb{R} \ : \ \exists \ q \in [0, q_{\max}] \, , \ p = P(q)\} \\ &\equiv P([0, q_{\max}]) \\ &\equiv [0, p_{\max}] \, .\end{aligned}$$

We see that since P is a continuous function that goes from one-dimensional aggregate-production set $\mathbb{R}_+$ to the set of prices $[0, p_{\max}] \subset \mathbb{R}$, this set turns out to be convex.

The utility function is now $u(i, \cdot, \cdot) : [0, q_{\max}] \times [0, p_{\max}] \to \mathbb{R}$:

$$u(i, s(i) \, , p) \equiv ps(i) - c_i(s(i)) \, .$$

Continuity of P implies that Theorem 2.10 still holds and so we can compute the Point-Rationalizable State set using the eductive procedure described by the right hand side of (2.7) using A instead of the integral. Since the strategy and state sets are unidimensional[23] we get the result of Corollary 2.12 and we know then that this set is a compact interval.

Furthermore, in this second approach to the example we have that the payoff function is affine in the state variable and so Proposition 2.24 holds and we then see that what we are actually calculating is in fact the set of Rationalizable States (Rationalizable Prices).

The equivalence between Rationalizable and Point-Rationalizable sets for the case of prices is obtained directly from Proposition 2.24. The question remains on whether this holds for the first approach. Clearly the payoff function is not necessarily affine on production quantities and so we can not apply this Proposition directly. However, without making any further assumptions we do have that the set of Rationalizable and Point-Rationalizable States in the “ aggregate production quantities” approach are the same. For this, note that we have already argued that the set of Point-Rationalizable States is an interval in $\mathbb{R}$. Then, since the model presents strategic substitutes the limits of this interval are actually the largest and smallest Rationalizable States (Guesnerie and Jara-Moroni 2007). This implies that the whole interval is the set of Rationalizable States.

[23]Otherwise we would have it if we had used only the integral and so aggregate production instead of prices.

4.3 Forecasts over the set of strategies

When we consider forecasts over the sets of strategies, players should have a prior over each of the other player's individual actions. A forecasts in this case then would be a profile of probability measures with support in the set of strategies S. The question is not trivial since what we would have in this case is a continuum of random variables indexed by the set of players and it is not clear how payoff should depend on this profile of probability measures. However, in the setting of the original paper by Schmeidler (1973) this technical issue can be overtaken since in this setting payoffs depend on the profiles of probability measures that represent mixed strategies, which are the same mathematical objects as forecasts. We will give first a loose description of how the eductive process should work when agents use forecasts over the set of strategies, to continue with a formal description for the case where S is a finite pure strategy set.

Given a set of strategy profiles $H \subseteq S^I$, consider the set of strategies that a player $i \in I$ may use in strategy profiles in H and denote it $H(i)$:

$$H(i) := \{\ y \in S\ :\ y = s(i)\,, \text{ and } \mathbf{s} \in H\}\,.$$

Ideally we would like to have forecasts on this set, and use the set $\mathcal{P}(H(i))$. Since we do not know whether $H(i)$ is a Borel set, we may use for instance the closure of $H(i)$, $\text{cl}\{H(i)\}$ and consider then $\mathcal{P}(\text{cl}\{H(i)\})$.

We will say that a (measurable) mapping $\mathbf{m} : I \to \mathcal{P}(S)$ is a *forecast profile over* H if $m(i)$ has support on $\text{cl}\{H(i)\}$ λ-a.e.. The question is, how should rational forecasts be generated?

If originally any strategy can be used, when thinking about possible actions taken by their rivals players should consider any possible forecast profile in the set of actions. This should generate for each player a set of best-replies-to-forecasts. The issue is still on how players use their forecasts to generate this set, but suppose we have this. Once all the players have done this exercise, we will have a correspondence that maps players to the set of strategies that represents all the possible strategy profiles that could be played reacting optimally to some forecast (where different players could be using different forecasts, recall that this is an issue of forecasting the forecasts of the others). This correspondence would be the result of a first iteration and should be the point of depart for the second iteration. Forecasts now should be profiles of probability measures where the support of each probability should be in the closure of the set associated to each player by this correspondence.

We define the mapping of best-reply-to-forecasts as the set of strategies that maximize some "expected" utility given a forecast profile on S, $\mathbb{B}\mathrm{r}(i,\,\cdot\,) : \mathcal{P}(S)^I \rightrightarrows S$:

$$\mathbb{B}\mathrm{r}(i, \mathbf{m}) := \operatorname{argmax}_{y\in S} \mathbb{E}_{\mathbb{P}(\mathbf{m})} \left[\pi(i, y, \mathbf{s})\right].$$

We use the notation $\mathbb{E}_{\mathbb{P}(\mathbf{m})}$ to represent that the payoff that is being maximized is an expected payoff where the expectation comes from the fact that players are using non-degenerate forecasts over the set of strategies. We note $\mathbb{P}(\mathbf{m})$ to indicate that the profile $\mathbf{m}$ induces in some sense a probability measure over the set of strategy profiles. We do not give an answer here on to how this is done.

Given a measurable set valued mapping $F : I \rightrightarrows S$, we can obtain, for each agent $i \in I$, the set of best-replies to forecasts over this mapping as:

$$\mathbb{B}\mathrm{r}(i, F) := \left\{ y \in S : \begin{array}{l} y \in \mathbb{B}\mathrm{r}(i, \mathbf{m}), \mathbf{m} \text{ is a fore-} \\ \text{cast profile over } F \end{array} \right\}. \tag{2.15}$$

Finally, we can define the process of elimination of non best-reply-to-forecasts, described in the previous paragraphs, with the mapping R that takes a set valued mapping $F : I \rightrightarrows S$ and returns a subset $R(F) \subseteq S^I$,

$$R(F) := \left\{ \mathbf{s} \in S^I : \begin{array}{l} \mathbf{s} \text{ is a measur-} \\ \text{able selection of} \\ i \rightrightarrows \mathbb{B}\mathrm{r}(i, F) \end{array} \right\}. \tag{2.16}$$

The process[24] described by equation (2.16) considers that strategy profiles that are "kept" are those that can be constructed from best replies of agents taking decisions considering any of the possible forecast profiles over F. Of course, as was the case before, on a same strategy profile $\mathbf{s}$ of $R(F)$ the strategies of two different agents can be best-responses to two different forecast profiles over F.

Definition 2.25. The set of Rationalizable Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$H \subseteq R(H)$$

and we note it $\mathbb{R}_S$.

For each player, $i \in I$, there will be a set of Rationalizable Strategies, namely the union, over all the rationalizable strategy profiles in $\mathbb{R}_S$, of the best response set of the considered player. That is, the set of Rationalizable Strategies for player $i \in I$ is,

$$\mathbb{R}_S(i) := \mathbb{B}\mathrm{r}(i, \mathbb{R}_S)$$

[24]The set $R(F)$ can itself be regarded as a set valued mapping from I to S. This has to be taken in account to have the set of Rationalizable Strategies well defined.

So now that we have presented the main ideas of rationalizability in terms of strategies, let's turn to a setting where the best-reply-to-forecast mapping has a concrete sense.

4.4 Games with a continuum of players and finite strategy set

In Schmeidler's formulation of a game with a continuum of players, payoff functions $\pi(i, \cdot, \cdot)$ are defined on the product set $\Delta \times \Delta^I$, where $\left\{e^1, \cdots, e^L\right\} \subseteq \mathbb{R}^L$ is a finite set of pure strategies that we identify with the canonical base of $\mathbb{R}^L$ and $\Delta \equiv \text{co}\left\{\left\{e^1, \cdots, e^L\right\}\right\}$ is the set of mixed strategies, the convex hull of $\left\{e^1, \cdots, e^L\right\}$ and the simplex in $\mathbb{R}^L$. The functions $\pi(i, \cdot, \cdot) : \Delta \times \Delta^I \to \mathbb{R}$ take in this setting the form:

$$\pi(i, y, \mathbf{m}) := y \cdot h(i)(\mathbf{m}) \tag{2.17}$$

where $h(i) : \Delta^I \to \mathbb{R}^L$ is an auxiliary vector utility function whose coordinate l gives the utility of player i when he chooses action e^l, $\mathbf{m}$ is a (mixed) strategy profile and $y \in \Delta$ is a (mixed) strategy of player i. If payoffs of players depend on the integral of the (mixed) strategy profile $\mathbf{m}$, $\int_I \mathbf{m}$, then we can say that Schmeidler's setting is ours with $S \equiv \mathcal{A} \equiv \Delta$. In this case the functions $h(i)$ can be regarded as depending only on the values of the integrals (as in our setting)[25]. As quoted by Schmeidler himself, the central result of his paper is the existence and purification theorem, Theorem 2.26 below, where the main assumption is precisely this last one. We state this theorem in the context of our framework.

Theorem 2.26 (Schmeidler 1973)**.** *If the following conditions are satisfied:*

1. *The functions $h(i)$ depend only on the integral of the mixed strategy profile $\boldsymbol{m}$,*
2. *The functions $\hat{h}(i) : \Delta \to \mathbb{R}^L$ such that $h(i)(\boldsymbol{m}) \equiv \hat{h}(i)\left(\int \boldsymbol{m}\right)$, are continuous,*
3. *For all $\boldsymbol{m} \in \Delta^I$ and all $l, k \in \{1, \ldots, L\}$ the set*

$$\left\{i \in I \ : \ h(i)^l(\boldsymbol{m}) > h(i)^k(\boldsymbol{m})\right\}$$

is measurable,

then there exists a Pure Strategy Nash Equilibrium of the game π.

This theorem motivates that we look at Schmeidler's formulation from a slightly different point of view. As we have already said, one possibility is to consider this setting in the context

[25] See Rath (1992) for a discussion on this matter.

of a game $\mathbf{u}$ where the set of strategies, S, is $S \equiv \Delta$. The implicit properties imposed on payoff functions in this definition imply the hypothesis of Theorem 2.26 and so we know that the results stated so far are true for the set of mixed strategies. However, if we keep focusing on the set of pure strategies, we can benefit from the structure of Schmeidler's formulation to have a well defined best-reply-to-forecast mapping. The payoff functions depend on the integral of the mixed strategy profile, in particular when we consider only pure strategies they also depend on the integral of pure strategy profiles. We can make then a difference between (Point-)Rationalizability in pure or mixed strategies. At this point we have to introduce some more notation: we will represent by a subscript on the corresponding set or operator whether we are considering pure or mixed strategies [26]. In Schmeidler's formulation we have $S_p \equiv \{e^1, \cdots, e^L\}$ and $S_m \equiv \Delta$. We continue to consider the game where $S \equiv \mathcal{A} \equiv \Delta$ and so $\mathcal{A}$ is the same as the mixed strategy set Δ which in turn is equal to the set of probability measures over the set of pure strategies $\mathcal{P}(S_p)$, and so we get:

$$S \equiv S_m \equiv \mathcal{A} \equiv \Delta \equiv \mathcal{P}\big(\{e^1, \cdots, e^L\}\big) \equiv \mathcal{P}(S_p)\,.$$

If S_p is a finite pure strategy set, then any forecast in the form of a probability distribution over a subset $Y \subseteq S_p$ can be considered as a point in $\mathcal{P}(Y) \equiv \operatorname{co}\{Y\} \subseteq \Delta \equiv \operatorname{co}\{S_p\}$. So a forecast profile would be a function $\mathbf{m} : I \to \operatorname{co}\{Y\}$. Since mixed strategies are the same mathematical objects as probability forecasts over the set of actions of each player, in the current setting we get to identify the expected utility $\mathbb{E}_{\mathbb{P}(\mathbf{m})}[\pi(i, y, \mathbf{s})]$ mentioned above, with the following expression:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}(\mathbf{m})}[\pi(i, y, \mathbf{s})] &\equiv y \cdot h(i)(\mathbf{m}) \\ &\equiv u\bigg(i, y, \int \mathbf{m}\bigg)\end{aligned} \tag{2.18}$$

Where now $\mathbb{E}_{\mathbb{P}(\,\cdot\,)}[\pi(i,\,\cdot\,, \mathbf{s})] : S_p \times \Delta^I \to \mathbb{R}$ is a function that depends on the *pure* strategy $y \in S_p$ and we interpret the profile of probability distributions $\mathbf{m} \in \Delta^I \equiv \mathcal{P}(S)^I$ as a forecast, under the hypothesis that $h(i)$ depends on $\mathbf{m}$ through the integral.

In this setting then we get that the set of forecast profiles on a mapping $F : I \rightrightarrows S_p$ can be described by the mapping $\operatorname{co}\{F\} : I \rightrightarrows \Delta$ defined as $\operatorname{co}\{F\}(i) := \operatorname{co}\{F(i)\}$. With this,

[26]For instance, the pointwise eductive procedure Pr can be defined on pure strategies:

$$Pr_p(H) \equiv \left\{ \mathbf{s} \in S_p^I : \begin{array}{l} \mathbf{s} \text{ is a measurable selection of the corre-} \\ \text{spondence } i \rightrightarrows \operatorname{Br}_p(i, H) \end{array} \right\}$$

where $H \subseteq S_p^I$ and $\operatorname{Br}_p(i,\,\cdot\,) : S_p^I \rightrightarrows S_p$ is defined by:

$$\operatorname{Br}_p(i, \mathbf{s}) \equiv \operatorname{argmax}\{\pi(i, y, \mathbf{s}) \;:\; y \in S_p\}$$

we can now consider six different rationalizable sets:

1. The set of Point-Rationalizable Pure Strategies $\mathbb{P}_{S_p}$
2. The set of Point-Rationalizable Mixed Strategies $\mathbb{P}_{S_m}$
3. The set of Rationalizable Pure Strategies $\mathbb{R}_{S_p}$
4. The set of Rationalizable Mixed Strategies $\mathbb{R}_{S_m}$
5. The set of Point-Rationalizable States $\mathbb{P}_{\mathcal{A}}$
6. The set of Rationalizable States $\mathbb{R}_{\mathcal{A}}$

Where the first three have been proved to be well defined and can be obtained from the eductive processes defined by optimal strategies and forecasts on the corresponding sets. The last two have been discussed with more detail in Subsections 3.2 and 4.1.

Theorem 2.27. *In a game $\boldsymbol{u}$ where $S \equiv \Delta$ we have,*

$$\forall\ t, \quad Pr_p^t\big(S_p^I\big) \equiv R_p^t\big(S_p^I\big)\,.$$

Proof.

From the relation in (2.18) we can see that to look at all forecast profiles is equivalent to look at all the integrals of such profiles. Moreover, we see that in equation (2.15), and considering F to be the constant mapping $F(i) \equiv S_p$, the set of forecast profiles over F is the set valued mapping $\mathrm{co}\,\{F\}$ as defined above. That is, to obtain $\mathrm{Br}_p\big(i, S_p^I\big)$ we are interested in the integral of F while the calculus of $\mathbb{B}\mathrm{r}_p(i,F)$ considers the integral of $\mathrm{co}\,\{F\}$. From Aumann (1965) the integral of the convex hull mapping is equal to the integral of the mapping itself and so we have:

$$\int_I F(i)\,\mathrm{di} \equiv \int_I \mathrm{co}\,\{F\}(i)\,\mathrm{di}.$$

So no matter whether we are considering point or standard forecasts we obtain the same set of states. In consequence we get that the set of maximizers is the same:

$$\mathrm{Br}_p\big(i, S_p^I\big) \equiv \mathbb{B}\mathrm{r}_p(i,F)$$

Thus we take measurable selections from the same mapping and so,

$$Pr_p\big(S_p^I\big) \equiv R_p\big(S_p^I\big)\,.$$

By induction over t we get that

$$Pr_p^t\big(S_p^I\big) \equiv R_p^t\big(S_p^I\big)\,.$$

■

Corollary 2.28. *In a game* $\boldsymbol{u}$ *where* $S \equiv \Delta$ *we have,*

$$\mathbb{R}_{S_p} \equiv \mathbb{P}_{S_p}$$

Corollary 2.28 says that in the setting where payoff functions depend on the integral of the mixed strategy profile, and we consider S to be the set of mixed strategies associated to a finite pure strategy set, then we get that Point-Rationalizability is equivalent to Rationalizability in terms of pure strategy profiles.

In this context we identify the set $\mathbb{P}_S$ with $\mathbb{P}_{S_m}$ and so from Theorem 2.14 we know that $\mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_{S_m})$. What can we say about the relation between $\mathbb{P}_{\mathcal{A}}$ and the (Point-)Rationalizable sets in pure strategies? An answer is given in Corollary 2.29.

Corollary 2.29. *In a game* $\boldsymbol{u}$ *where* $S \equiv \Delta$ *we have,*

$$\text{(i)}\quad \mathbb{P}_{\mathcal{A}} \equiv \bar{A}(\mathbb{P}_{S_m}) \quad \text{and} \quad \mathbb{P}_{S_m} \equiv \bar{B}(\mathbb{P}_{\mathcal{A}}) \equiv \left\{ \boldsymbol{m} \in S_m^I : \begin{array}{l} \boldsymbol{m} \text{ is a measurable selection of} \\ i \rightrightarrows B_m(i, \mathbb{P}_{\mathcal{A}}) \end{array} \right\};$$

$$\text{(ii)}\quad \mathbb{P}_{\mathcal{A}} \equiv \bar{A}\big(\mathbb{P}_{S_p}\big) \quad \text{and} \quad \mathbb{P}_{S_p} \equiv \left\{ \boldsymbol{s} \in S_p^I : \begin{array}{l} \boldsymbol{s} \text{ is a measurable selection of} \\ i \rightrightarrows B_p(i, \mathbb{P}_{\mathcal{A}}) \end{array} \right\}.$$

Proof.

Item (i) is the exact same result of Theorem 2.14.

Now note that the best response mappings $B_p(i,\cdot\,) : \mathcal{A} \rightrightarrows S_p$ and $B_m(i,\cdot\,) : \mathcal{A} \rightrightarrows S_m$ satisfy for $X \subseteq \mathcal{A}$:

$$\cup_{a\in X} B_p(i,a) \subseteq \cup_{a\in X} B_m(i,a) \equiv \cup_{a\in X} \operatorname{co}\{B_p(i,a)\} \subseteq \operatorname{co}\{\cup_{a\in X} B_p(i,a)\}\,,$$

and so we get

$$\tilde{Pr}_p(X) \equiv \int_I B_p(i,X)\,\mathrm{di} \subseteq \int_I B_m(i,X)\,\mathrm{di} \subseteq \int_I \operatorname{co}\{B_p(i,X)\}\,\mathrm{di} \equiv \tilde{Pr}_p(X)\,,$$

where the integral in the middle is $\tilde{Pr}_m(X)$. Point (ii) of the Corollary is then consequence of Theorem 2.14.

■

This is, in a game $\mathbf{u}$ with $S \equiv \Delta \equiv S_m$ we have that the set of Rationalizable Pure Strategies is equal to the set of Point-Rationalizable Pure Strategies, and these sets are paired with the set of Point-Rationalizable States which in turn is paired with the set of Point-Rationalizable Mixed Strategies.

Finally let's note that the hypothesis of Theorem 2.26 are implied by the assumptions on $\mathbf{u}$ when we consider the set S to be the set of mixed strategies of a finite strategy set game. Moreover, since we want to deal with rationalizability in terms of pure strategies, it is not enough to identify S with the finite set of pure strategies, since in that case we would not be asking that the utility functions depended on the mixed strategy profiles through their integral which is crucial for our results.

5 Comments and Conclusions

In this work we have formally introduced the concept of Rationalizability for models that use a continuum of agents. We have proposed a definition for Point-Rationalizable Strategies in the context of general games with a continuum of players, considering the original characterization for games with finite set of players, compact strategy sets and continuous utility functions; as the maximal subset of the strategy profiles set that satisfies being a fixed point of the process of elimination of non-best response strategies. When such models have the particularity that payoffs depend on other players' actions through an aggregate variable that cannot be unilaterally affected, we have defined as well the set of Point-Rationalizable States. This last setting is an important generalization of several models that explore Rational Expectations in economics such as models of currency attacks, stag hunts, standard markets, macroeconomic dynamics and global games.

We have given sufficient conditions that allow the (Point-)Rationalizable sets to be well defined and characterized. As in the case of finite player games, continuity properties of the payoff functions are crucial to assure the convergence of the process of elimination of non-best replies (the *eductive process*) to the rationalizable set. For the continuum of players case, an additional measurability assumption must be made on the mapping that associates players to their payoff functions to be able to have existence of equilibrium. It turns out that this same assumption is sufficient to assure the integrability of the set valued mapping that is used in the eductive process and, in consequence, to obtain the constructive characterization of the different rationalizbale sets introduced throughout the document.

The set of Point-Rationalizable Strategies is paired with the set of Point-Rationalizable States. We have shown that the set of Point-rationalizable States can be obtained, as in the case of finite player games with (Point-)Rationalizable Strategies, by eliminating unreasonable states. Moreover, this set is non-empty, convex and compact.

We have seen as well that for the most important application of Rationalizability in economic models, namely Strong Rationality, it is equivalent in terms of properties and more desirable in terms of tractability to use the state approach rather than the strategy profile approach.

To incorporate standard Rationalizability to our framework, we have formally defined Rationalizable States. We give a similar characterization for this set and we give a sufficient (but not at all general) condition on payoff functions, in order to have equivalence between standard and point Rationalizability.

In the particular case where the strategy sets are finite and payoff functions depend on the integral of the mixed strategy profile, we were able to formally define Rationalizable strategies and we have extended an equivalence result to Rationalizability vs. Point-Rationalizability in terms of pure strategy profiles, which in turn implies that in this setting the three concepts: Rationalizable Pure Strategies, Point-Rationalizable Pure Strategies and Point-Rationalizable States; give the same outcomes.

We have defined a key concept in a unified exploratory framework that encompasses a variety of economic models. With this, we have a general framework on which we can study general properties of equilibria such as (local) eductive stability of equilibria and applications to models with continuum of agents that feature strategic complementarities or substitutes (Cooper (1999), Chamley (2004), Guesnerie (2005), Guesnerie and Jara-Moroni (2007)).

CHAPTER 3

Strategic Complementarities vs Strategic Substitutabilities

Introductory Notes

In this Chapter we are interested on stability of equilibria. We use the framework presented in Chapter 2 to define and study the connection between two different stability concepts of rational expectations equilibria: Expectational Stability and Eductive Stability (or Strong Rationality). The main difference between these two concepts is the assumption of heterogeneity of expectations of agents for the case of Eductive Stability. This has already been done in a series of papers by Evans and Guesnerie (1993, 2003, 2005) in a more specific model with dynamic characteristics in the two most recent papers. In the present work we stick to the static situation and focus on strategic uncertainty. As Evans and Guesnerie claim, (local) Eductive Stability implies (local) Expectational Stability,[1] However, in general Eductive Stability is (strictly) more demanding than Expectational Stability. Nevertheless, Milgrom and Roberts (1990) prove that in a Supermodular (finite player) Game [2] with unique Nash Equilibrium (NE), then the latter is the unique Rationalizable Strategy profile (it would be Strongly Rational in our present terminology). In this light, we study (local) stability of equilibria of our model with a continuum of agents when endowed with the proper structure related to Strategic Complementarities (the supermodular case of Milgrom and Roberts (1990)) and Strategic Substitutabilities (the submodular case).

To introduce complementarity and substitutability we need first to give an order structure to our strategy and aggregate sets. We use some elements of lattice theory that we introduce

[1]In the Linear model, Local Eductive Stability is equivalent to Local Expectational Stability if agents are homogeneous Local Eductive Stability is (strictly) more demanding than Expectational Stability when agents are heterogeneous

[2]The definition of supermodular game is given in Subsection 5.1

before entering into context. The results in the second part of the paper (Sections 5 and further on) paper are presented under the hypothesis of these sets being the product of compact intervals of $\mathbb{R}$, however, they remain true if we consider the slightly more general case where they are complete lattices (see the definition below) and they are stated as such.

Lattices

All the following definitions and the consequences of embedding a model with these concepts can be found with more detail in Topkis (1998).

In a partially ordered set[3] $(E, \geq)$, the *interval* $[x, y]$ is the set $\{\ z \in E\ :\ x \leq z \leq y\}$ [4]. Given a nonempty subset $X \subseteq E$ an *upper bound* of X is an element $x \in E$ such that $x \geq x'$, $\forall\ x' \in X$, analogously a lower bound of X is an element $x \in E$ such that $x \leq x'$, $\forall\ x' \in X$. If $\bar{x}$ is an upper bound of X and $\bar{x} \leq x$ for any upper bound of X, then $\bar{x}$ is the *supremum* of X and we note it $\sup_E X$. The *infimum* of X, is defined analogously and we note it $\inf_E X$.

The set E is a *lattice* if for each two element subset $\{x, y\} \subseteq E$, the elements $\sup_E \{x, y\} \in E$ and $\inf_E \{x, y\} \in E$. The lattice E is a *complete lattice* if any nonempty subset $X \subseteq E$ has a greatest and smallest bound on E, that is $\sup_E X \in E$ and $\inf_E X \in E$. A subset X of E is a *sublattice* of E if for all $x, y \in X$, $\sup_E \{x, y\} \in X$ and $\inf_E \{x, y\} \in X$. The sublattice X is *subcomplete* if for any nonempty subset X' of X, $\sup_E \{X'\} \in X$ and $\inf_E \{X'\} \in X$.

For functions $f : E \to \mathbb{R}$, we say that f is *supermodular* if $\forall\ x, y \in E$,

$$f(x) + f(y) \leq f\Big(\inf_E \{x, y\}\Big) + f\Big(\sup_E \{x, y\}\Big)$$

Given two lattices S_1 and S_2, we say that a function $f : S_1 \times S_2 \to \mathbb{R}$ has increasing (decreasing) differences in it's two arguments if $\forall\ x \geq x'$ the function $f(x, y) - f(x', y)$ is increasing (decreasing) in y. This is $\forall\ y \geq y'$,

$$f(x, y) - f(x', y) \geq f(x, y') - f(x', y')$$

for increasing differences and

$$f(x, y) - f(x', y) \leq f(x, y') - f(x', y')$$

for decreasing differences.

[3]The pair $(E, \geq)$ is a partially ordered set if $\geq$ is a binary relation on E that is reflexive, transitive and antisymmetric.

[4]If $x \nleq y$ then $[x, y] = \emptyset$.

With $\geq$ we can induce a set ordering in the set of subsets of E, $\mathcal{P}(E)$, as follows: for $X, Y \subseteq E$ we say that X is greater than Y, noted $X \succeq Y$, if $\forall (x, y) \in X \times Y$, $\sup_E \{x, y\} \in X$ and $\inf_E \{x, y\} \in Y$. With this definition we are able to define the concept of increasing (decreasing) set valued mapping. We will say that a correspondence $F : E \rightrightarrows Y$ is *increasing* (*decreasing*) if $x \geq x'$ then $F(x) \succeq F(x')$ $(F(x) \preceq F(x'))$. Note that if F is single valued we obtain the usual definition of increasing (decreasing) function.

Full Paper. Current version: Guesnerie and Jara-Moroni (2007)

Expectational Coordination in a class of Economic Models : Strategic Substitutabilities versus Strategic Complementarities

Abstract

We consider an economic model that features : 1. a continuum of agents 2. an aggregate state of the world over which agents have an infinitesimal influence. We first propose a review, based on work by Jara-Moroni (2008b), of the connections between the *eductive viewpoint* that puts emphasis on *Strongly Rational Expectations equilibrium* and the standard game-theoretical rationalizability concepts. We explore the scope and limits of this connection depending on whether standard rationalizability versus point-rationalizability, or the local versus the global viewpoint, are concerned. In particular, we define and characterize the set of *Point-Rationalizable States* and prove its convexity. Also, we clarify the role of the heterogeneity of beliefs in general contexts of expectational coordination (see Evans and Guesnerie (2005)). Then, as in the case of strategic complementarities the study of some *best response* mapping is a key to the analysis, in the case of *unambiguous* strategic substitutabilities the study of some second iterate, and of the corresponding two-period cycles, allows to describe the point-rationalizable states. We provide application in microeconomic and macroeconomic contexts.

1 Introduction

Our purpose in this paper is twofold.

First, we attempt to bring in a similar light, the standard game theoretical viewpoint of coordination on rationalizable solutions and the related viewpoint adopted in the study of expectational coordination in economic contexts, as for example in Guesnerie (1992, 2002), Evans and Guesnerie (1993, 2003, 2005). In this work, as well as in most related work on expectational coordination in economic contexts, (Morris and Shin (1998), Chamley (1999, 2004)) as well as in the theory of crisis, economic agents are *non-atomic*, in the sense that they are too small to have a significant influence on the economic system, and the *eductive* reasoning that governs the evaluation of expectational stability refers to game-theoretical rationalizability ideas. Our aim of linking the "economic" and the "game-theoretical" views brings us to adopt the framework of a game with a continuum of agents and aggregators in the sense used by Rath (1992). Relying in particular on Jara-Moroni (2008b), we show the precise connections between the game-theoretical concepts of rationalizability, point-rationalizability and the *economic* concepts of *eductive stability*. We stress the convexity properties of the different sets of *rationalizable outcomes* that follow, in the continuum game, from Liapounov like theorems. We establish the connections between the concepts of IE-Stability, the different concepts of Strong Rationality as well as between their local counterparts that allow to select locally *eductively stable* equilibria.

Second, relying on this framework, we focus attention on two classes of economic problems. In the first one, aggregate strategic complementarities, we reassess and strengthen well known game-theoretical results concerning equilibria and rationalizable solutions. The second class of models has, on the contrary, aggregate strategic substitutabilities. All expectational properties obtained in the strategic complementarity case are shown to have counterparts here. In particular, the set of *Rationalizable states* is precisely located from *cycles of order 2* associated with the system. With differentiability assumptions we get stronger results and simple sufficient conditions assuring the existence of a global Strongly Rational Expectations or a unique rationalizable solution. Applications are given for example using the general equilibrium model of Guesnerie (2001).

The paper proceeds as follows. In Section 2, we introduce games with a continuum of players and we relate it to a class of economic models with a continuum of agents. We show how this setting may be viewed from a game theoretical point of view and we introduce the concepts of Economic equilibrium and Nash Equilibrium. In section 3 we formulate Rationalizability in this context. We introduce first the concepts of Point-Rationalizable Strategies. We present an economic version of Rationalizability introducing Point-Rationalizable States and Rationalizable States and we relate these concepts to the game theoretical ones. In section 4 we address the economic concepts of Iterative Expectational Stability and Eductive Stability using the tools defined in section 3. Then in Section 5 we successively focus attention on aggregate models with Strategic Complementarities or Strategic Substitutabilities. Our general results here are tightened when we examine the differentiable version of the model. In Section 7 we conclude.

2 The canonical model and concepts.

This Section aims at making a careful connection between the underlying game theoretical concepts and the economic application which they are solicited for. We present first a game theoretical framework that underlies the standard economic model with a continuum of agents, presented afterwards. We then introduce and compare the parallel tools used for the analysis.

2.1 Games with a continuum of players

We consider a *game with a continuum of players*. Non atomic games with continuum of players where first introduced by Schmeidler (1973). In these games the set of players is the measure space $(I, \mathcal{I}, \lambda)$, where I is the unit interval of $\mathbb{R}$, $I \equiv [0,1]$, and λ is the lebesgue measure. Each player chooses a strategy $s(i) \in S(i)$ and we take $S(i) \subseteq \mathbb{R}^n$. Strategy profiles in this setting are identified with integrable selections[5] of the set valued[6] mapping $i \rightrightarrows S(i)$. For simplicity, we will assume that all the players have the same compact strategy set $S(i) \equiv S \subset \mathbb{R}^n_+$. As a consequence, since S is compact, the set of meaningful strategy profiles is the set of measurable functions from I to S [7] noted from now on S^I.

In a game, players have payoff functions that depend on their own strategy and the complete profile of strategies of the player $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$. In our particular framework these functions depend, for each player, on his own strategy and an average of the strategies of all the other players. To obtain this average we use the integral of the strategy profile, $\int_I s(i)\,\mathrm{di}$. This implies that all the relevant information about the actions of the opponents is summarized by the values of the integrals, which are points in the set [8]

$$\mathcal{A} \equiv \int_I S(i)\,\mathrm{di}.$$

Hypothesis over the correspondence $i \rightrightarrows S(i)$ that assure that the set $\mathcal{A}$ is well defined can be found in Aumann (1965) or in Chapter 14 of Rockafellar and Wets (1998). In this case

[5] A selection is a function $\mathbf{s} : I \to \mathbb{R}^n$ such that $s(i) \in S(i)$.

[6] We use the notation $\rightrightarrows$ for set valued mappings (also referred to as correspondences), and $\to$ for functions.

[7] Equivalently, the set of measurable selections of the constant set valued mapping $i \rightrightarrows S$.

[8] Following Aumann (1965) we define for a correspondence $F : I \rightrightarrows \mathbb{R}^n$ its' integral, $\int_I F(i)\ \mathrm{di}$, as:

$$\int_I F(i)\ \mathrm{di} := \left\{ x \in \mathbb{R}^n \ : \ x = \int_I f(i)\ \mathrm{di} \text{ and } f \text{ is an integrable selection of } F \right\}$$

we get that $\mathcal{A}$ is a convex set (Aumann 1965). Moreover, since $S(i) \equiv S$ we have that [9]

$$\mathcal{A} \;\equiv\; \text{co}\,\{S\}\,. \tag{3.1}$$

Pay-offs $\pi(i,\,\cdot\,,\,\cdot\,)$ in this setting are evaluated from an auxiliary utility function $u(i,\,\cdot\,,\,\cdot\,) : S \times \text{co}\,\{S\} \to \mathbb{R}$ such that:

$$\pi(i, y, \mathbf{s}) \equiv u\Big(i, y, \int_I s(i)\ \text{di}\Big) \tag{3.2}$$

We assume:

C : *For all agent $i \in I$, $u(i,\,\cdot\,,\,\cdot\,)$ is continuous.*

HM : *The mapping that associates to each agent a utility function* [10] *is measurable.*

C is standard and does not deserve special comments. **HM** is technical but in a sense natural in this setting. Adopting both assumptions on utility functions put us in the framework of Rath (1992). We begin by giving a definition of Nash Equilibrium in this setting.

Definition 3.1. A (pure strategy) Nash Equilibrium of a game is a strategy profile $\mathbf{s}^* \in S^I$ such that:

$$\forall\, y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i)\ \text{di}\Big) \geq u\Big(i, y, \int s^*(i)\ \text{di}\Big)\,, \quad \forall\, i \in I\ \lambda\text{-a.e.} \tag{3.3}$$

It is useful to use the best reply correspondence $\text{Br}(i,\,\cdot\,) : S^I \rightrightarrows S$ defined as:

$$\text{Br}(i, \mathbf{s}) \;:=\; \text{argmax}_{y \in S}\, \pi(i, y, \mathbf{s})\,. \tag{3.4}$$

The correspondence $\text{Br}(i,\,\cdot\,)$ describes the optimal response set for player $i \in I$ facing a strategy profile $\mathbf{s}$.

In our setting, and considering the auxiliary function $u(i,\,\cdot\,,\,\cdot\,)$, we can define as well the optimal strategy correspondence $B(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S$ as the correspondence which associates to each point $a \in \mathcal{A}$ the set:

$$B(i, a) \;:=\; \text{argmax}_{y \in S}\, \{u(i, y, a)\}\,. \tag{3.5}$$

[9] Where co $\{X\}$ stands for the convex hull of a set X (see Rath (1992)).

[10] The set of functions for assumption **HM** is the set of real valued continuous functions defined on $S \times \text{co}\,\{S\}$ endowed with the sup norm topology.

Since, in this setting, $a = \int_I s(i)\ \mathrm{di}$, then $\mathrm{Br}(i, \mathbf{s}) = B(i, a)$, it follows that a Nash equilibrium is a strategy profile $\mathbf{s}^* \in S^I$ such that, $\forall\ i \in I$ λ-a.e., $s^*(i) \in \mathrm{Br}(i, \mathbf{s}^*)$, or equivalently, $s^*(i) \in B\big(i, \int s^*(i)\,\mathrm{di}\big)$ (see Proposition 3.4 below).

Under the previously mentioned hypothesis Rath shows that for every such game there exists a Nash Equilibrium.

Theorem 3.2 (Rath 1992)**.** *The above game has a (pure strategy) Nash Equilibrium.*

The proof of the Theorem is based on the Kakutani fixed point Theorem applied to what we call later the Cobweb Mapping, defined in (3.11). Indeed, a fixed point of such a correspondence determines an equilibrium of the game (see Proposition 3.4 below as well).

2.2 Economic System with a continuum of agents

We address now a class of stylized economic models in which there is a large number of small agents $i \in I$. In this economic system, there is an aggregate variable or signal that represents the *state* of the system. We call $\mathcal{A} \subseteq \mathbb{R}^K$ the set of all possible states of the economic system. Interaction of agents occurs through an aggregation operator, A, that to each strategy profile $\mathbf{s}$ associates a state of the model $a = A(\mathbf{s})$ in the set of states $\mathcal{A}$. The key feature of the system is that no agent can affect unilaterally the state of the system. That is, a change of the actions of only one, or a *small* group of agents, does not modify the value of the state of the system.

These features are those captured with the non-atomic game-theoretical framework described in the previous subsection. The so-called economic system is then naturally imbedded onto the just defined game with a continuum of players when we use as the aggregation operator A the integral[11] of the strategy profile $\mathbf{s}$:

$$A(\mathbf{s}) \equiv \int_I s(i)\,\mathrm{di}.$$

so that the state set $\mathcal{A}$ is $\mathrm{co}\,\{S\}$ (see equation 3.1 and the comments therein). This assures that $\mathcal{A}$ is a nonempty convex compact subset of $\mathbb{R}^n$ (i.e. $K = n$) (Aumann 1965).

[11]The aggregation operator can as well be the integral of the strategy profile with respect to any measure $\bar{\lambda}$ that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\bigg(\int_I s(i)\,f(i)\ \mathrm{di}\bigg)$$

where $G : \int_I S(i)\ \mathrm{d}\bar{\lambda}(i) \to \mathcal{A}$ is a continuous function and f is the density of the measure $\bar{\lambda}$ with respect to the lebesgue measure. However not all the results in this work remain true if we choose such a setting.

The variable $a \in \mathcal{A}$, that represents the state of the system, determines, along with each agents' own action, his payoff. For each agent $i \in I$ then, we use the payoff function $u(i, \cdot, \cdot) : S \times \mathcal{A} \to \mathbb{R}$ introduced in (3.2). Agents act to maximize this payoff function.

In a situation where agents act in ignorance of the actions taken by *the others* or, for what matters, of the value of the state of the system, they have to rely on forecasts. That is, their actions must be a best response to some subjective probability distribution over the space of aggregate data $\mathcal{A}$. Mathematically, actions have to be elements of the set of points that maximize expected utility, where the expectation is taken with respect to this subjective probability. We can consider then the best reply to forecasts correspondence $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ defined by:

$$\mathbb{B}(i, \mu) \quad := \quad \text{argmax}_{y \in S}\, \mathbb{E}_{\mu}\left[u(i, y, a)\right] \tag{3.6}$$

where $\mu \in \mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ is the space of probability measures over $\mathcal{A}$. Since the utility functions are continuous, problems (3.5) and (3.6) are well defined and have always a solution, so consequently the mappings $B(i, \cdot)$ and $\mathbb{B}(i, \cdot)$ take non-empty compact values for all $a \in \mathcal{A}$. Clearly $B(i, a) \equiv \mathbb{B}(i, \delta_a)$, where δ_a is the Dirac measure concentrated in a.

An equilibrium of this system is a state a^* generated by actions of the agents that are optimal reactions to this state. We denote $\Gamma(a) = \int_I B(i, a)\ \text{di}$.

Definition 3.3. An *equilibrium* is a point $a^* \in \mathcal{A}$ such that:

$$a^* \in \Gamma(a^*) \equiv \int_I B(i, a^*)\, \text{di} \equiv \int_I \mathbb{B}(i, \delta_{a^*})\, \text{di} \tag{3.7}$$

Assumptions **C** and **HM** assure that the integrals in Definition 3.3 are well defined [12]. The equilibrium conditions in (3.7) are standard description of self fulfilling forecasts. That is, in an equilibrium a^*, agents must have a self-fulfilling point forecast (Dirac measures) over a^*, i.e with the economic terminology, a perfect foresight equilibrium (see Guesnerie (1992)).

It is unsurprising that an *equilibrium* as defined in (3.7) has as a counterpart in the game-theoretical approach a *Nash Equilibrium* of the underlying game as defined in (3.3). Precisely :

Proposition 3.4. *For every (pure strategy) Nash Equilibrium $\boldsymbol{s}^*$ of the system's underlying game, there exists a unique equilibrium a^* given by $a^* := A(\boldsymbol{s}^*)$ and if a^* is an equilibrium of the system, then $\exists\ \boldsymbol{s}^* \in S^I$ that is a Nash Equilibrium of the underlying game.*

Proof. Indeed, if a^* satisfies (3.7), then there exists an integrable strategy profile $\mathbf{s}^*$ such that

[12] This is a consequence of Lemma 2.6 in Chapter 2, which is restated and proved in the appendix of this Chapter as Lemma 3.47, as in the original paper

$s^*(i) \in B(i, a^*)$ and $A(\mathbf{s}^*) = a^*$. That is $s^*(i) \in B\big(i, \int_I s^*(i) \text{ di}\big)$, or equivalently

$$\forall\, y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i) \;\text{ di}\Big) \geq u\Big(i, y, \int s^*(i) \;\text{ di}\Big)\,, \quad \forall\, i \in I \;\lambda\text{-a.e.}$$

Conversely, if

$$\forall\, y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i) \;\text{ di}\Big) \geq u\Big(i, y, \int s^*(i) \;\text{ di}\Big)\,, \quad \forall\, i \in I \;\lambda\text{-a.e.} \tag{3.8}$$

then $s^*(i) \in \text{Br}(i, \mathbf{s}) \equiv B\big(i, \int s^*(i) \;\text{ di}\big)\; \forall\, i \in I\; \lambda$-a.e.. Defining $a^* := \int_I s^*(i) \;\text{ di}$ we get that $a^* \in \int_I B(i, a^*)\,\text{di}$.

■

We will refer equivalently then, to equilibria as points $a^* \in \mathcal{A}$, representing *economic equilibria*, and $\mathbf{s}^* \in S^I$, as Nash Equilibria of the underlying game.

Theorem 3.5. *The stylized economic model has an equilibrium.*

Proof. It is the consequence of the proof of Theorem 3.2 and is related to the previous Proposition. ■

Example 3.6. Variant of Muth's (1961) model presented in Guesnerie (1992). In this example there is a group of farmers indexed by the unit interval. Farmers decide a positive production quantity $q(i)$ and get as payoff income from sales minus the cost of production: $pq(i) - C_i(q(i))$, where p is the price at which the good is sold. The price is obtained from the inverse demand (or price) function, evaluated in total aggregate production Q. We see that this model fits our framework.

We already said that the set of agents is the unit interval $I = [0, 1]$ and we endow it with the lebesgue measure. Strategies are production quantities, so strategy profiles are functions from the set of agents to the positive line $\mathbb{R}_+$ (i.e. $n = 1$), $\mathbf{q} : I \to \mathbb{R}_+$. The aggregate variable in this case is aggregate production. Agents can calculate their payoff by knowing aggregate production through the price function and deciding their production. So the aggregate state space is the positive line as well, $\mathbb{R}_+$ (i.e. $K = 1 = n$). The payoff of an agent is income from sales minus cost of production, the utility function is then $u(i, q, Q) = P(Q)\, q - C_i(q)$. Where $P : \mathbb{R}_+ \to \mathbb{R}_+$ is an inverse demand (or price) function that, given a quantity of good, gives the price at which this quantity is sold. If we suppose that P is bounded and attains the value 0 from a certain q_{max} on, then we get that the aggregate state set $\mathcal{A}$ is equal to the set of strategies $S(i) \equiv S$, and both are the interval $[0, q_{max}]$. The aggregation operator, the integral of the production profile $\mathbf{q}$, gives aggregate production $Q = \int_I q(i) \;\text{ di}$.

On this example we can make the observation that the state of he game could be chosen to be the price instead of aggregate production. This is not always the case if we want to obtain the properties stated further on in this work. However, since this example is one-dimensional, it is the case that most of the properties herein presented are passed on from the aggregate production set to the price set.

We are interested now on the plausibility of the equilibrium forecasts, or equivalently to the assessment of the strength of expectational coordination described here. Our assessment relies on the concepts of Rationalizability (Bernheim 1984; Pearce 1984) or on the derived concepts, in our economic framework, of Strong Rationality (Guesnerie 1992). In the next two sections then, we exploit the game-theoretical viewpoint to asses Rationalizability in the economic context.

3 Rationalizability and the “eductive learning viewpoint”.

3.1 Rationalizability : the game viewpoint.

Rationalizability is associated with the work of Bernheim (1984) and Pearce (1984). The set of Rationalizable Strategy profiles were defined and characterized in the context of games with a finite number of players, continuous utility functions and compact strategy spaces. It has been argued that Rationalizable strategy profiles are profiles that can not be discarded as outcomes of the game based on the premises of rationality of players, independence of decision making and common knowledge (see Tan and da Costa Werlang (1988)).

First, agents only use strategies that are best responses to their forecasts and so strategies in S that are never best response will never be used; second, agents know that other agents are rational and so know that the others will not use the strategies that are not best responses and so each agent may find that some of his remaining strategies may no longer be best responses, since each agent knows that all agents know, etc. . This process continues ad-infinitum. The set of Rationalizable solutions is such that it is a *fixed point* of the elimination process, and it is the maximal set that has such a property (Basu and Weibull 1991).

Rationalizability has been studied in games with finite number of players. In a game with a continuum of agents, the analysis has to be adapted. Following Jara-Moroni (2008b), and coming to our setting, in a game-theoretical perspective, the recursive process of elimination of non best responses, when agents have point expectations, is associated with the mapping $Pr : \mathcal{P}(S^I) \to \mathcal{P}(S^I)$ which to each subset $H \subseteq S^I$ associates the set $Pr(H)$ defined by:

$$Pr(H) := \left\{ \mathbf{s} \in S^I \ : \ \ \mathbf{s} \text{ is a measurable selection of } i \ \rightrightarrows \ \mathrm{Br}(i, H) \right\}. \tag{3.9}$$

The operator Pr represents the process under which we obtain strategy profiles that are constructed as the reactions of agents to strategy profiles contained in the set $H \subseteq S^I$. If it is known that the outcome of the game is in a subset $H \subseteq S^I$, with point expectations, the strategies of agent $i \in I$ are restricted to the set $\mathrm{Br}(i, H) \equiv \bigcup_{\mathbf{s} \in H} \mathrm{Br}(i, \mathbf{s})$ and so actual strategy profiles must be measurable selections of the set valued mapping $i \rightrightarrows \mathrm{Br}(i, H)$. It has to be kept in mind that strategies of different agents in a strategy profile in $Pr(H)$ can be the reactions to (possibly) different strategy profiles in H.

We then define :

Definition 3.7. The set of Point-Rationalizable[13] Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$H \equiv Pr(H). \tag{3.10}$$

and we note it $\mathbb{P}_S$.

Rationalizable Strategies should be obtained from a similar exercise but considering forecasts as probability measures over the set of strategies of the opponents. Loosely speaking each player should consider a profile of probability measures (his forecasts over each of his opponents play) and maximize some expected utility, expectation taken over an induced probability measure over the set of strategy profiles. A difficulty in a context with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the agent's name. There is no straightforward solution in any case. However, in our framework it is possible to bypass this difficulty. We present in the next section the concepts of Rationalizable States and Point-Rationalizable States, where forecasts and the process of elimination are now taken over the set of states $\mathcal{A}$.

3.2 Rationalizability : an "economic" viewpoint.

Before going to the rationalizability, it is useful to describe the Cobweb mapping, which we will refer to sometimes later as the Iterative Expectational process.

[13] Following Bernheim (1984) we refer as Point-Rationalizability to the case of forecasts as points in the set of strategies or states and plain Rationalizability to the case of forecasts as probability distributions over the corresponding set.

Cobweb Mapping and Equilibrium

Given the optimal strategy correspondence, $B(i, \cdot)$, defined in (3.5) we can define the *cobweb mapping*[14] $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$:

$$\Gamma(a) := \int_I B(i,a) \;\mathrm{di} \tag{3.11}$$

This correspondence represents the actual possible states of the model when all agents react to the same state $a \in \mathcal{A}$. Following Definition 3.3 we see that the equilibria of the economic system are identified with the fixed points of the cobweb mapping.

Definition 3.8. The set of *Aggregate Cobweb Tâtonnement Outcomes*, $\mathbb{C}_{\mathcal{A}}$, is defined by:

$$\mathbb{C}_{\mathcal{A}} := \bigcap_{t\geq 0} \Gamma^t(\mathcal{A})$$

where Γ^t is the tth iterate[15] of the correspondence Γ.

From the proof of Theorem 3.2 (see Rath (1992)) we get that in our framework the cobweb mapping Γ is upper semi continuous as a set valued mapping, with non-empty, compact and convex values $\Gamma(a)$.

State Rationalizability

Below we present the mathematical formulation of *Point-Rationalizable States* and *Rationalizable States*, and explore the relation between Point-Rationalizability and Rationalizability in our context. We aim at clarifying the different perspectives on equilibrium stability and the connections between the notions of local and global Strong Rationality (Section 3.2)). For the proofs of the results herein stated and a more detailed treatment the reader is referred to Jara-Moroni (2008b).

Analogously to what is done in subsection 3.1, given the optimal strategy correspondence defined in equation (3.5) we can define the process of non reachable or non generated states, considering forecasts as points in the set of states, as follows:

$$\tilde{Pr}(X) := \int_I B(i,X)\,\mathrm{di} \tag{3.12}$$

[14]The name cobweb mapping comes from the familiar cobweb tâtonnement although in this general context the process of iterations of this mapping may not necessarily have a cobweb-like graphic representation.

[15]This is:

$$\Gamma^0 := \mathcal{A} \quad \Gamma^{t+1} := \Gamma\big(\Gamma^t(\mathcal{A})\big)$$

This is, if initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$ we have that forecasts are constrained by X. Then, if expectations are restricted to point-expectations, agents deduce that the possible actions of each agent $i \in I$ are in the set $B(i, X) := \bigcup_{a \in X} B(i, a)$. Since all agents know this, each agent can only discard the strategy profiles $s \in S^I$ that are not a selection of the mappings that assign each agent to the these sets. Finally, they would conclude that the actual state outcome will be restricted to the set obtained as the integral of this set valued mapping.

Definition 3.9. The set of *Point-Rationalizable States* is the maximal subset $X \subseteq \mathcal{A}$ that satisfies the condition:

$$X \equiv \tilde{Pr}(X)$$

and we note it $\mathbb{P}_{\mathcal{A}}$.

We define similarly the set of Rationalizable States. The difference between Rationalizability and Point-Rationalizability is that in Rationalizability forecasts are no longer constrained to points in the set of outcomes. To assess Rationalizability we consider the correspondence $\mathbb{B}(i, \cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ defined in (3.6). The process of elimination of non expected-utility-maximizers is described with the mapping $\tilde{R} : \mathcal{B}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$:

$$\tilde{R}(X) := \int_I \mathbb{B}(i, \mathcal{P}(X)) \, \mathrm{di} \tag{3.13}$$

If it is common knowledge that the actual state is restricted to a borel subset $X \subseteq \mathcal{A}$, then agents will use strategies only in the set $\mathbb{B}(i, \mathcal{P}(X)) := \cup_{\mu \in \mathcal{P}(X)} \mathbb{B}(i, \mu)$ where $\mathcal{P}(X)$ stands for the set of probability measures whose support is contained in X. Forecasts of agents can not give positive weight to points that do not belong to X. Strategy profiles then will be selections of the correspondence $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$. The state of the system will be the integral of one of these selections.

Definition 3.10. The set of *Rationalizable States* is the maximal subset $X \subseteq \mathcal{A}$ that satisfies:

$$\tilde{R}(X) \equiv X \tag{3.14}$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

The difference between $\tilde{Pr}$ and $\tilde{R}$ is that the second operator considers expected utility maximizers and so for a given borel set $X \subseteq \mathcal{A}$ we have $\tilde{Pr}(X) \subseteq \tilde{R}(X)$. We get directly the result in Proposition 3.13 below.

Bypassing the game-theoretical difficulties occurring in games with a continuum of players, the states set approach provides a substitute for the Rationalizability concept.

Rationalizability : the game versus the economic viewpoint

Rationalizability in the context of the games with continuum of players that we are considering is studied in Jara-Moroni (2008b). Therein it is proved that, in our context, the set of Point-Rationalizable pure Strategies is paired with the set of Point-Rationalizable States; moreover, in the context of the original model of Schmeidler[16] these sets are also paired with the set of Rationalizable (pure) Strategies. We state the result that is pertinent to our framework.

Proposition 3.11. *We have:*

$$\begin{aligned} \mathbb{P}_S &\equiv \left\{ \boldsymbol{s} \in S^I \ : \ \boldsymbol{s} \text{ is a measurable selection of } i \ \rightrightarrows \ B(i, \mathbb{P}_{\mathcal{A}}) \right\} && (3.15) \\ \mathbb{P}_{\mathcal{A}} &\equiv \left\{ a \in \mathcal{A} \ : \ a = \int_I s(i)\,\mathrm{d}i \text{ and } \boldsymbol{s} \text{ is a measurable function in } \mathbb{P}_S \right\}. && (3.16) \end{aligned}$$

Equations (3.15) and (3.16) stress the equivalence for point-rationalizability between the state approach and the strategic approach in games with continuum of players : the sets of point-rationalizable states can be obtained from the set of point-rationalizable strategies and vice versa. In (3.15) we see that the strategy profiles in $\mathbb{P}_S$ are profiles of best responses to $\mathbb{P}_{\mathcal{A}}$. Conversely in (3.16) we get that the points in $\mathbb{P}_{\mathcal{A}}$ are obtained as integrals of the profiles in $\mathbb{P}_S$.

We will make use of Proposition 3.12 below, which provides, in the continuum of agents framework, a key technical property of the set of Point-Rationalizable States.

Proposition 3.12. *The set of Point-Rationalizable States can be computed as*

$$\mathbb{P}_{\mathcal{A}} \equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^{t}(\mathcal{A})$$

The set $\mathbb{P}_{\mathcal{A}}$, indeed obtains as the outcome of the iterative elimination of unreachable states.

[16]The functions $u(i,\cdot,\cdot)$ are defined on a **finite** strategy set S and depend on the integral of a **mixed** strategy profile.

4 Rationalizable outcomes, Equilibria and Stability

4.1 The global viewpoint.

We denote by $\mathbb{E} \subseteq \mathcal{A}$, the set of equilibria of the economic system. The inclusions below are unsurprising, in the sense that they reflect the decreasing strength of the expectational coordination hypothesis, when going from equilibria to Aggregate Cournot outcomes, then to Point-Rationalizable States, and finally to Rationalizable States.

Proposition 3.13. *We have:*

$$\mathbb{E} \subseteq \mathbb{C}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

The first inclusion is direct since fixed points of Γ are obtained as integrals of selections of the best response correspondence $i \rightrightarrows B(i, a^*)$ and so will not be eliminated during the process that characterizes the set $\mathbb{C}_{\mathcal{A}}$. We can obtain the two last inclusions of Proposition 3.13 noting that if a set satisfies $X \subseteq \tilde{Pr}(X)$ then it is contained in $\mathbb{P}_{\mathcal{A}}$ and equivalently if it satisfies $X \subseteq \tilde{R}(X)$ then it is contained in $\mathbb{R}_{\mathcal{A}}$. Then, the second inclusion is obtained from the fact that each point in $\mathbb{C}_{\mathcal{A}}$, as a singleton, satisfies $\{a^*\} \subseteq \tilde{Pr}(\{a^*\})$ and the third inclusion is true because the set $\mathbb{P}_{\mathcal{A}}$ satisfies $\mathbb{P}_{\mathcal{A}} \subseteq \tilde{R}(\mathbb{P}_{\mathcal{A}})$.

An important corollary of Proposition 3.12 is that the set of Point-Rationalizable States is convex. This is a specific and nice property of our setting with a continuum of agents.

Theorem 3.14.

The set of Point-Rationalizable States is well defined, non-empty, convex and compact.

The set of Rationalizable States is non-empty and convex.

Proof. The properties are obtained from the convexity of each of the sets that are involved in the intersection in the characterization of $\mathbb{P}_{\mathcal{A}}$ in Proposition 3.12. That is, $\mathbb{P}_{\mathcal{A}}$ is the intersection of a nested family of non-empty, compact, convex sets. Non-emptiness of $\mathbb{P}_{\mathcal{A}}$ is guaranteed by Proposition 3.12 along with Theorem 3.2, since an equilibrium would never be eliminated, and so there exists a point $a^* \in \mathcal{A}$ that belongs to every set $\tilde{Pr}^t(\mathcal{A})$. Proposition 3.13 implies the property for $\mathbb{R}_{\mathcal{A}}$, while its' convexity obtains from the definition.

■

In Evans and Guesnerie (1993), two stability concepts of Rational Expectations Equilibria are compared: Iterative Expectational Stability, based on the convergence of iterations of forecasts; and Strong Rationality, based on the uniqueness of the Rationalizable Outcomes (Guesnerie 1992) of an economic model. In what follows, we define these two concepts following the terminology of Guesnerie and Evans and Guesnerie, for our setting.

Definition 3.15. An equilibrium a^* is said to be *Globally Iterative Expectationaly Stable* if $\forall\ a^0 \in V$ any sequence $a^t \in \Gamma(a^{t-1})$ satisfies $\lim_{t\to\infty} a^t = a^*$ $(= \mathbb{E})$.

The terminology of Iterative Expectational Stability is adopted from the literature on expectational stability in dynamical systems (Evans and Guesnerie (1993, 2003, 2005)). It captures the idea that virtual coordination processes converge globally, under the implicit assumption that agents have *homogenous deterministic expectations.*

Definition 3.16. The equilibrium state a^* is (globally) *Strongly Point Rational* if

$$\mathbb{P}_{\mathcal{A}} \equiv \{a^*\} \ (= \mathbb{E}).$$

The idea is now hat virtual coordination processes converge globally, under the implicit assumption that agents have *heterogenous and deterministic expectations.*

Definition 3.17. The equilibrium state a^* is (globally) *Strongly Rational* if

$$\mathbb{R}_{\mathcal{A}} \equiv \{a^*\} \ (= \mathbb{E}).$$

Eductive coordination then obtains when agents have *heterogenous and stochastic expectations.*

Strong Rationality and Strong Point Rationality can be related to heterogeneous beliefs of agents. Both concepts refer to heterogenous forecasts of agents, (even if these agents were homogeneous (have the same utility function)). With Strong rationality, forecasts are based on stochastic expectations, when with Strong Point Rationalizability, we restrict attention to point expectations. When we turn to Iterative Expectational Stability (IE-Stability), we drop the possibility of heterogeneity of forecasts. The iterative process associated with IE-Stability is based on iterations of the cobweb mapping Γ which describe agents reactions to the same point forecast over the set of states.

It is straightforward that these concepts are increasingly demanding : Strong Rationality implies Strong Point Rationalizability that implies Iterative Expectational Stability.

We turn now to the local version of these concepts.

4.2 The local viewpoint.

We now give the local version of the above stability concepts.

Again, the definition of (local) IE-Stability (Lucas 1978; DeCanio 1979), stated below is similar to the one given in Evans and Guesnerie (1993)

Definition 3.18. An equilibrium a^* is said to be *Locally Iterative Expectationaly Stable* if there is a neighborhood $V \ni a^*$ such that $\forall\ a^0 \in V$ any sequence $a^t \in \Gamma(a^{t-1})$ satisfies $\lim_{t\to\infty} a^t = a^*$.

Definition 3.19. An equilibrium state a^* is *Locally Strongly Point Rational* if there exists a neighborhood $V \ni a^*$ such that the process governed by $\tilde{Pr}$ started at V generates a nested family that leads to a^*. This is, $\forall\ t \geq 1$,

$$\tilde{Pr}^t(V) \subset \tilde{Pr}^{t-1}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{Pr}^t(V) \equiv \{a^*\}\,.$$

Definition 3.20. An equilibrium state a^* is *Locally Strongly Rational* (Guesnerie 1992) if there exists a neighborhood $V \ni a^*$ such that the eductive process governed by $\tilde{R}$ started at V generates a nested family that leads to a^*. This is, $\forall\ t \geq 1$,

$$\tilde{R}^t(V) \subset \tilde{R}^{t-1}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{R}^t(V) \equiv \{a^*\}\,.$$

The connections between the concepts stressed in the next Proposition, are straightforward.

Proposition 3.21. *We have:*

(i) *a^* is (Locally) Strongly Rational* $\implies$ *a^* is (Locally) IE-Stable.*

(ii) *a^* is Locally Strongly Rational* $\implies$ *a^* is Locally Strongly Point Rational.*

A sufficient condition for the converse to be true is that there exist a neighborhood V of a^* such that for almost all $i \in I$, for any borel subset $X \subseteq V$:

$$\mathbb{B}(i, \mathcal{P}(X)) \subseteq \mathrm{co}\,\{B(i,X)\} \tag{3.17}$$

The proof of this and of the following Proposition are relegated to the appendix.

At a first glance the hypothesis in the second part of Proposition 3.21 appears to be very restrictive, however it involves only local properties of the agents' utility functions. It intuitively states that given a restriction on common knowledge (subsets of the set V), when agents evaluate all the possible actions to take when facing probability forecasts with support "close" to the equilibrium, these actions are somehow "not to far" or "surrounded" by the set of actions that could be taken when facing point forecasts ($\mathbb{B}(i,\mu) \subseteq \text{co}\{B(i,X)\}$ if $\text{supp}(\mu) \subseteq X$). The assumption is true in most applications and standard assumptions over utility functions imply it.

Condition (3.17) relates the individual reactions of agents facing non degenerate subjective forecasts, with their reactions when facing point (dirac) forecasts. A different approach can be overtaken when comparing the aggregate reaction of the system to common knowledge on the restriction of the possible outcomes. In this approach we are interested on the convergence of the process generated by point forecasts. If this convergence is sufficiently fast, then we say that the equilibrium is *Strictly* Locally Point Rational, and we may get that this convergence speed, drags the eductive process to the equilibrium as well.

For a positive number $\alpha > 0$ and a set $V \subseteq \mathcal{A}$ that contains a unique equilibrium a^* we will denote by V_α the set:

$$V_\alpha := \{x \in \mathcal{A} \ : \ x = \alpha(v - a^*)\,,\ v \in V\}$$

Definition 3.22. We say that an equilibrium state a^* is *Strictly Locally Point Rational* if it is Locally Strongly Point Rational and there is a number $\bar{k} < 1$ such that, $\forall\ 0 < \alpha \leq 1$,

$$\sup_{v \in \tilde{Pr}(V_\alpha)} \parallel v - a^* \parallel < \bar{k} \sup_{v' \in V_\alpha} \parallel v' - a^* \parallel .$$

Strict Locally Point Rationality assesses the idea of fast convergence of the point forecast process. Under this property, we have that $\tilde{Pr}(V) \subset V_{\bar{k}}$, with $\bar{k} < 1$, and so $\tilde{Pr}^t(V) \subset V_{\bar{k}^t}$.

Proposition 3.23. *If the utility functions are twice continuously differentiable, $a^* \in \text{int}\,\mathcal{A}$, $\mathbb{B}(i,\mu)$ is single valued for all μ with support in a neighborhood of a^* and $Du_{ss}(s,a)$ is non singular in an open set $V \ni a^*$, then*

$$a^* \textit{ is Locally Strongly Rational} \iff a^* \textit{ is Strictly Locally Point Rational.}$$

The idea of the proposition is that if the process governed by point forecasts is sufficiently fast, then, although the eductive process may be slower, it is anyhow dragged to the equilibrium state. This is, the eductive process may converge at a lower rate, it can not escape the force of $\tilde{Pr}$.

5 Economic games with strategic complementarities or substitutabilities.

5.1 Economic games with strategic complementarities.

In this section we want to study the consequences over expectational coordination and eductive stability of the presence of *Strategic Complementarities* in our Economic System with a Continuum of Agents. We will say that the economic system presents Strategic Complementarities if the individual best response mappings of the underlying game are increasing for each $i \in I$. That is, if we consider the general payoff functions $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$, the usual product order in $\mathbb{R}^n$ over S and the order $\geq_{S^I}$ defined by $\mathbf{s} \geq_{S^I} \mathbf{s}'$ if and only if $s(i) \geq s(i)'$ for almost all $i \in I$, over S^I, then we would like the mappings $\text{Br}(i, \cdot) : S^I \rightrightarrows S$ defined in (3.4) to be increasing for the induced set ordering in S. That is, if $\mathbf{s} \geq_{S^I} \mathbf{s}'$ then $\text{Br}(i, \mathbf{s}) \succeq \text{Br}(i, \mathbf{s}')$.

The most classical representation of complementarity in games is the theory of *supermodular games* as studied in Milgrom and Roberts (1990) and Vives (1990) (see as well Topkis (1998)). In a supermodular game, a normal form game with a finite number of players is embedded within a lattice structure.

A normal form game $\mathcal{G} := \left\langle I, (S_i)_{i \in I}, (\pi_i(\cdot, \cdot))_{i \in I} \right\rangle$ is supermodular if $\forall\ i \in I$:

1.A S_i is a complete lattice.

2.A $\pi_i(s_i, s_{-i})$ is order upper semi-continuous in s_i and order continuous in s_{-i}, with finite upper bound.

3.A $\pi_i(\cdot, s_{-i})$ is supermodular on s_i for all $s_{-i} \in S_{-i}$

4.A $\pi_i(s_i, s_{-i})$ has increasing differences in s_i and s_{-i}

We will understand strategic complementarity then, as supermodularity of the underlying game. Supermodularity (and of course submodularity as in the next section) could be studied in the context of games with continuum of agents with a broad generality using the strategic approach (using for instance the tools available from Riesz spaces). However, our present concern suggests to focus on the set of states and introduce strategic complementarities ideas directly in this framework. This last assertion is not at all superfluous since it is the fact that we work with a continuous of agents that allows to focus on forecasts over the set of aggregate states. Since agents can not affect the state of the system, all agents have forecasts over the same set, namely the set of states $\mathcal{A}$. This would not be possible in the context of *small* game since then the forecast of different agents would be in different sets, namely the set of aggregate values of *the others* which could well be a different set for each agent. Another difficulty is passing from strategies to states in terms of complementarity. Part of the work to be presented focuses on the possibility of inheritance by the state approach of the properties of complementarity (and substitutability in the next section). An important result related with this issue is treated in Lemma 3.48 in the appendix.

Our objective will then be to understand the consequences of the assumptions introduced on all the sets under scrutiny (equilibria, Cournot outcomes, Point-Rationalizable States, Rationalizable States).

Let us proceed as suggested and make, in the economic setting, the following assumptions over the strategy set S and the utility functions $u(i, \cdot, \cdot)$.

1.B S is the product of n compact intervals in $\mathbb{R}_+$.

2.B $u(i, \cdot, a)$ is supermodular for all $a \in \mathcal{A}$ and all $i \in I$.

3.B $\forall\, i \in I$, the function $u(i, y, a)$ has increasing differences in y and a. That is, $\forall\, y, y' \in S$, such that $y \geq y'$ and $\forall\, a, a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i, y, a) - u(i, y', a) \geq u(i, y, a') - u(i, y', a') \tag{3.18}$$

Assumption 2.B is straightforward. Assumption 1.B implies that the set of strategies is a complete lattice in $\mathbb{R}^n$. Since in our model we already assumed that the utility functions $u(i, \cdot, \cdot)$ are continuous, we obtain that in particular the functions $\pi(i, \cdot, \cdot)$ satisfy 2.A (endowing S^I with the weak topology for instance, this is of no relevance for what follows). Finally, if we look at $\mathcal{A}$ and S^I as ordered sets (with the product order of $\mathbb{R}^n$ in $\mathcal{A}$ and $\geq_{S^I}$ in S^I), we see that the aggregation mapping $A : S^I \to \mathcal{A}$ is increasing, and so assumption 3.B implies 4.A.

Proposition 3.24. *Under assumptions 1.B through 3.B, the mappings $B(i, \cdot)$ are increasing in a in the set $\mathcal{A}$, and the sets $B(i, a)$ are complete sublattices of S.*

Proof. The first property is a consequence of Theorem 2 in Milgrom and Roberts (1990) and the second part we apply Theorem 2.8.1 in Topkis (1998) considering the constant correspondence $S_a \equiv S \;\forall\, a \in \mathcal{A}$ ■

Definition 3.25. We name $\mathcal{G}$, an economic system such that S and $u(i, \cdot, \cdot)$ satisfy assumptions 1.B through 3.B.

One implication of our setting is that since S is a convex complete lattice, then $\mathcal{A} \equiv \mathrm{co}\{S\} \equiv S$ is as well a complete lattice. From now on we will refer to the supermodular setting as $\mathcal{G}$.

Proposition 3.26. *In $\mathcal{G}$ the correspondence Γ is increasing and $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$.*

Recall that the set of equilibria is $\mathbb{E} \subseteq \mathcal{A}$ and this is the set of fixed points of Γ. For a correspondence $F : \mathcal{A} \rightrightarrows \mathcal{A}$, we will denote the set of fixed points of F as E_F. Consequently $\mathbb{E} \equiv E_\Gamma$. We see now that under assumptions 1.B through 3.B we get the hypothesis of Proposition 3.27.

Proposition 3.27. *If $\mathcal{A}$ is a complete lattice, Γ is increasing, $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$, then $\mathbb{E} \neq \emptyset$ is a complete lattice.*

Proof. As a consequence of the Theorem 2.5.1 in Topkis (1998) E_Γ is a non-empty complete lattice. ■

In the previous proposition we have an existence result, but what is most important is that the set of equilibria has a complete lattice structure. In particular we know that there exist points $\underline{a}^* \in \mathcal{A}$ and $\bar{a}^* \in \mathcal{A}$ (that could be the same point) such that if $a^* \in \mathbb{E}$ is an equilibrium, then $\underline{a}^* \leq a^* \leq \bar{a}^*$.

The previous results tell us that when the economic system's underlying game is supermodular and since the aggregate mapping is monotone (in this case increasing), then we can apply Proposition 3.27 and work in a finite dimensional setting (the set $\mathcal{A}$) rather than infinite dimensional. We state this as a formal result in the next proposition.

Proposition 3.28. *In $\mathcal{G}$ we have*

$$\mathbb{P}_{\mathcal{A}} \subseteq \left[\inf_{E_\Gamma} \{E_\Gamma\}, \sup_{E_\Gamma} \{E_\Gamma\}\right]$$

and $\inf_{E_\Gamma} \{E_\Gamma\}$ and $\sup_{E_\Gamma} \{E_\Gamma\}$ are equilibria.

The proof is relegated to the appendix. The intuitive interpretation of the proof is as follows. Originally, agents know that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. Since the actual state is in the image through $\tilde{Pr}$ of $\mathcal{A}$, the monotonicity properties of the forecasts to state mappings allow agents to deduce that the actual state will be in fact greater than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and smaller than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, it suffices to consider the cases where all the agents having the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$. The eductive procedure then can be secluded on each iteration, only with iterations of Γ. Since Γ is increasing, we get an increasing sequence that starts at $\underline{a}^0$ and a deceasing sequence that starts at $\bar{a}^0$. These sequence converge and upper semi continuity of Γ implies that their limits are fixed points of Γ.

There are three key features to keep in mind, that lead to the conclusion. First, the fact that there exists a set $\mathcal{A}$ that, being a complete lattice and having as a subset the whole image of the mapping A, allows the eductive process to be initiated. Second, monotonic structure of the model implies that it suffices to use Γ to seclude, in each step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the structure of the model allow the process to converge. Now that we have proved this result for the Point-Rationalizable set, we can use the proof of Proposition 3.28 to get the same conclusion for the set of Rationalizable States. For this we use the following Lemma.

Lemma 3.29. *In $\mathcal{G}$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\, a \in \operatorname{supp}(\mu)$, then $\forall\, i \in I$*

$$B(i, a') \preceq \mathbb{B}(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\, a \in \operatorname{supp}(\mu)$, then $\forall\, i \in I$

$$B(i, a') \succeq \mathbb{B}(i, \mu)\,.$$

This is, if the forecast of an agent has support on points that are larger than a point $a' \in \mathcal{A}$, then his optimal strategy set is larger than the optimal strategy associated to a' (for the induced set ordering) and analogously for the second statement.

Proof. Observe first that supermodularity of $u(i, \cdot, a)$ is preserved[17] when we take expectation on a.

Now consider $y' \in B(i, a')$ and $y \in \mathbb{B}(i, \mu)$ we show that $\min\{y, y'\} \in B(i, a')$ and $\max\{y, y'\} \in \mathbb{B}(i, \mu)$. Since $y' \in B(i, a')$ we have that:

$$0 \leq u(i, y', a') - u(i, \min\{y, y'\}, a')\,.$$

[17]If $u(i, \cdot, a)$ is supermodular, then for $s, s' \in S$, we have for each $a \in \mathcal{A}$:

$$u(i, \min\{s, s'\}, a) + u(i, \max\{s, s'\}, a) - (u(i, s, a) + u(i, s', a)) \geq 0$$

Taking expectation we get the result.

Increasing differences of $u(i, y, a)$ in (y, a) implies that $\forall\, a \in \text{supp}(\mu)$,

$$u(i, y', a') - u(i, \min\{y, y'\}, a') \leq u(i, y', a) - u(i, \min\{y, y'\}, a)$$

and so if on the right hand side we take expectation with respect to μ we get

$$u(i, y', a') - u(i, \min\{y, y'\}, a') \leq \mathbb{E}_\mu[u(i, y', a)] - \mathbb{E}_\mu[u(i, \min\{y, y'\}, a)].$$

Supermodularity of $u(i, \cdot, a)$ implies that

$$\mathbb{E}_\mu[u(i, y', a)] - \mathbb{E}_\mu[u(i, \min\{y, y'\}, a)] \leq \mathbb{E}_\mu[u(i, \max\{y, y'\}, a)] - \mathbb{E}_\mu[u(i, y, a)]$$

and the last term is less or equal to 0 since $y \in \mathbb{B}(i, \mu)$.

All these inequalities together imply that $\max\{y, y'\} \in \mathbb{B}(i, \mu)$ and $\min\{y, y'\} \in B(i, a')$

The second statement is proved analogously.

■

The fact that the points $\sup_{E_\Gamma}\{E_\Gamma\}$ and $\inf_{E_\Gamma}\{E_\Gamma\}$ are equilibria, implies that they are Point-Rationalizable and Rationalizable states and so Proposition 3.28 states that the interval $\left[\inf_{E_\Gamma}\{E_\Gamma\}, \sup_{E_\Gamma}\{E_\Gamma\}\right]$ is the smallest interval that contains the set $\mathbb{P}_{\mathcal{A}}$. Considering Lemma 3.29 and the proof of Proposition 3.28 we get that this same interval contains tightly the set $\mathbb{R}_{\mathcal{A}}$.

Theorem 3.30. *In the economic system with Strategic Complementarities we have:*

(i) *The set of equilibria $\mathbb{E} \subseteq \mathcal{A}$ is complete lattice.*

(ii) *There exist a greatest equilibrium and a smallest equilibrium, that is $\exists\, \underline{a}^* \in \mathbb{E}$ and $\bar{a}^* \in \mathbb{E}$ such that $\forall\, a^* \in \mathbb{E}$, $\underline{a}^* \leq a^* \leq \bar{a}^*$.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex sets, tightly contained in the interval $[\underline{a}^*, \bar{a}^*]$. That is,*

$$\mathbb{P}_{\mathcal{A}} \quad \subseteq \quad \mathbb{R}_{\mathcal{A}} \quad \subseteq \quad \{\underline{a}^*\} + \mathbb{R}^n_+ \bigcap \{\bar{a}^*\} - \mathbb{R}^n_+$$

and $\bar{a}^* \in \mathbb{P}_{\mathcal{A}}$ and $\underline{a}^* \in \mathbb{P}_{\mathcal{A}}$.

Proof. Using Lemma 3.29 in the proof of Proposition 3.28 we can see that $\tilde{R}^t(\mathcal{A}) \subseteq [\underline{a}^t, \bar{a}^t]$ and so we get the result.

■

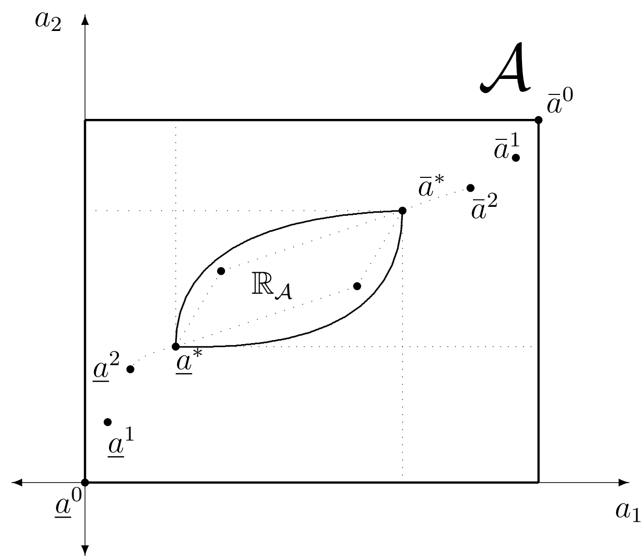


Figure 3.1: Strategic Complementarities for $\mathcal{A} \subset \mathbb{R}^2$ with four equilibria.

Convexity of $\mathbb{P}_{\mathcal{A}}$ implies the convex hull of $\mathbb{E}$ is contained in $\mathbb{P}_{\mathcal{A}}$, in particular the segment

$$\{ \ a \in \mathcal{A} \ : \ a = \alpha \underline{a}^* + (1 - \alpha) \, \bar{a}^* \quad \alpha \in [0, 1] \} \ \subseteq \ \mathbb{P}_{\mathcal{A}} \ \subseteq \ \mathbb{R}_{\mathcal{A}}$$

Let us also note :

Corollary 3.31. *If in $\mathcal{G}$ Γ has a unique fixed point a^*, then*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}} \equiv \{a^*\} \, .$$

Our results are unsurprising. In the context of an economic game with a continuum of agents, they mimic, in an expected way, the standards results obtained in a game-theoretical framework with a finite number of agents and strategic complementarities. Additional convexity properties reflect the use of a continuum setting.

We see from Theorem 3.30 and Corollary 3.31, that under the presence of strategic complementarity, uniqueness of equilibrium implies the success of the elimination of unreasonable states. The unique equilibrium is then Strongly Rational and this stability is global.

Corollary 3.32. *In $\mathcal{G}$, the three following statements are equivalent:*

(i) *an equilibrium a^* is Strongly Rational.*

(ii) *an equilibrium a^* is IE-Stable.*

(iii) *there exists a unique equilibrium a^*.*

Proof. From the definitions of both concepts of stability we see that Strong Rationality implies IE-Stability. The relevant part of the corollary is then that under Strategic Complementarities, we have the inverse. If a^* is IE-Stable, then from the proof of Proposition 3.28 we see that the sequences $\{\underline{a}^t\}_{t\geq 0}$ and $\{\bar{a}^t\}_{t\geq 0}$ must both converge to a^* and so we get that a^* is *Eductively Stable*.

■

This last statement may be interpreted as the fact that in the present setting, heterogeneity of expectations does not play any role in expectational coordination. This is a very special feature of expectational coordination as argued in Evans and Guesnerie (1993). Surprisingly enough, a similar feature appears in the next class of models under consideration.

5.2 Economic games with Strategic Substitutabilities

We turn to the case of *Strategic Substitutabilities*. We will say that the economic system presents Strategic Substitutabilities if the individual best response mappings of the underlying game are decreasing for each $i \in I$. That is, if $\mathbf{s} \geq_{S^I} \mathbf{s}'$ then $\mathrm{Br}(i, \mathbf{s}) \preceq \mathrm{Br}(i, \mathbf{s}')$. Where $\mathrm{Br}(i, \cdot\,)$ and $\geq_{S^I}$ are the same as in section 5.1.

To study the consequences of embedding our model in a setting of strategic substitutabilities we use the same structure as in the previous section except that we replace assumption 3.B with assumption 3.B' below.

1.B S is the product of n compact intervals in $\mathbb{R}_+$.

2.B $u(i, \cdot\,, a)$ is supermodular for all $a \in \mathcal{A}$

3.B' $u(i, y, a)$ has decreasing differences in y and a. That is, $\forall\ y,\ y' \in S$, such that $y \geq y'$ and $\forall\ a,\ a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i, y, a) - u(i, y', a) \leq u(i, y, a') - u(i, y', a') \tag{3.19}$$

Assumptions 1.B through 3.B' turn the underlying game of our model into a *submodular game*[18] with a continuum of agents. The relevant difference with the previous section is that now the monotonicity of the mapping A along with assumption 3.B' implies that the best response mappings are decreasing on the strategy profiles.

The following Propositions are the counterparts of Propositions 3.24 and 3.26.

[18]A submodular game is a game under assumptions 1.A to 3.A with assumption 4.A replaced by assumption 4.A': the payoff functions $\pi_i(s_i, s_{-i})$ have decreasing differences in (s_i, s_{-i})

Proposition 3.33. *Under assumptions 1.B, 2.B and 3.B', the mappings $B(i,\cdot)$ are decreasing in a in the set $\mathcal{A}$, and the sets $B(i,a)$ are complete sublattices of S.*

Definition 3.34. We name $\mathcal{G}'$, an economic system such that S and $u(i,\cdot,\cdot)$ satisfy assumptions 1.B, 2.B, and 3.B'.

Proposition 3.35. *In $\mathcal{G}$ the correspondence Γ is decreasing and $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$.*

We denote Γ^2 for the second iterate of the cobweb mapping, that is $\Gamma^2 : \mathcal{A} \rightrightarrows \mathcal{A}$, $\Gamma^2(a) := \cup_{a' \in \Gamma(a)} \Gamma(a')$.

Corollary 3.36. *In $\mathcal{G}$ the correspondence Γ^2 is increasing and $\Gamma^2(a)$ is subcomplete for each $a \in \mathcal{A}$.*

Proof. Is a consequence of Γ being decreasing.

■

The correspondence Γ^2 will be our main tool for the case of strategic substitutabilities. This is because, in the general context, the fixed points of Γ^2 are point-rationalizable just as the fixed points of Γ are. Actually, it is direct to see that the fixed points of any iteration of the mapping Γ are as well point-rationalizable. The relevance of strategic substitutabilities is that under their presence it suffices to use the second iterate of the cobweb mapping to seclude the set of point-rationalizable states. Using Proposition 3.27 we get that under assumptions 1.B, 2.B and 3.B', the set of fixed points of Γ^2, E_{Γ^2}, shares the properties that the set of equilibria $\mathbb{E}$ had under strategic complementarities.

Proposition 3.37. *The set of fixed points of Γ^2, E_{Γ^2} is a non empty complete lattice.*

Proof. Apply Proposition 3.27 to Γ^2.

■

The relevance of Proposition 3.37 is that, as in the case of strategic complementarities, under strategic substitutabilities it is possible to seclude the set of Point-Rationalizable States into a tight compact interval. This interval is now obtained from the complete lattice structure of the set of fixed points of Γ^2, which can be viewed, in a multi-period context, as cycles of order 2 of the system.

Proposition 3.38. *In $\mathcal{G}'$ we have*

$$\mathbb{P}_{\mathcal{A}} \subseteq \left[\inf_{E_{\Gamma^2}} \{E_{\Gamma^2}\}, \sup_{E_{\Gamma^2}} \{E_{\Gamma^2}\} \right]$$

and $\inf_{E_{\Gamma^2}} \{E_{\Gamma^2}\}$ *and* $\sup_{E_{\Gamma^2}} \{E_{\Gamma^2}\}$ *are point-rationalizable.*

The proof is relegated to the appendix. Keeping in mind the proof of Proposition 3.28, we can follow the idea of the proof of Proposition 3.38. As usual, common knowledge says that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. In first order basis then, the actual state is known to be in the image through $\tilde{Pr}$ of $\mathcal{A}$. Since now the cobweb mapping is decreasing, the structure of the model allows the agents to deduce that the actual state will be in fact smaller than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and greater than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, again it suffices to consider the cases where all the agents having the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$ and this will give $\underline{a}^1$, associated to $\bar{a}^0$, and $\bar{a}^1$, associated to $\underline{a}^0$. However, now we have a difference with the strategic complementarities case. In the previous section the iterations started in the lower bound of the state set were lower bounds of the iterations of the eductive process. As we see, this is not the case anymore. Nevertheless, here is where the second iterate of Γ gains relevance. In a second order basis, once we have $\underline{a}^1$ and $\bar{a}^1$ obtained as above, we can now consider the images through Γ of these points and we get new points $\bar{a}^2$, from $\underline{a}^1$, and $\underline{a}^2$, from $\bar{a}^1$, that are respectively upper and lower bounds of the second step of the eductive process. This is, in two steps we obtain that the iterations started at the upper (resp. lower) bound of the states set is un upper (resp. lower) bound of the second step of the eductive process. Moreover, the sequences obtained by the second iterates are increasing when started at $\underline{a}^0$ and decreasing when started at $\bar{a}^0$. The complete lattice structure of $\mathcal{A}$ again implies the convergence of the monotone sequences while Γ^2 inherits upper semi continuity from Γ. This implies that the limits of the sequences are fixed points of Γ^2.

The three key features that lead to the conclusion are analogous to the strategic complementarity case. First, $\mathcal{A}$ is a complete lattice that has as a subset its' image through the function A and thus allows the eductive process to be initiated. Second, monotonic structure of the model implies that it now suffices to use Γ^2 to seclude, every second step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the monotonic structure of the model allow the process to converge.

Note that, also as in the case of strategic complementarities, since the limits of the interval in Proposition 3.38 are point-rationalizable, this is the smallest interval that contains the set of point-rationalizable states.

Adapting the proof of Lemma 3.29 to the decreasing differences case, we obtain its' counterpart for the strategic substitutabilities case stated below.

Lemma 3.39. *In $\mathcal{G}'$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\ a \in \operatorname{supp}(\mu)$, then $\forall\ i \in I$*

$$B(i, a') \succeq \mathbb{B}(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\ a \in \operatorname{supp}(\mu)$, then $\forall\ i \in I$

$$B(i, a') \preceq \mathbb{B}(i, \mu)\,.$$

We are now able to state the main result of the strategic substitutabilities case.

Theorem 3.40. *In the economic system with Strategic Substitutabilities we have:*

(i) *There exists at least one equilibrium a^*.*

(ii) *There exist greatest and a smallest rationalizable strategies, that is $\exists\ \underline{a} \in \mathbb{R}_{\mathcal{A}}$ and $\bar{a} \in \mathbb{R}_{\mathcal{A}}$ such that $\forall\ a \in \mathbb{R}_{\mathcal{A}}$, $\underline{a} \leq a \leq \bar{a}$, where $\underline{a}$ and $\bar{a}$ are cycles of order 2 of the Cobweb mapping.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex.*

(iv) *The sets of Rationalizable and Point-Rationalizable States are tightly contained in the interval $[\underline{a}, \bar{a}]$. That is,*

$$\mathbb{P}_{\mathcal{A}} \quad \subseteq \quad \mathbb{R}_{\mathcal{A}} \quad \subseteq \quad \{\underline{a}\} + \mathbb{R}^n_+ \bigcap \{\bar{a}\} - \mathbb{R}^n_+$$

and $\bar{a} \in \mathbb{P}_{\mathcal{A}}$ and $\underline{a} \in \mathbb{P}_{\mathcal{A}}$.

Proof. Using Lemma 3.39 in the proof of Proposition 3.38 we can see that $\tilde{R}^{2t}(\mathcal{A}) \subseteq [\underline{a}^{2t}, \bar{a}^{2t}]$ and so we get the two first results.

The last assertion is a consequence of the general setting of Rath (1992). Theorem 3.2 gives the existence of equilibrium.

■

Summing up, we have that in the case of Strategic Substitutabilities we can still use the correspondence Γ (through its' second iterate) to seclude to an interval the sets of Point-Rationalizable and Rationalizable States. This inclusion is tight since the boundaries of this interval are in fact Point-rationalizable States.

Corollary 3.41. *If in $\mathcal{G}'$, Γ^2 has a unique fixed point a^*, then*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}} \equiv \{a^*\}.$$

Proof. Observe that both limits of the interval presented in Theorem 3.40, $\underline{a}$ and $\bar{a}$, are fixed points of Γ^2. Hence the result.

■

As opposed to the case of strategic complementarities, the optimistic equivalence result of Corollary 3.32 can not be directly obtained in the setting of strategic substitutabilities. If the sequences $\bar{b}^t$ and $\underline{b}^t$ defined in the proof of Proposition 3.38 converge to the same point, i.e. $\underline{b}^* = \bar{b}^* = a^*$, then a^* is the unique equilibrium of the system, it is strongly rational and IE-stable. However, under strategic substitutabilities there could well be a unique equilibrium that is not necessarily strongly rational. Think of the case of $\mathcal{A} \subset \mathbb{R}$, where a continuous decreasing function Γ has unique fixed point, that could well be part of a bigger set of Point-Rationalizable States (see figure 3.2).

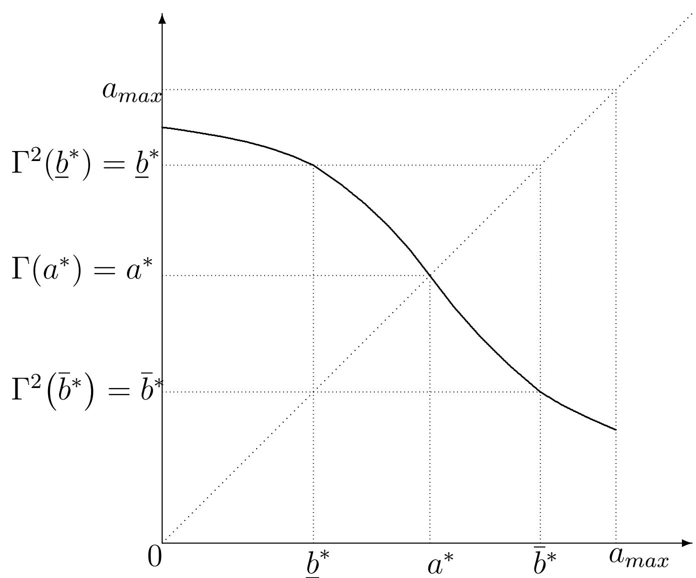


Figure 3.2: Strategic substitutes for $\mathcal{A} \equiv [0, a_{max}] \subset \mathbb{R}$. There exists a unique equilibrium and multiple fixed points for Γ^2

Corollary 3.42. *The following statements are equivalent.*

(i) *an equilibrium a^* is Strongly Rational.*

(ii) *an equilibrium a^* is IE-Stable.*

Again heterogeneity of expectations in a sense does not matter, for evaluating the quality of expectational coordination. However, here uniqueness of equilibrium does not assure its' global stability. We recover the intuitions stated in Guesnerie (2005).

6 The differentiable case.

Here, we add an assumption concerning the cobweb mapping Γ:

H1 $\Gamma : \mathcal{A} \to \mathcal{A}$ is a $\mathcal{C}^1$-differentiable function.

Remark 3.43. Note that from the definition of Γ, the vector-field $(a - \Gamma(a))$ points outwards on $\mathcal{A}$: formally, this means that if $p(a)$ is a supporting price vector at a boundary point of $\mathcal{A}$ $(p(a) \cdot \mathcal{A} \leq 0)$, then $p(a) \cdot (a - \Gamma(a)) \geq 0$. When, as in most applications $\mathcal{A}$ is the product of intervals for example $[0, M_h]$, this means $\Gamma_h(a) \geq 0$, whenever $a_h = 0$, and $M_h - \Gamma_h(a) \geq 0$, whenever $a_h = M_h$.

The jacobian of the function Γ, ∂Γ, can be obtained from the first order conditions of problem (3.5) along with (3.11).

$$\partial\Gamma(a) = \int_I \partial B(i, a) \ \text{di}$$

where $\partial B(i, a)$ is the jacobian of the optimal strategy (now) function. This jacobian is equal to:

$$\partial B(i, a) \equiv -[Du_{ss}(i, B(i, a), a)]^{-1} Du_{sa}(i, B(i, a), a) \tag{3.20}$$

where $Du_{ss}(i, B(i, a), a)$ is the matrix of second derivatives with respect to s of the utility functions and $Du_{sa}(i, B(i, a), a)$ is the matrix of cross second derivatives, at the point $(B(i, a), a)$.

6.1 The strategic complementarities case.

Under assumptions 1.B to 3.B, along with $\mathcal{C}^2$ differentiability of the functions $u(i, \cdot, \cdot)$, we get from (3.20) that the matrices $\partial B(i, a)$ are positive[19], and consequently so is $\partial\Gamma(a)$.

From the properties of positive matrices are well known. When there exists a positive vector x, such that $Ax < x$, the matrix A is said *productive* : its eigenvalue of highest

[19] It is a well know fact that increasing differences implies positive cross derivatives on $Du_{sa}(i, B(i, a), a)$ and it can be proved that for a supermodular function the matrix $-[Du_{ss}(i, \cdot, \cdot)]^{-1}$ is positive at $(B(i, a), a)$

modulus is positive and smaller than one. When a is one-dimensional, the condition says that the slope of Γ is smaller than 1.

In this special case, as well as in our more general framework, the condition has the flavor that actions do not react too wildly to expectations..

In this case, we obtain :

Theorem 3.44 (Uniqueness)**.** *If* $\forall\ a \in \mathcal{A}$*,* $\partial\Gamma(a)$ *is a productive matrix, then there exists a unique Strongly Rational Equilibrium.*

Proof.

Compute in any equilibrium a^* the sign of $\det[I - \partial\Gamma(a^*)]$. If $\partial\Gamma(a^*)$ is productive, its eigenvalue of highest modulus is real positive and smaller than 1. Hence the real eigenvalues of $[I - \partial\Gamma(a^*)]$ are all positive[20]. It follows that the sign of $\det[I - \partial\Gamma(a^*)]$ is the sign of the characteristics polynomial $\det\{[I - \partial\Gamma(a^*)] - \lambda I\}$ for $\lambda \to -\infty$, i.e is plus. The index of $\varphi(a) = a - \Gamma(a)$ is then $+1$. The Poincaré-Hopf theorem for vector fields pointing inwards implies that the sum of indices must be equal to $+1$, hence the conclusion of uniqueness. Strong Rationality follows from Corollary 3.32.

■

Our assumptions also have consequences for *eductive* stability.

Theorem 3.45 (Expectational coordination)**.** *If sign of* $\det[I - \partial\Gamma(a^*)]$ *is* $+$*, for some equilibrium* a^**, then* a^* *is locally eductively stable.*

Proof.

Take as initial local hypothetical CK neighborhood $\left\{\{a_-\} + \mathbb{R}^n_+\right\} \cap \left\{\{a_+\} - \mathbb{R}^n_+\right\}$ where $a_- < a^*$, $a_+ > a^*$, both being close to a^*.

The general argument of Proposition 3.28 works.

■

The above statements generalize in a reasonable way the intuitive findings easily obtainable from the one-dimensional model.

[20] It has at least one real eigenvalue, associated with the eigenvalue of highest modulus of $\Gamma(a^*)$.

6.2 The strategic substitutabilities case.

Let us go to the Strategic Substitutabilities case. We maintain the previous boundary assumptions.

When passing from 3.B to 3.B' we get that now the matrix $\partial\Gamma$ has negative[21] entries. And $I - \partial\Gamma(a)$ is a positive matrix. Again, it has only positive eigenvalues, whenever the positive eigenvalue of highest modulus of -$\partial\Gamma$ is smaller than 1.

Theorem 3.46. *Let us assume that* $\forall\ a_1,\ a_2 \in \mathcal{A}$, $\partial\Gamma\left(a_1\right)\partial\Gamma\left(a_2\right)$ *is productive,*

1- There exists a unique equilibrium.

2- It is globally Strongly Rational.

Proof.

The assumption implies that $\forall a \in \mathcal{A}$, $-\partial\Gamma\left(a\right)$ is productive.

Hence $I - \partial\Gamma$ is a positive matrix, and whenever the positive eigenvalue of highest modulus of $-\partial\Gamma$ is smaller than 1, it has only positive eigenvalues. Then its determinant is positive.

Then the above Poincaré-Hopf argument applies to the first and second iterate of Γ.

It follows that there exists a unique equilibrium and no two-cycle.

Then, Theorem 3.40 applies.

■

Also, as above one can show that the *productive* condition when it holds in one equilibrium ensures local strong rationality.

7 Comments and Conclusions

The Rational Expectations Hypothesis has been subject of scrutiny in recent years through the assessment of Expectational Coordination. Although the terminology is still fluctuating, the ideas behind what we call here *Strong Rationality* or *Eductive Stability* have been at the heart of the study of diverse macroeconomic and microeconomic models of standard markets

[21]Since the matrix $Du_{sa}(i, B(i,a), a)$ has only non-positive entries under strategic substitutabilities (see note 19).

with one or several goods, see Guesnerie, models of information transmission (Desgranges (2000), Desgranges and Heinemann (2005), Ben Porath-Heifetz).

In this work we aimed to address the subject of *eductive stability* with broad generality. We have presented a stylized framework that encompasses a significant class of economic models. We have made the connection between what may be called the *economic* viewpoint and a now standard line of research in game-theory: games with a continuous of players. The paper has assessed the connections between a number of tools with game-theoretical flavor which are available for analyzing the expectational *stability* or *plausibility* of equilibria in a so-called *economic* context with non-atomic agents. The presence of an aggregate variable in the model allowed for us to go back and forth between the economic and game-theoretical point of view, making the connection between the different approaches.

We have exhibited properties of what we called the set of Rationalizable and Point-Rationalizable States. The Rationalizable set is proved to be non-empty and convex as the set of Point-Rationalizable States, with this last set also being compact.

In this context, when the economy is dominated by strategic complementarities, we have derived results that reformulate the classical game-theoretical findings of Milgrom and Roberts and Vives.

In the opposite polar case of strategic substitutabilities, using the properties of the second iterate of the Cobweb mapping, we have exhibited results that parallel the first ones, while stressing however striking differences. For example, when in the strategic complementarities, uniqueness triggers all expectational stability criteria, this is no longer the case with strategic substitutabilities : uniqueness does not imply *expectational stability*, whatever the exact sense given to the assertion. Related remarks apply for local uniqueness that has different implications for local stability in the two cases under examination. We give then simple and appealing conditions implying uniqueness of equilibria and stability in the sense of Strong Rationality, although in this case the former does not imply the latter.

In all cases, it turns out that the *eductive process* that allows to obtain Point-Rationalizable and (locally) Strong Rationalizable States can be achieved tightly with the *iterative expectations process* or "Cobweb tâtonnement " used to explain Iterative Expectational Stability. In both circumstances, one may argue that heterogeneity of expectations makes no difference for expectational coordination. This is a most significant feature of these situations that strikingly contrast the general case studied in Evans and Guesnerie (2005). Many economic models that fit our framework, such as the one associated with the analysis of expectational stability in a class of general dynamical systems (Evans and Guesnerie 2005) have neither strategic complementarities nor substitutabilities. The complexity of the findings that has increased when going from the first case to the second one, will still increase. In this sense, we hope that these results provide a useful benchmark for a deeper understanding of the role of the heterogeneity of beliefs in expectational coordination. The beginning of the road map drawn from this paper should help to continue the route.

Appendix

Technical Lemmas

Lemma 3.47. *Under assumptions* ***C*** *and* ***HM****, for a closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$ is measurable and has compact values.*

Proof. See the proof of Lema 2.6 in page 24.

■

Lemma 3.48. *If $S \subset \mathbb{R}^n$ is a complete lattice for the product order in $\mathbb{R}^n$, then for a measurable correspondence $F : I \rightrightarrows S$ with nonempty, closed and subcomplete values, the functions $\underline{\mathbf{s}} : I \to S$ and $\bar{\mathbf{s}} : I \to S$, defined by*

$$\underline{s}(i) := \inf_S F(i)\,, \tag{3.21}$$

$$\bar{s}(i) := \sup_S F(i)\,, \tag{3.22}$$

are measurable selections of F.

Proof.

Since $F(i)$ is subcomplete, $\underline{s}(i)$ and $\bar{s}(i)$ belong to $F(i)$. We have to prove that $\underline{\mathbf{s}}$ and $\bar{\mathbf{s}}$ are measurable.

Since F is measurable, it has a Castaing representation. That is, there exists a countable family of measurable functions $\mathbf{s}^\nu : I \to \mathbb{R}^n$, $\nu \in \mathbb{N}$, such that $s^\nu(i) \in F(i)$ and,

$$F(i) \equiv \mathrm{cl}\{\{s^\nu(i) \;:\; \nu \in \mathbb{N}\}\}\,. \tag{3.23}$$

For $\underline{\mathbf{s}}$, consider then for each $\nu \in \mathbb{N}$ the set valued mappings $F^\nu : I \rightrightarrows \mathbb{R}^n$, defined by [22]

$$F^\nu(i) := F(i) \bigcap \;]-\infty, s^\nu(i)]$$

[22]The *interval* $]-\infty, x]$ is the set of points of $\mathbb{R}^n$ that are smaller than $x \in \mathbb{R}^n$, similarly $[x, +\infty[$ is the set of points in $\mathbb{R}^n$ that are greater than x.

Since F is measurable and closed valued, and we can write $]-\infty, s^\nu(i)] = s^\nu(i) - \mathbb{R}^n_+$ which is as well measurable and closed valued, the correspondences F^ν are measurable and closed valued [23].

Note that $\forall\ \nu \in \mathbb{N}$, $\underline{s}(i) \in F^\nu(i)$. Defining the closed valued correspondence $\underline{F} : I \rightrightarrows \mathbb{R}^n$:

$$\underline{F}(i) := \bigcap_{\nu \in \mathbb{N}} F^\nu(i)$$

we get then that $\underline{s}(i) \in \underline{F}(i)$. The correspondence $\underline{F}$ is as well measurable [23].

We now prove that actually $\underline{F}(i) \equiv \{\underline{s}(i)\}$, which completes the proof. Indeed, suppose that $y \in \underline{F}(i)$. Then, by definition of $\underline{F}$, $y \in F(i)$ and $y \leq s^\nu(i)$, $\forall\ \nu \in \mathbb{N}$. From equality (3.23) we get that any point in $F(i)$ can be obtained as the limit of a subsequence of $\{s^\nu(i)\ :\ \nu \in \mathbb{N}\}$, so in the limit the inequality is maintained, this is $\forall\ s \in F(i)$, $y \leq s$. That is, y is a lower bound for $F(i)$. This implies, by the definition of $\inf_S F(i)$, that $y \leq \inf_S F(i)$, but $y \in F(i)$, so $\inf_S F(i) \leq y$. Thus, $y \leq \underline{s}(i) = \inf_S F(i) \leq y$.

Analogous arguments applied to the mapping $\bar{F} : I \rightrightarrows \mathbb{R}^n$:

$$\bar{F}(i) := F(i) \ \bigcap \left(\bigcap_{\nu \in \mathbb{N}} [s^\nu(i)\,, +\infty[\right)$$

prove the statement for $\bar{\mathbf{s}}$.

■

Proof of Proposition 3.21

Proof.

For (i): note that

$$\Gamma(a) \equiv \int_I B(i,a)\,\mathrm{di} \equiv \begin{cases} \tilde{Pr}(\{a\}) \\ \int_I \mathbb{B}(i, \delta_{a^*})\,\mathrm{di} \equiv \tilde{R}(\{a\}) \end{cases}$$

and use Proposition 3.13.

For (ii) from Proposition 3.13 we see that we only need to prove that under condition 3.17:

[23] See Proposition 14.11 in Rockafellar and Wets (1998)

$$a^* \text{ is Locally Strongly Point Rational} \quad \Longrightarrow \quad a^* \text{ is Locally Strongly Rational.}$$

For a subset $X \subseteq \mathcal{A}$ call $\mathbb{P}(X) := \bigcap_{t\geq 0} \tilde{Pr}^t(X)$ and note that if $\mathbb{P}(X) \equiv \{a^*\}$ then $\forall\ X' \subseteq X$, $\mathbb{P}(X') \equiv \{a^*\}$.

Take V, the neighborhood of the Proposition. For a borel subset $X \subseteq V$ the hypothesis implies that the integral of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$ is contained in the integral of $i \rightrightarrows \operatorname{co}\{B(i,X)\}$. From Aumann (1965) we know that:

$$\int_I \operatorname{co}\{B(i,X)\}\,\mathrm{di} \equiv \int_I B(i,X)\,\mathrm{di}$$

and so

$$\tilde{R}(X) \equiv \int_I \mathbb{B}(i, \mathcal{P}(X))\,\mathrm{di} \subseteq \int_I B(i,X)\,\mathrm{di} \equiv \tilde{Pr}(X) \tag{3.24}$$

If a^* is Locally Strongly Point Rational then there exists a neighborhood V' such that $\mathbb{P}(V') = \{a^*\}$. So now take an open ball of radius $\varepsilon > 0$ around a^* that is contained in both V and V'. To ensure that the process for probability forecasts is well defined, we can take the closed ball of radius $\varepsilon/2$, $\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)$, that is strictly contained in the previous ball and of course in the intersection of both neighborhoods. In particular, we have that $\mathbb{P}\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big) \equiv \{a^*\}$ and that $\tilde{R}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big)$ is well defined and closed for all $t \geq 1$. The last assertion, along with (3.24), imply that for all $t \geq 1$ $\tilde{R}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big) \equiv \tilde{Pr}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big)$. We conclude that,

$$\bigcap_{t\geq 0} \tilde{R}^t\left(\mathrm{B}\left(a^*, \frac{\varepsilon}{2}\right)\right) \equiv \mathbb{P}\left(\mathrm{B}\left(a^*, \frac{\varepsilon}{2}\right)\right) \equiv \{a^*\}$$

■

Proof of Proposition 3.23

Proof.

We give the proof for the case where all the agents have the same utility function $u : S \times \mathcal{A} \to \mathbb{R}$.

Consider then a convex neighborhood V of a^* and the space of probability measures $\mathcal{P}(V)$. Take a probability measure with finite support, μ, in this space, this is $\mu = \sum_{l=1}^{L} \mu_l \delta_{a_l}$, with $\{a_l\}_{l=1}^{L} \subset V$. For this measure, under the differentiability hypothesis, we can prove that if the support of μ, $\{a_1, \ldots, a_L\}$, is contained in a ball [24] $\mathrm{B}(a^*, \varepsilon_1)$, then

$$\| \mathbb{B}(\mu) - B(\mathbb{E}_\mu [a])\| < \varepsilon_1^2.$$

Since $\mathbb{E}_\mu [a] \in V$ we get that $B(\mathbb{E}_\mu [a]) \in B(V)$. Using a density argument we may conclude that $\mathbb{B}(\mu)$ is "close" to $B(V) \subseteq \mathrm{co}\{B(V)\} \equiv \tilde{Pr}(V)$ for any measure in $\mathcal{P}(V)$. We can take then $\varepsilon_1 > 0$ small, related to the neighborhood V, such that,

$$\tilde{R}(V) \subset \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \tag{3.25}$$

From the hypothesis we get that we can choose a number $\bar{k} < k' < 1$ such that the following inclusions hold:

$$\tilde{Pr}(V) \subset V_{\bar{k}} \subset V_{k'} \subset V \tag{3.26}$$

$$\tilde{R}(V) \subset \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \subset V_{k'} \tag{3.27}$$

Moreover, taking the second iterate of $\tilde{R}$ starting at V, using (3.27) and (3.25) on $V_{k'}$,

$$\tilde{R}^2(V) \subset \tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big)$$

where ε_2 depends on k'. However it can be chosen in such a way that the following inclusions hold. Using (3.26) we get

$$\begin{aligned} \tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big) &\subset V_{\bar{k}k'} + \mathrm{B}\big(0, \varepsilon_2^2\big) \\ &\subset V_{k'^2}. \end{aligned}$$

We have then,

$$\tilde{R}^2(V) \subset V_{k'^2}$$

Using the same argument, choosing ε_t related to the powers of k', ${k'}^{t-1}$, we get that for all t,

$$\tilde{R}^t(V) \subset \tilde{Pr}(V_{k'^{t-1}}) + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{\bar{k}k'^{t-1}} + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{k'^t}$$

We conclude then that the eductive process converges to the equilibrium a^*.

■

[24] Since $\mathcal{A}$ is compact V is bounded.

Proof of Proposition 3.28

Proof.

From Propositions 3.27 and 3.26 we get that E_Γ is non empty and has a greatest and a smallest element.

Following the structure of the proof of Theorem 5 in Milgrom and Roberts we prove that $\tilde{Pr}^t(\mathcal{A})$ is contained in some interval $[\underline{a}^t, \bar{a}^t]$ and that the sequences $\underline{a}^t$ and $\bar{a}^t$ satisfy $\underline{a}^t \to \inf_{E_\Gamma}\{E_\Gamma\}$ and $\bar{a}^t \to \sup_{E_\Gamma}\{E_\Gamma\}$.

Define $\underline{a}^0$ and $\underline{a}^t$ as:

$$\underline{a}^0 := \inf \mathcal{A} \tag{3.28}$$

$$\underline{a}^t := \inf_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big) \qquad \forall\, t \geq 1 \tag{3.29}$$

- $\tilde{Pr}^t(\mathcal{A}) \subseteq [\underline{a}^t, +\infty[$

  Clearly it is true for $t = 0$.
  Suppose that it is true for $t \geq 0$. That is, $\underline{a}^t \leq a\ \forall\ a \in \tilde{Pr}^t(\mathcal{A})$. Since $B(i, \cdot)$ is increasing, we get that $B(i, \underline{a}^t) \preceq B(i, a)\ \forall\ a \in \tilde{Pr}^t(\mathcal{A})$. In particular $\forall\ y \in B(i, a)$ and $\forall\ a \in \tilde{Pr}^t(\mathcal{A})$, we have $\inf_S B(i, \underline{a}^t) \leq y$. From Lemma 3.48, the correspondence $i \rightrightarrows \inf_S B(i, \underline{a}^t)$ is measurable. This implies that for any measurable selection $\mathbf{s} \in S^I$ of $i \rightrightarrows B\Big(i, \tilde{Pr}^t(\mathcal{A})\Big)$,

$$\int \inf_S B\big(i, \underline{a}^t\big)\,\mathrm{d}i \leq \int s(i)\,\mathrm{d}i. \tag{3.30}$$

  Since $B(i, \underline{a}^t)$ is subcomplete, $\inf_S B(i, \underline{a}^t) \in B(i, \underline{a}^t)$ and so we get that:

$$\begin{aligned}\inf_{\mathcal{A}} \Gamma\big(\underline{a}^t\big) &\equiv \inf_{\mathcal{A}} \big\{\ b \in \mathcal{A}\ :\ \exists\ \mathbf{s} \text{ measurable selection of } i \rightrightarrows B\big(i, \underline{a}^t\big) \text{ such that, } b = A(\mathbf{s})\big\} \\ &\leq \int \inf_S B\big(i, \underline{a}^t\big)\end{aligned} \tag{3.31}$$

  We conclude then that

$$\underline{a}^{t+1} \equiv \inf_{\mathcal{A}} \Gamma\big(\underline{a}^t\big) \leq \int \inf_S B\big(i, \underline{a}^t\big) \leq a \qquad \forall\ a \in \tilde{Pr}^{t+1}(\mathcal{A})\,.$$

  The equality is the definition of $\underline{a}^{t+1}$, the first inequality comes from (3.31) and the second one is obtained from (3.30) and the definition of $\tilde{Pr}$.

- The sequence is increasing:

By definition of $\underline{a}^0$, $\underline{a}^0 \leq \underline{a}^1$. Suppose that $\underline{a}^{t-1} \leq \underline{a}^t$, then from Lemma 2.4.2 in Topkis (1998), $\underline{a}^t \equiv \inf_{\mathcal{A}} \Gamma(\underline{a}^{t-1}) \leq \inf_{\mathcal{A}} \Gamma(\underline{a}^t) \equiv \underline{a}^{t+1}$.

- The sequence has a limit and $\lim_{t\to+\infty} \underline{a}^t$ is a fixed point of Γ:

  Since the sequence is increasing and $\mathcal{A}$ is a complete lattice, it has a limit $\underline{a}^*$. Furthermore, since Γ is subcomplete, upper semi-continuity of Γ implies that $\underline{a}^* \in \Gamma(\underline{a}^*)$.

- $\underline{a}^* \equiv \inf_{E_\Gamma} \{E_\Gamma\}$:

  According to the previous demonstration, since the fixed points of Γ are in the set $\mathbb{P}_{\mathcal{A}}$, all fixed points must be in $[\underline{a}^*, +\infty[$ and so $\underline{a}^*$ is the smallest fixed point.

Defining $\bar{a}^0$ and $\bar{a}^t$ as:

$$\bar{a}^0 := \sup \mathcal{A} \tag{3.32}$$
$$\bar{a}^t := \sup_{\mathcal{A}} \Gamma\big(\bar{a}^{t-1}\big) \qquad \forall\, t \geq 1 \tag{3.33}$$

In an analogous way we obtain that $\mathbb{P}_{\mathcal{A}} \subseteq \;\;]-\infty, \bar{a}^*]$, with $\bar{a}^*$ being the greatest fixed point of Γ.

■

Proof of Proposition 3.38

Proof.

Following the proof of Proposition 3.28, consider the sequence $\{\underline{a}^t\}_{t=0}^{\infty}$ therein defined, but let us change the definition of $\underline{a}^t$ when t is odd to:

$$\underline{a}^t := \sup_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big)\,.$$

By the definition of $\underline{a}^0$, we know that $\forall\ a \in \mathcal{A}$, $a \geq \underline{a}^0$. Since the mappings $B(i,\cdot)$ are decreasing we have $B(i,\underline{a}^0) \succeq B(i,a)\ \forall\ a \in \mathcal{A}$ and in particular

$$\sup_S B\big(i,\underline{a}^0\big) \geq y \quad \forall\ y \in B(i,a)\ \ \forall\ a \in \mathcal{A}$$

Since $B(i,\underline{a}^0)$ is subcomplete $\sup_S B(i,\underline{a}^0) \in B(i,\underline{a}^0)$ and from Lemma 3.48 the function $i \to \sup_S B(i,\underline{a}^0)$ is measurable, so $\int \sup_S B(i,\underline{a}^0) \in \Gamma(\underline{a}^0)$, thus

$$\sup_{\mathcal{A}} \Gamma\big(\underline{a}^0\big) \geq \int \sup_S B\big(i,\underline{a}^0\big)\,\mathrm{di} \geq \int s(i)\,\mathrm{di}$$

for any measurable selection $\mathbf{s}$ of $i \rightrightarrows B(i,\mathcal{A})$. That is $\underline{a}^1 \geq a \; \forall \, a \in \tilde{Pr}^1(\mathcal{A})$; or equivalently,

$$\tilde{Pr}^1(\mathcal{A}) \subseteq \left]-\infty, \underline{a}^1\right].$$

A similar argument leads to conclude that $\tilde{Pr}^2(\mathcal{A}) \subseteq [\underline{a}^2, +\infty[$.

Let us define then the sequence $\underline{b}^t := \underline{a}^{2t}$, $t \geq 0$. This sequence satisfies:

1. $\tilde{Pr}^{2t} \subseteq \left[\underline{b}^t, +\infty\right[$. This can be obtained as above by induction over t.
2. $\left\{\underline{b}^t\right\}_{t\geq 0}$ is increasing.

As before, we get that $\left\{\underline{b}^t\right\}_{t\geq 0}$ has a limit $\underline{b}^*$. Since Γ is u.s.c. and $\mathcal{A}$ is compact, we obtain that the second iterate of Γ, Γ^2 is as well u.s.c.. Moreover, from Proposition 3.36, we get that $\underline{b}^t \in \Gamma^2\left(\underline{b}^{t-1}\right)$. This implies that $\underline{b}^*$ is a fixed point of Γ^2 and so it is a point-rationalizable state. Consequently we get

1. $\mathbb{P}_{\mathcal{A}} \subseteq [\underline{b}^*, +\infty[$
2. $\underline{b}^* \in \Gamma^2(\underline{b}^*)$ and $\underline{b}^*$ is a point-rationalizable state.

Considering the analogous sequence to obtain the upper bound for $\mathbb{P}_{\mathcal{A}}$:

$$\begin{aligned}
\bar{a}^0 &:= \sup \mathcal{A} & & \\
\bar{a}^t &:= \inf_{\mathcal{A}} \Gamma\left(\bar{a}^{t-1}\right) & & \text{when } t \text{ is odd} \\
\bar{a}^t &:= \sup_{\mathcal{A}} \Gamma\left(\bar{a}^{t-1}\right) & & \text{when } t \text{ is even}
\end{aligned} \tag{3.34}$$

We generate a decreasing sequence $\left\{\bar{b}^t\right\}_{t\geq 0}$ defined by $\bar{b}^t := \bar{a}^{2t}$, $t \geq 0$, who's limit $\bar{b}^*$, is a point-rationalizable state and an upper bound for $\mathbb{P}_{\mathcal{A}}$, that is:

1. $\mathbb{P}_{\mathcal{A}} \subseteq \left]-\infty, \bar{b}^*\right]$
2. $\bar{b}^* \in \Gamma^2\left(\bar{b}^*\right)$. Which implies that $\bar{b}^*$ is a point-rationalizable state.

As a summary, we get:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t\geq 0} \tilde{Pr}^t(\mathcal{A}) \subseteq \bigcap_{t\geq 0} \tilde{Pr}^{2t}(\mathcal{A}) \subseteq \left[\underline{b}^*, \bar{b}^*\right] \tag{3.35}$$

■

CHAPTER 4

The Cournot Outcome as the Result of Price Competition

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The Cournot Outcome as the Result of Price Competition

Abstract

In a homogeneous product duopoly with concave revenue and convex costs we study a two stage game in which, first, firms engage simultaneously in capacity (production) and, after production levels are made public, there is sequential price competition in the second stage. Randomizing the order of play in the price subgame, we can find: (i) that the Cournot outcome can be sustained as a pure strategy subgame perfect Nash equilibrium (SPNE) of the whole game, (ii) a SPNE in which firms produce strictly more than the Cournot outcome.

1 Introduction

In their paper of (1983), Kreps and Scheinkman state that the difference between Cournot and Bertrand competition is more than just the strategy space, but that timing of decisions is also relevant. To illustrate this, they study and solve a Bertrand like duopoly model of competition where timing of decision is inverted. Capacity decision is made simultaneously and before price decision (as opposed to Bertrand-Edgeworth models where the choice of capacity and price is interpreted as being simultaneous), and the low priced firm may not serve all the demand at her price (as it is in the Bertrand approach) due to capacity constraints. In a two stage game where firms first set simultaneously capacity and then engage in simultaneous price competition with demand rationed following the efficient rationing rule, the unique Subgame Perfect Nash Equilibrium (SPNE) has as outcome the Cournot quantities and prices. This result has been subject to criticism. Davidson and Deneckere (1986) argue that the Kreps and Scheinkman result depends strongly on the chosen rationing rule and that in fact, it is not likely that the Cournot outcome may raise as the result of the price game in most of the cases. However, their proof assumes value zero for the cost in the first (capacity-setting) stage. Moreover, for an important range of capacities the unique Nash Equilibrium of the price-setting subgame in Kreps and Scheinkman (1983) is in non degenerate mixed strategies, rendering the result hard to interpret and motivating a literature that argues that the only possible pure strategy Nash Equilibrium of price games must be the competitive outcome (see Allen and Hellwig (1986) or Dixon (1992)).

Related to the first issue Madden (1998) shows, in a slightly different framework, that for uniformly elastic demands the Kreps and Scheinkman result holds, even if proportional rationing is used. Related to the second one, we have that departing from the original constant marginal cost and no capacity constraints hypothesis (Bertrand 1883), we pass to models with voluntary trading constraint at the competitive supply or strictly increasing marginal costs (Edgeworth 1925), which lead to problems with existence of equilibrium in pure-strategies. Liberating the supply constraint to any possible quantity in order to restore pure-strategy equilibria existence when convex costs are present, we end up with the competitive outcome. These Bertrand-Edgeworth approaches to oligopolistic competition, however, still feature the main difference between Bertrand and Cournot models that Kreps and Scheinkman were assessing. Namely, the production decision is actually done after prices and demands are revealed. For instance, in Dixon (1992) the equilibrium-stated-intention of serving the whole demand can be interpreted as a non credible threat or a unsustainable engagement since once a firm realizes that it has the lowest price it is not rational to serve more than the competitive supply at the announced price. Thus we are back at the timing of decisions issue.

The objective of this note is then threefold: without changing any assumptions on primitives and maintaining the timing of decisions (prices after quantities), provide a setting where the Cournot outcome can be sustained as a SPNE in pure strategies of the duopoly game, give space for the possibility of finding non-Cournot outcomes sustained as SPNE in pure strategies and give a clear condition on rationing rules that rejects Cournot as the outcome of equilibrium.

We explore a setting where in a first stage firms set capacity simultaneously and in a second stage engage in sequential price competition where the order of play is assigned randomly. The possibility of obtaining clear pure strategy subgame perfect equilibria for each subgame will help to clearly understand the strategic interaction in the duopoly framework that results in the Cournot outcome, although competition in prices is assumed. We will see that under the same assumptions on primitives as in the seting of Kreps and Scheinkman the Cournot outcome is sustained as a subgame perfect equilibrium. We will also see a simple condition on rationing rules to reject the Cournot outcome in both the Kreps and Scheinkman and the current setting. A last new feature of the approach presented in this paper is the possibility of finding non Cournot outcomes sustained as SPNE without changing any of the assumptions on demand rationing (Davidson and Deneckere 1986; Dixon 1992), nor on concavity or continuity of involved functions nor cost structure (Vives 1986; Deneckere and Kovenock 1992; Deneckere and Kovenock 1996), but only based on uncertainty of leadership and the dynamic nature of the price subgame. We explore this issue and give conditions that allow for it to occur.

The paper provides a clear and more tractable story for the Cournot outcome as the result of price competition.

2 The Model

Consider a market with two identical firms a and b that produce a homogeneous good and engage in a two stage competitive situation. In a first stage, firms simultaneously and independently build capacity (which can be understood as a decision in production quantities). A vector of capacity quantities $k = (k_a, k_b)$ is understood as an available supply on the market of $k_a + k_b$ to be traded after the second stage. In the second stage, after learning how much capacity their opponent installed, firms set prices.

Aggregate demand for the firms outputs as a function of price is given by the function $D : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that $0 < D(0) < +\infty$, that there is an upper bound for the set where $D(p) > 0$, and that D is continuous and strictly decreasing in this set. With this we can consider the Price function $P : \mathbb{R}_+ \to [0, P(0)]$ defined by $P(q) := D^{-1}(q)$ for $q \in [0, D(0)]$ and $P(q) := 0$ for $q > D(0)$. The cost of producing is given by a function $C : \mathbb{R}_+ \to \mathbb{R}_+$.

We make the following assumptions:

1. The function $p \to pD(p)$ is strictly concave.

2. Function C is strictly increasing, convex, and $C(0) = 0$; P satisfies

$$P(0) > \lim_{k \to 0} C'(k) > 0.$$

With these elements we can define the Cournot best response mapping $r_C : \mathbb{R}_+ \to [0, D(0)]$ as:

$$r_C(k) := \begin{cases} \operatorname{argmax}_{x \in [0, D(0)-k]} \{P(k+x)\, x - C(x)\} & \text{if } k \leq D(0) \\ 0 & \text{if } k > D(0) \end{cases}$$

For simplicity we note $r(k) \equiv r_0(k)$. Assumptions 1 and 2 imply that r_C is strictly decreasing when it is positive, that the function $k \to r(k) + k$ is non-decreasing (strictly increassing in the interval where $r(k) > 0$) and that if we have two cost functions C_1 and C_2 with $C_1' > C_2'$ then $r_{C_1}(y) < r_{C_2}(y)\ \forall\, y \in [0, D(0)]$ such that $0 < r_{C_2}(y)$ (see Kreps and Scheinkman (1983)). Thus, for each function C as above there is a unique Cournot equilibrium outcome with price and production quantities (p_C^*, k_C^*, k_C^*). That is, k_C^* is defined by:

$$r_C(k_C^*) = k_C^*$$

and p_C^* is the price that solves:

$$D(p_C^*) = 2k_C^*$$

or equivalently

$$p_C^* = P(2k_C^*)$$

And if $C_1' > C_2'$ then $k_{C_1}^* < k_{C_2}^*$ and $p_{C_1}^* > p_{C_2}^*$. We will note k^* and p^* for k_0^* and p_0^* respectively. Finally we note $R(k)$ the value:

$$R(k) := P(r(k) + k)\, r(k),$$

the revenue associated with the best response to a quantity k when there is no production cost and p^m the zero cost monopoly price[1].

Net payoffs of the game that we will study are given by the function $\pi_i : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$

[1] $p^m = \operatorname{argmax} pD(p)$

$i \in \{a, b\}$:

$$\pi_i(k,p) := \begin{cases} p_i \min\{k_i, D(p_i)\} - C(k_i) & \text{if } p_i < p_j \\ p_i \min\left\{k_i, \max\left\{\frac{D(p_i)}{2}, D(p_i) - k_j\right\}\right\} - C(k_i) & \text{if } p_i = p_j \\ p_i \min\{k_i, \max\{0, D(p_i) - k_j\}\} - C(k_i) & \text{if } p_i > p_j \end{cases}$$

Kreps and Scheinkman (1983) show that[2] in this setting, the only subgame perfect Nash equilibrium outcome of a two stage game with *simultaneous* price competition in the second stage, is the Cournot outcome. Note that we are using, as in their work, the surplus maximizing rationing rule. The choice of rationing rule is not obvious. For instance, Davidson and Deneckere (1986) show that not all standard rationing rules lead to obtain Cournot price and quantity as the outcome of two stage game with simultaneous price setting. We address this issue as well in this work in a further section.

3 Equilibrium With Simultaneous Price-Setting Subgame

In this section we recall the equilibrium outcome of the whole game in the case were firms set prices simultaneously in the second stage subgame. We do this by solving the second stage price subgame and obtaining a reduced form game in capacities. The simultaneous price capacity constrained subgame has been analyzed by Levitan and Shubik (1972) for the case of linear demand, no cost and equal capacities; and with more generality by Kreps and Scheinkman (1983) and Osborne and Pitchik (1986). Their results are very well summarized by Deneckere and Kovenock (1992) in their Theorem 1 which we reproduce below as Proposition 4.2. In the price subgame, since capacity has already been set, cost is irrelevant and firms only care about maximizing their income from sales. To continue, we first need to define certain elements.

Definition 4.1. For $k_i > 0$, $(i,j) \in \{(a,b),(b,a)\}$. We define $\underline{p}(k_i, k_j)$ as follows:

(i) If $k_i \geq r(k_j)$ then let $\underline{p}(k_i, k_j)$ be the smallest solution[3] of the equation:

$$p \min\{k_i, D(p)\} = R(k_j) \tag{4.1}$$

(ii) If $k_i < r(k_j)$ then $\underline{p}(k_i, k_j) := P(k_a + k_b)$

[2]See Section 3.

[3]Concavity of $pD(p)$ and the fact that it takes value 0 in $p = 0$ and $p = P(0)$, assure that (4.1) has at most two solutions.

$R(k_j)$, the revenue for firm i when playing the best response to j in the zero cost Cournot game, is the maximum revenue that firm i can receive by being the high priced firm when having enough capacity. Indeed, revenue for i being the high priced firm is equal to $p_i \min\{k_i, \max\{0, D(p_i) - k_j\}\}$ which, if $k_i \geq r(k_j)$ turns into $p_i \max\{0, D(p_i) - k_j\}$ and is maximized at $p_i = P(r(k_j) + k_j)$ when $k_j \leq D(0)$ and at $\mathbb{R}_+$ when $k_j > D(0)$, with the value $P(r(k_j) + k_j)\, r(k_j) \equiv R(k_j)$ (if $k_j > D(0)$ then both $P(r(k_j) + k_j)$ and $r(k_j)$ equal 0). When $k_i \geq r(k_j)$, price $\underline{p}(k_i, k_j)$ equals the payoff of being the low priced firm to $R(k_j)$, that is, price $\underline{p}(k_i, k_j)$ makes firm i indifferent between being the high priced firm and the low priced firm. If $k_i < r(k_j)$, then the previous analysis does not make sense, but in this case payoff is bounded from above by pk_i for any $p \leq P(k_a + k_b)$ and if $p > P(k_a + k_b)$ then it is in the decreasing part of $p \max\{0, D(p) - k_j\}$. Roughly, if firm j sets a price lower than $\underline{p}(k_i, k_j)$ then firm i would rather set the highest between the residual demand monopoly price $P(r(k_j) + k_j)$ and the competitive price $P(k_a + k_b)$, and if j sets a higher price then firm i would rather undercut or match. Finally note that if $k_i \geq r(k_j)$ then $\underline{p}(k_i, k_j) > P(k_a + k_b)$.

Now we can state the result:

Proposition 4.2 (Kreps and Scheinkman (1983, Deneckere and Kovenock (1992))**.** *For each pair (k_i, k_j) such that $k_i \geq k_j$ for $(i, j) \in \{(a, b), (b, a)\}$, in terms of the simultaneous price-setting subgame we have three regions of interest:*

1. *If $k_j \geq D(0)$, the equilibrium is in pure strategies with both firms naming price zero and net zero profit.*
2. *If $k_i \leq r(k_j)$ then the equilibrium is in pure strategies, both firms name price $P(k_a + k_b)$ and net $k_a P(k_a + k_b)$ and $k_b P(k_a + k_b)$.*
3. *If $k_i > r(k_j)$, then the equilibrium is in non-degenerate mixed strategies with common support*
$$\left[\underline{p}(k_i, k_j)\, , P(r(k_j) + k_j)\right].$$
Equilibrium payoffs are $R(k_j)$ for firm i and $\underline{p}(k_i, k_j)\, k_j$ for firm j.

As it is very well known, case 1 is the standard Bertrand result. In case 2 capacity is binding over the residual demand monopoly payoff maximum and thus firms are not interested in being the high-priced one. Clear-market price is the highest they can set with both selling at full capacity. Case 3 is the Edgeworth-cycle case, where there are no pure strategy equilibria. Now that we have obtained the reduced form payoffs we can state that in the full game there is a unique equilibrium outcome, namely the Cournot outcome.

Proposition 4.3 (Kreps and Scheinkman (1983))**.** *In the two-stage game, the unique equilibrium outcome is given by: $k_a = k_b = k_C^*$ and $p_a = p_b = p_C^*$.*

Uniqueness is demonstrated in Kreps and Scheinkman (1983). To see that the Cournot outcome is an equilibrium let's look at the reduced form payoff for firm a given that her rival b sets capacity $k_b = k_C^*$, $\pi_a(k_a, k_C^*)$. From Proposition 4.2, and the fact that if $k_a \leq k_b = k_C^*$ $(= r_C(k_C^*) < r(k_C^*))$ then $k_b < r(k_a)$, we have:

$$\pi_a(k_a, k_C^*) = \begin{cases} k_a P(k_a + k_C^*) - C(k_a) & \text{if } k_a \leq r(k_C^*) \\ R(k_C^*) - C(k_a) & \text{if } k_a > r(k_C^*) \end{cases}$$

Clearly $\pi_a(k_a, k_C^*)$ is strictly decreasing for $k_a > r(k_C^*)$. Even more, from what we said in section 2, the function $k_a \to k_a P(k_a + k_C^*) - C(k_a)$ attains it's maximum at $r_C(k_C^*) < r(k_C^*)$ and thus $k_a = k_C^*$ is the best response to $k_b = k_C^*$ in the reduced form capacity-setting game.

4 Equilibria With Sequential Price-setting Subgame

We now turn to a variation in which price-setting is done sequentially, and the order of play is randomly determined. In this setting, firms will first (simultaneously) set capacity without knowing whether they will be leader or follower in the price-setting subgame. Once capacities are set, nature announces the leader, who proceeds to set her price. Knowing the value of this price, her rival proceeds to set her price. Once capacities and prices are set, trade takes place. Before capacities are determined, firms know that they have a probability of being leader in the price subgame. They maximize their expected profit, expectations coming from the random event of being leader or follower.

4.1 Price-setting subgame

We begin by solving the second stage subgame. Once capacities are set and nature has determined the order of play, firms engage in sequential price competition.

We will name by firm 1 the firm that sets the price first and firm 2 the one who goes second. We obtain a capacity constrained Stackelberg price game. This kind of game has been analyzed in Deneckere and Kovenock (1992), and more recently in Dastidar (2004), where capacity is assumed to be the competitive supply at the announced price. In both approaches, following Deneckere and Kovenock (1992), demand is assigned first to the follower and then to the leader in order to have no problems with upper semi-continuity of payoffs at their suppremum.

It is well known that in the Kreps and Scheinkman setting (splitting demand on tied prices) there is no subgame perfect equilibrium on pure strategies in the price setting subgame, since for many prices p set by the leader, the follower does not have a well defined best response. For many of these cases we can argue that firm 2 infinitesimally undercuts firm 1 when out of the equilibrium path, allowing us to find pure strategy Nash Equilibria of the subgame. This occurs when the high capacity firm acts as follower. Nevertheless, we still have the non existence problem when this firm is leader. There is a whole set of pairs (k_a, k_b) for which no equilibrium exists. Below we show that the continuity problem being solved by the Deneckere and Kovenock (1992) assumption on demand rationing, does not affect the economic conclusions of the model and effectively helps to clarify the behavior of the agents.

For the moment, the only modification from the setting of section 3 is to pass from simultaneous to sequential price-setting, without modifying the rationing rule. We summarize the results of this setting in Proposition 4.6 below. Later on, we will see that modifying the game as in Deneckere and Kovenock (1992), we can obtain very close reduced payoff functions justifying in this way the use of the modifications for what follows.

We begin by stating the following lemma, whose demonstration is the same as in the simultaneous subgame case, and can be found in Kreps and Scheinkman (1983).

Lemma 4.4. *In equilibrium, none of the firms will announce a price smaller than $P(k_a+k_b)$, since it is a dominated strategy.*

A very useful result regarding the limit prices $\underline{p}(k_i, k_j)$ defined in (4.1) is:

Lemma 4.5.
$$\left.\begin{aligned} \min\{D(0), k_i\} &> k_j \\ k_j &> r(k_i) \end{aligned}\right\} \quad \Longrightarrow \quad k_i R(k_i) < k_j R(k_j)$$

Moreover if $k_i, k_j > 0$,
$$\underline{p}(k_i, k_j) > \underline{p}(k_j, k_i)$$

Proof. For the first part see proof of Lemma 5 in Kreps and Scheinkman (1983). For the second part we have:
$$\underline{p}(k_i, k_j) \geq \frac{R(k_j)}{k_i} > \frac{R(k_i)}{k_j} = \underline{p}(k_j, k_i)$$

The first inequality comes from the definition of $\underline{p}(k_i, k_j)$ and the second one is the first part of the Lemma. The equality holds since we have that

$$\underline{p}(k_j, k_i) < P(r(k_i) + k_i) < P(r(k_j) + k_j) \leq P(k_j)$$

and so $\min\left\{k_j, D\big(\underline{p}(k_j, k_i)\big)\right\} = k_j$.

■

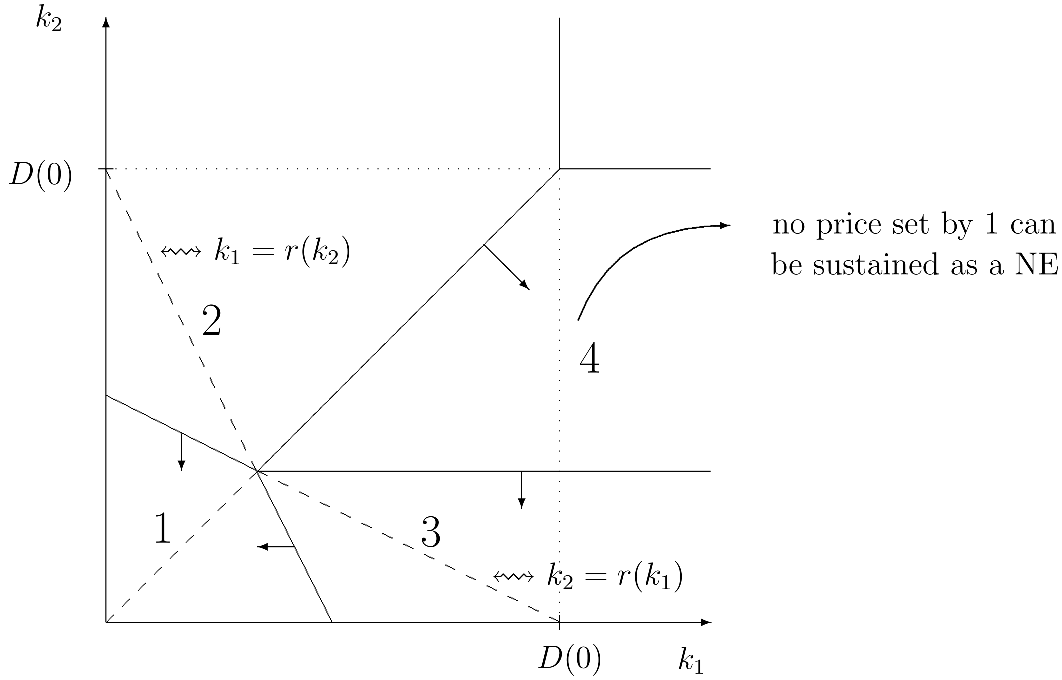

Figure 4.1: The four regions of Proposition 4.6. The arrows indicate at what region belong the boundaries. In region 4 there is no (pure strategy) Nash Equilibrium in the sequential price subgame.

Proposition 4.6. *With respect to the sequential price-setting subgame, if* $\min\{k_1, k_2\} < D(0)$ *we can distinguish four regions of interest:*

1. *If* $k_1 \leq r(k_2)$ *and* $k_2 \leq r(k_1)$*, there exists a pure strategy Nash Equilibrium (*NE*) with payoff* $\pi_1(k_1, k_2) = P(k_1 + k_2)k_1$ *for firm 1 and* $\pi_2(k_1, k_2) = P(k_1 + k_2)k_2$ *for firm 2. In equilibrium both firms set price* $P(k_1 + k_2)$*.*

2. *If* $k_1 < k_2$ *and* $k_2 > r(k_1)$*, then there exist a continuum of pure strategy Nash Equilibria indexed by* p_1^* *where firm 1 nets* $\pi_1(k_1, k_2) = p_1^* k_1$ *and firm 2 has payoff* $\pi_2(k_1, k_2) = R(k_1)$*.*
   In equilibrium, if $k_1 > 0$ *firm 1 sets price* $p_1^* \in \left[\max\left\{P(k_1 + k_2), \underline{p}(k_1, k_2)\right\}, \underline{p}(k_2, k_1)\right]$ *and firm 2 responds by setting* $P(r(k_1) + k_1)$*.*

3. *If* $k_1 > r(k_2)$ *and* $k_2 \leq r(k_2)$ *there exists a pure strategy* NE *with payoffs* $\pi_1(k_1, k_2) = P(r(k_2) + k_2)r(k_2) = R(k_2)$ *and* $\pi_2(k_1, k_2) = P(r(k_2) + k_2)k_2$*.*
   The outcome of the subgame is with both players announcing $P(r(k_2) + k_2)$*.*

4. *If* $k_1 \geq k_2$ *and* $k_2 > r(k_2)$*, then* **there is no pure strategy** NE **in the price-setting subgame***.*

Proof. For the first three cases we will give the NE strategies.

In 1 consider the strategy for firm 2 that matches the price of firm 1 when she sets $p_1 = P(k_1 + k_2)$ and undercuts for higher prices; and for firm 1 the strategy $p_1 = P(k_1 + k_2)$. With these strategies payoffs as function of their own prices is, for each firm:

$$\pi_i(p_i) = \begin{cases} p_i k_i, & \text{if } p_i \leq P(k_1 + k_2) \\ p_i \max\{0, D(p_i) - k_j\}, & \text{if } p_i > P(k_1 + k_2) \end{cases}$$

which is continuous and, since $k_i \geq r(k_j)$, it is maximized at $p_i = P(k_1 + k_2)$

In 2 if $k_1 = 0$ then firm 2 is a monopolist and the result holds. If $k_1 > 0$, from Lemma 4.5 we have:

$$\underline{p}(k_2, k_1) > \max\{P(k_1 + k_2), \underline{p}(k_1, k_2)\}.$$

Then, for a given $p_1^* \in \left[\max\{P(k_1 + k_2), \underline{p}(k_1, k_2)\}, \underline{p}(k_2, k_1)\right]$, consider the strategies where firm 2 sets the residual demand monopoly price $P(r(k_1) + k_1)$ when $p_1 \leq p_1^*$ and undercuts when $p_1 > p_1^*$, and firm 1 sets p_1^*. Since $p_1^* \leq \underline{p}(k_2, k_1)$, firm 2, facing p_1^*, maximizes it's payoff by setting the price $P(r(k_1) + k_1)$. For firm 1, given the strategy of firm 2 the payoff is:

$$\pi_1(p_1, p_2(p_1)) = \begin{cases} p_1 k_1, & \text{if } p_1 \leq p_1^* \\ p_1 \max\left\{0, (D(p_1) - k_2)\right\}, & \text{if} p_1^* < p_1 \end{cases}$$

Since $p_1^* \geq \underline{p}(k_1, k_2)$, we have:

$$\pi_1(p_1^*, p_2(p_1^*)) \geq R(k_2)$$

which gives the result.

In 3, if $k_2 = 0$ then firm 1 is monopolist and the result holds. If $k_2 > 0$ consider the strategies where firm 1 sets $P(r(k_2) + k_2)$ and firm 2 undercuts if $p_1 > P(2k_2)$ matches if $p_1 \in \left[\underline{p}(k_2, k_1), P(2k_2)\right]$ and takes the residual demand monopoly price $P(r(k_1) + k_1)$ if $p_1 < \underline{p}(k_2, k_1)$. Given the strategy of firm 2, payoff for firm 1 is:

$$\pi_1(p_1, p_2(p_1)) = \begin{cases} p_1 \min\left\{k_1, D(p_1)\right\} & \text{if } p_1 < \underline{p}(k_2, k_1) \\ p_1(D(p_1) - k_2) & \text{if } \underline{p}(k_2, k_1) \leq p_1 \leq P(2k_2) \\ p_1 \min\left\{0, D(p_1) - k_2\right\} & \text{if } p_1 > P(2k_2) \end{cases}$$

This function attains it's maximum at $p_1^* = P(r(k_2) + k_2) \in \left[\underline{p}(k_2, k_1), P(2k_2)\right]$. Indeed, if $k_2 \leq r(k_1)$ then $P(r(k_1) + k_1) \leq P(k_1 + k_2)$ and thus $\underline{p}(k_2, k_1) \leq P(k_1 + k_2) < P(2k_2)$; if $k_2 > r(k_1)$, applying the same reasoning as in the proof of 2 we have $\underline{p}(k_2, k_1) < \underline{p}(k_1, k_2) \leq P(r(k_2) + k_2) \leq P(2k_2)$. It remains to check that firm 2 is maximizing it's payoff given the strategy of firm 1. Since firm 1 sets price $P(r(k_2)+k_2) > P(r(k_1)+k_1)$, the only possibility for firm 2 to maximize it's payoff is to match firm 1's price (higher prices are in the decreasing part of the residual demand associated profit, and smaller prices are in the increasing part of the low priced firm profit). So we have to check that matching gives a maximum (that is, that firm 2's payoff is upper semi-continuous at p_1^*). Since $k_2 \leq r(k_2)$ we have that $\min\left\{k_2, \max\left\{\frac{r(k_2)+k_2}{2}, r(k_2)\right\}\right\} = k_2$, with what we obtain the result.

Finally, for 4 first note that we still have $\underline{p}(k_1,k_2) > \underline{p}(k_2,k_1)$, and that the previous equilibrium outcome can no longer be, since now firm 2 does not have a best response[4] to $p_1 = P(r(k_2)+k_2)$. Note as well that by setting $p_1 = P(r(k_2)+k_2)$ firm 1 can assure a payoff of at least $R(k_2)$. We now prove that no price set by firm 1 can be sustained as a Nash Equilibrium of the subgame.

- If $p_1^* > \max\left\{P(2k_2), \underline{p}(k_2,k_1)\right\}$ or $p_1^* = \underline{p}(k_2,k_1) > P(2k_2)$, then firm 2 does not have a best response.
- If $p_1^* < \underline{p}(k_2,k_1)$ or $p_1^* = \underline{p}(k_2,k_1) < P(2k_2)$, then we have $p_1^* < \underline{p}(k_1,k_2)$. If p_1^* were an equilibrium then the best response of firm 2 would be such that $p_2(p_1^*) = P(r(k_1)+k_1) > p_1^*$ and thus firm 1 would net $p_1^* k_1 < R(k_2)$. Then, setting price $p_1 = P(r(k_2)+k_2)$, firm 1 can assure striclty greater payoff. Thus, p_1^* cannot be a NE of the subgame.
- If $p_1^* = \underline{p}(k_2,k_1) = P(2k_2)$ then firm 2 has two possible best responses: p_1^* and $P(r(k_1)+k_1)$. If it sets the second price, then we are back in the preceding case. If it sets p_1^* we still have that the profit of firm 1 is strictly lower than $R(k_2)$ and then it can gain more by setting the residual demand monopoly price.
- Now suppose that $\left]P(2k_2), \underline{p}(k_2,k_1)\right] \neq \emptyset$ and consider p_1^* in this interval. Firm 2, facing such a price, sets $p_2(p_1^*) = p_1^*$ and firm 1 nets $p_1^*(D(p_1^*)-k_2) < R(k_2)$ and again by setting $p_1 = P(r(k_2)+k_2)$ firm 1 assures at least $R(k_2)$ and thus p_1^* is not an equilibrium.

■

[4]Since $k_2 > r(k_2)$, now

$$P(r(k_2)+k_2)\min\left\{k_2, \max\left\{\frac{r(k_2)+k_2}{2}, r(k_2)\right\}\right\}$$
$$= P(r(k_2)+k_2)\,\frac{r(k_2)+k_2}{2} \;<\; P(r(k_2)+k_2)\,k_2$$
$$= \lim_{p \nearrow P(r(k_2)+k_2)} pk_2.$$

In item 2, the actual equilibrium price depends obviously in the strategy of firm 2. Firm 2 can threaten to undercut a price higher to p_1^* and firm 1 will have to set this price, this would require a degree of communication between firms that we are not assuming. Of course, as usual in sequential interaction, if firm 1 sets price $\underline{p}(k_2, k_1)$, it would not be optimal for firm two to undercut this price following a previous threat, so firm 1 can actually force the highest profit equilibrium. Thus the most reasonable equilibrium is for firm 1 to set price $\underline{p}(k_2, k_1)$. This equilibrium is not in general subgame perfect since the actions chosen by firm 2 out of the equilibrium path are not necessarily optimal.

Remark 4.7. The relevant difference between cases 3 and 4 is that in 4 it is not optimal for firm 2 to set the price $P(r(k_2) + k_2)$ when firm 1 sets this price, because of the downward jump in its revenue function. Nevertheless, under a subgame perfect equilibrium logic, we can think of the strategy where firm 2 undercuts if firm 1's price is strictly greater than $\underline{p}(k_2, k_1)$ and takes the residual demand monopoly price if not. Given this strategy for firm two, it is optimal for firm 1 to set the price $P(r(k_2) + k_2)$. Then $\forall\, \varepsilon > 0$, there would exist $\delta > 0$ such that firm 2 could announce a price $P(r(k_2) + k_2) - \delta$ in order to secure a revenue $u_2^\delta(k_2) := (P(r(k_2) + k_2) - \delta)k_2$ that satisfies $P(r(k_2) + k_2)k_2 - \varepsilon < u_2^\delta(k_2) < P(r(k_2) + k_2)k_2$ and $\pi_1(k_1, k_2) = P(r(k_2) + k_2)r(k_2) = R(k_2)$. That is, a "reasonable" revenue outcome for this subgame would be $\pi_1(k_1, k_2) = P(r(k_2) + k_2)\, r(k_2) = R(k_2)$ and $\pi_2(k_1, k_2) = u_2^\delta(k_2) \xrightarrow[\varepsilon \to 0]{} P(r(k_2) + k_2)k_2$.

With Proposition 4.6 we can see that introducing the dynamic nature of the price subgame on the Kreps and Scheinkman setting it is possible to find pure strategy Nash equilibria in a subset of the region where there is none in the simultaneous case. That is when the high capacity firm acts as follower, but we still have the non existence problem when this firm is leader. Nevertheless, we have gained clarity about strategic price setting in the capacity constrained price game. However, the results are still not completely neat. Note that in the regions where there exists pure strategy Nash Equilibrium in the sequential subgame the leader can net the exact same profit as in the simultaneous subgame, while in case 4, given the previous remark, the simultaneous subgame payoff can be approximated for the leader. The follower can get a strictly greater payoff when being the small firm.

In order to continue analyzing the game, we want to have a setting where we are actually able to find Nash Equilibria for all pairs of capacities $(k_a, k_b) \in \mathbb{R}_+^2$. Of course it could be possible to find a Nash Equilibrium of the whole game by choosing appropriately the actions of the firms when out of the equilibrium path (in fact, as we will see later, the previous description is pretty accurate about how firms will strategically behave). But the aim of the paper is precisely to describe the strategic interaction at all levels in order to be able to neatly obtain the outcome of the whole game.

So now we will present the results when the variation of the sequential price-setting game is used. In this variation, following Deneckere and Kovenock (1992), demand splitting in tied prices is changed. In order to have upper semi-continuity of payoff functions at their suppremum, we will assume that on tied prices, all demand goes first to the follower (and is not equally shared as in Kreps and Scheinkman (1983)). Thus when matching a price p_1, firm 2 will net $\pi_2(k, p_1, p_1) = \lim_{p \nearrow p_1} \pi_2(k, p_1, p)$. Formally, tied prices revenues become,

$$\pi_1(k_1, k_2, p, p) := p \min \{k_1, \max \{0, D(p) - k_2\}\}$$
$$\pi_2(k_1, k_2, p, p) := p \min \{k_2, , D(p)\}$$

That is, given capacities (k_a, k_b) and price p_1, for firm 2 we use the *upper semi-continuous hull* of the payoff function of the Kreps and Scheinkman setting, and for firm 1, given p_2 we use the *lower semi-continuous hull* of the payoff function of the Kreps and Scheinkman setting.

With this “slight” modification the sequential price-setting subgame has always (meaning for all capacity pairs) a subgame perfect equilibrium in pure strategies. The main implication of the modification is that now player two has always a best response (since for all p_1 it's payoff as a function of price is *sup-compact* and upper semi-continuous). The strategy for firm 2 will simply be to match the price of firm 1 if it is greater or equal to $\underline{p}(k_2, k_1)$ and take the residual monopoly price if not. The outcome of the game will depend on the action of firm 1, since it's profit maximizing price, given the strategy of firm 2, may depend on the first stage choice of capacity through the relative values of the prices $\underline{p}(k_1, k_2)$ and $\underline{p}(k_2, k_1)$. Proposition 4.8 below is similar to Theorems 2 and 3 of Deneckere and Kovenock (1992). The difference is that here we explicitly give the payoff of the low priced firm in the subgame where capacities are in the range where the simultaneous price-setting game has equilibrium in mixed strategies.

Proposition 4.8. *With respect to the modified sequential price-setting subgame we can distinguish five regions of interest:*

1. *If $k_1 \leq r(k_2)$ and $k_2 \leq r(k_1)$, there is a unique pure strategy subgame perfect Nash Equilibrium (*SPNE*) in which both firms set price $P(k_1 + k_2)$. Payoffs are $\pi_1(k_1, k_2) = P(k_1 + k_2)k_1$ and $\pi_2(k_1, k_2) = P(k_1 + k_2)k_2$.*

2. *If $k_1 < k_2$, $k_1 < D(0)$ and $k_2 > r(k_1)$ then there is a unique* SPNE *where firm 1 sets price $\underline{p}(k_2, k_1)$ and firm 2 responds by setting $P(r(k_1) + k_1)$. Payoffs are $\pi_1(k_1, k_2) = \underline{p}(k_2, k_1)\, k_1$ and $\pi_2(k_1, k_2) = R(k_1)$*

3. *If $k_1 > k_2$, $k_2 < D(0)$ and $k_1 > r(k_2)$ then there there is a unique* SPNE *where both firms set price $P(r(k_2) + k_2)$. Payoffs are $\pi_1(k_1, k_2) = P(r(k_2) + k_2)r(k_2) = R(k_2)$ and $\pi_2(k_1, k_2) = P(r(k_2) + k_2)k_2$.*

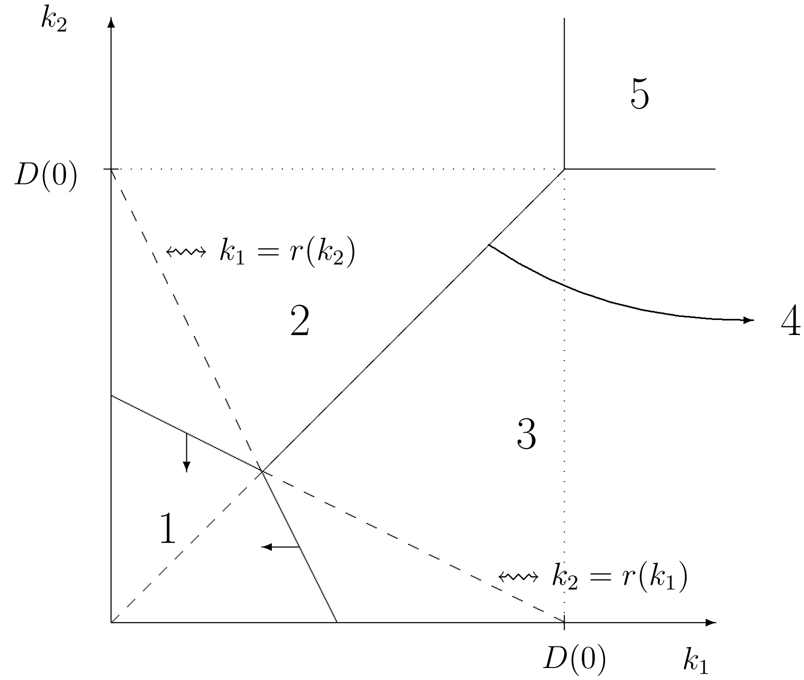

Figure 4.2: The five regions of interest in Proposition 4.8.

4. *If $k_1 = k_2 = k$, $k < D(0)$ and $k > r(k)$, then there are two* SPNE *in pure strategies. One where firm 1 sets $\underline{p}(k,k)$ and firm 2 responds by setting $P(r(k)+k)$ with payoffs $\pi_1(k,k) = \underline{p}(k,k)\,k = R(k)$ and $\pi_2(k,k) = R(k)$. And another where both firms set $P(r(k)+k)$ and net $\pi_1(k,k) = P(r(k)+k)r(k) = R(k)$ and $\pi_2(k,k) = P(r(k)+k)k$.*

5. *If $\min\{k_1, k_2\} \geq D(0)$, then there is a continuum of* SPNE *indexed by p_1^* where firm 1 sets $p_1^* \in \mathbb{R}_+$ and firm 2 sets $\min\{p_1^*, p^m\}$ if $p_1^* > 0$ and sets any price $p_2^* \in \mathbb{R}_+$ if $p_1^* = 0$. Payoffs are $\pi_1(k_1,k_2) = 0$ for firm 1 and $\pi_2(k_1,k_2) = \min\{p_1^*, p^m\}\, D(\min\{p_1^*, p^m\})$*

Proof. We solve by backward induction. Given the capacities set in the first stage, the best response mapping $p_2^*(\,\cdot \mid k_1, k_2) : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ for firm 2 as a function of price p_1, set by firm 1, is:

$$p_2^*(p_1|\; k_1, k_2) = \begin{cases} \mathbb{R}_+ & \text{if } p_1 = \underline{p}(k_2, k_1) = 0 \\ \max\{P(k_a + k_b)\,, P(r(k_1) + k_1)\} & \text{if } p_1 \leq \underline{p}(k_2, k_1) \text{ and} \\ & \underline{p}(k_2, k_1) > 0 \\ \min\{p_1, \max\{P(k_2)\,, p^m\}\} & \text{if } p_1 \geq \underline{p}(k_2, k_1) \end{cases}$$

For 1 $\underline{p}(k_2, k_1) = P(k_a + k_b)$ so Lemma 4.4 tells us that for any possible price set by firm 1 in equilibrium, firm 2 will match or undercut. Thus firm 1 gets the residual demand payoff which is maximized at $p_1^* = P(k_a + k_b)$.

For 2 given the strategy of firm 2, firm 1 nets the low priced firm payoff when setting $p_1 < \underline{p}(k_2, k_1)$ and the residual demand payoff when setting $p_1 > \underline{p}(k_2, k_1)$. Since $\underline{p}(k_1, k_2) < \underline{p}(k_2, k_1)$, the only possibility for firm 1 to have a maximizer is to have upper semi-continuity at $p_1 = \underline{p}(k_2, k_1)$. And this only obtains if firm 2 sets $P(r(k_1) + k_1)$.

In 3 we have $\underline{p}(k_1, k_2) > \underline{p}(k_2, k_1)$ and thus firm 1 maximizes it's payoff by setting $P(r(k_2) + k_2)$ at which firm 2 responds by matching.

In 5, $\underline{p}(k_2, k_1) = 0$, so if $p_1 > 0$ firm two will match or undercut and firm 1 will net 0. Then, firm 1 can set, in equilibrium, any price $p_1^* > 0$, and firm 2 will match or undercut depending on the value of p_1^*. For the special case of $p_1 = 0$, note that then firm 1 still nets 0 (so it is a possible strategy in equilibrium) and firm 2 nets 0 as well, so then firm 2 can actually set any price in the price set. Thus any pair $(0, p_2^*)$, $p_2^* \in \mathbb{R}_+$ can be sustained as a SPNE of the price game.

■

Suppose we have $k_a > k_b$ and $k_a > r(k_b)$, so that we stand in the case were in the simultaneous price game the equilibrium is in non degenerate mixed strategies and the residual monopoly profit for firm a is greater than the market clearing profit. If firm b is the leader, then it knows that if it sets a price higher than $\underline{p}(k_a, k_b)$ it will be undercut and then it would get the residual demand, but since $\underline{p}(k_b, k_a) < \underline{p}(k_a, k_b)$ firm b would net strictly more by setting a price $\underline{p}(k_b, k_a) < p_b \leq \underline{p}(k_a, k_b)$ since then firm a would rather take the residual demand. On the other hand, if firm b is the follower, since it is the small firm, in order to sell it's whole production by being the low priced firm, it can set a bigger price than firm a ($P(k_b) > P(k_a)$). Firm a then assumes it will be undercut and thus maximizes it's payoff, as the high priced firm, setting the residual demand monopoly price $P(r(k_b) + k_b)$. Then, firm b will set the same price (this is also true in the case $k_a = k_b = k > r(k)$ since the rule assigns first the demand to the follower and thus it will sell all it's capacity). Then firm a will obtain a payoff $R(k_b)$ with certainty and so it is indifferent between being leader or follower in the price setting game, while firm b would rather go second since it nets $\underline{p}(k_a, k_b)\, k_b$ if it is leader and $P(r(k_b) + k_b)\, k_b$ if it is follower and in this capacity region we have $P(r(k_b) + k_b) > \underline{p}(k_a, k_b)$. In Deneckere and Kovenock (1992) they address the "choice of roles" issue rising from this observation, since we could say that if leadership were endogenized, then the large firm would become leader. The small firm would wait for the announcement of the large firm, which would then set the corresponding price first. The authors focus then on games of timing of price announcement.

For the special case $k_a = k_b = k > r(k)$ we have two SPNE, one where firms set different prices and one where both firms set the residual demand monopoly price, but only the second one allows us to have upper semi-continuity of reduced payoff in the capacity setting game. Since the first firm is indifferent between the two equilibria, we turn to upper semi-continuity of payoffs to find possible non Cournot symmetric Nash equilibria of the reduced capacity setting game in the next section. Regarding the second of these equilibria, because of the rationing rule, the firm that announces it's price in second place will net a strictly greater payoff than the leader, therefore considering this SPNE we could say that both firms would prefer to wait for the announcement of the rival before announcing it's own price. The rationing rule assigns the demand first to the follower, which is the sole difference between roles for this strategy profile.

Proposition 4.8 ratifies what we had obtained in Proposition 4.6. Payoffs associated to the SPNE of the modified sequential price subgame are exactly the same as those associated to the equilibria of the original (sequential) one, explored in Proposition 4.6. Even more, in the region where there was no pure strategy Nash equilibrium in the price subgame (case 4 in Proposition 4.6) we now have existence and uniqueness (except for the line $k_1 = k_2$) of SPNE and the payoffs associated to this unique SPNE are precisely the ones mentioned in Remark 4.7. This outlines the power of the Deneckere and Kovenock (1992) modification of the sequential price game: changing the splitting rule on tied prices not only helps to solve the mathematical problem of upper semi-continuity of payoffs, but also does not depart from the intuitive outcome (nor in payoffs nor in strategies) of the price game in the original setting, allowing to formally obtain these intuitive results over price competition with capacity constraints when the small firm is follower. We see that the continuity problem being solved by the Deneckere and Kovenock assumption on demand splitting on tied prices, does not affect the economic conclusions of the model and effectively helps to clarify the behavior of the agents.

4.2 Capacity Setting Reduced Game

Now that we have obtained all the SPNE of the prices-setting subgame, we can study the capacity setting stage of the game. The usual approach would be to use reduced payoff functions obtained from the outcomes of the subgame for each possible capacity pairs of the first stage generating a capacity setting reduced game. The problem is that for the case of $k_a = k_b = k > r(k)$ we don't have a unique payoff for the follower. So situations regarding one of the firms setting a capacity k_i greater than k^* have to be studied carefully. Nevertheless, let's introduce the first stage capacity-setting game and the price-setting SPNE associated payoffs.

In this stage firms set capacities simultaneously. They maximize their expected payoff of setting capacity k_i knowing that the choice of the opponent will have an effect on the final profit. When setting capacity, firms do not know whether they will set their price first or second in the price-setting stage. This uncertainty may come, for instance, from the nature of the variable under scrutiny. Capacity building or stock production may occur long before price competition. Even more, once we enter the stage of price competition, and thinking in some kind of endogenous timing or role in the price game, we have that the only preference on roles is for the large firm being follower [5]. As we have said, the small firm is indifferent between announcing its price first or second. Thus it is not clear how long firms can wait before being forced to announce a price. We will assume then that there is an exogenous factor that determines the role in the price setting stage. Firms know, however, in the first stage of the game, that there is a probability λ_i of being leader in the price game for firm i, $i \in \{a, b\}$. In an incomplete information type of game, these distributions could represent the probability of being impatient in the second stage of the game. They can be interpreted as well as priors or beliefs of the firms about their role in the second stage price game. Similarly, they induce priors onto the possible payoff outcomes (not necessarily roles) of this subgame. Although we allow for the probabilities to be different for each firm and they could well depend on the installed capacities, it is better, to keep track of the exercise, to think of them as being equal to 1/2. As we will see below, the values of the probabilities λ_i are irrelevant for the purpose of telling a neat economic story behind the Cournot outcome as a result of price competition. These values become important when seeking for other equilibria of the game, an issue that we address as well in what follows. The interpretation of these distribution as priors may raise as well questions about their genesis. Questions for instance, of endogenous generation of priors, which would inspire an exercise closely related to the choice of roles (Hamilton and Slutsky 1990; Deneckere and Kovenock 1992; van Damme and Hurkens 2004).

The capacity setting reduced game has as players the two firms, with strategy space $\mathbb{R}_+$ and payoff given by the function $\pi_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$\pi_i(k_i, k_j) := \lambda_i \pi_1(k_i, k_j) + (1 - \lambda_i)\, \pi_2(k_j, k_i) - C(k_i) \qquad (i, j) \in \{(a, b), (b, a)\}$$

where π_1 and π_2 are the profit functions obtained from the equilibria of the sequential price-setting subgame described in Proposition 4.8.

[5] In the absence of naturally generated roles in sequential play (Gal-Or 1985; Amir and Stepanova 2006), Hamilton and Slutsky (1990) and van Damme and Hurkens (2004) study game forms where players chose when to play in a sequential game. When choosing the same timing it is assumed that the simultaneous subgame equilibrium payoffs are obtained, which we prefer to avoid in this note, as Deneckere and Kovenock (1992) do in their work.

Cournot Outcome

Regarding capacity setting competition, we first notice that the Cournot outcome is a Subgame Perfect Nash equilibrium of the two-stage game with sequential price-setting, as it was in the two-stage game with simultaneous price-setting stage case. If a firm sets the Cournot capacity, then the opponent gets the Cournot payoff up to a certain capacity, namely the Cournot best response for zero cost, and for greater capacities, since the higher capacity firm nets the residual demand monopoly payoff no matter being leader or follower in the second stage game, income remains fixed while cost increases.

Proposition 4.9. *The Cournot outcome defined by* $k_a = k_b = k_C^*$ *and* $p_a = p_b = p_C^*$ *can be sustained as a* SPNE *of the two-stage game.*

Proof. Set $k_b^* = k_C^*$. Then $k_b^* < k^*$ and thus Proposition 4.8 gives a unique reduced payoff for firm a:

$$\pi_a(k_a, k_C^*) = \begin{cases} P(k_a + k_C^*)\, k_a - C(k_a) & \text{if } k_a \leq r(k_C^*) \\ R(k_C^*) - C(k_a) & \text{if } k_a > r(k_C^*) \end{cases}$$

As in Proposition 4.3, $\pi_a(\,\cdot\,, k_C^*)$ is continuous and if $k_a > r(k_C^*)$ it is strictly decreasing. Moreover, the function $k_a \to k_a P(k_a + k_C^*) - C(k_a)$, the Cournot payoff function, attains it's maximum at $k_a^* = r_C(k_C^*) < r(k_C^*)$ and thus $k_a = k_b^*$ is the best response to $k_b^* = k_C^*$ in the reduced form capacity-setting game. Since the (unique) best response to k_C^* in the reduced form game is to set the same capacity and the game is symmetric, we conclude.

∎

So now we have a clear neat story behind the Cournot outcome as the result of price competition. If firms set small capacities, then price competition produces clear-market prices. Facing Cournot capacity of the rival, a firm that sets a high capacity will net an income that depends only on the capacity set by it's rival. The large firm if being leader in the price game knows it will be undercut. If being follower, the small firm will set the highest price possible at which it can sell it's whole capacity, that is, $\underline{p}(k_i, k_j)$ with $k_i > k_j$, the price that leaves the high firm indifferent between matching and taking the residual demand, so the large firm nets $R(k_j)$.

The Cournot outcome can be sustained as a subgame perfect equilibrium in pure strategies at all stages of the game.

Other Equilibria

Considering the possibility of other equilibria, note that setting the same capacity and it being smaller than the value of the zero cost Cournot equilibrium k^*, can not be an equilibrium since deviating to the Cournot best response maximizes payoff given the capacity of the rival. It is the same argument as in the previous Proposition. This rules out any pure-strategy Nash equilibrium with both capacities smaller than k^*.

However it could be possible to find symmetric Nash equilibria where firms engage in higher capacities. Suppose that the expected reduced payoff (expectation taken over the possibility of being leader or follower) in the first stage game, given that the rival plays a capacity $\bar{k}$ strictly greater than the zero cost Cournot equilibrium k^*, is increasing in the interval where there is a difference between being leader or follower, and that the expected reduced payoff of matching this capacity is greater (or equal) than the maximum payoff that could be attained by setting a capacity in the indifference (between being leader or follower) interval (if not void); and consider the following strategies:

> *In the first stage set $\bar{k}$. In the second stage, once roles are assigned, follow the subgame perfect strategies given in Proposition 4.8. For the special case where $k_a = k_b = k$ let the leader set the price $P(r(k) + k)$.*

Then, using price leadership as a coordination device, the outcome $\left(\bar{k}, \bar{k}\right)$ can be sustained as a SPNE of the game.

Indeed, once capacities are set, what follows has already been discussed. So the question is: why would a firm set this capacity given that the other one did? The answer comes from the possibility that the firm was follower in the second stage game because the follower can set a relatively high price:

$$P\left(2\bar{k}\right) < P\left(r\left(\bar{k}\right) + \bar{k}\right).$$

First, notice that setting a higher capacity can not be a best response since payoff is decreasing in that zone. Given the strategies in the price-setting subgame, upper semi-continuity of reduced payoff gives the existence of the best response in the interval $\left[0, \bar{k}\right]$.

Second, note that reduced payoff for the small firm is of the form $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)$ where $\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ is the expected price faced by a small firm. Expected price $\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ for firm a against capacity $\bar{k}$ is given by:

$$\mathbb{E}\left[p \mid \left(k_a, \bar{k}\right)\right] = \begin{cases} P\left(k_a + \bar{k}\right) & \text{if } k_a \leq r^{-1}\left(\bar{k}\right) \\ \lambda_a \underline{p}\left(\bar{k}, k_a\right) + (1 - \lambda_a) P(r(k_a) + k_a) & \text{if } r^{-1}\left(\bar{k}\right) \leq k_a < \bar{k} \end{cases}$$

so that payoff for the small firm is continuous in $\left[0, \bar{k}\right[$ and strictly greater than the Cournot payoff for capacities in the interval $\left]r^{-1}\left(\bar{k}\right), \bar{k}\right[$.

For the small firm as a second mover, when comparing[6] the maximum profit from setting a capacity k such that $k < r^{-1}\left(\bar{k}\right)$ and setting $\bar{k}$, the fall in price $(P\left(k+\bar{k}\right) \searrow P\left(r\left(\bar{k}\right)+\bar{k}\right) > P\left(\bar{k}+\bar{k}\right))$ and the increase of the cost $\left(C(k) \nearrow C\left(\bar{k}\right)\right)$ may be compensated by the increase in the volume of sales $\left(k \nearrow \bar{k}\right)$. The second mover price function, $P(r(k)+k)$, is less steep than the standard Cournot price function, allowing the possibility that payoff was increasing in the interval $\left]r^{-1}\left(\bar{k}\right), \bar{k}\right[$.

Second mover advantage has to be such that it drags sufficiently upwards the expected payoff $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)$ and the functions $k \to \mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right]$ and $k \to -C(k)$ need not be too strongly decreasing.

These two conditions can be written as find $\bar{k}$ such that:

$$P\left(r\left(\bar{k}\right)+\bar{k}\right)\left[\lambda_i r\left(\bar{k}\right)+(1-\lambda_i)\bar{k}\right] - C\left(\bar{k}\right) \geq \max_{k \in \left[0, r^{-1}\left(\bar{k}\right)\right]} P\left(k+\bar{k}\right) k - C(k) \tag{4.2}$$

$$\lim_{k \nearrow \bar{k}} \frac{\partial}{\partial k}\left[\mathbb{E}\left[p \mid \left(k, \bar{k}\right)\right] k - C(k)\right] \geq 0 \tag{4.3}$$

Condition (4.2) states that the payoff of matching is greater than the payoff of setting a capacity in the zone where there is no difference between being leader or follower, and condition (4.3) states that the left hand derivative[7] of the payoff function at the point $\bar{k}$ is positive which, along with concavity of payoff in the region where there is difference between being leader and follower and continuity at the point $k = r^{-1}(\bar{k})$, assures that payoff is maximized at $k = \bar{k}$.

We illustrate this situation with the following example:

Example 4.10. Consider the duopoly with price function $P(q) = 1 - q$, and cost $C(q) = cq$ for both firms. The Cournot best response function is then:

$$r_C(k) = \max\left\{\frac{1-c-k}{2}, 0\right\}.$$

The zero-cost Cournot equilibrium is given by $k^* \equiv 1/3$ and $k_C^* = (1-c)/3$.

If $\lambda_a = \lambda_b = 1/2$ and c is sufficiently small then we can find a continuum of equilibria along the line $k_a = k_b$. For instance for $c \equiv 0.05$, $\bar{k}_a = \bar{k}_b = \bar{k} = 0.3375$ can be sustained by a SPNE of the game and there is a continuum of Nash Equilibria along the line $k_a = k_b$ in a neighborhood of $\bar{k}$. Of course, k_C^* is not in this neighborhood.

[6] This comparison is relevant only if $\bar{k} \leq r(0)$.
[7] From the definition of $\underline{p}(k_i, k_j)$ when $k_j \nearrow k_i$.

Cournot Outcome and Contingent Demands

In Davidson and Deneckere (1986) Theorem 1 argues that the Cournot outcome can not emerge as an equilibrium of the two stage game discussed by Kreps and Scheinkman considering almost any rationing rule, in a certain class, different from the *surplus maximizing* rationing rule. This is relevant as well for this work since the central argument states that if we change the rationing rule then, given that in the first stage the set capacities were the Cournot quantities, the Cournot price is not a best response to itself, ruling out as well the Cournot outcome as the equilibrium of the sequential subgame. However the proof is made only for the case of zero cost of capacity. As we will see below their result, as stated in their work, is in fact only valid for this case and once there are costs a finer treatment becomes necessary. Madden (1998) addresses this matter for a special setting that is not completely compatible with the Kreps and Scheinkman or Davidson and Deneckere assumptions on the demand function, but that enlightens what really is relevant when trying to answer the question about how dependent are the results on the chosen rationing rule. Consider the class of contingent demands that satisfy:

1. $D_i(p_i|p_j)$ downward-sloping twice differentiable except in $p_i = p_j$.
2. $D_i(p_i|p_j) = D(p_i)$ for $p_i < p_j$
3. $D_i(p_i|p_j) = \max\left\{\frac{D(pi)}{2}, D(p_i) - k_j\right\}$ when $p_i = p_j$
4. $\max\{0, D(p_i) - k_j\} \leq D_i(p_i|p_j) \leq \max\{0, \min\{D(p_j) - k_j, D(p_i)\}\}$ for $p_i > p_j$

Both rationing rules: *proportional* and *surplus maximizing* satisfy points 1 to 4.

Suppose that contingent demand for the high priced firm is given by a $\mathcal{C}^1$ function $CD(\,\cdot\,|p_j, k_i, k_j) : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$D_i(p_i|p_j) = \begin{cases} D(p_i) & \text{if } p_i < p_j \\ \max\left\{\frac{D(p_i)}{2}, D(p_i) - k_j\right\} & \text{if } p_i = p_j \\ \max\{0, CD(p_i|p_j, k_j, k_j)\} & \text{if } p_i > p_j \end{cases}$$

and D_i satisfies conditions 1 to 4. Note that for D_i to satisfy conditions 1 to 4, we must have that $CD(p_j|p_j, k_j, k_j) = D(p_j) - k_j$. Using the notation $e^{CD}_{q,p}$ for the elasticity of demand $CD(\,\cdot\,|p^*_C, k^*_C, k^*_C)$ we can state the following proposition.

Proposition 4.11. *If:*

$$e^{CD}_{q,p}(p^*_C) > -1 \tag{4.4}$$

then the Cournot outcome does not rise as the equilibrium of the two stage game (sequential or simultaneous).

Proof. Suppose the two players set k_C^* in the first stage. If player j sets price p_C^* the payoff for player i as a function of price is:

$$\pi_i(p, p_C^*, k_C^*, k_C^*) = \begin{cases} pk_C^* & p \leq p_C^* \\ p \max\{0, CD(p|p_C^*, k_C^*, k_C^*)\} & p > p_C^* \end{cases}$$

This function is then continuous[8] and increasing to the left of p_C^*. The derivative to the right of p_C^* is

$$\frac{\partial}{\partial p}\pi_i(p, p_C^*, k_C^*, k_C^*) = p\ CD'(p|p_C^*, k_C^*, k_C^*) + CD(p|p_C^*, k_C^*, k_C^*)$$

thus we have:

$$\begin{aligned} \lim_{p \downarrow p_C^*} \frac{\partial}{\partial p}\pi_i(p, p_C^*, k_C^*, k_C^*) &= p_C^* CD'(p_C^*|p_C^*, k_C^*, k_C^*) + CD(p_C^*|p_C^*, k_C^*, k_C^*) \\ &= p_C^* CD'(p_C^*|p_C^*, k_C^*, k_C^*) + k_C^* \\ &= k_C^* \left(\frac{p_C^*}{k_C^*} CD'(p_C^*|p_C^*, k_C^*, k_C^*) + 1 \right) \\ &> 0 \end{aligned}$$

Then $p_C^* \notin \text{argmax}_p \{\pi_i(p, p_C^*, k_C^*, k_C^*)\}$.

■

The proof follows the same ideas behind Davidson and Deneckere's proof (1986). The condition implies that payoff as a function of price, is increasing at the Cournot price. In their result, the condition of $D_i(p_i|p_j)$ being locally distinct to the right from the surplus maximizing contingent demand at the point $p_i = p_j = p_C^*$ when $k_i = k_j = k_C^*$, is not sufficient when costs are not equal to zero and is a special case of our condition (4.4). Condition (4.4) is also in the line of Madden (1998), if the residual demand is not sufficiently elastic at the Cournot price, then the Cournot outcome can not emerge as an equilibrium of the two stage game.

[8] Since $CD(p|p_C^*, k_C^*, k_C^*)$ satisfies 2, $\lim_{p \downarrow p_C^*} CD(p|p_C^*, k_C^*, k_C^*) = k_C^*$.

5 Concluding Remarks

We have studied a three stage duopoly game where firms first set simultaneously capacities and then engage in sequential price competition. We have related the outcome of the sequential price game (Deneckere and Kovenock 1992) to the simultaneous price game (Kreps and Scheinkman 1983) and we have obtained that the Cournot outcome of the market can be sustained as a SPNE in pure strategies

Mixed strategies are a solution to the problem of existence in simultaneous price games. However, there is still not a consensus about what a mixed strategy outcome represents in a price game. Usual justifications for mixed strategy equilibria such as a population distribution from where individuals are randomly selected to play the game, as in evolutionary game theory do note seem appealing. Moreover, the fact that the pure strategies that are in the support of the mixed strategy distributions are not necessarily best responses to the pure strategies that are in the support of the rival's mixed strategy distribution, gives rise to the regret property of mixed strategies equilibria: once a realization of the distribution is played, players may not be satisfied with the outcome, which is somehow incoherent with Nash behavior or perfectness of equilibria (see for instance Vives (1999) and Friedman (1988) for a discussion on Mixed strategies Nash equilibria in oligopoly games).

We have proposed in this note an alternative approach to the problem of existence of equilibrium in pure strategies that is more in the line of Harsanyi (1973). Firms are uncertain of their role in the price stage of the game. They know that sequential price competition will take place, but since this competition is so far in the future, price leadership is unknown. As in an incomplete information game, players know a probability distribution over their roles in the price game and maximize expected payoff in the first stage. Price leadership serves then as a coordination device that allows to obtain equilibrium outcomes in pure strategies. It is important to notice that this result does not depend on the value of the probability distribution, but only in the characteristics of the involved functions. That is, on the structure of the market: cost of production, elasticity of demand. We have seen that suitably and slightly modifying the sharing rule on tied prices it is possible to formally reproduce the intuitively expected behavior of firms in the original setting (equal share at tied prices).

Finally we have made space for the possibility of finding other symmetric quantity outcomes sustained as SPNE depending only on primitives and on the probabilities of being leader in the price game. No changes with respect to the Kreps and Scheinkman (1983) setting have been made on cost or demand functions. The discussion is opened then on how to treat such probabilities (endogenously generated, parameters of the model, firm independent, etc.). Many lines of exploration can be taken: general n firm oligopoly, non symmetric cost structures of firms (Deneckere and Kovenock 1996) and effects of changes on contingent demand (Davidson and Deneckere 1986), to give some examples.

CHAPTER 5

Conclusions

The main contributions of this thesis to Economic Theory and Mathematical Economics are, on one hand the introduction of the concept of Rationalizability to a general class of games with a continuum of players, and on the other hand, bringing together in an original setting different areas of Economic Theory that remained until now relatively unrelated (large games, rational expectations in economics).

In the area of large games, the main contribution is the formal and general definition of the different sorts of Rationalizability in games with a continuum of players: Point-Rationalizable Strategy profiles in games with a continuum of players (Definition 2.5), Point-Rationalizable Sates in the particular case where payoffs depend on strategy profiles only through the value of the integral (Definition 2.7), the extension of Rationalizability to non-point forecasts in this same class of games by introducing Rationalizable States (Definition 2.21) and the definition of Rationalizable Strategy Profiles in games where the set of actions of the players are finite (Definition 2.25). Moreover, the main results are the characterizations of the sets of (Point-) Rationalizable States (Theorems 2.10 and 2.22).

In Chapter 3 we have assessed stability of equilibria in a class of economic models where forecasts of agents lie on the set of aggregate states. Relying on the general framework introduced in Chapter 2 and the definitions of Rationalizability therein presented, we have defined Strong Rationality (or Eductive Stability) as the uniqueness of the Rationalizable solutions of the economic setting. We have studied as well the relation between this stability concept and Iterative Expectational Stability in the general setting. The characterization of Rationalizability has allowed for us as well to explore the local viewpoint to expectational stability in this general setting as it has been done in more specific contexts in the literature. An equilibrium is defined to be Locally Strongly Rational (or Locally Eductively Stable) if it is the unique result of the eductive process described in Theorem 2.10 of Chapter 2.

The main results of Chapter 3 are Theorems 3.30 and 3.40 in which we have endowed the economic setting with a lattice structure in order to study the consequences over eductive stability of the presence of Strategic Complementarities and Strategic Substitutabilities. These results can be summarized as follows: under the presence of Strategic Complementarities, uniqueness of equilibrium assures it's eductive stability while this is not necessarily true for the case of Strategic Substitutes, where the study of the second iterate of a best-reply type mapping may give a clue about this matter. We have as well shown that the sets of Equilibria and Rationalizable States have some attractive mathematical structure. Moreover, through Corollaries 3.32 and 3.42 we have shown the equivalence of the two stability concepts under scrutiny in these two contexts.

Relevant mathematical results in these two Chapters, that risk to have remained unnoticed, are Lemmas 2.6, which assures the necessary measurability properties of the best response mapping as an operator over the players names, and 2.19, that expands continuity properties to the case of general forecasts, in Chapter 2; and Lemmas 3.39 and 3.29, where we show that the best-reply-to-forecasts mapping has appealing monotonicity properties, in Chapter 3 that rely on the original result of Lemma 3.48, where we show that it is possible to extract largest and smallest measurable selections from the best-reply mappings as functions of the players' names, in this same Chapter.

Finally, we have contributed in Chapter 4, with an interesting exercise, to the field of oligopolistic competition. In such exercise, we present a duopoly in which firms engage in capacity in a first stage and compete sequentially on prices on a second stage of a three stage game. Price leadership is randomized, endowing the game with symmetry and allowing the possibility of finding pure strategy subgame perfect equilibria, as it had not been done yet. We have obtained as a result that the Cournot outcome of the duopoly can be sustained as a Subgame Perfect Nash Equilibrium of the whole game, allowing to clearly interpret the economic findings trough the sequence of play. We have as well obtained a sub product: the possibility of finding non-Cournot outcomes, as a consequence of randomizing and the attractiveness of being second mover in the price-setting subgame. We have provided a sufficient condition for the existence of such equilibrium outcomes.

We expect that Chapters 2 and 3 altogether may open the scope of related research by providing a general framework, original tools and pertinent links and relations between the different elements that have been studied therein; that we hope will prove to be useful in what will follow.

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