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CREACIÓN DE VALOR Y REDUCCIÓN DE RIESGO MEDIANTE COORDINACIÓN DE DECISIONES FINANCIERAS Y OPERACIONALES

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL INDUSTRIAL

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El presente trabajo propone un modelo para la toma coordinada de decisiones financieras y operacionales en una empresa que paga dividendos. El objetivo de coordinar ambas áreas es maximizar el valor presente esperado de los pagos futuros de dividendos, el cual es un indicador claro del valor de la empresa. Utilizando un marco de tiempo continuo, se modela el dinero disponible en caja como un proceso browniano cuyos parametros pueden ser modificados mediante cambios en la estrategia operacional.

En un primer paso se estudia el problema de elección de estrategia operacional y de política de dividendos para una empresa sin deuda. Haciendo uso de las ecuaciones de Hamilton-Jacobi-Bellman se formula una solución analítica al problema. Se prueba que existe una frontera, análoga a la frontera eficiente de un mercado de acciones, en la que se encuentran las estrategias operacionales que deben ser implementadas. La empresa debe elegir la estrategia operacional de acuerdo a su riqueza inicial y luego implementar estrategias más riesgosas a medida que aumentan las reservas de dinero. La repartición de dividendos debe seguir una política en la que sólo se reparten dividendos cuando las reservas de dinero sobrepasan un cierto nivel. Toda riqueza generada por sobre este nivel 'optimo' de reserva debe ser distribuida a los inversionistas.

Por último se analiza un modelo estático en el que la estrategia operacional no puede ser alterada y la empresa debe elegir una política de repartición de dividendos además del monto de un préstamo de largo plazo. Esto se traduce en un problema de punto fijo en el que la empresa debe conocer la tasa de interés para elegir el monto del préstamo y a su vez el banco debe conocer el monto del préstamo para fijar una tasa de interés. A través de un ejercicio numérico se calcula el monto óptimo de deuda, el valor que esta deuda agrega, la tasa de interés asociada a la deuda y el tiempo esperado de vida de la empresa. Los resultados obtenidos muestran que para empresas con capital inicial limitado el apalancamiento puede a la vez agregar valor y ayudar a mantaner la empresa solvente por un mayor tiempo cuando la tasa libre de riesgo es lo suficientemente baja. Empresas con más capital inicial también pueden crear valor adquiriendo un poco de deuda, pero esto acortará su tiempo esperado de vida.

Abstract

This paper follows up on the paper by Radner and Shepp (1996)[14] which developed a model to make production and financing decisions in a coordinated manner within a continuous-time framework. We extend their model by adding the possibility of leverage. We consider a firm whose net earnings follow a brownian process which is influenced by the firm's operating strategy. The firm has to decide on the optimal operating policy as well as the leverage ratio and the dividend policy. In this first approach we consider a static model in which debt is long term and in which there is only one available operational strategy. We show that the coordination of financial and operational decisions can create significant value for cash-constrained firms. Results include an optimal dividend policy, optimal leverage ratio, optimal operational risk taking policy, and the value created by optimal decision making.

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Contents

1 Introduction

Most literature on the topics of operational and financial decisions treat both areas as if they were completely independent. In reality, this separation can usually be observed in successful large corporations. In established firms agents making operational decisions usually have no influence on financial policy. They apply traditional operations management techniques to make optimal operational decisions. The finance department tries to provide enough cash to carry out these optimal decisions and at the same time tries to keep investors happy by maintaining a steady flow of dividends. This dichotomy is supported by the famous Modigliani-Miller[13] theorem, which states that firms operating in perfect and complete capital markets need not to worry about coordination of operations and capital structure. However, the real world is far from the Modigliany-Miller world and the premises needed to support their results are rarely met. Markets are imperfect, with asymmetries of information, transaction costs, taxes, accounting and legal costs. The consequences of these market imperfections are specially dire when looking at firms that have limited working capital and that are trying to grow, such as startups. In these nascent, capital-constrained firms, financial and operational decisions are usually made by the same person or group of people and the interactions between the two areas are carefully monitored. Firms with little available cash can rarely implement optimal operational policies due to liquidity constraints. In turn, operational variables such as inventory levels can affect financial decisions such as borrowing (for example when financing operations through asset based loans). The goal of this paper is to study how the coordination of financial and operational decisions can aid to create value and reduce risk.

2 Related Research

In 1958 Modigliani and Miller[13] demonstrated that within a perfect capital market there is no need for firms to coordinate investment and operational decisions with financial and capital structure decisions. This widely spread result had a strong influence in research literature during the years that followed. In turn, most research in the topics of financial and operational policies address both in a separate manner. Recently there has been a change in this bias and there is a rapidly growing body of literature investigating the interactions between finance and operations in firms that have limited working capital and that operate in imperfect and incomplete capital markets.

This new stream of model-based research literature follows two main branches. On one hand we have discrete-time models that derive optimal coordinated financial and operational strategies in both static single-period models and dynamic multi-period frameworks. On the other hand we have continuous-time models which emulate the revenue stream of a firm by means of a diffusion-type model. The firm can affect the parameters (drift and volatility) of the diffusion through operational decisions and it can also influence it's risk exposure by means of it's dividend payments. Both approaches have cast light on the mechanisms of interaction between finance and operations and have provided interesting insights that could help budget-constrained firms reduce the risk of bankruptcy and create more value.

Li et al. (1997)[12], Hu and Sobel (2005)[11], Zhang and Sobel (2007)[17], Babich and Sobel (2002)[1], and Buzacott and Zhang (2004)[2] analyze discrete-time dynamic models of financially constrained firms facing both stochastic and deterministic demand. Li et al. (1997)[12] propose a multi-period framework in which a firm decides every period on how much to borrow (only short-term loans), how much to produce and how much dividend to issue before knowing the period's demand. The firms seeks to maximize the present value of dividends net of capital subscriptions. Two types of bankruptcy are envisioned: *reorganization* bankruptcy where the default causes a costly restructuring but operations continue, and *wipeout* bankruptcy which causes a permanent halt of operations. They explicitly characterize the optimal policy for a special case and find that the integration of the production/inventory problem with the cash flow and financial problems of a firm may be relevant and worthwhile specially for firms operating with thin budgets in volatile markets. Hu and Sobel (2005)[11] further expand the latter model by adding a capital structure (long-term debt) and find that optimal coordinated decisions, in comparison to decentralized decisions, yield lower inventories, require less working capital, make larger short-term loans, require less long-term debt, have a lower default risk, and yield higher expected dividends net of capital subscriptions. Zhang and Sobel (2007)[17] also extend the model originally developed by Li et al. (1997)[12] by considering a more complex inventory cost structure which includes ordering setup costs and smoothing costs. Babich and Sobel (2002)[1] propose an infinite-horizon discounted Markov decision process in which an IPO event is treated as a stopping time. The value of the IPO is modeled as a random variable whose distribution depends on the firm's current assets, its most recent sales revenue, and its most recent profits. Every period the firm must decide on capacity expansion, production, and loan size. They characterize an optimal capacity-expansion and financing policy so as to maximize the expected present value of the firm's IPO. Buzacott and Zhang (2004)[2] incorporate asset-based financing into production decisions by modeling the available cash in each period as a function of the firm's assets and liabilities. They characterize a strategy that maximizes the expected retained earnings at the end of a planning horizon which illustrates the importance of joint consideration of production and financing decisions when a firm is capital-constrained.

Other authors working with discrete-time models have taken a static, single-period approach that yields qualitatively different conclusions from the articles mentioned before. Xu and Birge (2004)[15], Xu and Birge (2005)[16], and Dada and Hu (2008)[6] are all examples in this line of research. Xu and Birge (2004)[15] analyze the relationship between production and financing decisions through a financially-constrained classical news vendor model. They take into account the effects of market imperfections such as taxes and bankruptcy costs and they conduct a sensibility analysis that illustrates how low-margin producers can take significant advantage in coordinating production and financing decisions. Xu and Birge (2005)[16] build on their previous model by incorporating the interest conflict between corporate managers and firm owners. They also take a look at some empirical support on their predicted relationship between product margin and market leverage. Dada and Hu (2008)[6] analyze a Stackelberg game between a bank (the leader) and a capital-constrained newsvendor (CCNV) (the follower). The bank must determine the interest rate to charge for the loan, taking into account the possibility of default, and the CCNV must decide on how much to borrow at the given interest rate. Results show that the CCNV ends up ordering a quantity that is smaller than the optimal fractile.

Most works that have used continuous-time models have derived from an original paper by Radner and Shepp (1996)[14] in which the revenue stream of a firm is modeled by means of a brownian motion. Since different operating strategies should yield different combinations of expected return and volatility, the brownian motion's parameters in their model (drift/volatility pair) can be chosen from a finite set (A) which emulates the operational strategies that are accessible to the firm. The firm must also decide on how much (if any) dividends to distribute to its shareholders and its objective is to maximize the expected present value of dividends. They show that the optimal dividend policy follows a bang-bang strategy in which no dividends should be distributed until the retained earnings reach a certain level. All excess above this level should be payed out as dividend. The firm's operating policy should choose drift/volatility pairs that belong to the upper extreme points of the convex hull of A, and it should choose pairs with higher risk as it accumulates more earnings. Other authors like Hjgaard and Taksar (1998a, 1998b, 1999)[8][9][10], Choulli, Taksar and Yu Zhou (2003)[5], and Cadenillas, Chouilli, Taksar and Zhang (2006)[3] have explored similar models where there is one (μ, σ) pair available, but the firm can affect it proportionally so it actually has access to all pairs $(u\mu, u\sigma)$, where $u \in [0, 1]$. This emulates an insurance company that can control the reinsurance rate (u). They have also expanded the model by introducing bounds on the dividend payout rate, transaction costs, and constant liability payments. He and Liang (2008)[7] also study the optimal control problem of the insurance company with proportional reinsurance policy, but unlike others, they consider the financing process by allowing the firm to issue equity. Cadenillas, Sarkar and Zapatero (2007)[4] also study the optimal dividend policy in a continuous-time framework, but for a firm whose cash reservoir follows a mean reverting process.

The objective of this dissertation is to develop a model like the one studied by Radner and Shepp (1996)[14] in a continuous-time framework where the firm must also make capital structure decisions such as long and short term debt. This issue is one of the most important matters that fall into the category of financial decisions. We will address this issue by taking both a static and a dynamic approach. In the static model debt levels can be chosen only at the start of operations (thus emulating long-term debt) and there will be only one available operational strategy. In the dynamic model both the operational strategy and leverage level may be modified at any point in time. Both approaches will help deepen the understanding of the interactions between finance and operations for different types of firms.

3 General Model Description

We consider a firm whose net revenues are uncertain and has access to a finite number of distinct 'operational strategies' that yield different expected rates of return and volatility. To emulate this we use the Bachelier additive model. For a certain strategy (i) the accumulated revenues (X_t) are expected to grow at a fixed rate (μ_i) given by the operational strategy that has been chosen. The variance parameter (σ_i) is also given by the operational strategy choice. The revenues follow the following diffusion :

$$\mathrm{d}X_t = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}\mathcal{B}_t$$

Revenues can either be accumulated in a cash reserve $(\hat{P}_t)$ or distributed to investors in the form of dividends. The total amount of dividends distributed up to time t is represented by V_t. At the beginning of operations the firm's cash reserve is composed of an initial amount of equity (P_0) and a long term debt (D) which will remain constant throughout the life of the firm. The firm pays interests on the debt at a rate fixed by the bank (ρ). If the cash reserve ever reaches the threshold 0 the firm is considered to be bankrupt and ceases to exist.

The firm's manager must decide on the amount of initial debt, the distribution of dividends and the operational policy which influences the stochastic process of the revenues. The manager acts in the best interest of the shareholders and makes his decisions seeking to maximize the expected present value of all future dividend payments.

The firm's operational policy is represented by the parameters μ and σ (drift and volatility) of the diffusion. The firm has N available operational strategies so there are as many possible (μ, σ) pairs. Operational strategy i ($i \in \{1, 2, .., N\}$) is asociated with the drift-volatility pair (μ_i, σ_i). The assets that the firm owns and uses to operate (machinery, warehouses, etc) are considered completely illiquid. Also, we do not consider the possibility of reinvestment of revenues on more illiquid assets that could make available new, and better, operational strategies. We also leave out of the model the addition of extra capital that could save the firm from bankruptcy. Since dividends must be nonnegative and debt cannot be modified the only way the firm's cash reserve grows is through revenues. The purpose of this paper is to study the optimal dividend/operational policy as well as the optimal leverage ratio of such a firm.

The mathematical formulation of the problem is as follows. We begin with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a Brownian motion $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)$. $\mathcal{V}$ represents the set of positive non-decreasing right-continuous processes and $\mathcal{T}$ the set of $\mathcal{F}_t$-stopping times, the time of bankruptcy is represented by the greek letter τ. A control policy $\pi = (V_t^\pi, \begin{pmatrix} \mathbf{1}_{C_1}^\pi \\ \mathbf{1}_{C_2}^\pi \\ \vdots \\ \mathbf{1}_{C_N}^\pi \end{pmatrix}, D; t \geq 0)$ where $C_i = \left\{ \bigcup_j \, [t_1^j, t_2^j[\right\}$ is considered admisible if V_t^π belongs to $\mathcal{V}$, all t_1^j and t_2^j belong to $\mathcal{T}$, $\bigcup C_i = [t_0, \tau[$, $\bigcap C_i = \{\phi\}$, and $D \geq 0$. $\prod$ denotes the set of all admisible controls. The first component of the control V_t^π therefore corresponds to the total amount of dividends that have been distributed up to time t, the second control component indicates which operational policy is used for all $t \in [t_0, \tau[$ (C_i are the time intervals at which operational strategy i will be applied), and finally the third component is the amount of longterm debt the firm will ask the bank for. The dynamics of the cash process $\hat{P}_t^\pi$ satisfy:

$$\begin{aligned} \mathrm{d}\hat{P}_t^\pi &= (\sum_{i=1}^N \hat{\mu}_i \mathbf{1}_{C_i}^\pi)\mathrm{d}t + (\sum_{i=1}^N \sigma_i \mathbf{1}_{C_i}^\pi)\mathrm{d}\mathcal{B}_t - \mathrm{dV}_\mathrm{t}^\pi, \text{ where:} \\ \hat{P}_t &= P_t + D, \\ \hat{\mu}_i &= \mu_i - \rho D, \end{aligned}$$

The value of the firm at time t is represented by $W(\hat{P}_t)$ which is defined as the expected present value of all future dividend payments. This translates into:

$$W(\hat{P}_t^\pi) = \mathbb{E}\left[\int_t^{\tau^\pi} e^{-rs} \mathrm{dV}_\mathrm{s}^\pi \right], \quad \tau^\pi = \inf\{t \geq 0; \hat{P}_t^\pi \leq 0\}, \quad \text{B.C: } W(0) = 0$$

The aim of this paper is to find the maximum possible value of the firm at any point in time, defined as:

$$W(\hat{P}_t) = \sup_{\pi \in \prod} W(\hat{P}_t^{\pi})$$

and the optimal policy (π^*) that maximizes the firm's value:

$$\pi^* : W(\hat{P_t^{\pi *}}) = W(\hat{P}_t)$$

4 Preliminary Results

In an independent work, we first studied the same problem addressed in Radner and Shepp (1996)[14]. This is a particular case of the model described above in which there is no debt ($D_t = 0, \forall t$). The results obtained and the procedure used to solve that problem will help the reader understand the procedure used to solve the more complicated scenario we wish to address ahead.

The problem consists in determining the optimal dividend and operational strategy so as to maximize the present value of the firm. This translates into:

$$\begin{aligned}
&\max_{V_t, i_t} \left\{ \mathbb{E}(\int_t^{\tau} e^{-rs} \mathrm{dV_s}) \right\} \\
&s.t: \\
&\mathrm{d}P_t = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}\mathcal{B} - \mathrm{dV_t} \\
&dV_t \geq 0, \text{ where:} \\
W(P_t) = \mathbb{E}(&\int_t^{\tau} e^{-rs} \mathrm{dV_s}), \quad \tau = inf\{t \geq 0; P_t \leq 0\}, \quad \text{B.C: } W(0) = 0
\end{aligned}$$

Now let :

$$W^*(\hat{P}_t) = \max_{V_t, i_t} \left\{ \mathbb{E}(\int_t^{\tau} e^{-rs} \mathrm{dV_s}) \right\}$$

Without loss of generality we will omit the $*$, and $W(P_t)$ will be the value of the firm at time t when the optimal dividend and operational policies are applied. By definition we have that:

$$e^{-r\varepsilon} W(P_\varepsilon) = \int_0^{\varepsilon} \mathrm{d}(e^{-rt} W(P_t)) + W(P_0)$$

Also:

$$\begin{aligned}
\mathrm{d}(e^{-rt}W(P_t)) &= -re^{-rt}W(P_t)\mathrm{d}t + e^{-rt}W'(P_t)\mathrm{d}P_t + \frac{e^{-rt}}{2}W''(P_t)\mathrm{d}\langle P_tP_t\rangle \\
&= \Big[-re^{-rt}W(P_t) + e^{-rt}W'(P_t)\mu + \frac{e^{-rt}\sigma^2}{2}W''(P_t)\Big]\mathrm{d}t + e^{-rt}W'(P_t)\sigma\mathrm{d}\mathcal{B} \\
&- e^{-rt}W'(P_t)\mathrm{dV_t} \\
&\Rightarrow \\
e^{-r\varepsilon}W(P_\varepsilon) &= \int_o^\varepsilon \Big[-re^{-rt}W(P_t) + e^{-rt}W'(P_t)\mu + \frac{e^{-rt}\sigma^2}{2}W''(P_t)\Big]\mathrm{d}t \\
&+ \int_0^\varepsilon e^{-rt}W'(P_t)\sigma\mathrm{d}\mathcal{B} - \int_0^\varepsilon e^{-rt}W'(P_t)\mathrm{dV_t} + W(P_0)
\end{aligned}$$

On the other hand we have that:

$$\begin{aligned}
W(P_0) &\geq \mathbb{E}\Big[\int_0^\varepsilon e^{-rt}\mathrm{dV_t} + e^{-r\varepsilon}W(P_\varepsilon)\Big] \\
&\geq W(P_0) + \mathbb{E}\Big(\int_0^\varepsilon \Big[-re^{-rt}W(P_t) + e^{-rt}W'(P_t)\mu + \frac{e^{-rt}\sigma^2}{2}W''(P_t)\Big]\mathrm{d}t\Big) \\
&+ \mathbb{E}\Big(\int_0^\varepsilon e^{-rt}W'(P_t)\sigma\mathrm{d}\mathcal{B}\Big) + \mathbb{E}\Big(\int_0^\varepsilon e^{-rt}(1 - W'(P_t))\mathrm{dV_t}\Big) && \Big/ - W(P_0) \\
0 &\geq \mathbb{E}\Big(\int_0^\varepsilon \Big[-re^{-rt}W(P_t) + e^{-rt}W'(P_t)\mu + \frac{e^{-rt}\sigma^2}{2}W''(P_t)\Big]\mathrm{d}t\Big) \\
&+ \mathbb{E}\Big(\int_0^\varepsilon e^{-rt}(1 - W'(P_t))\mathrm{dV_t}\Big) && \Big/ \lim_{\varepsilon\to 0}\frac{[\%]}{\varepsilon} \\
0 &\geq \Big\{\mathrm{d}V_t(1 - W'(P)) - rW(P) + \mu W'(P) + \frac{\sigma^2}{2}W''(P)\Big\} \Leftrightarrow
\end{aligned}$$

Equality can be reached with a dividend policy in which dividends are only given out when $W'(P) = 1$ (thus making the term $\mathrm{d}V_t(1 - W'(P))$ null for all value of P), and by choosing (μ, σ) so as to satisfy the following differential equation:

$$rW(P) = \mu_i W'(P) + \frac{{\sigma_i}^2}{2}W''(P) \tag{1}$$

It can be deduced (see Radner and Shepp (1996)[14] for a complete demonstration theorem) that the optimal dividend policy is $\mathrm{d}V_t = 0$ for all $P_t < P^*$, and $\mathrm{d}V_t = (P_t - P^*)$ for all $P_t > P^*$ where $W'(P^*) = 1$. We can interpret that there exists an optimum level of P which we wish to attain. If P is smaller than the optimum level then no dividends will be given out so as to reach the optimum level as soon as possible. Inversely if the level of P surpasses the optimum level dividends will be given out at an infinite rate so as to reach the optimum level instantaneously. We also know that the optimum level P^* satisfies $W'(P^*) = 1$. The value of $W(P_t)$ for $0 \leq P_t \leq P^*$ is given by the differential equations:

$$rW(P) = \mu_{i^*} W'(P) + \frac{{\sigma_{i^*}}^2}{2}W''(P), \forall P \in [P_k, P_{k+1}] \tag{2}$$

Where i^* is the optimal strategy $\forall P \in [P_k, P_{k+1}]$.

$$B.C : W(0) = 0,\ W'(P^*) = 1,\ W''(P^*) = 0$$

The solutions to these differential equations are given by:

$$\begin{aligned} W(P) &= Ae^{\alpha^+ P} + Be^{\alpha^- P} \\ &Where: \\ \alpha^+ &= \frac{-\mu_{i^*} + \sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} \\ \alpha^- &= \frac{-\mu_{i^*} - \sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} \end{aligned}$$

The procedure for determining the best operational strategies and the interval in which each of these strategies is optimal can be found in appendix A.

5 The Static Model

5.1 The firm's Strategy

Consider a particular case of the general model in which debt cannot be modified so as to emulate long-term debt. Also, available operational strategies are restricted to a single strategy represented by one (μ, σ) pair. This translates into the following equations:

$$\begin{aligned} \mathrm{d}\hat{P}_t &= \hat{\mu}\mathrm{d}t + \sigma \mathrm{d}\mathcal{B} - \mathrm{dV_t},\ where: \\ \hat{P}_t &= P_t + D, \\ \hat{\mu} &= \mu - \rho D, \qquad and \end{aligned}$$

$$dV_t \geq 0,\quad W(\hat{P}_t) = \mathbb{E}(\int_t^\tau e^{-rs}\mathrm{dV_s}),\quad \tau = inf\{t \geq 0; \hat{P}_t \leq 0\},\quad B.C:\ W(0) = 0$$

The problem is basically the same as the one presented in the preliminary results section, only the initial cash reservoir has been augmented by an amount D and the (μ/σ) point cloud has been reduced to a single point and translated by the vector $(-\rho D, 0)$. Since there is only one operational strategy it is the optimal one. It follows that the optimal dividend policy can be determined with the procedure described in the previous section, meaning that no dividends should be payed out until a certain threshold (P^*) is attained, and all excess above that level should be distributed to the shareholders. Both P^*, and D represent the firm's strategy:

For a fixed amount of debt D, the firm choses the dividend strategy dV_t that maximizes it's present value:

$$W^*(\hat{P}_t) = \max_{dV_t}\left\{\mathbb{E}(\int_t^{\tau} e^{-rs}\mathrm{dV_s})\right\}$$

Without loss of generality we will omit the $*$, and $W(P_t)$ will be the value of the firm at time t when the optimal dividend policy is applied. Also the ˆ in $\hat{P}$ and $\hat{\mu}$ will be omitted to make notation in this development less intricate. Following the procedure described in the previous section (see appendix B for a detailed step by step application of the procedure) we can write that:

$$W(P_t) = A\left[e^{\alpha^+ P_t} - e^{\alpha^- P_t}\right]$$

where:

$$\alpha^+ = \frac{-\mu + \sqrt{\mu^2 + 2r\sigma}}{\sigma^2}, \quad \alpha^- = \frac{-\mu - \sqrt{\mu^2 + 2r\sigma}}{\sigma^2}$$

$$P^* = \frac{ln([\frac{\alpha^-}{\alpha^+}]^2)}{\alpha^+ - \alpha^-} \qquad A = \frac{1}{\alpha^+ e^{\alpha^+ P^*} - \alpha^- e^{\alpha^- P^*}}$$

Now that we know how to explicitly calculate de value of $W(P_t)$, the firm chooses the optimal amount of debt (D^*) by maximizing the value of $W^($$P_0 + D)$:

$$D^* = \arg\max_D\left\{W^*(P_0 + D)\right\}$$

5.2 The Bank's Strategy

Suppose there exists a competitive banking market so the interest rate charged by the bank is the lowest interest rate that equals the expected present value of all interest payments to the value of the debt. The bank is then considered to be risk neutral and does not expect to make neither profit nor loss through the loan.

$$\begin{aligned} D &= \mathbb{E}(\int_0^{\tau} e^{-rs}\rho D\mathrm{ds}) \Leftrightarrow \\ \rho &= [\mathbb{E}(\int_0^{\tau} e^{-rs}\mathrm{ds})]^{-1} \Leftrightarrow \\ \rho &= \frac{r}{1 - \mathbb{E}(e^{-r\tau})} \end{aligned} \tag{3}$$

Since the time to bankruptcy (τ) is a random variable that depends on the dividend strategy (P^*) which in turn depends on the interest rate (ρ), equation (3) defines a fixed point problem. To solve this fixed point equation the value of the moment generating function of τ must first be expressed explicitly in terms of $(\mu, \sigma, D, \rho, r)$. We can use Ito's lemma to find out the moment generating function of τ:

Let $g(\hat{P}_t)$ be the solution to the differential equation:

$$-rg(\hat{P}_t) + \hat{\mu}g'(\hat{P}_t) + \frac{\sigma^2}{2}g''(\hat{P}_t) = 1 \tag{4}$$

We can then write:

$$e^{-r\tau}g(\hat{P}_\tau) = \int_0^\tau \mathrm{d}(e^{-rt}g(\hat{P}_t)) + g(\hat{P}_0) \tag{5}$$

By definition of τ we know that $\hat{P}_\tau = 0$, so (4) is equivalent to:

$$e^{-r\tau}g(0) = \int_0^\tau \mathrm{d}(e^{-rt}g(\hat{P}_t)) + g(\hat{P}_0)$$

Also:

$$\begin{aligned}
\mathrm{d}(e^{-rt}g(\hat{P}_t)) &= -re^{-rt}g(\hat{P}_t)\mathrm{d}t + e^{-rt}g'(\hat{P}_t)\mathrm{d}\hat{P}_t + \frac{e^{-rt}}{2}g''(\hat{P}_t)\mathrm{d}\langle\hat{P}_t\hat{P}_t\rangle \\
&= e^{-rt}\Big[-rg(\hat{P}_t) + \hat{\mu}g'(\hat{P}_t) + \frac{\sigma^2}{2}g''(\hat{P}_t)\Big]\mathrm{d}t + e^{-rt}g'(\hat{P}_t)\sigma\mathrm{d}\mathcal{B} - e^{-rt}g'(\hat{P}_t)\mathrm{dV_t} \quad \Big/ (4) \Rightarrow \\
&= e^{-rt}\mathrm{d}t + e^{-rt}g'(\hat{P}_t)\sigma\mathrm{d}\mathcal{B} - e^{-rt}g'(\hat{P}_t)\mathrm{dV_t} \\
&\Rightarrow \\
e^{-r\tau}g(0) &= \int_o^\tau e^{-rt}\mathrm{d}t + \int_0^\tau e^{-rt}g'(\hat{P}_t)\sigma(u)\mathrm{d}\mathcal{B} - \int_0^\tau e^{-rt}g'(\hat{P}_t)\mathrm{dV_t} + g(\hat{P}_0) \\
&= g(\hat{P}_0) + \frac{1}{r} - \frac{e^{-r\tau}}{r} + \int_0^\tau e^{-rt}g'(\hat{P}_t)\sigma(u)\mathrm{d}\mathcal{B} - \int_0^\tau e^{-rt}g'(\hat{P}_t)\mathrm{dV_t} \quad \Big/ \mathbb{E}()
\end{aligned}$$

$$g(0)\mathbb{E}[e^{-r\tau}] = g(\hat{P}_0) + \frac{1}{r} - \frac{\mathbb{E}[e^{-r\tau}]}{r} + \mathbb{E}[\int_0^\tau e^{-rt}g'(\hat{P}_t)\mathrm{dV_t}] \tag{6}$$

Now we set the border conditions on $g(\hat{P})$ as:

$$g(0) = 1, \ \textit{and} \ g'(P^*) = 0 \tag{7}$$

These border conditions make the integral in (6) equal to zero. This is because dV_t is zero for all $\hat{P}_t$ except $\hat{P}_t = P^*$, value at which we force $g'(\hat{P}_t)$ to be zero, thus making the integrand null for all value of t. We then have:

$$\mathbb{E}[e^{-r\tau}] = g(\hat{P}_0) + \frac{1}{r} - \frac{\mathbb{E}[e^{-r\tau}]}{r} \ \Leftrightarrow$$

$$\mathbb{E}[e^{-r\tau}] = \frac{1 + rg(\hat{P}_0)}{1 + r} \tag{8}$$

Where $g(\hat{P})$ is the solution to differential equation (4) with border conditions (7). Now we can write the fixed point equation (3) as:

$$\rho = \frac{1 + r}{1 - g(\hat{P}_0)} \tag{9}$$

with:

$$g(\hat{P}_0) = A_1 e^{\alpha^+ \hat{P}_0} + A_2 e^{\alpha^- \hat{P}_0} - \frac{1}{r}$$

where:

$$\alpha^+ = \frac{-\hat{\mu} + \sqrt{\hat{\mu}^2 + 2r\sigma}}{\sigma^2}, \qquad \alpha^- = \frac{-\hat{\mu} - \sqrt{\hat{\mu}^2 + 2r\sigma}}{\sigma^2}, \qquad P^* = \frac{ln([\frac{\alpha^-}{\alpha^+}]^2)}{\alpha^+ - \alpha^-},$$
$$A_1 = \frac{(1+r)\alpha^- e^{\alpha^- P^*}}{r(\alpha^- e^{\alpha^- P^*} - \alpha^+ e^{\alpha^+ P^*})}, \quad A_2 = \frac{-(1+r)\alpha^+ e^{\alpha^+ P^*}}{r(\alpha^- e^{\alpha^- P^*} - \alpha^+ e^{\alpha^+ P^*})}$$

5.3 Calculation of Interest Rate and Optimal Leverage

The interest rate ρ^* that satisfies equation (9) will depend on the amount of debt (D), the amount of initial equity (P_0), and the risk-free interest rate (r). Equation (3) tells us that in the best case scenario, with $\tau = \infty$, the value of ρ^* will be the same as the risk-free rate (r). This is the minumum value that ρ^* could possibly have. ρ^* is also smaller than μ/D. This is because a higher rate would make the value of $\hat{\mu}$ negative (which means the firm would not accept the loan), so we have two values that provide a minimum and maximum barrier that enable us to calculate the value of ρ^*. To do this, we calculate both sides of equation (9) for values of $\rho \in (r, \mu/D)$ until both sides have the same value. Once we have a way to calculate $\rho^*(P_0, D, r)$ we can also calculate the optimal amount of debt (D^*) the firm should ask the bank for. For each D for which there exists an associated value of ρ^* we can now calculate the value of $W(P_0 + D) - W(P_0)$ (The value added by debt), and look for it's maximum.

5.4 Variable Normalization

In this section we introduce some variable changes that will simplify the problem, enabling us to make an in-depth analysis with less variables to consider. The original problem was:

$$\begin{aligned} \mathrm{d}\hat{P}_t &= (\mu - \rho D)\mathrm{d}t + \sigma \mathrm{d}\mathcal{B}_t - \mathrm{dV_t}, \quad where: \\ \hat{P}_t &= P_t + D \\ W(P_t) &= \mathbb{E}(\int_t^\tau e^{-rs}\mathrm{dV_t}) \end{aligned} \tag{10}$$

Now we define:

$$\begin{aligned} s &= \left(\frac{\mu}{\sigma}\right)^2 t, & \tilde{P}_s &= \frac{\mu}{\sigma^2} P_{\frac{\sigma^2 s}{\mu^2}}, & \tilde{V}_s &= \frac{\mu}{\sigma^2} V_{\frac{\sigma^2 s}{\mu^2}}, \\ \tilde{D} &= \frac{\mu}{\sigma^2} D, & \tilde{\mathcal{B}}_s &= \frac{\mu}{\sigma} \mathcal{B}_{\frac{\sigma^2 s}{\mu^2}}, & \tilde{\rho} &= \frac{\sigma^2}{\mu^2}\rho, \\ \tilde{\tau} &= \frac{\sigma^2}{\mu^2}\tau, & \tilde{r} &= \frac{\sigma^2}{\mu^2} r, & \tilde{W}(\tilde{P}_s) &= \frac{\mu}{\sigma^2} W(\hat{P}_{\frac{\sigma^2 s}{\mu^2}}) \end{aligned} \tag{11}$$

(10) & (11) $\Rightarrow$

$$\begin{aligned} \mathrm{d}\check{P}_s &= (1-\tilde{\rho}\tilde{D})\mathrm{d}s + \mathrm{d}\tilde{\mathfrak{B}}_s - \mathrm{d}\tilde{\mathrm{V}}_s, \\ \check{P}_t &= \tilde{P}_t + \tilde{D} \\ \tilde{W}(\tilde{P}_s) &= \mathbb{E}\left(\int_s^{\tilde{\tau}} e^{-\tilde{r}s'}\mathrm{d}\tilde{\mathrm{V}}_{s'}\right) \end{aligned} \tag{12}$$

With these new, normalized variables we can write problem (12). This problem is analog to problem (10), for the special case $\mu = \sigma = 1$. Notice that $\tilde{\mathfrak{B}}_s$ still is a standard Brownian motion, so the problem really is the same as before, only now we have just 2 variables $(\tilde{r}, \tilde{P}_0)$ that affect the value of the optimal debt $(\tilde{D}^*)$.

5.5 Numerical Results

In this section we conduct a series of numerical simulations in order to understand how much leverage is optimum for a firm and how this leverage can affect the expected value, lifetime expectancy, and cost of capital. Tables 1 through 4, found at the end of this subsection, indicate the ideal leverage ratio, the value added by the optimal debt per unit of initial equity, the risk premium charged by the bank, and finally the expected time-to-default multiplier for different values of $\tilde{r}$ and $\tilde{P}_0$. With the variable normalization introduced earlier, it is possible to use these tables to find out the optimal values for any firm.

To illustrate the use of the tables we will provide an example. Suppose we have a firm that has $P_0 = 2$ initial equity and the risk-free rate is 11.25%. Let the drift and volatility of the revenue stream of the firm be $\mu = 1$ and $\sigma = 2$, respectively. The normalized initial equity of the firm $\tilde{P}_0$ is given by $\frac{1}{2^2} * 2 = 0.5$. The normalized risk free rate $\tilde{r}$ is given by $\frac{2^2}{1^2} * 0.1125 = 0.45$. With these values we can look up the information in the tables. In table 1 we can see the firm's optimal leverage ratio is 0.4. This means the firm should ask for a long term loan of $D = 0.4 * P_0 = 0.4 * 2 = 0.8$. In table 2 we can see that this debt will add 0.08 extra value per unit of initial equity. That is a total of $0.08 * P_0 = 0.08 * 2 = 0.16$ extra added value. In table 3 we can read the risk premium charged by the bank is 0.71, which means that the interest rate on the debt is equal to $11.25\% + 0.71 * 11.25\% = 19.24\%$. Finally in table 4 we can see that the expected time to default is going to be multiplied by 0.81 compared to the same firm without debt, so the firm is actually supposed to reach bankruptcy faster when leveraged. This may be counterintuitive, since a shorter time to bankruptcy is normally associated with a smaller present value. As we will explain, this is not always the case, specially when valuation of the firm is risk neutral.

Tables 1, 2, and 3 provide very intuitive results. In table 1 we can see that the optimal leverage diminishes when the value of P_0 increases, which is reasonable since the money from the bank will be less useful if the firm has more money of its own. The amount of leverage also decreases when the risk free rate increases. This is because when the cost of debt rises it becomes less attractive to incur in a big loan.

In table 2 we can see that the value added by debt per unit of initial equity behaves similarly to the optimal leverage. Debt provides the most value when the firm has a small starting capital and the risk free rate is small. This is because in this case debt can help both shorten the time needed to start distributing dividends and at the same time increase the expected time to default (see table 4).

Table 3 shows that the risk premium charged by the bank is very sensitive to the risk-free rate. For small values of risk-free rate, the value of the risk premium is small. It varies very little for different values of initial equity, increasing slightly for smaller amounts of initial equity. As the risk-free rate increases, the value of the risk premium becomes much more sensitive to the value of $\tilde{P}_0$, reaching very high values when $\tilde{P}_0$ is small. This is due to the fact that the firm becomes much more vulnerable to bankruptcy if the cost of debt rises when it has a small initial capital.

Finally table 4 shows us how the expected time to default changes with the acquisition of debt. For a fixed risk-free rate we can see that the value of the multiplier is convex with respect to $\tilde{P}_t$. When $\tilde{P}_t$ is big the optimal leverage is small so the time to default changes very little, diminishing slightly. The debt adds value because it shortens the expected time to distribute dividends, but the smaller effective drift ($\hat{\mu} = \mu - \rho D$), actually shortens the expected time to default. As P_t decreases the leverage increases and the expected time to default diminishes more, up to a certain point at which is starts increasing again. At this point there are two opposing forces that balance each other out. On one hand we have the "cushion" provided by the extra cash and on the other hand we have the effective drift ($\hat{\mu}$) that diminishes with debt. For small values of the risk-free rate the value of the multiplier can even become > 1 for very small quantities of initial wealth. In this case the "cushion" is far more important than the smaller effective drift and the loan can actually help the firm stay on business for a longer time.

$\tilde{P}_0$ \ $\tilde{r}$	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	1.05	1.15	1.25
0.250	37.65	7.99	2.99	1.29	0.61	0.31	0.17	0.09	0.05	0.03	0.02	0.01	0.01
0.375	19.13	3.79	1.31	0.52	0.22	0.10	0.05	0.02	0.01	0.00	0.00	0.00	0.00
0.500	10.90	1.98	0.61	0.21	0.08	0.03	0.01	0.00	0.00				
0.625	6.60	1.08	0.29	0.08	0.02	0.00	0.00						
0.750	4.14	0.60	0.13	0.02	0.00								
0.875	2.65	0.33	0.05	0.00									
1.000	1.72	0.17	0.01										
1.125	1.13	0.08	0.00										
1.250	0.74	0.04											
1.375	0.48	0.01											
1.500	0.31	0.00											
1.625	0.20												
1.750	0.12												
1.875	0.07												
2.000	0.04												
2.125	0.02												
2.250	0.01												
2.375	0.00												
2.500													

Table 1: Value Added by Optimal Leverage $\left(\frac{[W(P_0+D)-W(P_0)]}{P_0}\right)$

$\tilde{P}_0$ \ $\tilde{r}$	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	1.05	1.15	1.25
0.250	9.72	5.24	3.01	1.78	1.13	0.75	0.52	0.39	0.28	0.21	0.16	0.12	0.09
0.375	6.16	3.20	1.81	1.07	0.66	0.43	0.29	0.19	0.12	0.08	0.04	0.02	0.00
0.500	4.36	2.17	1.19	0.67	0.40	0.23	0.12	0.06	0.01	0.00	0.00	0.00	
0.625	3.29	1.56	0.78	0.41	0.21	0.08	0.00	0.00	0.00				
0.750	2.56	1.13	0.53	0.23	0.07	0.00							
0.875	2.04	0.81	0.34	0.09	0.00								
1.000	1.66	0.59	0.19	0.00									
1.125	1.35	0.42	0.07										
1.250	1.11	0.28	0.00										
1.375	0.90	0.16											
1.500	0.74	0.06											
1.625	0.60	0.00											
1.750	0.48												
1.875	0.38												
2.000	0.29												
2.125	0.22												
2.250	0.14												
2.375	0.08												
2.500	0.02												

Table 2: Optimal Leverage Ratio (D/P_0)

$\tilde{r}$ / $\tilde{P}_0$	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	1.05	1.15	1.25
0.250	0.04	0.23	0.53	0.88	1.19	1.44	1.62	1.76	1.86	1.94	2.00	2.05	2.09
0.375	0.04	0.22	0.46	0.69	0.89	1.03	1.14	1.21	1.27	1.31	1.34	1.37	0.00
0.500	0.04	0.20	0.40	0.57	0.71	0.80	0.86	0.91	0.94	0.00	0.00	0.00	
0.625	0.04	0.19	0.35	0.49	0.58	0.65	0.69	0.00	0.00				
0.750	0.04	0.18	0.32	0.42	0.49	0.00	0.00						
0.875	0.04	0.17	0.28	0.37	0.00								
1.000	0.04	0.16	0.26	0.00									
1.125	0.04	0.15	0.24										
1.250	0.04	0.14	0.00										
1.375	0.04	0.13											
1.500	0.04	0.13											
1.625	0.04	0.00											
1.750	0.04												
1.875	0.04												
2.000	0.03												
2.125	0.03												
2.250	0.03												
2.375	0.03												
2.500	0.03												

Table 3: Risk-Premium of Optimal Debt ($[\rho - r]/r$)

$\tilde{r}$ / $\tilde{P}_0$	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	1.05	1.15	1.25
0.250	1.83	1.23	0.97	0.86	0.81	0.79	0.80	0.80	0.82	0.84	0.86	0.88	0.89
0.375	1.39	0.99	0.84	0.79	0.79	0.81	0.82	0.86	0.89	0.92	0.95	0.98	1.00
0.500	1.18	0.89	0.80	0.79	0.81	0.85	0.90	0.94	0.99	1.00	1.00	1.00	
0.625	1.07	0.84	0.81	0.83	0.87	0.93	1.00	1.00	1.00				
0.750	1.00	0.83	0.82	0.87	0.95	1.00							
0.875	0.96	0.83	0.85	0.94	1.00								
1.000	0.93	0.85	0.90	1.00									
1.125	0.92	0.87	0.96										
1.250	0.91	0.90	1.00										
1.375	0.91	0.93											
1.500	0.91	0.97											
1.625	0.92	1.00											
1.750	0.92												
1.875	0.93												
2.000	0.94												
2.125	0.95												
2.250	0.97												
2.375	0.98												
2.500	0.99												

Table 4: Expected Time to Default Multiplier $\left(\frac{\mathbb{E}[\tau(P_0+D)]}{\mathbb{E}[\tau(P_0)]}\right)$

6 Summary and Further Research

This paper provides a model for making optimal coordinated financial and operational decisions that maximizes the expected present value of all future dividend payments. Using a continous-time framework, we model the available cash as a brownian process whose parameters can be modified through changes in the operational strategy.

First we analyze the problem of choosing between all available operational strategies and deciding on the dividend distribution rate for a firm with pure equity. This problem was studied independently of Radner and Shepp (1996)[14] who also addressed the same problem and arrived to the same conclusions. By means of the Hamilton-Jacobi-Bellman equations we are able to formulate an analytical solution to the problem. We prove that there exists an analog to the efficient frontier of a stock market and that only operational strategies belonging to this efficient frontier must be implemented. The firm should choose an initial operational strategy according to its initial wealth and then implement riskier operational strategies as its wealth increases. Dividends should follow a bang-bang distribution policy where no dividends are given out until a certain equity threshold is attained. All excess above this equity level should be distributed to stockholders.

Afterwards we study a static model in which the operational strategy can't be modified, and the firm must decide on the size of a long-term loan. This translates into a fixed point problem in which the firm has to know the interest rate to decide on the loan and the bank has to know the size of the loan to decide on the interest rate. We then conduct a numerical example and calculate the optimal amount of leverage, the value added by leverage, the risk premium of the optimal debt and the expected time-to-default multiplier of leverage. Our results show that for firms with limited capital, leverage can both add value and help stay in business for a longer time if the risk free rate is low enough. Firms with greater amounts of initial equity may also help create value by acquiring a small amount of debt, although this will actually shorten their expected lifetime.

There are many possible directions for future research. The most immediate extension of the problem studied would be to analyze a dynamic model where both debt and operational strategy may be modified at any point in time. This would allow modeling of growing firms that take on short-term loans to finance their expansion. Also it would be interesting to study how real operational decisions could be linked to the parameters of the diffusion used to model the firm's available cash. Clearly operational variables have an impact on the expected return and volatility of the firm's cash flow, but it is not clear how real operational variables such as inventory order quantity, for example, translate into the diffusion's parameters or even if the chosen diffusion is the best suited. Finally, it would also be interesting to study a model where the interest rate charged by the bank not only depends on the company's risk profile but also has a stochastic component to emulate changing economic cycles. Since debt in our model is long-term, it seems like it would be more appropriate to have a variable interest rate. Although results presented in this paper are mainly academic and overlook many of the real-world variables that come into play when making leverage/operational/dividend related decisions, they do provide an insight into the possibility of creating value through the coordination of finance and operations in a real-world environment.

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Appendix A

This sections provides the complete procedure involved in solving equation (1) presented in the preliminary results section. Let i^* be the control that attains the maximum in the expression

$$\mu_i W'(P) + \tfrac{\sigma_i^2}{2} W''(P), \quad \forall P \in [P_1, P_2]$$

We then have that:

$$rW(P) = \mu_{i^*} W'(P) + \tfrac{\sigma_{i^*}^2}{2} W''(P), \quad \forall P \in [P_1, P_2]$$

The solution to this differential equation is given by:

$$\begin{aligned} W(P) &= Ae^{\alpha^+ P} + Be^{\alpha^- P} \\ &Where: \\ \alpha^+ &= \frac{-\mu_{i^*} + \sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} \\ \alpha^- &= \frac{-\mu_{i^*} - \sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} \end{aligned}$$

When $P_1 = 0$ we have that

$$A = -B$$

And for $P_2 = P^*$ we have

$$\begin{aligned} W'(P^*) &= 1 \Leftrightarrow \\ A\alpha^+ e^{\alpha^+ P^*} + B\alpha^- e^{\alpha^- P^*} &= 1 \end{aligned}$$

We still have to specify a criteria for the election of the optimal operational strategy i^*. We have n possible values of i to choose from. Each i is associated to a pair $(\mu_i, \frac{\sigma_i^2}{2})$ and the function we wish to maximize is the scalar product between vectors $(\frac{\sigma^2}{2}, \mu)$ and $(W''(P), W'(P))$.

When P=0 we have that:

$$\begin{aligned} W(0) &= 0 \\ W'(0) &= A(\alpha^+ - \alpha^-) = \frac{2A\sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} \\ W''(0) &= A({\alpha^+}^2 - {\alpha^-}^2) = \frac{-4A\mu_{i^*}\sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^4} & \Rightarrow \\ i^* &= argmax\left(\frac{2A\mu_i\sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^2} - \frac{2A\sigma_i^2\mu_{i^*}\sqrt{\mu_{i^*}^2 + 2r\sigma_{i^*}^2}}{\sigma_{i^*}^4}\right) & \Leftrightarrow \\ i^* &= argmax\left(\sigma_i^2\left[\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i^*}}{\sigma_{i^*}^2}\right]\right) \end{aligned}$$

Therefore, thanks to the continuity of $W(P)$ we deduce that there exists an interval $[0, P_1]$ inside of which the vector $(\frac{\sigma^2}{2}, \mu)$ with the biggest ratio $\frac{\mu}{\sigma^2}$ will maximize the expression

$\left(\sigma_i^2\left[\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i^*}}{\sigma_{i^*}^2}\right]\right)$. This is due to the fact that the value of the expression when the maximum is attained is equal to 0, therefore the expression has to be negative for all i that does not maximize the expression. This is only achieved when i^* is the one with the highest value of $\frac{\mu_i}{\sigma_i^2}$.

We have found a way to determine the best operational strategy i when the value of P is close to 0. Now we wish to determine the interval in which this control i_1^* is optimum. Moreover, we wish to determine all controls that will be optimum and their respective intervals.

Let I be the set of all possible controls i from which we can choose. Let $\bar{I}$ be the set of all controls that will be optimum in some interval. We will show that $\bar{I}$ can be defined by:

$$\bar{I} = \left\{ i \in I / \exists w = (w_1, w_2) \neq 0, w_1 \leq 0, w_2 \geq 0, (\sigma_i^2, \mu_i) = argmax\{w_1\sigma^2 + w_2\mu\} \right\}$$

Graphically we can clearly see how $\bar{I}$ is analog to the efficient frontier of a stock market. In figure 1 all the points of I are represented by * and those belonging to $\bar{I}$ are connected by a line:

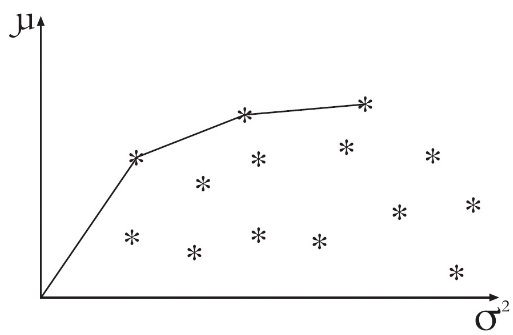

Figure 1: (σ^2, μ) Point Cloud and its Efficient Frontier

In $P = 0$ the vector $(\frac{\sigma^2}{2}, \mu)$ with the steepest slant is the optimum and it is perpendicular to the vector $(W''(P), W'(P))$. As the value of P increases the vector $(W''(P), W'(P))$ smoothly turns clockwise (this is demonstrated further ahead) and the control i_1^* is optimum until $((\frac{\sigma_{i_1^*}^2}{2}, \mu_{i_1^*}) - (\frac{\sigma_{i_2^*}^2}{2}, \mu_{i_2^*})) * (W''(P_1), W'(P_1)) = 0$. It is clear that the control i_2^* is that one associated to the vector $(\frac{\sigma^2}{2}, \mu)$ with the second biggest slant belonging to $\bar{I}$. The point P_1 in which we must switch from control i_1^* to control i_2^* is perfectly determined because it will not depend on the constant A. Indeed it can be isolated from the equation and we obtain:

$$P_1 = \frac{1}{\alpha_1^+ - \alpha_1^-} \ln\left[\left(\frac{\alpha_1^-}{\alpha_1^+}\right)\left(\frac{\alpha_1^+(\frac{\sigma_{i_1^*}^2 + \sigma_{i_2^*}^2}{2}) + \mu_{i_1^*} - \mu_{i_2^*}}{\alpha_1^-(\frac{\sigma_{i_1^*}^2 + \sigma_{i_2^*}^2}{2}) + \mu_{i_1^*} - \mu_{i_2^*}}\right)\right]$$

We then have that:

$$W(P) \;= A_2 e^{\alpha_2^+ P} + B_2 e^{\alpha_2^- P}, \quad \forall P \in [P_1, P_2]$$

The constants A_2 y B_2 are determined in function of the constant A by the equations of continuity of W y W'. In fact we have that:

$$\begin{aligned} A_{i+1}e^{\alpha^+_{i+1}P_i} &= \frac{W'_i - W_i\alpha^-_{i+1}}{\alpha^+_{i+1} - \alpha^-_{i+1}} \\ B_{i+1}e^{\alpha^-_{i+1}P_i} &= \frac{\alpha^+_{i+1}W_i - W'_i}{\alpha^+_{i+1} - \alpha^-_{i+1}} \end{aligned} \tag{A1}$$

This way we can calculate the value of P_2 for which we should change to control i_3^*. The method can be applied successively until we reach the value P^* which is determined by $W''(P^*) = 0$. Once P^* has been found we can determine the value of A that makes $W'(P^*) = 1$. The method exposed here allows the determination of optimal control and the intervals in which they should be used. We have only left to demonstrate that the vector $(W''(P), W'(P))$ indeed turns clockwise as the value of P increases and that W'' is continuous so that the method exposed here is valid.

Let θ be the angle formed by the vectors $(W''(P), W'(P))$ and $(-1, 0)$. To simplify notation let $Ae^{\alpha^+P} = X$ and $Be^{\alpha^-P} = Y$. We can write that

$$-\tan\theta = \frac{W'(P)}{W''(P)} \bigg/ \frac{d}{dP}$$

$$\begin{aligned} -\frac{d\tan\theta}{d\theta}\frac{d\theta}{dP} &= \frac{W''^2(P) - W'(P)W'''(P)}{W''^2(P)} \\ &= \frac{(\alpha^{+2}X + \alpha^{-2}Y)^2 - (\alpha^+X + \alpha^-Y)(\alpha^{+3}X + \alpha^{-3}Y)}{(\alpha^{+2}X + \alpha^{-2}Y)^2} \\ &= \frac{\alpha^+\alpha^-XY(2\alpha^+\alpha^- - \alpha^{+2} - \alpha^{-2})}{(\alpha^{+2}X + \alpha^{-2}Y)^2} \\ &= \frac{-\alpha^+\alpha^-XY(\alpha^+ - \alpha^-)^2}{(\alpha^{+2}X + \alpha^{-2}Y)^2} \Rightarrow \end{aligned}$$

$$\frac{d\theta}{dP} = \left(\frac{\alpha^+\alpha^-(\alpha^+ - \alpha^-)^2}{(\alpha^{+2}X + \alpha^{-2}Y)^2}\right)\left(\left[\frac{d\tan\theta}{d\theta}\right]^{-1}\right)\left(XY\right)$$

The first term is negative and the second term is positive so $\frac{d\theta}{dP}$ is positive if and only if XY is negative. As we will now show, XY is indeed always negative.

For easier notation purposes let us define $X_i = A_ie^{\alpha^+{}_iP_{i-1}}$ and $Y_i = B_ie^{\alpha^-{}_iP_{i-1}}$. For i=1 we have $X_1Y_1 = -{A_1}^2$ which is negative. Let us supose that X_iY_i is negative. We will now show that $X_{i+1}Y_{i+1}$ is also negative. We know that

$$\begin{aligned} X_{i+1} &= \frac{W'_i - W_i\alpha^-_{i+1}}{\alpha^+_{i+1} - \alpha^-_{i+1}} \\ Y_{i+1} &= \frac{\alpha^+_{i+1}W_i - W'_i}{\alpha^+_{i+1} - \alpha^-_{i+1}} \end{aligned}$$

So

$$
\begin{aligned}
X_{i+1}Y_{i+1} &= \frac{(\alpha_{i+1}^{+}W_i - W_i')(W_i' - W_i\alpha_{i+1}^{-})}{(\alpha_{i+1}^{+} - \alpha_{i+1}^{-})^2} \\
&= \frac{1}{(\alpha_{i+1}^{+} - \alpha_{i+1}^{-})^2}[\alpha_i^{+}X_i + \alpha_i^{-}Y_i - \alpha_{i+1}^{-}(X_i+Y_i)][\alpha_{i+1}^{+}(X_i+Y_i) - \alpha_i^{+}X_i - \alpha_i^{-}Y_i] \\
&= \frac{1}{(\alpha_{i+1}^{+} - \alpha_{i+1}^{-})^2}\Big(X_i^2(\alpha_i^{+} - \alpha_{i+1}^{-})(\alpha_{i+1}^{+} - \alpha_i^{+}) + Y_i^2(\alpha_{i+1}^{+} - \alpha_i^{-})(\alpha_i^{-} - \alpha_{i+1}^{-}) \\
&+ X_iY_i[(\alpha_i^{+} + \alpha_i^{-})(\alpha_{i+1}^{+} + \alpha_{i+1}^{-}) - 2(\alpha_{i+1}^{+}\alpha_{i+1}^{-} + \alpha_i^{+}\alpha_i^{-})]\Big)
\end{aligned}
$$

Now lets examine the signs of the terms. The first term $\frac{1}{(\alpha_{i+1}^{+} - \alpha_{i+1}^{-})^2}$ is positive. The second term can be divided into three terms that will be examined separately. The first is $X_i^2(\alpha_i^{+} - \alpha_{i+1}^{-})(\alpha_{i+1}^{+} - \alpha_i^{+})$. This term is negative if and only if $(\alpha_{i+1}^{+} - \alpha_i^{+})$ is negative.

$$
\begin{aligned}
(\alpha_{i+1}^{+} - \alpha_i^{+}) < 0 && \Leftrightarrow \\
0 > \sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}} - \frac{\mu_{i+1}}{\sigma_{i+1}^2} + \frac{\mu_i}{\sigma_i^2} - \sqrt{\left(\frac{\mu_i}{\sigma_i^2}\right)^2 + \frac{2r}{\sigma_i^2}} && \Leftrightarrow \\
\sqrt{\left(\frac{\mu_i}{\sigma_i^2}\right)^2 + \frac{2r}{\sigma_i^2}} > \sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}} + \left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right) && \Leftrightarrow \\
\frac{r}{\sigma_i^2} > \left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{r}{\sigma_{i+1}^2} - \frac{\mu_i}{\sigma_i^2}\frac{\mu_{i+1}}{\sigma_{i+1}^2} + \sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}}\left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right) && \Leftrightarrow \\
0 > r\left(\frac{1}{\sigma_{i+1}^2} - \frac{1}{\sigma_i^2}\right) + \left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)\left(\sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right) && \Leftrightarrow \\
r > \frac{\left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)}{\left(\frac{1}{\sigma_i^2} - \frac{1}{\sigma_{i+1}^2}\right)}\left(\sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)
\end{aligned}
$$

This is true for every $r > 0$. To prove this we can think of each side of the inequality as a function of r. For $r = 0$ both sides of the inequality are equal to 0. The left side has a slant equal to 1 and the right side's slant is given by $\frac{\left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)}{\left(\frac{1}{\sigma_i^2} - \frac{1}{\sigma_{i+1}^2}\right)\sigma_{i+1}^2\sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}}}$ This slant is decreasing with respect to r so it's maximum value is attained for $r = 0$. Since the maximum value of the slant is given by $\frac{\left(\frac{\mu_i}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)}{\left(\frac{\mu_{i+1}}{\sigma_i^2} - \frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)} < 1$ we can conclude that both functions of r start at the same value but the right side has a slant always smaller than the left side so it will be smaller than the left side for all $r > 0$. The inequality holds true for every $r > 0$, which lets us finally conclude that the term $X_i^2(\alpha_i^+ - \alpha_{i+1}^-)(\alpha_{i+1}^+ - \alpha_i^+)$ is negative. Now lets examine the sign of the second term $Y_i^2(\alpha_{i+1}^+ - \alpha_i^-)(\alpha_i^- - \alpha_{i+1}^-)$. This term is negative if and only if $(\alpha_i^- - \alpha_{i+1}^-) < 0$. Indeed

$$(\alpha_i^- - \alpha_{i+1}^-) < 0 \Leftrightarrow$$

$$\left[\sqrt{\left(\frac{\mu_{i+1}}{\sigma_{i+1}^2}\right)^2 + \frac{2r}{\sigma_{i+1}^2}} - \sqrt{\left(\frac{\mu_i}{\sigma_i^2}\right)^2 + \frac{2r}{\sigma_i^2}}\right] + \left(\frac{\mu_{i+1}}{\sigma_{i+1}^2} - \frac{\mu_i}{\sigma_i^2}\right) < 0 \Leftrightarrow$$

Which is true for all $r > 0$. Now the final term to be analyzed is $X_iY_i[(\alpha_i^+ + \alpha_i^-)(\alpha_{i+1}^+ + \alpha_{i+1}^-) - 2(\alpha_{i+1}^+\alpha_{i+1}^- + \alpha_i^+\alpha_i^-)]\Big)$. This term is negative if $(\alpha_i^+ + \alpha_i^-)(\alpha_{i+1}^+ + \alpha_{i+1}^-) > 0$. This is true because $(\alpha^+ + \alpha^-) = \frac{-2\mu}{\sigma^2}$ so we have $(\alpha_i^+ + \alpha_i^-)(\alpha_{i+1}^+ + \alpha_{i+1}^-) = \frac{4\mu_i\mu_{i+1}}{\sigma_i^2\sigma_{i+1}^2}$ which is positive. Since all three terms are negative, the sum is negative and we have shown that $X_{i+1}Y_{i+1}$ is negative if X_iY_i is negative. By induction we can say that X_iY_i is negative for all i and this proves that $\frac{d\sigma}{dP}$ is positive which is what we set out to prove in the first place.

Now we wil verify that W'' is continuous:

Because of (A1) we can write:

$$\begin{aligned}
W''_{k+1}(P_k) &= (\alpha_{k+1}^+)^2\frac{W'_k - W_i\alpha_{k+1}^-}{\alpha_{k+1}^+ - \alpha_{k+1}^-} + (\alpha_{k+1}^-)^2\frac{\alpha_{k+1}^+W_k - W'_k}{\alpha_{k+1}^+ - \alpha_{k+1}^-} \\
&= \frac{1}{\alpha_{k+1}^+ - \alpha_{k+1}^-}\Big(W'_k((\alpha_{k+1}^+)^2 - (\alpha_{k+1}^-)^2) - W_k\alpha_{k+1}^+\alpha_{k+1}^-(\alpha_{k+1}^+ - \alpha_{k+1}^-)\Big) \\
&= W'_k(\alpha_{k+1}^+ + \alpha_{k+1}^-) - W_k\alpha_{k+1}^+\alpha_{k+1}^- \\
&= \frac{2}{\sigma_{k+1}^2}(rW_k(P_k) - \mu_{k+1}W'_k(P_k))
\end{aligned}$$

On the other hand, because in point P_k we are indifferent between control i_k^* and control i_{k+1}^* we have that:

$$\begin{aligned}
rW_k(P_k) &= \mu_kW'_k(P_k) + \frac{\sigma_k^2}{2}W''_k(P_k) = \mu_{k+1}W'_k(P_k) + \frac{\sigma_{k+1}^2}{2}W''_k(P_k) \Rightarrow \\
W''_k(P_k) &= \frac{2}{\sigma_{k+1}^2}(rW_k(P_k) - \mu_{k+1}W'_k(P_k))
\end{aligned}$$

Which proves that indeed we have $W''_k(P_k) = W''_{k+1}(P_k)$.

Appendix B

This section provides the complete step by step procedure involved in solving the maximization problem presented in section 5.1 :

$$\max_{dV_t} \left\{ \mathbb{E}(\int_t^{\tau} e^{-rs} \mathrm{dV_s}) \right\}$$
$$s.t:$$
$$\mathrm{d}P_t = \mu \mathrm{d}t + \sigma \mathrm{d}\mathcal{B} - \mathrm{dV_t}$$
$$dV_t \geq 0, \ where:$$
$$W(P_t) = \mathbb{E}(\int_t^{\tau} e^{-rs} \mathrm{dV_s}), \quad \tau = inf\{t \geq 0; P_t \leq 0\}, \quad B.C: \ W(0) = 0$$

This is a particular case of the more general problem described in section 2 (Preliminary Results) in which there is only one available operational strategy. From the arguments presented in this section we know that the solution to this problem is given by the following differential equation:

$$rW(P) = \mu W'(P) + \frac{\sigma^2}{2} W''(P), \forall P \in [0, P^*] \tag{B1}$$
$$B.C: W(0) = 0, \ W'(P^*) = 1, \ W''(P^*) = 0$$

The solution to this differential equation is given by:

$$W(P) = Ae^{\alpha^+ P} + Be^{\alpha^- P}$$
$$Where:$$
$$\alpha^+ = \frac{-\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$
$$\alpha^- = \frac{-\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$

To find the values of parameters A, B, and P^* we need only to enforce the border conditions:

$$W(0) = 0 \Leftrightarrow A + B = 0 \Leftrightarrow$$
$$A = -B$$
$$W'(P^*) = 1 \Leftrightarrow A\left(\alpha^+ e^{\alpha^+ P^*} - \alpha^- e^{\alpha^- P^*}\right) = 1 \Rightarrow$$
$$A = \frac{1}{\left(\alpha^+ e^{\alpha^+ P^*} - \alpha^- e^{\alpha^- P^*}\right)}$$

$$
\begin{aligned}
W''(P^*) = 0 \Leftrightarrow \; A\big[(\alpha^+)^2 e^{\alpha^+ P^*} - (\alpha^-)^2 e^{\alpha^- P^*}\big] = 0 & \qquad \Leftrightarrow \\
(\alpha^+)^2 e^{\alpha^+ P^*} = (\alpha^-)^2 e^{\alpha^- P^*} & \qquad \Big/ \frac{1}{(\alpha^+)^2} \\
e^{\alpha^+ P^*} = [\frac{\alpha^-}{\alpha^+}]^2 e^{\alpha^- P^*} & \qquad \Big/ ln() \\
\alpha^+ P^* = ln([\frac{\alpha^-}{\alpha^+}]^2 e^{\alpha^- P^*}) & \qquad \Leftrightarrow \\
\alpha^+ P^* = ln([\frac{\alpha^-}{\alpha^+}]^2) + \alpha^- P^* & \qquad \Big/ - \alpha^- P^* \\
P^*(\alpha^+ - \alpha^-) = ln([\frac{\alpha^-}{\alpha^+}]^2) & \qquad \Big/ \frac{1}{\alpha^+ - \alpha^-} \\
P^* = \frac{ln([\frac{\alpha^-}{\alpha^+}]^2)}{\alpha^+ - \alpha^-} & \qquad \blacksquare
\end{aligned}
$$